Brownian Bridge:-

This is a stochastic process that is like a Brownian Motion except that with probability one it reches a specified point at a specified positive time. We first discuss Gaussian processes in general, the class to which the Brownian bridge belongs.

- Gaussian Processes:-

Defn: A Gaussian process $X(t), t \geq 0$ is a stochastic process that has the property that, for arbitrary times $0 < t_1 < t_2 < \cdots < t_n$, the random variables $X(t_1), X(t_2), \cdots, X(t_n)$ are jointly normally distributed.

A random vector $X = (X_1, \cdots, X_n)$ is jointly normal if it has joint density

$$f_X(\bar{x}) = \frac{1}{\sqrt{(2\pi)^n \det(C)}} \exp\left\{-\frac{1}{2}(\bar{x}-\mu)\, C^{-1} (\bar{x}-\mu)^{tr}\right\}.$$

$\bar{x} = (x_1, x_2, \cdots, x_n)$, $\mu = (\mu_1, \mu_2, \cdots, \mu_n)$ expectations

C - is the positive definite matrix of covariance.

The joint normal distribution is determined by their mean and covariance.

- Therefore the joint distribution of $X(t_1), X(t_2), \cdots X(t_n)$ is determined by the means and covariances of these random variables.

We denote the mean of $X(t)$ by $m(t)$ and for $s \geq 0$, $t \geq 0$ we denote the covariance of $X(s)$ and $X(t)$ by $c(s,t)$, i.e.,

$$m(t) = \mathbb{E}[X(t)] \ , \quad c(s,t) = \mathbb{E}[(X(s)-m(s))(X(t)-m(t))].$$

Example: ① Brownian motion $W(t)$ is a Gaussian process.

For $0 < t_1 < t_2 \cdots < t_n$ the increments

$$I_1 = W(t_1) \ , \ I_2 = W(t_2) - W(t_1) \ , \ \cdots \quad I_n = W(t_n) - W(t_{n-1}).$$

are independent and normally distributed. Writing

$$W(t_1) = I_1 \ , \ W(t_2) = \sum_{j=1}^{2} I_j \ , \ \cdots \quad W(t_n) = \sum_{j=1}^{n} I_j$$

{ If $\bar{X} = (X_1, \cdots X_n)$ is jointly normal then $\bar{Y} = A\bar{X}^{tr}$ is also jointly normal where A is a constant $n \times n$ matrix

$\Rightarrow$ The random variables $W(t_1), W(t_2), \cdots, W(t_n)$ are jointly normally distributed. (These random variables are not independent.)

The mean function for Brownian motion is

$$m(t) = \mathbb{E}[W(t)] = 0.$$

Let $0 \leq s \leq t$, then

$$\begin{aligned} c(s,t) &= \mathbb{E}[W(s)W(t)] \\ &= \mathbb{E}[W(s)(W(t)-W(s)) + W^2(s)] \\ &= \mathbb{E}[W(s)(W(t)-W(s))] + \mathbb{E}[W^2(s)] \end{aligned}$$

$$C(s,t) = 0 + s$$

since $W(s)$ and $W(t) - W(s)$ are independent and

$$\mathbb{E}[W(s)] = \mathbb{E}[W(t) - W(s)] = 0.$$

We conclude that $C(s,t) = s$ when $0 \le s \le t$

Similarly $C(s,t) = t$ when $0 \le t \le s$.

In general, we have $C(s,t) = s \wedge t$

where $s \wedge t = \min\{s, t\}$.

Example:- (2) Let $\Delta(t)$ be a non-random function of time, and define $I(t) = \int_0^t \Delta(s)\, dW(s)$.

where $W(t)$ is a Brownian motion. Then $I(t)$ is a Gaussian process.

For fixed $u \in \mathbb{R}$, the process

$$M_u(t) = \exp\left\{ u I(t) - \tfrac{1}{2} u^2 \int_0^t \Delta^2(s)\, ds \right\}$$

$$= \exp\left\{ u \int_0^t \Delta(s)\, dW(s) - \tfrac{1}{2} u^2 \int_0^t \Delta^2(s)\, ds \right\}.$$

is a Martingale.

Let $X(t) = u \int_0^t \Delta(s)\, dW(s) - \tfrac{1}{2} u^2 \int_0^t \Delta^2(s)\, ds$.

$$\Rightarrow dX(t) = u\Delta(t)\, dW(t) - \tfrac{1}{2} u^2 \Delta^2(t)\, dt$$

$$\Rightarrow dX(t)\, dX(t) = u^2 \Delta^2(t)\, dt.$$

Let $f(x) = e^x$, By Itô's formula

$$df(X(t)) = f'(X(t))\,dX(t) + \tfrac{1}{2} f''(X(t))\,dX(t)\,dX(t)$$

$$= e^{X(t)}\left[u\Delta(t)\,dW(t) - \tfrac{1}{2}u^2\Delta^2(t)\,dt + \tfrac{1}{2}u^2\Delta^2(t)\,dt\right]$$

$$= e^{X(t)}\,u\Delta(t)\,dW(t).$$

$\Rightarrow M_u(t)$ is a Martingale.

Hence $1 = M_u(0) = \mathbb{E}[M_u(t)] = e^{-\frac{1}{2}u^2\int_0^t \Delta^2(s)\,ds}\,\mathbb{E}[e^{uI(t)}]$

$$\Rightarrow \mathbb{E}[e^{uI(t)}] = e^{\frac{1}{2}u^2\int_0^t \Delta^2(s)\,ds}.$$

$I(t)$ is normally distributed with mean zero and variance $\int_0^t \Delta^2(s)\,ds$.

We must verify that, for $0 < t_1 < t_2 < \cdots < t_n$, the random variables $I(t_1), I(t_2), \cdots, I(t_n)$ are jointly normally distributed.

We show that for $0 < t_1 < t_2$ the two random increments $I(t_1) - I(0) = I(t_1)$ and $I(t_2) - I(t_1)$ are normally distributed and independent. For fixed $u_2 \in \mathbb{R}$, the martingale property of M_{u_2} implies that

$$M_{u_2}(t_1) = \mathbb{E}\left[M_{u_2}(t_2) \,\middle|\, \mathcal{F}(t_1)\right]$$

Now let $u_1 \in \mathbb{R}$ be fixed. Because $\dfrac{M_{u_1}(t_1)}{M_{u_2}(t_1)}$ is $\mathcal{F}(t_1)$-measurable, we obtain

$$M_{u_1}(t_1) = \mathbb{E}\left[\frac{M_{u_2}(t_2)\, M_{u_1}(t_1)}{M_{u_2}(t_1)} \Big| \mathcal{F}(t_1) \right]$$

$$= \mathbb{E}\left[\exp\left\{ u_1 I(t_1) + u_2 (I(t_2) - I(t_1)) - \tfrac{1}{2} u_1^2 \int_0^{t_1} \Delta^2(s)\,ds - \tfrac{1}{2} u_2^2 \int_{t_1}^{t_2} \Delta^2(s)\,ds \right\} \Big| \mathcal{F}(t_1) \right].$$

Now take expectations

$$1 = M_{u_1}(0) = \mathbb{E}\left[M_{u_1}(t_1) \right]$$

$$= \mathbb{E}\left[\exp\left\{ u_1 I(t_1) + u_2 (I(t_2) - I(t_1)) - \tfrac{1}{2} u_1^2 \int_0^{t_1} \Delta^2(s)\,ds - \tfrac{1}{2} u_2^2 \int_{t_1}^{t_2} \Delta^2(s)\,ds \right\} \right].$$

$$= \mathbb{E}\left[\exp\left\{ u_1 I(t_1) + u_2 (I(t_2) - I(t_1)) \right\} \right] \exp\left\{ -\tfrac{1}{2} u_1^2 \int_0^{t_1} \Delta^2(s)\,ds - \tfrac{1}{2} u_2^2 \int_{t_1}^{t_2} \Delta^2(s)\,ds \right\}.$$

where we used the fact that $\Delta^2(s)$ is non-random.

$$\Rightarrow \mathbb{E}\left[\exp\left\{ u_1 I(t_1) + u_2 (I(t_2) - I(t_1)) \right\} \right] = \exp\left\{ \tfrac{1}{2} u_1^2 \int_0^{t_1} \Delta^2(s)\,ds \right\} \exp\left\{ \tfrac{1}{2} u_2^2 \int_{t_1}^{t_2} \Delta^2(s)\,ds \right\}.$$

The right hand side is the product of the moment-generating function for a normal random variable with mean zero and variance $\int_0^{t_1} \Delta^2(s)\,ds$ and the moment-generating function for a normal random variable with mean zero and variance $\int_{t_1}^{t_2} \Delta^2(s)\,ds$.

It follows that

$$I(t_1) \sim N\left(0, \int_0^{t_1} \Delta^2(s)\,ds\right) \text{ and}$$

$$I(t_2) - I(t_1) \sim N\left(0, \int_{t_1}^{t_2} \Delta^2(s)\,ds\right)$$

and $I(t_1)$ and $I(t_2) - I(t_1)$ are independent.

$$m(t) = \mathbb{E}[I(t)] = 0.$$

and
$$\begin{aligned}
c(t_1, t_2) &= \mathbb{E}[I(t_1) I(t_2)] \\
&= \mathbb{E}\left[I(t_1)(I(t_2) - I(t_1)) + I^2(t_1)\right] \\
&= \mathbb{E}\left[I(t_1)(I(t_2) - I(t_1))\right] + \mathbb{E}\left[I^2(t_1)\right] \\
&= \mathbb{E}[I(t_1)]\,\mathbb{E}[(I(t_2) - I(t_1))] + \int_0^{t_1} \Delta^2(s)\,ds. \\
&= \int_0^{t_1} \Delta^2(s)\,ds.
\end{aligned}$$

For general case $c(s,t) = \int_0^{s \wedge t} \Delta^2(u)\,du$.

Brownian Bridge:-

Defn:- Let $W(t)$ be a Brownian motion. Fix $T>0$, we define the Brownian bridge from 0 to 0 on $[0,T]$ to be the process

$$X(t) = W(t) - t/T\, W(T), \quad 0 \le t \le T.$$

– Note that $t/T\, W(T)$ as a function of t is the line from $(0,0)$ to $(T, W(T))$