Therefore

$$C(t,x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d_-(\tau,x)} e^{-r\tau}\left(x\exp\{-\sigma\sqrt{\tau}\,y + (r-a-\tfrac{1}{2}\sigma^2)\tau\} - K\right) e^{-\frac{1}{2}y^2}\,dy$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d_-(\tau,x)} x\exp\{-\sigma\sqrt{\tau}\,y - (a+\tfrac{1}{2}\sigma^2)\tau - \tfrac{1}{2}y^2\}\,dy$$

$$- \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d_-(\tau,x)} e^{-r\tau} K e^{-\frac{1}{2}y^2}\,dy$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d_-(\tau,x)} x e^{-a\tau} \exp\{-\tfrac{1}{2}(y+\sigma\sqrt{\tau})^2\}\,dy - Ke^{-r\tau} N(d_-(\tau,x))$$

put $z = y + \sigma\sqrt{\tau}$ then

$$C(t,x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d_+(\tau,x)} x e^{-a\tau} e^{-z^2/2}\,dz - e^{-r\tau} K\, N(d_-(\tau,x)).$$

$$= x e^{-a\tau} N(d_+(\tau,x)) - Ke^{-a\tau} N(d_-(\tau,x)).$$

Lump payments of dividends:-

Consider $0<t_1<t_2< \cdots <t_n<T$. Think of $t_1, t_2, \ldots, t_n$ as the dividend paying dates in the asset. At each time t_j, the dividend paid is $a_j S(t_j-)$, where $S(t_j-)$ denotes the stock price just prior to the dividend payment. The stock price after dividend payment is the stock price before the dividend payment less the dividend payment

$$S(t_j) = S(t_j-) - a_j S(t_j-) = (1-a_j)\, S(t_j-).$$

We assume that a_j is an $\mathcal{F}(t_j)$ - measurable random variable taking value in $[0,1]$. If $a_j = 0$, no dividend is paid at time t_j. If $a_j = 1$, the full value of the stock is paid as a dividend at time t_j and the stock value zero thereafter. We set $t_0 = 0$ and $t_{n+1} = T$ and $a_0 = 0$ and $a_{n+1} = 0$. We assume that, between dividend payment dates, the stock price follows a generalized geometric Brownian motion:-

$$dS(t) = \alpha(t) S(t)\, dt + \sigma(t) S(t)\, dW(t), \quad t_j \le t < t_{j+1}, \; j = 0, 1, \ldots n.$$

Between dividend payment dates, the differential of the portfolio value corresponding to a portfolio process $\Delta(t), 0 \le t \le T$ is

$$\begin{aligned} dX(t) &= \Delta(t) dS(t) + R(t)[X(t) - \Delta(t)S(t)]dt \\ &= R(t)X(t)\,dt + (\alpha(t) - R(t))\Delta(t)S(t)\,dt + \sigma(t)\Delta(t)S(t)\,dW(t) \\ &= R(t)X(t)\,dt + \Delta(t)S(t)\sigma(t)[\Theta(t)\,dt + dW(t)] \end{aligned}$$

where $\Theta(t) = \dfrac{\alpha(t) - R(t)}{\sigma(t)}$ is the market price of risk.

At the dividend payment dates, the value of the portfolio stock holdings drops by $a_j \Delta(t_j) S(t_j-)$, but the portfolio collects the dividend $a_j \Delta(t_j) S(t_j-)$, and so the portfolio value does not jump. It follows that

$$dX(t) = R(t)X(t)\,dt + \Delta(t)S(t)\sigma(t)[\Theta(t)\,dt + dW(t)]$$

We define $d\widetilde{W}(t) = dW(t) + \Theta(t)dt$ and change to a measure $\widetilde{\mathbb{P}}$ under which $\widetilde{W}(t)$ is a Brownian motion, and obtain the risk-neutral pricing formula

$$D(t)X(t) = \widetilde{\mathbb{E}}\left[D(T)X(T) \mid \mathcal{F}(t)\right].$$

Here we price a European call under the assumption that σ, r and each a_j are constant, we have

$$dS(t) = rS(t)\,dt + \sigma S(t)\,d\widetilde{W}(t), \quad t_j \le t < t_{j+1}, \; j=0,1,2,\dots n.$$

Therefore

$$S(t_{j+1}^-) = S(t_j)\exp\left\{\sigma\left(\widetilde{W}(t_{j+1}) - \widetilde{W}(t_j)\right) + \left(r - \tfrac{1}{2}\sigma^2\right)(t_{j+1} - t_j)\right\}.$$

Hence, we have

$$\begin{aligned} S(t_{j+1}) &= (1 - a_{j+1})\, S(t_{j+1}^-) \\ &= (1 - a_{j+1})\, S(t_j)\exp\left\{\sigma\left(\widetilde{W}(t_{j+1}) - \widetilde{W}(t_j)\right) + \left(r - \tfrac{1}{2}\sigma^2\right)(t_{j+1} - t_j)\right\}. \end{aligned}$$

$$\Rightarrow \quad \frac{S(t_{j+1})}{S(t_j)} = (1 - a_{j+1})\exp\left\{\sigma\left(\widetilde{W}(t_{j+1}) - \widetilde{W}(t_j)\right) + \left(r - \tfrac{1}{2}\sigma^2\right)(t_{j+1} - t_j)\right\}.$$

It follows that

$$\begin{aligned} \frac{S(T)}{S(0)} = \frac{S(t_{n+1})}{S(t_0)} &= \prod_{i=0}^{n} \frac{S(t_{i+1})}{S(t_i)} \\ &= \prod_{i=0}^{n} (1 - a_{i+1})\exp\left\{\sigma \widetilde{W}(T) + \left(r - \tfrac{1}{2}\sigma^2\right)T\right\}. \end{aligned}$$

$$\Rightarrow \quad S(T) = S(0)\prod_{i=0}^{n}(1 - a_{i+1}) \cdot \exp\left\{\sigma \widetilde{W}(T) + \left(r - \tfrac{1}{2}\sigma^2\right)T\right\}.$$

This is the same formula we would have for the price at time T of a geometric Brownian motion not paying dividends if the initial price were $S(0)\prod_{i=0}^{n-1}(1 - a_{i+1})$ rather than $S(0)$.

For $dS(t) = rS(t)dt + \sigma S(t) d\widetilde{W}(t)$

$$S(T) = S(0) \cdot \exp\left\{ \sigma \widetilde{W}(T) + (r - \tfrac{1}{2}\sigma^2) T \right\}.$$

Therefore the price at time zero of a European call on this dividend - paying asset is

$$S(0) \prod_{i=0}^{n-1} (1 - a_{i+1})\, N(d_+) - e^{-rT} K\, N(d_-).$$

where

$$d_{\pm} = \frac{1}{\sigma \sqrt{T}} \left[\log \frac{S(0)}{K} + \sum_{i=0}^{n-1} \log(1 - a_{i+1}) + (r \pm \tfrac{1}{2}\sigma^2) T \right]$$

— A similar formula holds for the call price at time t between 0 and T

— In those cases, one includes only the terms $(1 - a_{i+1})$ corresponding to the dividend dates between t and T.