Proof:- claim $g(t, X(t))$, $0 \le t \le T$ is a martingale.

By previous theorem

$$g(t, X(t)) = \mathbb{E}[h(X(T)) | \mathcal{F}(t)]$$

Thus for $s < t$

$$\mathbb{E}[g(t, X(t)) | \mathcal{F}(s)] = \mathbb{E}[\mathbb{E}[h(X(T)) | \mathcal{F}(t)] | \mathcal{F}(s)]$$

$$= \mathbb{E}[h(X(T)) | \mathcal{F}(s)] = g(s, X(s)).$$

Hence the claim. Now

$$d(g(t, X(t))) = g_t(t, X(t))\,dt + g_x(t, X(t))\,dX(t) + \tfrac{1}{2} g_{xx}(t, X(t))\,dX(t)\,dX(t).$$

$$= g_t(t, X(t))\,dt + g_x(t, X(t))\,b(t, X(t))\,dt + g_x(t, X(t))\,\sigma(t, X(t))\,dW(t) + \tfrac{1}{2} g_{xx}(t, X(t))\,\sigma^2(t, X(t))\,dt$$

$$= [g_t(t, X(t)) + b(t, X(t))\,g_x(t, X(t)) + \tfrac{1}{2} g_{xx}(t, X(t))\,\sigma^2(t, X(t))]\,dt + g_x(t, X(t))\,\sigma(t, X(t))\,dW(t).$$

since $g(t, X(t))$ is a martingale so the dt term must be equal to 0. Thus we must have

$$g_t(t, x) + g_x(t, x)\,b(t, x) + \tfrac{1}{2}\sigma^2(t, x)\,g_x(t, x) = 0.$$

Theorem:- consider the stochastic differential equation

$$dX(t) = b(t, X(t))\,dt + \sigma(t, X(t))\,dW(t).$$

Let $h(y)$ be a Borel-measurable function. Define

$$f(t, x) = \mathbb{E}^{t,x}[e^{-r(T-t)} h(X(T))]$$

where r is a constant. Then $f(t, x)$ satisfies the partial differential equation $f_t(t, x) + b(t, x) f_x(t, x) + \tfrac{1}{2}\sigma^2(t, x) f_{xx}(t, x) = r f(t, x)$ with $f(T, x) = h(x)$.

Proof:- $f(t,x) = e^{-r(T-t)} g(t,x)$. Thus

$$f(t, X(t)) = e^{-r(T-t)} g(t, X(t))$$
$$= e^{-r(T-t)} \mathbb{E}[h(X(T)) | \mathcal{F}(t)]$$
$$= \mathbb{E}[e^{-r(T-t)} h(X(T)) | \mathcal{F}(t)].$$

For $s<t$

$$\mathbb{E}[f(t, X(t)) | \mathcal{F}(s)] = \mathbb{E}[\mathbb{E}[e^{-r(T-t)} h(X(T)) | \mathcal{F}(t)] | \mathcal{F}(s)].$$
$$= \mathbb{E}[e^{-r(T-t)} h(X(T)) | \mathcal{F}(s)].$$

But $f(s, X(s)) = \mathbb{E}[e^{-r(T-s)} h(X(T)) | \mathcal{F}(s)]$.

Therefore $f(t, X(t))$ is not a martingale. But

$$\mathbb{E}[e^{-rt} f(t, X(t)) | \mathcal{F}(s)] = \mathbb{E}[e^{-rt} \mathbb{E}[e^{-r(T-t)} h(X(T)) | \mathcal{F}(t)] | \mathcal{F}(s)].$$
$$= \mathbb{E}[e^{-rT} h(X(T)) | \mathcal{F}(s)]$$
$$= \mathbb{E}[e^{-rs} e^{-r(T-s)} h(X(T)) | \mathcal{F}(s)] = e^{-rs} f(s, X(s))$$

Thus $e^{-rt} f(t, X(t))$ is a martingale. Now

$$d(e^{-rt} f(t,X(t))) = e^{-rt} df(t, X(t)) - r e^{-rt} f(t, X(t))\, dt$$
$$= e^{-rt} [-r f(t,X(t)) + f_t(t,X(t)) + f_x(t, X(t))\, b(t,X(t))$$
$$+ \tfrac{1}{2} \sigma^2(t, X(t)) f_{xx}(t,X(t))] dt + e^{-rt} \sigma(t,X(t)) f_x(t,X(t))\, dW(t).$$

so by setting dt term eaual to 0 we have

$$f_t + b f_x + \tfrac{1}{2}\sigma^2 f_{xx} = rf \;\;\&\;\; f(T,x) = h(x).$$

Let $W(t) = (W_1(t), W_2(t))$ be a two-dimensional Brownian motion consider two stochastic differential equations.

$$dX_1(t) = \beta_1(t, X_1(t), X_2(t))\,dt + \gamma_{11}(t, X_1(t), X_2(t))\,dW_1(t) + \gamma_{12}(t, X_1(t), X_2(t))\,dW_2(t).$$

$$dX_2(t) = \beta_2(t, X_1(t), X_2(t))\,dt + \gamma_{21}(t, X_1(t), X_2(t))\,dW_1(t) + \gamma_{22}(t, X_1(t), X_2(t))\,dW_2(t).$$

$$X_1(0) = x_0^1 \ \& \ X_2(0) = x_0^2.$$

Let $h(y_1, y_2)$ be a Borel measurable function. we define

$$g(t, x_1, x_2) = \mathbb{E}^{t, x_1 x_2}\left[h(X_1(T), X_2(T))\right]$$

$$f(t, x_1, x_2) = \mathbb{E}^{t, x_1, x_2}\left[e^{-r(T-t)} h(X_1(T), X_2(T))\right].$$

Then
$$g_t + \beta_1 g_{x_1} + \beta_2 g_{x_2} + \tfrac{1}{2}(\gamma_{11}^2 + \gamma_{12}^2) g_{x_1 x_1} + (\gamma_{11}\gamma_{21} + \gamma_{12}\gamma_{22}) g_{x_1 x_2} + \tfrac{1}{2}(\gamma_{21}^2 + \gamma_{22}^2) g_{x_2 x_2} = 0$$

$$f_t + \beta_1 f_{x_1} + \beta_2 f_{x_2} + \tfrac{1}{2}(\gamma_{11}^2 + \gamma_{12}^2) f_{x_1 x_1} + (\gamma_{11}\gamma_{21} + \gamma_{12}\gamma_{22}) f_{x_1 x_2} + \tfrac{1}{2}(\gamma_{21}^2 + \gamma_{22}^2) f_{x_2 x_2} = rf$$

Also satisfy the terminal conditions

$$g(T, x_1, x_2) = f(T, x_1, x_2) = h(x_1, x_2) \quad \forall x_1, x_2.$$

Problem:- let $X(t)$ be the solution to the SDE

$$dX(t) = \alpha X(t)dt + \sigma dW(t), \quad X(0) = x_0.$$

where α, σ, x_0 are constant and $W(t)$ is a Brownian motion.

① Determine $\mathbb{E}[X(t)]$

② Determine $\mathrm{var}(X(t))$

③ Find the solution of the SDE.

solution:- ① $X(t) = x_0 + \alpha\int_0^t X(s)ds + \sigma\int_0^t dW(s).$

$$\Rightarrow \mathbb{E}[X(t)] = x_0 + \alpha\,\mathbb{E}\Big[\int_0^t X(s)ds\Big] + \sigma\,\mathbb{E}\Big[\int_0^t dW(s)\Big].$$

$$= x_0 + \alpha\,\mathbb{E}\Big[\int_0^t X(s)ds\Big].$$

which leads to the ordinary differential equation.

$$\frac{d}{dt}\mathbb{E}[X(t)] = \alpha\,\mathbb{E}[X(t)].$$

$$\& \quad \mathbb{E}[X(0)] = x_0.$$

which has the solution $\mathbb{E}[X(t)] = x_0 e^{\alpha t}$.

② $\mathrm{var}(X(t)) = \mathbb{E}[X(t)^2] - \big(\mathbb{E}[X(t)]\big)^2$

$$= \mathbb{E}[X^2(t)] - x_0^2 e^{2\alpha t}$$

$$dX^2(t) = 2X(t)dX(t) + dX(t)dX(t)$$

$$= 2X(t)\big(\alpha X(t)dt + \sigma dW(t)\big) + \sigma^2 dt$$

$$= (2\alpha X^2(t) + \sigma^2)dt + 2\sigma X(t)dW(t).$$

$$\mathbb{E}[X^2(t)] = x_0^2 + \mathbb{E}\Big[\int_0^t (2\alpha X^2(s) + \sigma^2)ds + 2\sigma\,\mathbb{E}\int_0^t X(s)dW(s)$$

$$= x_0^2 + \int_0^t 2\alpha\,\mathbb{E}[X_s^2]\,ds + \sigma^2 t$$

let $g(t) = \mathbb{E}[X^2(t)]$, the problem leads to the ordinary differential equation $\frac{d}{dt}g(t) = 2\alpha g(t) + \sigma^2$

$$\Rightarrow \frac{d}{dt}\left(\bar{e}^{2\alpha t} g(t)\right) = \bar{e}^{2\alpha t} \sigma^2$$

$$\Rightarrow g(t) = e^{2\alpha t}\left(x_0^2 + \sigma^2 \int_0^t \bar{e}^{2\alpha s} ds\right)$$

$$= x_0^2 e^{2\alpha t} + \sigma^2 \frac{e^{2\alpha t} - 1}{2\alpha}$$

so the variance is $var(X(t)) = \sigma^2 \frac{e^{2\alpha t} - 1}{2\alpha}$.

③ $$X(t) = e^{\alpha t}\left(x_0 + \int_0^t \bar{e}^{\alpha s} \sigma \, dW(s)\right).$$

Problem:- Let $K(x)$ be a non-negative, continuous function and let $f(x)$ be a bounded and continuous. suppose that $u(t,x)$ is a bounded function that satisfies the partial differential equation

$$\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2} - K(x) u \quad --- (*)$$

and the intial condition $u(0,x) = f(x)$.

Then the stochastic representation of the solution $u(t,x)$ is

$$u(t,x) = \mathbb{E}\left[\exp\left\{-\int_0^t K(W(s))ds\right\} f(W(t))\right],$$

where $W(t)$ is a Brownian motion started at x ($W(0)=x$).

Solution:- Fix $t>0$ and consider the stochastic process

$$Y(s) = \bar{e}^{R(s)} u(t-s, W(s)), \text{ where}$$

$$R(s) = -\int_0^s K(W(u))\, du.$$

$$dY(s) = -K(W(s)) \bar{e}^{R(s)} u(t-s, W(s))\, ds$$
$$- u_t(t-s, W(s)) \bar{e}^{R(s)} ds + u_x(t-s, W(s)) \bar{e}^{R(s)} dW(s)$$
$$+ \tfrac{1}{2} u_{xx}(t-s, W(s)) \bar{e}^{R(s)} ds.$$

$$dY(s) = e^{-R(s)}\left[-u_t(t-s, W(s)) - K(W(s))\,u(t-s, W(s)) + \tfrac{1}{2}u_{xx}(t-s, W(s))\right]ds + u_x(t-s, W(s))\,e^{-R(s)}\,dW(s).$$

Since u satisfies the PDE ⊛, the ds terms in the last expression sum to zero, leaving

$$dY(s) = u_x(t-s, W(s))\,e^{-R(s)}\,dW(s).$$

Thus $Y(s)$ is a martingale upto time t. Hence

$$Y(0) = u(t, W(0)) = \mathbb{E}[Y(t)]$$

$$\Rightarrow u(t,x) = \mathbb{E}\left[e^{-R(t)}u(0, W(t))\right]$$

$$= \mathbb{E}\left[e^{-R(t)}f(W(t))\right]$$

$$= \mathbb{E}\left[\exp\left\{-\int_0^t K(W(s))\,ds\right\}f(W(t))\right].$$