Thus we have the equation

$$B_t(t,T) + \alpha(t)B(t,T) - \tfrac{1}{2}\gamma(t)B^2(t,T) = -1.$$

and from (*) (**) we get

$$A_t(t,T) = \beta(t)B(t,T) - \tfrac{1}{2}\delta(t)B^2(t,T).$$

Proposition:- Assume that μ and σ are of the form

$$\begin{cases} \mu(t,r) = \alpha(t)r + \beta(t) \\ \sigma(t,r) = \sqrt{\gamma(t)r + \delta(t)}. \end{cases}$$

Then the model admits an ATS of the form (1), where A and B satisfy the system

$$\begin{cases} B_t(t,T) + \alpha(t)B(t,T) - \tfrac{1}{2}\gamma(t)B^2(t,T) = -1 \\ B(T,T) = 0 \end{cases}$$

$$\begin{cases} A_t(t,T) = \beta(t)B(t,T) - \tfrac{1}{2}\delta(t)B^2(t,T) \\ A(T,T) = 0 \end{cases}$$

Analytical Results for some standard Models:-

The vasicek Model :-

$$dr(t) = (b - ar(t))dt + \sigma dW(t)$$

Here $-\alpha(t) = a$, $\beta(t) = b$, $\sqrt{\delta(t)} = \sigma$, $\gamma(t) = 0$

$$\Rightarrow B_t(t,T) - aB(t,T) = -1, \quad B(T,T) = 0$$

$$\Rightarrow B(t,T) = \tfrac{1}{a}\{1 - e^{-a(T-t)}\}.$$

$$A_t(t,T) = b\,B(t,T) - \tfrac{1}{2}\sigma^2 B^2(t,T)\ , \ A(T,T) = 0$$

$$\Rightarrow A(t,T) = \frac{\sigma^2}{2}\int_t^T B^2(s,T)\,ds - b\int_t^T B(s,T)\,ds.$$

In the Vasicek Model bond prices are given by

$$p(t,T) = e^{A(t,T) - B(t,T)r}$$

where

$$B(t,T) = \frac{1}{a}\left\{1 - e^{-a(T-t)}\right\},$$

$$A(t,T) = \frac{\{B(t,T) - T + t\}(ab - \tfrac{1}{2}\sigma^2)}{a^2} - \frac{\sigma^2 B^2(t,T)}{4a}.$$

The Ho-Lee Model:-

$$dr(t) = \theta(t)\,dt + \sigma\,dW(t)$$

Here $\alpha(t) = 0$, $\beta(t) = \theta(t)$, $\sqrt{\delta(t)} = \sigma$ and $\gamma(t) = 0$.

Thus, for the Ho-Lee model the ATS equations become

$$B_t(t,T) = -1\ , \ B(T,T) = 0$$

$$\Rightarrow B(t,T) = T - t$$

and $\begin{cases} A_t(t,T) = \theta(t)B(t,T) - \tfrac{1}{2}\sigma^2 B^2(t,T) \\ A(T,T) = 0 \end{cases}$

$$\Rightarrow A(t,T) = \int_t^T \theta(s)(s-T)\,ds + \frac{\sigma^2}{2}\frac{(T-t)^3}{3}.$$

It now remains to choose θ such that the theoretical bond price at $t=0$, fit the observed initial term structure $\{p^*(0,T);\, T \geq 0\}$. Thus we want to find θ such that $p(0,T) = p^*(0,T)$ for all $T \geq 0$.

The CIR Model

$$dr(t) = (b - a r(t))dt + \sigma\sqrt{r(t)}\, dW(t)$$

Here $\alpha(t) = -a$, $\beta(t) = b$, $\sqrt{\gamma(t)} = \sigma$, $\delta(t) = 0$

Thus, the ATS equations are

$$\begin{cases} B_t(t,T) - aB(t,T) - \frac{1}{2}\sigma^2 B^2(t,T) = -1 \\ B(T,T) = 0 \end{cases}$$

and $$\begin{cases} A_t(t,T) = bB(t,T) \\ A(T,T) = 0 \end{cases}$$

The term structure for CIR model is given by

$$F^T(t,r) = A_0(T-t)\, e^{-B(T-t)r}$$

where

$$B(x) = \frac{2(e^{\gamma x} - 1)}{(\gamma + a)(e^{\gamma x} - 1) + 2\gamma}$$

$$A_0(x) = \left[\frac{2\gamma e^{(a+\gamma)(x/2)}}{(\gamma + a)(e^{\gamma x} - 1) + 2\gamma}\right]^{2ab/\sigma^2}$$

and $\gamma = \sqrt{a^2 + 2\sigma^2}$.

Properties of the short rate Models:-

The vasicek Model:-

$$dr(t) = a(c - r(t))dt + \sigma dW(t) \quad (a > 0) \quad \text{---} (1)$$

Assuming the spot rates are deterministic, we take $\sigma = 0$ and obtain the ODE

$$dr(t) = a(c - r(t))dt$$

$$\Rightarrow r(t) = c + (r(0) - b)e^{-at}$$

This means that if $r(0) > b$ then $r(t)$ is decreasing towards b, and if $r(0) < b$, then $r(t)$ is increasing towards ~~the~~ b. The term $\sigma dW(t)$ adds some "white noise to the process.

The solution of the equation (1) is given by

$$r(t) = \cancel{b}\, c + (r(0) - c)e^{-at} + \sigma e^{-at}\int_0^t e^{as} dW(s).$$

Thus $r(t)$ is a Gaussian process with mean and variance

$$\mathbb{E}(r(t)) = c + (r(0) - c)e^{-at}$$

$$Var(r(t)) = \frac{\sigma^2}{2a}\left(1 - e^{-2at}\right).$$

The Vasicek model has been criticized because it allows for negative interest rates and unbounded large rates.

The Cox-Ingersoll-Ross Model:-

$$dr(t) = a(b - r(t))\,dt + \sigma\sqrt{r(t)}\,dW(t), \quad a > 0$$

$$\Rightarrow r(t) = r(0) + abt - a\int_0^t r(s)\,ds + \sigma\int_0^t \sqrt{r(s)}\,dW(s).$$

Take the expectation we obtain

$$\mathbb{E}(r(t)) = r(0) + abt - a\int_0^t \mathbb{E}[r(s)]\,ds.$$

Let $\mu(t) = \mathbb{E}[r(t)]$. Then

$$\mu(t) = r(0) + abt - a\int_0^t \mu(s)\,ds.$$

$$\Rightarrow \mu'(t) = ab - a\mu(t) \Rightarrow d\left(e^{at}\mu(t)\right) = abe^{at}$$

$$\Rightarrow \mu(t) = b + e^{-at}(r(0) - b).$$

Now we compute the second moment $\mu_2(t) = \mathbb{E}[r^2(t)]$. By Itô's formula we have

$$\cancel{d(r^2(t)) = 2r(t)\,dr(t) + dr(t)\,dr(t)}$$

$$\cancel{= 2r(t)\,dr(t) + \sigma^2 r(t)\,dt}$$

$$d(r^2(t)) = 2r(t)\,dr(t) + dr(t)\,dr(t)$$

$$= 2r(t)\,dr(t) + \sigma^2 r(t)\,dt$$

$$= 2r(t)\left[a(b - r(t))\,dt + \sigma\sqrt{r(t)}\,dW(t)\right] + \sigma^2 r(t)\,dt$$

$$= \left[(2ab + \sigma^2)r(t) - 2ar^2(t)\right]dt + 2\sigma(r(t))^{3/2}\,dW(t)$$

$$\Rightarrow r^2(t) = r^2(0) + \int_0^t \left[(2ab + \sigma^2)r(s) - 2ar^2(s)\right]ds + 2\sigma\int_0^t r^{3/2}(s)\,dW(s).$$

Taking the expectation yields

$$\mu_2(t) = r^2(0) + \int_0^t \left[(2ab+\sigma^2)\mu(s) - 2a\mu_2(s)\right] ds.$$

$$\Rightarrow \mu_2'(t) = (2ab+\sigma^2)\mu(t) - 2a\mu_2(t)$$

$$\Rightarrow d\left(e^{2at}\mu_2(t)\right) = (2ab+\sigma^2)e^{2at}\mu(t)$$

$$\Rightarrow \mu_2(t) = r^2(0)e^{-2at} + (2ab+\sigma^2)\left[\frac{b}{2a}(1-e^{-2at}) + \frac{r(0)-b}{a}(1-e^{-at})e^{-at}\right].$$

Here $r(t)$ is not a Gaussian process

The main advantage of this model is that, it is not possible for the interest rates to become negative.

The Dothan model:-

$$dr(t) = ar(t)\,dt + \sigma r(t)\,dW(t)$$

$$\Rightarrow r(t) = r(0)\exp\left\{(a-\sigma^2/2)t + \sigma W(t)\right\}.$$

The distribution of $r(t)$ is log-normal, the mean and variance are

$$\mathbb{E}[r(t)] = r(0)e^{(a-\sigma^2/2)t}\,\mathbb{E}\left[e^{\sigma W(t)}\right] = r(0)e^{at}$$

$$\operatorname{var}(r(t)) = r^2(0)e^{2(a-\sigma^2/2)t}\operatorname{var}\left(e^{\sigma W(t)}\right)$$

$$= r^2(0)e^{2at}\left(e^{\sigma^2 t} - 1\right).$$