1. State whether the following statement is TRUE or FALSE:
   For a birth-death process (BDP) with strictly positive birth and death rates, the steady state probabilities will exist if there exists a positive integer M such that $\lambda_n < \mu_n$ for all $n \geq M$.

   Solution:

   Answer: TRUE

   We know that convergence of $1 + \sum_{n=1}^{\infty} \prod_{i=1}^{n} \frac{\lambda_{i-1}}{\mu_i}$ is the necessary and sufficient condition for the existence of a steady state distribution $\{p_n\}$ for the BDP. And, this would be the case if $\lambda_n/\mu_n < 1$ for $n \geq M$ for some M (as in this case the tail of the above series can be bounded by a geometric series).

2. State whether the following statement is TRUE or FALSE:
   In an $M/M/1$ queueing system, the steady state system size probabilities as well as the performance measures L and W remain the same even if the parameters λ and μ are replaced by 3λ and 3μ, respectively.

   Solution:

   Answer: FALSE

   Since $p_n = (1-\rho)\rho^n$ and $L = \rho/(1-\rho)$, these quantities are not affected by the replacement (as ρ remains the same), but since $W = 1/(\mu - \lambda)$, this would be affected by the change of parameters.

3. For a small computing system, the processing time per job is exponentially distributed with an average time of 2 minutes. Jobs arrive randomly according to a Poisson process at an average rate of one job every 3 minutes. Any number of jobs can wait in the system and are processed on a first-come-first-served basis. It is decided that when the work load increases to the level such that the average waiting time in the system reaches 10 minutes, the computing capacity will be increased. What is the average arrival rate of jobs per hour at which this will occur?

   Solution:

   Answer: 24 [Hint: Enter the answer as an integer.]

   Here, $\lambda = 1/3$ jobs per minute, $\mu = 1/2$ jobs per minute and $\rho = 2/3$. Therefore, we have that $L = 2$ jobs and $W = 6$ minutes. We want to compute λ such that $W = 10$. Solving $1/(\mu - \lambda) = 10$ we get $\lambda = 2/5$ jobs per minute which equals 24 jobs per hour.

4. In the small computing system given in the previous problem (Problem 3), let α denote the probability that an arriving job will require more than 12 minutes to complete the service in the system (i.e., the job turn-around time T exceeds 12 minutes). Then, $40\,\alpha$ equals _______.

   Solution:

   Answer: 5.40 **to** 5.43 [Hint: Use the original (not the increased ones) values for the parameters. Enter the answer in two decimals.]

   The system waiting time T is distributed as $Exp(\mu - \lambda)$ and hence $F_T(t) = 1 - e^{-t/6}$. Therefore $40\alpha = 40P\{T > 12\} = 40e^{-12/6} = 40(0.1353) = 5.4134$.

5. In an $M/M/c$ system (in equilibrium) with $c = 5, r = 4.5, p_5 = 0.045$, the expected number of customers in the system equals _______.

Solution:

Answer: 8.52 **to** 8.58 [Hint: Enter the answer in two decimals.]

We know that $L = r + \frac{r^c p_0}{c!}\frac{\rho}{(1-\rho)^2} = r + p_c\frac{\rho}{(1-\rho)^2}$. Here, $\rho = r/c = 0.9$ and therefore $L = 4.5 + 4.05 = 8.55$

6. In a small travel ticket booking centre with two identical counters, the customers arrive as a Poisson process with an average arrival rate of 15 per hour and the service time (in each counter) is exponentially distributed with a mean of 4 minutes per customer. The (steady state) expected sojourn time (W) of a customer (in minutes) equals _______.

Solution:

Answer: 5.25 **to** 5.40 [Hint: Enter the answer in two decimals.]

Given $\lambda = 15, \mu = 15, c = 2$. This gives $r = 1, \rho = 1/2$. From the expression for p_0, we get $p_0 = 1/3$ and hence $L_q = 1/3$. This gives $W_q = L_q/\lambda = 1/45$ and $W = W_q + (1/\mu) = 4/45$ hour or 5.3333 minutes.

7. Assume that messages arrive at a message switching centre in one of the outgoing communication lines as a Poisson process with an average rate of 180 messages per minute. The line has a transmission rate of 600 characters per second. Suppose that the message length follows an exponential distribution with an average length of 160 characters. Further, suppose that it is decided to have only a finite number of message buffers K (so that at any time there can only be at most $K+1$ messages in the centre). What is the optimum value of K such that the probability of all the buffers are filled at any particular time is less than 0.005?

Solution:

Answer: 16 [Hint: Enter an integer value.]

This system can be modelled as an $M/M/1/(K+1)$ queueing system, where each state represents the number of messages in the switching centre. Here, $\lambda = 3 \times 160 = 480$ characters per second (since 3 messages arrive per second), $\mu = 600$ characters per second, and hence $\rho = \frac{\lambda}{\mu} = 0.8$. Therefore, we need to obtain K, with $\rho = 0.8$, such that

$$p_{K+1} = \frac{(1-\rho)\rho^{K+1}}{1-\rho^{K+2}} < 0.005.$$

This means that $K > \dfrac{\ln(0.005/(1-0.995\rho))}{\ln \rho} - 1 = 15.6202$ and hence $K = 16$.

8. State whether the following statement is TRUE or FALSE:
In steady state, the average number of customers in the queue in an $M/M/1$ system is greater than that of the corresponding $M/M/1/K$ queueing system.

Solution:

Answer: TRUE

We know that $L_q = \rho^2/(1-\rho)$ for $M/M/1$. And, it is $L_q = \dfrac{\rho}{1-\rho} - \dfrac{\rho(K\rho^K + 1)}{1-\rho^{K+1}}$ for $M/M/1/K$ (with $\rho < 1$). It is now a simple exercise to show that $L_q^{(M/M/1)} - L_q^{(M/M/1/K)} > 0$.