Portfolio Dynamics

Let us consider a financial market with N different types of stocks

self-financing Portfolios in Discrete time:-

We consider a financial market living in discrete time on a ~~filtra~~ filtered probability space $(\Omega, \mathcal{F}, P, \mathcal{F}(t), t \geq 0)$, so we are allowed trade at discrete points in time $t = 0, 1, 2, \ldots$ On the market we can trade in N different assets with price processes $S^1, S^2, \ldots, S^N$. We will use the following notations

S_n^i = the price of one unit of asset No. i at time n

h_n^i = number of units of asset No. i bought at time n,

d_n^i = dividends from asset No. i at time n,

h_n = the portfolio $[h_n^1, \ldots, h_n^N]$,

c_n = consumption at time n,

V_n = the value of the portfolio h_{n-1} at time n.

The interpretation of the dividend process d is that if you are holding one unit of asset No. i during the interval $[n-1, n]$, then you obtain the amount d_n at time n.

Here we assume that the processes S, h, d, c are adapted to the filtration $\{\mathcal{F}(t), t \geq 0\}$.

Arbitrage Theory in continuous Time by Tomas Björk

Our model works as follows:-

1. At time n we buy the portfolio h_n at the price S_n, and the portfolio value V_n is defined as the value of h_n at time n. We then keep the portfolio until time $n+1$
2. We enter time $n+1$ carrying our old portfolio h_n with us.
3. At time $n+1$ we get our share of the dividend d_{n+1} and decide on the amount c_{n+1} to be consumed.
4. After consuming c_{n+1}, we re-balance the old portfolio h_n into the new portfolio h_{n+1}
5. we keep the portfolio h_{n+1} until time $n+2$.

We obtain the basic formula

$$V_n = \sum_{i=1}^{N} h_n^i S_n^i = h_n S_n \text{ (denotes scalar product)}.$$

Defn:- A self-financing portfolio supporting the consumption stream c is a portfolio ~~no~~ with no withdrawal of money (apart from dividends and consumption). In other words the purchase of a new portfolio, as well as all consumption, must be financed solely by the dividends obtained and/or by selling assets already in the portfolio.

self-financing condition in mathematical terms:-

1. At time $n+1$ the value of ~~the~~ our old portfolio is $h_n S_{n+1}$.

② At time $n+1$ we get the dividend d_{n+1}^{i} per unit of asset No. i. The total amount obtained from dividends is thus given by $h_n d_{n+1}$. We also choose to consume C_{n+1}.

③ We then buy the new portfolio h_{n+1} at the price S_{n+1}, so the cost of this new portfolio is $h_{n+1} S_{n+1}$.

④ The self-financing condition is the condition that, at time $n+1$, the cost of the new portfolio plus the consumption equals the value of the old portfolio plus dividends.

⑤ The self-financing budget constraint is thus given by the formula

$$h_{n+1} S_{n+1} + C_{n+1} = h_n S_{n+1} + h_n d_{n+1}.$$

In order to obtain the V-dynamics we introduce the following notation. For any sequence $\{x_n\}_{n=0}^{\infty}$ of real numbers, we define the operator Δ by

$$\Delta x_n = x_{n+1} - x_n.$$

Lemma:- For any pair of sequences of real numbers $\{x_n\}_{n=0}^{\infty}$ and $\{y_n\}_{n=0}^{\infty}$ we have the relations

$$\Delta (xy)_n = x_n \Delta y_n + y_{n+1} \Delta x_n$$

$$\Delta (xy)_n = y_n \Delta x_n + x_{n+1} \Delta y_n$$

$$\Delta (xy)_n = x_n \Delta y_n + y_n \Delta x_n + \Delta x_n \Delta y_n.$$

Recall that by definition

$$V_n = h_n S_n,$$

$$\Rightarrow \Delta V_n = \Delta(h_n S_n) = h_n \Delta S_n + S_{n+1} \Delta h_n.$$

We can write the budget contraint as

$$S_{n+1} \Delta h_n = h_n d_{n+1} - C_{n+1}.$$

Substituting this into the V dynamics gives us the following result.

<u>Proposition:-</u> The dynamics of a self-financing portfolio supporting the consumption stream c are given by

$$\Delta V_n = h_n \Delta S_n + h_n d_{n+1} - C_{n+1}$$

$$= \sum_{i=1}^{N} h_n^i \left\{ \Delta S_n^i + d_{n+1}^i \right\} - C_{n+1}.$$

<u>Defn:-</u> We define the cumulative dividend process D^i by

$$D_n^i = \sum_{K=1}^{n} d_K^i$$

We see that D_n^i is the sum of all dividends paid out over the time interval $[0,n]$ from one unit of asset No. i

$$d_{n+1}^i = \Delta D_n^i, \quad i = 1, 2, \ldots N.$$

<u>Proposition:-</u> For a self-financing portfolio supporting the consumption stream c, the dynamics are

$$\Delta V_n = h_n \Delta S_n + h_n \Delta D_n - C_{n+1}$$

$$= \sum_{i=1}^{N} h_n^i \left\{ \Delta S_n^i + \Delta D_n^i \right\} - C_{n+1}$$

where $D = (D^1, D^2, \ldots, D^N)$ is the cumulative vector dividend process.

Self-financing Portfolio in Continuous time:-

We now move to continuous time and considers a financial market with N assets. We will use the following notations

S_t^i = the price of one unit of asset No. i at time t,

h_t^i = number of units of asset No. i held at time t,

h_t = the portfolio $[h_t^1, \cdots, h_t^N]$,

D_t^i = the cumulative dividend process for asset No. i

c_t = consumption rate at time t,

V_t = the value of the portfolio h_t at time t,

We now go to the formal continuous time limit with the discrete time theory. We make the identifications

$$n \sim t$$
$$n+1 \sim t+dt$$
$$\Delta V_n \sim dV_t$$
$$\Delta S_n \sim dS_t$$
$$\Delta D_n \sim dD_t$$

Defn:- Consider an $\{\mathcal{F}(t), t \geq 0\}$ adapted N-dimensional price process S.

① A portfolio strategy is any $\{\mathcal{F}(t), t \geq 0\}$ adapted N-dimensional process $h(t) = (h_t^1, h_t^2, \cdots, h_t^N)$.

② The value process V^h corresponding to the portfolio h is given by $V_t^h = \sum_{i=1}^{N} h_t^i S_t^i$.

③ A consumption process is any $\{\mathcal{F}(t), t \geq 0\}$ adapted one-dimensional process c.

(4) A portfolio - consumption pair (h, c) is called self-financing if the value process V^h satisfies the condition

$$dV_t^h = \sum_{i=1}^{N} h_t^i \{dS_t^i + dD_t^i\} - c_t dt,$$

i.e., if $\quad dV_t^h = h_t dS_t + h_t dD_t - c_t dt.$

(5) The portfolio h is said to be Markovian if it is of the form $\quad h_t = h(t, S(t))$

for some function $h : \mathbb{R}_+ \times \mathbb{R}^N \to \mathbb{R}^N$.

Defn:- For a given portfolio h the corresponding relative portfolio or portfolio weights ω is given by

$$\omega_t^i = \frac{h_t^i S_t^i}{V_t^h}, \quad i = 1, 2, \cdots, N$$

where we have $\sum_{i=1}^{N} \omega_t^i = 1$.

Lemma:- A portfolio - consumption pair (h, c) is self-financing if and only if

$$dV_t^h = V_t^h \cdot \sum \omega_t^i \frac{dS_t^i + dD_t^i}{S_t^i} - c_t dt.$$

Remark:- If $D_t \equiv 0$ and $c_t \equiv 0$ then

$$dV_t^h = h_t dS_t = \sum_{i=1}^{N} h_t^i dS_t^i$$

and $$dV_t^h = V_t^h \cdot \sum_{i=1}^{N} \omega_t^i \frac{dS_t^i}{S_t^i}$$