(4) A portfolio - consumption pair (h, c) is called self-financing if the value process V^h satisfies the condition

$$dV_t^h = \sum_{i=1}^{N} h_t^i \{dS_t^i + dD_t^i\} - c_t dt,$$

i.e., if $\quad dV_t^h = h_t dS_t + h_t dD_t - c_t dt.$

(5) The portfolio h is said to be Markovian if it is of the form $\quad h_t = h(t, S(t))$

for some function $h : \mathbb{R}_+ \times \mathbb{R}^N \to \mathbb{R}^N$.

Defn:- For a given portfolio h the corresponding relative portfolio or portfolio weights ω is given by

$$\omega_t^i = \frac{h_t^i S_t^i}{V_t^h}, \quad i = 1, 2, \cdots, N$$

where we have $\displaystyle\sum_{i=1}^{N} \omega_t^i = 1$.

Lemma:- A portfolio - consumption pair (h, c) is self-financing if and only if

$$dV_t^h = V_t^h \cdot \sum \omega_t^i \frac{dS_t^i + dD_t^i}{S_t^i} - c_t dt.$$

Remark:- If $D_t \equiv 0$ and $c_t \equiv 0$ then

$$dV_t^h = h_t dS_t = \sum_{i=1}^{N} h_t^i dS_t^i$$

and $\displaystyle dV_t^h = V_t^h \cdot \sum_{i=1}^{N} \omega_t^i \frac{dS_t^i}{S_t^i}$

Let us consider a financial market consisting of only two assets: a risk free asset with price process $B(t)$ and a stock with price process $S(t)$.

The price process $B(t)$ is the price of a risk free asset it has the dynamics

$$dB(t) = r(t)B(t)dt$$

where $r(t)$ is any adapted process.

$$\Rightarrow B(t) = B(0) \exp\left\{ \int_0^t r(s)ds \right\}.$$

We assume that the stock price $S(t)$ is given by

$$dS(t) = S(t)\,\mu(t, S(t))dt + S(t)\,\sigma(t, S(t))\,dW(t).$$

where W is a Wiener Process (Brownian motion) and μ and σ are given functions. The function σ known as the volatility of S, and μ is the local mean rate of return of S.

<u>Def$\underline{n}$:-</u> consider a financial market with vector price process S. A contingent claim with date of maturity (exercise date) T, also called a T-claim, is any random variable $\mathcal{X} \in \mathcal{F}_T^S$, where $\mathcal{F}_T^S = \sigma\{S_t : 0 \le t \le T\}$. A contingent claim $\mathcal{X}$ is called a simple claim if it is of the form $\mathcal{X} = \phi(S(T))$. The function ϕ is called the contract function.

<u>Example:-</u> The European call is a simple contingent claim, for which the contract function is given by

$$\phi(x) = \max\{x - K, 0\}.$$

- our main problem is to determine a fair price for the claim, and we will use the notation $\Pi(t,\mathcal{X})$ for the price process of the claim $\mathcal{X}$, where we suppress the $\mathcal{X}$. In the case of a simple claim we will sometimes write $\Pi(t,\Phi)$.

Note that for any contingent claim $\mathcal{X}$ we have the relation

$$\Pi(T,\mathcal{X}) = \mathcal{X}.$$

and in the particular case of a simple claim $\mathcal{X}=\Phi(S(T))$

$$\Pi(T,\mathcal{X}) = \Phi(S(T)).$$

Defn:- An arbitrage possibility on a financial market is a self-financed portfolio h such that

$$V^h(0)=0$$

$$\mathbb{P}(V^h_T(T)\geq 0)=1 \text{ and } \mathbb{P}(V^h(T)>0)>0.$$

We say that the market is arbitrage free if there are no arbitrage possibilities.

Assumption:- We assume that the price process $\Pi(t,\mathcal{X})$ is such that there are no arbitrage possibilities on the market consisting of $(B(t), S(t), \Pi(t,\mathcal{X}))$.

We interpret an arbitrage possibility as a serious case of mispricing in the market, and our main assumption is that the market is efficient in the sense that no arbitrage is possible

- A natural question is how we can identify an arbitrage possibility.

Proposition:- suppose that there exists a self-financing portfolio h, such that the value process V^h has the dynamics

$$dV^h(t) = K(t)V^h(t)dt,$$

where K is an adapted process. Then it must hold that $K(t) = r(t)$ for all t, or there exists an arbitrage possibility.

Proof:- For simplicity assume that K and r are constant and $K > r$. Then we can borrow money from the bank at the rate r. This money is immediately invested in the portfolio strategy h where it will grow at the rate K with $K > r$.

Thus the net investment at $t = 0$ is zero, whereas our wealth at any time $t > 0$ will be positive. In other words we have an arbitrage.

On the other hand if $r > K$, we sell the portfolio h short and invest this money in the bank, and again there is an arbitrage.

The main point of the above is that if a portfolio has a value process whose dynamics contain no driving Wiener process, i.e., a locally risk free portfolio, then the rate of return of that portfolio must equal to the short rate of interest.

The Black-scholes Equation:-

We assume that the ~~given~~ market consists of two assets with price dynamics given by

$$dB(t) = r B(t)\, dt$$

$$dS(t) = S(t)\, \mu(t, S(t))\, dt + S(t)\, \sigma(t, S(t))\, dW(t)$$

We consider a simple contingent claim $\mathcal{X} = \Phi(S(T))$

We assume that this claim can be traded on a market and its price process $\Pi(t, \Phi)$ has the form

$$\Pi(t, \Phi) = F(t, S(t)) := \pi(t)$$

for some smooth function F.

$$d\pi(t) = dF(t, S(t)) = F_t(t, S(t))\, dt + F_x(t, S(t))\, dS(t) + \tfrac{1}{2} F_{xx}(t, S(t))\, dS(t)\, dS(t)$$

$$= \left(F_t + \mu S F_x + \tfrac{1}{2}\sigma^2 S^2 F_{xx}\right)(t, S(t))\, dt + (\sigma S F_x)(t, S(t))\, dW(t)$$

Then $\quad d\pi(t) = \mu_\pi(t)\, \pi(t)\, dt + \sigma_\pi(t)\, \pi(t)\, dW(t)$

where $\quad \mu_\pi(t) = \dfrac{F_t + \mu S F_x + \frac{1}{2}\sigma^2 S^2 F_{xx}}{F}$

$$\sigma_\pi(t) = \frac{\sigma S F_x}{F} \text{ (shorthand notation)}$$

$$= \frac{\sigma(t, S(t))\, S(t)\, F_x(t, S(t))}{F(t, S(t))}.$$

Let us now form a portfolio based on two assets: the underlying stock and the derivative asset. Denoting the relative portfolio by (u_S, u_π). Then the value V of the portfolio is

$$dV(t) = V(t)\left\{u_S[\mu dt + \sigma dW] + u_\pi[\mu_\pi dt + \sigma_\pi dW]\right\}$$

$$\Rightarrow dV(t) = V(t)\left[u_S\mu + u_\pi\mu_\pi\right]dt + V(t)\left[u_S\sigma + u_\pi\sigma_\pi\right]dW$$

The only restriction on the relative portfolio is that we must have $u_S + u_\pi = 1$

Let us define the relative portfolio by the linear system of equations

$$\left.\begin{array}{l} u_S + u_\pi = 1 \\ u_S\sigma + u_\pi\sigma_\pi = 0 \end{array}\right\} \Rightarrow \begin{array}{l} u_S = \dfrac{\sigma_\pi}{\sigma_\pi - \sigma} \\ u_\pi = \dfrac{-\sigma}{\sigma_\pi - \sigma} \end{array}$$

Then $dV(t) = V(t)\left[u_S\mu + u_\pi\mu_\pi\right]dt$

Thus we have obtained a locally riskless portfolio and the market is arbitrage free, hence we must have

$$u_S\mu + u_\pi\mu_\pi = r \quad \text{---} (*)$$

Also we have $u_S = \dfrac{\sigma_\pi}{\sigma_\pi - \sigma} = \dfrac{\sigma S F_x/F}{\sigma S F_x/F - \sigma} = \dfrac{S(t)F_x(t,S(t))}{S(t)F_x(t,S(t)) - F(t,S(t))}$. --- ①

$$u_\pi = \frac{-\sigma}{\sigma_\pi - \sigma} = \frac{-F(t,S(t))}{S(t)F_x(t,S(t)) - F(t,S(t))} \quad \text{---} ②$$

Now we substitute ① & ② into into ⊗, then we obtain

$$\frac{SF_x}{SF_x - F}\mu + \left(\frac{-\cancel{F}}{SF_x - F}\right)\left(\frac{F_t + \mu S F_x + \frac{1}{2}\sigma^2 S^2 F_{xx}}{\cancel{F}}\right) = r$$

$$\Rightarrow \cancel{\mu S F_x} - F_t - \cancel{\mu S F_x} - \tfrac{1}{2}\sigma^2 S^2 F_{xx} = rSF_x - rF$$

$$\Rightarrow F_t + rSF_x + \tfrac{1}{2}\sigma^2 S^2 F_{xx} = rF$$

Also, we must have the relation

$$\Pi(T, \Phi) = \Phi(S(T)).$$

So, F has to satisfy the following PDE

$$\left.\begin{aligned} F_t(t,x) + rxF_x(t,x) + \tfrac{1}{2}\sigma^2 x^2 F_{xx}(t,x) &= rF(t,x) \\ F(T,x) &= \Phi(x) \end{aligned}\right\}.$$

Definition:- We say that a T-claim $\mathcal{X}$ can be replicated, alternatively that it is reachable or hedgeable, if there exists a self-financing portfolio h such that

$$V_T^h = \mathcal{X}$$

In this case we say that h is a hedge against $\mathcal{X}$. Alternatively, h is called a replicating or hedging portfolio. If every contingent claim is reachable we say that the market is complete.

Meta-theorem:- Let M denote the numbers of underlying traded assets in the model excluding the risk-free asset, and let R denote the number of random sources. Generically we then have the following relations:-

① The model is arbitrage free iff $M \le R$

② The model is complete if and only if $M \ge R$

③ The model is complete and arbitrage free iff $M = R$.

As an example we take the Black-Scholes model, where we have one underlying asset S plus the risk-free asset so $M=1$. we have one driving wiener process, giving us $R=1$, so in fact $M=R$. using the meta-theorem above we expect the Black-scholes model to be arbitrage free ass well as complete and this is indeed the case.

Incomplete Market:-

We know from the meta-theorem that makets generically are imcomplete when there are more random sources than there are traded assets and this can occur in an ~~ifn~~ infinite number of ways, so there is no "canonical" way of writting down a model of an incomplete market. Here we study a particular type of imcomple market, namely a "factor model", i.e., a market where there are some non-traded underlying objects. Before we go on to the formal description of the model let us briefly recall what we may expect in an incomplete market model

- since the market is imcomplete we will not be able to hedge a generic contingent claim.
- In particular there will not be a unique price for a generic derivative.

Here we consider a simplest possible imcomplete market, namely a market where the only randomnesss comes from a stochastic process $X(t)$ which is not the price of a traded asset. The model is as follows

$$\begin{cases} dX(t) = \mu(t, X(t))\,dt + \sigma(t, X(t))\,dW(t) \\ dB(t) = r B(t)\,dt \end{cases}$$