Exotic options:-

European calls and puts are called vanilla or plain-vanilla options. Their payoffs depend only on the final value of the underlying asset. Options whose payoffs depend on the path of the underlying asset are called path-dependent or exotic. Here we consider only three types of exotic options, namely barrier options, lookback options and Asian options.

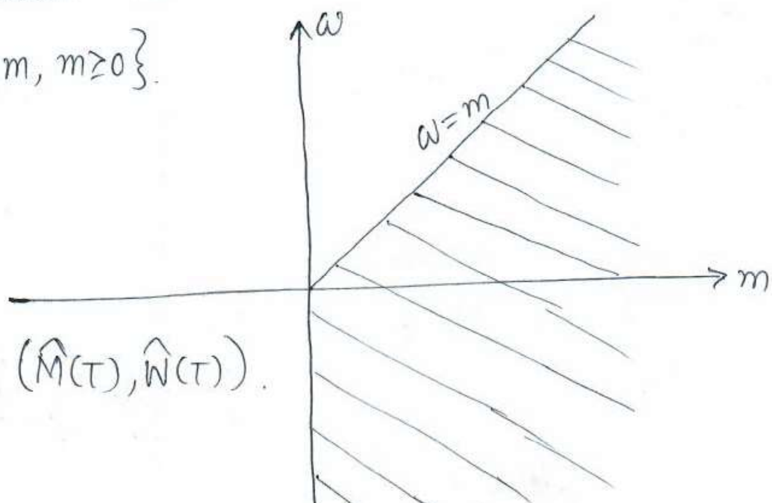
Maximum of Brownian Motion with Drift:-

Let $\widetilde{W}(t)$, $0 \le t \le T$ be a Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let α be a given number and define

$$\widehat{W}(t) = \alpha t + \widetilde{W}(t) \quad 0 \le t \le T$$

and $$\widehat{M}(T) = \max_{0 \le t \le T} \widehat{W}(t)$$

Because $\widehat{W}(0) = 0$, we have $\widehat{M}(T) \ge 0$, we also have $\widehat{W}(T) \le \widehat{M}(T)$. Therefore the random variables $(\widehat{M}(T), \widehat{W}(T))$ takes values in the set $\{(m, \omega) : \omega \le m, m \ge 0\}$.



Range of $(\widehat{M}(T), \widehat{W}(T))$.

Theorem:- The joint density under $\widetilde{\mathbb{P}}$ of the pair $(\widehat{M}(T), \widehat{W}(T))$ is

$$\widehat{f}_{\widehat{M}(T), \widehat{W}(T)}(m, \omega) = \frac{2(2m-\omega)}{T\sqrt{2\pi T}} e^{\alpha\omega - \frac{1}{2}\alpha^2 T - \frac{1}{2T}(2m-\omega)^2}, \quad \omega \le m, \ m \ge 0$$

and is zero for other values of m and ω.

Proof:- we define the exponential martingale

$$\hat{Z}(t) = e^{-\alpha \widehat{W}(t) - \frac{1}{2}\alpha^2 t} = e^{-\alpha \widehat{W}(t) + \frac{1}{2}\alpha^2 t}, \quad 0 \le t \le T.$$

and use $\hat{Z}(T)$ to define a new probability measure $\widehat{\mathbb{P}}$ by

$$\widehat{\mathbb{P}}(A) = \int_A Z(T)\, d\widetilde{\mathbb{P}}, \quad \forall A \in \mathcal{F}.$$

According to Girsanov's theorem, $\widehat{W}(t)$ is a Brownian motion under $\widehat{\mathbb{P}}$. We know that the joint density of $(\widehat{M}(T), \widehat{W}(T))$ under $\widehat{\mathbb{P}}$ is

$$\hat{f}_{\widehat{M}(T), \widehat{W}(T)}(m, \omega) = \frac{2(2m-\omega)}{T\sqrt{2\pi T}} e^{-\frac{1}{2T}(2m-\omega)^2}, \quad \omega \le m,\ m \ge 0.$$

and is zero for other values of m and ω.

Now

$$\widetilde{\mathbb{P}}\{\widehat{M}(T) \le m, \widehat{W}(T) \le \omega\} = \widetilde{\mathbb{E}}\left[\mathbb{I}_{\{\widehat{M}(T) \le m,\, \widehat{W}(T) \le \omega\}}\right]$$

$$= \widehat{\mathbb{E}}\left[\frac{1}{\hat{Z}(T)} \mathbb{1}_{\{\widehat{M}(T) \le m,\, \widehat{W}(T) \le \omega\}}\right]$$

$$= \widehat{\mathbb{E}}\left[e^{\alpha \widehat{W}(T) - \frac{1}{2}\alpha^2 T} \mathbb{1}_{\{\widehat{M}(T) \le m,\, \widehat{W}(T) \le \omega\}}\right]$$

$$= \int_{-\infty}^{\omega} \int_{-\infty}^{m} e^{+\alpha y - \frac{1}{2}\alpha^2 T} \hat{f}_{\widehat{M}(T), \widehat{W}(T)}(x, y)\, dx\, dy.$$

Therefore the density of $(\widehat{M}(T), \widehat{W}(T))$ under $\widetilde{\mathbb{P}}$ is.

$$\frac{\partial^2}{\partial m \partial \omega} \widetilde{\mathbb{P}}\{\widehat{M}(T) \le m, \widehat{W}(T) \le \omega\} = e^{\alpha \omega - \frac{1}{2}\alpha^2 T} \hat{f}_{\widehat{M}(T), \widehat{W}(T)}(m, \omega).$$

$$= \frac{2(2m-\omega)}{T\sqrt{2\pi T}} e^{\alpha \omega - \frac{1}{2}\alpha^2 T - \frac{1}{2T}(2m-\omega)^2}$$

Corollary:- we have

$$\widetilde{\mathbb{P}}\{\widehat{M}(T)\le m\} = N\left(\frac{m-\alpha T}{\sqrt{T}}\right) - e^{2\alpha m} N\left(\frac{-m-\alpha T}{\sqrt{T}}\right), \quad m\ge 0 \quad \text{---} (*)$$

and the density under $\widetilde{\mathbb{P}}$ of the random variable $\widehat{M}(T)$ is

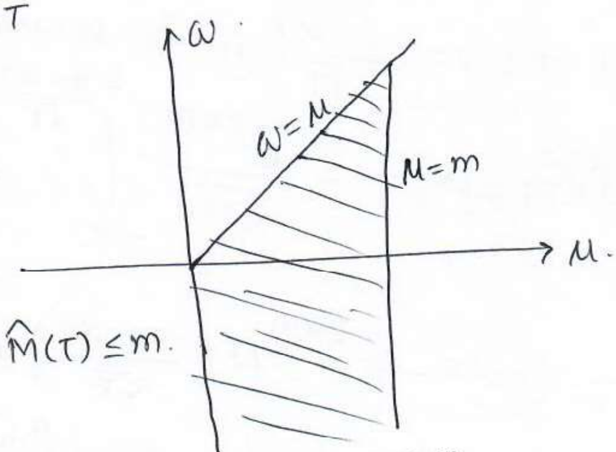
$$\widetilde{f}_{\widehat{M}(T)}(m) = \frac{2}{\sqrt{2\pi T}} e^{-\frac{1}{2T}(m-\alpha T)^2} - 2\alpha e^{2\alpha m} N\left(\frac{-m-\alpha T}{\sqrt{T}}\right), \quad m\ge 0.$$

and is zero for $m<0$. --- (**)

Proof:- $\widetilde{\mathbb{P}}\{\widehat{M}(T)\le m\}$.

$$= \int_{\omega=0}^{\omega=m}\int_{\mu=\omega}^{\mu=m} \frac{2(2\mu-\omega)}{T\sqrt{2\pi T}} e^{\alpha\omega-\frac{1}{2}\alpha^2 T-\frac{1}{2T}(2\mu-\omega)^2}\, d\mu\, d\omega$$

$$+ \int_{\omega=-\infty}^{\omega=0}\int_{\mu=0}^{\mu=m} \frac{2(2\mu-\omega)}{T\sqrt{2\pi T}} e^{\alpha\omega-\frac{1}{2}\alpha^2 T-\frac{1}{2T}(2\mu-\omega)^2}\, d\mu\, d\omega$$



The region $\widehat{M}(T)\le m$.

$$= -\int_0^m \frac{1}{\sqrt{2\pi T}} e^{\alpha\omega-\frac{1}{2}\alpha^2 T-\frac{1}{2T}(2\mu-\omega)^2}\Big|_{\mu=\omega}^{\mu=m} d\omega$$

$$-\int_{-\infty}^0 \frac{1}{\sqrt{2\pi T}} e^{\alpha\omega-\frac{1}{2}\alpha^2 T-\frac{1}{2T}(2\mu-\omega)^2}\Big|_{\mu=0}^{\mu=m} d\omega$$

$$= -\frac{1}{\sqrt{2\pi T}}\int_0^m e^{\alpha\omega-\frac{1}{2}\alpha^2 T-\frac{1}{2T}(2m-\omega)^2} d\omega + \int_0^m \frac{1}{\sqrt{2\pi T}} e^{\alpha\omega-\frac{1}{2}\alpha^2 T-\frac{1}{2T}\omega^2} d\omega$$

$$-\frac{1}{\sqrt{2\pi T}}\int_{-\infty}^0 e^{\alpha\omega-\frac{1}{2}\alpha^2 T-\frac{1}{2T}(2m-\omega)^2} d\omega + \int_{-\infty}^0 \frac{1}{\sqrt{2\pi T}} e^{\alpha\omega-\frac{1}{2}\alpha^2 T-\frac{1}{2T}\omega^2} d\omega.$$

$$= -\frac{1}{\sqrt{2\pi T}}\int_{-\infty}^{m} e^{\alpha\omega - \frac12\alpha^2 T - \frac{1}{2T}(2m-\omega)^2}\,d\omega + \frac{1}{\sqrt{2\pi T}}\int_{-\infty}^{m} e^{\alpha\omega - \frac12\alpha^2 T - \frac{1}{2T}\omega^2}\,d\omega .$$

Observe that

$$-\frac{1}{2T}(\omega-2m-\alpha T)^2 = -\frac{(2m-\omega)^2}{2T} + \alpha\omega - 2\alpha m - \tfrac12\alpha^2 T$$

and $-\frac{1}{2T}(\cancel{\alpha}\,\omega-\alpha T)^2 = -\frac{\omega^2}{2T} + \alpha\omega - \frac12\alpha^2 T$

Therefore

$$\widetilde{\mathbb{P}}(\hat{M}(T)\le m) = -\frac{e^{2\alpha m}}{\sqrt{2\pi T}}\int_{-\infty}^{m} e^{-\frac{1}{2T}(\omega-2m-\alpha T)^2}\,d\omega$$

$$+\frac{1}{\sqrt{2\pi T}}\int_{-\infty}^{m} e^{-\frac{1}{2T}(\omega-\alpha T)^2}\,d\omega .$$

we make the change of variable $y = \frac{\omega-2m-\alpha T}{\sqrt{T}}$ in the first integral and $y = \frac{\omega-\alpha T}{\sqrt{T}}$ in the second integral, we obtain

$$\widetilde{\mathbb{P}}(\hat{M}(T)\le m) = -\frac{e^{2\alpha m}}{\sqrt{2\pi}}\int_{-\infty}^{\frac{-m-\alpha T}{\sqrt{T}}} e^{-\frac12 y^2}\,dy + \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\frac{m-\alpha T}{\sqrt{T}}} e^{-\frac12 y^2}\,dy .$$

$$= -e^{2\alpha m}N\left(\frac{-m-\alpha T}{\sqrt{T}}\right) + N\left(\frac{m-\alpha T}{\sqrt{T}}\right). \quad \text{----} \ (1)$$

To obtain the density, we differentiate (1) w.r.t. m

$$\frac{d}{dm}\widetilde{\mathbb{P}}(\hat{M}(T)\le m) = N'\left(\frac{m-\alpha T}{\sqrt{T}}\right)\cdot\frac{1}{\sqrt{T}} - 2\alpha e^{2\alpha m}N\left(\frac{-m-\alpha T}{\sqrt{T}}\right)$$

$$- e^{2\alpha m}N'\left(\frac{-m-\alpha T}{\sqrt{T}}\right)\left(-\frac{1}{\sqrt{T}}\right)$$

$$= \frac{1}{\sqrt{2\pi T}}e^{-\frac{1}{2T}(m-\alpha T)^2} - 2\alpha e^{2m\alpha}N\left(\frac{-m-\alpha T}{\sqrt{T}}\right) + \frac{e^{2\alpha m}}{\sqrt{2\pi T}}e^{-\frac{1}{2T}(-m-\alpha T)^2}$$

The exponent in the third term is

$$2\alpha m - \frac{(m+\alpha T)^2}{2T} = \frac{4\alpha mT - m^2 - 2\alpha mT - \alpha^2T^2}{2T} = \frac{-(m-\alpha T)^2}{2T}$$

which is the exponent in the first term. Combining the first and third terms we obtain (**)

Barrier options:-

A Barrier options is a path-dependent option whose payoff at maturity depends on whether the underlying asset's price reaches some pre-defined barrier during the life of the option

Most common barrier options are namely up-and-out, up-and-in, down-and-out, down-and-in call or put options:

A down-and-out option has the barrier below the initial asset price and knocks out if the asset price fall below the barrier.

A up-and-out option has the barrier above the initial asset price and knocks out if the asset price cross above the barrier (i.e., it becomes worthless).

Let $S(t)$, $0 \le t \le T$ be the underlying asset price process. Define

$$M_S = \max_{0 \le t \le T} S(t) \quad \text{and} \quad m_S = \min_{0 \le t \le T} S(t).$$

- B is the pre-determined barrier

For an up-and-out call option

- $S(0) < B$
- payoff at maturity is $= (S(T)-K)^+ \mathbb{1}_{\{M_S \le B\}}$.

For down-and-out call option

- $S(0) > B$
- payoff at maturity is $= (S(T)-K)^+ \mathbb{1}_{\{m_S \ge B\}}$.

A knock-in option comes into existence option only when the underlying asset price reaches a barrier.

For up-and-in call option

- $S(0)$ ~~B~~ $< B$
- pay-off at maturity is $= (S(T)-K)^+ \mathbb{1}\{M_S \geq B\}$.

For down-and-in call option

- $S(0) > B$
- pay-off at maturity is $= (S(T)-K)^+ \mathbb{1}\{m_S \leq B\}$.

- <u>up-and-out call:-</u>

Our underlying asset is geometric Brownian motion

$$dS(t) = r S(t) dt + \sigma S(t) d\widetilde{W}(t)$$

Where $\widetilde{W}(t)$, $0 \leq t \leq T$ is a Brownian motion under the risk-neutral measure $\widetilde{\mathbb{P}}$. Consider a European call, expiring at time T, with strike price K and up-and-out barrier B. We assume that $K < B$, otherwise the option must be knock out (in order to be in the money and hence could only pay off zero). The solution to the stochastic differential equation for the asset price is

$$S(t) = S(0) e^{\sigma \widetilde{W}(t) + (r - \frac{1}{2}\sigma^2)t} = S(0) e^{\sigma \widehat{W}(t)}$$

where $\widehat{W}(t) = \alpha t + \widetilde{W}(t)$, and $\alpha = \frac{1}{\sigma}(r - \frac{1}{2}\sigma^2)$

We define $\widehat{M}(T) = \max_{0 \leq t \leq T} \widehat{W}(t)$, so.

$$\max_{0 \leq t \leq T} S(t) = S(0) e^{\sigma \widehat{M}(T)}$$

The option knocks out if and only if $S(0) e^{\sigma \widehat{M}(T)} > B$

The pay off of the option is $\overset{V(T)}{=} (S(T)-K)^+ \mathbb{1}_{\{S(0)e^{\sigma \widehat{M}(T)} \leq B\}}$.

$$V(T) = \left(S(0)e^{\sigma\widehat{W}(T)} - K\right)^+ \mathbb{1}_{\{S(0)e^{\sigma\widehat{M}(T)} \le B\}}.$$

$$= \left(S(0)e^{\sigma\widehat{W}(T)} - K\right) \mathbb{1}_{\{S(0)e^{\sigma\widehat{W}(T)} \ge K\}} \mathbb{1}_{\{S(0)e^{\sigma\widehat{M}(T)} \le B\}}.$$

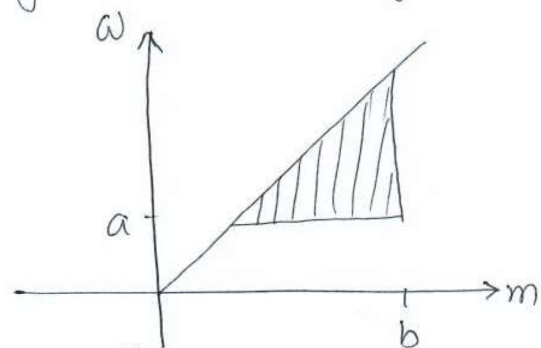
$$= \left(S(0)e^{\sigma\widehat{W}(T)} - K\right) \mathbb{1}_{\{S(0)e^{\sigma\cdot\widehat{W}(T)} \ge K,\ S(0)e^{\sigma\widehat{M}(T)} \le B\}}.$$

$$= \left(S(0)e^{\sigma\widehat{W}(T)} - K\right) \mathbb{1}_{\{\widehat{W}(T) \ge a,\ \widehat{M}(T) \le b\}}.$$

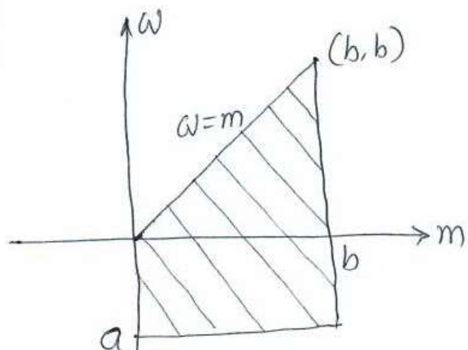
where $a = \frac{1}{\sigma}\log\frac{K}{S(0)}$ and $b = \frac{1}{\sigma}\log B/S(0)$.

The risk-neutral price at time zero of the up-and-out call is $V(0) = \widetilde{\mathbb{E}}\left[e^{-rT}V(T)\right]$.

If $a \ge 0$, we must integrate over the region $\{(m,\omega) : a \le \omega \le m \le b\}$



If $a < 0$, we integrate over the region $\{(m,\omega) : a \le \omega \le m,\ 0 \le m \le b\}$



In both cases the region can be described as
$\{(m,\omega) ;\ a \le \omega \le b,\ \omega^+ \le m \le b\}$.

Note that here $S(0) \le B \Rightarrow b \ge 0$.