MODULE 9: Queueing Networks (contd...)

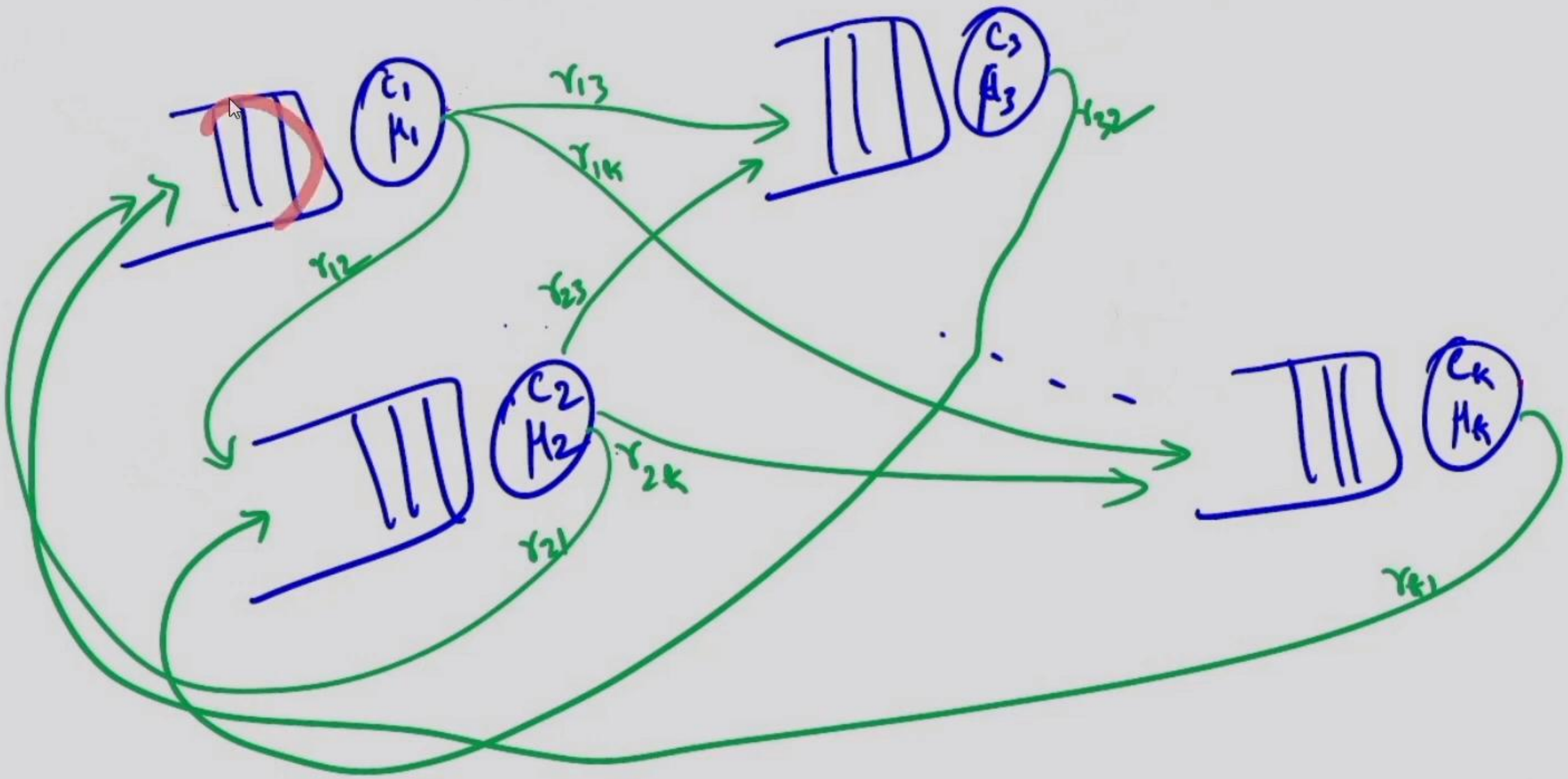
LECTURE 32

Closed Jackson Networks

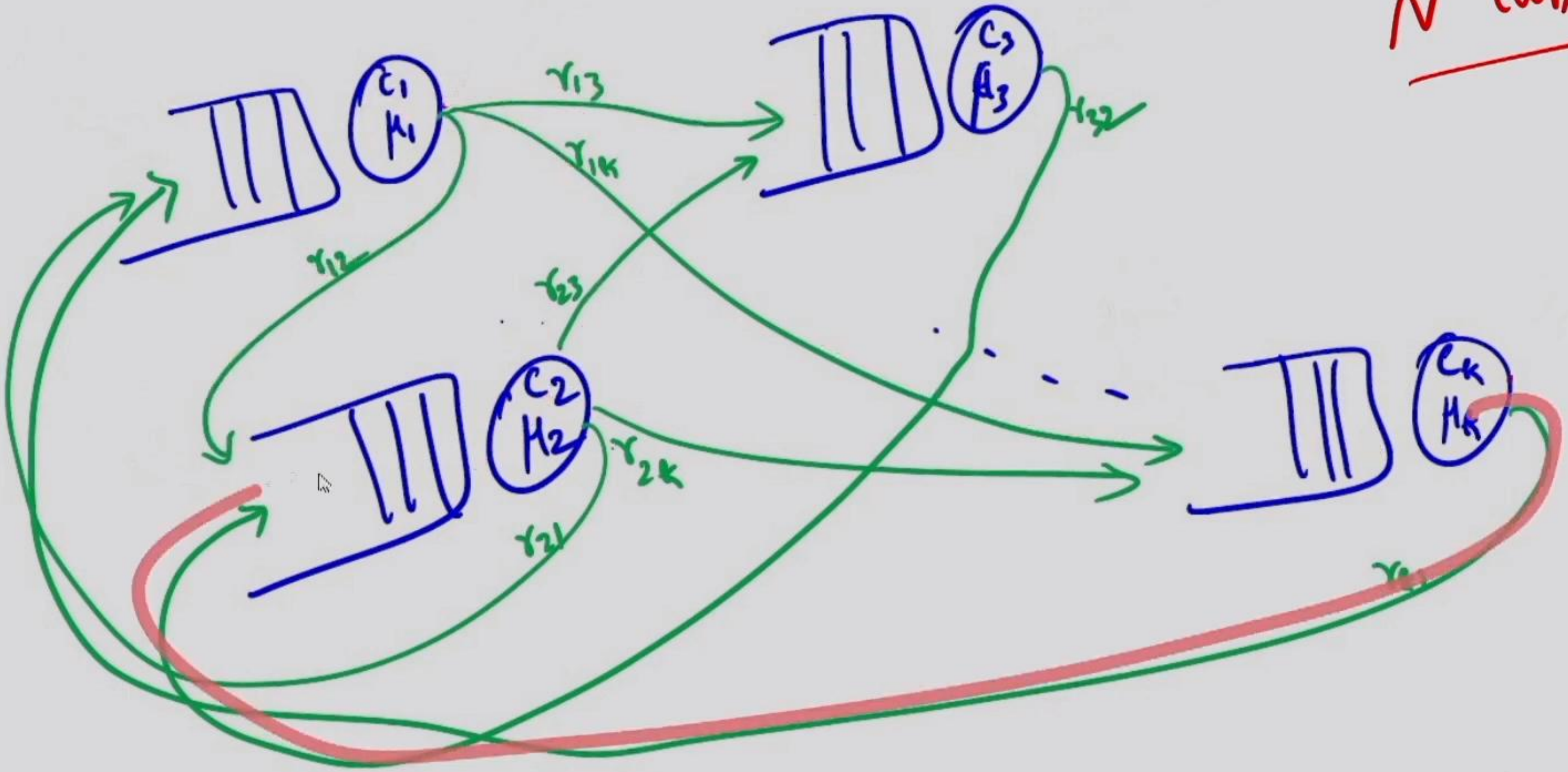
Closed Jackson Networks

- A general Jackson network with $\gamma_i = 0$ and $r_{i0} = 0,\ \forall i$, is called as a closed Jackson network. This is also known as a Gordon-Newell network.
- The simple finite-source queue (machine repairmen problem) is a closed network, with two nodes (one representing operating machines and the other the repair facility).
  ▶ Here, $i = 1, 2; j = 0, 1, 2; r_{12} = r_{21} = 1$ and all other r_{ij}'s are zero (in a general Jackson network).
- In a closed queueing network, a fixed number of customers continually circulate in the system getting routed from one node to another. New customers do not enter the system nor do customers leave the system. In case of multiple classes, it must be closed for all classes.
- Such a network is equivalent to a finite source queueing system of N (say) items that continuously travel inside the network.

- We now have a setup for closed Jackson networks as:
  - A network of k service nodes.
  - Service rate (exponential) at node i is μ_i, with c_i servers at node i.
  - Routing probability is r_{ij}, $1 \le i, j \le k$ (independent of the system state).
  - There are N customers in total in the network. No new customers arrive to the network and no customer departs from the system.
  - No limit on queue capacity at any node (no blocking).
- We have a Markovian system and the state of the system can be described via N_i's, where N_i is the random variable for number of customers (in queue and in service) at node i in steady-state.
  - ▶ Note that $N_1 + N_2 + \cdots + N_k = N$.


c1 μ1
r13
r1K
r12
r23
c2 μ2
r2K
r21
c3 μ3
r32
cK μK
rK1

N customers



- As usual, we want the joint distribution $P\{N_1 = n_1, \ldots, N_k = n_k\} = p_{n_1,n_2,\ldots,n_k}$ from which we can obtain other required quantities.
- The notation for the k-component vector is as follows:

State	Simplified Notation
$n_1, n_2, \ldots, n_i, \ldots, n_j, \ldots, n_k$	$\overline{n}$
$n_1, n_2, \ldots, n_i + 1, \ldots, n_j - 1, \ldots, n_k$	$\overline{n}; i^+ j^-$

- Assume for now that $c_i = 1, \forall i$ (i.e., single server at each node).
- The stochastic balance (global) equation for state $\bar{n}$ with $n_i \geq 1, \forall i$ is:

$$\sum_{j=1}^{k} \sum_{\substack{i=1 \\ (i \neq j)}}^{k} \mu_i r_{ij} p_{\bar{n};i^+j^-} \quad = \quad \sum_{i=1}^{k} \mu_i (1 - r_{ii}) p_{\bar{n}}$$

▶ The above will also hold for the case $n_i = 0$ if we set terms with negative subscripts and terms containing μ_i for which $n_i = 0$ to zero.

- Since this network is a special case of a general Jackson network, so we have a product-form solution

$$p_{\bar{n}} = C\rho_1^{n_1}\rho_2^{n_2}\dots\rho_k^{n_k},$$

where $\rho_i = \lambda_i/\mu_i$ must satisfy the balance equations for flow at each node i, so that the flows into and out of node i are equal.

- Therefore, the traffic equations now become

$$\lambda_i = \mu_i\rho_i = \sum_{j=1}^{k} \lambda_j r_{ji} = \sum_{j=1}^{k} \mu_j r_{ji}\rho_j, \quad i = 1, 2, \dots, k.$$

- As earlier, we assume that the routing matrix R is irreducible and non-absorbing.
- But, one of the traffic equations is redundant (as $\sum_i \lambda_i$ is fixed). We can set one ρ_i equal to 1 when solving $\mu_i\rho_i = \sum_{j=1}^{k} \mu_j r_{ji}\rho_j$.
- *Exercise: Verify that the product-form solution satisfies the balance equations.*

- For this closed network, C is not a pruduct of terms and must be evaluated from

$$\sum_{n_1+n_2+\cdots+n_k=N} C\rho_1^{n_1}\rho_2^{n_2}\cdots\rho_k^{n_k} = 1$$

$$\implies C = \left(\sum_{n_1+n_2+\cdots+n_k=N} \rho_1^{n_1}\rho_2^{n_2}\cdots\rho_k^{n_k}\right)^{-1} = C(N) \quad \text{(say)}$$

The constant C is written as $C(N)$ to denote the fact that is a function of N. It is also the case that the solution is often written in terms of $C^{-1}(N) = G(N)$ and therefore, the solution is

$$p_{n_1,n_2,\ldots,n_k} = \frac{1}{G(N)}\rho_1^{n_1}\rho_2^{n_2}\cdots\rho_k^{n_k},$$

where $G(N) = \sum_{n_1+n_2+\cdots+n_k=N} \rho_1^{n_1}\rho_2^{n_2}\cdots\rho_k^{n_k}$ is the normalization constant.

♦ Here, observe that the joint distribution is not a product of marginals.

The Multi-Server Case

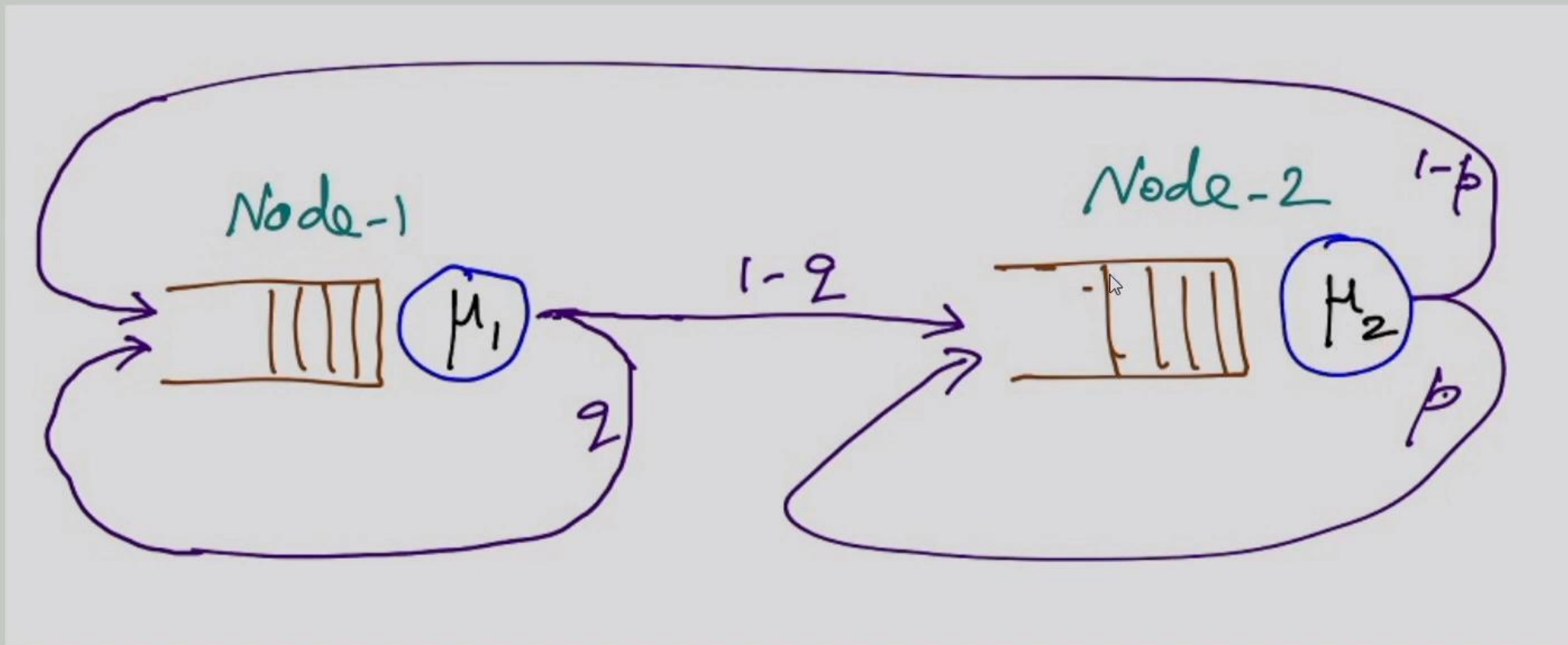
- The single channel at each node closed network can be extended to c_i-servers at node-i. The solution becomes

$$p_{n_1,n_2,\dots,n_k} = \frac{1}{G(N)} \prod_{i=1}^{k} \frac{\rho_i^{n_i}}{a_i(n_i)},$$

where $a_i(n_i) = \begin{cases} n_i! & n_i < c_i \\ c_i! c_i^{n_i - c_i} & n_i \geq c_i \end{cases}$ and $G(N) = \displaystyle\sum_{n_1+n_2\cdots+n_k=N} \prod_{i=1}^{k} \frac{\rho_i^{n_i}}{a_i(n_i)}.$

Example (Two-Node Closed Queueing Network)

- Consider the following two-node single-server-at-each-node closed queueing network with a total of M customers in the network:



09_QNetworks2.pdf *

Create account Login

9/24

Example

- The steady state joint probability distribution is given by

$$p_{M-m,m} = \frac{1}{G(M)} \rho_1^{M-m} \rho_2^m, \qquad m = 0, 1, 2, \ldots, M.$$

We must find ρ_i from $\mu_i \rho_i = \sum_{j=1}^{k} \mu_j r_{ji} \rho_j$. Here, the routing matrix R is $R = \begin{pmatrix} q & 1-q \\ 1-p & p \end{pmatrix}$ and hence the traffic equations becomes

$$\mu_1 \rho_1 = \mu_1 q \rho_1 + \mu_2 (1-p) \rho_2$$
$$\mu_2 \rho_2 = \mu_1 (1-q) \rho_1 + \mu_2 p \rho_2$$

Since these equations are linearly dependent, as we know already, we can set one of the ρ_i's to 1 and solve for the other.
Set $\rho_1 = 1$. Then, from the second equation, we get $\rho_2 = \dfrac{1-q}{1-p} \dfrac{\mu_1}{\mu_2}$.

09_QNetworks2.pdf * | Create account | Login | 10/24

Example

We thus have the steady state solution for the closed network as

$$p_{M-m,m} = \frac{1}{G(M)}\rho_2^m = \frac{1}{G(M)}\left(\frac{(1-q)\mu_1}{(1-p)\mu_2}\right)^m, \quad m = 0, 1, 2, \ldots, M,$$

where the normalizing constant $G(M)$ is given by $G(M) = \sum_{m=0}^{M} \rho_2^m = \frac{1-\rho_2^{M+1}}{1-\rho_2}$.

Performance Measures:

The probability that node-1 is busy equals $1 - p_{0,M} = 1 - \frac{1}{G(M)}\rho_2^M = \frac{G(M-1)}{G(M)}$.

Similarly, the probability that node-2 is busy equals $1 - p_{M,0} = 1 - \frac{1}{G(M)} = \rho_2 \frac{G(M-1)}{G(M)}$.

The average number of customers in node-2 and node-1 are

$$L_2 = \sum_{m=0}^{M} \frac{m\rho_2^m}{G(M)} = \frac{\rho_2}{G(M)} \sum_{m=0}^{M} m\rho_2^{m-1} = \frac{\rho_2}{(1-\rho_2)^2 G(M)}\left[1 - \rho_2^{M+1} - (1-\rho_2)(M+1)\rho_2^M\right]$$

$$L_1 = M - L_2$$