1. Customers arrive at station 1 according to a Poisson process at a rate of 10 per hour, get served sequentially at stations $1, 2, 3$ and 4 in that order and depart after getting served at station 4. The mean service time is 10 minutes at station 1, 15 minutes at station 2, 5 minutes at station 3, and 20 minutes at station 4, all of them being exponentially distributed. The first station has two servers, the second has three servers, the third has a single server, while the last has four servers. Then, if α denotes the steady state probability that there are no customers in station 3 and station 4, then $125\,\alpha$ equals (approximately)

   (A) 1.0926
   (B) 0.4438
   (C) 0.0827
   (D) 2.1829

   Solution:

   Answer: (B)

   The given network is series or tandem open Jackson network with parameters as follows: $k = 4$ nodes, $\gamma_1 = 10$, $\gamma_2 = \gamma_3 = \gamma_4 = 0$, $\mu_1 = 6, \mu_2 = 4, \mu_3 = 12, \mu_4 = 3$ (all per hour), $c_1 = 2, c_2 = 3, c_3 = 1, c_4 = 4$. Here, $r_1 = 10/6, r_2 = 10/4, r_3 = 10/12, r_4 = 10/3$ and $\rho_1 = 10/12, \rho_2 = 10/12, \rho_3 = 10/12, \rho_4 = 10/12$. Since $\rho_i < 1$ for all i, the network is stable.

   By Jackson theorem, each node can be treated as an $M/M/c_i$ node and performance measures can be obtained. Using the formula for p_0 in an $M/M/c$ queue given by

   $$p_0 = \left[\frac{r^c}{c!(1-\rho)} + \sum_{i=0}^{c-1} \frac{r^i}{i!}\right]^{-1},$$

   we get p_0 for the four nodes as $0.0909, 0.0449, 0.1667, 0.0213$. The probability that the station 3 and 4 are empty equals $\alpha = (0.1667)(0.0213) = 0.003551$ and hence $125\alpha = 0.4438$.

2. Consider the four-station queueing network problem given in Question 1 above. The mean number of customers in the network equals (approximately)

   (A) 16.47
   (B) 18.09
   (C) 23.09
   (D) 28.08

   Solution:

   Answer: (C)

   The given network is series or tandem open Jackson network with parameters as follows: $k = 4$ nodes, $\gamma_1 = 10$, $\gamma_2 = \gamma_3 = \gamma_4 = 0$, $\mu_1 = 6, \mu_2 = 4, \mu_3 = 12, \mu_4 = 3$ (all per hour), $c_1 = 2, c_2 = 3, c_3 = 1, c_4 = 4$. Here, $r_1 = 10/6, r_2 = 10/4, r_3 = 10/12, r_4 = 10/3$ and $\rho_1 = 10/12, \rho_2 = 10/12, \rho_3 = 10/12, \rho_4 = 10/12$. Since $\rho_i < 1$ for all i, the network is stable.

   By Jackson theorem, each node can be treated as an $M/M/c_i$ node and performance measures can be obtained. Using the formula for $M/M/c$ queue given by

   $$L = r + \left(\frac{r^c\rho}{c!(1-\rho)^2}\right) p_0, \quad \text{with } p_0 = \left[\frac{r^c}{c!(1-\rho)} + \sum_{i=0}^{c-1} \frac{r^i}{i!}\right]^{-1},$$

we get $L_1 = 5.4545, L_2 = 6.0112, L_3 = 5, L_4 = 6.6219$. Hence, $L = L_1 + L_2 + L_3 + L_4 = 23.0876$.

3. State whether the following statement is TRUE or FALSE:
Consider the four-station queueing network problem given in Question 1 above. The output process from station 3 is a Poisson process with rate 10 per hour.

Solution:

Answer: TRUE.

Referring to the lectures, it can be observed that the network is a series network with no feedback, and satisfies the conditions of Burke's theorem. The output from each node (and therefore the input to the next node) is again Poisson with the same rate as the input process. And, this is true for node 3 too.

4. Consider a three-node open Jackson Network as described below. The external (Poisson process) arrival rate at node 1 is 25 per hour and there are no external arrivals to other nodes. There are two servers at each node. The mean (exponentially distributed) service time at node 1 is 3 minutes, at node 2 it is 2 minutes, and at node 3 it is 150 seconds. After completing the service at node 1, all the customers move to node 2 for service and then to node 3 for further service. After completing the service at node 3, the probability of return to the first node 0.30 and the probability of exiting the network is 0.70. Then, the total mean flow rate to node 2 equals

(A) $\frac{85}{2}$

(B) $\frac{65}{2}$

(C) 25

(D) $\frac{250}{7}$

Solution:

Answer: (D)

This is Jackson network with $k = 3$, $\gamma_1 = 25, \gamma_2 = \gamma_3 = 0, \mu_1 = 20, \mu_2 = 30, \mu_3 = 24, c_1 = c_2 = c_3 = 2$, $R = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0.30 & 0 & 0 \end{bmatrix}$ and $r_{30} = 0.70$. The solution to the traffic equations is given by $\boldsymbol{\lambda} = \boldsymbol{\gamma}(I - R)^{-1} = (250/7, 250/7, 250/7) = (35.7143, 35.7143, 35.7143)$. That is, $\lambda_2 = 250/7$.

5. Consider the three-node open Jackson network given in Question 4 above. If W_1, W_2, W_3 are the expected waiting time at nodes $1, 2, 3$, respectively, then which one of the following is correct?

(A) $W_1 > W_3 > W_2$

(B) $W_1 > W_2 > W_3$

(C) $W_2 > W_1 > W_3$

(D) $W_3 > W_2 > W_1$

Solution:

Answer: (A)

This is Jackson network with $N = 3$, $\gamma_1 = 25, \gamma_2 = \gamma_3 = 0, \mu_1 = 20, \mu_2 = 30, \mu_3 = 24, c_1 = c_2 = c_3 = 2$, $R = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0.30 & 0 & 0 \end{bmatrix}$ and $r_{30} = 0.70$. The traffic equations yields $\boldsymbol{\lambda} = \boldsymbol{\gamma}(I - R)^{-1} = (35.7143, 35.7143, 35.7143)$. Using the formula of $M/M/2$ queue, we get $L_1 = 8.8050, L_2 = 1.8437, L_3 = 3.3336$. Hence, $W_1 = \frac{L_1}{\lambda_1} = 14.8$ min, $W_2 = \frac{L_2}{\lambda_2} = 3.1$ min, $W_3 = \frac{L_3}{\lambda_3} = 5.6$ min. Hence, $W_1 > W_3 > W_2$.

6. A restaurant serves two take-away dishes, pasta and noodles. There are two separate single-server counters, one for pasta and one for noodles. Arrival of customers to the restaurant follow a Poisson process with a mean rate of 20 per hour, with 60% going for pasta counter and 40% going for noodles counter on arrival. 20% of those who finish service at the pasta counter go next to the noodles counter; the other 80% leave the restaurant. Similarly, 10% of those who finish service from the noodles counter go next to the pasta counter; the other 90% leave the restaurant. No customer will revisit the counter that he has already visited. It takes on average 4 minutes to fulfill a pasta order and 5 minutes to fulfil a noodles order, both the service times being exponential. What is the total mean flow rate into the pasta counter? [Hint: Multi-class network]

(A) 12.8

(B) 12

(C) 10.8

(D) 14.4

Solution:

Answer: (A)

The given situation can be modelled by two-class two-node open Jackson network. Class-1 goes first to the pasta counter and Class-2 goes first to the noodles counter. Node-1 represents pasta counter and node-2 represents noodles counter. Under usual notations we have $\lambda = 20/h$, $\mu_1 = 15/h$, $\mu_2 = 12/h$, $\gamma_1 = 20 * 60/100 = 12$, $\gamma_1^{(1)} = 12$, $\gamma_1^{(2)} = 0$, and $\gamma_2 = 20 * 40/100 = 8$, $\gamma_2^{(1)} = 0$, $\gamma_2^{(2)} = 8$, with the routing matrices

$$R^{(1)} = \begin{bmatrix} 0 & 0.2 \\ 0 & 0 \end{bmatrix} \qquad R^{(2)} = \begin{bmatrix} 0 & 0 \\ 0.1 & 0 \end{bmatrix}$$

Solving the traffic equations for the two classes, we get

$$(\lambda_1^{(1)}, \lambda_1^{(2)}) = (\gamma_1^{(1)}, \gamma_1^{(2)}) \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0.2 \\ 0 & 0 \end{bmatrix} \right\}^{-1} \implies (\lambda_1^{(1)}, \lambda_1^{(2)}) = (12, 2.4)$$

$$(\lambda_2^{(1)}, \lambda_2^{(2)}) = (\gamma_2^{(1)}, \gamma_2^{(2)}) \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0.1 & 0 \end{bmatrix} \right\}^{-1} \implies (\lambda_2^{(1)}, \lambda_2^{(2)}) = (0.8, 8)$$

Therefore, we have the total mean flow rates into two nodes as $(\lambda_1, \lambda_2) = (\lambda_1^{(1)} + \lambda_1^{(2)}, \lambda_2^{(1)} + \lambda_2^{(2)}) = (12 + 0.8, 2.4 + 8) = (12.8, 10.4)$. Thus, $\lambda_1 = 12.8$.

7. Consider the pasta-noodles restaurant problem given in Question 6 above. How many (approximately) customers on average are there in the restaurant?

   (A) 5.818

   (B) 6.5

   (C) 10.724

   (D) 12.318

Solution:

Answer: (D)

For each node, we can use $M/M/1$ formula to calculate performance measures. So, $L_1 = \frac{\lambda_1}{\mu_1 - \lambda_1} = \frac{12.8}{15 - 12.8} = 5.818$ and $L_2 = \frac{\lambda_2}{\mu_2 - \lambda_2} = \frac{10.4}{12 - 10.4} = 6.5$. Thus, $L = L_1 + L_2 = 12.318$.

8. Consider the pasta-noodles restaurant problem given in Question 6 above. What is the average sojourn time (in minutes) for a customer at the noodles counter?

   (A) 23.5

   (B) 30

   (C) 37.5

   (D) 44

Solution:

Answer: (C)

We have that $W_2 = \frac{L_2}{\lambda_2} = \frac{6.5}{10.4} = 0.625h. = 37.5$ minutes.