DEPARTMENT OF MATHEMATICS
INDIAN INSTITUTE OF TECHNOLOGY GUWAHATI

MA 597 Queueing Theory and Applications January - May 2019

Exercise Sheet - 2

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1. Three children (denoted by $1, 2, 3$) arranged in a circle play a game of throwing a ball to one another. At each stage the child having the ball is equally likely to throw it into any one of the other two children. Suppose that X_0 denotes the child who had the ball initially and X_n $(n \geq 1)$ denotes the child who had the ball after n throws. Show that $\{X_n\}$ forms a Markov chain. Find P and draw the transition probability diagram. Assuming an initial distribution $(1/2, 1/4, 1/4)$, calculate $P\{X_2 = 1|X_0 = 1\}, P\{X_2 = 2|X_0 = 3\}, P\{X_0 = 3|X_2 = 2\}$, and also the probability that the child who had originally the ball will have it after two throws.

   Also, find P if the number of children is m (≥ 3). Observe that for this P the column sums are also unity. Such a matrix is called a doubly stochastic matrix. Show that the equilibrium distribution of such a DTMC is discrete uniform.

2. Show that a state i in a DTMC is recurrent if and only if $\sum_{n=1}^{\infty} p_{ii}^{(n)} = \infty$, and deduce that recurrence is a class property. Consider an one-dimensional random walk on the set of integers $\mathbb{Z}$ and find out when it is recurrent.

3. Consider a system with two components. We observe the state of the system every hour. A given component operating at time n has probability p of failing before the next observation at time $n+1$. A component that was in a failed condition at time n has a probability r of being repaired by time $n+1$, independent of how long the component has been in a failed state. The component failures and repairs are mutually independent events. Let X_n be the number of components in operation at time n. $\{X_n, n \geq 0\}$ is a discrete-time homogeneous Markov chain with state space $S = \{0, 1, 2\}$. Determine its transition probability matrix P, and draw the transition probability diagram. Does the steady state probability distribution exist? If so, obtain it.

4. Trials are performed in sequence. If the last two trials were successes, then the next trial is a success with probability 0.8; otherwise the next trial is a success with probability 0.5. In the long run, what proportion of trials are successes?

5. Let $\{X_n, n \geq 0\}$ denote an ergodic Markov chain with limiting probabilities π_i. Define the process $\{Y_n, n \geq 1\}$ by $Y_n = (X_{n-1}, X_n)$, that is, Y_n keeps track of the last two states of the original chain. Is $\{Y_n, n \geq 1\}$ a Markov chain? If so, determine its transition probabilities and find $\lim_{n\to\infty} P\{Y_n = (i, j)\}$.