Martingale Models for the short rate:-

~~Here we~~ In the literature there are a large number of proposals on how to specify the Q-dynamics of r. We present a list of the most popular short rate models

① Vasicek

$$dr(t) = (b - ar(t))dt + \sigma\, dW(t), \quad (a>0)$$

② Cox-Ingersoll-Ross (CIR)

$$dr(t) = (b - ar(t))dt + \sigma\sqrt{r(t)}\, dW(t)$$

③ Dothan

$$dr(t) = ar\,dt + \sigma r\, dW(t)$$

④ Black-Derman-Toy

$$dr(t) = \theta(t) r(t)\, dt + \sigma(t) r(t)\, dW(t)$$

(5) Ho-Lee

$$dr(t) = \theta(t)\, dt + \sigma\, dW(t)$$

(6) Hull-White (extended Vasicek)

$$dr(t) = [\theta(t) - ar(t)]dt + \sigma\, dW(t) \quad (a(t)>0)$$

⑦ Hull-White (extended CIR)

$$dr(t) = [\theta(t) - a\, r(t)]dt + \sigma\sqrt{r(t)}\, dW(t) \quad (a(t)>0).$$

Remark:- The term structure, as well as the prices of all other interest rate derivatives, are completely determined by specifying the r-dynamics under the martingale measure Q.

As we saw, the term structure (i.e., the family of bond price processes) will, together with all other derivatives, be completely determined by the general term structure equation

$$\begin{cases} F_t + \{\mu - \sigma\lambda\} F_r + \frac{1}{2}\sigma^2 F_{rr} - rF = 0 \\ F(T, r) = \phi(r) \end{cases}$$

as soon as we have specified the following objects:

- The drift term μ
- The diffusion term σ
- The market price of risk λ.

We recall that the term $\mu - \sigma\lambda$ is precisely the drift term of the short rate under the martingale measure. ~~More~~ In particular the Q-dynamics of $r(t)$ is given by

$$dr(t) = \{\mu - \lambda\sigma\} dt + \sigma dW^Q(t).$$

Affine term structures:-

Defn:- If the term structure $\{p(t,T);\ 0 \le t \le T,\ T > 0\}$ has the form

$$p(t,T) = F(t, r(t); T),$$

where F has the form $F(t,r,T) = e^{A(t,T) - B(t,T)r}$,

where A and B are deterministic fns, then the model is said to possess an affine term structure (ATS).

Assume that we have the Q-dynamics

$$dr(t) = \mu(t, r(t))\,dt + \sigma(t, r(t))\,dW(t)$$

and assume that this model actually possesses an ATS.

since F must solve the term structure equation, we thus obtain

$$A_t(t,T)F - B_t(t,T)rF + \mu(-B(t,T))F$$
$$+\tfrac{1}{2}\sigma^2 B^2(t,T)F - rF = 0$$

$$\Rightarrow A_t(t,T) - (1 + B_t(t,T))r - \mu(t,r)B(t,T)$$
$$+\tfrac{1}{2}\sigma^2(t,r)B^2(t,T) = 0 \quad \text{---} \; (*)$$

The boundary value $F(T,r;T) = 1$ implies

$$A(T,T) = 0 = B(T,T)$$

Assume that μ and σ have the form

$$\mu(t,r) = \alpha(t)r + \beta(t)$$
$$\sigma(t,r) = \sqrt{\gamma(t)r + \delta(t)}$$

Then $(*)$ transforms into

$$A_t(t,T) - \beta(t)B(t,T) + \tfrac{1}{2}\delta(t)B^2(t,T)$$
$$- \left\{1 + B_t(t,T) + \alpha(t)B(t,T) - \tfrac{1}{2}\gamma(t)B^2(t,T)\right\}r = 0. \quad \text{---} \; (**)$$

This equation holds for all t, T and r, so let us consider it for a fixed choice of T and t. Since the equation holds for all values of r the coefficient of r must be equal to zero.

Thus we have the eauation

$$B_t(t,T) + \alpha(t)B(t,T) - \tfrac{1}{2}\gamma(t)B^2(t,T) = -1.$$

and from ⊛ ⊛⊛ we get

$$A_t(t,T) = \beta(t)B(t,T) - \tfrac{1}{2}\delta(t)B^2(t,T).$$

Proposition:- Assume that μ and σ are of the form

$$\begin{cases} \mu(t,r) = \alpha(t)r + \beta(t) \\ \sigma(t,r) = \sqrt{\gamma(t)r + \delta(t)}. \end{cases}$$

Then the model admits an ATS of the form ①, where A and B satisfy the system

$$\begin{cases} B_t(t,T) + \alpha(t)B(t,T) - \tfrac{1}{2}\gamma(t)B^2(t,T) = -1 \\ B(T,T) = 0 \end{cases}$$

$$\begin{cases} A_t(t,T) = \beta(t)B(t,T) - \tfrac{1}{2}\delta(t)B^2(t,T) \\ A(T,T) = 0 \end{cases}$$

Analytical Results for some standard Models:-

The vasicek Model :-

$$dr(t) = (b - ar(t))dt + \sigma dW(t)$$

Here $-\alpha(t) = a$, $\beta(t) = b$, $\sqrt{\delta(t)} = \sigma$, $\gamma(t) = 0$

$$\Rightarrow B_t(t,T) - aB(t,T) = -1, \quad B(T,T) = 0$$

$$\Rightarrow B(t,T) = \tfrac{1}{a}\left\{1 - e^{-a(T-t)}\right\}.$$