1. In an $M/G/1$ queueing system with usual notations, and with R denoting the residual service time, which of the following is/are true?

   (A) $W_q = \dfrac{\rho E(R)}{1-\rho}$

   (B) $W_q = \dfrac{E(R)}{1-\rho}$

   (C) $L_q = \dfrac{\rho E(R)}{1-\rho}$

   (D) $L_q = \dfrac{E(R)}{1-\rho}$

   Solution:

   Answer: (A)
   Follows from the lecture notes, since $E(R) = E(S^2)/(2E(S))$.

2. The service is given in an $M/G/1$ queueing system in the following manner. When a job enters into service, a coin is tossed. The probability of obtaining a head in the toss is p and tail is $q(= 1 - p)$. If tail comes, the job will not enter in service (service time zero). If head comes, its service time follows an $Exp(q)$ distribution. The average service time $E(S)$ and the variance of service time σ_B^2 are given, respectively, by

   (A) $\dfrac{p}{1-p}$ and $\dfrac{p(2-p)}{(1-p)^2}$.

   (B) $\dfrac{q}{1-q}$ and $\dfrac{p(2-p)}{(1-p)^2}$.

   (C) $\dfrac{p}{1-p}$ and $\dfrac{q(2-p)}{(1-q)^2}$.

   Solution:

   Answer: (A)
   The service time density $b(x)$ is given by $b(x) = q\delta(x) + pqe^{-qx}, \quad x \geq 0$, where $\delta(\cdot)$ is the Dirac delta function or the unit impulse function, and therefore

   $$E(S) = 0 \times q + \frac{1}{q} \times p = \frac{p}{1-p}, \quad \text{and} \quad E(S^2) = 0 \times q + \frac{2}{q^2} \times p = \frac{2p}{(1-p)^2}$$

   $$\implies Var(S) = \sigma_B^2 = \frac{2p}{(1-p)^2} - \frac{p^2}{(1-p)^2} = \frac{p(2-p)}{(1-p)^2}.$$

3. Consider the $M/G/1$ system given in Question 2 above. Assuming the rate of Poisson arrivals is λ and with $\rho = \lambda E(S)$, what is the mean line delay (W_q)?

   (A) $\dfrac{\rho(2-q)}{1-q-\lambda q}$.

   (B) $\dfrac{\rho}{1-p-\lambda p}$.

(C) $\dfrac{\lambda p(2-p)}{2p(1-p-\lambda p)}$.

Solution:

Answer: (B)

$$\text{We know that } W_q = \frac{\lambda E(S^2)}{2(1-\rho)}, \quad \rho = \lambda E(S) = \frac{\lambda p}{1-p}, \quad E(S^2) = \frac{2p}{(1-p)^2}$$
$$W_q = \frac{\lambda \frac{2p}{(1-p)^2}}{2\left(1 - \frac{\lambda p}{1-p}\right)} = \frac{\rho}{1-p-\lambda p}$$

4. We have an $M/D/1$ system for which mean service time is 4 minutes. Then residual service time follows
   - (A) Uniform distribution
   - (B) Poisson distribution
   - (C) Erlang distribution
   - (D) Exponential distribution

Solution:

Answer: (A)

The service time CDF is $B(x) = \begin{cases} 0 & x < 4 \\ 1 & x \geq 4 \end{cases}$. The PDF of the residual service time is

$$b(x) = \frac{1 - B(x)}{E(S)} = \begin{cases} \frac{1}{4} & 0 < x < 4 \\ 0 & \text{otherwise} \end{cases}$$

5. A small company renting cars has 6 cars available. The costs (depreciation, insurance, maintenance, etc.) are 60 rupees per car per day. Customers arrive according to a Poisson process with a rate of 5 customers per day. A customer rents a car for a duration of time with a mean of 1.5 days. Renting a car costs 110 rupees per day. Arriving customers for which no car is available are lost (they will go to another company). Determine the mean profit (in rupees) per day (approximately).
   - (A) 150
   - (B) 167
   - (C) 182
   - (D) 210

Solution:

Answer: (B)

The given system can be modeled as $M/G/c/c$ system with $\lambda = 5$ /day, $E(S) = 1.5$ days, $c = 6$. Using Blocking probability formula $B(c, r) = \frac{r^c/c!}{\sum_{i=0}^{c} r^i/i!}$, $r = \lambda E(S)$, we get $B(6, 7.5) = 0.3615$. By Little's law, $L = r(1 - B(c, r)) = 4.78875$. Mean profit per day is $L * 110 - 6 * 60 = 166.7$.

6. State whether the following statement is TRUE or FALSE.
The departure point system size probabilities are equal to the general time system size probabilities in an $M/G/1$ system.

Solution:

Answer: TRUE, from the lectures.

7. Consider an $H_2/M/1$ system in which the interarrival time is either $Exp(1)$ distribution with probability $\alpha_1 = 1/3$ or $Exp(2)$ distribution with probability $\alpha_2 = 1 - \alpha_1$. The service rate is $\mu = 3$. Then, the single root r_0 of the characteristic equation (required to determine the arrival-point probabilities) is equal to ________

Solution:

Answer: Range:$[0.51, 0.55]$ [Hint: Enter the answer in two decimals]
For $H_2/M/1$, we know Laplace-Stieltjes transform is

$$A^*(s) = \frac{\alpha_1\lambda_1}{s+\lambda_1} + \frac{\alpha_2\lambda_2}{s+\lambda_2} = \frac{1}{3} \times \frac{1}{s+1} + \frac{2}{3} \times \frac{2}{s+2}$$

since $r_0 = A^*(\mu - \mu r_0) = A^*(3 - 3r_0)$ for $G/M/1, \mu = 3$ we find

$$r_0 = \frac{1}{3} \times \frac{1}{4-3r_0} + \frac{2}{3} \times \frac{2}{5-3r_0}$$

$$(r_0 - 1)(9r_0^2 - 18r_0 + 7) = 0$$

on solving it we have three roots $r_0 = 1, 1 + \sqrt{2}/3, 1 - \sqrt{2}/3$,
since $0 < r_0 < 1$, we have $r_0 = 1 - \sqrt{2}/3 = 0.5286$.

8. Consider the $H_2/M/1$ system given in Question 7 above. What is the (approximate) average number of customers in system at arrival points ($L^{(A)}$)?

(A) 1.12

(B) 2.23

(C) 2.75

(D) 1.64

Solution:

Answer: (A)

$$L^{(A)} = \frac{r_0}{1-r_0} = \frac{0.5286}{0.4714} = 1.12.$$