$$\Rightarrow F_t + r s F_x + \tfrac{1}{2}\sigma^2 s^2 F_{xx} = rF$$

Also, we must have the relation

$$\Pi(T,\Phi) = \Phi(S(T)).$$

So, F has to satisfy the following PDE

$$\left.\begin{aligned} F_t(t,x) + r x F_x(t,x) + \tfrac{1}{2}\sigma^2 x^2 F_{xx}(t,x) &= rF(t,x) \\ F(T,x) &= \phi(x) \end{aligned}\right\}.$$

<u>Definition:-</u> We say that a T-claim $\mathcal{X}$ can be replicated, alternatively that it is reachable or hedgeable, if there exists a self-financing portfolio h such that

$$V_T^h = \mathcal{X}$$

In this case we say that h is a hedge against $\mathcal{X}$. Alternatively, h is called a replicating or hedging portfolio. If every contingent claim is reachable we say that the market is complete.

<u>Meta-theorem:-</u> Let M denote the numbers of underlying traded assets in the model excluding the risk-free asset, and let R denote the number of random sources. Generically we then have the following relations:-

① The model is arbitrage free iff $M \le R$

② The model is complete if and only if $M \ge R$

③ The model is complete and arbitrage free iff $M = R$.

As an example we take the Black-Scholes model, where we have one underlying asset S plus the risk-free asset so $M=1$. we have one driving wiener process, giving us $R=1$, so in fact $M=R$. using the meta-theorem above we expect the Black-scholes model to be arbitrage free ass well as complete and this is indeed the case.

Incomplete Market:-

We know from the meta-theorem that makets generically are imcomplete when there are more random sources than there are traded assets and this can occur in an ~~ifn~~ infinite number of ways, so there is no "canonical" way of writing down a model of an incomplete market. Here we study a particular type of imcomple market, namely a "factor model", i.e., a market where there are some non-traded underlying objects. Before we go on to the formal description of the model let us briefly recall what we may expect in an incomplete market model

- since the market is imcomplete we will not be able to hedge a generic contingent claim.
- In particular there will not be a unique price for a generic derivative.

Here we consider a simplest possible imcomplete market, namely a market where the only randomnesss comes from a stochastic process $X(t)$ which is not the price of a traded asset. The model is as follows

$$\begin{cases} dX(t) = \mu(t, X(t))\,dt + \sigma(t, X(t))\,dW(t) \\ dB(t) = r\,B(t)\,dt \end{cases}$$

Now we consider a given contingent claim, written in terms of the process $X(t)$. We define the T- claim $\mathcal{X}$ by

$$\mathcal{X} = \phi(X(T))$$

where ϕ is some given function, and our problem is that of ~~sty~~ studying the price process $\Pi(t, \mathcal{X})$ for this claim.

Example:- Let $X(t)$ be the temperature at time t at the place Manali

- suppose now that you want to go to Manali for a holiday at the particular time T, but your fear that it will be unpleasantly cold when you visit manali.
- Then it may be wise to buy "holiday insurance" i.e., a contract which pays you a certain amount of money if the weather is unpleasant at a ~~spe~~ prespecified time in a prespecified place. If the contract function ϕ above may have the form

$$\phi(x) = \begin{cases} 10{,}000 & \text{if } x \leq 10 \\ 0 & \text{if } x > 10 \end{cases}$$

i.e., if the temperature at time T is below $10^\circ C$ you will obtain 10,000 Rs from the insurance company, where as you will get nothing if the temperature exceeds $10^\circ C$.

Now consider two fixed T- claim, $\mathcal{X}, \mathcal{Y}$ of the form

$$\mathcal{X} = \phi(X(T)),$$

$$\mathcal{Y} = \rho(X(T))$$

where ϕ and ρ are given real valued functions.

The project is to find out how the price of these two derivatives must be related to each other in order to avoid arbitrage possibilities on the derivative market.

Assume that the prices of the claims are of the form

$$\Pi(t, x) = F(t, X(t))$$
$$\Pi(t, y) = G(t, X(t))$$

where F and G are smooth real valued functions.

By Itô formula

$$dF = \mu_F F\,dt + \sigma_F F\,dW$$
$$dG = \mu_G G\,dt + \sigma_G G\,dW$$

Here the processes μ_F and σ_F are given by

$$\mu_F = \frac{F_t + \mu F_x + \frac{1}{2}\sigma^2 F_{xx}}{F}, \quad \sigma_F = \frac{\sigma F_x}{F}$$

and correspondingly for μ_G and σ_G.

Now form a self-financing portfolio based on F and G, with portfolio weights denoted by u_F and u_G respectively. Then

$$dV = V\left\{u_F \cdot \frac{dF}{F} + u_G \cdot \frac{dG}{G}\right\}.$$

$$\Rightarrow dV = V\left\{u_F \cdot \mu_F + u_G \cdot \mu_G\right\}dt + V\left\{u_F \cdot \sigma_F + u_G \cdot \sigma_G\right\}dW$$

In order to make this portfolio locally riskless we must choose u_F and u_G such that $u_F \cdot \sigma_F + u_G \cdot \sigma_G = 0$ and we must have

$$u_F + u_G = 1.$$

The solution to this system is given by

$$u_F = \frac{-\sigma_G}{\sigma_F - \sigma_G}, \quad u_G = \frac{\sigma_F}{\sigma_F - \sigma_G}.$$

Thus, we have

$$dV = V\left\{\frac{\mu_G \cdot \sigma_F - \mu_F \cdot \sigma_G}{\sigma_F - \sigma_G}\right\} dt$$

We have thus created a locally risk-free asset, so ~~using~~ absense of arbitrage must implies the equation

$$\frac{\mu_G \cdot \sigma_F - \mu_F \cdot \sigma_G}{\sigma_F - \sigma_G} = r$$

$$\Rightarrow \frac{\mu_F - r}{\sigma_F} = \frac{\mu_G - r}{\sigma_G}$$

The important fact to notice about this equation is that the left-hand side does not depend on the choice of G, while the right-hand side does not depend ~~on~~ upon the choice of F. We have proved the following central result

<u>Proposition:-</u> Assume that the market is arbitrage free. Then there exists a universal process $\lambda(t)$ such that

$$\frac{\mu_F - r}{\sigma_F} = \lambda$$

regardless of the specific choice of the derivative F.

λ is called the market price of risk.

From the above proposition, we have

$$\frac{\mu_F - r}{\sigma_F} = \lambda$$

$$\Rightarrow \frac{F_t + \mu F_x + \frac{1}{2}\sigma^2 F_{xx} - rF}{\sigma F_x} = \lambda$$

$$\Rightarrow F_t + (\mu - \lambda\sigma) F_x + \frac{1}{2}\sigma^2 F_{xx} = rF$$

Furthermore it is clear that we must have the boundary condition $F(T, x) = \phi(x) \quad \forall x$

Proposition:- (Pricing equation)

Assuming absence of arbitrage, the pricing function $F(t,x)$ of the T-claim $\Phi(X(T))$ solves the following boundary value problem

$$\begin{cases} F_t(t,x) + \{\mu(t,x) - \lambda(t,x)\,\sigma(t,x)\} F_x(t,x) + \frac{1}{2}\sigma^2(t,x)\, F_{xx}(t,x) \\ \qquad = rF(t,x) \\ F(T,x) = \phi(x) \end{cases}$$

- In order to solve it we have to know r, $\mu(t,x)$, $\sigma(t,x)$, $\phi(x)$, $\lambda(t,x)$
- r, $\mu(t,x)$, $\sigma(t,x)$ and $\phi(x)$ are specified within the model
- The market price of risk λ is not specified within the model
- Let us fix the benchmark claim $P(X(T))$
- Assume that the price process $G(t,x)$ for $P(X(T))$ is specified exogenously
- Then we can compute the market price of risk by the above formula.
- The price of an ~~ab~~ arbitrary claim is uniquely determine by the price G.

Propn:- Assuming absence of arbitrage, the pricing function $F(t,x)$ of the T-claim $\phi(X(T))$ is given by the formula

$$F(t,x) = e^{-r(T-t)} \widehat{\mathbb{E}}\left[\phi(X(T)) \mid X(t)=x\right].$$

The dynamics of X under the martingale measure $\widehat{\mathbb{P}}$ is give by

$$dX(t) = \{\mu(t, X(t)) - \lambda(t, X(t))\, \sigma(t, X(t))\}dt + \sigma(t, X(t))\, d\widetilde{W}(t).$$

where $\widehat{W}$ is a Wiener process under $\widehat{\mathbb{P}}$.

$$\text{set } Z(t) = \exp\left\{-\int_0^t \lambda(u)\, dW(u) - \tfrac{1}{2}\int_0^t \lambda^2(u)\, du\right\}.$$

$$\widetilde{W}(t) = W(t) + \int_0^t \lambda(u)\, du, \quad \text{set } Z = Z(T) \text{ then } \mathbb{E}[Z]=1.$$

We define a new probability measure $\widehat{\mathbb{P}}$ by the formula

$$\widehat{\mathbb{P}}(A) = \int_A Z(\omega)\, d\mathbb{P}(\omega) \quad \forall A \in \mathcal{F}.$$

$$\begin{aligned} dX(t) &= \mu(t, X(t))\, dt + \sigma(t, X(t))\, dW(t) \\ &= \mu(t, X(t))\, dt + \sigma(t, X(t))\left[dW(t) + \lambda(t)\, dt\right] - \lambda(t)\, \sigma(t, X(t))\, dt \\ &= \left(\mu(t, X(t)) - \lambda(t, X(t))\, \sigma(t, X(t))\right)dt + \sigma(t, X(t))\, d\widetilde{W}(t). \end{aligned}$$

- Note that there is a one-one correspondence between the martingale measure and the market price of risk.
- choosing a particular λ is equavalent to choosing a particular martingale measure $\widehat{\mathbb{P}}$
- who chooses the martingale measure?
- The market price of risk is determined on the market by the agents in the market.

- Let us take a concrete model as given
- We assume that we know the exact form of μ and σ.
- We need to know the market price of risk $\lambda(t,x)$
- Assume some contracts $\Phi_i(X(T))$ $i=1,2,\cdots n$ are already traded in the market.
- Let us assume that we want to choose our market price of risk from a parameterized family of functions. i.e., we assume that λ is of the form

$$\lambda = \lambda(t,x;\beta) \quad , \beta \in \mathbb{R}^k$$

We carry out the following scheme. We are standing at time $t=0$

- compute the theoretical pricing function $F^i(t,x)$ for the claims $\phi_1, \phi_2, \cdots \phi_n$. This is done by solving pricing PDE for each contract and the result will depend on the parameter β, so

$$F^i = F^i(t,x,\beta) \quad , \; i=1,2,\cdots n.$$

- By observing today's value of the underlying process, say $X(0)=x_0$ we can compute today's theoretical price of the contracts as

$$\pi^i(0,\beta) = F^i(0,x_0,\beta)$$

- Now go to the market and observe the actually traded price for the contracts, $\pi^{i*}(0)$.
- We now choose the "implied" parameter β^* such that

$$\pi^{i*}(0) \simeq \pi^i(0,\beta^*) \quad , \; i=1,2,\cdots n.$$

- one way, out of many, is to determine β^* by solving the Least squares minimization problem

$$\min_{\beta \in \mathbb{R}^k} \left[\sum_{i=1}^{n} \{ \pi^i(0,\beta) - \pi^{i*}(0) \}^2 \right]$$

Here $\lambda(t,x,\beta^*)$ is not a theoretical one, but an empirical one.