MODULE 10: Semi-Markovian Queueing Systems

LECTURE 36

Renewal Processes

Renewal Processes

Definition

Let $\{X_n, n = 1, 2, \ldots\}$ be a sequence of non-negative independent and identically distributed random variables with distribution function F and finite mean μ. Define the sequence $\{S_n, n \geq 0\}$ by

$$S_0 = 0, \quad S_n = S_{n-1} + X_n = X_1 + X_2 + \cdots + X_n, \quad n \geq 1$$

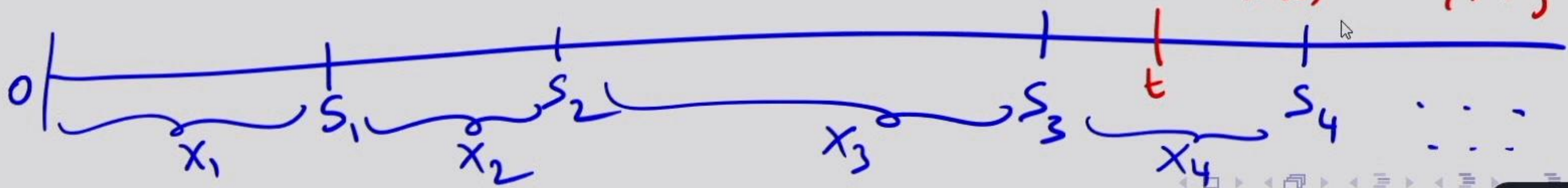
The random variable S_n is called the nth renewal time, while the time duration X_n is called the nth renewal interval. Further, define the random variable of the number of renewals until time t by

$$N(t) = \sup\{n : S_n \leq t\}$$

Then the continuous-time process $\{N(t), t \geq 0\}$ is called a renewal process with distribution F (or generated or induced by the distribution F).

We may also say that $\{X_n\}$ defines a renewal process.

$N(t) = \max\{1,2,3\} = 3$



- If $S_n = t$ for some n, then a renewal is said to occur at t and hence S_n gives the time (epoch) of the nth renewal, and is called nth renewal (or regeneration) epoch. The random variable $N(t)$ gives the number of renewals occurring in $[0, t]$.
- The random variable X_n gives the inter-event time (or waiting time) between $(n-1)$th and nth renewals. The inter-event times are independently and identically distributed.
- The Poisson process is the unique renewal process with the Markov property. This generalization of the Poisson process is obtained by removing the restriction of exponential distributed holding times and by considering that the inter-event times as IID nonnegative random variables with an arbitrary distribution.

Example

Consider a stage in an industrial process relating to production of a certain component in batches. Immediately on completion of production of a batch, that of another batch is undertaken. Suppose that the times taken to produce successive batches are IID random variables with distribution F. We get a renewal process with distribution F.

- We will always assume that $P\{X_i = 0\} = 0$. The strong law of large numbers implies that $S_n/n \to \mu$ with probability one as $n \to \infty$. Hence $S_n < t$ cannot hold for infinitely many n and thus $N(t)$ is finite with probability one.
- We have the distribution function of S_n, for $n \geq 1$ as $F_n(x) = P\{S_n \leq x\} = F^{n*}(x)$, where F^{n*} is the n-fold convolution of F with itself.
  - ▶ The above follows from the fact that, if X and Y are independent and distributed according to CDFs F and G, respectively, then

$$P\{X + Y \leq t\} = F * G(t) = \int_0^t G(t-u)dF(u), \quad \text{for all } t \geq 0.$$

- It can be shown that $\lim_{t\to\infty} \frac{N(t)}{t} = \frac{1}{\mu}$ holds with probability one. Therefore, the quantity $1/\mu$ (i.e., the inverse of the mean length of a renewal interval) is called the **rate** of the renewal process.

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Renewal Function and Renewal Equation

- Observe that $\{N(t) \geq n\} \Leftrightarrow \{S_n \leq t\}$ (or equivalently $\{N(t) < n\} \Leftrightarrow \{S_n > t\}$). Therefore, the distribution of $N(t)$ is given by

$$
\begin{aligned}
p_n(t) = p\left\{N(t) = n\right\} &= P\left\{N(t) \geq n\right\} - P\left\{N(t) \geq n+1\right\} \\
&= P\left\{S_n \leq t\right\} - P\left\{S_{n+1} \leq t\right\} = F_n(t) - F_{n+1}(t) \\
&= F^{n*}(t) - F^{(n+1)*}(t).
\end{aligned}
$$

- The function $M(t) = E(N(t))$ is called the renewal function of the renewal process with distribution F. The renewal function plays a fundamental role in renewal theory. The expected number of renewals in $[0, t]$ is given by

$$
\begin{aligned}
M(t) = \sum_{n=0}^{\infty} n p_n(t) &= \sum_{n=0}^{\infty} n\left\{F^{n*}(t) - F^{(n+1)*}(t)\right\} = \sum_{n=1}^{\infty} F^{n*}(t) \\
&= F(t) + \sum_{n=1}^{\infty} F^{(n+1)*}(t).
\end{aligned}
$$

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- Now, observe that $\sum_{n=1}^{\infty} F^{(n+1)*}(t) = \sum_{n=1}^{\infty} \int_0^t F^{n*}(t-x)dF(x) = \int_0^t \left\{ \sum_{n=1}^{\infty} F^{n*}(t-x) \right\} dF(x)$, assuming that interchange of summation and integration is valid.
- Substituting the above in $M(t)$ above, we get the fundamental equation of renewal theory or renewal equation given by

$$\boxed{M(t) = F(t) + \int_0^t M(t-x)dF(x).}$$

- Renewal theorems (elementary renewal theorem, Blackwell's theorem, key renewal theorem) involving limiting behaviour of $M(t)$ are powerful results in renewal theory and are important from the point of view of applications (Refer to any standard text, like Ross).

Residual and Excess Lifetimes

- We consider two random variables of interest in renewal theory. For any given $t > 0$, there corresponds a unique $N(t)$ such that

$$S_{N(t)} \leq t < S_{N(t)+1} \quad \{\text{i.e. } t \text{ falls in the interval } X_{N(t)+1}\}$$

- **The residual (or excess) lifetime** at time t is given by the time from t to the next renewal epoch, i.e.

$$Y(t) = S_{N(t)+1} - t \quad \{\text{It is also called forward recurrence time at } t\}$$

- **The spent (or current) lifetime or age** time t is given by the the time to t since the last renewal epoch, i.e.

$$Z(t) = t - S_{N(t)} \quad \{\text{It is also called backward recurrence time at } t\}$$

- The total lifetime at t (or length of the lifetime containing t) is given by

$$Y(t) + Z(t) = S_{N(t)+1} - S_{N(t)} = X_{N(t)+1}.$$

- These random variables $Y(t)$ and $Z(t)$ arise naturally in queueing contexts (e.g., arrivals, departures).
- *Definition:* A non-negative RV X (and also its CDF F) is called **lattice** if there is a positive number $d > 0$ with $\sum_{n=0}^{\infty} P\{X = nd\} = 1$. If X is lattice, then the largest such number d is called the period of X (and F).
- **The distribution of** $Y(t)$ can be obtained as

$$P\{Y(t) \leq x\} = F(t+x) - \int_0^t [1 - F(t+x-y)]dM(y), \quad x \geq 0 \quad \text{[and } 0 \text{ for } x \leq 0\text{]}.$$

If F is non-lattice, then the limiting distribution of $Y(t)$ is

$$P\{Y \leq x\} = \lim_{t\to\infty} P\{Y(t) \leq x\} = \frac{1}{\mu}\int_0^x [1 - F(y)]dy, \quad x \geq 0.$$

- Noting that $\{Y(t) > x\} = \{Z(t-x) > x\}$, the distribution of $Z(t)$ can be deduced as

$$P\{Z(t) \le x\} = \begin{cases} 0, & x \le 0 \\ F(t) - \int_0^{t-x} [1 - F(t-y)] dM(y), & 0 < x \le t \\ 1, & x > t \end{cases}$$

  If F is non-lattice, then the limiting distribution of $Z(t)$ is

$$P\{Z \le x\} = \lim_{t\to\infty} P\{Y(t) \le x\} = \frac{1}{\mu} \int_0^x [1 - F(y)] dy, \quad x \ge 0.$$

- When these exist, the two limiting distributions Y and Z are identical. It can be easily verified that for exponential X_i, the distributions of $Y(t)$ and $Z(t)$ are again exponential with the same mean $\mu = E(X_i)$.
- The mean of Y and Z can obtained as $E(Y) = E(Z) = \dfrac{E(X_i^2)}{2E(X_i)}$.
- If F is a lattice distribution, then the distributions of $Y(t)$ and $Z(t)$ have no limits for $t \to \infty$ except in some special cases.

Some Generalizations of (Ordinary) Renewal Process

- **Delayed (modified) Renewal Process:** First, suppose that the first inter-arrival time X_1 (i.e. time from the origin upto the first renewal) has a distribution G which is different from the common distribution F of the remaining inter-arrival times $X_2, X_3....$ i.e. the initial distribution G is different from subsequent common distribution F. We then get what is known as a modified or delayed renewal process. Such a situation arises when the component used at $t = 0$ is not new. When $G \equiv F$, the modified process reduces to the ordinary renewal process.
- **Alternating renewal processes.** Consider a stochastic process $\{X(t), t \geq 0\}$ with state space $\{0, 1\}$. Suppose the process starts in state 1 (also called the 'up' state). It stays in that state X_1 amount of time and then jumps to state 0 (also called the 'down' state). It stays in state 0 for Y_1 amount of time and then goes back to state 1. This process repeats forever, with X_n being the nth up time, and Y_n the nth down time. The nth up time followed by the nth down time is called the nth cycle.

Example

we consider the working of a component, the lifetime (or time to failure) being given by a sequence $\{X_n\}$ of IID random variables, on the assumption that the detection of failure and repair or replacement of the failed component take place instantaneously. Here the corresponding system has only one state-the working state and a renewal occurs at the termination of a working state (ar failure af a component).

Consider now that the detection and repair or replacement of a failed item are not instantaneous and that the time taken to do so is a random variable. The system then has two states-the working state and the repair state (during which repair of the failed component or search for a new one is under way). Here the two sequences of states-the working states and the repair (failed) state alternate. Suppose that the duration of the working states (or lifetimes or limes to failure) are given by a sequence IID random variables and the duration of repair states (times taken to repair or search) are given by a sequence of IID random variables. We have then an alternating renewal processes or two-stage renewal process.