1. State whether the following statement is TRUE or FALSE.
Let $\{X_n, n = 1, 2, \dots\}$ defines a renewal process. Let the probability density function of X_n be denoted by $f(t)$ and its Laplace transform be denoted by $\bar{f}(s)$. Then, the Laplace transform of the renewal function of the renewal process is given by $\dfrac{\bar{f}(s)}{s[1-\bar{f}(s)]}$.

Solution:

Answer: TRUE.

We know that $M(t) = \sum_{n=1}^{\infty} F^{n*}(t)$. Taking LT on both sides, and using the properties of the LT, we get

$$\bar{M}(s) = \sum_{n=1}^{\infty} \frac{(\bar{f}(s))^n}{s} = \frac{\bar{f}(s)}{s[1-\bar{f}(s)]}.$$

2. State whether the following statement is TRUE or FALSE.
Let $\{X_n, n = 1, 2, \dots\}$ defines a renewal process, where X_n is a continuous random variable with the Laplace transform of its probability density function given by $\left(\dfrac{\lambda}{s+\lambda}\right)^2$. Then, the renewal function of the renewal process is given by $\dfrac{\lambda t}{2} - \dfrac{1}{4}[1-e^{-2\lambda t}]$.

Solution:

Answer: TRUE.

The LT of the renewal function in terms of the LT of the PDF is given, after substitution, by

$$\bar{M}(s) = \frac{\bar{f}(s)}{s[1-\bar{f}(s)]} = \frac{\lambda^2}{s[(s+\lambda)^2-\lambda^2]} = \frac{\lambda}{2s}\left(\frac{1}{s} - \frac{1}{s+2\lambda}\right) = \frac{\lambda}{2s^2} - \frac{1}{4}\left(\frac{1}{s} - \frac{1}{s+2\lambda}\right),$$

by partial fractions. Inversion of the above yields the required result.

3. State whether the following statement is TRUE or FALSE.
A semi-Markov process is a generalization of a CTMC.

Solution:

Answer: TRUE, by definition.

4. At a post office, customers arrive according to a Poisson process with a rate of 60 customers per hour. Half of the customers have a service time that is the sum of a fixed time of 15 seconds and an exponentially distributed time with a mean of 15 seconds. The other half have an exponentially distributed service time with a mean of one minute. Then, the mean waiting time in the queue (in minutes) in steady state equals

(A) $\dfrac{33}{15}$

(B) $\dfrac{25}{7}$

(C) $\dfrac{37}{16}$

(D) $\frac{40}{17}$

Solution:

Answer: (C)
Here $\lambda = 1/\text{min}$. Let $X \sim Exp(4), Y \sim Exp(1)$. If S be the total service time, then

$$E(S) = \frac{1}{2}E\left(1/4 + X\right) + \frac{1}{2}E(Y) = \frac{3}{4} \text{ min}$$
$$E(S^2) = \frac{1}{2}E\left(1/4 + X\right)^2 + \frac{1}{2}E(Y^2) = \frac{1}{2} * \frac{5}{16} + \frac{1}{2} * 2 = \frac{37}{32} \text{ min}$$

Here $\rho = 3/4$. Then mean waiting time in queue is $W_q = \dfrac{\lambda E(S^2)}{2(1-\rho)} = \dfrac{37}{16}$ minutes.

5. In a heavy machine shop, the overhead crane is 75% utilized. Time study observations gave the average slinging time as 10.5 minutes with a standard deviation of 8.8 minutes. What is the delay (in minutes) in getting into the service?

(A) 35.35

(B) 20.45

(C) 18.57

(D) 26.81

Solution:

Answer: (D)
Given $\rho = 0.75 = \frac{\lambda}{\mu}$, where $\mu = 1/10.5 = 5.71/$ hr, we have $\lambda = 4.29/$ hr. Now $\sigma = 8.8$ min=0.1467 hr. Then

$$W_q = \frac{\lambda^2\sigma^2 + \rho^2}{2\lambda(1-\rho)} = 0.446887 \;\; hr. = 26.8132 \;\; min.$$

6. Consider the heavy machine shop problem given in Question 5 above. If the average service time is reduced to 8 minutes with a standard deviation of 6 minutes, how much reduction (in minutes), over the delay obtained in Question 5, will occur on average in the delay of getting served?

(A) 10.5

(B) 15.6

(C) 18.5

(D) 20

Solution:

Answer: (C)
Here, $\mu = 1/8/$ min $= 7.5/$ hr., and $\sigma = 6/60 = 0.1$ hr. In the new setup, using the previous relations, we get $\rho^{(new)} = 0.571$, and $W_q^{(new)} = 0.1386$ hr.= 8.32 min. Hence, the reduction $26.8 - 8.32$ min= 18.5 min. Here, utilization of the crane is reduced to 57.1%.

7. State whether the following statement is TRUE or FALSE.
   With usual notations, the steady state waiting time in the queue in an $M/G/1$ queueing system equals $\dfrac{1+\mu^2\sigma_B^2}{2}\dfrac{\rho}{1-\rho}$.

   Solution:

   Answer: FALSE.
   We have that $W_q = \dfrac{\rho^2/\lambda + \lambda\sigma_B^2}{2(1-\rho)} = \dfrac{1+\mu^2\sigma_B^2}{2}\dfrac{\rho}{\mu-\lambda}$.

8. State whether the following statement is TRUE or FALSE.
   Consider a computer network node in which requests for data arrive in a Poisson process at the rate of 0.5 per unit time. Assume that the data retrieval (service) takes a constant amount of one unit of time. Then, the PGF of the steady state departure-time distribution of the number of requests in the node is given by $\Pi(z) = \dfrac{(z-1)e^{-(1-z)/2}}{2[z-e^{-(1-z)/2}]}$.

   Solution:

   Answer: TRUE.
   We can model this system as an $M/D/1$ queue and use the techniques developed in this section for its analysis. We have $k_j = e^{-0.5}\frac{0.5^j}{j!}$, the PGF of k_j is given by

$$
\begin{aligned}
K(z) &= \sum_{j=0}^{\infty} e^{-0.5}\frac{0.5^j}{j!}z^j = e^{-0.5(1-z)},\\
\Pi(z) &= \frac{0.5(1-z)e^{-0.5(1-z)}}{e^{-0.5(1-z)}-z}.
\end{aligned}
$$