MODULE 8: Queueing Networks

LECTURE 30

Queueing Networks with Blocking, Open Jackson Networks

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Queueing Networks with Blocking

- Blocking occurs when a customer cannot enter the queue because of queue capacity limitation.
- In a single queue, the effect of blocking is that the customer has to leave the system without service. Such a loss may not be a practical option in a network of queues.
- The blocking characteristics may be of different types and some of them are:
  ▶ Rejection Blocking: The blocked customer is forced to leave the system (can happen in open network only).
  ▶ Transfer Blocking: The blocked unit waits at the current node (keeping the server engaged) until it can move to the next destination node.
  ▶ Repetitive Service Blocking: The blocked job goes for another service at the current node (and repeats, if necessary) until it can move to the destination node.
- The analysis of queueing networks with blocking is much more complex and they may not have a product-form solution.

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A Two-Node Series Queue with Blocking

- Consider a very simple two station, single-server at each station series network, where no queue is allowed to form at any of the station.
  ▶ Arrivals follow $PP(\lambda)$ and service times at the two stations are $Exp(\mu_1)$ and $Exp(\mu_2)$, respectively.
  ▶ The blocking policy is 'transfer blocking'.
  ▶ The system can be represented as $M/M/1/1 \to \bullet/M/1/1$

- Possible system states are: $(0,0), (1,0), (0,1), (1,1)$ and $(b,1)$
  ♦ While the first four describes the number in each node, the state $(b,1)$ describes the situation where a customer finished service at node 1 is waiting for the server at node 2 to become free.
- If p_{n_1,n_2} denotes the steady state probability of the system, then they satisfy

$$\lambda p_{0,0} = \mu_2 p_{0,1}$$
$$\mu_1 p_{1,0} = \mu_2 p_{1,1} + \lambda p_{0,0}$$
$$(\lambda + \mu_2) p_{0,1} = \mu_1 p_{1,0} + \mu_2 p_{b,1}$$
$$(\mu_1 + \mu_2) p_{1,1} = \lambda p_{0,1}$$
$$\mu_2 p_{b,1} = \mu_1 p_{1,1}$$

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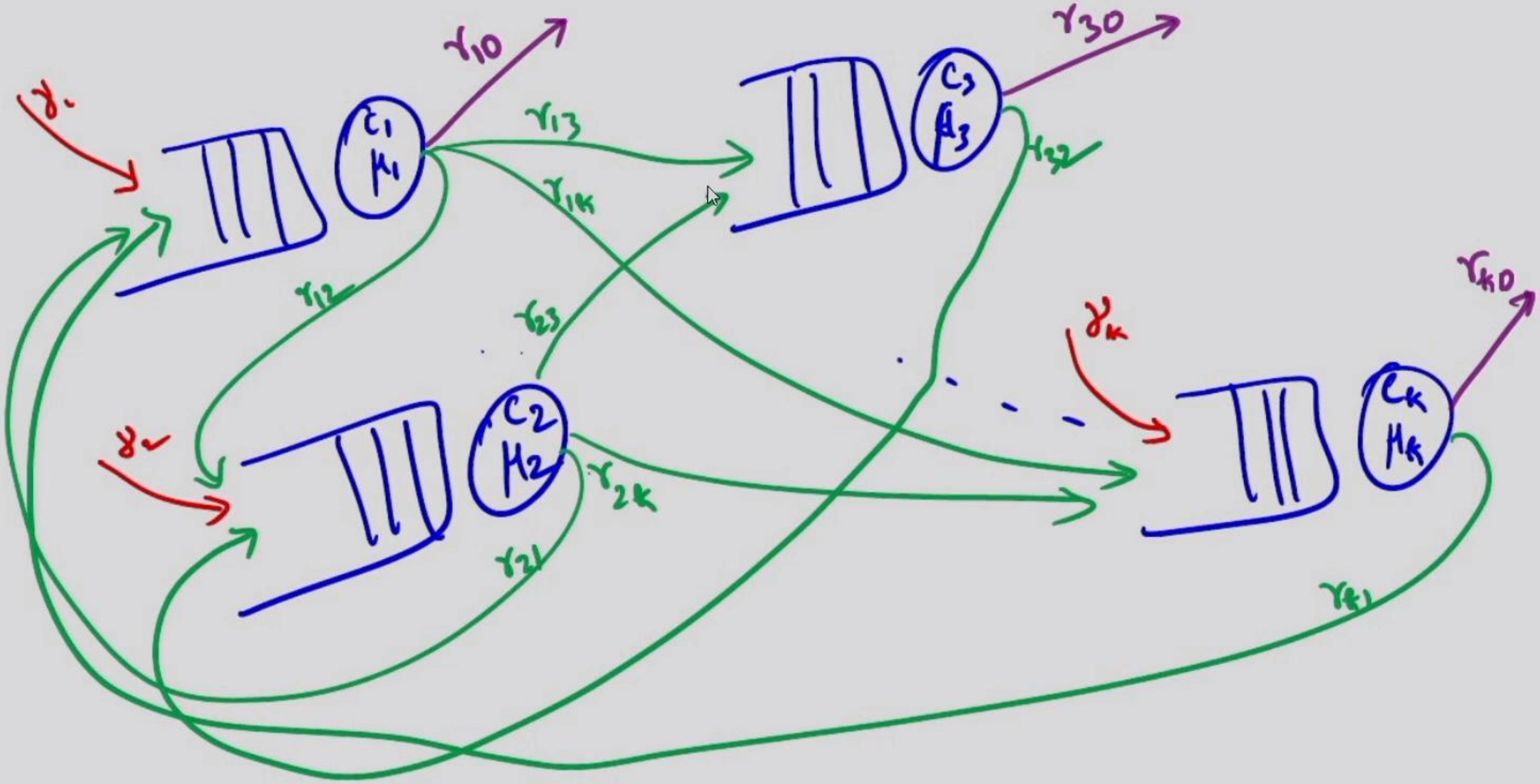
- As usual, all other probabilities can be expressed in terms of $p_{0,0}$ and the normalization condition can be used to determine $p_{0,0}$.
- The problem expands if we allow a positive but finite queue capacity in front of each node or if we allow more nodes.
- The complexity results from having to write the balance equations for each possible system state (though conceptually they can be handled as it is similar to what we have seen so far).
- For large but finite set of equations (as above), numerical techniques for solving these system of equations can also be employed.

Open Jackson Networks

- We now consider the general open Jackson networks as described earlier.
  - A network of k service nodes.
  - Arrival at node i according to a Poisson process with rate γ_i.
  - Service rate (exponential) at node i is μ_i, with c_i servers at node i.
  - Routing probability is r_{ij} (independent of the system state), with r_{i0} denoting the probability of exiting the network from node i.
  - No limit on queue capacity at any node (no blocking).
- We have a Markovian system and the state of the system can be described via N_i's, where N_i is the random variable for number of customers (in queue and in service) at node i in steady-state.
- As usual, we want the joint distribution $P\{N_1 = n_1, \ldots, N_k = n_k\} = p_{n_1,n_2,\ldots,\ldots n_k}$ from which we can obtain other required quantities.


γ1
r10
r13
r1K
r12
C1
μ1
C3
μ3
r30
r32
γ2
C2
μ2
r23
r2K
r21
γK
rK0
CK
μK
rK1

- The notation for the k-component vector is as follows:

State	Simplified Notation
$n_1, n_2, \ldots, n_i, \ldots, n_j, \ldots, n_k$	$\overline{n}$
$n_1, n_2, \ldots, n_i + 1, \ldots, n_j, \ldots, n_k$	$\overline{n}; i^+$
$n_1, n_2, \ldots, n_i - 1, \ldots, n_j, \ldots, n_k$	$\overline{n}; i^-$
$n_1, n_2, \ldots, n_i + 1, \ldots, n_j - 1, \ldots, n_k$	$\overline{n}; i^+ j^-$

- Assume for now that $c_i = 1, \forall i$ (i.e., single server at each node).
- The stochastic balance (global) equation for state $\bar{n}$ with $n_i \geq 1, \forall i$ is:

$$\sum_{i=1}^{k} \gamma_i p_{\bar{n};i^-} + \sum_{\substack{j=1 \\ (i \neq j)}}^{k} \sum_{i=1}^{k} \mu_i r_{ij} p_{\bar{n};i^+j^-} + \sum_{i=1}^{k} \mu_i r_{i0} p_{\bar{n};i^+} = \sum_{i=1}^{k} \mu_i (1 - r_{ii}) p_{\bar{n}} + \sum_{i=1}^{k} \gamma_i p_{\bar{n}}$$

▶ The above will also hold for the case $n_i = 0$ if we set terms with negative subscripts and terms containing μ_i for which $n_i = 0$ to zero.

- Jackson (1957, 1963) showed that the solution to the balance equations is in a '$product form'$':

$$p_{\bar{n}} = C\rho_1^{n_1}\rho_2^{n_2}\dots\rho_k^{n_k}$$

  - This definition of product form is less restrictive and C need not separate into a product.
- We will give Jackson's solution and then show that it satisfies the balance equations. This is also known as Jackson's theorem for open networks.
- Let λ_i be the total mean flow rate into node i (external and rerouted). Given γ_i's and r_{ij}'s, to satisfy equilibrium flow balance at each node, we have the '*traffic equations*' given by

$$\lambda_i = \gamma_i + \sum_{j=1}^{k} \lambda_j r_{ij}$$

or, in vector-matrix form $\boldsymbol{\lambda} = \boldsymbol{\gamma} + \boldsymbol{\lambda} R.$ $(R$: Routing Matrix$)$

The solution to the traffic equations is $\boldsymbol{\lambda} = \boldsymbol{\gamma}(I - R)^{-1}$. The inverse of $I - R$ exists as long as there is at least one node for exit and no node is totally absorbing.

- Define $\rho_i = \frac{\lambda_i}{\mu_i}$. The network will be at equilbrium if each of these nodes are at equilibrium which can happen only if $\rho_i < 1$ for $i = 1, 2, \ldots k$.
- The steady state solution to the balance equations is

$$p_{\bar{n}} = p_{n_1, n_2, \ldots, n_k} = \left[(1-\rho_1)\rho_1^{n_1}\right]\left[(1-\rho_2)\rho_2^{n_2}\right]\ldots\left[(1-\rho_k)\rho_k^{n_k}\right] \qquad n_i \geq 0,\ i = 1, 2, \ldots, k$$

which is a true product of marginal distributions.
- Once the traffic equations have been solved, the individual nodes may be considered in isoloation.
- The network acts as if each node could be viewed as an independent $M/M/1$ queue (even though that is really not the case) and the joint distribution can be written as a product of marginal distributions.
- The flow into each node behave as if it is Poisson, even though they may not be really Poisson in nature (i.e., if there is a feedback in the network).

- To verify that the solution, we first show that $p_{\bar{n}} = C\rho^{n_1}\rho^{n_2}\ldots\rho^{n_k}$ satisfies the balance equations and then $C = \prod_{i=1}^{k}(1-\rho_i)$. Plugging this into the balance equations gives us

$$\sum_{i=1}^{k}\frac{\gamma_i\mu_i}{\lambda_i} + \sum_{j=1}^{k}\sum_{\substack{i=1 \\ (i\neq j)}}^{k}\mu_i r_{ij}\frac{\lambda_i\mu_j}{\lambda_j\mu_i} + \sum_{i=1}^{k}\mu_i r_{i0}\frac{\lambda_i}{\mu_i} \quad =? \quad \sum_{i=1}^{k}(\mu_i - \mu_i r_{ii} + \gamma_i)$$

From the traffic equations, we have

$$\lambda_j = \gamma_j + \sum_{\substack{i=1 \\ (i\neq j)}}^{k} r_{ij}\lambda_i + r_{jj}\lambda_j \quad \Rightarrow \quad \sum_{\substack{i=1 \\ (i\neq j)}}^{k} r_{ij}\lambda_i = \lambda_j - \gamma_j - r_{jj}\lambda_j.$$

Substituting this into the above, we get

$$\sum_{i=1}^{k}\frac{\gamma_i\mu_i}{\lambda_i} + \sum_{j=1}^{k}\frac{\mu_j}{\lambda_j}(\lambda_j - \gamma_j - r_{jj}\lambda_j) + \sum_{i=1}^{k}\mu_i r_{i0}\frac{\lambda_i}{\mu_i} \quad =? \quad \sum_{i=1}^{k}(\mu_i - \mu_i r_{ii} + \gamma_i)$$

$$\Rightarrow \quad \sum_{i=1}^{k}\left(\frac{\gamma_i\mu_i}{\lambda_i} + \frac{\mu_i}{\lambda_i}(\lambda_i - \gamma_i - r_{ii}\lambda_i) + \lambda_i r_{i0}\right) \quad =? \quad \sum_{i=1}^{k}(\mu_i - \mu_i r_{ii} + \gamma_i)$$

- Finally, we get

$$\sum_{i=1}^{k} \lambda_i r_{i0} \quad =? \quad \sum_{i=1}^{k} \gamma_i. \qquad \leftarrow \text{TRUE!}$$

This means that the total flow out of the network equals the total flow in which must be true for the steady-state to hold.

- Now, C can be obtained from

$$\sum_{n_k=0}^{\infty} \cdots \sum_{n_2=0}^{\infty} \sum_{n_1=0}^{\infty} C \rho_1^{n_1} \rho_2^{n_2} \ldots \rho_k^{n_k} = 1$$

$$\implies C = \prod_{i=1}^{k} (1 - \rho_i) \qquad \rho_i < 1, \quad i = 1, 2, \ldots, k$$

Thus, the solution is verified.

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Performance Measures

- For single-channel node open Jackson network considered here, we have

$$L_i = \frac{\rho_i}{1-\rho_i} \quad \text{and} \quad W_i = \frac{L_i}{\lambda_i} \qquad \text{for node } i$$

♦ This is because of the product form of the joint distribution, and does not imply that the nodes are truly $M/M/1$.

- The expected total number of customers in the network is $\sum_{i=1}^{k} L_i$.
- The expected total wait in the network for any customer before its final departure is

$$W = \frac{\sum_i L_i}{\sum_i \gamma_i} \quad \text{(Little's formula for the entire network)}$$