MA 373 : Financial Engineering II

January - May 2022

Department of Mathematics, Indian Institute of Technology Guwahati

Total Marks: 60   Mid-Semester Examination   Duration: Two Hours

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- Answer **all** questions.
- Justify all your answers. Answers without justification carry no marks.

Consider the standard Black-Scholes model. Our underlying risky asset is geometric Brownian motion

$$dS(t) = 3\ S(t)dt + 2\ S(t)d\tilde{W}(t),\ S(0) = 1,$$

where $\tilde{W}(t), 0 \le t \le T$ is a Brownian motion under the risk neutral measure $\tilde{\mathbb{P}}$.

1. (i) Find the solution $S(t)$ of this equation. Is $\{S(t), t \ge 0\}$ a martingale under the risk neutral measure $\tilde{\mathbb{P}}$?

   (ii) Determine a real number $\alpha(\neq 0)$ such that $\{S^{\alpha}(t), t \ge 0\}$ is a martingale under the risk neutral measure $\tilde{\mathbb{P}}$.

   (iii) Let τ be the exit time of $S(t)$ out of the interval $(1/2, 2)$ i.e., $\tau = \inf\{t > 0 : S(t) \notin (1/2, 2)\}$. Compute $\tilde{\mathbb{P}}(X(\tau) = 2)$.

   [5+5+5]

2. The stochastic average of stock prices between 0 and t is defined by

$$X(t) = \frac{1}{t}\int_0^t S(u)d\tilde{W}(u)$$

   (i) Find $dX(t)$, $\tilde{\mathbb{E}}[X(t)]$ and $Var[X(t)] =: \tilde{\mathbb{E}}[(X(t) - \tilde{\mathbb{E}}[X(t)])^2]$.
   (ii) Show that $2X(t) = R(t) - 3A(t)$, where $R(t) = \frac{S(t)-S(0)}{t}$ is the raw average of the stock price and

$$A(t) = \frac{1}{t}\int_0^t S(u)du$$

   is the continuous arithmetic average.
   (iii) Suppose at time t we have $S(t) = x \ge 0$ and $\int_0^t S(u)du = y \ge 0$. Find the price at time t of a derivative which pays at maturity $X(T)$.

   [9+4+6]

3. Let $0 < K_1 < K_2$. Find the price of a financial derivative which pays at maturity Rs. 1 if $K_1 \le S(t) \le K_2$ and zero otherwise. This is a box-bet and its payoff is given by

$$V(T) = \begin{cases} 1 & \text{if } K_1 \le S(t) \le K_2 \\ 0 & \text{otherwise.} \end{cases}$$

   (Find the price in terms of normal distribution function)

   [8]

4. Let $Y(t) = \min_{0 \le u \le t} S(u)$ and $0 = t_0 < t_1 < \cdots < t_m = T$ be a partition of $[0, T]$. Find

$$\lim_{\|\pi_m\| \to 0} \sum_{j=1}^{m} (Y(t_j) - Y(t_{j-1}))(S(t_j) - S(t_{j-1})),$$

where $\|\pi_m\| = \max_{j=1,2,\cdots,m}(t_j - t_{j-1})$. [5]

5. Let $X(t),\ Y(t)$ satisfy the following system of SDE's

$$\begin{aligned} dX(t) &= \alpha X(t)dt + Y(t)d\tilde{W}(t),\ X(0) = x_0 \\ dY(t) &= \alpha Y(t)dt - X(t)d\tilde{W}(t),\ Y(0) = y_0, \end{aligned}$$

where x_0, y_0 are real constants. Show that $R(t) = X^2(t) + Y^2(t)$ is deterministic (non-random). [10]

6. Let $X(t) = 2t + 3\tilde{W}(t)$ and $Y(t) = 2t + \tilde{W}(t)$. If $X(t)$ and $Y(t)$ model the prices of two stocks, which one would you like to own? [3]