Proof:- Note that for any T- claim $\mathcal{X}$ we have

$$\Pi(t,\mathcal{X}) = S(t)\,\mathbb{E}^S\left[\frac{\mathcal{X}}{S(T)}\,\Big|\,\mathcal{F}(t)\right] \quad (S \text{ as the numeriare})$$

$$= p(t,T)\,\mathbb{E}^T\left[\mathcal{X}\,|\,\mathcal{F}(t)\right] \quad (p(t,T) \text{ as the numeriare}).$$

where $\mathbb{E}^T$ denotes integration w.r.t Q^T.

Now let $\mathcal{X} = r(T)$, then we have

$$\Pi(t,\mathcal{X}) = \mathbb{E}^Q\left[r(T)\,e^{-\int_t^T r(s)\,ds}\,\Big|\,\mathcal{F}(t)\right]$$

$$= p(t,T)\,\mathbb{E}^T\left[r(T)\,|\,\mathcal{F}(t)\right].$$

$$\Rightarrow \mathbb{E}^T\left[r(T)\,|\,\mathcal{F}(t)\right] = \frac{1}{p(t,T)}\,\mathbb{E}^Q\left[r(T)\,e^{-\int_t^T r(s)\,ds}\,\Big|\,\mathcal{F}(t)\right].$$

$$= -\frac{1}{p(t,T)}\,\mathbb{E}^Q\left[\frac{\partial}{\partial T}\,e^{-\int_t^T r(s)\,ds}\,\Big|\,\mathcal{F}(t)\right]$$

$$= -\frac{1}{p(t,T)}\,\frac{\partial}{\partial T}\,\mathbb{E}^Q\left[e^{-\int_t^T r(s)\,ds}\,\Big|\,\mathcal{F}(t)\right]$$

$$= -\frac{p_T(t,T)}{p(t,T)} = f(t,T)$$

An alternative view of the money Account:-

Let us consider a self-financing portfolio which at each time t consists entirely of bonds maturing x units of time later - At time t the portfolio thus consists only of bonds with maturity $t+x$. So the dynamics for this portfolio is given by

$$dV(t) = V(t)\cdot 1\cdot \frac{dp(t,t+x)}{p(t,t+x)}$$

where the constant 1 indicates that the weight of the t+x bond in the portfolio equals one

Note that

$$\frac{dp(t,t+x)}{p(t,t+x)} = \left\{r(t) + A(t,t+x) + \tfrac{1}{2} S^2(t,t+x)\right\} dt + S(t,t+x)\, dW(t).$$

Letting $x \to 0$, gives us

$$\lim_{x \to 0} A(t,t+x) = 0$$

$$\lim_{x \to 0} S(t,t+x) = 0 \quad \& \quad \lim_{x \to 0} p(t,t+x) = 1$$

$$\Rightarrow \; dV(t) = r(t) V(t)\, dt$$

which we recognize as the dynamics of the money account.

Fixed Coupon bonds :-

The simplest coupon bond is the fixed coupon bond. This is a bond which at some intermediary points in time, will provide predetermined payments (coupons) to the holder of the bond.

The formal description is as follows

- Fixed a number of dates $T_0, T_1, \ldots T_n$. Here T_0 is the emission date of the bond, whereas $T_1, \ldots T_n$ are the coupon dates.

- At T_i, $i = 1, 2, \ldots n$, the owner of the bond receives the deterministic coupon c_i

- At time T_n the owner receives the face value K.

The coupon bond can be replicated by holding a portfolio of zero coupon bonds with maturities $T_i, i=1,2,\dots n$. More precisely we will hold c_i zero coupon bonds of maturity T_i, $i=1,2,\dots n$, and $K+c_n$ bonds with maturity T_n, so the price $p(t)$ at time $t<T_1$ of the coupon bond is given by

$$p(t) = K\cdot p(t,T_n) + \sum_{i=1}^{n} c_i \cdot p(t,T_i).$$

Very often the coupons are determined in terms of return. The return for the ith coupon is typically quoted as a simple rate acting on the face value K, over the period $[T_{i-1},T_i]$. For example if the ith coupon has a return equal to r_i and the face value K, this mean that

$$c_i = r_i(T_i - T_{i-1})K.$$

For a standardized coupon bond $T_i = T_0 + i\delta$ and $r_i = r$

The price of a such bond will be given by

$$p(t) = K\left(p(t,T_n) + r\delta\sum_{i=1}^{n} p(t,T_i)\right).$$

Floating Rate bonds:-

Here we consider a simple floating rate bonds, where the coupon rate r_i is set to the spot LIBOR rate $L(T_{i-1},T_i)$

Thus
$$c_i = (T_i - T_{i-1})L(T_{i-1},T_i)K$$

Note that $L(T_{i-1}, T_i)$ is determined at time T_{i-1}, but c_i is not delivered until time T_i.

Now we compute the value of this bond at time $t < T_0$, when $T_i - T_{i-1} = \delta$ & $K = 1$. By the definition of the LIBOR rate we have

$$c_i = \delta \, \frac{1 - p(T_{i-1}, T_i)}{\delta \, p(T_{i-1}, T_i)} = \frac{1}{p(T_{i-1}, T_i)} - 1$$

The value at t, of the term -1 (paid out at T_i) is of course equal to $-p(t, T_i)$ and it remains to compute the value of the term $\frac{1}{p(T_{i-1}, T_i)}$, which is paid out at T_i.

This is easily done through the following arguments:

- Buy at time t one T_{i-1} bond. This will cost $p(t, T_{i-1})$
- At time T_{i-1} you will receive the amount 1.
- Invest this unit amount in T_i-bond. This will give you exactly $\frac{1}{p(T_{i-1}, T_i)}$ bonds.
- At T_i the bonds will mature, each at face value 1. Thus at time T_i you will obtain the amount $\frac{1}{p(T_{i-1}, T_i)}$

Thus the value at t of obtaining $\frac{1}{p(T_{i-1}, T_i)}$ at T_i is given by $p(t, T_{i-1})$ and the value at t of the coupon c_i is $p(t, T_{i-1}) - p(t, T_i)$.

Hence the valuation formula for the floating rate bond is

$$p(t) = p(t, T_n) + \sum_{i=1}^{n} \left[p(t, T_{i-1}) - p(t, T_i) \right] = p(t, T_0).$$

In particular if $t = T_0$ then $p(T_0) = 1$.

Interest Rate Swaps:-

This is basically a scheme where you exchange a payment stream at a fixed rate of interest, known as the swap rate, for a payment stream at a floating rate (typically a LIBOR rate).

Here we will study the forward swap settled in arrears, which is defined as follows. We denote the principal by K and the swap rate by R. Assume that $T_i = T_0 + \delta i$ and payments occurs at the dates $T_1 \cdots T_n$ (not at T_0). If you swap a fixed rate for a floating rate (in this case the LIBOR spot rate), then at time T_i, you will receive the amount

$$K \delta L(T_{i-1}, T_i)$$

which is exactly $K c_i$, where c_i is the ith coupon for the floating rate bond. At T_i you will pay the amount $K \delta R$.

The net cash flow at T_i is thus given by

$$K \delta [L(T_{i-1}, T_i) - R],$$

Using our results from the floating rate bond, we can compute the value at $t < T_0$ of this cash flow as

$$K p(t, T_{i-1}) - K(1+\delta R) p(t, T_i).$$

The total value $\Pi(t)$, at t, of the swap is given by

$$\Pi(t) = K \sum_{i=1}^{n} \left[p(t, T_{i-1}) - (1+\delta R) p(t, T_i) \right]$$

$$= K p(t, T_0) - K \sum_{i=1}^{n} d_i \, p(t, T_i)$$

when $d_i = R\delta$, $i = 1, 2, \ldots n-1$

$d_n = (1 + R\delta)$.

The question is how the swap rate R is determined. By definition it is chosen such that the value of the swap equals zero at the time when the contract is made. If we assume that the contract is written at $t=0$, then the swap rate is given by

$$R = \frac{p(0, T_0) - p(0, T_n)}{\delta \sum_{i=1}^{n} p(0, T_i)}$$

In the case when $T_0 = 0$, this formula reduces to

$$R = \frac{1 - p(0, T_n)}{\delta \sum_{i=1}^{n} p(0, T_i)}.$$

Yield and Duration:-

Consider a zero coupon T-bond with market price $p(t,T)$. We now look for the bond's "internal rate of interest" i.e., the constant short rate of interest which will give the same value to this bond as the value given by the market. Denoting this value of the short rate by y, then

$$p(t,T) = e^{-y(T-t)} \cdot 1$$

where the factor 1 indicates the face value of the bond.

Defn: The continuously compounded zero coupon yield $y(t,T)$ is given by $$y(t,T) = -\frac{\log p(t,T)}{(T-t)}.$$

For a fixed t, the function $T \to y(t,T)$ is called the (zero coupon) yield curve.

Note that the yield $y(t,T)$ is nothing more than the spot rate for the interval $[t,T]$

Now let us consider a fixed coupon bond with market value $p(t)$ at time t.

Defn:- The yield to maturity, $y(t,T)$ of a fixed coupon bond at time t, with market price p, and payments c_i at T_i for $i=1,2,--n$, is defined as the value of y which solves the equation

$$p(t) = \sum_{i=1}^{n} c_i e^{-y(T_i-t)}.$$