1. State whether the following statement is TRUE or FALSE:
   If $X_1 \sim Exp(\lambda_1)$ and $X_2 \sim Exp(\lambda_2)$, then $P\{X_1 < X_2\} = \dfrac{\lambda_1}{\lambda_1 + \lambda_2}$.

Solution:

**Answer: TRUE**

By conditioning on $X_1$,

$$P(X_1 < X_2) = \int_0^\infty [1 - F_2(t)]\, dF_1(t) = \int_0^\infty e^{-\lambda_2 t}\lambda_1 e^{-\lambda_1 t} dt = \frac{\lambda_1}{\lambda_1 + \lambda_2}.$$

2. Inquiries arrive at an average rate of 15 inquiries per minute and as per a Poisson process. Let $\alpha$ denote the probability that in an 1-minute period, 3 inquiries arrive during the first 10 seconds and 4 inquiries arrive during the last 15 seconds. Then $100\,\alpha$ equals _______.

Solution:

**Answer: Range:** 4.10 **to** 4.20 [Hint: Enter the answer in two decimals.]

The arrival rate per second is $\lambda = \dfrac{15}{60} = \dfrac{1}{4}$. The probability of interest is

$$\begin{aligned} P\{N(10) = 3, N(60) - N(45) = 4\} &= P\{N(10) = 3\}P\{N(60) - N(45) = 4\} \quad \text{(independent increment)} \\ &= P\{N(10) = 3\}P\{N(15) = 4\} \qquad \text{(stationary increment)} \\ &= \frac{\left(\frac{10}{4}\right)^3 e^{-(10/4)}}{3!} \frac{\left(\frac{15}{4}\right)^4 e^{-(15/4)}}{4!} = 0.041423. \end{aligned}$$

Thus, $100\,\alpha = 4.1423$.

3. Let $\{A(t), t \geq 0\}$ and $\{B(t), t \geq 0\}$ be two independent Poisson processes with rates $\alpha$ and $\beta$, respectively. Then, the mean and variance (at time $t$), respectively, of the process $\{A(t) - B(t), t \geq 0\}$ are

   (A) $(\alpha + \beta)t$ and $(\alpha + \beta)t$
   (B) $(\alpha + \beta)t$ and $(\alpha - \beta)t$
   (C) $(\alpha - \beta)t$ and $(\alpha + \beta)t$
   (D) $(\alpha - \beta)t$ and $(\alpha - \beta)t$

Solution:

**Answer: (C)**

Let $C(t) = A(t) - B(t)$. Then the PGF of $C(t)$ is (using independence of $A$ and $B$)

$$P(z) = E(z^{C(t)}) = E(z^{A(t)})E((1/z)^{B(t)}) = e^{-(\alpha+\beta)t}e^{\alpha t z + \beta t/z},$$

from which we obtain $P'(1) = (\alpha - \beta)t$ and $P''(1) = (\alpha + \beta)t + (\alpha - \beta)^2 t^2$. Therefore, $E[C(t)] = (\alpha - \beta)t$ and $Var(C(t)) = (\alpha + \beta)t$.

4. State whether the following statement is TRUE or FALSE:
If $\{N(t), t \geq 0\}$ is a Poisson process with rate $\lambda$, the correlation coefficient between $N(s)$ and $N(s+u)$, where $s, u > 0$, equals $\sqrt{\dfrac{u}{s+u}}$.

Solution:

**Answer: FALSE**

For a Poisson process, we have $E(N(t)) = \lambda t, Var(N(t)) = \lambda t$ and $E(N^2(t)) = \lambda t + (\lambda t)^2$. Using the independent increment property, we have

$$E[N(s)N(s+u)] = E[N(s)(N(s+u) - N(s))] + E[N^2(s)] = \lambda s \lambda u + \lambda s + \lambda^2 s^2$$

and hence the autocovariance between $N(s)$ and $N(s+u)$ is obtained as $Cov(N(s), N(s+u)) = \lambda s$. Therefore, the autocorrelation function is given by

$$\rho_{s,s+u} = \frac{Cov(N(s), N(s+u))}{\sqrt{Var(N(s))Var(N(s+u))}} = \sqrt{\frac{s}{s+u}}.$$

5. Suppose that patients arrive at a hospital according to a Poisson process with a rate of 10 per hour. If you know that one patient has arrived between 10:00am and 11:00am on a particular day, then the probability that he/she would have arrived between 10:10am and 10:30am equals _______.

Solution:

**Answer: Range:** 0.32 **to** 0.34 [Hint: Enter the answer in two decimals.]

Using the independent increments property and the conditional distribution of arrival times of a Poisson process (see the lectures), we see that the time of arrival of the patient is uniformly distributed between 0 and 60 minutes. Hence, the required probability is $1/3 = 0.3333$.

6. Which of the following matrices cannot be a generator matrix for a CTMC?

(A) $\begin{bmatrix} -4 & 3 & 1 \\ 3 & -6 & 2 \\ 2 & 3 & -5 \end{bmatrix}$

(B) $\begin{bmatrix} -7 & 7 & 0 \\ 0 & 0 & 0 \\ 3 & 3 & -6 \end{bmatrix}$

(C) $\begin{bmatrix} 5 & -5 & 0 \\ -1 & 3 & -2 \\ 0 & -1 & 1 \end{bmatrix}$

(D) $\begin{bmatrix} -6 & 4 & 2 \\ 1 & -8 & 7 \\ 4 & 5 & -9 \end{bmatrix}$

Solution:

**Answer: (A) and (C)**

For $Q$ to be a generator matrix, we need to have $q_{ii} \leq 0, q_{ij} \geq 0, j \neq i$ and $\sum_j q_{ij} = 0, \forall i$. Only (B) and (D) satisfy these, while (A) and (C) do not.

7. Let $\mathbf{P}(t) = ((P_{ij}(t)))$ denote the matrix of the transition probability functions of a CTMC on the state space $S = \{1, 2\}$ and with the generator (rate) matrix $Q = \begin{bmatrix} -a & a \\ b & -b \end{bmatrix}$. Then $P_{11}(t)$ and $P_{22}(t)$, respectively, are

(A) $\dfrac{a}{a+b} + \dfrac{b}{a+b}e^{-(a+b)t}$ and $\dfrac{b}{a+b} + \dfrac{a}{a+b}e^{-(a+b)t}$

(B) $\dfrac{b}{a+b} + \dfrac{a}{a+b}e^{-(a+b)t}$ and $\dfrac{a}{a+b} + \dfrac{b}{a+b}e^{-(a+b)t}$

(C) $\dfrac{a}{a+b} - \dfrac{b}{a+b}e^{-(a+b)t}$ and $\dfrac{b}{a+b} - \dfrac{a}{a+b}e^{-(a+b)t}$

(D) $\dfrac{b}{a+b} - \dfrac{a}{a+b}e^{-(a+b)t}$ and $\dfrac{a}{a+b} - \dfrac{b}{a+b}e^{-(a+b)t}$

Solution:

**Answer: (B)**

The forward Kolmogorov equations, for $i = 1, 2$, are

$$P'_{i1}(t) = -aP_{i1}(t) + bP_{i2}(t), \quad P'_{i2}(t) = aP_{i1}(t) - bP_{i2}(t)$$

which gives us $P'_{11}(t) - (a+b)P_{11}(t) = b$ and $P'_{22}(t) + (a+b)P_{22}(t) = a$. Solving these differential equations using the conditions $P_{11}(0) = 1 = P_{22}(0) = 1$, we get

$$P_{11}(t) = \frac{b}{a+b} + \frac{a}{a+b}e^{-(a+b)t} \quad \text{and} \quad P_{22}(t) = \frac{a}{a+b} + \frac{b}{a+b}e^{-(a+b)t}.$$

8. Which of the following statements is/are true for CTMCs?

(A) The sojourn time (or holding time or residence time) at (a non-absorbing) state $i$ of a CTMC is exponentially distributed with parameter $-q_{ii}$.

(B) If a positive recurrent CTMC starts off with its stationary distribution, then it retains that distribution as its state distribution forever after.

(C) There exists an irreducible CTMC wherein all the states are not positive recurrent.

(D) A birth-death process will always have a stationary distribution.

Solution:

**Answer: (A), (B) and (C)**

By the properties of CTMCs that we have seen in the lectures, it follows that (A), (B) and (C) are true, while (D) is not true (Take constant rates $\lambda$ and $\mu$ with $\lambda > \mu$).