Bonds and Interest Rates

Definition:- A zero coupon bond with maturity date T, also called a T-bond, is a contract which guarantees the holder one dollar to be paid on the date T. The price at time t of a bond with maturity date T is denoted by $p(t,T)$.

We now make an assumption to guarantee the existence of a sufficiently rich and regular bond market.

Assumption:- We assume the following

- There exists a (frictionless) market for T-bonds for every $T > 0$.
- The relation $p(t,t) = 1$ holds for all t.
- For each fixed t, the bond price $p(t,T)$ is differentiable with respect to time of maturity

- For a fixed value of t, $p(t,T)$ is a function of T. This function provides the prices, at the fixed time t, for bonds of all possible maturities. The graph of this function is called "the bond price curve at t" or "the term structure at t". For each t, $p(t,T)$ will be differentiable w.r.t T.
- For a fixed maturity T, $p(t,T)$ will be a stochastic process, This process gives the prices, at different times, of the bond with fixed maturity T and the trajectory will typically be very irregular

(like a Wiener process).

The bond market is different from any other market that we have consider so far, in the sense that the bond market contains an infinite number of assets (one bond type for each time of maturity). The basic goal in interest rate theory is roughly that of investigating the relations between all these different bonds. More precisely we pose the following general problems, ~~to be~~

- What is a reasonable model for the bond market above?
- Which relations must hold between the price processes for bonds of different maturities, in order to guarantee an arbitrage free bond market?
- Is it possible to derive arbitrage free bond prices from a specification of the dynamics of the short rate of interest?
- Given a model for the bond market, how do you compute prices of interest rate derivatives, such as a European call option on an underlying bond?

Interest rates:-

suppose that we are standing at time t and let us fix two other points in time S and T, with $t < S < T$. The immediate project is to write a contract at time t which allows us to make an investment of 1 Rs at time S, and to have a deterministic rate of return, determined at the contract time t, over the interval $[S,T]$. This can be achived as follows:-

- At time t we sell one S-bond. This will give us $p(t,S)$ Rs.
- Buy exactly $P(t,S)/P(t,T)$ T-bonds. Thus our net investment at time t equals zero.
- A time S the S-Bond matures, so we are obliged to pay out 1 Rs.
- At time T the T-Bond mature, so we will receive the amount $P(t,S)/p(t,T)$ Rs.
- The net effect is that, an investment of 1 Rs at time S has yielded $p(t,s)/p(t,T)$ Rs at time T.
- Thus, at time t, we have made a contract guaranteeing a riskless rate of interest over the future time interval $[S,T]$. such an interest rate is called a forward rate.
- The simple forward rate (or LIBOR rate) L, is the solution to the equation

$$1+(T-S)L=\frac{p(t,S)}{p(t,T)}$$

the continuous compounded forward rate R is the solution to the equation

$$e^{R(T-S)}=\frac{p(t,S)}{p(t,T)}.$$

Definition:- (1) The simple forward rate for $[S,T]$ contracted at t, henceforth ~~of~~ referred to as the LIBOR forward rate, is defined as

$$L(t;S,T) = -\frac{p(t,T) - p(t,S)}{(T-S)\,p(t,T)}$$

(2) The simple spot rate for $[S,T]$, henceforth referred to as the LIBOR spot rate, is defined as

$$L(S,T) = -\frac{p(S,T) - 1}{(T-S)\,p(S,T)}$$

(3) The continuously compounded forward rate for $[S,T]$ contracted at t is defined as

$$R(t,S,T) = -\frac{\log p(t,T) - \log p(t,S)}{(T-S)}$$

(4) continuously compounded ~~for~~ spot rate $R(S,T)$, for the period $[S,T]$ is defined as

$$R(S,T) = -\frac{\log p(S,T)}{(T-S)}.$$

(5) The instantaneous forward rate with maturity T, contracted at t, is defined by

$$f(t,T) = -\frac{\partial \log p(t,T)}{\partial T}$$

(6) The instantaneous short rate at time t is defined by

$$r(t) = f(t,t).$$

Note that spot rates are forward rates where the time of contracting coincides with the start of the interval over which the interest rate is effective, i.e., $t=S$.

The instantaneous forward rate, is the limit of the continuously compounded forward rate when $S \to T$. It can be interpreted as the riskless rate of interest, contracted at t, over the infinitesimal interval $[T, T+dT]$.

Defn:- The money market account process is defined by

$$B(t) = \exp\left\{\int_0^t r(s)\,ds\right\},$$

i.e.,
$$\begin{cases} dB(t) = r(t)\,B(t)\,dt \\ B(0) = 1 \end{cases}$$

The interpretation of money market account is the same as before, i.e., you may think of it as describing a bank with a stochastic short rate.

Lemma:- For $t \le s \le T$ we have

$$p(t,T) = p(t,s)\exp\left\{-\int_s^T f(t,u)\,du\right\}.$$

and in particular

$$p(t,T) = \exp\left\{-\int_t^T f(t,u)\,du\right\}.$$

Proof:-
$$f(t,u) = -\frac{\partial}{\partial u}\log p(t,u) \quad \text{For } t \le u \le T$$

$$\Rightarrow \int_s^T f(t,u)\,du = -\int_s^T d(\log p(t,u))$$

$$\Rightarrow p(t,T) = p(t,s)\exp\left\{-\int_s^T f(t,u)\,du\right\}.$$

If $s = t$ then

$$p(t,T) = \exp\left\{-\int_t^T f(t,u)\,du\right\} \text{ (since } p(t,t) = 1\text{)}.$$

If we wish to make a model for the bond market, it is obvious that this can be done in many different ways

- We may specify the dynamics of the short rate (and then try to derive bond prices using arbitrage arguments).
- We may directly specify the dynamics of all possible bonds.
- We may specify the dynamics of all possible forward rates and then use the above Lemma in order to obtain bond prices

All these approaches are related to each other

- Relation between $df(t,T)$, $dp(t,T)$ and $r(t)$

We will consider dynamics of the following form

Short rate dynamics

$$dr(t) = a(t)\,dt + b(t)\,dW(t) \quad \text{-----} \quad (1)$$

Bond price dynamics

$$dp(t,T) = p(t,T)\,m(t,T)\,dt + p(t,T)\,v(t,T)\,dW(t) \quad \text{--} \quad (2)$$

Forward rate dynamics

$$df(t,T) = \alpha(t,T)\,dt + \sigma(t,T)\,dW(t). \quad \text{----} \quad (3)$$

The processes $a(t)$, $b(t)$, $m(t,T)$, $v(t,T)$, $\alpha(t,T)$, $\sigma(t,T)$ are adapted processes.

We will study the formal relations which must hold between bond prices and interest rates. ~~and~~