Black - Scholes - Merton formula:-

$$dS(t) = \alpha S(t)\,dt + \sigma S(t)\,dW(t) \longrightarrow \text{stock price}$$

$r \rightarrow$ interest rate

$$BSM(\tau, x, K, r, \sigma) = x N\left(d_+(\tau, x)\right) - e^{-r\tau} K N\left(d_-(\tau, x)\right)$$

where $d_\pm(\tau, x) = \frac{1}{\sigma\sqrt{\tau}}\left[\log \frac{x}{K} + \left(r \pm \frac{1}{2}\sigma^2\right)\tau\right]$, $\tau = T - t$

τ - time to expiration.

Continuously paying Dividend:-

Consider a stock, modeled as a generalized geometric Brownian motion, that pay dividends continuously over time at a rate $A(t)$ per unit time. Here $A(t)$, $0 \le t \le T$ is a non-negative adapted process. Dividends paid by a stock reduce its value and so we shall take as our model of the stock price

$$dS(t) = \alpha(t) S(t)\,dt + \sigma(t) S(t)\,dW(t) - A(t) S(t)\,dt$$

- $\alpha(t)$, $\sigma(t)$ and $R(t)$ are all assumed to be adapted processes.
- If an agent holds the stock then agent receives both the capital gain or loss due to stock price movements and the continuous paying dividend.

Now consider an agent who holds $\Delta(t)$ shares of stock at time t, then the portfolio value $X(t)$ satisfies

$$dX(t) = \Delta(t)\,dS(t) + \Delta(t)A(t)S(t)\,dt + R(t)\left[X(t) - \Delta(t)S(t)\right]dt$$

$$= R(t)X(t)\,dt + \left(\alpha(t) - R(t)\right)\Delta(t)S(t)\,dt + \sigma(t)\Delta(t)S(t)\,dW(t)$$

$$= R(t)X(t)\,dt + \Delta(t)S(t)\sigma(t)\left[\Theta(t)\,dt + dW(t)\right] \quad \text{---} \; (1)$$

where $\Theta(t) = \dfrac{\alpha(t) - R(t)}{\sigma(t)}$

is the ~~risk~~ market price of risk.

We define

$$\widetilde{W}(t) = W(t) + \int_0^t \Theta(u)\,du.$$

and use Girsanov's theorem to change to a measure $\widetilde{\mathbb{P}}$ under which $\widetilde{W}$ is a Brownian motion. So we may rewrite (1) as

$$dX(t) = R(t)X(t)\,dt + \Delta(t)S(t)\sigma(t)\,d\widetilde{W}(t)$$

The discounted portfolio value satisfies

$$d\left(X(t)D(t)\right) = \Delta(t)D(t)S(t)\sigma(t)\,d\widetilde{W}(t).$$

where $D(t) = \exp\left\{-\int_0^t R(s)\,ds\right\}$ is the discounted process.

In particular, under the risk-neutral measure $\widehat{\mathbb{P}}$ the discounted portfolio process is a martingale, thus

$$D(t)V(t) = \widehat{\mathbb{E}}\left[D(T)V(T) \mid \mathcal{F}(t)\right], \quad 0 \le t \le T$$

where the price of the derivative at time t is $V(t)$ and $V(T)$ is an $\mathcal{F}(T)$-measurable random variable.

If we wish to hedge a short position in a derivative security paying $V(T)$ at time T, where $V(T)$ is an $\mathcal{F}(T)$-measurable random variable. we will need to choose the initial capital $X(0)$ and a portfolio $\Delta(t)$, $0 \le t \le T$ so that $X(T) = V(T)$.

Because $X(t)D(t)$ is a martingale under $\widetilde{\mathbb{P}}$, we must have.

$$D(t)X(t) = \widetilde{\mathbb{E}}\left[D(T)V(T) \mid \mathcal{F}(t)\right], \quad 0 \le t \le T.$$

The value $X(t)$ of this portfolio at each time t is the value (price) of the derivative security at that time, which we denote by $V(t)$. Hence we obtain.

$$D(t)V(t) = \widetilde{\mathbb{E}}\left[D(T)V(T) \mid \mathcal{F}(t)\right], \quad 0 \le t \le T.$$

The evolution of the underlying stock under the risk-neutral measure $\widetilde{\mathbb{P}}$, is given by

$$dS(t) = [R(t) - A(t)]S(t)\,dt + \sigma(t)S(t)\,d\widetilde{W}(t).$$

$$\Rightarrow S(t) = S(0)\exp\left\{\int_0^t \sigma(u)\,d\widetilde{W}(u) + \int_0^t \left[R(u) - A(u) - \tfrac{1}{2}\sigma^2(u)\right]du\right\}.$$

The process $e^{\int_0^t A(u)\,du} D(t)S(t) = \exp\left\{\int_0^t \sigma(u)\,d\widetilde{W}(u) - \tfrac{1}{2}\int_0^t \sigma^2(u)\,du\right\}$.

is a martingale.

under the risk-neutral measure, the stock does not have mean rate of return $R(t)$, and consequently the discounted stock-price is not a martingale.

Continuously paying dividend with constant coefficient:-

Here we assume that volatility σ, the interest rate r, and the dividend rate a are constant, the stock price at time t, is given by

$$S(t) = S(0) \exp\left\{ \sigma \widetilde{W}(t) + (r - a - \tfrac{1}{2}\sigma^2)t \right\}.$$

For $0 \le t \le T$, we have

$$S(T) = S(t) \exp\left\{ \sigma(\widetilde{W}(T) - \widetilde{W}(t)) + (r - a - \tfrac{1}{2}\sigma^2)(T-t) \right\}.$$

According to the risk-neutral pricing formula, the price at time t, of a European call is

$$V(t) = \widetilde{\mathbb{E}}\left[e^{-r(T-t)} (S(T) - K)^+ \middle| \mathcal{F}(t) \right].$$

To evaluate this, we first compute

$$c(t,x) = \widetilde{\mathbb{E}}\left[e^{-r(T-t)} \left(x \exp\left\{ \sigma(\widetilde{W}(T) - \widetilde{W}(t)) + (r-a-\tfrac{1}{2}\sigma^2)(T-t) \right\} - K \right)^+ \right]$$

$$= \widetilde{\mathbb{E}}\left[e^{-r\tau} \left(x \exp\left\{ -\sigma\sqrt{\tau}\, Y + (r-a-\tfrac{1}{2}\sigma^2)\tau \right\} - K \right)^+ \right]$$

where $\tau = T - t$ and $Y = -\dfrac{\widetilde{W}(T) - \widetilde{W}(t)}{\sqrt{(T-t)}}$

is standard normal under $\widetilde{\mathbb{P}}$. We define

$$d_\pm(\tau, x) = \frac{1}{\sigma\sqrt{\tau}}\left[\log x/K + (r - a \pm \tfrac{1}{2}\sigma^2)\tau \right]$$

Note that the integrand is non-zero if and only if $Y < d_-(\tau, x)$.

Therefore

$$C(t,x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d_-(\tau,x)} e^{-r\tau}\left(x\exp\{-\sigma\sqrt{\tau}\,y + (r-a-\tfrac{1}{2}\sigma^2)\tau\} - K\right) e^{-\frac{1}{2}y^2}\,dy$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d_-(\tau,x)} x\exp\{-\sigma\sqrt{\tau}\,y - (a+\tfrac{1}{2}\sigma^2)\tau - \tfrac{1}{2}y^2\}\,dy$$

$$- \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d_-(\tau,x)} e^{-r\tau} K e^{-\frac{1}{2}y^2}\,dy$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d_-(\tau,x)} x e^{-a\tau} \exp\{-\tfrac{1}{2}(y+\sigma\sqrt{\tau})^2\}\,dy - Ke^{-r\tau}N(d_-(\tau,x))$$

put $z = y + \sigma\sqrt{\tau}$ then

$$C(t,x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d_+(\tau,x)} x e^{-a\tau} e^{-z^2/2}\,dz - e^{-r\tau}K\,N(d_-(\tau,x)).$$

$$= xe^{-a\tau}N(d_+(\tau,x)) - Ke^{-a\tau}N(d_-(\tau,x)).$$

Lump payments of dividends:-

Consider $0<t_1<t_2<\cdots<t_n<T$. Think of $t_1, t_2, \cdots t_n$ as the dividend paying dates in the asset. At each time t_j, the dividend paid is $a_j S(t_j-)$, where $S(t_j-)$ denotes the stock price just prior to the dividend payment. The stock price after dividend payment is the stock price before the dividend payment less the dividend payment

$$S(t_j) = S(t_j-) - a_j S(t_j-) = (1-a_j)\,S(t_j-).$$