martingale measure corresponding to the numeraire S_1.

Let us use the pricing part of the above theorem for an arbitrary choice of T-claim $\mathcal{X}$. we have

$$\Pi(0,\mathcal{X}) = S_0(0)\,\mathbb{E}^0\left[\mathcal{X}/S_0(T)\,|\,\mathcal{F}(0)\right] = S_0(0)\,\mathbb{E}^0\left[\mathcal{X}/S_0(T)\right]$$

$$\Pi(0,\mathcal{X}) = S_1(0)\,\mathbb{E}^1\left[\mathcal{X}/S_1(T)\right]$$

Let $L_0^1(T)$ be the Radon-Nikodym derivative

$$L_0^1(T) = \frac{dQ^1}{dQ^0} \text{ on } \mathcal{F}(T).$$

i.e., $$Q^1(A) = \int_A L_0^1(T)\,dQ^0 \quad \forall A \in \mathcal{F}(T).$$

We can write

$$\Pi_0(0,\mathcal{X}) = S_1(0)\,\mathbb{E}^0\left[\frac{\mathcal{X}}{S_1(T)}\cdot L_0^1(T)\right]$$

$$\Rightarrow S_0(0)\,\mathbb{E}^0\left[\mathcal{X}/S_0(T)\right] = S_1(0)\,\mathbb{E}^0\left[\mathcal{X}/S_1(T)\,L_0^1(T)\right].$$

for all T-claims $\mathcal{X}$. We thus deduce that

$$\Rightarrow \frac{S_0(0)}{S_0(T)} = \frac{S_1(0)}{S_1(T)}\cdot L_0^1(T).$$

$$\Rightarrow L_0^1(T) = \frac{S_1(T)}{S_0(T)}\cdot\frac{S_0(0)}{S_1(0)}.$$

which is our candidate as a Radon-Nikodym derivative.

The Radon-Nikodym derivative process is

$$L_0^1(t) = \mathbb{E}^0\left[L_0^1(T)\,|\,\mathcal{F}(t)\right]$$

$$= \frac{S_0(0)}{S_1(0)}\cdot\frac{S_1(t)}{S_0(t)}$$

since $\frac{S_1(t)}{S_0(t)}$ is a Q^0 martingale.

Proposition:- Assume that Q^0 is a martingale measure for the numeraire S_0 and assume that S_1 is a positive asset price process such that $S_1(t)/S_0(t)$ is a Q^0 martingale. Define Q^1 on $\mathcal{F}(T)$ by the Radon-Nikodym derivative process $L_0^1(t)$.

Then Q^1 is a martingale measure for S_1.

Proof:- $Q^1 \sim Q_0$

We have to show that $\frac{S_i(t)}{S_1(t)}$ is a Q^1- martingale

We know that $\frac{S_i(t)}{S_0(t)}$ is a Q^0- martingale.

$$\mathbb{E}^1\left[\frac{S_i(t)}{S_1(t)} \Big| \mathcal{F}(s)\right] = \frac{1}{L_0^1(s)} \mathbb{E}^0\left[L_0^1(t)\cdot\frac{S_i(t)}{S_1(t)} \Big| \mathcal{F}(s)\right]$$

$$\left\{\begin{array}{l} \text{Let } \widetilde{\mathbb{P}}(A) = \int_A Z\,d\mathbb{P} \quad \forall\, A \in \mathcal{F}(T) \text{ and } Z(t) = \mathbb{E}[Z/\mathcal{F}(t)] \\ \text{Let } Y \text{ be an } \mathcal{F}(t)\text{-measurable random variable. Then} \\ \widetilde{\mathbb{E}}[Y|\mathcal{F}(s)] = \frac{1}{Z(s)} \mathbb{E}[YZ(t)|\mathcal{F}(s)] \text{ for } 0 \le s \le t \le T \end{array}\right\}$$

$$= \frac{1}{L_0^1(s)} \mathbb{E}^0\left[\frac{S_0(0)}{S_1(0)}\cdot\frac{\cancel{S_1(t)}}{S_0(t)}\cdot\frac{S_i(t)}{\cancel{S_1(t)}} \Big| \mathcal{F}(s)\right]$$

$$= \frac{S_0(0)}{S_1(0)}\cdot\frac{1}{L_0^1(s)} \mathbb{E}^0\left[\frac{S_i(t)}{S_0(t)} \Big| \mathcal{F}(s)\right]$$

$$= \frac{S_0(0)}{S_1(0)}\cdot\frac{1}{L_0^1(s)}\cdot\frac{S_i(s)}{S_0(s)} = \frac{S_i(s)}{S_1(s)}$$

$\Rightarrow$ Q^1 is a martingale measure for S_1.

Some Examples:-

We will now present some examples to illustrate the usefulness of the change of numeraire techique.

Linearly Homogeneous Contracts:-

A typical example whean a change of numeraire is so useful occurs when dealing with derivatives defined in terms of several underlying assets. Assume for example that we are given two asset prices S_1 and S_2 and that the contract $\mathcal{X}$ to be priced is of the form $\mathcal{X} = \Phi(S_1(t), S_2(t))$, where Φ is a given linearly homogeneous function, i.e.,

$$\Phi(\lambda x, \lambda y) = \lambda \cdot \Phi(x, y).$$

for all x, y and all $\lambda > 0$. using the standard risk-neutral machinery with B as the numeraire, and denoting the risk-neutral martingale measure by Q we would have to compute the price as

$$\Pi(t, \mathcal{X}) = \mathbb{E}^{Q}\left[e^{-\int_t^T r_s \, ds} \Phi(S_1(t), S_2(t)) \,\middle|\, \mathcal{F}(t) \right]$$

which essentially amounts to the calculation of a triple integral. If we instead use S_1^{Φ} as numeraire, with martingal measure Q^1 we have

$$\Pi(t, \mathcal{X}) = S_1(t)\, \mathbb{E}^{Q^1}\left[\frac{\Phi(S_1(T), S_2(T))}{S_1(T)} \,\middle|\, \mathcal{F}(t) \right]$$

$$= S_1(t)\, \mathbb{E}^{Q^1}\left[\varphi(Z_2(T)) \,\middle|\, \mathcal{F}(t) \right]$$

where $\varphi(z) = \Phi(1, z)$ and $Z_2(t) = S_2(t)/S_1(t)$.

In this formula we note that the factor $S_1(t)$ is the price of the traded asset S_1 at time t, so, it can be directly observed on the market. Thus the computational work is reduced to computing a single integral.

Exchange Option:-

Assume that we have two stocks S_1 and S_2, with price processes of the following form under the probability measure $\mathbb{P}$.

$$dS_1(t) = \alpha_1 S_1(t)\,dt + \sigma_1 S_1(t)\,d\bar{W}(t)$$
$$dS_2(t) = \alpha_2 S_2(t)\,dt + \sigma_2 S_2(t)\,d\bar{W}(t)$$

Here $\alpha_1, \alpha_2 \in \mathbb{R}$ and $\sigma_1, \sigma_2 \in \mathbb{R}^2$, $\bar{W}$ is a two-dimensional standard Wiener process under $\mathbb{P}$, we assume absence of arbitrage.

The T-claim to be priced is an exchange option, which gives the holder the right, but not the obligation, to exchange on S_2 share for one S_1 share at time T. Formally this means that the claim is given by $\mathcal{X} = \max[S_2(T) - S_1(T), 0]$.

Using homogeneity, the price is given by

$$\Pi(t, \mathcal{X}) = S_1(t)\,\mathbb{E}^1\left[\frac{\max[S_2(T) - S_1(T), 0]}{S_1(T)} \,\middle|\, \mathcal{F}(t)\right]$$
$$= S_1(t)\,\mathbb{E}^1\left[\max[Z_2(T) - 1, 0] \,\middle|\, \mathcal{F}(t)\right]$$

with $Z_2(t) = S_2(t)/S_1(t)$, $\mathbb{E}^1$ denoting expectation under Q^1.

We now see that the expectation above is in fact the value of a European call option on Z_2 with strike price $K=1$ in a world with zero short rate.

We now have to compute the Q^1 dynamics of Z_2

$$dZ_2(t) = d\left(\frac{S_2(t)}{S_1(t)}\right) = \frac{1}{S_1(t)} dS_2(t) + S_2(t) d\left(\frac{1}{S_1(t)}\right) + dS_2(t) d\left(\frac{1}{S_1(t)}\right).$$

$$d\left(\frac{1}{S_1(t)}\right) = -\frac{1}{S_1^2(t)} dS_1(t) + \frac{1}{2}\frac{2}{S_1^3(t)} dS_1(t)\, dS_1(t)$$

$$= -\frac{\alpha_1}{S_1(t)} dt - \frac{\sigma_1}{S_1(t)} d\bar{W}(t) + \frac{1}{S_1(t)} \|\sigma_1\|^2 dt$$

$$dZ_2(t) = Z_2(t)(- \cdots) dt + Z_2(t)(\sigma_2 - \sigma_1) d\bar{W}(t)$$

where we do not care about dt-terms. Under Q^1 we know that Z_2 is a martingale and since the volatility terms do not change under a Girsanov transformation we obtain directly the Q^1 dynamics as

$$dZ_2(t) = Z_2(t)(\sigma_2 - \sigma_1) d\bar{W}^1(t)$$

where $\bar{W}^1$ is Q^1- Wiener process. We can write this as

$$dZ_2(t) = Z_2(t)\, \sigma\, d\bar{W}^1(t)$$

where W^1 is a scalar Q^1- Wiener process and $\sigma = \|\sigma_2 - \sigma_1\|$

Using the Black- Scholes formula with zero short rate, unit strike price and volatility σ, the price of the exchange option is given by the formula

$$\pi(t,x) = S_1(t)\left\{Z_2(t)\,N(d_1) - N(d_2)\right\}$$

$$= S_2(t)\,N(d_1) - S_1(t)\,N(d_2)$$

where $$d_1 = \frac{1}{\sigma\sqrt{T-t}}\left\{\ln\left(\frac{S_2(t)}{S_1(t)}\right) + \tfrac{1}{2}\sigma^2(T-t)\right\}.$$

$$d_2 = d_1 - \sigma\sqrt{T-t}.$$