Using the martingale property of the discounted stock price we can replace $\widetilde{\mathbb{E}}[S(u)]$ with $e^{r(t-u)}S(t)$ which yields

$$\widetilde{\mathbb{E}}\left[\int_0^T S(u)\,du \,\Big|\, \mathcal{F}(t)\right] = \int_0^t S(u)\,du + S(t)\int_t^T e^{-r(t-u)}\,du.$$

$$= \int_0^t S(u)\,du + \frac{S(t)}{r}\left(e^{r(T-t)} - 1\right)$$

It follows that

$$e^{-r(T-t)}\,\widetilde{\mathbb{E}}\left[\frac{1}{T}\int_0^T S(u)\,du \,\Big|\, \mathcal{F}(t)\right] = e^{-r(T-t)}\frac{1}{T}\int_0^t S(u)\,du + \frac{S(t)}{rT}\left(1-e^{-r(T-t)}\right)$$

and $$v(t,x,y) = y\,\frac{e^{-r(T-t)}}{T} + x\,\frac{1-e^{-r(T-t)}}{rT}.$$

$$\frac{\partial v}{\partial t} = (ry-x)\frac{e^{-r(T-t)}}{T}, \quad \frac{\partial v}{\partial x} = \frac{1-e^{-r(T-t)}}{rT}, \quad \frac{\partial v}{\partial y} = \frac{e^{-r(T-t)}}{T}.$$

$$v(t,0,y) = y\,\frac{e^{-r(T-t)}}{T}.$$

$$v(T,x,y) = y/T.$$

$$\Delta(t) = \frac{1-e^{-r(T-t)}}{rT}.$$

This quantity is non-random, since it only depends on time but not on the current value of $S(t)$ or its history.

To price this call option we create a portfolio process whose value at time T is

$$X(T) = \frac{1}{T}\int_0^T S(u)\,du - K$$

We begin with a non-random function of time $\gamma(t)$, $0 \le t \le T$, which will be number of shares of the risky asset held by our portfolio.
Also assume that $d\gamma(t)\,d\gamma(t) = 0 = d\gamma(t)\,dS(t)$

~~This implies d(~~ We take $\gamma(t)$ to be

$$\gamma(t) = \frac{1}{rT}\left(1 - e^{-r(T-t)}\right), \quad 0 \le t \le T.$$

and $$X(0) = \frac{1}{rT}\left(1 - e^{-rT}\right) S(0) - e^{-rT} K.$$

Then $d\gamma(t)\,d\gamma(t) = 0$ and $d\gamma(t)\,dS(t) = 0$.

This implies that

$$d(\gamma(t)S(t)) = \gamma(t)\,dS(t) + S(t)\,d\gamma(t).$$

$$d\left(e^{r(T-t)}\gamma(t)S(t)\right) = e^{+r(T-t)}\,d(\gamma(t)S(t)) - r e^{r(T-t)}\gamma(t)S(t)\,dt$$

$$= e^{r(T-t)}\gamma(t)\,dS(t) + e^{r(T-t)}S(t)\,d\gamma(t) - r e^{r(T-t)}\gamma(t)S(t)\,dt$$

$$\Rightarrow e^{r(T-t)}\gamma(t)\left(dS(t) - rS(t)\,dt\right) = d\left(e^{r(T-t)}\gamma(t)S(t)\right) - e^{r(T-t)}S(t)\,d\gamma(t)$$

The portfolio value evolves according to the equation

$$dX(t) = \gamma(t)\,dS(t) + r\left(X(t) - \gamma(t)S(t)\right)dt$$

$$= rX(t)\,dt + \gamma(t)\left(dS(t) - rS(t)\,dt\right)$$

$$\Rightarrow d\left(e^{r(T-t)}X(t)\right) = -r e^{r(T-t)}X(t)\,dt + e^{r(T-t)}\,dX(t)$$

$$= e^{r(T-t)}\left(dX(t) - rX(t)\,dt\right)$$

$$= e^{r(T-t)}\gamma(t)\left(dS(t) - rS(t)\,dt\right)$$

$$= d\left(e^{r(T-t)}\gamma(t)S(t)\right) - e^{r(T-t)}S(t)\,d\gamma(t).$$

At time zero, we buy $\frac{1}{rT}(1-e^{-rT})$ share of the risky asset. which costs $\frac{1}{rT}(1-e^{-rT})S(0)$. But our initial capital is insufficient to do this, and we borrow $e^{-rT}K$ from money market account. For $0 \le t \le T$ our

$$\Delta(t) = \gamma(t) = \frac{1}{Tr}\left(1-e^{-r(T-t)}\right)$$

$$\Rightarrow d\gamma(t) = -\frac{1}{T}e^{-r(T-t)}$$

Now
$$e^{r(T-t)}X(t) = e^{rT}X(0) + \int_0^t d\left(e^{r(T-u)}\gamma(u)S(u)\right) - \int_0^t e^{r(T-u)}S(u)\,d\gamma(u)$$

$$= \frac{1}{rT}e^{rT}(1-e^{-rT})S(0) - K + e^{r(T-t)}\gamma(t)S(t)$$
$$- \frac{1}{rT}e^{rT}\left(1-e^{-rT}\right)S(0) + \frac{1}{T}\int_0^t S(u)\,du.$$

$$= -K + e^{r(T-t)}\gamma(t)S(t) + \frac{1}{T}\int_0^t S(u)\,du.$$

Therefore
$$X(t) = \frac{1}{rT}\left(1-e^{-r(T-t)}\right)S(t) + e^{-r(T-t)}\frac{1}{T}\int_0^t S(u)\,du$$
$$- e^{-r(T-t)}K, \quad 0 \le t \le T. \quad --- (*)$$

In particular
$$X(T) = \frac{1}{T}\int_0^T S(u)\,du - K.$$

as desired, and $V(T) = X^+(T) = \max\{X(T), 0\}$.

The price of the Asian call at time t is

$$V(t) = \widetilde{\mathbb{E}}\left[e^{-r(T-t)}V(T) \,\middle|\, \mathcal{F}(t)\right] = \widetilde{\mathbb{E}}\left[e^{-r(T-t)}X^+(T) \,\middle|\, \mathcal{F}(t)\right].$$

Let us define

$$Y(t) = \frac{X(t)}{S(t)}.$$

$$d\left(\frac{1}{S(t)}\right) = -\frac{1}{(S(t))^2} dS(t) + \frac{1}{(S(t))^3} dS(t)\, dS(t)$$

$$= -\frac{1}{[S(t)]^2}\left[r S(t) dt + \sigma S(t)\, d\widetilde{W}(t)\right] + \frac{1}{S^3(t)} \sigma^2 S^2(t)\, dt$$

$$= \frac{+1}{S(t)}\left[(-r + \sigma^2) dt - \sigma\, d\widetilde{W}(t)\right]$$

$$dX(t) = r X(t) dt + \gamma(t)\left(dS(t) - r S(t)\right) dt$$

$$= r X(t) dt + \gamma(t) \sigma S(t)\, d\widetilde{W}(t)$$

$$dY(t) = \frac{dX(t)}{S(t)} + X(t)\, d\left(\frac{1}{S(t)}\right) + dX(t)\, d\left(\frac{1}{S(t)}\right).$$

$$= \frac{1}{S(t)}\left[\cancel{r X(t)} dt + \gamma(t) \sigma S(t)\, d\widetilde{W}(t)\right]$$

$$+ \frac{X(t)}{S(t)}\left[(\sigma^2 - \cancel{r}) dt - \sigma\, d\widetilde{W}(t)\right] - \frac{\sigma^2}{S(t)} \gamma(t) S(t)\, dt$$

$$= \gamma(t) \sigma\, d\widetilde{W}(t) - \sigma^2 \gamma(t)\, dt + Y(t)\sigma\left[\sigma dt - d\widetilde{W}(t)\right]$$

$$= -\gamma(t)\sigma\left[\sigma dt - d\widetilde{W}(t)\right] + Y(t)\sigma\left[\sigma dt - d\widetilde{W}(t)\right]$$

$$= \sigma\left[\gamma(t) - Y(t)\right]\left[d\widetilde{W}(t) - \sigma dt\right].$$

$Y(t)$ is not a martingale under $\widetilde{\mathbb{P}}$. We set

$$\widetilde{W}^S(t) = \widetilde{W}(t) - \sigma t$$

then $dY(t) = \sigma\left[\gamma(t) - Y(t)\right] d\widetilde{W}^S(t)$. — (1)

According to Girsanov's theorem $\widetilde{W}^S(t)$, $0 \le t \le T$ is a B.M under the probability measure $\widetilde{\mathbb{P}}^S$ defined by

$$\widetilde{\mathbb{P}}^S(A) = \int_A Z(T)\, d\widetilde{\mathbb{P}}, \quad A \in \mathcal{F}.$$

where $Z(t) = \exp\left\{\sigma \widetilde{W}(t) - \frac{1}{2}\sigma^2 t\right\} = \frac{e^{-rt} S(t)}{S(0)}.$

under the probability measure $\widetilde{\mathbb{P}}^S$, the process $Y(t)$ is a Markov. It is given by the SDE ① and because $\gamma(t)$ is non-random.

NOW

$$V(t) = e^{rt}\,\widetilde{\mathbb{E}}\left[e^{-rT}X^+(T)\,\middle|\,\mathcal{F}(t)\right]$$

$$= \frac{S(t)}{e^{-rt}S(t)}\,\widetilde{\mathbb{E}}\left[e^{-rT}S(T)\left(\frac{e^{-rT}X(T)}{e^{-rT}S(T)}\right)^+\,\middle|\,\mathcal{F}(t)\right]$$

$$= \frac{S(t)}{Z(t)}\,\widetilde{\mathbb{E}}\left[Z(T)\,Y^+(T)\,\middle|\,\mathcal{F}(t)\right]$$

$$= S(t)\,\widetilde{\mathbb{E}}^S\left[Y^+(T)\,\middle|\,\mathcal{F}(t)\right].$$

Given Y is $\mathcal{F}(T)$-measurable $\widetilde{\mathbb{E}}^S[Y|\mathcal{F}(t)] = \frac{1}{Z(t)}\widetilde{\mathbb{E}}[YZ(T)|\mathcal{F}(t)]$.

where $\widetilde{\mathbb{E}}^S$ denotes the expectation under the probability measure $\widetilde{\mathbb{P}}^S$. Because $Y(t)$ is Markov under $\widetilde{\mathbb{P}}^S$, there must be some function $g(t,y)$ such that

$$g(t,Y(t)) = \widetilde{\mathbb{E}}^S\left[Y^+(T)\,\middle|\,\mathcal{F}(t)\right].$$

Then $g(T,y) = y^+$, $y \in \mathbb{R}$.

$$\gamma(t) = \frac{1}{Tr}\left(1 - e^{-r(T-t)}\right)$$

$$\Rightarrow d\gamma(t) = -\frac{1}{T}e^{-r(T-t)}.$$

$$X(t) = \frac{1}{rT}\left(1 - e^{-r(T-t)}\right)S(t) + e^{-r(T-t)}\frac{1}{T}\int_0^t S(u)\,du$$

$$- e^{-r(T-t)}K \quad, 0 \le t \le T \;\cdots\; (*).$$

Note that $g(t, Y(t))$ is a martingale under $\widetilde{\mathbb{P}}^S$ and.

$$d(g(t,Y(t))) = g_t(t,Y(t))\,dt + g_y(t,Y(t))\,dY(t) + \tfrac{1}{2} g_{yy}(t,Y(t))\,dY(t)\,dY(t)$$

$$= \left[g_t(t,Y(t)) + \tfrac{1}{2}\sigma^2(\gamma(t)-Y(t))^2 g_{yy}(t,Y(t))\right]dt + \sigma(\gamma(t)-Y(t))\,g_y(t,Y(t))\,d\widetilde{W}^S(t).$$

We conclude that $g(t,y)$ satisfies the partial differential equation

$$g_t(t,y) + \tfrac{1}{2}\sigma^2(\gamma(t)-y)^2 g_{yy}(t,y) = 0, \quad 0 \le t \le T,\ y \in \mathbb{R}. \quad \text{--- (2)}$$

Theorem:- (Večeř). For $0 \le t \le T$, the price $V(t)$ at time t of the Asian call option is

$$V(t) = S(t)\, g\left(t, \frac{X(t)}{S(t)}\right)$$

where $g(t,y)$ satisfies (2) and $X(t)$ is given by (*). The boundary condition are $g(T,y) = y^+$

$$\lim_{y \to -\infty} g(t,y) = 0, \quad \lim_{y \to \infty}[g(t,y) - y] = 0, \quad 0 \le t \le T.$$

When $y(t)$ is very negative, the probability that $Y(T)$ is also negative is near one and therefore the probability $Y^+(T) = 0$ is near one. This ~~caug~~ causes $g(t, Y(t))$ to be near zero.

When $y(t)$ is large, the ~~the~~ probability that $Y(T) > 0$ is near one.

Therefore $g(t,Y(t)) \cong \widetilde{\mathbb{E}}^S[Y(T) \mid \mathcal{F}(t)] = Y(t)$

Because $Y(T)$ is a martingale under $\widetilde{\mathbb{P}}^S$.