MODULE 9: Queueing Networks (contd...)

LECTURE 35

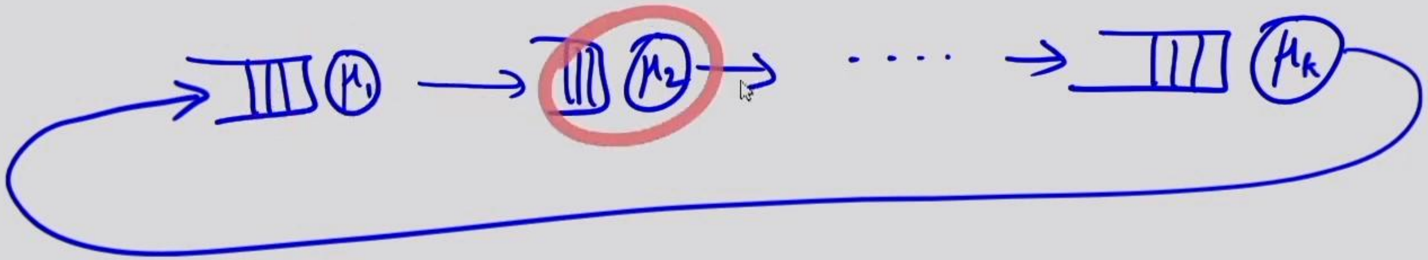
Cyclic Queueing Networks, Extensions of Jackson Networks

Cyclic Queueing Networks

- Taylor and Jackson (1954) introduced the concept of cyclic queue as a model for analyzing the flow of aircraft engines from operation to maintenance to available for operation. There have been many works since then with different application contexts.
- A cyclic queue is a sort of a series queue in a circle, where the output of the last node feeds back to the first node.
- If we consider a closed network of k nodes such that

$$r_{ij} = \begin{cases} 1, & j = i+1,\ 1 \le i \le k-1 \\ 1, & i = k,\ j = 1 \\ 0, & \text{otherwise} \end{cases}$$

  then we have a cyclic queue.
- Since this is a special case of closed queueing network, the results of closed queueing networks apply here.



- Assume that there is a single server at each node.
- We have the joint system size distribution as

$$p_{n_1,n_2,\ldots,n_k} = C\rho_1^{n_1}\rho_2^{n_2}\cdots\rho_k^{n_k}$$

- The traffic equations $\mu_i\rho_i = \sum_{j=1}^{k}\mu_j r_{ji}\rho_j$ on substitution of r_{ij}'s give

$$\mu_i\rho_i = \begin{cases} \mu_{i-1}\rho_{i-1}, & i = 2,3,\ldots,k, \\ \mu_k\rho_k, & i = 1. \end{cases}$$

Thus, we have

$$\rho_i = \begin{cases} \left(\frac{\mu_{i-1}}{\mu_i}\right)\rho_{i-1}, & i = 2,3,\ldots k \\ \frac{\mu_k}{\mu_1}\rho_k, & (i = 1). \end{cases}$$

- That is,

$$\rho_2 = \frac{\mu_1}{\mu_2}\rho_1, \quad \rho_3 = \frac{\mu_2}{\mu_3} \times \ \rho_2 = \frac{\mu_1}{\mu_3}\rho_1, \ldots, \rho_{k-1} = \frac{\mu_1}{\mu_{k-1}}\rho_1, \quad \rho_k = \frac{\mu_1}{\mu_k}\rho_1.$$

- Since one ρ can be set equal to one, we set $\rho_1 = 1$.
- Then the solution is

$$p_{n_1,n_2,\ldots,n_k} = \frac{1}{G(N)} \frac{\mu_1^{N-n_1}}{\mu_2^{n_2}\mu_3^{n_3}\ldots\mu_k^{n_k}},$$

  where $G(N)$ can be computed as earlier (naive or convolution algorithm).
- Multiple-server case can aso be treated similarly.
- The machine repairmen problem is really a two-node cyclic queue.

Example

Consider a cyclic queue with two nodes and $N = K$ circulating jobs.
The first node (node 1) has an exponential server with rate λ, and the second node (node 2) has an exponential server with rate μ.
This can be viewed as a special case of the two-node closed queueing network considered earlier (with $\mu_1 = \lambda, \mu_2 = \mu, p = q = 0, M = K$).
The traffic equations becomes

$$\lambda\rho_1 = \mu\rho_2$$
$$\mu\rho_2 = \lambda\rho_1$$

Since these equations are linearly dependent, as we know already, we can set one of the ρ_i's to 1 and solve for the other. Set $\rho_1 = 1$. Then, from the second equation, we get $\rho_2 = \dfrac{\lambda}{\mu}$.
We thus have the steady state solution for the closed network as

$$p_{K-m,m} = \frac{1}{G(K)}\rho_2^m = \frac{1}{G(K)}\left(\frac{\lambda}{\mu}\right)^m, \quad m = 0, 1, 2, \ldots, K,$$

where the normalizing constant $G(K)$ is given by $G(K) = \sum_{m=0}^{K} \rho_2^m = \dfrac{1 - \rho_2^{K+1}}{1 - \rho_2}$.

Example

Taking into account both the cases of $\lambda = \mu$ and $\lambda \neq \mu$, we get the complete steady state solution as

$$p_{K-m,m} = \begin{cases} \dfrac{(1-\rho_2)\rho_2^m}{1-\rho_2^{K+1}}, & \rho_2 \neq 1, \\ \dfrac{1}{K+1}, & \rho = 1, \end{cases} \qquad m = 0, 1, 2, \ldots, K.$$

Recall that the steady state system size distribution for an $M/M/1/K$ system is also given by the above expression.
Thus, a two-node cyclic queue may be considered as equivalent to an $M/M/1/K$ system.

Extensions of Jackson Networks

- There have been many directions in which Jackson networks have been generalized (like state-dependent exogenenous arrivals, state dependent service, travel times between nodes, etc.)
- The most significant and useful form of extension of Jackson network was to the case of multiple classes of customers, considered by Baskett, Chandy, Muntz, and Palacios (1975). These are called BCMP networks.
- A BCMP network considers different service disciplines as well as a number $Q(\geq 1)$ of classes of customers. The setup is:
  - k nodes in a closed network with $Q(\geq 1)$ classes of customers.
  - Customers circulate in the network and may change class as they move from one node to another.
  - A customer of class r completing service at node i next goes to node j as a class s customer is denoted by $p_{ir,js}$. Hence, the routing matrix $R = (p_{ir,js}),\ 1 \leq i, j \leq k,\ 1 \leq r, s \leq Q$.
  - The service rate of a customer of class r at node i is μ_{ir}.

- A product-form solution is obtained in a BCMP network for the following cases of queueing disciplines:
  – Processor sharing
  – ample service
  – LCFS with preemptive-resume servicing

  The exogeneous Poisson input can be state dependent and the service distributions can be phase-type.
- A product-form solution is also obtained in a BCMP network for c-server FCFS nodes, but with service times of all classes must be IID exponential (i.e., all customer types look alike).
- Kelly (1975, 1976,1979) considered some other significant generalizations of Jacksons networks and that included extensions to most general queueing disciplines. He obtained product form solution with Erlang service distributions too and conjectured for more general distributions (which was proved later).

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- At the same time as these theoretical developments were happening, attention was drawn towards efficient procedures/algorithms for obtaining the computational results.
- Interest and research continue in the computational aspects of Jaskson networks and its extensions because of their immense importance for modelling a variety of systems in various fields (computer, communication, logistics, etc.).