MODULE 8: Queueing Networks

LECTURE 31

Waiting Times and Multiple Classes in Open Jackson Networks

- **Multi-Server Case:** The results of the single-server at each node can be generalized to open Jackson networks with c_i channels at node i. The joint distribution of number in the network will still be given in a product form as:

$$= p_{n_1} p_{n_2} \cdots p_{n_k}$$

$$p_{\bar{n}} = p_{n_1, n_2, \ldots, n_k} = \prod_{i=1}^{k} \frac{r_i^{n_i}}{a_i(n_i)} p_{0i} \qquad (r_i = \frac{\lambda_i}{\mu_i})$$

where $a_i(n_i) = \begin{cases} n_i!, & n_i < c_i \\ c_i^{n_i - c_i} c_i!, & n_i \geq c_i \end{cases}$ and p_{0i} is determined such that $\sum_{n_i=0}^{\infty} p_{0i} \frac{r_i^{n_i}}{a_i(n_i)} = 1.$

- The network again acts as if each node were an independent $M/M/c_i$.
- The performance measures can be obtained in a similar matter as that of the single-channel network.

Waiting Time Distributions and Output Processes

- Though it is tempting to conclude that the waiting time distribution at a node should be the same as that of $M/M/c$, this is not necessarily true.
- Recall that in $M/M/c$, we relied on the fact $p_n = q_n$ which relied on Poisson input.
- Here the arrivals to nodes are not Poisson in general (because of feedbacks).
- Unless the network is a feedforward one, and in this case the arrivals to a node is Poisson, nothing can be concluded about actual waiting time distributions.
  - ▶ But the mean values at nodes satisfy Little's law.
- Even with feedforward networks and with multiple servers, the sojourn times can be complex.
  - ▶ Feedback and bypassing cause complexity problems with sojourn times.

- One may also be interested in output processes from individual nodes.
  - ▶ For series or feedforward networks, as we have seen, the flows between nodes and to the outside are truly Poisson.
  - ▶ Feedback destroys Poisson flows, but Jackson's solution still holds.

- Despite the above difficulties, the system size results are quite neat and Jackson networks have been useful in modelling a variety of network situations.

Example (Three-Node Call Center)

- Calls arrive in a Poisson fashion with a mean rate of 35 per hour at a three-node telephone system of an insurance company.
- Upon calling, there are two options: 1-for claims and 2-for policy service. The caller's listening, decision and button-pressing time is exponential with a mean of 30 seconds.
- Only one call at a time can be processed and other calls wait in queue.
- Estimates suggest that 55% of the calls are related to claims and 45% are policy service calls.
- The claims processing node has 3 parallel servers and policy service node has 7 parallel servers, with both following exponential service time distributions with mean of 6 minutes and 20 minutes, respectively.
- All buffers in front of nodes can hold as many calls as come into the queues.
- About 2% of the customers finishing claims go on to the policy service and 1% vice-versa.
- What is the average queue sizes in front of each node and the total average time a customer spends in the system?

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Example (contd...)

- With 2 denoting the claims node and 3 the policy node, the routing matrix R is

$$R = \begin{pmatrix} 0 & 0.55 & 0.45 \\ 0 & 0 & 0.02 \\ 0 & 0.01 & 0 \end{pmatrix}$$

And, $\gamma_1 = 35/h,\ \gamma_2 = \gamma_3 = 0, \quad c_1 = 1,\ \mu_1 = 120/h, \quad c_2 = 3,\ \mu_2 = 10/h, \quad c_3 = 7,\ \mu_3 = 3/h$

- For solving the traffic equations, we have

$$(I - R)^{-1} = \begin{pmatrix} 1 & 0.5546 & 0.4611 \\ 0 & 1.0002 & 0.02 \\ 0 & 0.01 & 1.0002 \end{pmatrix} \text{ and hence } \boldsymbol{\lambda} = \boldsymbol{\gamma}(I - R)^{-1} = (35, 19.411, 16.138).$$

Then the offered loads are: $r_1 = \dfrac{35}{120} = 0.292, r_2 = \dfrac{19.411}{10} = 1.941, r_3 = \dfrac{16.132}{3} = 5.379.$

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Open Jackson Networks with Multiple Customer Classes

- A customer of one type has a different routing probability matrix than a customer of another type.
- Solve the traffic equations separately for each customer type and then add the resulting λ's.
- Let $R^{(t)}$ be the routing probability matrix for a customer of type t $(t = 1, 2, \ldots, n)$.
- Solve $\boldsymbol{\lambda}^{(t)} = \boldsymbol{\gamma}^{(t)} + \boldsymbol{\lambda}^{(t)} R^{(t)}$ to get $\boldsymbol{\lambda}^{(t)}$ for each customer type t. Then, $\boldsymbol{\lambda} = \sum_t \boldsymbol{\lambda}^{(t)}$.
- We proceed, as before, to obtain L_i for each node using $M/M/c$ results.
- Noting that all customer types have the same average waiting time, since they have identical service time distributions and wait in the same FCFS queue, the average waiting time at each node can be obtained via Little's law. Similar is the case with average system sojourn time.
- We can also obtain the average system size for customer type t at node i as

$$L_i^{(t)} = \frac{\lambda_i^{(t)}}{\lambda_i^{(1)} + \lambda_i^{(2)} + \cdots + \lambda_i^{(n)}} L_i$$

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Example

Example (Three-Node Network - revisited)

- Recall the previous example in which it was implicit that a customer can revisit the previously visited nodes, and this is not realistic in the given scenario.
- The way around is: Customers who first go to claims are $type-1$ customers and customers who first go to policy service are $type-2$ customers. Then the two routing matrices are:

$$R^{(1)} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & .02 \\ 0 & 0 & 0 \end{pmatrix}, \qquad R^{(2)} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & .01 & 0 \end{pmatrix}$$

- Since 55% of the arrivals are $type-1$ (the rest $type-2$), we have $\gamma_1^{(1)} = 19.25$ and $\gamma_1^{(2)} = 15.75$.
- Solving the traffic equations simultaneously, we get

$$\lambda_1^{(1)} = 19.25, \quad \lambda_2^{(1)} = 19.25, \quad \lambda_3^{(1)} = 0.385$$
$$\lambda_1^{(2)} = 15.75, \quad \lambda_2^{(2)} = 0.1575, \quad \lambda_3^{(2)} = 15.75$$

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Example (contd...)

- Adding to get the total flows gives $\boldsymbol{\lambda} = (35, 19.408, 16.135)$.
  - ▶ Observe the slightly lower flows in nodes 2 and 3 (compared to earlier).
- Using $M/M/c$ results, we get

$$L_1 = 0.412, \quad L_2 = 2.705, \quad L_3 = 6.777, \quad \text{and} \;\; L = 9.894$$

- The average system sojourn time is again $W = \dfrac{9.894}{35} = 0.283h$ or about 17 minutes.
- Similarly one can find the average number of each type of customer at each node as:

$$L_1^{(1)} = \frac{19.25}{19.25 + 15.75} L_1 = 0.227, L_2^{(1)} = \frac{19.25}{19.4075} L_2 = 2.683, L_3^{(1)} = \frac{0.385}{16.135} L_3 = 0.162,$$
$$L_1^{(2)} = \frac{15.75}{35} L_1 = 0.185, L_2^{(2)} = \frac{0,1575}{19.4075} L_2 = 0.022, L_3^{(2)} = \frac{15.75}{16.135} L_3 = 6.616.$$