exercise policy depends only on the value of $S(t)$ and not on the time variable t. The owner of the put should exercise as soon as $S(t)$ fall "far enough" below K. In other words, it is reasonable to expect that the optimal exercise policy is of the form

"Exercise the put as soon as $S(t)$ falls to the level L_*"

we have two questions to answer

(i) What is the value of L_* and how do we know it corresponds to optimal exercise?

(ii) What is the value of the put?

<u>Theorem</u>:- (Laplace transform for first passage time of drifted Brownian motion). Let $\widetilde{W}(t)$ be a Brownian motion under a probability measure $\widetilde{\mathbb{P}}$, let μ be a real number, and let m be a positive number. Define $X(t) = \mu t + \widetilde{W}(t)$ and set

$$\tau_m = \min\{t \geq 0 : X(t) = m\},$$

if $X(t)$ never reaches the level m, then we interpret τ_m to be ∞. Then

$$\widetilde{\mathbb{E}}\left[e^{-\lambda \tau_m}\right] = e^{-m(-\mu + \sqrt{\mu^2 + 2\lambda})} \text{ for all } \lambda > 0,$$

where we interpret $e^{-\lambda \tau_m}$ to be zero if $\tau_m = \infty$.

The underlying asset price process $S(t)$ is given by

$$dS(t) = r S(t)\, dt + \sigma S(t)\, d\widetilde{W}(t).$$

$$\Rightarrow S(t) = S(0) \exp\left\{ \sigma \widetilde{W}(t) + (r - \tfrac{1}{2}\sigma^2)t \right\}.$$

Suppose the owner sets a positive level $L < K$ and resolves to exercise the put the first time the stock price falls to L. If $S(0) \le L$ then she exercises immediately (at time zero) and the value of the put in this case is $v_L(S(0)) = K - S(0)$. If $S(0) > L$, she exercises at the stopping time

$$\tau_L = \min\{ t \ge 0 ;\ S(t) = L \}.$$

At the time of exercise, the put pays $K - S(\tau_L) = K - L$. Thus the value of the put under this exercise strategy to be

$$v_L(S(0)) = (K-L)\, \widetilde{\mathbb{E}}\left[e^{-r\tau_L} \right] \text{ for all } S(0) \ge L.$$

Lemma:- The function $v_L(x)$ is given by the formula

$$v_L(x) = \begin{cases} K - x, & 0 \le x \le L \\ (K-L)\left(\frac{x}{L}\right)^{-\frac{2r}{\sigma^2}}, & x \ge L \end{cases}$$

Proof:- If $S(0) \le L$, then owner exercises at time $t = 0$ and ~~in~~ the value of the put in this case is $v_L(x) = K - x$.

If $S(0) = x > L$. ~~But~~ The stopping time τ_L is the first time $S(t)$ reaches the level L.

But $S(t) = L$ if and only if

$$x \exp\{\sigma \widetilde{W}(t) + (r - \tfrac{1}{2}\sigma^2)t\} = L$$

$$\Leftrightarrow \; -\widetilde{W}(t) - \tfrac{1}{\sigma}(r - \tfrac{1}{2}\sigma^2)t = \tfrac{1}{\sigma}\log \tfrac{x}{L}$$

Now apply the theorem with $\lambda = r$ and $\mu = -\frac{1}{\sigma}(r - \frac{1}{2}\sigma^2)$ and $m = \frac{1}{\sigma}\log\frac{x}{L}$. Note that

$$\mu^2 + 2\lambda = \tfrac{1}{\sigma^2}(r^2 - r\sigma^2 + \tfrac{1}{4}\sigma^4) + 2r$$
$$= \tfrac{1}{\sigma^2}(r^2 + r\sigma^2 + \tfrac{1}{4}\sigma^4) = \frac{1}{\sigma^2}(r + \sigma^2/2)^2.$$

Therefore

$$-\mu + \sqrt{\mu^2 + 2\lambda} = \frac{1}{\sigma}(r - \tfrac{1}{2}\sigma^2) + \frac{1}{\sigma}(r + \sigma^2/2) = \frac{2r}{\sigma}$$

This implies that

$$\widetilde{\mathbb{E}}[e^{-r\tau_L}] = \exp\left\{-\tfrac{1}{\sigma}\log\tfrac{x}{L}\cdot\frac{2r}{\sigma}\right\} = \left(\tfrac{x}{L}\right)^{-2r/\sigma^2}$$

Hence $v_L(x) = (K-L)\left(\frac{x}{L}\right)^{-2r/\sigma^2}$ for $x \geq L$.

Therefore, among those exercise policies of the form

"Exercise the put as soon as $S(t)$ falls to the level L",

the best one is obtained by choosing $L = L_*$. We expect therefore that $v_{L_*}(x)$ is the price of the put $v_*(x)$.

First we determine the value of L_*. We note that

$$v_L(x) = (K-L)L^{2r/\sigma^2}(x)^{-2r/\sigma^2} \text{ for all } x \geq L.$$

L_* is the value of L that maximizes this quantity when we hold x fixed.

Thus we define

$$g(L) = (K-L)L^{2r/\sigma^2} \Rightarrow g(0)=0 \;\&\; \lim_{L\to+\infty} g(L) = -\infty$$

$$g'(L) = -L^{2r/\sigma^2} + \frac{2r}{\sigma^2}(K-L)L^{2r/\sigma^2-1}$$

$$= -\frac{2r+\sigma^2}{\sigma^2}L^{2r/\sigma^2} + \frac{2r}{\sigma^2}KL^{2r/\sigma^2-1}$$

$$g'(L)=0 \Rightarrow L_* = \frac{2r}{2r+\sigma^2}K. \text{ and } 0<L_*<K$$

$$g(L_*) = \frac{\sigma^2}{2r+\sigma^2}\left(\frac{2r}{2r+\sigma^2}\right)^{2r/\sigma^2} K^{\frac{2r+\sigma^2}{\sigma^2}} > 0.$$

Therefore, we have

$$v_{L_*}(x) = \begin{cases} K-x, & 0\le x\le L_* \\ (K-L_*)\left(x/L_*\right)^{-2r/\sigma^2}, & x\ge L_*, \end{cases}$$

$$\Rightarrow \quad v'_{L_*}(x) = \begin{cases} -1 & 0\le x\le L_* \\ -(K-L_*)\dfrac{2r}{\sigma^2 x}\left(x/L_*\right)^{-2r/\sigma^2}, & x\ge L_*. \end{cases}$$

Note that $v'_{L_*}(L_*^+) = -\frac{2r}{\sigma^2 L_*}(K-L_*) = -1 = v'_{L_*}(L_*^-)$

The derivative of $v_{L_*}(x)$ is continuous at $x=L_*$.

$$v''_{L_*}(x) = \begin{cases} 0 & 0\le x< L_* \\ (K-L_*)\dfrac{2r(2r+\sigma^2)}{\sigma^4 x^2}\left(\dfrac{x}{L_*}\right)^{-2r/\sigma^2}, & x>L_* \end{cases}$$

$$v''_{L_*}(L_*^-) = 0 \text{ and } v''_{L_*}(L_*^+) = (K-L_*)\frac{2r(2r+\sigma^2)}{\sigma^4 L_*^2} > 0.$$

The second derivative of $v_{L_*}(x)$ has a jump at $x=L_*$.

For $x>L_*$

$$r v_{L_*}(x) - rx\, v'_{L_*}(x) - \tfrac{1}{2}\sigma^2 x^2 v''_{L_*}(x)$$

$$= (K - L_*)\left(r + \frac{2r^2}{\sigma^2} - \frac{r(2r+\sigma^2)}{\sigma^2}\right)\left(\frac{x}{L_*}\right)^{-2r/\sigma^2} = 0$$

On the other hand, for $0 \le x < L_*$,

$$r v_{L_*}(x) - r x v'_{L_*}(x) - \tfrac{1}{2}\sigma^2 x^2 v''_{L_*}(x) = r(K - x) + r x = rK.$$

In particular, we see that $v_{L_*}(x)$ satisfies the so-called linear complementarity conditions

$$v(x) \ge (K - x)^+ \text{ for all } x \ge 0, \quad \text{--- (a)}$$

$$r v(x) - r x v'(x) - \tfrac{1}{2}\sigma^2 x^2 v''(x) \ge 0 \text{ for all } x \ge 0, \quad \text{-- (b)}$$

and for each $x \ge 0$, equality holds in either (a) or (b). -- (c)

The linar complementanity conditions (a)-(c) determine the function $v_{L_*}(x)$. More precisely, the function $v_{L_*}(x)$ is the only bounded continuous function having a continuous derivative that satisfies these conditions:

Probabilistic Characterization of the put price

Theorem:- Let $S(t)$ be the stock price given by

$$dS(t) = r S(t)\,dt + \sigma S(t)\,d\widetilde{W}(t)$$ and let

$$\tau_{L_*} = \min\{t \ge 0 : S(t) = L_*\}.$$ Then $e^{-rt} v_{L_*}(S(t))$ is a supermartingale under $\widetilde{\mathbb{P}}$, and the stopped process $e^{-r(t\wedge\tau_{L_*})} v_{L_*}(S(t\wedge\tau_{L_*}))$ is a martingale.

Proof: $d(e^{-rt} v_{L_*}(S(t)))$

$$= e^{-rt}\left[-r v_{L_*}(S(t))\,dt + v'_{L_*}(S(t))\,dS(t) + \tfrac{1}{2} v''_{L_*}(S(t))\,dS(t)\,dS(t)\right].$$

$$= e^{-rt}\left[-r v_{L_*}(S(t)) + rS(t)\, v'_{L_*}(S(t)) + \tfrac{1}{2}\sigma^2 S^2(t)\, v''_{L_*}(S(t))\right]dt$$
$$+ e^{-rt}\sigma S(t)\, v'_{L_*}(S(t))\, d\widetilde{W}(t).$$

The dt term in this expression is either 0 or $-rK$, depending on whether $S(t) > L_*$ or $S(t) < L_*$. If $S(t) = L_*$, $v''_{L_*}(S(t))$ is undefined, but the probability $S(t) = L_*$ is zero so this does not matter. Thus we have

$$d\left(e^{-rt} v_{L_*}(S(t))\right) = -e^{-rt} rK\, \mathbb{1}_{\{S(t)<L_*\}}\, dt + e^{-rt}\sigma S(t)\, v'_{L_*}(S(t))\, d\widetilde{W}(t). \quad \text{---} \circledast$$

Because the dt term is less than or equal to zero, $e^{-rt} v_{L_*}(S(t))$ is a supermartingale

If $S(0) > L_*$, then prior to the time τ_{L_*} when the stock price first reaches L_*, the dt term is zero and hence $e^{-r(t\wedge\tau_{L_*})} v(S(t\wedge\tau_{L_*}))$ is a martingale. In particular $\circledast$ implies that

$$\int_0^{t\wedge\tau_{L_*}} d\left(e^{-rt} v_{L_*}(S(t))\right) = \int_0^{t\wedge\tau_{L_*}} -e^{-rt} rK\, \mathbb{1}_{\{S(t)<L_*\}} + \int_0^{t\wedge\tau_{L_*}} e^{-rt}\sigma S(t)\, v'_{L_*}(S(t))\, d\widetilde{W}(t)$$

$$\Rightarrow e^{-r(t\wedge\tau_{L_*})} v_{L_*}(S(t\wedge\tau_{L_*})) = v_{L_*}(0) + \int_0^{t\wedge\tau_{L_*}} e^{-ru}\sigma S(u)\, v'_{L_*}(S(u))\, d\widetilde{W}(u).$$

Itô-integrals are martingales, hence the Itô-integral above stopped at the stopping time τ_{L_*} is also a martingale.