American Derivative Securities:-

Stopping Times:-

Definition:- A stopping time τ is a random variable taking values in $[0,\infty]$ and satisfying

$$\{\tau \le t\} \in \mathcal{F}(t) \text{ for all } t \ge 0.$$

Example:- (First passage time for a continuous process).

Let $X(t)$ be a continuous process with continuous paths, let m be a number, and let

$$\tau_m = \min\{t \ge 0 : X(t) = m\}.$$

This is the first time the process $X(t)$ reaches the level m. If $X(t)$ never reaches the level m, then we interpret τ_m to be ∞.

To show mathematically that τ_m is a stopping time. Let $t \ge 0$ be given. We need to show that $\{\tau_m \le t\}$ is in $\mathcal{F}(t)$.

If $t=0$, then $\{\tau_m \le t\} = \{\tau = 0\}$ is either Ω or ϕ depending on whether $X(0) = m$ or $X(0) \ne m$. In either case $\{\tau_m \le 0\} \in \mathcal{F}(0)$.

We consider the case $t > 0$. Suppose $\omega \in \Omega$ satisfies $\tau_m(\omega) \le t$. Then there is some number $s \le t$ such that $X(s,\omega) = m$. For each positive integer n, there is an open interval of time containing s for which the process X is in $(m - 1/n, m + 1/n)$. In this interval, there is a rational number $q \le s \le t$. Therefore ω is in the set

$$A = \bigcap_{n=1}^{\infty} \bigcup_{0 \le q \le t \, (q \text{ rational})} \{m - 1/n < X(q) < m + 1/n\}.$$

we have shown that $\{\tau \le t\} \subset A$.

On the other hand, if $\omega \in A$, then for every positive integer n there is a rational number $q_n \le t$ such that

$$m - 1/n < X(q_n, \omega) < m + 1/n .$$

The infinite sequence $\{q_n\}_{n=1}^{\infty}$ must have an accumulation point s in the closed, bounded interval $[0,t]$. In other words, there must exist a number $s \in [0,t]$ and a subsequence $\{q_{n_k}\}_{k=1}^{\infty}$ such that $\lim_{k\to\infty} q_{n_k} = s$. But

$$m - \frac{1}{n_k} < X(q_{n_k}, \omega) < m + \frac{1}{n_k} \quad \forall\, k = 1, 2, \cdots$$

Letting $k \to \infty$ in these inequalities and using the fact that X has continuous paths, we see that $X(s,\omega) = m$. It follows that $\tau_m(\omega) \le t$. We have shown that $A \subset \{\tau_m \le t\}$. Hence $A = \{\tau_m \le t\}$.

Because X is adapted to the filtration $\mathcal{F}(t)$, for each positive integer n and rational $q \in [0,t]$, the set

$$\{m - 1/n < X(q) < m + 1/n\}.$$

is in $\mathcal{F}(q) \subset \mathcal{F}(t)$. Because there are only countably many rational number q in $[0,t]$, they can arranged in a sequence, and the union

$$B_n = \bigcup_{0 \le q \le t,\, q \text{ rational}} \{m - 1/n < X(q) < m + 1/n\}.$$

is really a union of a sequence of sets in $\mathcal{F}(t)$. Therefore $B_n \in \mathcal{F}(t)$.

Because B_n is in $\mathcal{F}(t)$ for every positive integer n, the intersection $\bigcap_{n=1}^{\infty} B_n = A$ is also in $\mathcal{F}(t)$.

We have already shown that $A = \{\tau_m \leq t\}$. We conclude that $\{\tau_m \leq t\} \in \mathcal{F}(t)$. Hence τ_m is a stopping time.

Suppose that we have an adapted process $X(t)$ and a stopping time τ. We define the stopped process $X(t \wedge \tau)$, where $\wedge$ denotes the minimum of two quantities (i.e., $t\wedge\tau = \min\{t,\tau\}$). The stopped process $X(t\wedge\tau)$ agree with $X(t)$ upto time τ, and thereafter it is frozen at the value $X(\tau)$.

Optional Sampling Theorem:-

A martingale (supermartingale, submartingale) stopped at a stopping time is a martingale (supermartingale, submartingale) respectively.

Perpetual American put:-

The simplest interesting American option is perpetual American put.

The underlying asset has the price process $S(t)$ given by

$$dS(t) = rS(t)\,dt + \sigma S(t)\,d\widetilde{W}(t),$$

where $r, \sigma > 0$ and $\widetilde{W}(t)$ is a Brownian motion under the risk-neutral probability measure $\widetilde{\mathbb{P}}$.

The perpetual American put pays $K-S(t)$ if it is exercised at time t. This is its intrinsic value.

Definition:- Let $\mathcal{S}$ be the set of all stopping times. The price of the perpetual American put is defined to be

$$v_*(x) = \max_{\tau \in \mathcal{S}} \widetilde{\mathbb{E}}\left[e^{-r\tau}(K-S(\tau))\right]$$

where $x = S(0)$ is the initial stock price. In ~~case~~ the event that $\tau = \infty$ we interpret $e^{-r\tau}(K-S(\tau))$ to be zero.

- The owner of the perpetual American ~~call~~ put can choose an exercise time τ, subject only to the condition that she may not look ahead to determine when to exercise.
- The mathematical formulation of this restriction is that τ must be a stopping time.
- The price of the option at time zero is the risk-neutral expected payoff of the option, discounted from the exercise time back to time zero.

The owner of the perpetual American put can exercise at any time. In particular, there is no expiration date. This makes every date like every other date. Because every date is like every other date, it is reasonable to expect that optimal exercise policy depends only on the value of $S(t)$ and not on the time variable t. The owner of the put should exercise as soon as $S(t)$ fall "far enough" below K. In ~~the~~ other words, it is reasonable to expect that the optimal exercise policy is of the form

"Exercise the put as soon as $S(t)$ falls to the level L_*"

we have two questions to answer

(i) What is the value of L_* and how do we know it corresponds to optimal exercise?

(ii) What is the value of the put?

Theorem:- (Laplace transform for first passage time of drifted Brownian motion). Let $\widetilde{W}(t)$ be a Brownian motion under a probability measure $\widetilde{\mathbb{P}}$, let μ be a real number, and let m be a positive number. Define $X(t) = \mu t + \widetilde{W}(t)$ and set

$$\tau_m = \min\{t \geq 0 : X(t) = m\},$$

if $X(t)$ never reaches the level m, then we interpret τ_m to be ∞. Then

$$\widetilde{\mathbb{E}}[e^{-\lambda \tau_m}] = e^{-m(-\mu + \sqrt{\mu^2 + 2\lambda})} \text{ for all } \lambda > 0,$$

where we interpret $e^{-\lambda \tau_m}$ to be zero if $\tau_m = \infty$.