Consider the SDE

$$dS(t) = \mu(t) S(t) dt + \sigma(t) S(t) dW(t)$$
$$S(0) = S_0$$

where $|\mu(t)| + |\sigma(t)| \leq K$

Here $b(t,x) = \mu(t) x$ and $\sigma(t,x) = \sigma(t) x$

Both the conditions for uniqueness and existence are satisfied. Now by Ito-formula.

$$d(\ln S(t)) = \frac{1}{S(t)} dS(t) + \left(-\frac{1}{2S^2(t)}\right) dS(t)\, dS(t).$$

$$= \frac{1}{S(t)}\left[\mu(t) S(t) dt + \sigma(t) S(t) dW(t)\right]$$
$$- \frac{1}{2S^2(t)} \sigma^2(t) S^2(t)\, dt$$

$$= \sigma(t)\, dW(t) + \left(\mu(t) - \tfrac{1}{2}\sigma^2(t)\right) dt$$

Thus $\ln S(T) - \ln S(0) = \int_0^T \sigma(t)\, dW(t) + \int_0^T \left(\mu(t) - \tfrac{1}{2}\sigma^2(t)\right) dt$

$$\Rightarrow S(T) = S_0 \exp\left\{\int_0^T \sigma(t)\, dW(t) + \int_0^T \left(\mu(t) - \tfrac{1}{2}\sigma^2(t)\right) dt\right\}.$$

Consider a first order ordinary differential equation

$$\frac{dx(t)}{dt} = f(t) x(t) + g(t)$$

$x(0) = x$, where $f(t)$ is a continuous function

To solve this equation we multiply both side by Integreating factor $e^{-\int_0^t f(s) ds}$, we get

$$e^{-\int_0^t f(s) ds}\left[\frac{dx(t)}{dt} - f(t) x(t)\right] = g(t)\, e^{-\int_0^t f(s) ds}$$

Let $h(t) = e^{-\int_0^t f(s) ds}$, then

$$\frac{d}{dt}\left(h(t)x(t)\right) = h(t)g(t)$$

The solution of the above equation is

$$h(t)x(t) = x + \int_0^t h(s)g(s)\,ds$$

$$\text{or } \quad x(t) = x[h(t)]^{-1} + [h(t)]^{-1}\int_0^t h(s)g(s)\,ds.$$

By a Linear SDE, we mean a SDE of the form

$$\begin{cases} dx(t) = (\phi(t)x(t) + \theta(t))\,dW(t) + (f(t)x(t) + g(t))\,dt \\ x(0) = x. \end{cases}$$

$$H(t) = e^{-Y(t)}.$$

$$Y(t) = \int_0^t \phi(s)\,dW(s) + \int_0^t f(s)\,ds - \tfrac{1}{2}\int_0^t \phi^2(s)\,ds.$$

Now (1) $d(H(t)X(t)) = X(t)\,dH(t) + H(t)\,dX(t) + dH(t)\,dX(t).$

$$dH(t) = -e^{-Y(t)}dY(t) + \tfrac{1}{2}e^{-Y(t)}dY(t)\,dY(t).$$

$$= -H(t)\left[f(t)\,dt + \phi(t)\,dW(t) - \tfrac{1}{2}\phi^2(t)\,dt\right]$$

$$+\tfrac{1}{2}H(t)\phi^2(t)\,dt.$$

$$= -H(t)f(t)\,dt - H(t)\phi(t)\,dW(t) + H(t)\phi^2(t)\,dt.$$

Thus $dH(t)\,dX(t) = -H(t)\phi(t)[\phi(t)X(t) + \theta(t)]\,dt.$

Therefore we get

$$d(X(t)H(t)) = H(t)\big[-\cancel{X(t)f(t)}\,dt - \cancel{X(t)\phi(t)}\,dW(t) + \cancel{X(t)\phi^2(t)}\,dt$$

$$+ (\cancel{\phi(t)X(t)} + \theta(t))\,dW(t) + (\cancel{f(t)X(t)} + g(t))\,dt$$

$$- \cancel{\phi^2(t)X(t)}\,dt - \phi(t)\theta(t)\,dt\big].$$

$$d(X(t)H(t)) = H(t)\left[\theta(t)dW(t) + g(t)\,dt - \theta(t)\phi(t)\,dt\right].$$

$$\Rightarrow H(t)X(t) = x + \int_0^t H(s)\theta(s)dW(s) + \int_0^t H(s)\{g(s) - \theta(s)\phi(s)\}ds$$

Thus $$X(t) = x e^{Y(t)} + \int_0^t e^{Y(t)-Y(s)}\theta(s)dW(s) + \int_0^t e^{Y(t)-Y(s)}\{g(s) - \theta(s)\phi(s)\}ds.$$

Example:- $\begin{cases} dX(t) = \mu X(t)dt + \sigma dW(t). \\ X(0) = x_0. \end{cases}$

Thus $f(t) = \mu$, $g(t) = 0$, $\phi(t) = 0$, $\theta(t) = \sigma$

Hence $Y(t) = \mu t$.

Therefore:- $X(t) = x_0 e^{\mu t} + \int_0^t e^{\mu(t-s)}\sigma\, dW(s).$

Example:- $\mathbb{P}(X_t = Y_t) = 1$ but $\mathbb{P}(X(t) = Y(t) : \forall t \geq 0) = 0.$

consider a positive random variable Z with a continuous distribution. put $X(t) \equiv 0$ and let

$$Y_t = \begin{cases} 0 & ; t \neq Z \\ 1 & ; t = Z \end{cases}$$

we have $\mathbb{P}(X(t) = Y(t)) = \mathbb{P}(Z \neq t) = 1$

But $\mathbb{P}(X(t) = Y(t) : \forall t \in [0,T]) = 0.$

let $\Omega = [0,T]$ and Z is a random variable uniformly distributed over the interval $[0,T]$.

Define $Y_t(\omega) = \begin{cases} 1 & \text{if } t = \omega \\ 0 & \text{if } t \neq \omega. \end{cases}$

consider the SDE

$$dX(t) = b(t, X(t))dt + \sigma(t, X(t))\, dW(t)$$

Let $h(y)$ be a Borel-measurable function. Define

$$g(t,x) = \mathbb{E}\left[h(X(T)) \mid X(t)=x\right] := \mathbb{E}^{t,x}\left[h(X(T))\right]$$

Assume that $\mathbb{E}\left[h(X(T)) \mid X(t)=x\right] < \infty \ \forall t \in [0,T] \ \& \ x \in \mathbb{R}$

Theorem:- Let $X(u)$, $u \geq 0$, be a solution to the SDE with intial condition given at 0. Then for $0 \leq t \leq T$

$$\mathbb{E}\left[h(X(T)) \mid \mathcal{F}(t)\right] = g(t, X(t)).$$

Corollary:- solutions to stochastic differential equations are Markov processes

The following theorem relates SDE's and PDE's

Theorem:- (Feynman-Kac). Consider the stochastic differential equation

$$dX(t) = b(t, X(t))dt + \sigma(t, X(t))\, dW(t).$$

Let $h(y)$ be a Borel-measurable function. Define the function

$$g(t,x) = \mathbb{E}^{t,x}\left[h(X(T))\right]$$

Then $g(t,x)$ satisfies the partial differential equation

$$g_t(t,x) + b(t,x)\, g_x(t,x) + \tfrac{1}{2}\sigma^2(t,x)\, g_{xx}(t,x) = 0.$$

and the terminal condition

$$g(T,x) = h(x) \ \forall x.$$

Proof:- claim $g(t, X(t)),\ 0 \le t \le T$ is a martingale.

By previous theorem

$$g(t, X(t)) = \mathbb{E}[h(X(T)) \mid \mathcal{F}(t)]$$

Thus for $s<t$

$$\mathbb{E}[g(t, X(t)) \mid \mathcal{F}(s)] = \mathbb{E}[\mathbb{E}[h(X(T)) \mid \mathcal{F}(t)] \mid \mathcal{F}(s)]$$

$$= \mathbb{E}[h(X(T)) \mid \mathcal{F}(s)] = g(s, X(s)).$$

Hence the claim. Now

$$d(g(t, X(t))) = g_t(t, X(t))\, dt + g_x(t, X(t))\, dX(t) + \tfrac{1}{2} g_{xx}(t, X(t))\, dX(t)\, dX(t).$$

$$= g_t(t, X(t))\, dt + g_x(t, X(t))\, b(t, X(t))\, dt + g_x(t, X(t))\, \sigma(t, X(t))\, dW(t) + \tfrac{1}{2} g_{xx}(t, X(t))\, \sigma^2(t, X(t))\, dt$$

$$= [g_t(t, X(t)) + b(t, X(t))\, g_x(t, X(t)) + \tfrac{1}{2} g_{xx}(t, X(t))\, \sigma^2(t, X(t))]\, dt + g_x(t, X(t))\, \sigma(t, X(t))\, dW(t).$$

since $g(t, X(t))$ is a martingale so the dt term must be equal to 0. Thus we must have

$$g_t(t,x) + g_x(t,x)\, b(t,x) + \tfrac{1}{2} \sigma^2(t,x)\, g_x(t,x) = 0.$$

Theorem:- consider the stochastic differential equation

$$dX(t) = b(t, X(t))\, dt + \sigma(t, X(t))\, dW(t).$$

Let $h(y)$ be a Borel-measurable function. Define

$$f(t,x) = \mathbb{E}^{t,x}[e^{-r(T-t)} h(X(T))]$$

where r is a constant. Then $f(t,x)$ satisfies the partial differential equation $f_t(t,x) + b(t,x) f_x(t,x) + \tfrac{1}{2}\sigma^2(t,x) f_{xx}(t,x) = r f(t,x)$

with $f(T, x) = h(x)$.