Module 6: General Markovian Queueing Systems

Lecture 21

Queues with Bulk Service

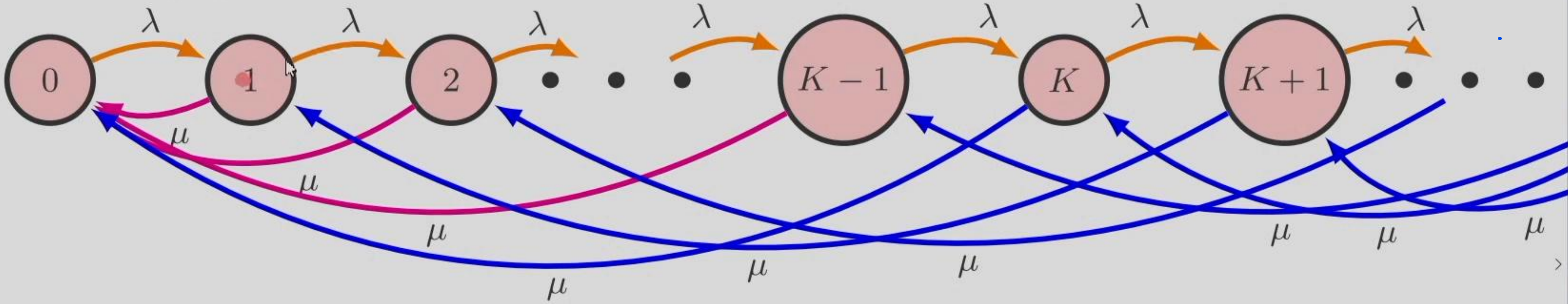
Bulk Service Queues ($M/M^{[Y]}/1$ Queues)

- We consider a single-server Markovian queue with *bulk service*.
  - Mass transit vehicles and carriers are natural batch servers.
- All the assumptions of M/M/1 are applicable with an additional assumption that the customers are served in groups/batches of size K.
- We consider two variations based on the system behavoiur when there are less than K in the system.
  - **Full Batch Model**:- Here the server processes exactly K customers at a time. If less than K customers are in the system, then the server remains idle until there are K customers in the system, at which point the server processes the K customers simultaneously. The service times are $Exp(\mu)$ and is the same for all the customers in a batch.
    For example, a small ferry service crossing the river only when it is full.
  - **Partial Batch Model**:- Here the server can process partial batches up to a maximum size of K. As before, customers are served K at a time, but now if there are less than K in the system, the server begin service on these customers. Furthermore, when there are less than K in service, new arrivals immediately enter service up to the limit K and finish with the others, regardless of the entry time into service. Service times are $Exp(\mu)$.
    For example, a guided tour or a puppet/movie show.

- There are other variants of the group service rule, the prominent one being the **general bulk service rule** (Neuts, 1967) in which the service is rendered with a minimum batch size of L and with a maximum batch size of K ($L \leq K$).
  - If there are fewer than L customers in the system, then the server remains idle until there are L customers whereupon all L enters the service.
  - If there are L or more, but less than K customers waiting, then all of them served together.
  - If there are more than K customers waiting, then a group of K enters the service.
  - The corresponding model is typically denoted as $M/M(L,K)/1$.
  - Examples include airport shuttle or a lift in a building (ground floor) or a traffic flow (where a minor road merges into a major road).
  - Systems where the customers can join or access an ongoing service batch anytime before the service of the batch is completed, provided the specified maximum service batch size is not reached, are known as systems with *accessible* batch service (otherwise they are called non-accessible).
- Another variant is that of the size of a batch being a random variable (depending on the unfilled capacity of the server).
- We consider only the $M/M^{[K]}/1$ full-batch and partial-batch (with accessible batch service) models only (as above).

$M/M^{[K]}/1$ Partial-Batch Model

- The underlying model is a CTMC model as described below.



- The stochastic balance equations are

$$\lambda p_0 = \mu p_1 + \mu p_2 + \cdots + \mu p_{K-1} + \mu p_K$$
$$(\lambda + \mu) p_n = \mu p_{n+K} + \lambda p_{n-1}, \quad n \geq 1.$$

- The general equation above can be rewritten in operator notation as

$$[\mu D^{K+1} - (\lambda+\mu)D + \lambda]p_n = 0, \quad n \geq 0.$$

- If $(r_1, r_2, \ldots, r_{K+1})$ are the roots of the operator or characteristic equation, then

$$p_n = \sum_{i=1}^{K+1} C_i r_i^n, \qquad n \geq 0.$$

Now, $\sum_{n=0}^{\infty} p_n = 1$ implies that each r_i must be less than one or $C_i = 0$ for r_i not less than one.

- From Rouche's theorem, it can be found that there is exactly one root (say, r_0) in $(0,1)$, under the condition for the stability of the system $\rho = \frac{\lambda}{K\mu} < 1$. (Exercise! Hint: Take $g = \mu r^{K+1} + \lambda$ and $f = -(\lambda+\mu)r$.) This implies that

$$p_n = C r_0^n, \qquad n \geq 0, \quad 0 < r_0 < 1.$$

Now, $\sum_{n=0}^{\infty} p_n = 1 \Rightarrow C = p_0 = 1 - r_0$ and hence

$$p_n = (1-r_0) r_0^n, \quad n \geq 0, \quad 0 < r_0 < 1.$$

Performance Measures

- The performance metrics can be obtained in the usual manner (noting that the steady-state distribution is geometric, like in $M/M/1$).
- We get

$$L = \frac{r_0}{1 - r_0}, \quad W = \frac{L}{\lambda} = \frac{r_0}{\lambda(1 - r_0)}$$

and, using Little's Law:

$$W_q = W - \frac{1}{\mu}, \quad L_q = L - \frac{\lambda}{\mu}.$$

Alternatively, directly computing from the steady-state probabilities, we get

$$L_q = r_0^K L, \quad \text{which is equal to } L_q = L - \frac{\lambda}{\mu}.$$

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Example

- A drive-it-through-yourself car wash facility installs a new machinery that permits the washing of two cars at once (and one if no other cars wait).
- A car that arrives while a single car is being washed joins the wash and finishes with the first car.
- There is no waiting capacity limitation.
- Arrivals are Poisson with mean 15 per hour. Time to wash a car is exponentially distributed with a mean of 6 minutes.
- The given data are: $\lambda = 15/h, \quad \mu = 10/h,$ and $K = 2$. The characteristic equation is

$$10r^3 - 25r + 15 = 5(2r^3 - 5r + 3) = 0.$$

Observing that one root is 1, we get

$$2r^2 + 2r - 3 = 0 \quad \Longrightarrow r = (-2 \pm \sqrt{28})/4.$$

Thus, we get $r_0 = 0.8229$, choosing the root less than 1. Therefore,

$$L = \frac{0.8229}{0.1771} = 4.6458\ cars \quad L_q = 4.6458 - \frac{15}{10} = 3.1458\ cars.$$

$M/M^{[K]}/1$ Full-Batch Model

- In this model, we assume that the batch size must be exactly K for the server to start the service, and if not, the server waits until such time to start.
- The stochastic balance equations are modified accordingly to yield

$$\begin{aligned} \lambda p_0 &= \mu p_K \\ \lambda p_n &= \mu p_{n+K} + \lambda p_{n-1}, \quad 1 \leq n < K \\ (\lambda + \mu) p_n &= \mu p_{n+K} + \lambda p_{n-1}, \quad n \geq K \end{aligned}$$

- The third equation is identical to that of the partial-batch model and hence, proceeding on similar lines, we get

$$p_n = C r_0^n, \qquad n \geq K-1, \quad 0 < r_0 < 1.$$

  But, obtaining C (and p_0) is a bit complicated here.
- From the first equation, we have

$$p_K = \frac{\lambda}{\mu} p_0 = C r_0^K, \quad C = \frac{\lambda p_0}{\mu r_0^K} \text{ and therefore } p_n = \frac{p_0 \lambda r_0^{n-K}}{\mu} \qquad n \geq K-1.$$

- Now, to get p_0, we use the remaining $K-1$ equations given by

$$\mu p_{n+K} = \lambda p_n - \lambda p_{n-1}, \qquad 1 \le n < K.$$

Using the geometric form of p_n when $n \ge K-1$ and substituting for p_{n+K} in the above equation, we get

$$p_0 r_0^n = p_n - p_{n-1}, \qquad 1 \le n < K.$$

These equations can be solved by iteration starting with $n = 1$. Alternatively, we observe that these are nonhomogeneous linear difference equations whose solutions are

$$p_n = C_1 + C_2 r_0^n.$$

Direct substitution into the above equation implies that $C_2 = -\frac{p_0 r_0}{(1-r_0)}$. The boundary condition at $n = 0$ implies that $C_1 = p_0 - C_2$. This gives

$$p_n = \begin{cases} \dfrac{p_0(1-r_0^{n+1})}{1-r_0}, & 1 \le n < K-1 \\ \dfrac{p_0 \lambda r_0^{n-K}}{\mu}, & n \ge K-1 \end{cases}$$

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- We can now determine p_0 using the usual boundary condition $\sum_{n=0}^{\infty} p_n = 1$ and we have

$$p_0 = \left(1 + \sum_{n=1}^{K-1} \frac{1 - r_0^{n+1}}{1 - r_0} + \frac{\lambda}{\mu} \sum_{n=K}^{\infty} r_0^{n-K}\right)^{-1}$$

$$= \left(1 + \frac{K-1}{1-r_0} - \frac{r_0^2(1 - r_0^{K-1})}{(1 - r_0^2)} + \frac{\lambda}{\mu(1-r_0)}\right)^{-1}$$

$$= \left(\frac{\mu r_0^{K+1} - (\lambda+\mu) r_0 + \lambda + \mu K(1-r_0)}{\mu(1-r_0)^2}\right)^{-1}$$

But we know that r_0 satisfy the characteristic equation and this means that

$$\mu r_0^{K+1} - (\lambda + \mu) r_0 + \lambda = 0$$

and thus we obtain finally

$$p_0 = \frac{\mu(1-r_0)^2}{\mu K(1-r_0)} = \frac{1-r_0}{K}.$$

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