It follows that

$$I(t_1) \sim N\left(0, \int_0^{t_1} \Delta^2(s)\,ds\right) \text{ and}$$

$$I(t_2) - I(t_1) \sim N\left(0, \int_{t_1}^{t_2} \Delta^2(s)\,ds\right)$$

and $I(t_1)$ and $I(t_2) - I(t_1)$ are independent.

$$m(t) = \mathbb{E}[I(t)] = 0.$$

and
$$\begin{aligned}
c(t_1, t_2) &= \mathbb{E}[I(t_1) I(t_2)] \\
&= \mathbb{E}\left[I(t_1)(I(t_2) - I(t_1)) + I^2(t_1)\right] \\
&= \mathbb{E}\left[I(t_1)(I(t_2) - I(t_1))\right] + \mathbb{E}\left[I^2(t_1)\right] \\
&= \mathbb{E}[I(t_1)]\,\mathbb{E}\left[(I(t_2) - I(t_1))\right] + \int_0^{t_1} \Delta^2(s)\,ds. \\
&= \int_0^{t_1} \Delta^2(s)\,ds.
\end{aligned}$$

For general case $c(s,t) = \int_0^{s \wedge t} \Delta^2(u)\,du$.

Brownian Bridge:-

Defn:- Let $W(t)$ be a Brownian motion. Fix $T > 0$, we define the Brownian bridge from 0 to 0 on $[0,T]$ to be the process

$$X(t) = W(t) - t/T\, W(T), \quad 0 \le t \le T.$$

- Note that $t/T\, W(T)$ as a function of t is the line from $(0,0)$ to $(T, W(T))$

- $X(0) = X(T) = 0$
- $X(t)$ is not adapted to the filtration $\mathcal{F}(t)$ generated by the Brownian motion $W(t)$.

For $0 < t_1 < t_2 < \cdots < t_n < T$, the random variables

$$X(t_1) = W(t_1) - t_1/T\, W(T), \quad \cdots \quad X(t_n) = W(t_n) - t_n/T\, W(T).$$

are jointly normal because $W(t_1), \cdots, W(t_n), W(T)$ are jointly normal. Hence the Brownian Bridge from 0 to 0 is a Gaussian process.

Its mean function is

$$m(t) = \mathbb{E}[X(t)] = \mathbb{E}[W(t) - t/T\, W(T)] = 0$$

For $s, t \in (0, T)$ the covariance function is

$$\begin{aligned} C(s,t) &= \mathbb{E}\left[\left(W(s) - s/T\, W(T)\right)\left(W(t) - t/T\, W(T)\right)\right] \\ &= \mathbb{E}[W(s)W(t)] - t/T\, \mathbb{E}[W(s)W(T)] - s/T\, \mathbb{E}[W(t)W(T)] \\ &\quad + \frac{st}{T^2}\mathbb{E}[W^2(T)]. \\ &= s \wedge t - \frac{ts}{T} - \frac{ts}{T} + \frac{st}{T^2} \cdot T \\ &= s \wedge t - \frac{st}{T}. \end{aligned}$$

Definition:- Let $W(t)$ be a Brownian motion. Fix $T > 0$, $a \in \mathbb{R}$ and $b \in \mathbb{R}$. We define the Brownian bridge from a to b on $[0, T]$ to be the process

$$X^{a \to b} = a + \frac{(b-a)t}{T} + X(t), \quad 0 \le t \le T,$$

where $X(t) = X^{0\to 0}$ is the Brownian bridge from 0 to 0.

$$X^{a\to b}(0) = a \text{ and } X^{a\to b}(T) = b.$$

Adding a non-random function to a Gaussian process gives us another Gaussian process. The mean function is

$$m^{a\to b}(t) = \mathbb{E}\left[X^{a\to b}(t)\right] = a + \frac{(b-a)}{T}t$$

covariance function is

$$c^{a\to b}(s,t) = \mathbb{E}\left[\left(X^{a\to b}(s) - m^{a\to b}(s)\right)\left(X^{a\to b}(t) - m^{a\to b}(t)\right)\right]$$

$$= \mathbb{E}\left[X(s)X(t)\right]$$

$$= s\wedge t - \frac{st}{T}.$$

Brownian Bridge as a Scaled stochastic Integral:-

We cannot write the Brownian bridge as a stochastic integral of a deterministic integrand because the variance of the Brownian Bridge,

$$\mathbb{E}\left[X^2(t)\right] = C(t,t) = t - t^2/T = \frac{(T-t)t}{T}$$

increases for $0 \le t \le T/2$ and decreases for $T/2 \le t \le T$.

The variance of $I(t) = \int_0^t \Delta(s)ds$ is $\int_0^t \Delta^2(s)ds$ which is non-decreasing in t.

However, we can obtain a process with the same distribution as the Brownian bridge from 0 to 0 as a scaled integral. In particular, consider

$$Y(t) = (T-t)\int_0^t \frac{1}{T-u}\,dW(u), \quad 0 \le t \leqslant T$$

The integral $I(t) = \int_0^t \frac{1}{T-u}\,dW(u)$ is a Gaussian process for $t \in [0,T]$.

For $0<t_1<t_2<\cdots<t_n<T$, the random variables $Y(t_1) = (T-t_1)I(t_1)$, $Y(t_2) = (T-t_2)I(t_2)$, $\cdots$ $Y(t_n) = (T-t_n)I(t_n)$ are jointly normal because $I(t_1), \cdots, I(t_n)$ are jointly normal. In particular $Y(t)$ is a Gaussian process.

The mean and covariance function of I are $m^I(t) = 0$.

$$C^I(s,t) = \int_0^{s\wedge t} \frac{1}{(T-u)^2}\,du = \frac{1}{T-(s\wedge t)} - \frac{1}{T} \quad \forall\, s,t \in [0,T).$$

This means that the mean function for Y is $m^Y(t) = 0$.

To compute the covariance function for Y, we assume for the moment that $0 \le s \le t \le T$. so that

$$C^I(s,t) = \frac{1}{T-s} - \frac{1}{T} = \frac{s}{T(T-s)}.$$

Then $C^Y(s,t) = \mathbb{E}\left[(T-s)(T-t)\,I(s)I(t)\right]$

$$= (T-s)(T-t)\frac{s}{T(T-s)} = \frac{(T-t)s}{T} = s - \frac{st}{T}.$$

In general

$$c^Y(s,t) = s \wedge t - \frac{st}{T} \quad \text{for } s,t \in [0,T)$$

This is the same covariance formula we obtained for the Brownian bridge. Because the mean and covariance function for a Gaussian process completely determine the distribution of the process. We conclude that the process Y has the same distribution as Brownian bridge from 0 to 0 on $[0,T]$.

We now consider the variance

$$\mathbb{E}[Y^2(t)] = c^Y(t,t) = \frac{t(T-t)}{T}$$

Note that as $t \uparrow T$, this variance converges to 0. In other words, as $t \uparrow T$, the random process $Y(t)$, which has mean zero, has a variance that converges to zero. This observation suggests that it makes sense to define $Y(T) = 0$. If we do that, then $Y(t)$ is continuous at $t = T$. We summarize this discussion with the following theorem.

Theorem:- Define the process

$$Y(t) = \begin{cases} (T-t)\int_0^t \frac{1}{T-u}\, dW(u) & \text{for } 0 \le t < T \\ 0 & \text{for } t = T. \end{cases}$$

Then $Y(t)$ is a continuous Gaussian process on $[0,T]$ and has mean and covariance functions

$$m^Y(t) = 0 \quad , t \in [0,T].$$

$$c^Y(s,t) = s \wedge t - \frac{st}{T} \text{ for all } s,t \in [0,T].$$

In particular, the process $Y(t)$ has the same distribution as the Brownian bridge from 0 to 0 on $[0,T]$.

Note that $Y(t)$ is adapted to the filtration generated by the Brownian motion $W(t)$

$$dY(t) = \int_0^t \frac{1}{T-u}\, dW(u) \cdot d(T-t) + (T-t) \cdot d\int_0^t \frac{1}{T-u}\, dW(u)$$

$$= -\int_0^t \frac{1}{T-u}\, dW(u) \cdot dt + (T-t) \cdot \frac{1}{(T-t)}\, dW(t)$$

$$= -\frac{Y(t)}{(T-t)}\, dt + dW(t)$$

Multidimensional Distribution of the Brownian Bridge:-

We fix $a \in \mathbb{R}$ and $b \in \mathbb{R}$ and let $X^{a\to b}(t)$ denote the Brownian bridge from a to b on $[0,T]$. We also fix $0 = t_0 < t_1 < \cdots < t_n < T$. We compute the joint density of $X^{a\to b}(t_1), \cdots, X^{a\to b}(t_n)$.