SHORT RATE MODELS

Here our main goal is ~~bon~~ broadly to investigate the relationship which must hold in an arbitrage free market between the price processes of bonds with different maturities.

- how to model an arbitrage free family of zero coupon bond prices processes $\{p(\cdot, T);\ T \geq 0\}$.

- Note that the price, $p(t,T)$, should depend upon the behavior of the short rate r over the interval $[t,T]$. Let

$$dr(t) = \mu(t, r(t))\,dt + \sigma(t, r(t))\,dW(t).$$

We assume that there is a market for T-bonds for every choice of T and the market is arbitrage free. We assume furthermore that for every T, the price of a T-bond has the form

$$p(t,T) = F(t, r(t); T) = F^T(t, r(t)).$$

where F is a smooth function.

Then $F^T(T, r) = 1 \quad \forall r$.

Fix two times of maturity s and T. Then by Itô formula

$$dF^T = F_t^T(t, r(t))\,dt + F_r^T(t, r(t))\,dr(t) + \tfrac{1}{2} F_{rr}^T(t, r(t))\,dr(t)dr(t).$$

$$= F_t^T(t, r(t))\,dt + \mu(t, r(t))\,F_r^T(t, r(t))\,dt$$

$$+ \sigma(t, r(t))\,F_r^T(t, r(t))\,dW(t) + \tfrac{1}{2} F_{rr}^T(t, r(t))\,\sigma^2(t, r(t))\,dt$$

$$= F^T \alpha^T\,dt + F^T \sigma^T\,dW(t)$$

where

$$\alpha^T = \frac{F_t^T + \mu F_r^T + \frac{1}{2}\sigma^2 F_{rr}^T}{F^T}$$

$$\sigma^T = \frac{\sigma F_r^T}{F^T}.$$

Now we consider a self-financing portfolio based on T-bond and S-bond with portfolio weights (u^S, u^T). Then we have the following portfolio value dynamics

$$dV(t) = V(t)\left\{u^T \frac{dF^T}{F^T} + u^S \frac{dF^S}{F^S}\right\}.$$

$$\Rightarrow \quad dV(t) = V(t)\left\{u^T\alpha^T + u^S\alpha^S\right\}dt + V(t)\left\{u^T\sigma^T + u^S\sigma^S\right\}dW(t).$$

We choose u^T and u^S such that

$$u^T + u^S = 1 \quad \text{and} \quad u^T\sigma^T + u^S\sigma^S = 0$$

Then

$$dV(t) = V(t)\left\{u^T\alpha^T + u^S\alpha^S\right\}dt$$

and $\quad u^T = -\dfrac{\sigma^S}{\sigma^T - \sigma^S}$ and $u^S = \dfrac{\sigma^T}{\sigma^T - \sigma^S}$

$$\Rightarrow dV(t) = V(t)\left\{\frac{\alpha^S\sigma^T - \sigma^S\alpha^T}{\sigma^T - \sigma^S}\right\}dt$$

Absence of arbitrage implies that $\dfrac{\alpha^S\sigma^T - \sigma^S\alpha^T}{\sigma^T - \sigma^S} = r(t)$

$$\Rightarrow \frac{\alpha^S(t) - r(t)}{\sigma^S(t)} = \frac{\alpha^T(t) - r(t)}{\sigma^T(t)}$$

Proposition:- Assume that market is arbitrage free. Then there exists a process $\lambda(t, r(t))$ such that

$$\frac{\alpha^T(t) - r(t)}{\sigma^T(t)} = \lambda(t) \quad [\text{market price of risk}].$$

regardless of the specific choice of the ~~bond~~. maturity time T.

Proposition:- (Term structure equation)

In an arbitrage free bond market, the bond pricing function F^T will satisfy the term structure equation

$$\begin{cases} F_t^T + \{\mu - \lambda\sigma\} F_r^T + \frac{1}{2}\sigma^2 F_{rr}^T - rF^T = 0 \\ F^T(T, r) = 1 \end{cases}$$

Proof:-
$$\alpha^T = \frac{F_t^T + \mu F_r^T + \frac{1}{2}\sigma^2 F_{rr}^T}{F^T}$$

and
$$\sigma^T = \frac{\sigma F_r^T}{F^T}$$

By substituting α^T and σ^T into $\frac{\alpha^T(t) - r(t)}{\sigma^T(t)} = \lambda(t)$ and after some algebraic manipulations we will get the above equation.

Now one can obtain more information from the term structure equation by applying the Feynman-Kac representation.

Proposition:- (Risk-neutral valuation)

Bond prices are given by the formula $p(t,T) = F^T(t, r(t))$ where

$$F^T(t,r) = \mathbb{E}_t^Q\left[e^{-\int_t^T r(s)\,ds} \times 1 \,\middle|\, r(t) = r\right].$$

The dynamics of r under the martingale measure Q is given by

$$\begin{cases} dr(s) = \{\mu - \lambda\sigma\}ds + \sigma dW^Q(s) \\ r(t) = r \end{cases}$$

where W^Q is a Wiener process under Q.

Proposition:- Let X be a contingent T-claim of the form $X = \Phi(r(T))$. In an arbitrage free market, let the price process $\Pi(t,\Phi)$ be given as $\Pi(t,\Phi) = F(t, r(t))$, where F solves the boundary value problem

$$\begin{cases} F_t + \{\mu - \sigma\lambda\}F_r + \tfrac{1}{2}\sigma^2 F_{rr} - rF = 0 \\ F(T,r) = \Phi(r) \end{cases}$$

Furthermore F has the stochastic representation

$$F(t,r;T) = \mathbb{E}^Q\left[\exp\left\{-\int_t^T r(s)\,ds\right\}\Phi(r(T)) \,\middle|\, r(t) = r\right]$$

where the dynamics of r under the risk-neutral measure Q is given by

$$dr(t) = \{\mu - \sigma\lambda\}dt + \sigma dW^Q(t).$$