CHANGE OF NUMERAIRE

- A Numeraire is the unit of account in which other assets are denominated
- one usually takes the numeraire to be the currency of a ~~cont~~ country

consider a given financial market with the risk-free asset B (money market account)

$$S_0(t) := B(t) = e^{\int_0^t r(s)\,ds}$$

where $r(t)$ is the interest rate process.

There are n-primary assets in the model and their prices satisfy equation

$$dS_i(t) = \alpha_i(t) S_i(t)\,dt + S_i(t) \sum_{j=1}^{d} \sigma_{ij}(t)\,dW_j(t)$$

$$= \alpha_i(t) S_i(t)\,dt + S_i(t)\,\sigma_i(t)\,d\bar{W}(t)$$

under the probability measure $\mathbb{P}$. Here $\bar{W}$ is a multidimensional standard wiener process.

We assume that $\exists$ a risk-neutral measure Q^0 (martingale measure). (i.e., $\exists$ a ~~multi~~ d-dimensional process $\theta(t) = (\theta_1(t), \cdots, \theta_d(t))$ satisfying the market price of risk-equation).

under Q^0 the discounted price process are martingale

In more formal terms we have the following theorem.

Theorem:- The market model is free of arbitrage if and only if there exists a martingale measure $Q^0 \sim \mathbb{P}$ such that the processes

$$\frac{S_0(t)}{S_0(t)}, \frac{S_1(t)}{S_0(t)}, \cdots, \frac{S_n(t)}{S_0(t)}$$

are martingales under Q^0.

- An arbitrage-free price system for all T-claim $\mathcal{X}$ is given by the formula

$$\pi(t,\mathcal{X}) = S_0(t)\,\mathbb{E}^0\left[\frac{\mathcal{X}}{S_0(T)}\middle|\mathcal{F}(t)\right]$$

where $\mathbb{E}^0$ denotes expectation under Q^0.

Lemma:- Let B be any strictly positive Itô-process, and define the normalized process Z with numeraire B, by $Z = S/B$. Then h is S-self-financing if and only if h is Z-self-financing, i.e.,

$$dV^S(t,h) = h(t)\,dS(t)$$

if and only if

$$dV^Z(t,h) = h(t)\,dZ(t).$$

Proof:- $Z = S/B$ and $V^Z = \dfrac{V^S}{B}$

Let h be S-self-financing then

$$dV^S(t) = h(t)\,dS(t) \text{ and } V^S = h(t)S(t).$$

$$dV^Z = \tfrac{1}{B}\,dV^S + V^S\,d(1/B) + dV^S\,d(1/B)$$

$$\Rightarrow dV^Z = \tfrac{1}{B}\, h(t)\, dS(t) + h(t) S(t)\, d(1/B) + h(t)\, dS(t)\, d(1/B) \quad \text{--- (1)}$$

Now, $$dZ(t) = \frac{1}{B}\, dS(t) + S(t)\, d(1/B) + dS(t)\, d(1/B) \quad \text{--- (2)}$$

(1) & (2) $\Rightarrow$ $dV^Z = h(t)\, dZ(t)$

$\Rightarrow$ h is Z-self-financing.

Now assume that $h(t)$ is Z-self-financing. Then

$$dV^Z(t) = h(t)\, dZ(t) \text{ and } V^Z(t) = h(t) Z(t).$$

Note that $V^S = V^Z B$

$$\Rightarrow dV^S = B\, dV^Z + V^Z dB + dV^Z dB$$
$$= B\, h(t)\, dZ(t) + h(t) Z(t)\, dB + h(t)\, dZ(t)\, dB(t) \quad \text{--- (3)}$$

and $S = ZB$

$$\Rightarrow dS = B\, dZ(t) + Z(t)\, dB(t) + dZ(t)\, dB(t) \quad \text{--- (4)}$$

(3) & (4) $\Rightarrow$ $dV^S = h(t)\, dS(t)$

Therefore $h(t)$ is S-self-financing.

Changing the Numeraire:-

Suppose that for a specific ~~nu~~ numeraire S_0 we have determine corresponding martingale measure Q^0 and the associated dynamics of the asset prices.

- Suppose now we want to change the ~~numerian~~ numeraire from S_0 to say S_1. Then we want to find the appropriate Girsanov tranformation which will take us from Q^0 to Q^1, where Q^1 is the martingale measure corresponding to the numeraire S_1.

Let us use the pricing part of the above theorem for an arbitrary choice of T-claim $\mathcal{X}$. we have

$$\Pi(0,\mathcal{X}) = S_0(0)\,\mathbb{E}^0\left[\mathcal{X}/S_0(T) \,\middle|\, \mathcal{F}(0)\right] = S_0(0)\,\mathbb{E}^0\left[\mathcal{X}/S_0(T)\right]$$

$$\Pi(0,\mathcal{X}) = S_1(0)\,\mathbb{E}^1\left[\mathcal{X}/S_1(T)\right]$$

Let $L_0^1(t)$ be the Radon-Nikodym derivative

$$L_0^1(T) = \frac{dQ^1}{dQ^0} \text{ on } \mathcal{F}(T).$$

$$\text{i.e.,} \quad Q^1(A) = \int_A L_0^1(T)\,dQ^0 \quad \forall A \in \mathcal{F}(T).$$

We can write

$$\Pi_0(0,\mathcal{X}) = S_1(0)\,\mathbb{E}^0\left[\frac{\mathcal{X}}{S_1(T)}\cdot L_0^1(T)\right]$$

$$\Rightarrow S_0(0)\,\mathbb{E}^0\left[\mathcal{X}/S_0(T)\right] = S_1(0)\,\mathbb{E}^0\left[\mathcal{X}/S_1(T)\, L_0^1(T)\right].$$

for all T-claims $\mathcal{X}$. We thus deduce that

$$\Rightarrow \frac{S_0(0)}{S_0(T)} = \frac{S_1(0)}{S_1(T)}\cdot L_0^1(T).$$

$$\Rightarrow L_0^1(T) = \frac{S_1(T)}{S_0(T)}\cdot\frac{S_0(0)}{S_1(0)}.$$

which is our candidate as a Radon-Nikodym derivative.

The Radon-Nikodym derivative process is

$$L_0^1(t) = \mathbb{E}^0\left[L_0^1(T) \,\middle|\, \mathcal{F}(t)\right]$$

$$= \frac{S_0(0)}{S_1(0)}\cdot\frac{S_1(t)}{S_0(t)}$$

since $\frac{S_1(t)}{S_0(t)}$ is a Q^0 martingale.