Theorem:- Define the process

$$Y(t) = \begin{cases} (T-t)\int_0^t \frac{1}{T-u}\, dW(u) & \text{for } 0 \le t < T \\ 0 & \text{for } t = T. \end{cases}$$

Then $Y(t)$ is a continuous Gaussian process on $[0,T]$ and has mean and covariance functions

$$m^Y(t) = 0 \quad , t \in [0,T].$$

$$c^Y(s,t) = s \wedge t - \frac{st}{T} \text{ for all } s,t \in [0,T].$$

In particular, the process $Y(t)$ has the same distribution as the Brownian bridge from 0 to 0 on $[0,T]$.

Note that $Y(t)$ is adapted to the filtration generated by the Brownian motion $W(t)$

$$dY(t) = \int_0^t \frac{1}{T-u}\, dW(u) \cdot d(T-t) + (T-t) \cdot d\int_0^t \frac{1}{T-u}\, dW(u)$$

$$= -\int_0^t \frac{1}{T-u}\, dW(u) \cdot dt + (T-t) \cdot \frac{1}{(T-t)}\, dW(t)$$

$$= -\frac{Y(t)}{(T-t)}\, dt + dW(t)$$

Multidimensional Distribution of the Brownian Bridge:-

We fix $a \in \mathbb{R}$ and $b \in \mathbb{R}$ and let $X^{a\to b}(t)$ denote the Brownian bridge from a to b on $[0,T]$. We also fix $0 = t_0 < t_1 < \cdots < t_n < T$. We compute the joint density of $X^{a\to b}(t_1), \cdots, X^{a\to b}(t_n)$.

Brownian bridge from a to b has the mean function

$$m^{a\to b}(t) = a + \frac{(b-a)t}{T} = \frac{(T-t)}{T}a + \frac{bt}{T}.$$

and covariance function

$$c(s,t) = s\wedge t - \frac{st}{T}.$$

For $0 \le s \le t \le T$, $c(s,t) = s - \frac{st}{T} = \frac{s(T-t)}{T}$

set $\mathcal{T}_j = T - t_j$ and $\mathcal{T}_0 = T$,

$$Z_j = \frac{X^{a\to b}(t_j)}{\mathcal{T}_j} - \frac{X^{a\to b}(t_{j-1})}{\mathcal{T}_{j-1}}.$$

Because $X^{a\to b}(t_1), \cdots X^{a\to b}(t_n)$ are jointly normal, so are $Z(t_1), \cdots Z(t_n)$. We compute

$$\mathbb{E}[Z_j] = \frac{1}{\mathcal{T}_j}\mathbb{E}\left[X^{a\to b}(t_j)\right] - \frac{1}{\mathcal{T}_{j-1}}\mathbb{E}\left[X^{a\to b}(t_{j-1})\right]$$

$$= \frac{a}{T} + \frac{bt_j}{T\mathcal{T}_j} - \frac{a}{T} - \frac{bt_{j-1}}{T\mathcal{T}_{j-1}}$$

$$= \frac{bt_j(T-t_{j-1}) - bt_{j-1}(T-t_j)}{T\mathcal{T}_j\mathcal{T}_{j-1}} = \frac{b(t_j - t_{j-1})}{\mathcal{T}_j\mathcal{T}_{j-1}}$$

$$\mathrm{var}(Z_j) = \frac{1}{\mathcal{T}_j^2}\mathrm{var}(X^{a\to b}(t_j)) - \frac{2}{\mathcal{T}_j\mathcal{T}_{j-1}}\mathrm{cov}(X^{a\to b}(t_j), X^{a\to b}(t_{j-1})) + \frac{1}{\mathcal{T}_{j-1}^2}\mathrm{var}(X^{a\to b}(t_{j-1}))$$

$$= \frac{1}{\mathcal{T}_j^2}c(t_j,t_j) - \frac{2}{\mathcal{T}_j\mathcal{T}_{j-1}}c(t_j,t_{j-1}) + \frac{1}{\mathcal{T}_{j-1}^2}c(t_{j-1},t_{j-1}).$$

$$= \frac{t_j}{T\mathcal{T}_j} - \frac{2t_{j-1}}{T\mathcal{T}_{j-1}} + \frac{t_{j-1}}{T\mathcal{T}_{j-1}} = \frac{t_j(T-t_{j-1}) - 2t_{j-1}(T-t_j) + t_{j-1}(T-t_j)}{T\mathcal{T}_j\mathcal{T}_{j-1}}$$

$$= \frac{t_j - t_{j-1}}{\mathcal{T}_j\mathcal{T}_{j-1}}$$

For $i < j$

$$\mathrm{cov}(Z_i, Z_j) = \frac{1}{\mathcal{T}_i\mathcal{T}_j}c(t_i,t_j) - \frac{1}{\mathcal{T}_i\mathcal{T}_{j-1}}c(t_i,t_{j-1}) - \frac{1}{\mathcal{T}_{i-1}\mathcal{T}_j}c(t_{i-1},t_j) + \frac{1}{\mathcal{T}_{i-1}\mathcal{T}_{j-1}}c(t_{i-1},t_{j-1})$$

$$\operatorname{cov}(Z_i, Z_j) = \frac{t_i(T-t_j)}{T\mathcal{T}_i\mathcal{T}_j} - \frac{t_i(T-t_{j-1})}{T\mathcal{T}_i\mathcal{T}_{j-1}} - \frac{t_{i-1}(T-t_j)}{T\mathcal{T}_{i-1}\mathcal{T}_j} + \frac{t_{i-1}(T-t_{j-1})}{T\mathcal{T}_{i-1}\mathcal{T}_{j-1}}$$

$$= \frac{t_i}{T\mathcal{T}_i} - \frac{t_i}{T\mathcal{T}_i} - \frac{t_{i-1}}{T\mathcal{T}_{i-1}} + \frac{t_{i-1}}{T\mathcal{T}_{i-1}} = 0.$$

We conclude that the normal random variables $Z_1, \cdots, Z_n$ are independent, their joint density is

$$f_{Z_1 \cdots Z_n}(z_1, \cdots, z_n) = \exp\left\{-\frac{1}{2}\sum_{j=1}^{n}\left(z_j - \frac{b(t_j - t_{j-1})}{\mathcal{T}_j\mathcal{T}_{j-1}}\right)^2\right.$$

We make the change of variables

$$z_j = \frac{x_j}{\mathcal{T}_j} - \frac{x_{j-1}}{\mathcal{T}_{j-1}}, \quad j = 1, 2, \cdots n$$

Then, we have

$$\frac{\partial z_j}{\partial x_j} = \frac{1}{\mathcal{T}_j}, \quad j = 1, 2, \cdots n$$

$$\frac{\partial z_j}{\partial x_{j-1}} = -\frac{1}{\mathcal{T}_{j-1}}, \quad j = 2, \cdots n.$$

and all other partial derivatives are zero. This leads to the Jacobian matrix

$$J = \begin{bmatrix} \frac{1}{\mathcal{T}_1} & 0 & \cdots & 0 \\ -\frac{1}{\mathcal{T}_1} & \frac{1}{\mathcal{T}_2} & \cdots & 0 \\ 0 & 0 & \cdots & \frac{1}{\mathcal{T}_n} \end{bmatrix} \Rightarrow \det J = \prod_{j=1}^{n} \frac{1}{\mathcal{T}_j}$$

By using the change of variables, we obtain the density for $X^{a\to b}(t_1), \cdots, X^{a\to b}(t_n)$

$$f_{X^{a\to b}(t_1),\dots,X^{a\to b}(t_n)}(x_1,x_2,\dots x_n) = \sqrt{\frac{T}{T-t_n}}\prod_{j=1}^{n}\frac{1}{\sqrt{(2\pi(t_j-t_{j-1}))}}$$

$$\exp\left\{-\frac{1}{2}\sum_{j=1}^{n}\frac{(x_j-x_{j-1})^2}{t_j-t_{j-1}} - \frac{(b-x_n)^2}{2(T-t_n)} + \frac{(b-a)^2}{2T}\right\}.$$

$$= \frac{p(T-t_n, x_n, b)}{p(T,a,b)}\prod_{j=1}^{n} p(t_j-t_{j-1}, x_{j-1}, x_j)$$

where $p(\tau,x,y) = \frac{1}{\sqrt{2\pi\tau}}\exp\left\{-\frac{(y-x)^2}{2\tau}\right\}$.

is the transition density for Brownian motion.

Brownian Bridge as a Conditioned Brownian motion:-

Let $0=t_0<t_1<t_2<\dots<t_n<T$ be given. The joint density of $W(t_1),\dots W(t_n), W(T)$ is

$$f_{W(t_1),\dots W(t_n),T}(x_1,x_2,\dots x_n,b) = p(T-t_n,x_n,b)\prod_{j=1}^{n}p(t_j-t_{j-1}, x_{j-1},x_j) \quad --- (*)$$

where $W(0)=x_0=a$.

Because $p(t_1-t_0, x_0,x_1) = p(t_1,a,x_1)$ is the density for the Brownian motion going from $W(0)=a$ to $W(t_1)=x_1$ in the time between $t=0$ and $t=t_1$. Similarly, $p(t_2-t_1, x_1,x_2)$ is the density for going from $W(t_1)=x_1$ to $W(t_2)=x_2$ between time $t=t_1$ and $t=t_2$.

The joint density for $W(t_1)$ and $W(t_2)$ is then the product

$$p(t_1, a, x_1)\, p(t_2 - t_1, x_1, x_2).$$

continuing in this way, we obtain the joint density for $W(t_1), W(t_2), \ldots W(t_n), W(T)$ given by ⊛

The marginal density of $W(T)$ is $p(T, a, b)$

Therefore, the density of $W(t_1), \ldots W(t_n)$ conditioned on $W(T) = b$ is thus

$$\frac{p(T - t_n, x_n, b)}{p(T, a, b)} \prod_{j=1}^{n} p(t_j - t_{j-1}, x_{j-1}, x_j)$$

and this is $f_{X^{a\to b}(t_1) \ldots X^{a\to b}(t_n)}(x_1, \ldots x_n)$.

Hence Brownian bridge from a to b on $[0, T]$ is a Brownian motion $W(t)$ on this time interval, starting at $W(0) = a$ and conditioned to arrive at b at time T. (i.e., conditioned on $W(T) = b$).

Let us define

$$M^{a\to b}(T) = \max_{0 \le t \le T} X^{a\to b}(t)$$

Corollary:- The density of $M^{a\to b}(T)$ is

$$f_{M^{a\to b}(T)}(y) = \frac{2(2y - b - a)}{T} e^{-\frac{2}{T}(y-a)(y-b)}, \quad y > \max\{a, b\}$$

Proof:- Let $M(t) = \max_{0 \le s \le t} W(s)$

The conditional distribution of $M(t)$ given $W(t) = \omega$ is

$$f_{M(t)/W(t)}(m/\omega) = \frac{2(2m-\omega)}{t} e^{-\frac{2m(m-\omega)}{t}}, \quad \omega \le m,\ m > 0.$$

Define $M^{0\to\omega}(T) = \max_{0 \le t \le T} X^{0\to\omega}(t)$

$$f_{M^{0\to\omega}(T)}(m) = f_{M(t)/W(T)}(m/\omega) = \frac{2(2m-\omega)}{T} e^{-\frac{2m(m-\omega)}{T}},$$

$$\omega < m\ ;\ m > 0.$$

The density of $f_{M^{a\to b}(T)}(y)$ can be obtained by translating from the initial condition $W(0) = a$ to $W(0) = a$.

In particular if we replace m by $y-a$ and replace ω by $b-a$ then we will get the result.