Module 11: Semi-Markovian Queueing Systems (contd...)

Lecture 40

M/G/1 Queues: Waiting Times and Busy Period

M/G/1 Queues : Waiting Times

- Question: Are there relationships, along the lines of $L = \lambda W$, between higher order moments? Or, between distributions (equivalently, LSTs)?
- The stationary distribution for the $M/G/1$ can be written in terms of waiting-time CDF as

$$p_n = \pi_n = \frac{1}{n!}\int_0^\infty (\lambda t)^n e^{-\lambda t} dF_T(t), \qquad n \geq 1$$

▶ This is so because the system size under FCFS will equal n at an arbitrary departure point if there have been n (Poisson) arrivals during the departure's system wait.

- If we multiply the above equation by z^n and sum over n

$$P(z) = \sum_{n=0}^{\infty} p_n z^n = \int_0^\infty e^{-\lambda t} \sum_{n=0}^{\infty} \frac{(\lambda t z)^n}{n!} dF_T(t) = \int_0^\infty e^{-\lambda t(1-z)} dF_T(t)$$

That is, $\boxed{P(z) = F_T^*\left[\lambda(1-z)\right]}$ (Here, $F_T^*(s)$ is the LST of $F_T(t)$) (1)

- By repeated differentiation of $P(z) = F_T^*[\lambda(1-z)]$, we can now find the relationship between moments of system size and system wait. By chain rule, we have that

$$\frac{d^k P(z)}{dz^k} = (-1)^k \lambda^k \left. \frac{d^k F_T^*(u)}{du^k} \right|_{u=\lambda(1-z)}$$

$$= (-1)^k \lambda^k (-1)^k \; E\left[T^k e^{-Tu}\right]\Big|_{u=\lambda(1-z)}$$

- Let $L_{(k)}$ denote the kth factorial moment of the system size and W_k the regular kth moment of the system waiting time. Then

$$\boxed{L_{(k)} = \left. \frac{d^k P(z)}{dz^k} \right|_{z=1} = \lambda^k W_k.} \qquad (2)$$

$\underline{k=2 \text{ case}}$:
$E[N(N-1)] = \lambda^2 E(T^2)$

The above is a generalization of Little's law.

- We now look at the relationship between the distributions.
- Recall that, in the $M/M/1$ queue, we saw that the system waiting time distribution can be written in terms of the service time distribution as

$$F_T(t) = (1-\rho)\sum_{n=0}^{\infty}\rho^n B^{(n+1)}(t), \tag{3}$$

where $B^{(n+1)}(t)$ is the $(n+1)$-fold convolution of the exponential CDF $B(t)$.
▶ Memoryless property helped to get the above result, taking care of the situation that the arrivals catch the server in the middle of a serving period with probability ρ.
▶ We do not have memoryless property now, and hence we require an alternative approach to derive a comparable result for $M/G/1$.

- We start with deriving a simple relationship between the LSTs of the service and waiting times, $B^*(s)$ and $F_T^*(s)$. We know that $P(z) = F_T^*\left[\lambda(1-z)\right]$ and, from our earlier results, we also know that

$$P(z) = \Pi(z) = \frac{(1-\rho)(1-z)K(z)}{K(z)-z}.$$

But the PGF $K(z)$ of $\{k_i\}$ is given by

$$K(z) = \int_0^\infty e^{-\lambda t} \sum_{n=0}^{\infty} \frac{(\lambda t z)^n}{n!} dB(t) = \int_0^\infty e^{-\lambda t(1-z)} dB(t).$$

$$\text{That is, } \boxed{K(z) = B^*[\lambda(1-z)].} \tag{4}$$

- We will now put all of these together.

- Putting all these three facts together, we obtain

$$F_T^*[\lambda(1-z)] = \frac{(1-\rho)(1-z)B^*[\lambda(1-z)]}{B^*[\lambda(1-z)] - z}$$

Equivalently, $$\boxed{F_T^*(s) = \frac{(1-\rho)sB^*(s)}{s-\lambda[1-B^*(s)]}.} \quad (5)$$

- Now, since $T = T_q + S$, we know, from the convolution property of transforms, that $F_T^*(s) = F_{T_q}^*(s)B^*(s)$. Thus,

$$\boxed{F_{T_q}^*(s) = \frac{(1-\rho)s}{s-\lambda[1-B^*(s)]}.} \quad (6)$$

Ex. Take exponential & derive result similar to M/M/1.

- Now, expanding the right hand side as a geometric series (since $(\lambda/s)\left[1 - B^*(s)\right] < 1$),

$$F^*_{T_q}(s) = (1-\rho)\sum_{n=0}^{\infty}\left(\frac{\lambda}{s}\left[1 - B^*(s)\right]\right)^n = (1-\rho)\sum_{n=0}^{\infty}\rho^n\left(\frac{\mu}{s}\left[1 - B^*(s)\right]\right)^n.$$

- It can be seen that $\mu[1 - B^*(s)]/s$ is the LST of the residual service time distribution

$$R(t) = \mu\int_0^t [1 - B(x)]dx. \qquad \text{[Recall our renewal theory results.]}$$

▶ $R(t)$ is the CDF of the remaining service time of the customer being served at the time of an arbitrary arrival, given that the arrival occurs when the server is busy.

- We thus obtain

$$F^*_{T_q}(s) = (1-\rho)\sum_{n=0}^{\infty}\rho^n\left[R^*(s)\right]^n.$$

After term-by-term inversion utilizing the convolution property, we obtain

$$F_{T_q}(t) = (1-\rho)\sum_{n=0}^{\infty}\rho^n R^{(n)}(t),$$

a result similar to the $M/M/1$ case.

Busy Period Analysis

- Determination of the distribution of the busy period for $M/G/1$ is difficult than that of $M/M/1$.
- But it not that much difficult to find the LST of the busy period, from which the moments can be obtained.
- Let $G(x)$ denote the CDF of the busy period X of an $M/G/1$ with service CDF $B(t)$.
- We condition X on the length of the first service time inaugurating the busy period.
  ♦ Each arrival during the first service time of the busy period generates its own busy period.
  Then,

$$G(x) = \int_0^x \Pr\{\text{ busy period generated by all arrivals during } t \leq x - t | \text{first service time} = t\} dB(t)$$

$$= \int_0^x \sum_{n=0}^{\infty} \frac{e^{-\lambda t}(\lambda t)^n}{n!} G^{(n)}(x-t) dB(t) \tag{7}$$

where $G^{(n)}(x)$ is the n-fold convolution of $G(x)$.
Next, let $G^*(s)$ be the LST of $G(x)$, and $B^*(s)$ be the LST of $B(t)$.

- Then, by taking the transform of both sides of the above, it is found that

$$G^*(s) = \int_0^\infty \int_0^x \sum_{n=0}^\infty e^{-sx} \frac{e^{-\lambda t}(\lambda t)^n}{n!} G^{(n)}(x-t)dB(t)dx.$$

- Changing the order of integration

$$\begin{aligned} G^*(s) &= \int_0^\infty \int_t^\infty \sum_{n=0}^\infty e^{-sx} \frac{e^{-\lambda t}(\lambda t)^n}{n!} G^{(n)}(x-t)dx\, dB(t) \\ &= \int_0^\infty \sum_{n=0}^\infty \frac{e^{-\lambda t}(\lambda t)^n}{n!} \int_t^\infty e^{-sx} G^{(n)}(x-t)dx\, dB(t) \end{aligned}$$

- Applying a change of variables $y = x - t$ to the inner integral gives

$$G^*(s) = \int_0^\infty \sum_{n=0}^\infty \frac{e^{-\lambda t}(\lambda t)^n}{n!} \left(e^{-st} \int_0^\infty e^{-sy} G^{(n)}(y)dy \right) dB(t)$$

- By the convolution property.

$$G^*(s) = \int_0^\infty \sum_{n=0}^{\infty} \frac{e^{-\lambda t}(\lambda t)^n}{n!} e^{-st} \left[G^*(s)\right]^n dB(t) = \int_0^\infty e^{-\lambda t} e^{\lambda t G^*(s)} e^{-st} dB(t)$$

That is, $$\boxed{G^*(s) = B^*[s + \lambda - \lambda G^*(s)].} \qquad (8)$$

- Hence, the mean length of the busy period is found as

$$E[X] = -\left.\frac{dG^*(s)}{ds}\right|_{s=0} \equiv G^{*\prime}(0),$$

where $G^{*\prime}(s) = B^{*\prime}\left[s + \lambda - \lambda G^*(s)\right]\left[1 - \lambda G^{*\prime}(s)\right].$

- Therefore,

$$E[X] = -B^{*\prime}\left[\lambda - \lambda G^*(0)\right]\left[1 - \lambda G^{*\prime}(0)\right] = -B^{*\prime}(0)\{1 + \lambda E[X]\},$$

or

$$E[X] = -\frac{B^{*\prime}(0)}{1 + \lambda B^{*\prime}(0)}.$$

Because $B^{*\prime}(0) = -1/\mu$,

$$E\,[X] = \frac{1/\mu}{1 - \lambda/\mu} = \frac{1}{\mu - \lambda}.$$

- This is exactly the same result we obtained earlier for $M/M/1$.