solution to the Black-scholes-Merton equation:-

Black-scholes-Merton partial differential equation is

$$c_t(t,x) + r x c_x(t,x) + \tfrac{1}{2}\sigma^2 x^2 c_{xx}(t,x) = r c(t,x), \quad t \in [0,T), \; x \geq 0$$

Terminal condition

$$c(T,x) = (x-K)^+$$

We use the notation $BSM(\tau, x, K, r, \sigma)$

$$c(t,x) = BSM(\tau, x, K, r, \sigma) = x N(d_+(\tau,x)) - K e^{-r\tau} N(d_-(\tau,x))$$

where $d_\pm(\tau,x) = \frac{1}{\sigma\sqrt{\tau}}\left[\log x/K + (r \pm \sigma^2/2)\tau\right]$.

τ, and x denote the time to expiration and the current stock price respectively. The parameters K, r, and σ are the strike price, the interest rate and the stock volatility, respectively

The Greeks:- The derivatives of the function $c(t,x)$ with respect to various variables are called the Greeks

$$\text{Delta} = \Delta = \frac{\partial c(t,x)}{\partial x} = N(d_+(\tau,x))$$

$$\text{Theta} = \Theta = \frac{\partial c}{\partial t} = -rK e^{-r(T-t)} N(d_-(T-t,x)) - \frac{\sigma x}{2\sqrt{T-t}} N'(d_+(T-t,x))$$

Because both N and N' are always positive, delta is always positive and theta is always negative.

$$\text{Gamma} = \Gamma = \frac{\partial^2 c}{\partial x^2} = N'(d_+(\tau,x)) \frac{\partial}{\partial x}(d_+(\tau,x))$$

$$= \frac{1}{\sigma x \tau} N'(d_+(\tau,x)).$$

Γ is always positive

$$\text{vega} = \nu = \frac{\partial c}{\partial \sigma} = x N'(d_+(\tau,x))\sqrt{\tau}$$

vega is always positive

$$\text{Rho} = \rho = \frac{\partial c}{\partial r} = K\tau e^{-r\tau} N(d_-(\tau,x)).$$

In order to actually perform all of the calculations of the Greeks, we need to recall that

$$N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-y^2/2}\, dy$$

$$\Rightarrow N'(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}.$$

$$c(t,x) = BSM(\tau, x, K, r, \sigma) = xN(d_1) - Ke^{-r\tau}N(d_2)$$

$$d_1 = \frac{\log x/K + (r + \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}, \quad d_2 = d_1 - \sigma\sqrt{\tau}.$$

Note that

$$\frac{xN'(d_1)}{Ke^{-r\tau}N'(d_2)} = \frac{x e^{-d_1^2/2}}{Ke^{-r\tau} e^{-d_2^2/2}} = \frac{x}{Ke^{-r\tau}} e^{\frac{d_2^2 - d_1^2}{2}}$$

$$= \frac{x}{Ke^{-r\tau}} e^{-\frac{1}{\cancel{\sigma^2\tau}}\left[(\log x/K + r\tau)\cancel{\sigma^2\tau}\right]}$$

$$= \frac{x}{K} e^{+r\tau} \cdot K/x \cdot e^{-r\tau} = 1.$$

$$\Rightarrow xN'(d_1) = Ke^{-r\tau}N'(d_2). \quad \text{---- (1)}$$

$$d_1 = \frac{1}{\sigma\sqrt{\tau}}\left[\log x/K + (r + \tfrac{1}{2}\sigma^2)\tau\right]$$

$$\frac{\partial d_1}{\partial x} = \frac{1}{x\sigma\sqrt{\tau}}, \quad \frac{\partial d_1}{\partial r} = \frac{\sqrt{\tau}}{\sigma}$$

$$\frac{\partial d_1}{\partial \sigma} = \frac{\sigma^2\tau - [\log x/k + (r+\tfrac{1}{2}\sigma^2)\tau]}{\sigma\sqrt{\tau}} = -\frac{d_2}{\sigma}$$

and $$\frac{\partial d_1}{\partial \tau} = \frac{-\log(x/k) + (r+\tfrac{1}{2}\sigma^2)\tau}{2\sigma\tau^{3/2}}$$

since $d_2 = d_1 - \sigma\sqrt{\tau}$, therefore

$$\frac{\partial d_2}{\partial x} = \frac{1}{x\sigma\sqrt{\tau}}, \quad \frac{\partial d_2}{\partial r} = \frac{\sqrt{\tau}}{\sigma}, \quad \frac{\partial d_2}{\partial \sigma} = -\frac{d_2}{\sigma} - \sqrt{\tau}$$

and $$\frac{\partial d_2}{\partial \tau} = \frac{-\log(x/k) + (r-\tfrac{1}{2}\sigma^2)\tau}{2\sigma\tau^{3/2}}$$

Delta:- $C = xN(d_1) - Ke^{-r\tau}N(d_2)$

$$\Delta = \frac{\partial C}{\partial x} = N(d_1) + xN'(d_1)\frac{\partial d_1}{\partial x} - Ke^{-r\tau}N'(d_2)\frac{\partial d_2}{\partial x}$$

$$= N(d_1) + xN'(d_1)\cdot\frac{1}{x}\frac{1}{\sigma\sqrt{\tau}} - Ke^{-r\tau}N'(d_2)\frac{1}{x\sigma\sqrt{\tau}}$$

$$= N(d_1) + \frac{1}{x\sigma\sqrt{\tau}}\left[xN'(d_1) - Ke^{-r\tau}N'(d_2)\right]$$

$$= N(d_1) \quad (\text{by } 1)$$

Gamma:- $\Delta = N(d_1)$

$$\Gamma = \frac{\partial^2 C}{\partial x^2} = \frac{\partial \Delta}{\partial x} = N'(d_1)\frac{\partial d_1}{\partial x} = N'(d_1)\frac{1}{x\sigma\sqrt{\tau}}$$

Rho:- $C = xN(d_1) - Ke^{-r\tau}N(d_2)$

$$\rho = \frac{\partial C}{\partial r} = xN'(d_1)\frac{\partial d_1}{\partial r} + K\tau e^{-r\tau}N(d_2) - Ke^{-r\tau}N'(d_2)\frac{\partial d_2}{\partial r}$$

$$= \frac{x\sqrt{\tau}}{\sigma}N'(d_1) + K\tau e^{-r\tau}N(d_2) - Ke^{-r\tau}\frac{\sqrt{\tau}}{\sigma}N'(d_2)$$

$$\rho = \frac{\sqrt{\tau}}{\sigma}\left[x N'(d_1) - K e^{-r\tau} N'(d_2)\right] + K\tau e^{-r\tau} N(d_2)$$

$$= K\tau e^{-r\tau} N(d_2) \quad (\text{by } 1)$$

Theta:- $\Theta = \dfrac{\partial c}{\partial \tau} = K r e^{-r\tau} N(d_2) + \dfrac{\sigma}{2\sqrt{\tau}} K e^{-r\tau} N'(d_2).$

Vega $= \dfrac{\partial c}{\partial \sigma} = \sqrt{\tau} K e^{-rT} N'(d_2)$

Implied volatility

In the Black-scholes model the input data consists of x, r, T, t, K, σ. Out of these parameters r, T, t, K can be observed directly, the problem of obtaining an estimate of the volatility σ.

Denoting the price of the option by p, the strike price by K, today's observed value of the underlying stock by x. Then by Black-scholes pricing formula

$$p = C(x, t, T, r, \sigma, K).$$

we then solve the above equation for σ.

In other words, we try to find the value σ which the market has implicitly used for valuing the option. The value σ is called the implied volatility.

Implied volatility is different for options having different strike price In fact, this implied volatility is generally a convex function of strike price This curve is known as the volatility smile.