1.) $dS(t) = 3S(t)\,dt + 2S(t)\,d\hat{W}(t), \quad S(0) = 1$

This is a linear SDE with $f(t) = 3,\ g(t) = 0,\ \phi(t) = 2$

$\theta(t) = 0$

(i) $\therefore\ Y(t) = \int_0^t \phi(s)\,dW(s) + \int_0^t f(s)\,ds - \frac{1}{2}\int_0^t \phi^2(s)\,ds$

$= \int_0^t 2\,dW + \int_0^t 3\,ds - \frac{1}{2}\int_0^t 4\,ds$

$= 2W(t) + 3t - 2t = 2W(t) + t$

$S(t) = S(0)\,e^{Y(t)} = S(0)\,e^{2W(t)+t}$

$dS(t) = 3S(t)\,dt + 2S(t)\,d\hat{W}(t)$

$\because$ of dt term $S(t)$ is not a martingale under $\hat{\mathbb{P}}$

(ii) $f(t, x) = x^{\alpha}$

$$df = \frac{\partial f}{\partial x} dx + \frac{1}{2} \frac{\partial^2 f}{\partial x^2} (dx)^2$$

$$= \alpha x^{\alpha-1} dx + \frac{1}{2} \alpha(\alpha-1) x^{\alpha-2} (dx)^2$$

$$df(t, S(t)) = d(S^{\alpha}) = \alpha S^{\alpha-1} dS + \frac{1}{2} \alpha(\alpha-1) S^{\alpha-2} (dS)^2$$

$$= \alpha S^{\alpha-1} (3S\,dt + 2S\,d\hat{W}) + \frac{1}{2} \alpha(\alpha-1) S^{\alpha-2} (4S^2 dt)$$

$$= 2\alpha S^{\alpha} d\hat{W} + dt(3\alpha S^{\alpha} + 2\alpha(\alpha-1) S^{\alpha}).$$

For martingale dt coefficient $= 0$

$$\therefore 3\alpha + 2\alpha(\alpha-1) = 0$$

$$3\alpha + 2\alpha^2 - 2\alpha = 0$$

$$2\alpha^2 + \alpha = 0$$

$$2\alpha(\alpha + \tfrac{1}{2}) = 0$$

$$\alpha = 0, -\tfrac{1}{2}.$$

$\because \alpha \neq 0$

$\therefore \alpha = -\frac{1}{2}$

2. $X(t) = \frac{1}{t}\int_0^t S(u)\, d\hat{W}(u)$

(i) taking $\int_0^t S(u)\, d\hat{W}(u) = P \Rightarrow X(t) = \frac{1}{t}P$

$$\boxed{dX(t)} = P\left(\frac{-1}{t^2}\right)dt + \frac{1}{t}dP + dP\, d\left(\frac{1}{t}\right)$$

$$= \frac{-P(t)}{t^2}dt + \frac{1}{t}S(t)\, d\hat{W}(t) - \frac{S(t)\, d\hat{W}(t)\, dt}{t^2}$$

$$= \boxed{-\frac{X(t)\, dt}{t} + \frac{1}{t}S(t)\, d\hat{W}(t)}$$

$$\hat{E}[X(t)] = \frac{1}{t}E\Big[\underbrace{\int_0^t S(u)\, d\hat{W}(u)}\Big] = 0$$

$\because$ this is an ito integral.

$$\text{var}[X(t)] = \hat{E}[X^2(t)] - (\hat{E}[X])^2$$

$$= \hat{E}[X^2(t)]$$

$$= \frac{1}{t^2}\hat{E}\left[\left(\int_0^t S(u)\, d\hat{W}(u)\right)^2\right]$$

$$= \frac{1}{t^2}\int_0^t \hat{E}[S^2(u)]\, du$$

(By ito isometry)

$$\left[\hat{E}[S^2(t)] = S^2(0)\, e^{2\mu t + \sigma^2 t}\right]$$

$$\therefore \text{var}[X(t)] = \frac{1}{t^2}S^2(0)\int_0^t e^{(2\mu + \sigma^2)s}\, ds$$

$$\text{var}[X(t)] = \frac{S^2(0)}{t^2}\frac{\left(e^{(2\mu+\sigma^2)t} - 1\right)}{(2\mu + \sigma^2)}$$

(2ii) $ds = S(\sigma dW + \alpha dt)$

$$S(t) - S(0) = \int_0^t S\sigma \, dW + \int_0^t \alpha S \, du$$

dividing by t.

$$\frac{S(t) - S(0)}{t} = \frac{\sigma}{t}\int_0^t S \, dW + \frac{\alpha}{t}\int_0^t S \, du.$$

$$R(t) = \sigma X(t) + \alpha A(t)$$

$$\sigma X(t) = R(t) - \alpha A(t)$$

(3) Claim: The payoff $V(T) = \begin{cases} 1 & \text{if } K_1 \le S(t) \le K_2 \\ 0 & \text{otherwise} \end{cases}$

can be obtained by buying one cash or nothing contract of strike K_1 and selling one with strike K_2 with same time to expire.

Proof: Payoff of cash or nothing contract with strike K $= \begin{cases} 1 & S(T) \geq K \\ 0 & S(T) < K \end{cases}$

X_1 = cash or nothing at K_1

X_2 = cash or nothing at K_2 $\quad \underline{K_1 \le K_2}$

Our Portfolio $= X_1 - X_2$

If $S(T) < K_1 \Rightarrow$ Payoff $(X_1) = 0$

Payoff $(X_2) = 0$

$\therefore$ Payoff (portfolio) $= 0$

If $K_1 \le S(T) < K_2 \Rightarrow$ Payoff $(X_1) = 1$

Payoff $(X_2) = 0$

Payoff (Portfolio) $= 1$

If $K_2 \le S(T) \Rightarrow$ Payoff $(X_1) = 1$

Payoff $(X_2) = 1$

Payoff (Portfolio) $= 0$

$\therefore$ Payoff of our portfolio matches $V(T)$ except at T. But $\tilde{P}(S(T) = K_2) = 0$

So it doesn't matter.

$$\text{Price } V(t) = \text{Price}(X_1) - \text{Price}(X_2)$$

$$= \cancel{e^{-r(T-t)}\left(N(d_2)(K_1)\right)}$$

$$= e^{-r(T-t)}\left(N(d_2(K_1)) - N(d_2(K_2))\right)$$

$$d_2(K) = \frac{1}{\sigma\sqrt{\tau}}\left[\log(x/K) + \left(r \pm \frac{\sigma^2}{2}\right)\tau\right]$$

4. $\lim_{\|\pi_m\| \to 0} \sum_{j=1}^{m} (Y(t_j) - Y(t_{j-1}))(S(t_j) - S(t_{j-1})) = P$

$$P \le \lim_{\|\pi_m\| \to 0} \max (S(t_j) - S(t_{j-1})) \sum_{j=1}^{m} (Y(t_j) - Y(t_{j-1}))$$

This inequality holds because Y is continuous and non decreasing in t.

$$= \lim_{\|\pi_m\| \to 0} \max (S(t_j) - S(t_{j-1}))(Y(T) - Y(0))$$

~~Since~~ $\because S(t)$ is continuous, $\therefore \lim_{\|\pi_m\| \to 0} \max (S(t_j) - S(t_{j-1})) \to 0$.

$$\therefore \lim_{\|\pi_m\| \to 0} \max (S(t_j) - S(t_{j-1}))(Y(T) - Y(0)) = 0$$

5.

$dX(t) = \alpha X(t)dt + Y(t)d\hat{W}(t), \quad X(0) = x_0$

$dY(t) = \alpha Y(t)dt - X(t)d\hat{W}(t), \quad Y(0) = y_0$

Let $f(t, x) = x^2$

Applying ito's formula.–

$$df(t,x) = \left(\frac{\partial f}{\partial x}\right)dx + \left(\frac{\partial f}{\partial t}\right)dt + \frac{1}{2}\left(\frac{\partial^2 f}{\partial x^2}\right)dx\,dx.$$

$$= 2x\,dx + \frac{1}{2}2\,dx\,dx$$

$$= 2x\,dx + (dx)^2 \cancel{dx}$$

$$df(t, X(t)) = 2X(t)\,dX(t) + (dX(t))^2$$

$$dX^2(t) = 2X(\alpha X\,dt + Y\,d\hat{W}) + Y^2\,dt$$

$$dX^2(t) = (2\alpha X^2 + Y^2)\,dt + 2XY\,d\hat{W} \quad -(1)$$

$$df(t, Y(t)) = 2Y(t)\,dY(t) + (dY(t))^2$$

$$dY^2(t) = 2Y(t)\left[\alpha Y(t)\,dt - X(t)\,d\hat{W}(t)\right] + X^2\,dt$$

$$dY^2(t) = (2\alpha Y^2 + X^2)\,dt - 2XY\,d\hat{W} \quad -(2)$$

Summing (1) and (2) $\Rightarrow$

$$dX^2 + dY^2 = \left(2\alpha(X^2+Y^2) + (X^2+Y^2)\right)dt$$

$$d(X^2+Y^2) = (2\alpha+1)(X^2+Y^2)\,dt$$

Taking $Z(t) = X^2(t) + Y^2(t) \Rightarrow Z(0) = x_0^2 + y_0^2$

$$dZ(t) = \cancel{(2\alpha+1)(X^2+Y^2)}\ (2\alpha+1)Z(t)\,dt$$

This is a linear SDE, ~~and its solution will be~~

with $\phi = 0, \theta = 0, f = 2(\alpha+1), g = 0$

~~St~~ ~~$Z(t) = Z_0 e$~~

$$Y'(t) = \int_0^t (2\alpha+1)\,ds = (2\alpha+1)t$$

$$\therefore Z(t) = (x_0^2 + y_0^2)\,e^{(2\alpha+1)t}$$

$\therefore \quad X^2(t) + Y^2(t) = (x_0^2 + y_0^2)\, e^{(2\alpha+1)t}$

$\because$ RHS is a determinitic value.

$\therefore$ $R(t) = X^2(t) + Y^2(t)$ is deterministic.