Asian Options:-

An option whose payoff includes a time average of the underlying asset price.

Our underlying asset is

$$dS(t) = rS(t)\,dt + \sigma S(t)\,d\widetilde{W}(t),$$

where $\widetilde{W}(t)$ is a Brownian motion under the risk-neutral measure $\widetilde{\mathbb{P}}$. Consider a Asian call option whose payoff at time T is

$$V(T) = \left(\frac{1}{T}\int_0^T S(t)\,dt - K\right)^+$$

where the strike price K is a non-negative constant.

By risk-neutral pricing formula

$$V(t) = \widetilde{\mathbb{E}}\left[e^{-r(T-t)}V(T)\,\middle|\,\mathcal{F}(t)\right], \quad 0 \le t \le T.$$

The usual iterated conditioning argument show that $e^{-rt}V(t)$ $0 \le t \le T$, is a martingale under $\widetilde{\mathbb{P}}$.

$$\text{Let } Y(t) = \int_0^t S(u)\,du.$$

$$\text{Then } dY(t) = S(t)\,dt.$$

And the pair of processes $(S(t), Y(t))$ is a two-dimensional Markov process. and $V(T) = \left(\frac{1}{T}Y(T) - K\right)^+$

This implies that there exist some function $v(t, x, y)$ such that

$$v(t, S(t), Y(t)) = \widetilde{\mathbb{E}}\left[e^{-r(T-t)}\left(\frac{1}{T}Y(T) - K\right)^+ \,\middle|\, \mathcal{F}(t)\right]$$

$$= \widetilde{\mathbb{E}}\left[e^{-r(T-t)}V(T)\,\middle|\,\mathcal{F}(t)\right].$$

Theorem:- The Asian call price function $v(t,x,y)$ satisfies the partial differential equation

$$v_t(t,x,y) + rx\,v_x(t,x,y) + x\,v_y(t,x,y) + \tfrac{1}{2}\sigma^2 x^2 v_{xx}(t,x,y)$$
$$= r\,v(t,x,y) \;,\; 0 \le t \le T \;,\; x \ge 0 \;,\; y \in \mathbb{R}.$$

and the boundary condition

$$v(t,0,y) = e^{-r(T-t)}\left(\tfrac{y}{T} - K\right)^+$$

$$\lim_{y\to-\infty} v(t,x,y) = 0 \;,\; 0 \le t < T,\; x \ge 0.$$

$$v(T,x,y) = \left(\tfrac{y}{T} - K\right)^+ \;,\; x \ge 0,\; y \in \mathbb{R}.$$

Proof:- Note that $dS(t)\,dY(t) = dY(t)\,dY(t) = 0$.

Now $d\left(e^{-rt} v(t, S(t), Y(t))\right)$

$$= e^{-rt}\left[-rv\,dt + v_t\,dt + v_x\,dS + v_y\,dY + \tfrac{1}{2} v_{xx}\,dS\,dS\right]$$
$$= e^{-rt}\left[-rv + v_t + rS\,v_x + S\,v_y + \tfrac{1}{2}\sigma^2 S^2 v_{xx}\right]dt$$
$$+ e^{-rt}\sigma S\, v_x\, d\widetilde{W}(t).$$

In order for this to be a martingale, the dt term must be zero, which implies

$$v_t(t,S(t),Y(t)) + r\,S(t)\,v_x(t,S(t),Y(t)) + S(t)\,v_y(t,S(t),Y(t))$$
$$+ \tfrac{1}{2}\sigma^2 S^2(t)\,v_{xx}(t,S(t),Y(t)) = r\,v(t,S(t),Y(t)).$$

If $S(t) = 0$ and $Y(t) = y$ for some value of t, then $S(u) = 0$ for all $u \in [t,T]$, and so $Y(u)$ is constant on $[t,T]$.

Therefore $Y(T) = y$ and $V(T) = \left(\tfrac{y}{T} - K\right)^+$. Hence

$$v(t,0,y) = e^{-r(T-t)}\left(\tfrac{y}{T} - K\right)^+.$$

Mathematically there is no problem with allowing y to be negative. If at time t we set $Y(t) = y$, then

$$Y(T) = y + \int_t^T S(u)\,du$$

Even if y is negative, this make sense, and in this case we could still have $Y(T) > 0$ or even $\frac{1}{T}Y(T) - K > 0$, so that the call expires in the money.

If $Y(t) = y$, $S(t) = x$ and holding x fixed and let $y \to -\infty$ then $Y(T)$ approaches $-\infty$ and $V(T) \to 0$

$$\lim_{y \to -\infty} v(t, x, y) = 0$$

$$v(T, x, y) = (y/T - K)^+, \quad x \geq 0, \; y \in \mathbb{R}.$$

Change of Numeraire:-

We first consider the case of an Asian call option with payoff $V(T) = \left(\frac{1}{T}\int_0^T S(t)\,dt - 0\right)^+ = \frac{1}{T}\int_0^T S(t)\,dt$ (with $K = 0$).

To price this call we create a portfolio process whose value at time T is

$$X(T) = \frac{1}{T}\int_0^T S(u)\,du.$$

Note that $\widetilde{\mathbb{E}}\left[\int_0^T S(u)\,du \,\middle|\, \mathcal{F}(t)\right] = \widetilde{\mathbb{E}}\left[\int_0^t S(u)\,du \,\middle|\, \mathcal{F}(t)\right] + \widetilde{\mathbb{E}}\left[\int_t^T S(u)\,du \,\middle|\, \mathcal{F}(t)\right]$

$$= \int_0^t S(u)\,du + \int_t^T \widetilde{\mathbb{E}}[S(u)]\,du$$

$$\widetilde{\mathbb{E}}[S(u)] = \widetilde{\mathbb{E}}\left[\widetilde{\mathbb{E}}[S(u) \mid \mathcal{F}(t)]\right] = \widetilde{\mathbb{E}}\left[S(t)\,e^{-r(t-u)}\right].$$

using the martingale property of the discounted stock price we can replace $\widetilde{\mathbb{E}}[S(u)]$ with $e^{r(u-t)}S(t)$ which yields

$$\widetilde{\mathbb{E}}\left[\int_0^T S(u)\,du \,\Big|\, \mathcal{F}(t)\right] = \int_0^t S(u)\,du + S(t)\int_t^T e^{-r(t-u)}\,du.$$

$$= \int_0^t S(u)\,du + \frac{S(t)}{r}\left(e^{r(T-t)} - 1\right)$$

It follows that

$$e^{-r(T-t)}\,\widetilde{\mathbb{E}}\left[\frac{1}{T}\int_0^T S(u)\,du \,\Big|\, \mathcal{F}(t)\right] = e^{-r(T-t)}\frac{1}{T}\int_0^t S(u)\,du + \frac{S(t)}{rT}\left(1-e^{-r(T-t)}\right)$$

and $$v(t,x,y) = y\,\frac{e^{-r(T-t)}}{T} + x\,\frac{1-e^{-r(T-t)}}{rT}.$$

$$\frac{\partial v}{\partial t} = (ry - x)\frac{e^{-r(T-t)}}{T}, \quad \frac{\partial v}{\partial x} = \frac{1-e^{-r(T-t)}}{rT}, \quad \frac{\partial v}{\partial y} = \frac{e^{-r(T-t)}}{T}.$$

$$v(t,0,y) = y\,\frac{e^{-r(T-t)}}{T}.$$

$$v(T,x,y) = y/T.$$

$$\Delta(t) = \frac{1-e^{-r(T-t)}}{rT}.$$

This quantity is non-random, since it only depends on time but not on the current value of $S(t)$ or its history.

To price this call option we create a portfolio process whose value at time T is

$$X(T) = \frac{1}{T}\int_0^T S(u)\,du - K$$