Exercise 1

Thursday, February 3, 2022 11:34 PM

Ex-1

2.a) $dY(t) = b(t, Y(t))\,dt + \sigma(t, Y(t))\,dW(t)$

$\cancel{dX(t)}$

$\cancel{dX(t)}$

$dX = X\,dt + dW, \quad X(0) = x_0$

$dX = dW + X\,dt$

$\phi = 0 \quad \theta = 1 \quad f = 1 \quad g = 0$

$Y = \int_0^t 0\,dW + \int_0^t ds - \frac{1}{2}\int 0\,ds$

$Y = t$

$H(t) = e^{-Y(t)} = e^{-t}$

$dX = X\,dt + dW$

$d(e^{-t}X) = (de^{-t})X + e^{-t}\,dX + d(e^{-t})\,dX$

$= -e^{-t}X + e^{-t}(\cancel{X\,dt} + dW)$

$\cancel{-e^{-t}dt(X\,dt + dW)}$

$= -\cancel{e^{-t}X\,dt}$

$d(e^{-t}X) = e^{-t}\,dW$

$e^{-t}X - x_0 = \int_0^t e^{-s}\,dW(s)$

$X(t) = x_0 e^t + \int_0^t e^{t-s}\,dW + \cancel{\int_0^t e^{t-s}}$

$\boxed{X(t) = x_0 e^t + \int_0^t e^{t-s}\,dW}$

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2. b) ~~dX(t)~~ $dX = -X dt + e^{-t} dW$

$\theta(t) = e^{-t}$ ~~$g = -x$~~ $f = -1$

$Y = \int_0^t 0\, dW + \int_0^t -1\, ds + 0 = -t$

~~$X(t) = x$~~

$X(t) = x_0 e^{-t} + \int_0^t e^{-t+s} e^{-s} dW$

$= x_0 e^{-t} + e^{-t} W(t)$

$X(t) = e^{-t}(x_0 + W(t))$

2 c.) $dX = r\,dt + \alpha X\, dW$, $X(0) = x_0$

$\phi = \alpha \quad g = r$

~~$X = x_0$~~

$Y = \int_0^t \alpha\, dW = \alpha W$

$X = x_0 e^{\alpha W} + \cancel{\int_0^t e^{\alpha W(t) - \alpha W(s)}} + \int_0^t e^{\alpha(W(t) - W(s))} r\, ds$

$= x_0 e^{\alpha W} + \int_0^t e^{\alpha(W(t) - W(s))} r\, ds$

2d) $dX = \frac{X}{2} dt + X dW$, $X(0) = 1$

$f = \frac{1}{2} \quad \phi = 1$

~~$X(t) =$~~ $Y = \int_0^t dW(t) + \frac{t}{2}$

$X = e^{W + t/2}$

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e) $dX = -\frac{1}{(1+t)} X dt + \frac{1}{(1+t)} dW$, $X(0) = 0$

$f = \frac{-1}{(1+t)} \quad \theta = \frac{1}{(1+t)}$

$Y = \int_0^t \frac{-1}{(1+t)} dt$

$= -\left[\ln(1+t)\right]_0^t$

$Y(t) = -\ln(1+t)$

$X(t) = \int_0^t \frac{e^{-\ln(1+t) + \ln(1+s)}}{(1+t)} dW$

1. $Y(t) \rightarrow dY(t) = b(t, Y(t))\, dt + \sigma(t, Y(t))\, dW$

$\cancel{Y(t) = W(t}$

a) $Y = W + 4t$

$f(t, x) = x + 4t$

$df = 1\, dx + 4\, dt$

$dY = dW + 4\, dt$

b) $Y = W^2$

$dY = 2W\, dW + dt$

$f(t, x) = x^2$

$df = 2x\, dx + \frac{1}{2} 2 (dx)^2$

$df(t, W) = 2W\, dW + dt = dY$

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(c) $Y(t) = t^2 W - 2\int_0^t s W(s)\, ds$

$dY = t^2 dW + (2tW - 2tW)dt$

$= t^2 dW$

(d) $Y = e^W + t^2 + 1$

$dY = e^W dW + 2t\, dt + \frac{e^W}{2} dt$

(e) $Y = \left(\frac{W}{3} + \alpha\right)^3$

$= \cancel{3}\left(\frac{W}{3} + \alpha\right)\frac{1}{\cancel{3}} dW + \frac{1}{3}\frac{1}{2}\cancel{(dW)^2}\, dt$

$= \left(\frac{W}{3} + \alpha\right) dW + \frac{dt}{6}$

(f) $Y = e^{ct + \alpha W}$

$dY = \alpha e^{ct+\alpha W} dW + c e^{ct+\alpha W} dt + \frac{\alpha^2}{2} e^{ct+\alpha W} dt$

$= e^{ct+\alpha W}\left(\alpha dW + c\,dt + \frac{\alpha^2}{2} dt\right)$

(g) $e^{\int_0^t h(s)\,dW(s) - \frac{1}{2}\int_0^t h^2 ds}$

(c) $Y(t) = t^2 W - 2\int_0^t s W\, ds$

$X(t) = \int_0^t s W ds$ $\qquad$ $Y(t) = t^2 W - 2X$

$dY = d(t^2 W) - 2dX$

$dX = \cancel{sW}\; tW\, dt$ $\qquad$ $= t^2 dW + \cancel{2tW dt} - \cancel{2tW dt}$

$= t^2 dW$

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g) $Y(t) = e^{\int_0^t h(s)\,dW(s) - \frac{1}{2}\int_0^t h^2\,ds}$

$Y(t) = e^{X(t) - \frac{1}{2}Z(t)}$

$X(t) = \int_0^t h(s)\,dW(s)$

$Z(t) = \int_0^t h^2(s)\,ds$

$dX = h\,dW \qquad X - \frac{Z}{2} = Q$

$dZ = h^2\,dt$

$P = X - \frac{Z}{2}$

$Y(t) = e^{P(t)}$

$f(t, x) = e^x$

$df = e^x\,dx + \frac{1}{2}e^x(dx\,dx)$

$dY = e^{P(t)}\,dP + \frac{1}{2}e^P\,(dP\,dP)$

$= e^{P(t)}\left(h\,dW - \cancel{\frac{h^2\,dt}{2}}\right) + \frac{e^P}{2}\left(\cancel{h^2}\,dt\right)$

$= e^{P(t)}\,h\,dW$

$= e^{X - \frac{Z}{2}}\,h\,dW$

$dY(t) = e^{\int_0^t h(s)\,dW(s) - \frac{1}{2}\int_0^t h^2(s)\,dW}$

$dW(t)$

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③ $Z(t) = \frac{1}{X(t)}$ $\quad dX(t) = \mu X(t)dt + \sigma X(t)dW(t)$

$, X(0) = x_0$

$f(t,x) = \frac{1}{x}$

$df = -\frac{1}{x^2}dx + \frac{2}{x^3 2}(dx\,dx)$

$df(t; X(t)) = d\left(\frac{1}{X(t)}\right) = d(Z(t)) = -\frac{1}{X^2(t)}dX + \frac{1}{2X^3}dX\,dX$

$= -\frac{1}{X^2}(\mu X dt + \sigma X dW) + \frac{1}{2X^3}(\sigma^2 X^2)dt$

$= -\frac{1}{X}(\mu dt + \sigma dW) + \frac{1}{X}\left(\frac{\sigma^2}{2}\right)dt$

$= -\frac{1}{X}\left(\sigma dW + \left(\mu - \frac{\sigma^2}{2}\right)dt\right)$

4.) $dX = (m - X)dt + \sigma dW, \quad X(0) = x_0$

$= f = -1 \quad g = m \quad \theta = \sigma \quad \phi = 0$

$\cancel{X(t) = x_0}\ Y(t) = \int_0^t -1\,ds = -t$

$X(t) = x_0 e^{-t} + \int_0^t e^{-t+s}\sigma\,dW + \int_0^t e^{-t+s} m\,ds$

$= x_0 e^{-t} + \sigma e^{-t}\int_0^t e^{s}dW + me^{-t}\int_0^t e^s ds$

$= x_0 e^{-t} + \sigma e^{-t}\int_0^t e^s dW + me^{-t}(e^t - 1)$

$= x_0 e^{-t} + \sigma e^{-t}\int_0^t e^s dW + m - me^{-t}$

$= m + \left(x_0 - m + \sigma\int_0^t e^s dW\right)e^{-t}$

$f = xe^t$

$\int df = \int e^t dx + xe^t dt$

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(4) $dX(t) = (m - X(t))\,dt + \sigma\, dW(t), \quad X(0) = x_0$

$\cancel{X(t) =}\ Y(t) = m - X(t)$

$dY(t) = -dX(t)$

$-dY(t) = Y(t)\,dt + \sigma\, dW(t), \quad X(0) = x_0 \quad Y(0) = m - x_0$

$dY(t) = -Y(t)\,dt + \sigma\, dW(t)$

$\cancel{\phi = -1} \quad f(t) = -1 \quad g(t) = 0 \quad \phi = 0 \quad \theta = \sigma$

$Z(t) = -t$

$\cancel{Y(t) = (m - x_0) e^{Z}}$

$Y(t) = (m - x_0) e^{-t} + \int_0^t e^{-t+s} \sigma\, dW + \int$

$Y(t) = (m - x_0) e^{-t} + \sigma e^{-t} \int_0^t e^{s}\, dW(s)$

$m - X(t) = (m - x_0) e^{-t} + \sigma e^{-t} \int_0^t e^{s}\, dW$

$X(t) = m - (m - x_0) e^{-t} - \sigma e^{-t} \int_0^t e^{s}\, dW$

$E[X(t)] = m - (m - x_0) e^{-t}$

$\cancel{X(t) = m - m}$

$$\begin{aligned}
var(X(t)) &= var\left[m - (m - x_0) e^{-t} - \sigma e^{-t} \int_0^t e^{s}\, dW\right] \\
&= var\left[\cancel{(m - x_0) e^{-t}} + \sigma e^{-t} \int_0^t e^{s}\, dW\right] \\
&= var\left[\sigma e^{-t} \int_0^t e^{s}\, dW\right] \\
&= \sigma^2 var\left[e^{-t} \int_0^t e^{s}\, dW\right] \\
&= \sigma^2 \int_0^t e^{2s-2t}\, dt \\
&= \sigma^2 e^{-2t} \frac{(e^{2s})_0^t}{2} = \frac{\sigma^2 e^{-2t}}{2}\left[e^{2t} - 1\right] \\
&= \frac{\sigma^2}{2}\left[1 - e^{-2t}\right]
\end{aligned}$$

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9. $dX(t) = \mu(t)\,dt + \sigma(t)\, dW(t), \quad X(0) = x_0$

$\phi = 0 \quad f = 0 \quad \theta = \sigma$

$Y = 0 \qquad g = \mu$

$X(t) = x_0 + \int_0^t \sigma(t)\, dW(t) + \int_0^t \mu(t)\, dt$

$E[X(t)] = x_0 + \int_0^t \mu(t)\, dt \qquad E[X_{n+1} | \mathcal{F}_n] \geq X_n$

$$\begin{aligned}
E[I(t) | \mathcal{F}(s)] &= E[I(t) - I(s) | \mathcal{F}_s] + I(s) \\
&= E[I(t) - I(s)] + I(s) \\
&= \int_s^t \mu(t)\, dt + I(s)
\end{aligned}$$

$E[I(t) | \mathcal{F}_s] - I(s) = \int_s^t \mu(t)\, dt \geq 0$

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$\therefore E[I(t) | \mathcal{F}_s] \geq I(s)$

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7.) $dX(t) = \alpha X(t)dt + Y(t)dW(t)$, $X(0) = x_0$
$dY(t) = \alpha Y(t)dt - X(t)dW(t)$, $Y(0) = y_0$

(i) Compute $E[X(t)]$, $E[Y(t)]$, $Cov(X(t), Y(t))$

$$X(t) = x_0 + \int_0^t \alpha X(t)dt + \int_0^t Y(t)dW(t)$$

$$Y(t) = y_0 + \int_0^t \alpha Y(t)dt - \int_0^t X(t)dW(t)$$

$$E[X(t)] = x_0 + E\left[\int_0^t \alpha X dt\right]$$
$$E[Y(t)] = y_0 + E\left[\int_0^t \alpha Y dt\right]$$

$$\frac{d}{dt}E[X(t)] = E[\alpha X]$$

$$\frac{d}{dt}E[X(t)] = \alpha E[X(t)]$$

$dz = \alpha z dt$

$\frac{dz}{z} = \alpha dt$

$\ln(z) - \ln(z(0)) = \alpha t$

$\ln(z) = \ln(z(0)) + \alpha t$

$z = e$

$$E[X(t)] = x_0 e^{\alpha t}$$
$$E[Y(t)] = y_0 e^{\alpha t}$$

$dX(t)$

$dX + dY = \alpha(X+Y)dt + (Y-X)dW$

$d(X+Y) = \alpha(X+Y)dt + (Y-X)dW$

$e^{-\alpha t}(d(X+Y) - \alpha(X+Y)dt) = e^{-\alpha t}(Y-X)dW$

$d((X+Y)e^{-\alpha t}) = e^{\alpha t}(Y-X)dW$

$(X+Y)e^{-\alpha(t)} - (x_0+y_0) = \int_0^t e^{\alpha s}(Y-X)dW$

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$d(XY) = Y(dX) + (dY)X + dY\,dX$

$d(XY) = Y(\alpha X dt + Y dW) + (\alpha Y dt - X dW)X$

$- XY\,dt$

$= dt(\alpha XY + \alpha XY - XY) + dW(Y^2 - X^2)$

$d(XY) = dt\,XY(2\alpha - 1) + dW(Y^2 - X^2)$

$XY - x_0 y_0 = \int_0^t XY(2\alpha - 1)\,ds + \int_0^t (Y^2 - X^2)\,dW$

$E[XY] = x_0 y_0 + E\int_0^t XY(2\alpha - 1)\,ds$

$\frac{d}{dt} E[XY] = E[XY](2\alpha - 1)$

$E[XY] = x_0 y_0 e^{(2\alpha - 1)t}$

$Cov(X, Y) = x_0 y_0 e^{(2\alpha - 1)t} - x_0 y_0 e^{2\alpha t}$

$= \cancel{x_0 y_0 e^{2\alpha(1+t) - 1}}$

$= x_0 y_0 \left[e^{(2\alpha - 1)t} - e^{2\alpha t} \right]$

$= x_0 y_0 e^{2\alpha t} \left[\frac{1}{e} - 1 \right]$

$= x_0 y_0 e^{2\alpha t} \left[\frac{1 - e}{e} \right]$

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$X(t) = (1-t)\, W\left(\frac{t}{1-t}\right)$

Q To check if a process is a brownian bridge.

Ans ① Check $X(0) = X(T)$

② Prove it is a gaussian process.

Linear combination of the variables in $X(t_i)$ reduces to linear comb. of variables $W(t_i)$ and hence have normal distribution.

③ Mean $= 0$ or $E[X(t)] = 0$

④ If $s, t \in [0,1]$ and $s < t$ then

$\frac{s}{1-s} < \frac{t}{1-t}$ $\quad Cov(X(s), X(t)) = s \wedge t - \frac{st}{T}$

⑤ $t \to X(t)$ is continuous with prob. 1 on $[0,1]$ continuity at 1

$X(t) = (1-t)\, W(t/(1-t)) \to 0$ as $t \to 1$

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⑩ By ito's formula have –

$$dX(t) = dh(W_1(t), W_2(t)) = \frac{\partial h}{\partial x_1}(W_1(t), W_2(t))\, dW_1(t)$$

$$+ \frac{\partial h}{\partial x_2}(W_1(t), W_2(t))\, dW_2 + \frac{1}{2}\frac{\partial^2 h}{\partial x_1^2}(W_1(t), W_2(t))\, dW_1\, dW_1$$

$$+ \frac{1}{2}\frac{\partial^2 h}{\partial x_2^2}(W_1(t), W_2(t))\, dW_2\, dW_2 + \underbrace{\frac{\partial^2 h}{\partial x_1 \partial x_2}(W_1(t), W_2(t))\, dW_1\, dW_2}_{=0}$$

$\because W_1, W_2$ are independent

$\because h$ is harmonic

$$dX(t) = dh(W_1(t), W_2(t)) = \frac{\partial h}{\partial x_1}\, dW_1 + \frac{\partial h}{\partial x_2}\, dW_2$$

$$X(t) = X(0) + \int_0^t \frac{\partial h}{\partial x_1}(W_1(s), W_2(s))\, dW_1(s)$$

$$+ \int_0^t \frac{\partial h}{\partial x_2}(W_1(s), W_2(s))\, dW_2(s)$$

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$$dX(t) = \frac{b - X(t)}{1-t}\,dt + dW$$

$$Y(t) = X(t) - b$$

$$dY = \frac{-Y(t)}{1-t}\,dt + dW$$

$$g(t) = 0 \quad f(t) = -\frac{1}{1-t} \quad \phi = 0 \quad \theta = 1$$

$$Y(t) = \int_0^t -\frac{1}{(1-t)}\,dt$$

$$= \left[\ln(1-t)\right]_0^t$$

$$= \ln(1-t) \qquad Y(0) = X(0) - b = a - b$$

$$e^{Y(t)} = 1 - t$$

$$Y(t) = (a-b)(1-t) + \int_0^t \frac{(1-t)}{(1-s)}\,dW$$

$$Y(t) = (a-b)(1-t) + (1-t)\int_0^t \frac{dW(s)}{(1-s)}$$

$$X(t) = a(1-t) + bt + (1-t)\int_0^t \frac{dW(s)}{(1-s)}$$

(7)

$$dX = \alpha X\,dt + Y\,dW \qquad X(0) = x_0$$
$$dY = \alpha Y\,dt - X\,dW \qquad Y(0) = y_0$$

$$\cancel{dX^2 = 2X\,dX + X}$$

$$dX^2 = 2X\,dX + 2(dX)^2$$
$$dX^2 = 2X(\alpha X\,dt + Y\,dW) + 2Y^2\,dt$$
$$dX^2 = (2\alpha X^2 + 2Y^2)\,dt + 2YX\,dW \quad \text{—— (1)}$$
$$dY^2 = 2Y\,dY + 2(dY)^2$$
$$= 2Y(\alpha Y\,dt - X\,dW) + 2(X^2\,dt)$$
$$= 2\alpha Y^2\,dt - 2XY\,dW + 2X^2\,dt \quad \text{—— (2)}$$
$$= (2\alpha Y^2 + 2X^2)\,dt - 2XY\,dW$$
$$d(X^2+Y^2) = \left(2\alpha(X^2+Y^2) + 2(X^2+Y^2)\right)dt$$

$$\int_t \frac{d(X^2+Y^2)}{(X^2+Y^2)} = \int 2(\alpha+1)\,dt \qquad X^2+Y^2 = Z$$

$$\int_0 \frac{dZ}{Z} = 2(\alpha+1)t \qquad Z(0) = x_0^2 + y_0^2$$

$$[\ln(Z)]_0^t = 2(\alpha+1)t$$

$$\ln\left(\frac{t}{x_0^2+y_0^2}\right) = 2(\alpha+1)t$$

$$\frac{t}{x_0^2+y_0^2} = e^{2(\alpha+1)t}$$

$$\boxed{t = (x_0^2+y_0^2)\,e^{2(\alpha+1)t}}$$

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(8) $dX(t) = (h(t) + g(t)X(t))dt + f(t)dW(t)$

$X(0) = x_0$

$$d\left(e^{-\int_0^t g(t)\,dt} X(t)\right) = h(t)dt + f(t)dW(t)$$

$$e^{-\int_0^t g(t)\,dt} = Y(t)$$

$$dXY = h(t)dt + f(t)\,dW(t)$$

$$X(t)Y(t) - x_0 = \int_0^t h(s)\,ds + \int_0^t f(s)\,dW(s)$$

$$X(t)Y(t) = x_0 + \int_0^t h(s)\,ds + \int_0^t f(s)\,dW(s)$$

$$X(t) = \frac{x_0 + \int_0^t h(s)\,ds + \int_0^t f(s)\,dW(s)}{Y(t)}$$

~~X(t)~~ Integral of non deterministic Integrand in Gaussian. Adding and Multiplying a non random function to a Gaussian process ~~to~~ gives up another gaussian.

Practice

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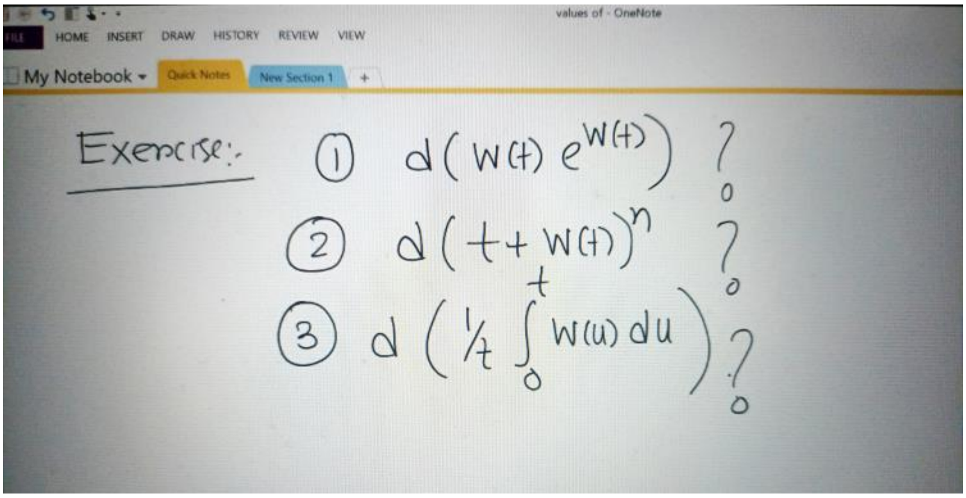

Exercise 2

Friday, February 25, 2022 10:53 AM

1 (i) $v(t,x,y) = \dfrac{y\, e^{-r(T-t)}}{T} + \dfrac{x\left(1-e^{-r(T-t)}\right)}{rT}$

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$dS = Sr\,dt + \sigma S\,d\tilde{W}$

$(dS)^2 = \sigma^2 \,dt$

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(ii) $\underbrace{d(e^{-rt} v(t, S(t), Y(t))}_{} \rightarrow$ Martingale

$= -re^{-rt} v(t, S(t), Y(t)) + e^{-rt}\left[v_x\, dt + v_x\, dS + v_y\, dY + \dfrac{v_{xx}}{2}\, dS\, dS \right]$

$= e^{-rt}\left[-rv + v_t\, dt + v_x\, dS + v_y\, dY + \dfrac{v_{xx}}{2} S^2 \sigma^2\, dt \right]$

$(rS\,dt + \sigma S\, d\tilde{W})$

$= e^{-rt}\left[-rv\,dt + v_t\, dt + v_x \cancel{dS} + v_y\, dY + \dfrac{v_{xx}}{2}\sigma^2 S^2\, dt \right]$

$= e^{-rt}\left[-rv\,dt + v_t\, dt + v_x rS\, dt + v_x \sigma S\, d\tilde{W} + v_y\, dY + \dfrac{v_{xx}}{2}\sigma^2 S^2\, dt \right]$

$= e^{-rt}\left[\underbrace{\left(-rv + v_t + v_x rS + v_y S(t) + \dfrac{v_{xx}}{2}\sigma^2 S^2 \right)}_{=0} dt + v_x \sigma S\, d\tilde{W} \right]$

$rv = v_t + r v_x x + v_y x + \dfrac{1}{2}\sigma^2 x^2 v_{xx}$

If $S(t) = 0$, $Y(T) = Y(t) = y$

$V(T) = \left(\dfrac{y}{T}\right)$

$V(t) = e^{-r(T-t)} \dfrac{y}{T} = v(t, 0, y)$

~~$V(T)$~~ $v(T, x, y) = \dfrac{y}{T}$

(iii)

$$v(t,x,y) = \frac{y e^{-r(T-t)}}{T} + \frac{x(1-e^{-r(T-t)})}{rT}$$

$$\frac{\partial v}{\partial x} = \frac{1-e^{-r(T-t)}}{rT} = \Delta(t)$$

(iv)

$$\boxed{\begin{aligned} dX &= \Delta dS + r(X-\Delta S)dt \\ &= rXdt + \Delta(dS - rSdt) \end{aligned}} \quad (1)$$

$$\cancel{dX - rXdt = (\Delta)(dS - rSdt)}$$

$$\cancel{dX = rXdt + \Delta(dS - rSdt)}$$

$$d(\Delta S(t)) = \Delta dS + (d\Delta) S$$

$$\begin{aligned} d(e^{r(T-t)}\Delta S) &= e^{r(T-t)} d(\Delta S) + \Delta S e^{r(T-t)}(-r)dt \\ &= e^{r(T-t)}[\Delta dS + S d\Delta - \Delta r S dt] \end{aligned}$$

$$d(e^{r(T-t)}\Delta S) - S d\Delta\, e^{r(T-t)} = e^{r(T-t)}[\Delta dS - \Delta r S dt] \quad (2)$$

$$\begin{aligned} d(e^{r(T-t)}X(t)) &= e^{r(T-t)}(-r)X(t) + e^{r(T-t)} dX \\ &= e^{r(T-t)}[-rX(t)dt + dX] \quad \boxed{\text{using 1}} \\ &= e^{r(T-t)}(\Delta)(dS - rSdt) \\ &= d(e^{r(T-t)}\Delta S) - S d\Delta\, e^{r(T-t)} \quad \boxed{\text{using 2}} \end{aligned}$$

$$\cancel{X(T) = e^{rT}X(0) = \Delta(T)S(T) = \Delta(0)S(0)e^{rT}}$$

$$e^{r(T-t)}X(t) = \underline{e^{rT}}X(0) + \int_0^t d\left(e^{r(T-t)}\Delta(u)S(u)\right) - \int_0^t e^{r(T-u)}S(u)\,d\Delta(u)$$

$$= \cancel{e^{r(T-t)}\Delta(t)S(t)} - \cancel{\frac{e^{rT}(1-e^{-rT})}{rT}S(0)} + e^{r(T-t)}\Delta(t)S(t) - \cancel{\frac{e^{rT}(1-e^{-rT})}{rT}S(0)} + \frac{1}{T}\int_0^t S(t)\,dt$$

$$\Delta(u) = \frac{1-e^{-r(T-u)}}{rT}$$

$$d\Delta = -\cancel{\frac{e^{-r(T-u)}}{rT}\,r} = -\frac{e^{-r(T-u)}}{T}$$

$$e^{r(T-t)}X(t) = e^{r(T-t)}\Delta(t)S(t) + \frac{1}{T}\int_0^t S(t)\,dt$$

$\boxed{t=T}$

$$X(T) = \cancel{\Delta t}\ \frac{1}{T}\int_0^t S(t)\,dt$$

4. $X(t) = W(t) - tW(1)$ $\qquad E[X] = E[W(t)] - tE[W(1)]$

$Y(t) = X^2$ $\qquad\qquad\qquad\qquad = 0 - 0$

$$E[Y(t)] = E[X^2] = \text{var}(X) + (E[X])^2$$
$$= t - t^2 + (0)^2$$
$$= t - t^2$$

$$E[W_{(t)}^2 + t^2W^2(1) - 2t\,\underline{W(t)W(1)}]$$
$$= t + t^2 - 2t\,\cancel{(0)}\,t$$
$$= t + t^2 - 2t^2 = t - t^2$$

(4) $W(t) - t\,W(1)$

$W(t) - t(W(1) - W(t) + W(t))$

$W(t) - t\,W(t) - t(W(1) - W(t))$

$W(t)[1-t] - t[W(1) - W(t)]$

$a = W(t)(1-t)$

$b = t(W(1) - W(t))$ $\quad E[a] = 0$

$E(a-b)^4 \quad N(0, 1-t) \quad E[b^2] = (1-t)t^2 \quad E[b] = 0$

$= \cancel{E[a^4 - 3a^3b + 3a^2b^2]} \quad E[a^2] = t(1-t)^2$

$= E[a^4 - 4a^3b + 6a^2b^2 - 4ab^3 + b^4]$

$2\ \frac{4\times 3}{2}$ $\quad = E[a^4 + 6a^2b^2 + b^4]$

$= 6t^3(1-t)^3 + E[a^4 + b^4]$

$= 6t^3(1-t)^3 + 3t^2(1-t)^4 + 3t^4(1-t)^2$

$= 3t^2(1-t)^2[t(1-t) + (1-t)^2 + t^2]$

$= 3t^2(1-t)^2[t + 1 + t^2 - 2t]$

$= 3t^2(1-t)^2[t^2 - t + 1]$

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MGF of $N(\mu, \sigma^2) =$

$= e^{\mu t + \sigma^2 t^2/2}$

$\cancel{e^{\mu t + \frac{1}{2}\sigma^2 t^2/2}[(\mu) + \sigma^2 2t]}$

$a = W(t)(1-t)$

$b = t(W(1) - W(t))$

$\cancel{a = N(0, t}$

$E[a^4] = E[(1-t)^4 W^4(t)]$

$= (1-t)^4 E[W^4(t)] = 3(1-t)^4 t^2$

$E[b^4] = 3t^4(1-t)^2$

$W_{(t)} \sim N(0, t)$

~~MGF(t)~~

$MGF(s) = e^{\mu s + \sigma^2 s^2/2}$

$= e^{t^2 s^2/2}$

$\frac{d^4(e^{t^2 s^2/2})}{ds^4} = \frac{d^3}{ds^3}\left(e^{t^2 s^2/2} s t^2\right)$

$= t^2 \frac{d^3}{ds^3}\left(s\,e^{t^2 s^2/2}\right)$

$= t^2\left[\frac{d^2}{ds^2}\left(e^{t^2 s^2/2} + s^2 t^2 e^{t^2 s^2/2}\right)\right]$

$= t^2\left[\frac{d^2}{ds^2}\left(e^{t^2 s^2/2}\right) + \frac{d^2}{ds^2}\left(s^2 t^2 e^{t^2 s^2/2}\right)\right]$

$= t^2\left[\frac{d}{ds}\left[s t^2 e^{t^2 s^2/2}\right] + \frac{d}{ds}\left(2s t^2 e^{t^2 s^2/2} + s^2 t^2 e^{t^2 s^2/2} s t^2\right)\right]$

$= t^2\left[t^2 \cancel{e^{t^2 s^2/2}} t s + 2t^2\right]$

$= t^2[3t^2] = 3t^4$

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$MGF(s) = E[e^{sT}] = e^{x/\sigma^2(\alpha - \sqrt{\mu^2 - 2s\sigma^2})}$

5(c)

$$\frac{d}{ds}\left(e^{x/\sigma^2(\alpha - \sqrt{\alpha^2 - 2s\sigma^2})}\right)$$

$$= e^{x/\sigma^2(\alpha - \sqrt{\alpha^2 - 2s\sigma^2})} \frac{x}{\cancel{\sigma^2}} \frac{1 \cdot \cancel{2}\cancel{\sigma^2}}{\cancel{2}\sqrt{\mu^2 - 2s\sigma^2}}$$

$$= \frac{x\, e^{x/\sigma^2(\alpha - \sqrt{\alpha^2 - 2s\sigma^2})}}{\sqrt{\alpha^2 - 2s\sigma^2}}$$

$$= \frac{x\, e^{x/\sigma^2(\alpha - \alpha)}}{\alpha} = \frac{x}{\alpha}$$

$$\frac{d}{ds} MGF = \frac{x\, e^{\frac{x}{\sigma^2}(\alpha - \sqrt{\alpha^2 - 2s\sigma^2})}}{\sqrt{\alpha^2 - 2s\sigma^2}}$$

$$\frac{d^2}{ds^2} MGF = \frac{x^2}{(\alpha^2 - 2s\sigma^2)} \cancel{e^{\frac{x}{\sigma^2}(\alpha - \sqrt{\alpha^2 - 2s\sigma^2})}}$$

$$+ x \cancel{e^{\frac{x}{\sigma^2}(\alpha - \sqrt{\alpha^2 - 2s\sigma^2})}} \left(\frac{+1 \cdot + \cancel{2}\sigma^2}{\cancel{2}\left(\sqrt{\alpha^2 - 2s\sigma^2}\right)^3}\right)$$

$$= \frac{x^2}{\alpha^2} + \frac{\sigma^2 x}{\alpha^3}$$

$$E[T^2] = \frac{x}{\alpha^2}\left(x + \frac{\sigma^2}{\alpha}\right)$$

$$var[T] = \cancel{\frac{x^2}{\alpha^2}} + \frac{\sigma^2 x}{\alpha^3} - \cancel{\frac{x^2}{\alpha^2}}$$

$$= \frac{\sigma^2 x}{\alpha^3}$$

Date __/__/____

6) $X(t) = 2t + 3W(t)$ $\quad \alpha = 2$

$Y(t) = 2t + W(t)$ $\quad \alpha = 2$

$E[\tau] = \frac{x}{\alpha}$

(a) Because Same $E[\tau] \rightarrow$ This is because of same α

(b) ~~risk α σ~~ α

$var(X(t)) = var(2t + 3W(t))$

$= E[X^2] - (E[X])^2$

$= E[4t^2 + 9W^2 + 12tW] - (2t)^2$

$= 9t + 0 = 9t$

$var(Y(t)) = E[Y^2] - (E(Y))^2$

$= E(4t^2 + W^2 + 4tW) - 4t^2$

$= t$

X(t) is more risky for same return

I would like to own Y

(7) $E(T_1) = \frac{9}{4} = 2.25$

Yes

$E[T_2] = \frac{14}{5} = 2.8$

Date __/__/__

(3) $\frac{1}{T_2 - T_1} \int_{T_1}^{T_2} S(u)\,du = V(T_2)$

$dS(u) = rS(u)\,du + \sigma S(u)\,d\tilde{W}(u)$

$$e^{-r(T_2-t)} \tilde{E}[V(T_2) \mid \mathcal{F}(t)] = e^{-r(T_2-t)} \tilde{E}_t[V(T_2)]$$

$$= e^{-r(T_2-t)} \frac{1}{T_2 - T_1} \int_{T_1}^{T_2} \tilde{E}_t[S(u)]\,du$$

$$= \frac{e^{-r(T_2-t)}}{T_2 - T_1} S(t) \int_{T_1}^{T_2} e^{r(u-t)}\,du$$

$$= \frac{e^{-r(T_2-t)}}{T_2 - T_1} S \left[\frac{e^{r(u-t)}}{r} \right]_{T_1}^{T_2}$$

$$= \frac{e^{-r(T_2-t)}}{(T_2 - T_1)r} S \left(e^{r(T_2-t)} - e^{r(T_1-t)} \right)$$

$$= \frac{S}{r(T_2-T_1)} \left(1 - e^{r(T_1-T_2)} \right)$$

$$= \frac{S}{r(T_2-T_1)} \left(1 - e^{-r(T_2-T_1)} \right)$$

2. $V(T) = \frac{1}{T} \int_0^T S(u)\,du - K \qquad dV(T) = S(u)\,du$

$$\cancel{e^{-r(T-t)}}\, \tilde{E}[V(T) \mid \mathcal{F}(t)] = \tilde{E}[V(T) \mid \mathcal{F}(t)]$$

$$= \tilde{E}_t[V(T)]$$

$$= \tilde{E}_t\left[\frac{1}{T} \left(\int_0^t S(u)\,du + \int_t^T S(u)\,du \right) - K \right]$$

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$$\tilde{E}_t\left[\frac{1}{T}\left[\int_0^t S(u)\,du + \int_t^T S(u)\,du\right]\right] - K$$

$$\frac{1}{T}\tilde{E}_t\left[\int_0^t S(u)\,du\right] + \frac{1}{T}\tilde{E}_t\left[\int_t^T S(u)\,du\right] - K$$

$$\frac{1}{T}\left[\int_0^t S(u)\,du\right] + \frac{1}{T}\int_t^T \tilde{E}_t[S(u)]\,du - K$$

$$v(t,x,y) = \frac{y}{T} + \frac{1}{T}\int_t^T x\,\cancel{e^{r(u-t)}}\,du - K$$

$$= \frac{y}{T} + \frac{x(T-t)}{T} - K$$

$$= \frac{y}{T} + x\left(1-\frac{t}{T}\right) - K$$

$$v(0,S(0),0) = 0 + S(0)(1-0) - K$$

$$\boxed{\begin{aligned} X(0) &= S(0) - K \\ \gamma(t) &= v_x = 1-\frac{t}{T} \\ X(t) &= v(t,x,y) = \frac{y}{T} + x\left(1-\frac{t}{T}\right) - K \end{aligned}}$$

Date ___/___/_____

(8) $S(t) = S(0)\, e^{\sigma \tilde{W} + (r - \frac{\sigma^2}{2})t}$

$X(t) = \ln\left(\frac{S(t)}{S(0)}\right) = \sigma \tilde{W} + \left(r - \frac{\sigma^2}{2}\right)t = \alpha t + \sigma W(t)$

$\alpha = \left(r - \frac{\sigma^2}{2}\right)$

(a) $E[T_2] = \frac{x}{\alpha} = \frac{\ln 2}{\left(r - \frac{\sigma^2}{2}\right)} = \frac{2\ln(2)}{2r - \sigma^2}$

$var(T_2) = \frac{\sigma^2 x}{\alpha^3} = \frac{8\sigma^2 \ln(2)}{(2r - \sigma^2)^3}$

b) NO

(b) $\alpha = 0.15 \cancel{- \frac{0.2 \times 0.2}{2}} \cancel{= 0.15 - 0.1 \times 0.2}$

$\cancel{= 0.15 - 0.02}$

$\cancel{= 0.13}$

$E[T_2] = \frac{\ln(2)}{0.15}$

$= 4.621$ years

(9)
(a) $dX = -\frac{1}{t^2}\int_0^t S(u)\,dW(u) + \frac{1}{t}S(t)\,dW(t)$

(Doubt)

(9)
(b) $dS = S(\sigma\, d\tilde{W} + \alpha\, dt)$

$\frac{S(t) - S(0)}{t} = \frac{\sigma}{t}\int_0^t S\, dW + \frac{1}{t}\int_0^t S\alpha\, du$

$R(t) = \sigma X(t) + \alpha A(t)$

$\sigma X(t) = R(t) - \alpha A(t)$

10-

(ii) $$\lim_{\|\pi_m\|\to 0} \sum_{j=1}^{m} \left(Y(t_j) - Y(t_{j-1})\right)\left(S(t_j) - S(t_{j-1})\right)$$

$$\le \lim_{\|\pi_m\|\to 0} \max\left(S(t_j) - S(t_{j-1})\right) \lim_{\|\pi_m\|\to 0} \sum_{j=1}^{m} \left(Y(t_j) - Y(t_{j-1})\right)$$

As

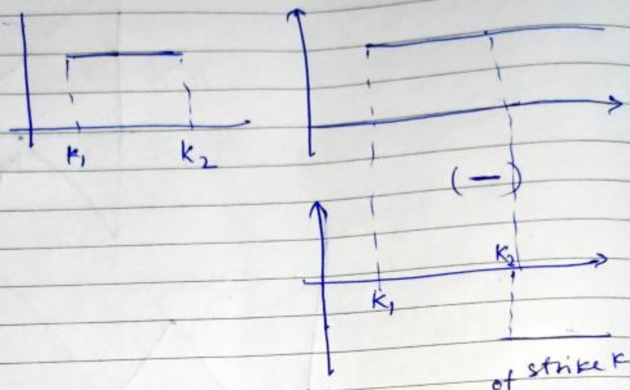
$$= \lim_{\|\pi_m\|\to 0} \max\left(S(t_j) - S(t_{j-1})\right).\left(Y(t_m) - Y_{t_0}\right)$$

$$= \lim_{\|\pi_m\|\to 0} \max\left(S(t_j) - S(t_{j-1})\right)\left(Y(T) - Y(0)\right)$$

Because $S(t)$ is cont.

$= 0$

(iii)



Buy one cash or nothing contract of strike K_1 and sell one, with strike K_2

$$v(T) = CON_{K_1}(T) - CON_{K_2}(T)$$

$$v(t) = CON_{K_1}(t) - CON_{K_2}(t)$$

$$= e^{-r(T-t)}\left(N(d_2)_{K_1} - N(d_2)_{K_2}\right)$$

$$= e^{-r(T-t)}\left[\frac{1}{\sigma\sqrt{\tau}}\left[\log\frac{K_2}{K_1}\right]\right]$$

(iv)

$$V(T) = \begin{cases} K & \text{if } \tau_b < T \\ 0 & \text{if } \tau_b \geq T \end{cases}$$

$$\tau_b = \inf \{ t \geq 0 \, ; \, S(t) = b \}$$

$$V(T) = K \, I_{\{\tau_b < T\}}$$

$$V(t) = \widetilde{E}[V(T) \mid \mathcal{F}_t] \, e^{-r(T-t)}$$

$$V(0) = e^{-rT} K \, \widetilde{\mathbb{P}}(\tau_b < t)$$

$$= e^{-rT} K \int_0^t f_{\tau_b}(y) \, dy$$

$$\cancel{\ln\left(\frac{S(t)}{S(0)}\right) = r\,dt + \left(r - \frac{\sigma^2}{2}\right) d\widetilde{W}}$$

$$\ln\left(\frac{S(t)}{S(0)}\right) = \sigma \widetilde{W} + \left(r - \frac{\sigma^2}{2}\right) t$$

$$\cancel{x = \ln\left(\frac{[illegible]}{S(0)}\right)}$$

$$\cancel{\ln\left(\frac{K}{S(0)}\right)}$$

$$f_{\tau_b}(y) = \frac{\ln\left(\frac{x}{S(0)}\right)}{\sigma\sqrt{2\pi y^3}} \exp\left\{ - \frac{\left(\ln\left(\frac{x}{S(0)}\right) - \alpha y\right)^2}{2\sigma^2 y} \right\}$$

$y > 0$

$$\alpha = \left(r - \frac{\sigma^2}{2}\right)$$

10-a)

7.3.

Proof. We note $S_T = S_0 e^{\sigma \widehat{W}_T} = S_t e^{\sigma(\widehat{W}_T - \widehat{W}_t)}$, $\widehat{W}_T - \widehat{W}_t = (\widetilde{W}_T - \widetilde{W}_t) + \alpha(T-t)$ is independent of $\mathcal{F}_t$, $\sup_{t \le u \le T}(\widehat{W}_u - \widehat{W}_t)$ is independent of $\mathcal{F}_t$, and

$$\begin{aligned} Y_T &= S_0 e^{\sigma \widehat{M}_T} \\ &= S_0 e^{\sigma \sup_{t\le u\le T} \widehat{W}_u} 1_{\{\widehat{M}_t \le \sup_{t\le u\le T} \widehat{W}_t\}} + S_0 e^{\sigma \widehat{M}_t} 1_{\{\widehat{M}_t > \sup_{t\le u\le T} \widehat{W}_u\}} \\ &= S_t e^{\sigma \sup_{t\le u\le T}(\widehat{W}_u - \widehat{W}_t)} 1_{\{\frac{Y_t}{S_t} \le e^{\sigma \sup_{t\le u\le T}(\widehat{W}_u - \widehat{W}_t)}\}} + Y_t 1_{\{\frac{Y_t}{S_t} \le e^{\sigma \sup_{t\le u\le T}(\widehat{W}_u - \widehat{W}_t)}\}}. \end{aligned}$$

So $E[f(S_T, Y_T)|\mathcal{F}_t] = E[f(x\frac{S_{T-t}}{S_0}, x\frac{Y_{T-t}}{S_0} 1_{\{\frac{y}{x} \le \frac{Y_{T-t}}{S_0}\}} + y 1_{\{\frac{y}{x} \le \frac{Y_{T-t}}{S_0}\}})]$, where $x = S_t$, $y = Y_t$. Therefore $E[f(S_T, Y_T)|\mathcal{F}_t]$ is a Borel function of (S_t, Y_t).

□

Exercise 3

Sunday, April 3, 2022 8:54 PM

1. (i) Payoff call $= S(t) - K$
$= 110 - 100 = 10\$$

(ii) Yes because $\tilde{E}[e^{-rT} h(S(T)) | \mathcal{F}(t)] \geq e^{-rt} h(S(t))$

(iii) Submartingale property.

(iv) Exercise immediately.

2. $C(S(t), t, K_1, T) = \tilde{E}[e^{-r(T-t)}(S(T) - K_1)^+ | \mathcal{F}(t)]$

$C(\quad " \quad K_2, ") \quad \tilde{E}[e^{-r(T-t)}(S(T) - K_2)^+ | \mathcal{F}(t)]$

$(S(T) - K_1)^+ \geq (S(T) - K_2)^+$

$e^{-r(T-t)}(S(T) - K_1)^+ \geq e^{-r(T-t)}(S(T) - K_2)^+$

$\tilde{E}[e^{-r(T-t)}(S(T) - K_1)^+ | \mathcal{F}(t)] \geq \tilde{E}[e^{-r(T-t)}(S(T) - K_2)^+ | \mathcal{F}(t)]$

$C(S(t), t, K_1, T) \geq C(S(t), t, K_2, T)$

~~$C(K_1) + K_1 < C(K_2) + K_2$~~

$C(K_1) \geq C(K_2)$

$S(T) - K_1 - (S(t) - K_2)$

$C(K_1) - C(K_2) = \tilde{E}[e^{-r(T-t)}[(S(T) - K_1)^+ - (S(T) - K_2)^+] | \mathcal{F}(t)]$

$= \tilde{E}[e^{-r(T-t)}[0 \, I_{(S(T) < K_1)} + (S(T) - K_1) I_{(K_1 < S(T) < K_2)}$

$+ (K_2 - K_1) I_{(K_2 \leq S(T))}]$

$S(T) - K_1 < K_2 - K_1$

$C(K_1) - C(K_2) \leq \tilde{E}[e^{-r(T-t)}(K_2 - K_1) | \mathcal{F}(t)] = e^{-r(T-t)}(K_2 - K_1)$

$\leq K_2 - K_1$

5(a)

$b = L = S(t) = S(0)\exp\left\{\sigma\tilde{W} + \left(r - \frac{\sigma^2}{2}\right)t\right\}$

$\cancel{\ln\left(\frac{S(t)}{S(0)}\right)}$

$\ln\left(\frac{L}{S(0)}\right) = \sigma\tilde{W} + \left(r - \frac{\sigma^2}{2}\right)t$

$\frac{1}{\sigma}\ln\left(\frac{L}{S(0)}\right) = \tilde{W} + \frac{1}{\sigma}\left(r - \frac{\sigma^2}{2}\right)t.$

$\mu = \left(r - \frac{\sigma^2}{2}\right)\frac{1}{\sigma}$

$m = \frac{1}{\sigma}\ln\left(\frac{L}{S(0)}\right)$

$\cancel{\mu^2 = \frac{r^2}{\sigma^2} + \frac{\sigma^2}{4} - r + 2r}$

$\mu^2 + 2r = \left(\frac{r}{\sigma} - \frac{\sigma}{2}\right)^2 + 2r$

$= \left(\frac{r}{\sigma}\right)^2 + \left(\frac{\sigma}{2}\right)^2 + 2\frac{r}{\sigma}\frac{\sigma}{2}$

$= \left(\frac{r}{\sigma} + \frac{\sigma}{2}\right)^2$

$$E\, e^{-r\tau_m} = e^{-(-\frac{1}{\sigma}(r-\frac{\sigma^2}{2}) + \frac{r}{\sigma} + \frac{\sigma}{2})(\frac{1}{\sigma}\ln(\frac{b}{S(0)}))}$$

$$= e^{-(\cancel{-\frac{r}{\sigma}} + \frac{\sigma}{2} + \cancel{\frac{r}{\sigma}} + \frac{\sigma}{2})(\frac{1}{\sigma}\ln(\frac{b}{S(0)}))}$$

$$= e^{-\cancel{\sigma}(\frac{1}{\cancel{\sigma}}\ln(\frac{b}{S(0)}))}$$

$$= e^{\ln\frac{S(0)}{b}} = \frac{S(0)}{b}$$

5(b) $S(t) = S(0)\exp\{\sigma \tilde{W} + (r - a - \frac{\sigma^2}{2})t\} = b$

$$\sigma \tilde{W} + (r - a - \frac{\sigma^2}{2})t = \ln(\frac{b}{S(0)})$$

$$\tilde{W} + \frac{1}{\sigma}(r - a - \frac{\sigma^2}{2})t = \frac{1}{\sigma}\ln(\frac{b}{S(0)})$$

$$\mu = \frac{1}{\sigma}(r - a - \frac{\sigma^2}{2})$$

$$m = \frac{1}{\sigma}\ln(\frac{b}{S(0)})$$

$$E[e^{-r\tau_m}] = e^{-m(-\mu + \sqrt{\mu^2 + 2\lambda})}$$

$$= e^{-\frac{1}{\sigma}\ln(\frac{b}{S(0)})(-\frac{1}{\sigma}(r - a - \frac{\sigma^2}{2}) + \sqrt{\mu^2 + 2r})}$$

$$\mu^2 + 2r = \frac{1}{\sigma^2}(r - a - \frac{\sigma^2}{2})^2 + 2r$$

$$= (\frac{r}{\sigma} - \frac{\sigma}{2} - \frac{a}{\sigma})^2 + 2r$$

$$= (\frac{r}{\sigma} - \frac{\sigma}{2})^2 + \frac{a^2}{\sigma^2} - \frac{2a}{\sigma}(\frac{r}{\sigma} - \frac{\sigma}{2}) + 2r$$

$$= (\frac{r}{\sigma} + \frac{\sigma}{2})^2 + \frac{a^2}{\sigma^2} - \frac{2a}{\sigma}(\frac{r}{\sigma} - \frac{\sigma}{2})$$

$$= \cancel{e^{-\ln(\frac{b}{S(0)})}}\ \cancel{e^{\ln(\frac{S(0)}{b})}}\ \left(\frac{S(0)}{b}\right)^{(-\frac{1}{\sigma^2}(r - a - \frac{\sigma^2}{2}) + \frac{1}{\sigma}\sqrt{\mu^2 + 2r})}$$

$$= \left(\frac{S(0)}{b}\right)^{(\frac{1}{2} - (\frac{r-a}{\sigma^2})) + \sqrt{\frac{\mu^2 + 2r}{\sigma^2}}}$$

$$\mu = \frac{1}{\sigma}\left(r - a - \frac{\sigma^2}{2}\right) \qquad m = \frac{1}{\sigma}\ln\left(\frac{b}{S(0)}\right)$$

$$\frac{\mu^2 + 2r}{\sigma^2} = \frac{\frac{1}{\sigma^2}\left(r - a - \frac{\sigma^2}{2}\right)^2}{\sigma^2} + \frac{2r}{\sigma^2}$$

$$= \left(\frac{r-a}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2r}{\sigma^2}$$

$$\therefore \underline{\text{Price}} \quad \left(\frac{S(0)}{b}\right)^{\frac{1}{2} - \left(\frac{r-a}{\sigma^2}\right) + \sqrt{\left(\frac{r-a}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2r}{\sigma^2}}}$$

(ii) $h_1(0) = \frac{1}{2} - \frac{r}{\sigma^2} + \sqrt{\left(\frac{r}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2r}{\sigma^2}}$

$$= \cancel{\frac{1}{2}} - \cancel{\frac{r}{\sigma^2}} + \cancel{\frac{r}{\sigma^2}} \cdot \cancel{0} + \frac{1}{2}$$

$$= 1$$

$$h(\delta) = \frac{1}{2} - \frac{(r-\delta)}{\sigma^2} + \sqrt{\left(\frac{r-\delta}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2r}{\sigma^2}}$$

$$h'(\delta) = \frac{1}{\sigma^2} + \frac{1}{2}\,\frac{2\left(\frac{r-\delta}{\sigma^2} - \frac{1}{2}\right)\left(-\frac{1}{\sigma^2}\right)}{\sqrt{\left(\frac{r-\delta}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2r}{\sigma^2}}}$$

$$\frac{r-\delta}{\sigma^2} - \frac{1}{2}$$

$$\frac{r - \sigma^2}{2\sigma^2} - \frac{2\delta}{2\sigma^2}$$

$$\frac{r - 2\sigma^2 - 2\delta}{2\sigma^2}$$

$$\frac{1}{\sigma^2} + \frac{\left(-\frac{1}{\sigma^2}\right)\left(\left(\frac{r-\delta}{\sigma^2} - \frac{1}{2}\right)\right)}{\sqrt{\left(\frac{r-\delta}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2r}{\sigma^2}}}$$

$$\frac{1}{\sigma^2}\left[1 - \frac{t}{\sqrt{t^2 + \frac{2r}{\sigma^2}}}\right]$$

$$-1 < \frac{-t}{\sqrt{t^2 + \frac{2r}{\sigma^2}}} < 1$$

$$0 < 1 - \frac{t}{\sqrt{t^2 + \frac{2r}{\sigma^2}}} < 2$$

$$\therefore \underline{\underline{h'(\delta) > 0}}$$

$S_t = S(0)e^{(r-\frac{\sigma^2}{2})t + \sigma\tilde{W}}$

(c) $b > 0$, $b < S(0)$

$$b = S(t) = S(0)\, e^{(r-\frac{\sigma^2}{2})t + \sigma\tilde{W}}$$

$$\ln\left(\frac{b}{S(0)}\right) = \left(r - \frac{\sigma^2}{2}\right)t + \sigma\tilde{W}$$

$$\frac{1}{\sigma}\ln\left(\frac{S(0)}{b}\right) = -\frac{1}{\sigma}\left(r - \frac{\sigma^2}{2}\right)t - \tilde{W}$$

$$\mu = -\frac{1}{\sigma}\left(r - \frac{\sigma^2}{2}\right)$$

$$m = \frac{1}{\sigma}\ln\left(\frac{S(0)}{b}\right)$$

$$\tilde{E}\left[e^{-r\tau_m}\right] = e^{-m(-\mu+\sqrt{\mu^2+2r})}$$

$$= e^{-\frac{1}{\sigma}\ln\left(\frac{S(0)}{b}\right)(}$$

$$2r + \mu^2 = \frac{1}{\sigma^2}\left(r - \frac{\sigma^2}{2}\right)^2 + 2r$$

$$= \frac{1}{\sigma^2}\left(r + \frac{\sigma^2}{2}\right)^2$$

$$\sqrt{\mu^2+2r} = \frac{1}{\sigma}\left(r + \frac{\sigma^2}{2}\right)$$

$$-\mu + \sqrt{\mu^2+2r} = \frac{1}{\sigma}\left(r + \frac{\sigma^2}{2}\right) + \frac{1}{\sigma}\left(r - \frac{\sigma^2}{2}\right)$$

$$= \frac{2r}{\sigma}$$

$$= e^{-\ln\left(\frac{S(0)}{b}\right)\left(\frac{2r}{\sigma^2}\right)}$$

$$= \left(\frac{b}{S(0)}\right)^{\frac{2r}{\sigma^2}}$$

(d) $b = S(t) = S(0)\, e^{\left(r - a - \frac{\sigma^2}{2}\right)t + \sigma \hat{W}}$

$$-\frac{\ln}{\sigma}\left(\frac{b}{S(0)}\right) = \frac{1}{\sigma}\left(r - a - \frac{\sigma^2}{2}\right) t \;\; \hat{W}$$

$$m = \frac{-\ln}{\sigma}\left(\frac{b}{S(0)}\right) = \frac{\ln}{\sigma}\left(\frac{S(0)}{b}\right)$$

$$\mu = -\frac{1}{\sigma}\left(r - a - \frac{\sigma^2}{2}\right)$$

$$e^{-r\lambda m} = e^{-m\left(-\mu + \sqrt{\mu^2 + 2r}\right)}$$

$$\mu^2 + 2r = \frac{1}{\sigma^2}\left(r - a - \frac{\sigma^2}{2}\right)^2 + 2r$$

$$e^{\frac{1}{\sigma}\ln\left(\frac{S(0)}{b}\right)\left(\mu - \sqrt{\frac{1}{\sigma^2}\left(r - a - \frac{\sigma^2}{2}\right)^2 + 2r}\right)}$$

$$\left(\frac{S(0)}{b}\right)^{\left(\frac{1}{\sigma^2}\left(r - a - \frac{\sigma^2}{2}\right) - \sqrt{\left(\frac{r-a}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2r}{\sigma^2}}\right)}$$

$$\left(\frac{S(0)}{b}\right)^{-\frac{r-a}{\sigma^2} + \frac{1}{2} - \sqrt{\left(\frac{r-a}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2r}{\sigma^2}}}$$

$$f(b) = \tilde{E}\left[e^{-r\tau_b}\left(S(\tau_b)-K\right)^{+}\right]$$

$$= \tilde{E}\left[e^{-r\tau_b}\right](b-K)$$

$$= \left(\frac{S(0)}{b}\right)^{h_1}(b-K) \quad \text{by part (b)}$$

(ii) $f(b) = (b-K)\left(\frac{S(0)}{b}\right)^{h_1}$

$$\frac{df(b)}{db} = \cancel{\left(\frac{S(0)}{b}\right)^{h_1}} + (b-K)h_1 \cancel{\left(\frac{S(0)}{b}\right)^{h_1-1}} \cdot \frac{-S(0)}{b \cdot b} = 0$$

$$1 + (b-K)h_1 \cdot \left(-\frac{1}{b}\right) = 0$$

$$1 = \left(\frac{b-K}{b}\right)h_1$$

$$b = bh_1 - Kh_1$$

$$Kh_1 = b(h_1 - 1)$$

$$\frac{Kh_1}{h_1 - 1} = b$$

(iii)

(iii) Price of perpetual Call $= \max f(b)$

$$= f(b^*)$$

$$= \frac{K}{(h_1 - 1)} \left(\frac{S(0)}{b^*} \right)^{h_1}$$

(f) $g(b) = E\left[e^{-r\tau_b}(K-b)\right]$

$$= (K-b)\, E\left[e^{-r\tau_b}\right]$$

$$= (K-b)\left(\frac{S(0)}{b}\right)^{h_2}$$

$$h_2 = \frac{1}{2} - \left(\frac{r-\delta}{\sigma^2}\right) - \sqrt{\left(\frac{r-\delta}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2r}{\sigma^2}}$$

$g'(b) = 0 \Rightarrow \cancel{(K-b)}($

$$(-1)\cancel{\left(\frac{S(0)}{b}\right)^{h_2}} + (K-b)\, h_2 \cancel{\left(\frac{S(0)}{b}\right)^{h_2-1}} \cdot -\frac{\cancel{S(0)}}{\cancel{b}\, b}$$

$$= -1 + \frac{(K-b)(-h_2)}{b}$$

$$-1 = \frac{h_2(b-K)}{b}$$

$$b = h_2 b - K h_2$$

$$h_2 K = b(h_2 - 1)$$

$$b = \frac{h_2 K}{h_2 - 1}$$

(iii) Price of PP $= \max g(b)$

$= g(b^*)$

$= (K - b^*)\left(\frac{S(0)}{b^*}\right)^{h_2}$

$= \left(K - \frac{K h_2}{h_2 - 1}\right)\left(\frac{S(0)}{b^*}\right)^{h_2}$

$= K\left[\frac{1}{1-h_2}\right]\left(\frac{S(0)}{b}\right)^{h_2}$

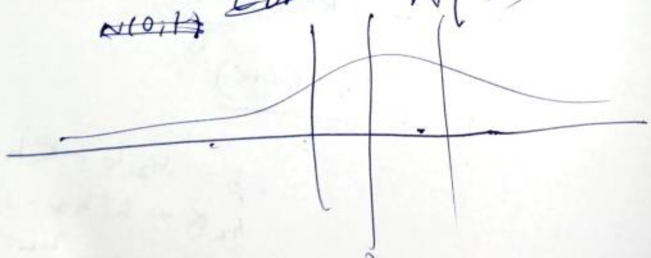
(g) $S(t) < b$

$S(0)e^{\sigma W + (r - \frac{\sigma^2}{2})t} < b$

$\sigma W + (r - \frac{\sigma^2}{2})t < \ln \frac{b}{S(0)}$

$W(t) < \frac{\ln\left(\frac{b}{S(0)}\right) - \left(r - \frac{\sigma^2}{2}\right)t}{\sigma} = \rho$

~~N(0,1)~~ ~~CDF~~ $N(\rho)$



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Previous year

Monday, April 11, 2022 12:07 PM

Q1 $\quad r = -2t \quad H = e^{2t}$

(i) $d(Hr) = H(4dt + 3\,dW)$

$$\int_0^t d(e^{2t} r) = \int_0^t 4e^{2t}dt + \int_0^t 3e^{2t}\,dW$$

$$e^{2t} r(t) - r(0) = \frac{4}{2}\left[e^{2s}\right]_0^t + \int_0^t 3e^{2s}\,dW(s)$$

$$e^{2t} r(t) - \cancel{2} = 2\left[e^{2t} - 1\right] + \int_0^t 3e^{2s}\,dW(s)$$

$$e^{2t} r(t) = 2e^{2t} + \int_0^t 3e^{2s}\,dW(s)$$

$$r(t) = 2 + e^{-2t}\int_0^t 3e^{2s}\,dW(s)$$

$$= 2 + 3e^{-2t}\int_0^t e^{2s}\,dW(s)$$

$$I = \int_0^t e^{2s}\,dW(s)$$

$$I \sim N\left(0, \int_0^t e^{4s}\,dt\right)$$

$$\sim N\left(0, \frac{1}{4}(e^{4t}-1)\right)$$

$$r(t) \sim N\left(2, \frac{9e^{-4t}}{4}(e^{4t}-1)\right)$$

$$\sim N\left(2, \frac{9}{4}(1-e^{-4t})\right)$$

(ii) Limiting Disⁿ = $N(2, 9/4)$

(iii) $e^{2t} r(t) - r(0) = 2[e^{2t} - 1] + \int_0^t 3 e^{2s} \, dW(s)$

$r(t) = e^{-2t} r(0) + e^{-2t} 2 (e^{2t} - 1) + e^{-2t} \int_0^t 3 e^{2s} \, dW(s)$

$= e^{-2t} \underbrace{r(0)}_{N(2, \frac{9}{4})} + 2(1 - e^{-2t}) + 3e^{-2t} \underbrace{\int_0^t e^{2s} \, dW(s)}_{N\left(0, \frac{(e^{4t}-1)}{4}\right)}$

$N\left(0, \frac{9}{4}(1-e^{-4t})\right)$

$= N\left(2e^{-2t}, \frac{9}{4} e^{-4t}\right)$

$= N\left(2e^{-2t}, \frac{9}{4} e^{-4t}\right) + N\left(2 - 2e^{-2t}, \frac{9}{4}(1 - e^{-4t})\right)$

$= N\left(\frac{\cancel{4 - 2e^{-2t}}}{2}, \frac{9}{4}\right) \to \underline{\underline{N(2, 9/4)}}$

$aN(b, c) = N(ab, a^2 c)$

(iv) $r(0) = 2$ $\quad P(r(t) < 0)$

$P\left(N(2, \frac{9}{4}(1-e^{-4t})) < 0\right)$

$= CDF(0) \quad e^{-\frac{(x-2)^2}{2 \cdot \frac{9}{4} \cdot (1-e^{-4t})}} \, dx$

$= \int_{-\infty}^{0} \frac{1}{\sqrt{2\pi \times \frac{9}{4} \times (1-e^{-4t})}}$

$e^{-\frac{(x-2)^2}{2 \cdot \frac{9}{4}}}$

$\lim_{t \to 0} P(r(t) < 0) = \int_{-\infty}^{0} \frac{1}{\sqrt{2\pi \times \frac{9}{4}}}$

(a) (i)

$$\mu^2 + 2r = \frac{1}{\sigma^2}\left(r - \frac{\sigma^2}{2} - \delta\right)^2 + 2r$$

$$-\frac{1}{\sigma}\ln\left(\frac{S}{b}\right) \cdot \sqrt{\frac{1}{\sigma^2}\left(r - \frac{\sigma^2}{2} - \delta\right)^2 + 2r}$$

$$\mu = -\frac{1}{\sigma}\left(r - \frac{\sigma^2}{2} - \delta\right)$$

$$m = \frac{1}{\sigma}\ln\left(\frac{S}{b}\right)$$

$$e^{-m\left(-\mu + \sqrt{\mu^2 + 2r}\right)}$$

$$= e^{\left(-\frac{1}{\sigma}\ln\left(\frac{S}{b}\right)\right)\left[+\frac{1}{\sigma}\left(r - \frac{\sigma^2}{2} - \delta\right) + \sqrt{\frac{1}{\sigma^2}\left(r - \frac{\sigma^2}{2} - \delta\right)^2 + 2r}\right]}$$

$$\left(\frac{S}{b}\right)^{-\frac{1}{\sigma}\left[\frac{1}{\sigma}\left(r - \frac{\sigma^2}{2} - \delta\right) + \sqrt{\left(\sigma^2\left(\frac{r-\delta}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2r\sigma^2}{\sigma^2}\right)}\right]}$$

$$\left(\frac{S}{b}\right)^{\left[-\frac{1}{\sigma^2}\left(r - \frac{\sigma^2}{2} - \delta\right) - \frac{1}{\cancel{\sigma}}\cancel{\sigma}\sqrt{\left(\frac{r-\delta}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2r}{\sigma^2}}\right]}$$

$$\left(\frac{S}{b}\right)^{\left[\frac{1}{2} - \frac{(r-\delta)}{\sigma^2} - \sqrt{\left(\frac{r-\delta}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2r}{\sigma^2}}\right]}$$

$$\left(\frac{1}{2} - \frac{r-\delta}{\sigma^2}\right) = t$$

$$\boxed{\left(\frac{S}{b}\right)^{t - \sqrt{t^2 + \frac{2r}{\sigma^2}}}}$$

ii) $g(b) = (K-b)\left(\frac{5}{b}\right)^{h_2}$

$$g'(b) = (-1)\cancel{\left(\frac{5}{b}\right)^{h_2}} + (K-b)\, h_2 \cancel{\left(\frac{5}{b}\right)^{(h_2-1)}} \left(-\frac{5}{b^2}\right) = 0$$

$$\left(\frac{5}{b}\right)$$

$$(K-b)\, h_2 \left(\frac{-\cancel{5}}{b^2}\right) = \frac{\cancel{5}}{\cancel{b}}$$

$$(K-b)\, h_2 = -b$$

$$Kh_2 = b(h_2-1)$$

$$\frac{Kh_2}{h_2-1} = b$$

(iii) $$g(b^*) = \left(K - \frac{Kh_2}{h_2-1}\right)\left(\frac{5(h_2-1)}{Kh_2}\right)^{h_2}$$

$$= K\left[\frac{\cancel{h_2}-1-\cancel{h_2}}{h_2-1}\right]\left[\frac{5(h_2-1)}{Kh_2}\right]^{h_2}$$

$$= -\frac{K}{h_2-1}\left(\frac{5}{b^*}\right)^{h_2}$$

$$\boxed{= \frac{K}{1-h_2}\left(\frac{5}{b^*}\right)^{h_2}}$$

Exercise 4

Friday, April 29, 2022 6:46 PM

Q1 — Ex. 7.1

Q2 Ex 7.5

Q3 Ex 7.6

Q4 (i) Ex 15.1 (ii)

Q5 Ex ~~8~~ 23.1

Q6 Ex. 23.2

Q7 E 22.5

Q8 Ex: 24.1

Q9 Ex 24.2

Q10 24.8

①

(i) $dB(t) = r(t)B(t)dt$

$dS(t) = S(t)(\alpha\, dt + \sigma\, d\tilde{W})$

$X = \phi(S(T))$

$\Pi(t) \rightarrow$ arb. free price process ~~by $\Pi(t)$~~

$\tilde{P} \rightarrow$ risk neutral measure.

$\rightarrow$ PI under RNM –

$$d\Pi(t) = r\Pi(t)\,dt + g(t)\,d\tilde{W}(t)$$

Adelwishet

$dS(t) = S(t)(r\,dt + \sigma\, d\tilde{W})$

$\Pi(t) = F(t, S(t))$

By ito's –

$$d\Pi(t) = \left[\frac{\partial F}{\partial t} + rS(t)\frac{\partial F}{\partial S} + \frac{1}{2}\sigma^2 S^2(t)\frac{\partial^2 F}{\partial S^2}\right]dt + \sigma S(t)\frac{\partial F}{\partial S}\, d\tilde{W}(t)$$

$$d\Pi(t) = \frac{\partial \Pi}{\partial x}dx + \frac{\partial \Pi}{\partial t}dt + \frac{\partial^2 \Pi}{\partial x^2}(dx)^2$$

$$= \frac{\partial \Pi}{\partial S}\left(S(t)(r\,dt + \sigma\, d\tilde{W})\right) + \frac{\partial \Pi}{\partial t}dt + \frac{1}{2}\frac{\partial^2 \Pi}{\partial S^2}\sigma^2 S^2 \,dt$$

$$= \left(\frac{\partial \Pi}{\partial S}S(t)r + \frac{\partial \Pi}{\partial t} + \frac{1}{2}\frac{\partial^2 \Pi}{\partial S^2}\sigma^2 S^2\right)dt + \frac{\partial \Pi}{\partial S}\sigma S(t)\, d\tilde{W}$$

By BSE $\quad f_t + rx\, f_s + \frac{1}{2} s^2 \sigma^2 f_{ss} = rF$

$$d\Pi = r\Pi\, dt + g(t)\, d\tilde{W} \qquad g(t) = \sigma S\frac{\partial f}{\partial S}$$

(ii) $Z(t) = \frac{\Pi(t)}{B(t)}$

$$dZ(t) = \frac{d\Pi}{B(t)} + \Pi(t)\, d\left(\frac{1}{B(t)}\right) + d\Pi\, d\left(\frac{1}{B(t)}\right)$$

$$= \frac{r\Pi\, dt + g(t)\, d\hat{W}}{B(t)} - \frac{\Pi(t)}{B^2(t)}\, d(B(t)) + \cancel{r\Pi dt}\,(g(t)\, d\hat{W})\left(\frac{1}{B^2}\right) dB$$

$$= \frac{\cancel{r\Pi\, dt} + g(t)\, d\hat{W}}{B(t)} - \cancel{\frac{\Pi(t)}{B^2(t)}\, rB(t)\, dt} + 0$$

$\because$ B has no dW term.

$$= \frac{g(t)\, d\hat{W}}{B(t)}$$

$$= \frac{\sigma S(t) \frac{\partial F}{\partial s}\, d\hat{W}}{B(t)}$$

(2) Std BSM and a claim $\overset{\text{Bin. op.}}{X} = \phi(S(T))$

$$\phi(s) = \begin{cases} K & \text{if } s \in [\alpha, \beta] \\ 0 & \text{ow} \end{cases}$$

Arbitrage free price $\Pi(t) = e^{-r(T-t)} E^Q_{t,s}[\phi(S(T))]$

Let $K = 1$

$$E^Q_{t,s}(\phi(T)) = E^Q[\phi(T) \mid S(t) = s]$$
$$= Q[S(T) \in [\alpha, \beta]]$$
$$= Q[\alpha \le S(T) \le \beta]$$

$$S(T) \le \beta$$

$\cancel{E^Q_t[\ln(S(T))]} =$

$$S(T) = S(0)\, e^{\sigma W(T) + \left(r - \frac{\sigma^2}{2}\right)T}$$

$$\frac{S(T)}{S(t)} = e^{\sigma(\hat{W}(T) - \hat{W}(t)) + \left(r - \frac{\sigma^2}{2}\right)(T-t)}$$

$$\ln\left(S(T)\right) = \ln(S(t)) + \sigma(\hat{W}(T) - \hat{W}(t))$$

$$P\left(S(T) \leq \beta\right)$$

$$P\left(S_0 e^{\sigma \hat{W}(T) + \left(r - \frac{\sigma^2}{2}\right)T} \leq \beta\right)$$

$$P\left(\sigma \hat{W}(T) + \left(r - \frac{\sigma^2}{2}\right)T \leq \ln\left(\frac{\beta}{S_0}\right)\right)$$

$$P\left(\frac{\hat{W}(T)}{\sqrt{T}} \leq \frac{\ln\left(\frac{\beta}{S_0}\right) - \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right)$$

$N(0,T)$ $N(0,1)$

$$N\left(\frac{\ln\left(\frac{\beta}{S_0}\right) - \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right)$$

$$\alpha \leq S(T)$$

$$\alpha \leq S_0 e^{\sigma \hat{W}(T) + \left(r - \frac{\sigma^2}{2}\right)T}$$

$$\ln\left(\frac{\alpha}{S_0}\right) \leq \sigma W(T) + \left(r - \frac{\sigma^2}{2}\right)T$$

$$\frac{\ln\left(\frac{\alpha}{S_0}\right) - \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} \leq \frac{\hat{W}(T)}{\sqrt{T}} \quad N(0,1)$$

$$\cancel{1 - N(}$$

$$N\left(\frac{-\ln\left(\frac{\alpha}{S_0}\right) + \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right)$$

$$\leq \frac{\hat{W}(T)}{\sqrt{T}} \leq \frac{\ln\left(\frac{\beta}{S_0}\right) - \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}$$

$$\cancel{E^{Q}_{t,s}\left[\phi(T)\right] = e^{-r(T-t)}}$$

$$\underbrace{\frac{\ln\left(\frac{K}{S_0}\right) - \left(r - \frac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}}}_{B} \leq \frac{W(T)}{\sqrt{t}} \leq \underbrace{\frac{\ln\left(\frac{b}{S_0}\right) - \left(r - \frac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}}}_{A}$$

$$\frac{W(T)}{\sqrt{t}} \to N(0,1)$$

$$\Pi(t) = e^{-r(T-t)}\left(N(A) - N(B)\right)$$

---

(3) $X = \dfrac{S(T_1)}{S(T_0)}$

$$dS = S(r\,ds + \sigma\, d\hat{W})$$

$$S(T_1) = S(t)\, e^{\sigma(\hat{W}(T_1) - \hat{W}(t)) + \left(r - \frac{\sigma^2}{2}\right)(T_1 - t)}$$

$$S(T_0) = S(t)\, e^{\sigma(\hat{W}(T_0) - \hat{W}(t)) + \left(r - \frac{\sigma^2}{2}\right)(T_0 - t)}$$

$$\frac{S(T_1)}{S(T_0)} = e^{\sigma(\hat{W}(T_1) - \hat{W}(T_0)) + \left(r - \frac{\sigma^2}{2}\right)(T_1 - T_0)}$$

$$E^{Q}_{s,t}\left[\frac{S(T_1)}{S(T_0)}\right] = e^{\left(r - \frac{\sigma^2}{2}\right)(T_1 - T_0)}\, E\left[e^{\sigma(\hat{W}(T_1) - \hat{W}(T_0))}\right]$$

$$= e^{\left(r - \frac{\sigma^2}{2}\right)(T_1 - T_0)}\, e^{\frac{(T_1 - T_0)}{2}\sigma^2}$$

$$\boxed{E[e^{tx}] = e^{\mu t + \frac{\sigma^2}{2}t^2}} \qquad = e^{r(T_1 - T_0)}$$

$$\Pi(t) = e^{r(T_1 - T_0) - r(T_1 - t)}$$

$$= e^{-r(t + T_0)}$$

(4) $dx = \alpha\, dt + \sigma\, dW$

To convert it to Q dynamics

$d\hat{W} = W + \int_0^t \lambda\, du \qquad \lambda = MPR$

$d\hat{W} = dW + \lambda\, dt$

$dx = \alpha\, dt + \sigma(d\hat{W} - \lambda\, dt)$

$= (\alpha - \lambda\sigma)\, dt + \sigma\, d\hat{W}$

$F(t, x(t))$

$dF = F_x\, dX + F_t\, dt + \frac{1}{2} F_{xx} (dX)^2$

$= F_x(\overset{(\alpha-\lambda\sigma)}{\cancel{\alpha}}\, dt + \sigma\, d\hat{W}) + F_t\, dt + \frac{1}{2} F_{xx}\sigma^2\, dt$

$= \underbrace{\left(F_x(\alpha - \lambda\sigma) + F_t + \frac{1}{2} F_{xx}\sigma^2\right)}_{rF} dt + F_x \sigma\, d\hat{W}$

(ii) $dZ = \frac{d\pi}{B} - \frac{\pi\, dB}{B^2}$

$= \frac{rF\, \cancel{dt} + \sigma F_x\, d\hat{W}}{B} - \frac{\pi(r\, \cancel{dt})}{B}$

$= \frac{\sigma F_x}{B}\, d\hat{W}$

(5) $dr = \alpha\, dt + \sigma\, dW$

$X = \phi(r(T))$

corr. PP $\pi(t)$

$\pi(t) = F(t, r(t))$

$\hat{W} = W + \int_0^t \lambda\, du$

$d\hat{W} = dW + \lambda\, dt$

$dr = \alpha\, dt + \sigma(d\hat{W} - \lambda\, dt)$

$= (\alpha - \lambda\sigma)\, dt + \sigma\, d\hat{W}$

5.) $dr = \alpha\, dt + \sigma\, dW$

$X = \phi(r(T))$

corresponding pp $\Pi(t)$

$\Pi(t) = F(t, r(t))$

$d\Pi = F_t\, dt + F_x\, dr + \frac{1}{2} F_{xx} (dr)^2$

$= F_t\, dt + F_x((\alpha - \lambda\sigma)\, dt + \sigma\, dW) + \frac{1}{2} F_{xx} \sigma^2\, dt$

$= \left(F_t + (\alpha-\lambda\sigma) F_x + \frac{1}{2} F_{xx} \sigma^2\right) dt + F_x\, \sigma\, dW$

$= rF\, dt + F_x \sigma\, dW$

(ii) $dZ = \frac{d\Pi}{B} - \frac{\Pi\, dB}{B^2}$

$= \frac{rF\, dt + \sigma F_x\, d\hat{W}}{B} - \frac{\Pi(r\, dt)}{B}$

$= \frac{\sigma F_x}{B}\, d\hat{W}$

(ii)

(6) Ex 23.2

(a) $p(t,T) = E^Q\left[\exp\left\{-\int_t^T r(s)\, ds\right\}\right]$

$f(t,T) = -\frac{\partial \ln(p(t,T))}{\partial T}$

$= -\frac{\partial}{\partial T} \ln\left(E^Q\left[\exp\left\{-\int_t^T r(s)\, ds\right\}\right]\right)$

$= \frac{E^Q\left[\exp\left\{-\int_t^T r(s)\, ds\right\} \left[r(T)\right]\right]}{E_t^Q\left[\exp\left\{-\int_t^T r(s)\, ds\right\}\right]}$

(b) $r(t) = f(t,t)$

$= \frac{E^Q[r(t)]}{E_t^Q[1]} = r(t)$

(7) (a)

$$p(0,T) = e^{-y(0,T)T}$$

$$f(0,T) = -\frac{\partial \ln(p(0,T))}{\partial T}$$

$$= -\frac{1}{p(0,T)} \frac{\partial}{\partial T}\left(e^{-y(0,T)T}\right)$$

$$= \frac{1}{\cancel{p(0,T)}} \cancel{e^{-y(0,T)T}} \left(y(0,T) + T\frac{\partial y(0,T)}{\partial T}\right)$$

$$= \cancel{p(0,T)}\ y(0,T) + T\frac{\partial y(0,T)}{\partial T}$$

(7) (b) ZC yield curve in an f^n of T.

then $\frac{\partial y(0,T)}{\partial T} \geq 0$

$$f(0,T) - y(0,T) \geq 0$$

$$f(0,T) \geq y(0,T)$$

$$p_T(t) = K\,p(t,T_n) + \sum_{i=1}^{n} c_i\,p(t,T_i)$$

$$p_T(0) = K\,p(0,T_n) + \sum_{i=1}^{n} c_i\,p(0,T_i)$$

$$Ke^{-y(0,T_n)T} + \sum_{i=1}^{n} c_i e^{-y(0,T_i)T_i} = Ke^{-y_M(0,T_n)T} + \sum_{i=1}^{n} c_i e^{-y_M(0,T_i)T_i}$$

$\cancel{K}\,P$

$$y(0,T_n) > y(0,T_{n-1}) \ldots > y(0,T_1)$$

$$Ke^{-y(0,T_n)T} + \sum_{i=1}^{n} c_i e^{-y(0,T_i)T_i}$$

$$= Ke^{-y_m(0,T_n)T} + \sum_{i=1}^{n} c_i e^{-y_m(0,T_n)T_i}$$

$$\geq Ke^{-y(0,T_n)(T)} + \sum_{i=1}^{n} c_i e^{-y(0,T_n)T_i}$$

$$y_m(0,T_n) \leq y(0,T_n)$$

$\therefore$

(8) (a) $dr = (b - ar)\,dt + \sigma\,dW$

$$Y = \int_0^t -a\,dt + 0 + 0$$

$$= -at$$

$$H = e^{at}$$

$$dHX = e^{at}(b\,dt + \sigma\,dW)$$

$$e^{at}r(t) - r(0) = \int_0^t b e^{as}\,ds + \int_0^t \sigma e^{as}\,dW(s)$$

$$r(t) = r(0)e^{-at} + \frac{e^{-at}}{a} b\,(e^{at} - 1) + e^{-at}\sigma \int_0^t e^{as}\,dW$$

$$r(t) = r(0)e^{-at} + \frac{b}{a}(1 - e^{-at}) + e^{-at}\sigma \int_0^t e^{as}\,dW$$

(8a)

$$r(t) = r(0)e^{-at} + \frac{b}{a}(1 - e^{-at}) + \sigma e^{-at}\int_0^t e^{as}\,dW$$

$$r(t) \sim N\left(r(0)e^{-at} + \frac{b}{a}(1 - e^{-at}),\ \sigma^2 e^{-2at}\underbrace{\int_0^t e^{2as}\,ds}\right)$$

$$\left(\frac{e^{2as}}{2a}\right)_0^t$$

$$\frac{e^{2at} - 1}{2a}.$$

$$r(t) \sim N\left(r(0)e^{-at} + \frac{b}{a}(1 - e^{-at}),\ \frac{\sigma^2}{2a}(1 - e^{-2at})\right)$$

(8b) $r(t) \sim N\left(\underset{\downarrow}{\frac{b}{a}}, \frac{\sigma^2}{2a}\right)$

mean reversion level

(8c) $E(r(t)) = \frac{b}{a}(1 - e^{-at}) + e^{-at} E(r(0))$

$$= \frac{b}{a}(1 - e^{-at}) + e^{-at}\left(\frac{b}{a}\right)$$

$$= \frac{b}{a}$$

$$var(r(t)) = var(r(0))e^{-2at} + \frac{\sigma^2}{2a}(1 - e^{-2at})$$

$$= \frac{\sigma^2}{2a}e^{-2at}$$

$$= \underline{\frac{\sigma^2}{2a}}$$

$t < u$

(9) $$r(u) = r(t)\,\underbrace{e^{-a(u-t)}}_{D(u)} + \underbrace{\frac{b}{a}\left(1 - e^{-a(u-t)}\right) + \sigma \int_t^u e^{-a(u-s)}\,dW}_{f(u)}$$

$$p(t,T) = E^{Q}_{t,r}\left[e^{-\int_t^T r(u)\,du}\right]$$

$$Z(t,T) = -\int_t^T r(u)\,du.$$

$D(u)$ is deterministic & $f(u)$ is stochastic and both are non dependent on $\underline{r}$

$$p(t,T) = E^{Q}_{tr}\left[e^{-\int_t^T r(u)\,du}\right]$$

$$= E^{Q}_{tr}\left[e^{-\int_t^T \left(r(t)D(u) + f(u)\right)du}\right]$$

$$e^{-\left(\int_t^T D(u)\,du\right) r(t)}\; E^{Q}_{tr}\left[e^{-\int_t^T f(u)\,du}\right]$$

$$E^{Q}_{t,r}\left[e^{-\int_t^T f(u)\,du}\right] = e^{A(t,T)}$$

$$B(t,T) = \int_t^T D(u)\,du$$

24.8

(10) $dY = (2aY + \sigma^2)\,dt + 2\sigma\sqrt{Y}\,dW \quad Y(0) = y_0$

$Z(t) = \sqrt{Y(t)}$ ST Z satisfies linear stoch. diff eq^n.

~~dZ = f~~

$f(t,x) = \sqrt{x}$

$df = \frac{1}{2\sqrt{x}}\,dx + \frac{1}{2}\left(\frac{1}{2} \cdot -\frac{1}{2} x^{-\frac{3}{2}}\right) (dx)^2$

$= \frac{1}{2\sqrt{x}}\,dx - \frac{1}{8} \cdot \frac{1}{x\sqrt{x}}\,(dx)^2$

$=$

$df(t,Y) = \frac{dY}{2\sqrt{Y}} - \frac{1}{8}\frac{1}{Y\sqrt{Y}}(dY)^2$

$= \frac{(2aY + \sigma^2)\,dt + 2\sigma\sqrt{Y}\,dW}{2\sqrt{Y}} - \frac{4}{8}\frac{\sigma^2 Y\,dt}{Y\sqrt{Y}}$

$= \frac{(2aY + \sigma^2)}{2\sqrt{Y}}dt + \frac{2\sigma\,dW}{2} - \frac{4}{8}\frac{\sigma^2\,dt}{\sqrt{Y}}$

$= \frac{2\sigma\,dW}{2} + a\sqrt{Y} + \frac{\sigma^2}{2\sqrt{Y}}\,dt - \frac{4\sigma^2}{8}\frac{dt}{\sqrt{Y}}$

$= \sigma\,dW + a\sqrt{Y}\,dt + \frac{\sigma^2\,dt}{\sqrt{Y}}\left(\frac{3}{8}\right)$

$dZ = \sigma\,dW + aZ\,dt$