MA 373 : Financial Engineering II

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Exercises 4

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1. Consider the standard Black-Scholes model and a T-claim $\mathcal{X}$ of the form $\mathcal{X} = \Phi(S(T))$. Denote the corresponding arbitrage free price process by $\Pi(t)$

   (i) Show that, under the risk-neutral measure $\tilde{\mathbb{P}}$, $\Pi(t)$ has a rate of return equal to the short rate of interest r. In other words show that $\Pi(t)$ has a differential of the form
   $$d\Pi(t) = r\Pi(t)dt + g(t)d\tilde{W}(t)$$
   (ii) Show that, under the risk-neutral measure $\tilde{\mathbb{P}}$, the process $Z(t) = \frac{\Pi(t)}{B(t)}$ is a martingale.

2. Consider the standard Black-Scholes model and a T-claim $\mathcal{X}$ of the form $\mathcal{X} = \Phi(S(T))$, where
   $$\Phi(s) = \begin{cases} K & \text{if } s \in [\alpha, \beta] \\ 0 & \text{otherwise.} \end{cases}$$
   ($\mathcal{X}$ is called binary option). Determine the arbitrage free price. The pricing formula will involve the standard Gaussian cumulative distribution function N.

3. Consider the standard BlackScholes model. Derive the arbitrage free price process for the claim $\mathcal{X}$ where $\mathcal{X}$ is given by $\mathcal{X} = \frac{S(T)}{S(T_0)}$. The times T_0 and T are given and the claim is paid out at time T.

4. Consider the simplest possible incomplete market, namely a market where the only randomness comes from a scalar stochastic process whichis not the price of a traded asset $X(t)$, with $\mathbb{P}$ dynamics given by
   $$dX(t) = \alpha(t, X(t))dt + \sigma(t, X(t))dW(t).$$
   Now Consider a T-claim $\mathcal{X} = \Phi(X(T))$ with pricing function $\Pi(t) = F(t, x)$.

   (i)Prove that dF under any risk-neutral measure Q has the form
   $$dF = rFdt + \{\cdots\}d\tilde{W}(t),$$
   where $\tilde{W}$ is a Q-Wiener process.

   (ii) Show that, under risk-neutral measure Q, the process $Z(t) = \frac{\Pi(t)}{B(t)}$ is a martingale.

5. We take as given an interest rate model with the following $\mathbb{P}$-dynamics for the short rate.
   $$dr(t) = \alpha(t, r(t))dt + \sigma(t, r(t))dW(t).$$
   Now consider a T-claim of the form $\mathcal{X} = \Phi(r(T))$ with corresponding price process $\Pi(t)$.

   (i) Show that, under any risk-neutral measure Q, the price process $\Pi(t)$ has a rate of return equal to the short rate of interest. In other words, show that the stochastic differential of $\Pi(t)$ is of the form
   $$d\Pi(t) = r(t)\Pi(t)dt + \sigma_\Pi \Pi(t)d\tilde{W}(t).$$
   (ii) Show that the normalized price process $Z(t) = \frac{\Pi(t)}{B(t)}$ is a Q-martingale.

6. (i) Assuming that we are allowed to differentiate under the expectation sign, show that the forward rates

$$f(t,T) = \frac{\mathbb{E}^Q_{t,r(t)}[r(T)\exp\{-\int_t^T r(s)ds\}]}{\mathbb{E}^Q_{t,r(t)}[\exp\{-\int_t^T r(s)ds\}]}$$

(ii) Check that indeed $r(t) = f(t,t)$.

7. Let $\{y(0,T); T \geq 0\}$ denote the zero coupon yield curve at $t = 0$. Assume that, apart from the zero coupon bonds, we also have exactly one fixed coupon bond for every maturity T. We make no particular assumptions about the coupon bonds, apart from the fact that all coupons are positive, and we denote the yield to maturity, again at time $t = 0$, for the coupon bond with maturity T, by $y_M(0,T)$. We now have three curves to consider: the forward rate curve $f(0,T)$, the zero coupon yield curve $y(0,T)$, and the coupon yield curve $y_M(0,T)$. The object of this exercise is to see how these curves are connected.

   (a) Show that

$$f(0,T) = y(0,T) + T\frac{\partial y(0,T)}{\partial T}.$$

   (b) Assume that the zero coupon yield curve is an increasing function of T. Show that this implies the inequalities

$$y_M(0,T) \leq y(0,T) \leq f(0,T), \forall T,$$

   (with the opposite inequalities holding if the zero coupon yield curve is decreasing).

8. Consider the Vasicek model, where we always assume that $a > 0$.

   (a) Solve the Vasicek SDE explicitly, and determine the distribution of $r(t)$.

   (b) As $t \to \infty$, the distribution of $r(t)$ tends to a limiting distribution. Show that this is the Gaussian distribution $N[\frac{b}{a}, \frac{\sigma}{\sqrt{2a}}]$. Thus we see that, in the limit, r will indeed oscillate around its mean reversion level $\frac{b}{a}$.

   (c) Now assume that $r(0)$ is a random variable, independent of the Brownian motion W, and by definition having the Gaussian distribution obtained in (b). Show that this implies that $r(t)$ has the limit distribution in (b), for all values of t.

9. Show directly that the Vasicek model has an affine term structure. Use the characterization of $p(t,T)$ as an expected value, insert the solution of the SDE for r, and look at the structure obtained.

10. The object of this exercise is to indicate why the CIR model is connected to squares of linear diffusions. Let Y be given as the solution to the following SDE.

$$dY(t) = (2aY(t) + \sigma^2)dt + \sigma\sqrt{Y(t)}dW(t),\ Y(0) = y_0.$$

    Define the process $Z(t)$ by $Z(t) = \sqrt{Y(t)}$. Show that $Z(t)$ satisfies a linear stochastic differential equation.