Stochastic Differential Equations:-

A stochastic differential equation (SDE) is an equation of the form

$$\left.\begin{aligned} dX(t) &= b(t, X(t))\,dt + \sigma(t, X(t))\,dW(t), \\ X(0) &= z \end{aligned}\right\} \quad \cdots (1)$$

Here $b(t,x)$ and $\sigma(t,x)$ are given functions and $z \in \mathbb{R}$.

The mathematical interpretation of the SDE is that $X(t)$ is a solution of the integral equation

$$X(t) = X(0) + \underbrace{\int_0^t b(s, X(s))\,ds}_{\text{Lebesgue integral}} + \underbrace{\int_0^t \sigma(s, X(s))\,dW(s)}_{\text{Itô integral}}.$$

<u>Q1:</u> Does there exist a solution?

and if there is a solution then is it unique?

<u>Q2:</u> How to solve such a differential equation?

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $\{W(t), t \geq 0\}$ a Brownian motion (BM) defined on it. Let $\mathcal{F}(t)$ be the filtration generated by $W(t)$. i.e., $\mathcal{F}(t) = \sigma\{W(s) : s \leq t\}$.

<u>Definition:-</u> A solution of the SDE above is a continuous stochastic process $X(t)$, $0 \leq t \leq T$ with the following properties

(i) $X(t)$ is adapted to the filtration $\mathcal{F}(t)$.

(ii) $\mathbb{P}(X(0)=x)=1$.

(iii) $\mathbb{E}\int_0^T |b(t,X(t))|\,dt<\infty$, $\mathbb{E}\int_0^T |\sigma^2(t,X(t))|\,dt<\infty$.

(iv) $X(t)=X(0)+\int_0^t b(s,X(s))\,ds+\int_0^t \sigma(s,X(s))\,dW(s)$,

$0\le t\le T$. a.s.

Definition:- The SDE above is said to have a unique solution if X and $\tilde{X}$ are two solutions then

$$\mathbb{P}\left[X(t)=\tilde{X}(t),\ 0\le t\le T\right]=1.$$

Theorem:- suppose that the coefficient $b(t,x)$, $\sigma(t,x)$ satisfy the global Lipschitz and linear growth conditions

$$|b(t,x)-b(t,y)|+|\sigma(t,x)-\sigma(t,y)|\le K|x-y| \quad \text{---} (*)$$

$$|b(t,x)|+|\sigma(t,y)|\le K(1+|x|) \quad \text{---} (**)$$

for some positive constant K. Then the SDE (1) has a unique solution and $\mathbb{E}\int_0^T |X(t)|^2\,dt<\infty$.

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consider deterministic differential equations.

$$\left.\begin{array}{l} dX(t)=X^2(t)\,dt \\ X(0)=1 \end{array}\right\} \text{--- (2)} \quad \& \quad \left.\begin{array}{l} dX(t)=3X^{2/3}(t)\,dt \\ X(0)=0 \end{array}\right\}.$$

For the first one $b(t,x)=x^2$, does not satisfy the linear growth condition. For the second one $b(t,x)=3x^{2/3}$ does not satisfy the Lipschitz condition.

In the first case the solution is

$$X(t) = \frac{1}{1-t}$$

But this explodes as $t \to 1$.

The linear growth condition ensures that the solution does not explode in a finite time.

For the second case, there are infinitely many solution in fact for any $a>0$

$$X(t) = \begin{cases} 0 & \text{for } t \le a \\ (t-a)^3 & \text{for } t > a \end{cases}$$

is a solution. Thus Lipschitz condition ensures uniqueness.

Gronwall's Inequality:- Let $f(\cdot)$ be a continuous function such that $f(t) \le C + K\int_0^t f(s)ds$ for $0 \le t \le T$, where C is a constant and K is a positive constant. Then

$$f(t) \le Ce^{Kt} \text{ for } 0 \le t \le T.$$

Proof:- Define $g(t) = C + K\int_0^t f(s)\,ds \quad \forall t \in [0,T]$.

Now by fundamental theorem of calculus,

$$g'(t) = Kf(t)$$

Note that $g(t) = C + K\int_0^t f(s)ds \ge f(t) \quad \forall t \in [0,T]$.

$$\Rightarrow g'(t) \le Kg(t) \Rightarrow e^{-Kt}g'(t) - Ke^{-Kt}g(t) \le 0.$$

$$\Rightarrow \frac{d}{dt}\left(e^{-Kt} g(t)\right) \le 0.$$

$$\Rightarrow \left(e^{-Kt} g(t)\right) \downarrow.$$

$$\Rightarrow e^{-Kt} g(t) - g(0) \le 0 \Rightarrow g(t) \le g(0) e^{Kt}$$

$$\Rightarrow f(t) \le C e^{Kt}.$$

Proof of uniqueness:- suppose $\exists$ two solutions $X_1(t)$ & $X_2(t)$. Then

$$X_1(t) = x + \int_0^t b(s, X_1(s))\,ds + \int_0^t \sigma(s, X_1(s))\,dW(s).$$

$$X_2(t) = x + \int_0^t b(s, X_2(s))\,ds + \int_0^t \sigma(s, X_2(s))\,dW(s).$$

Thus, $\mathbb{E}|X_1(t) - X_2(t)|^2$

$$= \mathbb{E}\left| \int_0^t \{b(s, X_1(s)) - b(s, X_2(s))\}\,ds + \int_0^t \{\sigma(s, X_1(s)) - \sigma(s, X_2(s))\}\,dW(s)\right|^2$$

$$\le 2\Big[\mathbb{E}\left|\int_0^t \{b(s, X_1(s)) - b(s, X_2(s))\}\,ds\right|^2$$

$$+ \mathbb{E}\left|\int_0^t \{\sigma(s, X_1(s)) - \sigma(s, X_2(s))\}\,dW(s)\right|^2$$

$$\le 2\Big[t\,\mathbb{E}\int_0^t |b(s, X_1(s)) - b(s, X_2(s))|^2\,ds$$

$$+ \mathbb{E}\int_0^t |\sigma(s, X_1(s)) - \sigma(s, X_2(s))|^2\,ds\Big]$$

(By Holder inequality and Ito-Isometry)

Note that $|b(s, X_1(s)) - b(s, X_2(s))|^2 \le K^2 |X_1(s) - X_2(s)|^2$ and

$$|\sigma(s, X_1(s)) - \sigma(s, X_2(s))|^2 \le K^2 |X_1(s) - X_2(s)|^2.$$

Therefore

$$\mathbb{E}|X_1(t)-X_2(t)|^2 \le 2K^2(1+t)\,\mathbb{E}\int_0^t |X_1(s)-X_2(s)|^2 ds.$$

Let $f(t)=\mathbb{E}\left[|X_1(t)-X_2(t)|^2\right]$

Then $$f(t) \le 2K^2(1+t)\int_0^t f(s)\,ds.$$

So by Gronwall's inequality $f(t)=0 \quad \forall t\ge 0$.

Hence $\mathbb{E}|X_1(t)-X_2(t)|^2=0$.

$\Rightarrow \mathbb{P}\left[X_1(t)=X_2(t)\right]=1$ for each $t\in[0,T]$.

$\Rightarrow \mathbb{P}\left[X_1(t)=X_2(t)\ \forall t\in \mathbb{Q}\cap[0,T]\right]=1$.

where $\mathbb{Q}$ denotes the set of rational numbers.

Note that $t\to X_1(t)$ is continuous and

$t\to X_2(t)$ is also continuous

Therefore $t\longrightarrow |X_1(t)-X_2(t)|$ is also continuous. Hence

$$\mathbb{P}\left(X_1(t)=X_2(t)\ \forall t\in[0,T]\right)=1.$$

Hence we have uniqueness.

For ~~exist~~ the proof of the existence (see Bernt Oksendal Stochastic Differential equations)