$\Delta(t)$ shares of the underlying asset is given by

$$d(e^{-rt}X(t)) = e^{-rt}\,\sigma S(t)\Delta(t)\,d\widetilde{W}(t)$$

At least theoretically, if an agent begins with a short position in the up-and-out call and with initial capital $X(0) = v(0, S(0))$, the usual delta-hedging formula

$$\Delta(t) = v_x(t, S(t))$$

will cause her portfolio value $X(t)$ to track the option value $v(t, S(t))$ up to time ρ of knock out or up to expiration T, whichever comes first.

Lookback Options:-

An option whose payoff is based on the maximum or minimum of the underlying asset price attains over some interval of time prior to expiration is called a lookback option. Examples

$V(T) = \max\left[\max_{t\le T} S(t) - K, 0\right]$, $V(T) = \max\left[K - \min_{t\le T} S(t), 0\right]$, $V(T) = S(T) - \min_{t\le T} S(t)$.

Here we consider floating strike lookback option. The payoff of this option is given by $V(T) = Y(T) - S(T)$, at expiration time T,

where $S(t) = S(0)e^{\sigma \widehat{W}(t)}$, $\widehat{W}(t) = \alpha t + \widetilde{W}(t)$, $\alpha = \frac{1}{\sigma}(r - \frac{1}{2}\sigma^2)$

with $\widehat{M}(t) = \max_{0\le u\le t} \widehat{W}(u)$ and $Y(t) = \max_{0\le u\le t} S(t) = S(0)e^{\sigma \widehat{M}(t)}$

$S(t) = rS(t)\,dt + \sigma S(t)\,d\widetilde{W}(t)$, $\widetilde{W}(t)$ is a Brownian motion under risk-neutral measure $\widetilde{\mathbb{P}}$.

$$S(t) = e^{(r - \frac{1}{2}\sigma^2)t + \sigma \widetilde{W}(t)}$$

Let $t \in [0,T]$ be given. At time t, the risk-neutral price of the lookback option is

$$V(t) = \widehat{\mathbb{E}}\left[e^{-r(T-t)}\left(Y(T) - S(T)\right) \mid \mathcal{F}(t)\right].$$

Because the pair of process $(S(t), Y(t))$ has the Markov property, there must exist a function $v(t,x,y)$ such that

$$V(t) = v(t, S(t), Y(t)).$$

For $0 \le t < T$ and $\tau = T-t$, we observe that

$$Y(T) = S(0)\, e^{\sigma \widehat{M}(t)} e^{\sigma(\widehat{M}(T) - \widehat{M}(t))} = Y(t)\, e^{\sigma(\widehat{M}(T) - \widehat{M}(t))}$$

If $\max\limits_{t \le u \le T} \widehat{W}(u) > \widehat{M}(t)$ (ie, if $\widehat{W}(t)$ attains a new maximum in $[t,T]$), then

$$\widehat{M}(T) - \widehat{M}(t) = \max_{t \le u \le T} \widehat{W}(u) - \widehat{M}(t).$$

On the other hand, if $\max\limits_{t \le u \le T} \widehat{W}(u) \le \widehat{M}(t)$, then $\widehat{M}(T) = \widehat{M}(t)$ and.

$$\widehat{M}(T) - \widehat{M}(t) = 0.$$

In either case, we have

$$\widehat{M}(T) - \widehat{M}(t) = \left[\max_{t \le u \le T} \widehat{W}(u) - \widehat{M}(t)\right]^+$$

$$= \left[\max_{t \le u \le T} \left(\widehat{W}(u) - \widehat{W}(t)\right) - \left(\widehat{M}(t) - \widehat{W}(t)\right)\right]^+$$

$$\Rightarrow \sigma\left(\widehat{M}(T) - \widehat{M}(t)\right) = \left[\max_{t \le u \le T} \sigma\left(\widehat{W}(u) - \widehat{W}(t)\right) - \sigma\left(\widehat{M}(t) - \widehat{W}(t)\right)\right]^+$$

$$= \left[\max_{t \le u \le T} \sigma\left(\widehat{W}(u) - \widehat{W}(t)\right) - \log\left(Y(t)/S(t)\right)\right]^+$$

Therefore

$$V(t) = e^{-r\tau}\, \widehat{\mathbb{E}}\left[Y(t) \exp\left\{\left[\max_{t \le u \le T} \sigma\left(\widehat{W}(u) - \widehat{W}(t)\right) - \log \frac{Y(t)}{S(t)}\right]^+\right\} \Big| \mathcal{F}(t)\right].$$

$$- e^{rt}\, \widehat{\mathbb{E}}\left[e^{-rT} S(T) \mid \mathcal{F}(t)\right]$$

Because the discounted asset price is a martingale under $\widehat{\mathbb{P}}$

the second term is

$$-e^{rt}e^{-rt}S(t) = S(t)$$

For the first term, we can take out what is known to obtain

$$e^{-r\tau}Y(t)\,\widehat{\mathbb{E}}\left[\exp\left\{\left[\max_{t\le u\le T}\sigma(\widehat{W}(u)-\widehat{W}(t)) - \log\frac{Y(t)}{S(t)}\right]^{+}\right\}\,\Big|\,\mathcal{F}(t)\right]$$

Because $Y(t)$ and $S(t)$ are $\mathcal{F}(t)$-measurable and

$\max\limits_{t\le u\le T}\sigma(\widehat{W}(u)-\widehat{W}(t))$ is independent of $\mathcal{F}(t)$, we can use the independent Lemma to write the conditional expectation as $g(S(t), Y(t))$, where

$$g(x,y) = \widehat{\mathbb{E}}\left[\exp\left\{\left[\max_{t\le u\le T}\sigma(\widehat{W}(u)-\widehat{W}(t)) - \log y/x\right]^{+}\right\}\right]$$

Therefore we have

$$V(t) = e^{-r\tau}Y(t)\,g(S(t),Y(t)) - S(t) \quad \text{or}$$

$$v(t,x,y) = e^{-r\tau}y\,g(x,y) - x.$$

To compute the function $g(x,y)$. Note that

$\max\limits_{t\le u\le T}(\widehat{W}(u)-\widehat{W}(t))$ has the same distribution under $\widehat{\mathbb{P}}$ as

$\max\limits_{0\le u\le \tau}(\widehat{W}(u)-\widehat{W}(0)) = \widehat{M}(\tau)$. Hence

$$g(x,y) = \widehat{\mathbb{E}}\left[\exp\left[\sigma\widehat{M}(\tau) - \log y/x\right]^{+}\right]$$

$$= \widehat{\mathbb{P}}\left[\widehat{M}(\tau) \le \tfrac{1}{\sigma}\log y/x\right] + x/y\,\widehat{\mathbb{E}}\left[e^{\sigma\widehat{M}(\tau)}\mathbb{1}_{\{\widehat{M}(\tau)\ge\frac{1}{\sigma}\log y/x\}}\right].$$

Theorem:- Let $v(t,x,y)$ denote the price at time t of the floating strike lookback option under the assumption that $S(t)=x$ and $Y(t)=y$. Then $v(t,x,y)$ satisfies the B-S-M partial differential equation

$$v_t(t,x,y) + rx\,v_x(t,x,y) + \tfrac{1}{2}\sigma^2 x^2 v_{xx}(t,x,y) = r\,v(t,x,y).$$

in the region $\{(t,x,y) : 0 \le t < T,\ 0 \le x \le y\}$ and satisfies the boundary conditions

$$v(t,0,y) = e^{-r(T-t)}y, \quad 0 \le t \le T,\ y \ge 0.$$

$$v_y(t,y,y) = 0, \quad 0 \le t \le T,\ y > 0.$$

$$v(T,x,y) = y - x, \quad 0 \le x \le y.$$

$$e^{-rt}V(t) = e^{-rt}v(t, S(t), Y(t)) \text{ is a martingale under } \widetilde{\mathbb{P}}.$$

Y is continuous and nondecreasing in t. Let $0 = t_0 < t_1 < t_2 \dots < t_m = T$ be a partition of $[0,T]$. Then

$$\sum_{j=1}^{m} \left(Y(t_j) - Y(t_{j-1})\right)^2$$

$$\le \max_{j=1,\dots m} \left(Y(t_j) - Y(t_{j-1})\right) \sum_{j=1}^{m} \left(Y(t_j) - Y(t_{j-1})\right)$$

$$= \max_{j=1,2..m} \left(Y(t_j) - Y(t_{j-1})\right) \cdot \left(Y(T) - Y(0)\right) \xrightarrow{\|\Pi\| \to 0} 0$$

Because $Y(t)$ is continuous. ($Y(t)$ has zero quadratic variation).

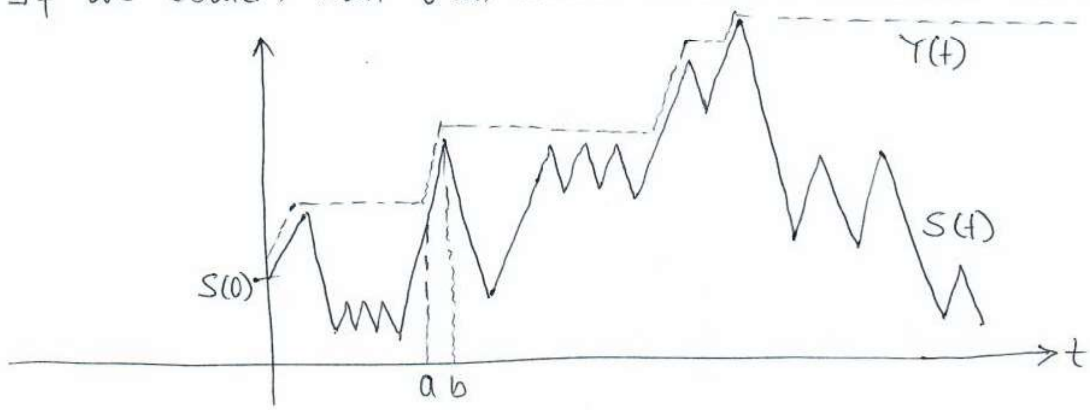
$$\therefore dY(t)\,dY(t) = 0.$$

Exercise:- Prove that $dY(t)\,dS(t) = 0$.

We can not write $Y(t)$ as

$$Y(t) = Y(0) + \int_0^t \theta(u)\,du$$

If we could, then $\theta(u)$ would be zero whenever $Y(t)$ is flat.



Let $[a,b]$ be an interval where $Y(t)$ is strictly increasing such an interval can occur only if $S(t)$ is strictly increasing on the interval, then $S(t)$ would accumulate zero quadratic variance on the interval (By the previous Argument). But $dS(t)\,dS(t) = \sigma^2 S^2(t)\,dt$ is positive for all t. Therefore $\theta(u) = 0$ for Lebesgue almost every u in $[0,T]$. This implies that $Y(t) = Y(0)$ for $0 \le t \le T$. But in fact $Y(t) > Y(0)$ for all $t > 0$. We conclude that $Y(t)$ can not be represented in the above form.

By Itô's-formula, we have

$$d\left(e^{-rt} v(t, S(t), Y(t))\right) = e^{-rt}(-r)\, v(t, S(t), Y(t))\,dt$$

$$+ e^{-rt}\Big[v_t(t, S(t), Y(t))\,dt + v_x(t, S(t), Y(t))\,dS(t)$$

$$+ v_y(t, S(t), Y(t))\,dY(t) + \tfrac{1}{2} v_{xx}(t, S(t), Y(t))\,dS(t)\,dS(t)$$

$$+ \tfrac{1}{2} v_{yy}(t, S(t), Y(t))\,dY(t)\,dY(t) + v_{xy}(t, S(t), Y(t))\,dS(t)\,dY(t)\Big].$$

$$= e^{-rt}\Big[-r\,v(t,S(t),Y(t)) + v_t(t,S(t),Y(t)) + r\,S(t)\,v_x(t,S(t),Y(t))$$
$$+ \tfrac{1}{2}\sigma^2 S^2(t)\,v_{xx}(t,S(t),Y(t))\Big]dt + e^{-rt}\,v_y(t,S(t),Y(t))\,dY(t)$$
$$+ e^{-rt}\,\sigma\,S(t)\,v_x(t,S(t),Y(t))\,d\widetilde{W}(t).$$

In order to have a martingale, the dt term must be zero and this gives us the Black-scholes-Merton equation. The term $e^{-rt} v_y(t,S(t),Y(t))\,dY(t)$ must also be zero. Note that $dY(t)$ term is naturally zero on the 'flat spots' of $Y(t)$ (ie., when $S(t)<Y(t)$). At the times when $Y(t)$ increases, which are the times when $S(t)=Y(t)$ the term $e^{-rt} v_y(t,S(t),Y(t))$ must be zero, because $dY(t)$ is positive This gives us $v_y(t,y,y)=0$, $y\geq 0$, $0\leq t\leq T$.

If any time t, we have $S(t)=0$, then we will have $S(T)=0$ and Y will be constant on $[t,T]$ if $Y(t)=y$ then $Y(T)=y$ and the price of the option at time t is

$$V(t) = \widetilde{\mathbb{E}}\left[e^{-r(T-t)}(y-0)\right] = y\,e^{-r(T-t)}.$$

Remark:- The discounted value of a portfolio that each time t holds $\Delta(t)$ shares of the underlying asset is given by

$$d\left(e^{-rt}X(t)\right) = e^{-rt}\sigma S(t)\,\Delta(t)\,d\widetilde{W}(t)$$

Also note that

$$d\left(e^{-rt}v(t,S(t),Y(t))\right) = e^{-rt}\sigma S(t)\,v_x(t,S(t),Y(t))\,d\widetilde{W}(t)$$

This implies that the delta-hedging formula is

$$\Delta(t) = v_x(t,S(t),Y(t)).$$