19012306C

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Q1 $Y(t) = e^{2t+4W(t)}$

~~dX(t)~~ Let $X(t) = 2t + 4W(t)$

$\therefore Y(t) = e^{X(t)}$

$dX(t) = 2dt + 4dW(t)$

Take $f(t,x) = e^{x}$

$\therefore df(t,x) = e^{x}dx + \frac{e^{x}}{2}(dx)^2$

$df(t, X(t)) = d(e^{X(t)}) = d(Y(t)) = e^{X(t)}dX(t) + \frac{e^{X(t)}}{2} dX(t)dX(t)$

$= e^{X(t)}[2dt + 4dW(t)] + e^{X(t)}\left[\frac{16}{2} dW(t)\,dW(t)\right]$

$= e^{X(t)}[10dt + 4dW(t)]$

$= e^{2t+4W(t)}[10dt + 4dW(t)]$

$\therefore dY(t) = 10e^{2t+4W(t)}dt + 4e^{2t+4W(t)}dW(t).$

ABHISHEK AGRAHARI 190123066

2. $dX(t) = -\frac{1}{2(1-t)} X(t)\,dt + \sqrt{1-t}\,dW(t)$, $X(0)=0$

i) Above equation is a linear SDE -

On comparing it with standard linear SDE -

$$dX(t) = (\phi(t)X(t) + \theta(t))\,dW(t) + (f(t)X(t) + g(t))\,dt$$

we get

$$f(t) = \frac{-1}{2(1-t)} \qquad g(t) = 0$$

$$\phi(t) = 0 \qquad \theta(t) = \sqrt{1-t}$$

$$\therefore Y(t) = \int_0^t \phi(s)\,dW(s) + \int_0^t f(s)\,ds - \frac{1}{2}\int_0^t \phi^2(s)\,ds.$$

$$= \int_0^t -\frac{1}{2(1-s)}\,ds = +\frac{1}{2}\left[\ln(1-s)\right]_0^t$$

$$= +\frac{1}{2}\left[\ln(1-t)\right]$$

$$\therefore X(t) = \int_0^t e^{\frac{1}{2}(\ln(1-t) - \ln(1-s))} \sqrt{1-s}\,dW(s)$$

$$= \int_0^t e^{\ln\sqrt{\frac{1-t}{1-s}}} \sqrt{1-s}\,dW(s)$$

$$= \int_0^t \sqrt{1-t}\,dW(s)$$

$$= \sqrt{1-t}\,W(t)$$

(ii) Yes $X(t)$ is a gaussian process as $W(t)$ is gaussian and $\sqrt{1-t}$ is deterministic f^n.

(iii) $E[X(t)] = \sqrt{1-t}\,E[W(t)] = 0$

ABHISHEK AGRAHARI 190123066.

(iii) $var(X(t)) = E[(X(t) - E[X(t)])^2]$

$= E[(\sqrt{1-t}\, W(t))^2]$

$= E[(1-t) W^2(t)]$

$= (1-t) E[W^2(t)]$

$= (1-t) var[W(t)]$

$= (1-t) t = t - t^2$

$\therefore$ ~~$var(X(t)) = var(W(t)) - t^2$~~

$\therefore$ ~~$var(X(t)) \le var(W(t))$~~

$var(X^{0\to 0}) = c(t,t) = t - \frac{t^2}{T} = t - t^2$ $\boxed{\because T = 1}$

$\therefore var(X(t)) = var(X^{0\to 0})$

(iv) $cov(X(t), X(s)) = E[(X(t) - E[X(t)])(X(s) - E[X(s)])]$

$= E[X(t) X(s)]$

$= E[\sqrt{1-t}\sqrt{1-s}\, W(t) W(s)]$

$= \sqrt{1-t}\sqrt{1-s}\, E[W(t) W(s)]$

$= \sqrt{1-t}\sqrt{1-s}\, cov[W(t), W(s)]$ Let $s < t$.

$= \sqrt{1-t}\sqrt{1-s}\,(s)$

$\neq cov(X^{0\to 0}(t), X^{0\to 0}(s)) = s - st$

$\therefore cov(X(t), X(s)] \neq cov[X^{0\to 0}(t), X^{0\to 0}(s)]$

$\therefore$ $X(t)$ is not a Brownian bridge from 0 to 0 on $[0,1]$

ABHISHEK AGRAHARI 190123066.

Q3 comparing with ~~Using the~~ Feynmac Kac ~~Rep~~ Representation.

$b(t,x) = 0$, $\sigma = \text{constant}$.

$h(x) = x^2$ and $g(t,x) = u(t,x)$

$$g(t,x) = E^{t,x}[h(X(T))]$$

$$u(t,x) = E[X^2(T) \mid X(t) = x]$$

$$dX(t) = b(t, X(t))\,dt + \sigma(t, X(t))\,dW(t)$$

$$dX(t) = 0 + \sigma\,dW(t)$$

$$X(T) - X(t) = \int_t^T \sigma\,dW(s)$$

$$X(T) = x + \int_t^T \sigma\,dW(s)$$

$$\therefore X^2(T) = x^2 + \left[\int_t^T \sigma\,dW(s)\right]^2 + 2x\int_t^T \sigma\,dW(s)$$

$$E[X^2(T) \mid X(t) = x] = x^2 + E\left[\left(\int_t^T \sigma\,dW\right)^2\right]$$

$$= x^2 + \int_t^T \sigma^2\,ds \quad \text{[using ito isometry]}$$

$$= x^2 + \sigma^2[T-t]$$

hence $u(t,x) = x^2 + \sigma^2(T-t)$