$$= (K - L_*)\left(r + \frac{2r^2}{\sigma^2} - \frac{r(2r+\sigma^2)}{\sigma^2}\right)\left(\frac{x}{L_*}\right)^{-2r/\sigma^2} = 0$$

On the other hand, for $0 \le x < L_*$,

$$r v_{L_*}(x) - r x v'_{L_*}(x) - \tfrac{1}{2}\sigma^2 x^2 v''_{L_*}(x) = r(K-x) + r x = rK.$$

In particular, we see that $v_{L_*}(x)$ satisfies the so-called linear complementarity conditions

$$v(x) \ge (K-x)^+ \text{ for all } x \ge 0, \quad \text{--- (a)}$$

$$r v(x) - r x v'(x) - \tfrac{1}{2}\sigma^2 x^2 v''(x) \ge 0 \text{ for all } x \ge 0, \quad \text{--- (b)}$$

and for each $x \ge 0$, equality holds in either (a) or (b). --- (c)

The linar complementanity conditions (a)-(c) determine the function $v_{L_*}(x)$. More precisely, the function $v_{L_*}(x)$ is the only bounded continuous function having a continuous derivative that satisfies these conditions:

Probabilistic Characterization of the put price

Theorem:- Let $S(t)$ be the stock price given by

$$dS(t) = r S(t)\,dt + \sigma S(t)\,d\widetilde{W}(t)$$ and let

$\tau_{L_*} = \min\{t \ge 0 : S(t) = L_*\}$. Then $e^{-rt} v_{L_*}(S(t))$ is a supermartingale under $\widetilde{\mathbb{P}}$, and the stopped process $e^{-r(t\wedge\tau_{L_*})} v_{L_*}(S(t\wedge\tau_{L_*}))$ is a martingale.

Proof: $d(e^{-rt} v_{L_*}(S(t)))$

$$= e^{-rt}\left[-r v_{L_*}(S(t))\,dt + v'_{L_*}(S(t))\,dS(t) + \tfrac{1}{2} v''_{L_*}(S(t))\,dS(t)\,dS(t)\right].$$

$$= e^{-rt}\left[-r v_{L_*}(S(t)) + rS(t)\, v'_{L_*}(S(t)) + \tfrac{1}{2}\sigma^2 S^2(t)\, v''_{L_*}(S(t))\right]dt$$

$$+ e^{-rt}\sigma\, S(t)\, v'_{L_*}(S(t))\, d\widetilde{W}(t).$$

The dt term in this expression is either 0 or $-rK$, depending on whether $S(t) > L_*$ or $S(t) < L_*$. If $S(t) = L_*$, $v''_{L_*}(S(t))$ is undefined, but the probability $S(t) = L_*$ is zero so this does not matter. Thus we have

$$d\left(e^{-rt} v_{L_*}(S(t))\right) = -e^{-rt} rK \, \mathbb{1}_{\{S(t) < L_*\}}\, dt + e^{-rt}\sigma\, S(t)\, v'_{L_*}(S(t))\, d\widetilde{W}(t). \quad \text{---} \circledast$$

Because the dt term is less than or equal to zero, $e^{-rt} v_{L_*}(S(t))$ is a supermartingale

If $S(0) > L_*$, then prior to the time τ_{L_*} when the stock price first reaches L_*, the dt term is zero and hence In particular $\circledast$ implies that $e^{-r(t\wedge\tau_{L_*})} v(S(t\wedge\tau_{L_*}))$ is a martingale.

$$\int_0^{t\wedge\tau_{L_*}} d\left(e^{-rt} v_{L_*}(S(t))\right) = \int_0^{t\wedge\tau_{L_*}} -e^{-rt} rK\, \mathbb{1}_{\{S(t) < L_*\}} + \int_0^{t\wedge\tau_{L_*}} e^{-rt}\sigma S(t) v'_{L_*}(S(t))\, d\widetilde{W}(t)$$

$$\Rightarrow e^{-r(t\wedge\tau_{L_*})} v_{L_*}(S(t\wedge\tau_{L_*})) = v_{L_*}(0) + \int_0^{t\wedge\tau_{L_*}} e^{-ru}\sigma\, S(u)\, v'_{L_*}(S(u))\, d\widetilde{W}(u).$$

Itô-integrals are martingales, hence the Itô-integral above stopped at the stopping time τ_{L_*} is also a martingale.

Corollary: Recall that $\mathcal{S}$ is the set of all stopping times. we have

$$v_{L*}(x) = \max_{\tau \in \mathcal{S}} \widetilde{\mathbb{E}}\left[e^{-r\tau}(K - S(\tau))\right],$$

where $x = S(0)$ is the initial stock price. In other words, $v_{L*}(x)$ is the perpetual American put price.

Proof:- Because $e^{-rt} v_{L*}(S(t))$ is a supermartingale under $\widetilde{\mathbb{P}}$. By optional sampling theorem for any $\tau \in \mathcal{S}$ we have

$$v_{L*}(x) = v_{L*}(S(0)) \geq \widetilde{\mathbb{E}}\left[e^{-r(t\wedge\tau)} v_{L*}(S(t\wedge\tau))\right]$$

Because $v_{L*}(S(t\wedge\tau))$ is bounded, we may let $t \to \infty$ and using the Dominated Convergence Theorem, we conclude that

$$v_{L*}(x) \geq \widetilde{\mathbb{E}}\left[e^{-r\tau} v_{L*}(S(\tau))\right] \geq \widetilde{\mathbb{E}}\left[e^{-r\tau}(K - S(\tau))\right] \quad \forall \tau \in \mathcal{S}$$

$$\left(\text{Since } v_{L*}(x) \geq (K-x)^+ \ \forall x\right)$$

$$\Rightarrow v_{L*}(x) \geq \max_{\tau \in \mathcal{S}} \widetilde{\mathbb{E}}\left[e^{-r\tau}(K - S(\tau))\right].$$

Note that $e^{-r(t\wedge\tau_{L*})} v(S(t\wedge\tau_{L*}))$ is a martingale under $\widetilde{\mathbb{P}}$.

Thus

$$v_{L*}(x) = v_{L*}(S(0)) = \widetilde{\mathbb{E}}\left[e^{-r(t\wedge\tau_{L*})} v_{L*}(S(t\wedge\tau_{L*}))\right]$$

Letting $t \to \infty$ and using DCT, we obtain

$$v_{L*}(x) = \widetilde{\mathbb{E}}\left[e^{-r\tau_{L*}} v_{L*}(S(\tau_{L*}))\right]$$

Since $e^{-r\tau_{L*}} v_{L*}(S(\tau_{L*})) = e^{-r\tau_{L*}} v_{L*}(L*) = e^{-r\tau_{L*}}(K - S(\tau_{L*}))$ if $\tau_{L*} < \infty$.

Hence, we have

$$v_{L_*}(x) = \widetilde{\mathbb{E}}\left[e^{-r\tau_{L*}}(K - S(\tau_{L*}))\right].$$

It follows that

$$v_{L_*}(x) \le \max_{\tau \in \mathcal{S}} \widetilde{\mathbb{E}}\left[e^{-r\tau}(K - S(\tau))\right].$$

Therefore, we have

$$v_{L_*}(x) = \max_{\tau \in \mathcal{S}} \widetilde{\mathbb{E}}\left[e^{-r\tau}(K - S(\tau))\right].$$

<u>Corollary:</u> consider an agent with initial capital $X(0) = v_{L_*}(S(0))$, the initial perpetual American put price. suppose this agent uses the portfolio process $\Delta(t) = v'_{L_*}(S(t))$ and consumes cash at rate $C(t) = rK \mathbb{1}_{\{S(t) < L_*\}}$. Then the value $X(t)$ of the agent's portfolio agrees with the option price $v_{L_*}(S(t))$ for all times t until the option is exercised. In particular $X(t) \ge (K - S(t))^+$ for all t until the option is exercised, so the agent can pay off a short option position regardless of when the option is exercised.

<u>Proof:-</u> The differential of the agent's portfolio value process is

$$dX(t) = \Delta(t)\,dS(t) + r(X(t) - \Delta(t)S(t))\,dt - C(t)\,dt$$

so,

$$\begin{aligned} d(e^{-rt}X(t)) &= e^{-rt}\left(-rX(t)\,dt + dX(t)\right) \\ &= e^{-rt}\left(\Delta(t)\,dS(t) - r\Delta(t)S(t)\,dt - C(t)\,dt\right) \\ &= e^{-rt}\left(\Delta(t)\sigma S(t)\,d\widetilde{W}(t) - C(t)\,dt\right). \end{aligned}$$

substituting $\Delta(t) = v'_{L_*}(S(t))$ and $C(t) = rK\,\mathbb{I}_{\{S(t)<L_*\}}$.

we have

$$d(e^{-rt}X(t)) = e^{-rt}\left(\sigma S(t)\, v'_{L_*}(S(t))\,d\widetilde{W}(t) - rK\,\mathbb{I}_{\{S(t)<L_*\}}dt\right)$$

$$= d\left(e^{-rt} v_{L_*}(S(t))\right)$$

Integrating both side and using $X(0) = v_{L_*}(S(0))$, we obtain

$$X(t) = v_{L_*}(S(t)) \text{ for all } t \text{ prior to exercise.}$$

The linear complementarity conditions ⓐ-ⓒ determine the function $v_{L_*}(x)$. Condition ⓒ says that if we divide the half-line $[0,\infty)$ into two sets, the stopping set

$$\mathcal{I} = \{x \geq 0;\ v_{L_*}(x) = (K-x)^+\}$$

and the continuation set

$$\mathcal{C} = \{x \geq 0;\ v_{L_*}(x) > (K-x)^+\}.$$

The equality holds in ⓑ for $x \in \mathcal{C}$. If the initial stock price is in $\mathcal{I}$, the the owner of the put can get full value by exercising it immediately. On the other hand, if the stock price is in $\mathcal{C}$, then the put is more valuable than its intrinsic value, and the owner of the put can capture this extra value by waiting until the stock price enters $\partial\mathcal{I}$ to exercise. The time of entry into the set $\partial\mathcal{I}$ is in fact τ_{L_*}.

The three linear complementarity conditions have counter parts that can be stated probabilistically rather than analytically.

Let $V(t) = e^{-rt} v(S(t))$ be the value of the perpetual American put. The stochastic process $V(t)$ satisfies the following three conditions

(i) $V(t) \geq (K - S(t))^+$ for all $t \geq 0$,

(ii) $e^{-rt} V(t)$ is a supermartingale under $\widetilde{\mathbb{P}}$, and

(iii) there exists a stopping time τ_* such that

$$V(0) = \widetilde{\mathbb{E}}\left[e^{-r\tau_*}(K - S(\tau_*))^+\right].$$

These three conditions determine the value of $V(0)$.

Finite-expiration American Put:-

Here, we consider an American put on a stock whose price is given by

$$dS(t) = rS(t)\,dt + \sigma S(t)\,d\widetilde{W}(t),$$

but now the put has a finite expiration time T.

<u>Definition</u>:- Let $0 \leq t \leq T$ and $x \geq 0$ be given. Assume $S(t) = x$ and $\mathcal{F}_u^{(t)} = \sigma\{S(v) : t \leq v \leq u\}$., $t \leq u \leq T$ and let $\mathcal{S}_{t,T}$ denote the set of all stopping times for the the filtration $\mathcal{F}_u^{(t)}$, $t \leq u \leq T$ taking values in $[t, T]$ or taking the value ∞. In other words a stopping time a stopping time in $\mathcal{S}_{t,T}$ makes the decision to stop at a time $u \in [t, T]$ based only on the path

of the stock price between times t and u.

The price at time t of the American put expiring at time T is defined to be

$$v(t,x) = \max_{\tau \in \mathcal{S}_{t,T}} \widetilde{\mathbb{E}}\left[e^{-r(\tau-t)} (K - S(\tau)) \,\middle|\, S(t) = x \right]$$

In the event that $\tau = \infty$, we interpret $e^{-r\tau}(K - S(\tau))$ to be zero. This is the case when the put expires unexercised.

Analytical characterization of the Put Price

The finite-expiration American put price function $v(t,x)$ satisfies the linear complementarity conditions

$$v(t,x) \geq (K-x)^+ \text{ for all } t \in [0,T],\ x \geq 0, \quad \text{--- (a)}$$

$$rv(t,x) - v_t(t,x) - rx\, v_x(t,x) - \tfrac{1}{2}\sigma^2 x^2 v_{xx}(t,x) \geq 0$$

$$\text{for all } t \in [0,T),\ x \geq 0 \text{ and} \quad \text{--- (b)}$$

for each $t \in [0,T)$ and $x \geq 0$, equality holds in either (a) or (b)

The set $\{(t,x);\ 0 \leq t \leq T,\ x \geq 0\}$ can be divided into two regions, the stopping set

$$\mathcal{S} = \{(t,x) : v(t,x) = (K-x)^+\}$$

and the continuation set

$$\mathcal{C} = \{(t,x);\ v(t,x) > (K-x)^+\}.$$

Theorem:- Let $S(t)$, $t \leq u \leq T$, be the stock price starting at $S(t) = x$. Let

$$\tau_* = \min\{u \in [t,T] : v(u, S(u)) = (K - S(u))^+\}.$$