MODULE 10: Semi-Markovian Queueing Systems

LECTURE 37

Regenerative Processes, Semi-Markov Processes

Regenerative Process

Definition

A stochastic process $\{X(t), t \geq 0\}$ is called a regenerative process if there exists time points $0 = T_0 < T_1 < T_2 < ...$ such that, for $n \geq 1$,

- the process $\{X(T_n + t), t \geq 0\}$ is independent of the process $\{X(t), 0 \leq t < T_n\}$.
- the processes $\{X(T_n + t), t \geq 0\}$ and $\{X(t), t \geq 0\}$ have the same joint distribution.

The T_n's are called regeneration epochs (times or points) and the lengths $T_1 - T_0, T_2 - T_1, \ldots$ are called regeneration cycles. T_n 's are IID random variables and $\{T_n, n \geq 0\}$ defines a renewal process. The renewal process is said to be embedded in $\{X(t)\}$ at the epochs $T_1, T_2 \ldots$. Every time a renewal occurs a cycle is said to be completed.

Thus, a regenerative process is a stochastic process with time points starting from which the process is a probabilistic replica if the whole process starting at 0.

Example

A renewal process is regenerative, with T_n representing the time of the nth renewal.

Example

A recurrent Markov chain is a regenerative process, with T_n being the time of nth recurrence.

Example

In an $M/G/c$ queue, whenever the queue is empty, all servers are idle and only the arival process has an effect on the future. Thus, the system process regenerates at the points T_n of the system becoming idle for the nth time. The durations $T_{n+1} - T_n$ are IID. Hence, the $M/G/c$ system process is a regenerative process.

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Example (Alternating Renewal Process)

Another example of a regenerative process is an alternating renewal process. Such a process can be envisaged by considering that a system can be in one of two possible states - say, 0 and 1. Initially, it is at state 0 and remains at that state for a time Y_1, and then a change of state to state 1 occurs in which it remains for a time Z_1, after which it again goes to state 0 for a time Y_2 and then goes to state 1 for a time Z_2 and so on. That is, its movement could be denoted by $0 \to 1 \to 0 \to 1 \ldots$. (For the initial state of 1, the movement sequence is $1 \to 0 \to 1 \to 0 \ldots$)

- Suppose that $\{Y_n\}, \{Z_n\}$ are two sequences of IID random variables and that Y_n and Z_n need not be independent, Let

$$T_n = T_{n-1} + Y_n + Z_n, \quad n = 1, 2, \ldots$$

Then at time T_1 the process restarts itself, and so also at times $T_2, T_3 \ldots$. The interval $T_n - T_{n-1}$ denotes a complete cycle, and the process restarts itself after each complete cycle.

- Let $E[Y_n] = E[Y], \quad E(Z_n) = E(Z)$. Then the long-run proportions of time that the system is at states 0 and 1 are, respectively, are

$$p_0 = \lim_{t\to\infty} P\{X(t) = 0\} = \frac{E(Y)}{E(Y) + E(Z)}$$

and $$p_1 = \lim_{t\to\infty} P\{X(t) = 1\} = \frac{E(Z)}{E(Y) + E(Z)} = 1 - p_0.$$

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Use in Queueing

- The results of the alternating renewal process example given above have an important application in queueing theory.
- Consider a single-server queueing system such that an arriving customer is immediately taken for service if the server is free, but joins a waiting line if the server is busy.
- The system can be considered to be in two states (idle or busy) according to whether the server is idle or busy.
- The idle and busy states alternate and together constitute a cycle of an alternating renewal process. A busy period starts as soon as a customer arrives before an idle server and ends at the instant when the server becomes free for the first time.
- The epochs of commencement of busy periods are regeneration points. Let I_n and B_n denote the lengths of nth idle and busy periods, respectively, and let

$$E(I_n) = E(I) \quad \text{and} \quad E(B_n) = E(B).$$

- Then the long-run proportion of time that the server is idle equals

$$p_0 = \frac{E(I)}{E(I) + E(B)} \tag{1}$$

and the long-run proportion of time that the server is busy equals

$$p_1 = \frac{E(B)}{E(I) + E(B)}. \tag{2}$$

- *Remark:* If the arrival process is Poisson with mean λt, then it follows (from its lack of memory property) that an idle period is exponentially distributed with mean $1/\lambda$, i.e, $E(I) = 1/\lambda$. Then when p_0 or p_1 is known, $E(B)$ can be found.

Markov Renewal Processes and Semi-Markov Processes

We consider a special class of regenerative processes that is important for the analysis of many queueing systems. And, this class generalizes Markov processes and renewal processes at the same time.

Definition

Let S denote a countable state space. For every $n = 0, 1, 2, \dots$, let X_n denote a RV on S and T_n a nonnegative RV such that $0 = T_0 < T_1 < T_2 < \dots$ and $\sup_{n\to\infty} T_n = \infty$ almost surely. Define the process $\{Y(t), t \geq 0\}$ by

$$Y(t) = X_n \quad \text{for} \quad T_n \leq t < T_{n+1}$$

for all $t \geq 0$. If

$$P\{X_{n+1} = j, T_{n+1} - T_n \leq u | X_0, \dots, X_n, T_0, \dots, T_n\} = P\{X_{n+1} = j, T_{n+1} - T_n \leq u | X_n\}$$

hods for all $n = 0, 1, 2, \dots, j \in S$, and $u \geq 0$, then $\{Y(t), t \geq 0\}$ is called a semi-Markov process on S.

The sequence of random variables $\{(X_n, T_n), n \geq 0\}$ is called the embedded Markov renewal chain (or simply the Markov renewal process).

- The semi-Markov process is *homogeneous* if

$$Q_{ij}(t) = P\{X_{n+1} = j, T_{n+1} - T_n \leq t \,|\, X_n = i\}$$

is independent of n. We consider only this case.

- By definition, a semi-Markov process is a pure jump process and hence the sample paths are step functions.
- By construction, the semi-Markov process is determined by the embedded Markov renewal chain and vice versa.
- Let $\{Y(t), t \geq 0\}$ is a homogeneous Markov process with S and parameters $\lambda_i, i \in S$ for the exponential holding times. The embedded Markov chain $\{X_n\}$ has the transition matrix $P = (p_{ij})$. Then $\{Y(t)\}$ is a semi-Markov process with $Q_{ij}(t) = p_{ij}\left(1 - e^{-\lambda_i t}\right)$ for all $i, j \in S$. Thus, for a Markov process, the distribution of $T_{n+1} - T_n$ is exponential and independent of the state entered at time T_{n+1}. These are the two features for which the semi-Markov process is a generalization of the Markov process on a discrete state space (i.e., CTMC).

- It can be shown easily that, for a semi-Markov process $\{Y(t), t \geq 0\}$ with embedded Markov renewal chain $\{(X_n, T_n), n \geq 0\}$, the chain $\{X_n, n \geq 0\}$ is a Markov chain (called embedded Markov chain). And, we denote its TPM by $P = (p_{ij})_{i,j \in S}$. Then the following relation holds for all $i, j \in S$:

$$p_{ij} = P\{X_{n+1} = j | X_n = i\} = \lim_{t \to \infty} Q_{ij}(t).$$

- According to its embedded Markov chain $\{X_n\}$ we call a semi-Markov process irreducible, recurrent or transient.
  An irreducible recurrent semi-Markov process is regenerative, as one can fix any initial state $i \in S$ and the times of visiting this state to be a renewal process.
- Define $F_{ij}(t) = Q_{ij}(t)/p_{ij}$ for all $t \geq 0$ and $i, j \in S$ if $p_{ij} > 0$, while $F_{ij}(t) = 0$ otherwise. Then, this can be interpreted as

$$F_{ij}(t) = P\{T_{ij} \leq t\} = P\{T_{n+1} - T_n \leq t | X_n = i, X_{n+1} = j\},$$

  i.e, this is the distribution function of T_{ij}, the conditional sojourn time at state i given that the next transition is to state j.
  Then, the unconditional sojourn time at state i equals $\tau_i = \sum_j p_{ij} T_{ij}$.

Example

A pure-birth process is a special type of Markov renewal process with

$$
\begin{aligned}
Q_{ij}(t) &= 1 - e^{a_i t}, \quad j = i+1, \\
&= 0, \quad \text{otherwise}
\end{aligned}
$$

Then

$$
\begin{aligned}
p_{ij} &= 1, \quad j = i+1, \\
&= 0 \quad \text{otherwise} \\
F_{ij}(t) = Q_{ij}(t), &\quad \tau_i = T_{ij}, \quad j = i+1
\end{aligned}
$$

- A Markov renewal process becomes a Markov process when the transition times are independent exponential and are independent of the next state visited.
- It becomes a Markov chain when the transition times are all identically equal to 1.
- If the state space S is trivial, i.e., there is only one state, then the increments are IID. And, in this case, $\{T_n, n \geq 0\}$ defines a renewal process.
- Semi-Markov processes are used in the study of certain queueing systems.

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Asymptotic Behaviour

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- Let $p_k = \lim_{t\to\infty} P\{Y(t) = k\}$ for $k \in S$.
- Suppose that the embedded Markov chain $\{X_n\}$ is irreducible and positive recurrent with stationary distribution $\{\nu_j, j \in S\}$. That is, $\nu_j = \lim_{n\to\infty} p_{ij}^{(n)}$ exists and are given as the unique nonnegative solution of

$$\nu_j = \sum_{k\in S} \nu_k p_{kj}, \quad j \in S, \quad \sum_j \nu_j = 1.$$

Then, we would expect that p_k to be proportional to $\nu_k \mu_k$, i.e.,

$$p_k = \frac{\nu_k \mu_k}{\sum_{j\in S} \nu_j \mu_j},$$

where $\mu_k = E(\tau_k)$ is the expected sojourn time in state k until the next transition happens at time T_{n+1}.

(Refer to any standard text for a proof)

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