MODULE 11: Semi-Markovian Queueing Systems (contd...)

LECTURE 43

G/M/1 Queues

11_GM1 queues.pdf

General Input, Single Server (G/M/1) Queues

- All assumptions of $M/M/1$ are in place, except the assumption of Poisson arrivals.
- Interarrival times follow a general distribution and are IID.
- We are then dealing with a $G/M/1$ queueing system in steady state.
- As in $M/G/1$, we consider an embedded Markov chain approach here too.
- Here, we consider the system at arrival times in contrast with the $M/G/1$ model where we considered th system at departure times.
- Let $X_n = N(t_n-)$ denote the number of customers in the system just prior to the arrival of the nth customer. Then,

$$X_{n+1} = X_n + 1 - B_n, \quad \text{if } B_n \leq X_n + 1,\ X_n \geq 0,$$

where B_n is the number of customers served during the interarrival time $t_{n+1} - t_n$ between the n^{th} and $(n+1)^{st}$ arrivals. Since $t_{n+1} - t_n$'s are IID, they have a common CDF by $A(t)$ and B_n's are also IID.

- Since B_n does not depend on the past history of the queue, given X_n at the time of nth arrival, $\{X_0, X_1, X_2, \ldots\}$ is a Markov chain.
- Define the probability that there are exactly k service completions between two consecutive arrivals (given that $\geq k$ just prior to the first arrival) as

$$b_k = \int_0^\infty \frac{e^{-\mu t}(\mu t)^k}{k!} dA(t)$$

That is, $b_k = P\{B_n = k | X_n \geq k\}$.

- The single-step transition probability matrix for the embedded Markov chain $\{X_0, X_1, X_2, \ldots\}$ is

$$P = ((p_{ij})) = \begin{array}{c c} & \begin{array}{cccccc} 0 & 1 & 2 & \cdots & & \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ \vdots \end{array} & \begin{bmatrix} 1 - b_0 & b_0 & 0 & 0 & 0 & \cdots \\ 1 - \sum_{k=0}^{1} b_k & b_1 & b_0 & 0 & 0 & \cdots \\ 1 - \sum_{k=0}^{2} b_k & b_2 & b_1 & b_0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \end{bmatrix} \end{array}$$

G/M/1-type matrix

- The Markov chain is irreducible and aperiodic. It can be shown that it is positive recurrent when $\sum_{n=1}^{\infty} nb_n > 1$. Then the chain is ergodic and a steady state solution exists and denote by $\mathbf{a} = \{a_n\}, \quad n = 0, 1, 2, \ldots$ the probability vector that an arrival finds n in the system.
- We then have the usual stationary equations as

$$\mathbf{a}P = \mathbf{a} \quad \text{and} \quad \mathbf{ae} = 1$$

which can be written explicitly as

$$a_i = \sum_{k=0}^{\infty} a_{i+k-1} b_k, \quad i \geq 1,$$

$$a_0 = \sum_{j=0}^{\infty} a_j \left(1 - \sum_{k=0}^{j} b_k\right). \tag{1}$$

- We now use the operators method to solve the equations. Let $Da_i = a_{i+1}$. For $i \geq 1$, we can write the above equation as

$$a_i - (a_{i-1}b_0 + a_i b_1 + a_{i+1}b_2 + \dots) = 0$$
$$\implies \quad a_{i-1}(D - b_0 - Db_1 - D^2 b_2 - D^3 b_3 - \dots) = 0$$

- The characteristic equation for this difference equation is

$$z - b_0 - zb_1 - z^2 b_2 - z^3 b_3 - \cdots = 0, \quad i.e., \ \sum_{n=0}^{\infty} b_n z^n = z.$$

- Since $\{b_n\}$ is a probability distribution, the LHS is the PGF and hence the above becomes

$$\boxed{\beta(z) = \sum_{n=0}^{\infty} b_n z^n = z.}$$

- As in the case of $M/G/1$, it can be shown that $\beta(z) = A^*[\mu(1-z)]$, where A^* is LST of the interarrival-time CDF, and hence the above equation can be written as

$$\boxed{z = A^*[\mu(1-z)].}$$

- If we can find solutions of the characteristic equation then we can determine $\{a_n\}$.
- Assume that $\rho = \frac{\lambda}{\mu} < 1$. Then it can be shown that there is exactly one real root, say r_0, in $(0, 1)$ (We will prove this below). Hence, we have

$$a_n = Cr_0^n, \quad n \geq 0.$$

As usual, C can be determined to be $C = 1 - r_0$, giving us the the steady state arrival-point system size distribution as

$$a_n = (1 - r_0)r_0^n, \quad n \geq 0, \ \lambda/\mu < 1.$$

- We will now prove that r_0 is the single root of the characteristic equation.

- We consider the two sides of the equation $\beta(z) = \sum_{n=0}^{\infty} b_n z^n = z$ separately as

$$y = \beta(z) \quad \text{and} \quad y = z.$$

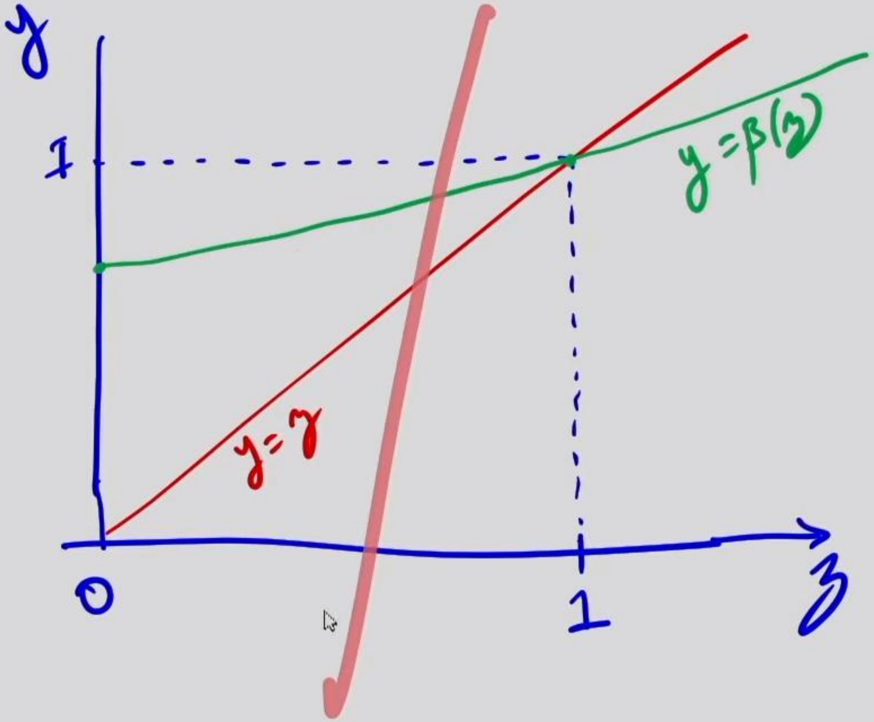
- Observe that $0 < \beta(0) = b_0 < 1$ and $\beta(1) = \sum_{n=0}^{\infty} b_n = 1$.
- Also, $\beta(z)$ is monotonically nondecreasing and convex because

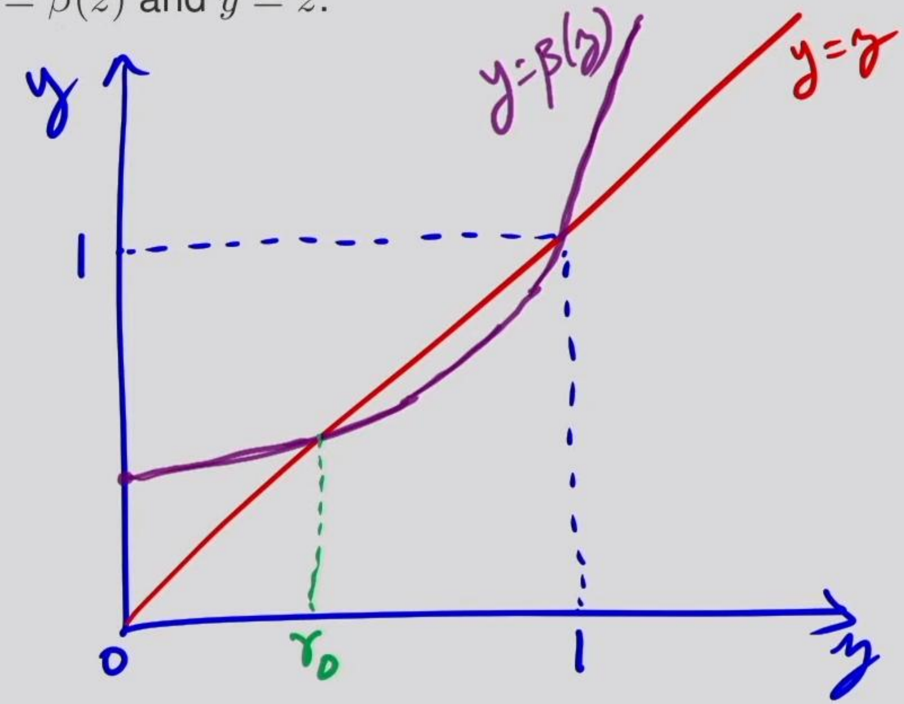
$$\beta'(z) = \sum_{n=1}^{\infty} n b_n z^{n-1} \geq 0,$$

$$\beta''(z) = \sum_{n=1}^{\infty} n(n-1) b_n z^{n-2} \geq 0.$$

- Further, since the service times are exponential, each b_n is strictly positive (i.e., $b_n > 0$ for $n \geq 0$), and this implies that $\beta(z)$ is strictly convex.

- There are two possible cases for the graphs of $y = \beta(z)$ and $y = z$:





- Either there are no intersections in $(0, 1)$, or there is exactly one intersection in $(0, 1)$.
  - The latter case occurs when $\beta'(1) = E[\text{number served during interarrival time}] = \mu/\lambda > 1$.
  - That is, when $\rho = \lambda/\mu < 1$, there is exactly one root r_0 in $(0, 1)$.

- We can show that r_0 is the only complex root with absolute value less than one, by using Rouche's theorem.
- Assume that $\beta'(1) = 1/\rho > 1$.
  Let $f(z) = -z$ and $g(z) = \beta(z)$.
  Because $g(1) = 1$ and $g'(1) > 1$, we have $g(1-\epsilon) < 1-\epsilon$ for small enough $\epsilon > 0$.
  Consider the set z such that $|z| = 1-\epsilon$. By the triangle inequality,

$$|g(z)| \leq \sum_{n=0}^{\infty} b_n |z|^n = g(1-\epsilon) < 1-\epsilon = |f(z)|.$$

  By Rouche's theorem, $f(z) = -z$ and $f(z) + g(z) = -z + \beta(z)$ have the same number of roots within the contour $|z| = 1-\epsilon$.
  Since ϵ can be made arbitrarily small, there is exactly one complex root of $z = \beta(z)$ whose absolute value is less than one.
  Thus, it must be the real root r_0 found earlier.

- Finding r_0 involves numerical procedures, but it is readily obtainable.
  - ▶ For example, the method of successive substitution

$$z^{(k+1)} = \beta(z^{(k)}), \quad k = 0, 1, 2, \ldots, \; 0 < z^{(0)} < 1$$

  is guaranteed to converge because of the shape of $\beta(z)$.

- In summary, the steady state arrival-point system size distribution is

$$a_n = (1 - r_0) r_0^n, \quad n \geq 0, \; \rho = \lambda/\mu < 1.$$

- Note the analogy between the above and that of $M/M/1$ (keeping in mind the fact the above are arrival-point probabilities, and not general-time probabilities).
  - ▶ Therefore, using a similar formula of $M/M/1$, we can get the expected measures only at arrival time.
- Unlike $M/G/1$, here for $G/M/1$, it is not true here that $a_n = p_n$ in general.
  - ▶ But, there exists a relationship between them and it can be obtained as $p_n = \rho a_{n-1}$ for $n \geq 1$.
  - ▶ In fact $a_n = p_n$ here if and only if the arrivals are Poisson, i.e., $G = M$.

- We use a superscript (A) to denote the particular measure of effectiveness is taken relative to arrival points only. We have

$$L^{(A)} = \frac{r_0}{1-r_0}, \quad L_q^{(A)} = \frac{r_0^2}{1-r_0}$$

- The line delay and the system-waiting-time distribution functions, $F_{T_q}(t)$ and $F_T(t)$ can also be obtained from $M/M/1$ by replacing ρ with r_0 to yield

$$F_{T_q}(t) = 1 - r_0 e^{-\mu(1-r_0)t}, \quad t \geq 0,$$
$$F_T(t) = 1 - e^{-\mu(1-r_0)t}, \quad t \geq 0,$$

with mean values

$$W_q = \frac{r_0}{\mu(1-r_0)}, \quad \text{and} \quad W = \frac{1}{\mu(1-r_0)}.$$

Note that the probability that a customer does not have to wait is given by $1 - r_0$.
♦ The above results refer to the distribution of the waiting times as observed by customers arriving to the system.

Example (M/M/1)

For exponentially distributed interarrival times, we have $A^*(s) = \lambda/(s+\lambda)$.

Hence, the equation $z = A^*(\mu - \mu z)$ reduces to $z = \dfrac{\lambda}{\lambda + \mu - \mu z} \Rightarrow (z-1)(\lambda - \mu z) = 0$.

Thus, the desired root is $r_0 = \rho = \lambda/\mu$ and hence the arrival-point distribution is

$$a_n = (1-\rho)\rho^n, \quad n = 0, 1, 2, \ldots.$$

Note that this is also the steady state solution of $M/M/1$, as expected.

Example

Suppose that the interarrival time consist of two exponential phases, the first phase with parameter μ and the second one with parameter 2μ, where μ is also the parameter of the exponential service time. The Laplace-Stieltjes transform of the interarrival time is then

$$A^*(s) = \frac{2\mu^2}{(s+\mu)(s+2\mu)}.$$

Hence, the equation $z = A^*(\mu - \mu z)$ reduces to

$$z = \frac{2\mu^2}{(2\mu - \mu z)(3\mu - \mu z)} = \frac{2}{(2-z)(3-z)} \Rightarrow (z-1)(z-2-\sqrt{2})(z-2+\sqrt{2}) = 0.$$

Thus, the desired root is $r_0 = 2 - \sqrt{2} = 0.5858$ and hence the arrival-point distribution is

$$a_n = (\sqrt{2}-1)(2-\sqrt{2})^n, \quad n = 0, 1, 2, \ldots.$$

11_GM1 queues.pdf *

Create account Login

13 / 23

Example

Suppose that there is a single-server queueing system with exponentially distributed service times, but there is no basis for assuming either exponential or E_k as interarrival times.
From data, it was observed that a k-point distribution fits well the interarrival times. That is,

$$P\{\text{interarrival time } = t_i\} = a(t_i) = a_i, \quad 1 \leq i \leq k.$$

We must find the root r_0 of the characteristic equation $z = A^*[\mu(1-z)] = \sum_{i=1}^{k} a_i e^{-\mu t_i (1-z)}$.

For illustration, assume a 3-point distribution $a_1 = a(2) = 0.2, a_2 = a(3) = 0.7, a_3 = a(4) = 0.1$ (assuming time in minutes).
One can then find the root r_0 of $z = A^*[\mu(1-z)]$ using successive substitution and then obtain the performance measures.
Exercise: Complete the exercise for the above assuming that $1/\mu = 2$ minutes. The root will be $r_0 = 0.467$.

Multiserver $G/M/c$ Queues

- Unlike the case of $M/G/c$ case, the embedded Markov chain approach is still applicable in the case of $G/M/c$.
- The analysis procedure remains the same, with the major exception of the value of b_n and its effect on the embedded matrix and the root-finding problem.
- The mean service rate is either $n\mu$ or $c\mu$ depending on the state, and hence b_n will now depend on both i and j.
- One can obtain, after some lengthier steps, the arrival-point system size probabilities (and so is the line-delay distribution).
- Other extensions of $G/M/1$ can also be thought of (Refer to Cohen(1982) for more details).