1. Which of the following statements is/are true?

   (A) A sequence of independent and identically distributed random variables forms a Markov chain.

   (B) In a birth-death process, the duration of time the process stays in state 5 is exponentially distributed.

   (C) A Markov process is a semi-Markov process.

   (D) If $\{W_t, t \geq 0\}$ is a Brownian motion, then $Cov(W_4, W_7) = 3$.

   Solution:

   Answer: (A), (B) and (C) [Hint: Pick all the correct statements]

   (A) is true, as for an IID sequence, the Markov property is trivially true. (B) is true, as BDP is special case of Markov. (C) is true by definition.

   For (D): For $s < u$, we have that $Cov(W_s, W_u) = E(W_s W_u)$, as $E(W_t) = 0$ for all t. But $W_u = W_s + (W_u - W_s)$ and this implies that $Cov(W_s, W_u) = E(W_s^2) + E(W_s(W_u - W_s)) = s$. Therefore, $Cov(W_4, W_7) = 4$, thus proving that (D) is false.

2. A Markov process is said to be a DTMC if

   (A) the parameter space is continuous and the state space is discrete.

   (B) the parameter space is discrete and the state space is continuous.

   (C) both the parameter space and the state space are discrete.

   (D) both the parameter space and the state space are continuous.

   Solution:

   Answer: (C), by definition. [Hint: Pick the correct statement]

3. Which of the following conditions would imply that $P = ((p_{ij}))_{i,j \in S}$, where $S = \{1, 2, 3, \dots\}$, will be a doubly stochastic matrix?

   (A) $p_{ij} \geq 0, \ \forall\, i, j \in S, \quad \sum_{j \in S} p_{ij} = 1, \ \forall\, i \in S \quad \text{and} \quad \sum_{i \in S} p_{ij} = 1, \ \forall\, j \in S.$

   (B) $p_{i,i+1} = 1, \ \forall\, i \in S \quad \text{and} \quad p_{ij} = 0, \ \forall\, i, j \in S, j \neq i + 1.$

   (C) $p_{ij} \geq 0, \ \forall\, i, j \in S \quad \text{and} \quad \sum_{j \in S} p_{ij} = 1, \ \forall\, i \in S.$

   (D) $p_{ii} = 1, \ \forall\, i \in S \quad \text{and} \quad p_{ij} = 0, \ \forall\, i, j \in S, j \neq i.$

   Solution:

   Answer: (A) and (D) [Hint: Pick all the correct statements]

   For a doubly stochastic matrix, the entries need to be nonnegative, and both the row and columns sums should equal unity. Only (A) and (D) satisfy these conditions.

4. State whether the following statement is TRUE or FALSE:
   A Markov chain is completely specified by its initial distribution and the transition probability matrix.

Solution:

Answer: TRUE, by referring to details in the lecture.

5. Consider a Markov chain with state space $S = \{0, 1, 2, 3\}$ and with the transition probability matrix P given by

$$P = \begin{bmatrix} 2/3 & 0 & 1/3 & 0 \\ 1 & 0 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 \\ 1/4 & 1/4 & 1/4 & 1/4 \end{bmatrix}.$$

Which of the following statements is/are true?

(A) State 1 is recurrent and periodic.

(B) State 0 is recurrent and aperiodic.

(C) State 3 is recurrent and aperiodic.

(D) State 2 is recurrent and aperiodic.

Solution:

Answer: (B) and (D) [Hint: Pick all the correct statements]

Draw the transition probability diagram to see that $\{0, 2\}$ is the only closed communicating class which must be recurrent (being a finite chain), while the states 1 and 3 are transient. Also, the period of the class $\{0, 2\}$ is 1 and hence aperiodic. Hence, only (B) and (D) are true.

6. Consider a Markov chain with state space $S = \{1, 2, 3\}$ and with the transition probability matrix P given by

$$P = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 1/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Which of the following statements is/are true?

(A) Markov chain is irreducible and positive recurrent.

(B) Markov chain is reducible.

(C) State 3 is an absorbing state.

(D) States 1 and 2 form a closed communicating class.

Solution:

Answer:(B), (C) and (D) [Hint: Pick all the correct statements]

Draw the transition probability diagram to see that there are two closed communicating classes, namely $\{1, 2\}$ and $\{3\}$, implying that the chain is reducible. State 3 is absorbing as $p_{33} = 1$. Hence, only (B), (C) and (D) are true.

7. Which of the following statements is/are true for a Markov chain?

(A) State i is said to be a recurrent if $\sum_{n=1}^{\infty} f_{ii}^{(n)} = 1$.

(B) State i is said to be a transient if $\sum_{n=0}^{\infty} p_{ii}^{(n)} = 1$.

(C) State i is said to be a recurrent if $\sum_{n=0}^{\infty} p_{ii}^{(n)} = 200$.

(D) State i is said to be a transient if $\sum_{n=1}^{\infty} f_{ii}^{(n)} = 0.5$.

Solution:

Answer:(A), (B) and (D) [Hint: Pick all the correct statements]

By definition, state i is recurrent (or transient) depending on $\sum_{n=1}^{\infty} f_{ii}^{(n)}$ equals 1 (or less than 1). Similarly, state i is recurrent (or transient) depending on $\sum_{n=0}^{\infty} p_{ii}^{(n)} = \infty$ (or $< \infty$). Hence, only (A), (B) and (D) are correct.

8. A particle moves on a circle through points which have been labelled $1, 2, 3, 4, 5$ (in a clockwise order). At each step, it has a probability 0.3 of moving to the right (clockwise) one point and 0.7 of moving to the left (anti-clockwise) one point. Let X_n denote its location on the circle after the n^{th} step. If $X_0 = 1$ and if $\alpha = P\{X_1 = 2, X_2 = 3, X_3 = 4, X_4 = 3\}$, then $275\,\alpha$ equals _______.

Solution:

Answer: Range: 5.10 **to** 5.30 [Hint: Enter the answer in two decimals]

Given $X_0 = 1$, using Markov property, we have

$$\begin{aligned}
\alpha =& P\{X_1 = 2, X_2 = 3, X_3 = 4, X_4 = 3\} \\
=& P\{X_4 = 3 | X_3 = 4\} P\{X_3 = 4 | X_2 = 3\} P\{X_2 = 3 | X_1 = 2\} P\{X_1 = 2 | X_0 = 1\} \\
=& (0.7)(0.3)^3 = 0.0189.
\end{aligned}$$

Then $275\alpha = 5.1975$.