MA 373 : Financial Engineering II

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Exercises 3

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1. Suppose that a trader holds an American call option on a non-dividend-paying stock with strike price \$100 with expiry time 6 months from now. The continuously compounded interest rate is 5% and the current stock price is \$110.

   (i) What happens if the trader exercises the call option now?

   (ii) Does the trader have an advantage if he exercises the option at expiry time?

   (iii) What would be the trader's strategy if he knew the stock price was going to fall?

2. Consider two identical American call options $C(S(t), t : K_1, T)$ and $C(S(t), t : K_2, T)$ having strike prices K_1 and K_2 with $K_1 < K_2$, $S(t)$ is the spot price at time t and $T > t$ is the expiry time. Show that

$$0 \leq C(S(t), t : K_1, T) - C(S(t), t : K_2, T) \leq K_2 - K_1.$$

3. Consider two identical American call options $P(S(t), t : K_1, T)$ and $P(S(t), t : K_2, T)$ having strike prices K_1 and K_2 with $K_1 < K_2$, $S(t)$ is the spot price at time t and $T > t$ is the expiry time. Show that

$$0 \leq P(S(t), t : K_2, T) - P(S(t), t : K_1, T) \leq K_2 - K_1.$$

4. Let the prices of two American call options with strikes \$50 and \$60 be \$2.50 and \$3.00, respectively where both options have the same time to expiry.

   (i) Is the no-arbitrage condition violated?

   (ii) Suggest a spread position so that the portfolio will ensure an arbitrage opportunity.

5. Consider the standard Black-Scholes model. Our underlying risky asset is geometric Brownian motion

$$dS(t) = rS(t)dt + \sigma S(t)d\tilde{W}(t), \; S(0) = 1,$$

   where $\tilde{W}(t), 0 \leq t \leq T$ is a Brownian motion under the risk neutral measure $\tilde{\mathbb{P}}$ and r, σ are strictly positive constants. Assume that $r > \frac{\sigma^2}{2}$.

   (a) Let $b > 0$ such that $b > S(0)$. Consider a contract that pays \$1 at the time when the stock reaches the barrier b for the first time. Show that the value of the contract at time t = 0 is

$$\frac{S(0)}{b}.$$

   (b) Assume the stock pays continuous dividends at the constant rate $\delta > 0$ and let $b > 0$ such that $b > S(0)$.

      (i) Consider a contract that pays \$1 at the time when the stock reaches the barrier b for the first time. Show that the value of the contract at time t = 0 is

$$\left(\frac{S(0)}{b}\right)^{h_1},$$

      where

$$h_1 = \frac{1}{2} - \frac{r-\delta}{\sigma^2} + \sqrt{\left(\frac{r-\delta}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2r}{\sigma^2}}.$$

(ii) Show that $h_1(\delta)$ is an increasing function for $\delta > 0$ and $h_1(0) = 1$.

(c) Let $b > 0$ such that $b < S(0)$. Consider a contract that pays \$1 at the time when the stock reaches the barrier b for the first time. Show that the value of the contract at time t = 0 is

$$\Big(\frac{S(0)}{b}\Big)^{\frac{-2r}{\sigma^2}}.$$

(d) Assume the stock pays continuous dividends at the constant rate $\delta > 0$ and let $b > 0$ such that $b < S(0)$. Consider a contract that pays \$1 at the time when the stock reaches the barrier b for the first time. Show that the value of the contract at time t = 0 is

$$\Big(\frac{S(0)}{b}\Big)^{h_2},$$

where

$$h_2 = \frac{1}{2} - \frac{r-\delta}{\sigma^2} - \sqrt{\Big(\frac{r-\delta}{\sigma^2} - \frac{1}{2}\Big)^2 + \frac{2r}{\sigma^2}}.$$

(e) Assume the stock pays continuous dividends at the constant rate $\delta > 0$ and let $b > 0$ such that $b > S(0)$.

(i) Assume a perpetual call is exercised whenever the stock reaches the barrier b from below. Show that the discounted value at time t = 0 is

$$f(b) = (b-K)\Big(\frac{S(0)}{b}\Big)^{h_1},$$

where

$$h_1 = \frac{1}{2} - \frac{r-\delta}{\sigma^2} + \sqrt{\Big(\frac{r-\delta}{\sigma^2} - \frac{1}{2}\Big)^2 + \frac{2r}{\sigma^2}}.$$

(ii) Show that the maximum value of $f(b)$ is realized for $b^* = K\frac{h_1}{h_1-1}$.

(iii) Prove the price of perpetual call

$$f(b^*) = \frac{K}{h_1 - 1}\Big(\frac{S(0)}{b^*}\Big)^{h_1},$$

(f) Assume the stock pays continuous dividends at the constant rate $\delta > 0$ and let $b > 0$ such that $b < S(0)$.

(i) Assume a perpetual put is exercised whenever the stock reaches the barrier b from above. Show that the discounted value at time t = 0 is

$$g(b) = (K-b)\Big(\frac{S(0)}{b}\Big)^{h_2},$$

where

$$h_2 = \frac{1}{2} - \frac{r-\delta}{\sigma^2} - \sqrt{\Big(\frac{r-\delta}{\sigma^2} - \frac{1}{2}\Big)^2 + \frac{2r}{\sigma^2}}.$$

(ii) Show that the maximum value of $g(b)$ is realized for $b^* = K\frac{h_2}{h_2-1}$.

(iii) Prove the price of perpetual put

$$f(b^*) = \frac{K}{1 - h_2}\Big(\frac{S(0)}{b^*}\Big)^{h_2},$$

(g) Let $0 < b < S(0)$ and let $t > 0$ fixed. Show that the following inequality holds

$$\tilde{\mathbb{P}}\Big(S(t) < b\Big) \leq \exp\Big\{-\frac{1}{2\sigma^2 t}\Big[\ln\Big(\frac{S(0)}{b}\Big) + (r - \frac{\sigma^2}{2})t\Big]^2\Big\}$$