of the stock price between times t and u.

The price at time t of the American put expiring at time T is defined to be

$$v(t,x) = \max_{\tau \in \mathcal{S}_{t,T}} \widetilde{\mathbb{E}}\left[e^{-r(\tau - t)} (K - S(\tau)) \,\middle|\, S(t) = x \right]$$

In the event that $\tau = \infty$, we interpret $e^{-r\tau}(K - S(\tau))$ to be zero. This is the case when the put expires unexercised.

Analytical characterization of the Put Price

The finite-expiration American put price function $v(t,x)$ satisfies the linear complementarity conditions

$$v(t,x) \geq (K-x)^+ \text{ for all } t \in [0,T],\ x \geq 0, \quad \text{--- (a)}$$

$$rv(t,x) - v_t(t,x) - rx\, v_x(t,x) - \tfrac{1}{2}\sigma^2 x^2 v_{xx}(t,x) \geq 0$$

$$\text{for all } t \in [0,T),\ x \geq 0 \text{ and} \quad \text{--- (b)}$$

for each $t \in [0,T)$ and $x \geq 0$, equality holds in either (a) or (b)

The set $\{(t,x);\ 0 \leq t \leq T,\ x \geq 0\}$ can be divided into two regions, the stopping set

$$\mathcal{S} = \{(t,x) : v(t,x) = (K-x)^+\}$$

and the continuation set

$$\mathcal{C} = \{(t,x);\ v(t,x) > (K-x)^+\}.$$

Theorem:- Let $S(t)$, $t \leq u \leq T$, be the stock price starting at $S(t) = x$. Let

$$\tau_* = \min\{u \in [t,T] : v(u, S(u)) = (K - S(u))^+\}.$$

Then $e^{-ru} v(u, S(u))$, $t \le u \le T$ is a supermartingale under $\widetilde{\mathbb{P}}$, and the stopped process $e^{-r(u \wedge \tau_*)} v(u, S(u \wedge \tau_*))$, $t \le u \le T$ is a martingale.

Proof:- By Itô-formula, we have

$$d[e^{-ru} v(u, S(u))]$$

$$= e^{-ru}\Big[-rv(u, S(u))\,du + v_u(u, S(u))\,du + v_x(u, S(u))\,dS(u) + \tfrac{1}{2} v_{xx}(u, S(u))\,dS(u)\,dS(u)\Big]$$

$$= e^{-ru}\Big[-rv(u, S(u)) + v_u(u, S(u)) + rS(u)v_x(u, S(u)) + \tfrac{1}{2}\sigma^2 S^2(u) v_{xx}(u, S(u))\Big]du + e^{-ru}\sigma S(u) v_x(u, S(u))\,d\widetilde{W}(u).$$

Note that for $(t, x) \in \mathcal{I}$, we have $v(t, x) = K - x$ and.

$$rv(t, x) - v_t(t, x) - rxv_x(t, x) - \tfrac{1}{2}\sigma^2 x^2 v_{xx}(t, x) = rK$$

Therefore the du term is $-e^{-ru} rK \mathbb{1}_{\{S(u) < L(T-u)\}}$, where $x = L(T-t)$ function forms the boundary between $\mathcal{C}$ and $\mathcal{I}$ and belongs to $\mathcal{I}$. This is nonpositive and so $e^{-ru} v(u, S(u))$ is a supermartingale under $\widetilde{\mathbb{P}}$. In fact, starting from $u = t$ and up until time τ_*, we have $S(u) > L(T-u)$, so the du term is zero. Therefore, the stopped process

$$e^{-r(u \wedge \tau_*)} v(u \wedge \tau_*, S(u \wedge \tau_*)), \quad t \le u \le T$$

is a martingale.

Corollary:- consider an agent with initial capital $X(0) = v(0, S(0))$ the initial finite-expiration put price. suppose this agent uses the portfolio process $\Delta(u) = v_x(u, S(u))$ and consumes cash at rate $C(u) = rK \mathbb{1}_{\{S(u) < L(T-u)\}}$ per unit time. Then $X(u) = v(u, S(u))$ for all time u between $u = 0$ and the time the option is exercised or expires. In particular $X(u) \geq (K - S(u))^+$ for all time u until the option is exercised or expires.

Remark:- The supermartingale property for $e^{-rt} v(t, S(t))$ implies

that $e^{-r(t\wedge\tau)} v(t\wedge\tau, S(t\wedge\tau)) \geq \widetilde{\mathbb{E}}\left[e^{-r(T\wedge\tau)} v(T\wedge\tau, S(T\wedge\tau)) \middle| \mathcal{F}(t)\right]$

For $\tau \in \mathcal{S}_{t,T}$, we have $t \wedge \tau = t$ and $T \wedge \tau = \tau$ if $\tau < \infty$ and $T\wedge\tau = T$ if $\tau = \infty$. Therefore, for $\tau \in \mathcal{S}_{t,T}$,

$$e^{-rt} v(t, S(t)) \geq \widetilde{\mathbb{E}}\left[e^{-r\tau} v(\tau, S(\tau)) \mathbb{1}_{\{\tau<\infty\}} + e^{-rT} v(T, S(T)) \mathbb{1}_{\{\tau=\infty\}} \middle| \mathcal{F}(t)\right]$$

$$\geq \widetilde{\mathbb{E}}\left[e^{-r\tau} v(\tau, S(\tau)) \middle| \mathcal{F}(t)\right] \quad --- (*)$$

where we interpret $e^{-r\tau} v(\tau, S(\tau)) = 0$ if $\tau = \infty$

Now $v(t, S(t)) \geq (K - S(t))^+ \geq K - S(t)$ ~~implies~~ imply that

$$\widetilde{\mathbb{E}}\left[e^{-r\tau} v(\tau, S(\tau)) \middle| \mathcal{F}(t)\right] \geq \widetilde{\mathbb{E}}\left[e^{-r\tau}(K - S(\tau)) \middle| \mathcal{F}(t)\right].$$

Thus, we have

$$e^{-rt} v(t, S(t)) \geq \widetilde{\mathbb{E}}\left[e^{-r\tau}(K - S(\tau)) \middle| \mathcal{F}(t)\right].$$

Because $S(t)$ is a Markov process, the right hand side is a function of t and $S(t)$. If we denote the value of $S(t)$ by x, we may write the above equation as

$$e^{-rt} v(t,x) \geq \widetilde{\mathbb{E}}\left[e^{-r\tau}(K-S(\tau)) \,\middle|\, S(t)=x\right] \quad --- (**)$$

Since above equation holds for every $\tau \in \mathcal{S}_{t,T}$, we obtain

$$v(t,x) \geq \max_{\tau \in \mathcal{S}_{t,T}} \widetilde{\mathbb{E}}\left[e^{-r(\tau-t)}(K-S(\tau)) \,\middle|\, S(t)=x\right].$$

For the reverse inequality, we recall that the stopped process

$e^{-r(t\wedge\tau_*)}\, v(t\wedge\tau_*, S(t\wedge\tau_*))$ is a martingale, where

$$\tau_* = \min\left\{u \in [t,T],\ v(u,S(u)) = (K-S(u))^+\right\}.$$

Note that $v(\tau_*, S(\tau_*)) = K - S(\tau_*)$ if $\tau_* < \infty$.

Replacing τ by τ_* in $(*)$ we make the first inequality into an equality. If $\tau_* = \infty$, we have $(T, S(T)) \in \mathcal{C}$ (i.e $S(T) > K$) so $v(T, S(T))\mathbb{1}_{\{\tau_*=\infty\}} = 0$. This makes the second inequality in $(*)$ into an equality. Finally, because $v(\tau, S(\tau)) = K - S(\tau)$ on $\mathbb{1}_{\{\tau<\infty\}}$,

Thus, we have

$$\widetilde{\mathbb{E}}\left[e^{-r\tau_*} v(\tau_*, S(\tau_*)) \,\middle|\, \mathcal{F}(t)\right] = \widetilde{\mathbb{E}}\left[e^{-r\tau_*}(K-S(\tau_*)) \,\middle|\, \mathcal{F}(t)\right].$$

Hence $(**)$ becomes

$$v(t,x) = \widetilde{\mathbb{E}}\left[e^{-r(\tau_*-t)}(K-S(\tau_*)) \,\middle|\, S(t)=x\right].$$

$$\Rightarrow v(t,x) = \max_{\tau \in \mathcal{S}_{t,T}} \widetilde{\mathbb{E}}\left[e^{-r(\tau-t)}(K-S(\tau)) \,\middle|\, S(t)=x\right].$$

American Call:-

Consider a stock whose price process $S(t)$ is given by

$$dS(t) = rS(t)\,dt + \sigma S(t)\,d\widetilde{W}(t),$$

$r, \sigma > 0$ and $\widetilde{W}(t)$ is a BM under the risk-neutral measure $\widetilde{\mathbb{P}}$.

Lemma:- Let $h(x)$ be a convex function $x \geq 0$ satisfying $h(0)=0$. Then the discounted intrinsic value $e^{-rt}h(S(t))$ of the American derivative security that pays $h(S(t))$ upon exercise is a submartingale:

Proof:- Because $h(x)$ is convex for $0 \leq \lambda \leq 1$ and $0 \leq x_1 \leq x_2$ we have

$$h((1-\lambda)x_1 + \lambda x_2) \leq (1-\lambda)h(x_1) + \lambda h(x_2).$$

Taking $x_1 = 0$ and $x_2 = x$ and using the fact that $h(0)=0$ we obtain that

$$h(\lambda x) \leq \lambda h(x) \quad \forall\ x \geq 0,\ 0 \leq \lambda \leq 1.$$

For $0 \leq u \leq t \leq T$, we have $0 \leq e^{-r(t-u)} \leq 1$, thus

$$\widetilde{\mathbb{E}}\left[e^{-r(t-u)} h(S(t)) \,\middle|\, \mathcal{F}(u)\right] \geq \widetilde{\mathbb{E}}\left[h\left(e^{-r(t-u)} S(t)\right) \,\middle|\, \mathcal{F}(u)\right]$$

The conditional Jensen's inequality implies

$$\widetilde{\mathbb{E}}\left[h\left(e^{-r(t-u)}S(t)\right)\middle|\mathcal{F}(u)\right] \geq h\left(\widetilde{\mathbb{E}}\left[e^{-r(t-u)}S(t)\middle|\mathcal{F}(u)\right]\right)$$

$$= h\left(e^{ru}\,\widetilde{\mathbb{E}}\left[e^{-rt}S(t)\middle|\mathcal{F}(u)\right]\right).$$

Because $e^{-rt}S(t)$ is a martingale under $\widetilde{\mathbb{P}}$, we have

$$h\left(e^{ru}\widetilde{\mathbb{E}}\left[e^{-rt}S(t)\,|\,\mathcal{F}(u)\right]\right) = h\left(e^{ru}e^{-ru}S(u)\right) = h(S(u)).$$

Hence, we obtain

$$\widetilde{\mathbb{E}}\left[e^{-r(t-u)}h(S(t))\,|\,\mathcal{F}(u)\right] \geq h(S(u)). \quad \text{------} \; (*)$$

$$\Rightarrow \widetilde{\mathbb{E}}\left[e^{-rt}h(S(t))\,|\,\mathcal{F}(u)\right] \geq e^{-ru}h(S(u)).$$

This is the submartingale property for $e^{-rt}h(S(t))$.

<u>Theorem:-</u> Let $h(x)$ be a nonnegative, convex function of $x \geq 0$ with $h(0)=0$. Then the price of the American derivative security expiring at time T and having intrinsic value $h(S(t))$, $0 \leq t \leq T$, is the same as the price of the European derivative security paying $h(S(T))$ at expiration T.

<u>Proof:-</u> Replace t by T in $(*)$, we obtain

$$\widetilde{\mathbb{E}}\left[e^{-r(T-u)}h(S(T))\,|\,\mathcal{F}(u)\right] \geq h(S(u)), \quad 0 \leq u \leq T$$

In other words, the European derivative security price always dominates the intrinsic value of the American derivative security. This show that the option to exercise early is worthless, and the price of the American derivative security agrees with the price of the European security

<u>Remark:-</u> Take $h(x) = (x-K)^+$

$\Rightarrow$ Price of an American call is the same as the price of European call.