MA 373 : Financial Engineering II

January - May 2022

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Exercises 2

February 21, 2022

1. Consider a zero-strike Asian call option whose payoff at time T is

$$V(T) = \frac{1}{T}\int_0^T S(u)du.$$

(i) Suppose at time t we have $S(t) = x \geq 0$ and $\int_0^t S(u)du = y \geq 0$. Use the fact that $e^{-ru}S(u)$ is a martingale under the risk neutral measure $\tilde{\mathbb{P}}$ to compute

$$e^{-r(T-t)}\tilde{\mathbb{E}}\Big[\frac{1}{T}\int_0^T S(u)du|\mathcal{F}(t)\Big].$$

Call your answer $v(t,x,y)$.

(ii) Verify that $v(t,x,y)$ satisfies the Black-Scholes-Merton equation

$$v_t(t,x,y)+rxv_x(t,x,y)+xv_y(t,x,y)+\frac{1}{2}\sigma^2x^2v_{xx}(t,x,y) = rv(t,x,y)0 \leq t < T, x \geq 0, y \geq 0.$$

and the boundary conditions

$$v(t,0,y) = e^{-r(T-t)}\frac{y}{T}$$

and

$$v(T,x,y) = \frac{y}{T}$$

(iii) Determine explicitly the process $\Delta(t) = v_x(t,x,y)$, and observe that it is not random.

(iv) Use the Ito-Doeblin formula to show that if you begin with initial capital $X(0) = v_x(0,S(0),0)$ and at each time hold $\Delta(t)$ shares of the underlying asset, investing or borrowing at the interest rate r in order to do this, then at time T the value of your portfolio will be

$$X(T) = \frac{1}{T}\int_0^T S(u)du.$$

2. Consider the continuously sampled a derivative security with payoff function

$$V(T) = \frac{1}{T}\int_0^T S(u)du - K,$$

but assume now that the interest rate is $r = 0$. Find an initial capital $X(0)$ and a nonrandom function $\gamma(t), 0 \leq t \leq T$, which will be the number of shares of risky asset held by our portfolio so that

$$X(T) = \frac{1}{T}\int_0^T S(u)du - K$$

still holds. Give the formula for the resulting process $X(t), 0 \le t \le T$, in term of underlying asset price and K.

3. Consider a new derivative, the Mean with effective period given by $[T_1, T_2]$ the holder of a Mean contract will, at the date of maturity T_2, obtain the amount

$$\frac{1}{T_2 - T_1} \int_{T_1}^{T_2} S(u)du$$

Determine the arbitrage free price, at time t, of the Mean contract where $t < T_1$.

4. Let $X(t) = W(t) - tW(1),\ 0 \le t \le 1$ be a Brownian bridge fixed at 0 and 1. Let $Y(t) = X^2(t)$. Find $E[Y(t)]$ and $Var(Y(t))$.

5. Consider the Brownian motion with drift $X(t) = \alpha t + \sigma W(t)$, with $\alpha, \sigma > 0$. Let $\tau = \inf\{t > 0 : X(t) = x\}$ denote the first hitting time of the barrier x, with $x > 0$.
(a) Then prove that $\mathbb{E}[e^{-s\tau}] = e^{\frac{x}{\sigma^2}(\mu - \sqrt{2s\sigma^2 + \mu^2})}$ for $s > 0$ mu is alpha
(b) Then show that density function of τ is given by you can leave for the time being

$$f_\tau(y) = \frac{x}{\sigma\sqrt{2\pi y^3}} \exp\{-\frac{(x - \alpha y)^2}{2\sigma^2 y}\},\ y > 0 \qquad \cdots (1)$$

(c) Find the mean and variance of τ.

6. Let $X(t) = 2t + 3W(t)$ and $Y(t) = 2t + W(t)$.
(a) Show that the expected times for $X(t)$ and $Y(t)$ to reach any barrier $x > 0$ are the same.
(b) If $X(t)$ and $Y(t)$ model the prices of two stocks, which one would you like to own?

7. Does $4t + 2W(t)$ hit 9 faster (in expectation) than $5t + 3W(t)$ hits 14?

8. Consider the doubling time of a stock $T_2 = \inf\{t > 0; S(t) = 2S(0)\}$.
(a) Find $E[T_2]$ and $Var(T_2)$. Do these values depend on the initial value of the stock?
(b) The expected return of a stock is $\boxed{\alpha = 0.15}$ r? and its volatility $\sigma = 0.20$. Find the expected time when the stock doubles its value.

9. The stochastic average of stock prices between 0 and t is defined by

$$X(t) = \frac{1}{t} \int_0^t S(u)dW(u),$$

where $\{W(t)\}_{t \ge 0}$ is Brownian motion.
(a) Find $dX(t), E[X(t)]$ and $Var(X(t))$
(b) Show that $\sigma X(t) = R(t) - \alpha A(t)$, where $R(t) = \frac{S(t) - S(0)}{t}$ is the raw average of the stock price and

$$A(t) = \frac{1}{t} \int_0^t S(u)du$$

is the continuous arithmetic average.

10. Let

$$S(t) = S(0)\exp\{\alpha t + \sigma \tilde{W}(t)\},\ \alpha = (r - \frac{\sigma^2}{2})$$

be the geometric Brownian motion, where $\tilde{W}(t), 0 \le t \le T$ is a Brownian motion under the risk neutral measure $\tilde{\mathbb{P}}$ and $Y(t) = \max_{0 \le u \le t} S(u)$.

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(i) Show that the pair of process $(S(t), Y(t))$ is Markov. We must show that whenever $0 \le t \le T$ and $f(x, y)$ is a measurable function, there exists another function $g(x, y)$ such that

$$\tilde{\mathbb{E}}[f(S(T), Y(T))|\mathcal{F}(t)] = g(S(t), Y(t)).$$

(ii) Let $0 = t_0 < t_1 < \cdots < t_m = T$ be a partition of $[0, T]$. Show that

$$\lim_{\|\pi_m\| \to 0} \sum_{j=1}^{m} (Y(t_j) - Y(t_{j-1}))(S(t_j) - S(t_{j-1})) = 0,$$

where $\|\pi_m\| = \max_{j=1,2,\cdots,m}(t_j - t_{j-1})$.

(iii) Let $0 < K_1 < K_2$. Find the price at time t of a derivative which pays at maturity

$$V(T) = \begin{cases} 1 & \text{if } K_1 \le S(T) \le K_2 \\ 0 & \text{otherwise.} \end{cases}$$

(iv) Let $\tau_b = \inf\{t > 0 : S(t) = b\}$. Find the price of a contract whose payoff is given by

$$V(T) = \begin{cases} K & \text{if } \tau_b < T \\ 0 & \text{if } \tau_b \ge T \end{cases}$$

(Hint: using equation (1) calculate the density function for τ_b)