If we wish to make a model for the bond market, it is obvious that this can be done in many different ways

- We may specify the dynamics of the short rate (and then try to derive bond prices using arbitrage arguments).
- We may directly specify the dynamics of all possible bonds.
- We may specify the dynamics of all possible forward rates and then use the above Lemma in order to obtain bond prices

All these approaches are related to each other

– Relation between $df(t,T)$, $dp(t,T)$ and $r(t)$

We will consider dynamics of the following form

Short rate dynamics

$$dr(t) = a(t)\,dt + b(t)\,dW(t) \quad \text{-----} \quad (1)$$

Bond price dynamics

$$dp(t,T) = p(t,T)\,m(t,T)\,dt + p(t,T)\,v(t,T)\,dW(t) \quad \text{--} \quad (2)$$

Forward rate dynamics

$$df(t,T) = \alpha(t,T)\,dt + \sigma(t,T)\,dW(t). \quad \text{----} \quad (3)$$

The processes $a(t)$, $b(t)$, $m(t,T)$, $v(t,T)$, $\alpha(t,T)$, $\sigma(t,T)$ are adapted processes.

We will study the formal relations which must hold between bond prices and interest rates. ~~and~~

Proposition:- (i) If $p(t,T)$ satisfies (2), then the forward rate dynamics we have

$$df(t,T) = \alpha(t,T)\,dt + \sigma(t,T)\,dW(t)$$

where α and σ are given by

$$\begin{cases} \alpha(t,T) = v_T(t,T)\cdot v(t,T) - m_T(t,T) \\ \sigma(t,T) = -v_T(t,T) \end{cases}$$

(ii) If $f(t,T)$ satisfies ③ then the short rate satisfies

$$dr(t) = a(t)\,dt + b(t)\,dW(t)$$

where

$$\begin{cases} a(t) = f_T(t,t) + \alpha(t,t) \\ b(t) = \sigma(t,t) \end{cases}$$

(iii) If $f(t,T)$ satisfies ③ then $p(t,T)$ satisfies

$$dp(t,T) = p(t,T)\left\{r(t) + A(t,T) + \tfrac{1}{2}\|S(t,T)\|^2\right\}dt + p(t,T)\,S(t,T)\,dW(t)$$

where $\|\cdot\|$ denotes the Euclidean norm, and

$$\begin{cases} A(t,T) = -\int_t^T \alpha(t,s)\,ds \\ S(t,T) = -\int_t^T \sigma(t,s)\,ds. \end{cases}$$

Proof:- From the Ito formula we have

$$d\log p(t,T) = \frac{1}{p(t,T)}\,dp(t,T) - \frac{1}{2}\frac{1}{p^2(t,T)}\,dp(t,T)\,dp(t,T).$$

$$= m(t,T)\,dt + v(t,T)\,dW(t) - \tfrac{1}{2}v^2(t,T)\,dt$$

$$\Rightarrow \log p(t,T) - \log p(0,T) = \int_0^t m(u,T)\,du + \int_0^t v(u,T)\,dW(u) - \tfrac{1}{2}\int_0^t v^2(u,T)\,du$$

$$\Rightarrow \frac{\partial}{\partial T}\log p(t,T) - \frac{\partial}{\partial T}\log p(0,T) = \int_0^t m_T(u,T)\,du + \int_0^t v_T(u,T)\,dW(u) - \tfrac{1}{2}\int_0^t v_T(u,T)\,v(u,T)\,du.$$

$$\Rightarrow -\frac{\partial}{\partial T}\log p(t,T) = f(0,T) - \int_0^t m_T(u,T)\,du - \int_0^t v_T(u,T)\,dW(u) + \tfrac{1}{2}\int_0^t v_T(u,T)\,v(u,T)\,du.$$

$$\Rightarrow f(t,T) - f(0,T) = \int_0^t \alpha(u,T)\,du + \int_0^t \sigma(u,T)\,dW(u)$$

$$\Rightarrow \begin{cases} \alpha(t,T) = v_T(t,T)\,v(t,T) - m_T(t,T) \\ \sigma(t,T) = -v_T(t,T) \end{cases}$$

(ii)
$$df(u,t) = \alpha(u,t)\,du + \sigma(u,t)\,dW(u)$$

$$\Rightarrow f(t,t) - f(0,t) = \int_0^t \alpha(u,t)\,du + \int_0^t \sigma(u,t)\,dW(u).$$

$$\Rightarrow r(t) = f(t,t) = f(0,t) + \int_0^t \alpha(u,t)\,du + \int_0^t \sigma(u,t)\,dW(u).$$

Now we can write

$$\alpha(u,t) = \alpha(u,u) + \int_u^t \alpha_T(u,s)\,ds$$

$$\sigma(u,t) = \sigma(u,u) + \int_u^t \sigma_T(u,s)\,ds.$$

Putting this into above equation, we have

$$r(t) = f(0,t) + \int_0^t \alpha(s,s)\,ds + \int_0^t\int_s^t \alpha_T(s,u)\,du\,ds + \int_0^t \sigma(s,s)\,dW(s) + \int_0^t\int_s^t \sigma_T(s,u)\,du\,dW(s).$$

Changing the order of integration we obtain

$$r(t) = f(0,t) + \int_0^t \alpha(s,s)\,ds + \int_0^t \int_0^s \alpha_T(s,u)\,ds\,du + \int_0^t \sigma(s,s)\,dW(s).$$

$$+ \int_0^t \int_0^s \sigma_T(s,u)\,dW(s)\,du.$$

$$\Rightarrow dr(t) = df(0,t) + \alpha(t,t)\,dt + \int_0^t \alpha_T(s,t)\,ds + \sigma(t,t)\,dW(t)$$

$$+ \int_0^t \sigma_T(s,t)\,dW(s).$$

$$\text{and } f(t,t) = f(0,t) + \int_0^t \alpha(u,t)\,du + \int_0^t \sigma(u,t)\,dW(u).$$

$$\Rightarrow dr(t) = \left[\alpha(t,t) + f_T(t,t)\right]dt + \sigma(t,t)\,dW(t)$$

$$\Rightarrow \begin{cases} a(t) = f_T(t,t) + \alpha(t,t) \\ b(t) = \sigma(t,t) \end{cases}$$

(iii) using the definition of the forward rates we may write

$$p(t,T) = e^{Y(t,T)}$$

where Y is given by

$$Y(t,T) = -\int_t^T f(t,s)\,ds.$$

From the Itô formula, we obtain the bond dynamics as

$$dp(t,T) = p(t,T)\,dY(t,T) + \tfrac{1}{2}\,p(t,T)\,(dY(t,T))^2$$

Now $\quad dY(t,T) = -d\left(\int_t^T f(t,s)\,ds\right)$

$$= -\frac{\partial}{\partial t}\left(\int_t^T f(t,s)\,ds\right)dt - \int_t^T df(t,s)\,ds.$$

$$= f(t,t)\,dt - \int_t^T \alpha(t,s)\,dt\,ds - \int_t^T \sigma(t,s)\,dW(t)\,ds.$$

$$dY(t,T) = \left[r(t) - \int_t^T \alpha(t,s)\,ds\right]dt - \int_t^T \sigma(t,s)\,ds\,dW(t).$$

$$= \left(r(t) + A(t,T)\right)dt + S(t,T)\,dW(t)$$

$$dp(t,T) = p(t,T)\left\{r(t) + A(t,T) + \tfrac{1}{2}\|S(t,T)\|^2\right\}dt$$

$$+ p(t,T)\,S(t,T)\,dW(t).$$

since $(dY(t,T))^2 = \|S(t,T)\|^2 dt$

Here $\begin{cases} A(t,T) = -\int_t^T \alpha(t,s)\,ds \\ S(t,T) = -\int_t^T \sigma(t,s)\,ds \end{cases}$

Note that the above results hold, regardless of the measure under consideration, and in particular we do not assume that markets are free of arbitrage.

Using the T-bond as Numeraire:-

suppose that we are given a specified bond market model with a fixed risk neutral martingale measure Q. For a fixed time of maturity T we now choose the zero coupon bond maturing at T as our new numeraire. We denote the price at t of a T-bond by $p(t,T)$

Defn:- For a fixed T, the T-forward measure Q^T is defined as the martingale measure for the numeraire process $p(t,T)$.

Let Q be the risk-neutral martingale measure with the money account B as the numeraire for our model. We have the following explicit description for Q^T.

Proposition:- If Q denotes the risk neutral martingale measure with the money account B as the numeraire, then the following hold:-

① The Radon-Nikodym derivative process

$$L_t^T = \frac{dQ^T}{dQ} \text{ on } \mathcal{F}(t),\ 0 \le t \le T$$

is given by

$$L_t^T = \frac{p(t,T)}{B(t)} \cdot \frac{1}{p(0,T)}$$

Proof:- The result follows immediately from the chapter change of Numeraire with $Q^T = Q^1$ and $Q^0 = Q$.

$$\left(L_1^0(t) = \frac{S_1(t)}{S_1(0)} \cdot \frac{S_0(0)}{S_0(t)} \qquad S_1(t) = p(t,T) \ \&\ S_0(t) = B(t). \right)$$

Lemma:- Assume that, for all $T>0$ we have $r(t)/B(t) \in L^1(Q)$. Then for every fixed T, the forward rate process $f(t,T)$ is a Q^T martingale and in particular we have

$$f(t,T) = \mathbb{E}^T\left[r(T) \mid \mathcal{F}(t) \right].$$

Proof:- Note that for any T- claim $\mathcal{X}$ we have

$$\Pi(t, \mathcal{X}) = S(t)\, \mathbb{E}^S\left[\frac{\mathcal{X}}{S(T)} \,\Big|\, \mathcal{F}(t) \right] \quad (S \text{ as the numeriare})$$

$$= p(t,T)\, \mathbb{E}^T\left[\mathcal{X} \,|\, \mathcal{F}(t) \right] \quad (p(t,T) \text{ as the numeriare}).$$

where $\mathbb{E}^T$ denotes integration w.r.t Q^T.

Now let $\mathcal{X} = r(T)$, then we have

$$\Pi(t, \mathcal{X}) = \mathbb{E}^Q\left[r(T)\, e^{-\int_t^T r(s)\,ds} \,\Big|\, \mathcal{F}(t) \right]$$

$$= p(t,T)\, \mathbb{E}^T\left[r(T) \,|\, \mathcal{F}(t) \right].$$

$$\Rightarrow \mathbb{E}^T\left[r(T) \,|\, \mathcal{F}(t) \right] = \frac{1}{p(t,T)}\, \mathbb{E}^Q\left[r(T)\, e^{-\int_t^T r(s)\,ds} \,\Big|\, \mathcal{F}(t) \right].$$

$$= -\frac{1}{p(t,T)}\, \mathbb{E}^Q\left[\frac{\partial}{\partial T} e^{-\int_t^T r(s)\,ds} \,\Big|\, \mathcal{F}(t) \right]$$

$$= -\frac{1}{p(t,T)} \frac{\partial}{\partial T}\, \mathbb{E}^Q\left[e^{-\int_t^T r(s)\,ds} \,\Big|\, \mathcal{F}(t) \right]$$

$$= -\frac{p_T(t,T)}{p(t,T)} = f(t,T)$$

An alternative view of the money Account:-

Let us consider a self-financing portfolio which at each time t consists entirely of bonds maturing x units of time later - At time t the portfolio thus consists only of bonds with maturity $t+x$. So the dynamics for this portfolio is given by

$$dV(t) = V(t) \cdot 1 \cdot \frac{dp(t, t+x)}{p(t, t+x)}$$