Consider a call option with maturity date T, strike price K with (28)
the underlying stock price $S(t)$ given by

$$dS(t) = rS(t)dt + \sigma S(t) d\widetilde{W}(t).$$

The payoff at maturity time is $V(T) = \max\{(S_T - K, 0\}$

$$V(t) \cancel{= C(t, S(t))} = \widetilde{\mathbb{E}}\left[e^{-r(T-t)} V(T) \mid \mathcal{F}(t)\right]$$

Proposition:- The price of a European call option at time t is given by $C(t,x) = x N(d_1) - K e^{-r(T-t)} N(d_2)$

$$d_1 = d_2 + \sigma\sqrt{T-t}$$

$$= \frac{\ln x/K + (r + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}$$

$$N(u) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{u} e^{-x^2/2} dx.$$

cash-or-nothing contract

$$V(T) = \begin{cases} 1 & \text{if } S(T) \geq K \\ 0 & \text{if } S(T) < K. \end{cases}$$

substituting $S(t) = e^{X(t)}$, the payoff becomes

$$V(T) = \begin{cases} 1 & \text{if } X(T) \geq \ln K \\ 0 & \text{if } S(T) < \ln K. \end{cases}$$

$$S(T) = S(t)\, e^{(r - \sigma^2/2)(T-t) + \sigma(\widetilde{W}(T) - \widetilde{W}(t))}.$$

$X(T)$ has the normal distribution

$$X(T) \sim N\left(\ln S(t) + (r - \sigma^2/2)(T-t),\ \sigma^2(T-t)\right)$$

$$V(t) = e^{-r(T-t)}\,\widehat{\mathbb{E}}\left[V(T)\,|\,\mathcal{F}(t)\right]$$

$$= e^{-r(T-t)} \int_{-\infty}^{\infty} V_T(x)\, p(x)\, dx$$

$$= e^{-r(T-t)} \int_{\ln K}^{+\infty} \frac{1}{\sqrt{2\pi}\,\sigma\sqrt{T-t}}\, e^{-\frac{\left[x-\ln S(t)-(r-\sigma^2/2)(T-t)\right]^2}{2\sigma^2(T-t)}}\, dx$$

Let $y = \dfrac{x-\ln S(t)-(r-\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}$, then

$$= e^{-r(T-t)} \int_{-d_2}^{+\infty} \frac{1}{\sqrt{2\pi}}\, e^{-y^2/2}\, dy = e^{-r(T-t)} \int_{-\infty}^{d_2} \frac{1}{\sqrt{2\pi}}\, e^{-y^2/2}\, dy$$

$$= e^{-r(T-t)}\, N(d_2).$$

Exercise:- $V(T) = \begin{cases} S(T) & \text{if } S(T) \geq K \\ 0 & \text{otherwise} \end{cases}$

show that the price of the contract time time t is $V(t) = S(t)N(d_1)$.

~~If the~~ The superposition principle:-

If the payoff of a derivative $f(T)$, can be written as a linear combination of payoffs

$$f(T) = \sum_{i=1}^{n} c_i V_i(T), \text{ with } c_i \text{ constants.}$$

then the price at time t is given by

$$f(t) = \sum_{i=1}^{n} c_i v_i(t)$$

where $v_i(t)$ is the price at time t of a derivative that pays at maturity $V_i(T)$.

$$f(t) = e^{-r(T-t)}\,\widehat{\mathbb{E}}\left[f(T)\,|\,\mathcal{F}(t)\right]$$

$$= e^{-r(T-t)}\,\widehat{\mathbb{E}}\left[\sum_{i=1}^{n} c_i V_i(T)\,|\,\mathcal{F}(t)\right]$$

$$= e^{-r(T-t)} \sum_{i=1}^{n} c_i\, \widehat{\mathbb{E}}\left[V_i(T)\,|\,\mathcal{F}(t)\right] = \sum_{i=1}^{n} c_i v_i(t)$$

Call option:-

$$C(T) = \max\{S(T) - K, 0\}$$
$$= V_1(T) - K V_2(T).$$

with $V_1(T) = \begin{cases} S(T) & \text{if } S(T) \geq K \\ 0 & \text{if } S(T) < K \end{cases}$ $\quad V_2(T) = \begin{cases} 1 & \text{if } S(T) \geq K \\ 0 & \text{if } S(T) < K. \end{cases}$

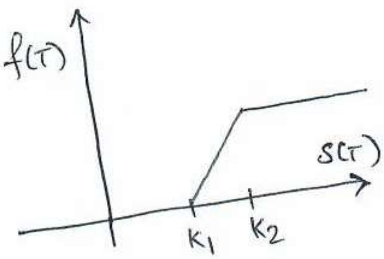
$$C(t) = V_1(t) - K V_2(t)$$
$$= S(t) N(d_1) - K e^{-r(T-t)} N(d_2).$$

Packages:- Packages are derivatives whose payoffs are linear combinations of payoff of options, cash and underlying asset.

- They can priced using the superposition principle.

The Bull spread:- Let $0 < K_1 < K_2$. A derivative with the payoff

$$f(T) = \begin{cases} 0 & \text{if } S(T) \leq K_1 \\ S(T) - K_1 & \text{if } K_1 < S(T) \leq K_2 \\ K_2 - K_1 & \text{if } K_2 < S(T). \end{cases}$$



is called a bull spread

$$f(T) = C_1(T) - C_2(T).$$

$C_1(T) = \begin{cases} 0 & \text{if } S(T) \leq K_1 \\ S(T) - K_1 & \text{if } K_1 < S(T) \end{cases}$ $\quad C_2(T) = \begin{cases} 0 & \text{if } S(T) \leq K_2 \\ S(T) - K_2 & \text{if } S(T) > \end{cases}$

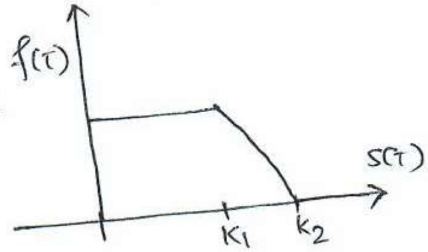
using the superposition principle, the price of a bull spread at time t is

$$f(t) = C_1(t) - C_2(t)$$
$$= S(t) N(d_1(K_1)) - K_1 e^{-r(T-t)} N(d_2(K_1)) - \left(S(t) N(d_1(K_2)) - K_2 e^{-r(T-t)} N(d_2(K_2$$

with
$$d_2(K_i) = \frac{\ln S(t) - \ln K_i + (r - \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}, \quad d_1(K_i) = d_2(K_i) + \sigma\sqrt{T-t}, \quad i=1,$$

The Bear spread:- let $0<K_1<K_2$. A derivative with the payoff

$$f(T)=\begin{cases} K_2-K_1 & \text{if } S(T)\le K_1 \\ K_2-S(T) & \text{if } K_1<S(T)\le K_2 \\ 0 & \text{if } K_2<S(T). \end{cases}$$

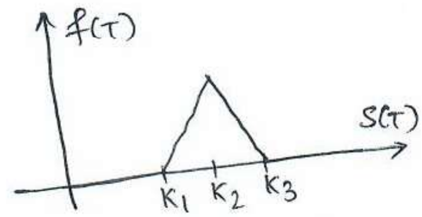


is called a bear spread.

Exercise:- Find the price of a bear spread at time t with $t<T$. (write down the payoff as a difference of two puts, one with strike price K_1 and the other with strike price K_2)

The Butterfly spread:- Let $0<K_1<K_2<K_3$ with $K_2=\frac{K_1+K_3}{2}$. A Butterfly spread is a derivative with the payoff given by

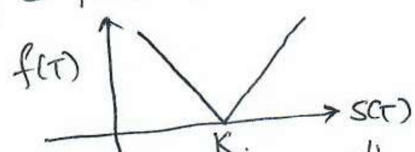
$$f(T)=\begin{cases} 0 & \text{if } S(T)\le K_1 \\ S(T)-K_1 & \text{if } K_1<S(T)\le K_2 \\ K_3-S(T) & \text{if } K_2\le S(T)<K_3 \\ 0 & \text{if } K_3\le S(T). \end{cases}$$



Exercise! Find the price of a butterfly spread at time t with $t<T$

(Hint $f(T)=C_1+C_3-2C_2$, where C_i is a call with strike price K_i.

straddles:- A derivative with the payoff $f(T)=|S(T)-K|$ is called a straddles.



Exercise ⓐ show that the payoff of a straddle can be written as

$$f(T)=\begin{cases} K-S(T) & \text{if } (S(T)\le K. \\ S(T)-K & \text{if } S(T)>K. \end{cases}$$

ⓑ Find the price of a straddle at time t, with $t<T$.