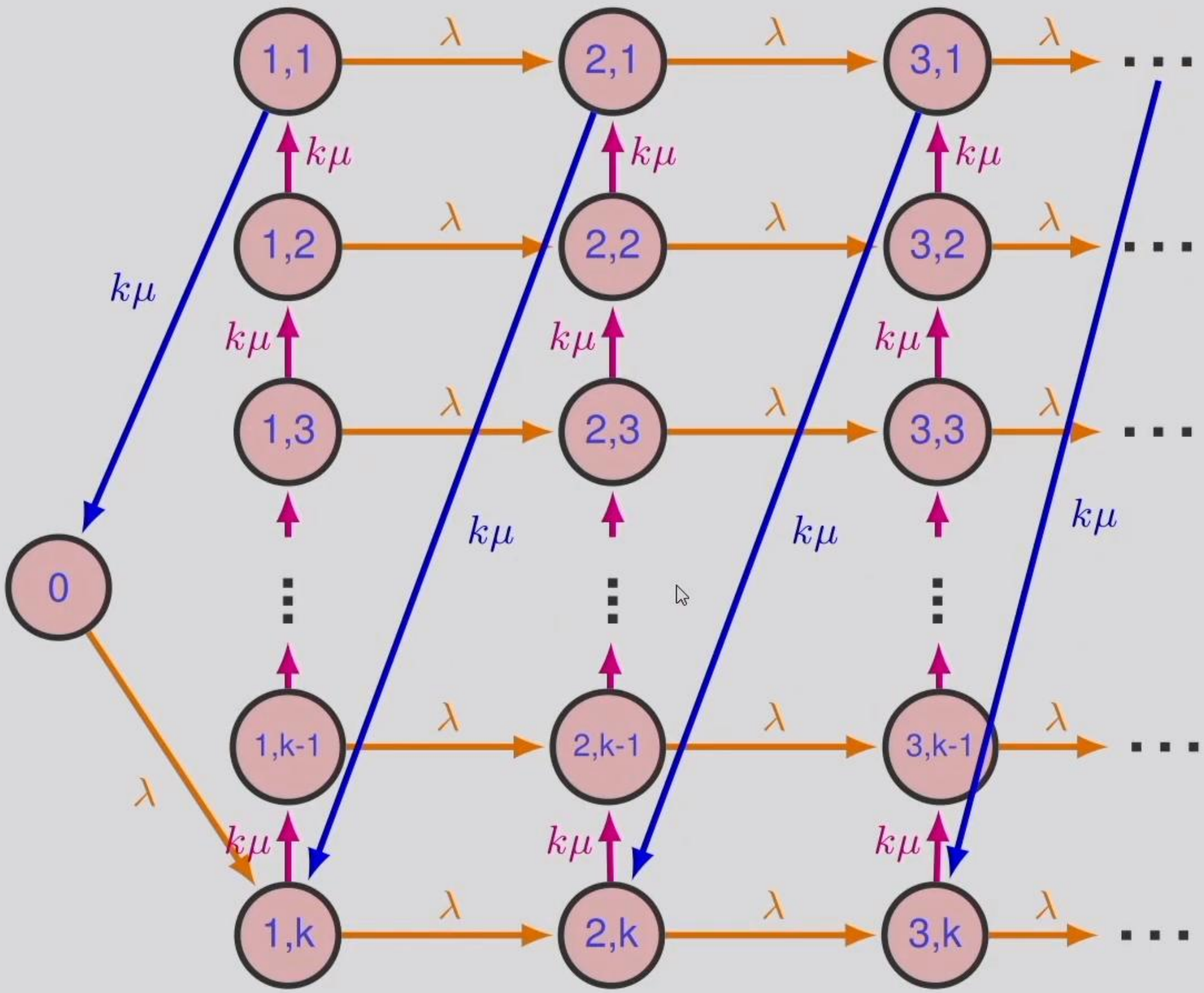
Erlang Service Model $(M/E_k/1)$

- All the assumptions of an $M/M/1$ model (in equilibrium) are in place, except that the service time distribution is now an Erlang type-k (E_k) distribution (with mean $\frac{1}{\mu}$).
- The service time duration may be thought of as consisting of k independent and identical exponential phases (or stages), each with mean $\frac{1}{k\mu}$ (even if there are not phases, it is convenient to think this way).
- Since the overall service rate is assumed to be μ, the rate of each service phase is $k\mu$.
- The system state can be described by a two-dimensional CTMC with state space $S = \{(n, i) : n \geq 0, 1 \leq i \leq k\}$, where n denotes the number of customers in the system and i is the current phase of the service for the customer currently in service.
- The phases are numbered backwards, so k is the first phase of service and 1 is the last phase.
- The total number of phases left when the system is in state (n, i) equals $(n-1)k + i$.



1,1
2,1
3,1
1,2
2,2
3,2
1,3
2,3
3,3
0
1,k-1
2,k-1
3,k-1
1,k
2,k
3,k
λ
$k\mu$

- We can write the following steady-state balance equations

$$
\begin{aligned}
(\lambda + k\mu)p_{n,i} &= k\mu p_{n,i+1} + \lambda p_{n-1,i}, \quad n \geq 2, \quad 1 \leq i \leq k-1 \\
(\lambda + k\mu)p_{n,k} &= k\mu p_{n+1,1} + \lambda p_{n-1,k}, \quad n \geq 2 \\
(\lambda + k\mu)p_{1,i} &= k\mu p_{1,i+1}, \quad 1 \leq i \leq k-1 \\
(\lambda + k\mu)p_{1,k} &= k\mu p_{2,1} + \lambda p_0, \\
\lambda p_0 &= k\mu p_{1,1}
\end{aligned}
$$

- These equations are not that easy to handle due to their bivariate nature.
- Interestingly, one can relate this Erlangian service queue to a constant bulk input model $M^{[K]}/M/1$, where each input unit brings in $K = k$ phases and the (phase) service rate μ is replaced by $k\mu$.
- One can utilize this equivalence of $M/E_k/1$ and $M^{[k]}/M/1$ to get the performance measures easily.

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Performance Measures

- To determine the average wait in queue W_q for the $M/E_k/1$ queue:
  - First observe that the average number of phases in the $M/E_k/1$ queue is the same as the average number of customers in the analogous $M^{[X]}/M/1$ queue (which was equal to $\frac{k+1}{2}\frac{\rho}{1-\rho}$).
- Here, in the Erlangian model, the service $k\mu$ replaces μ in the bulk arrival model and hence $\rho = \frac{k\lambda}{k\mu} = \frac{\lambda}{\mu}$.
  - Since the average time to process each phase is $\frac{1}{k\mu}$, the average wait in queue is the average number of phases in the system multiplied by $\frac{1}{k\mu}$. We get

$$W_q = \frac{1+\frac{1}{k}}{2}\frac{\rho}{\mu(1-\rho)}, \quad \rho = \frac{\lambda}{\mu}.$$

- It follows in the usual manner that

$$L_q = \lambda W_q = \frac{1+\frac{1}{k}}{2}\frac{\rho^2}{1-\rho}, \quad L = L_q + \rho, \quad W = \frac{L}{\lambda} = W_q + \frac{1}{\mu}, \quad \rho = \frac{\lambda}{\mu}.$$

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Steady-State Solution

- We will now try to get the steady-state probabilities for which we use the fact that state (n, i) can be transformed equivalent to $(n-1)k+i$ in a single-variable system.
- Mapping (n, i) to $(n-1)k+i$, the steady-state balance equations can be rewritten as

$$0 = -(\lambda + k\mu)p_{(n-1)k+i} + k\mu p_{(n-1)k+i+1} + \lambda p_{(n-2)k+i}, \qquad n \geq 1, \quad 1 \leq i \leq k$$
$$0 = -\lambda p_0 + k\mu p_1$$

where any p with a negative subscript is assumed to be zero.

- The above system can be rewritten again in a simplified manner (starting from $n = 1, i = 1$ and proceeding sequentially) as

$$0 = -(\lambda + k\mu)p_n + k\mu p_{n+1} + \lambda p_{n-k}, \qquad n \geq 1$$
$$0 = -\lambda p_0 + k\mu p_1$$

which is the same set of equations as for the bulk arrival queue for a constant batch size k and service rate $k\mu$.

- Defining $P(z) = \sum_{n=0}^{\infty} p_n z^n$ and proceeding as earlier, we get

$$P(z) = \frac{k\mu p_0(1-z)}{k\mu - (\lambda + k\mu)z + \lambda z^{k+1}}.$$

- Using $P(1) = 1$ to get $p_0 = \frac{\mu-\lambda}{\mu} = 1 - \rho, \quad \rho = \frac{\lambda}{\mu}$, we obtain

$$P(z) = \frac{k\mu(1-\rho)(1-z)}{k\mu - (\lambda + k\mu)z + \lambda z^{k+1}}$$

from which we can obtain $\{p_n\}$ by expanding $P(z)$ as a power series in z (through a partial fraction expanstion).

- Now, the p_n's obtained above gives the probability of the number of phases in the system. If $p_n^{(c)}$ denotes the probability that the number of customers in the system is n, then

$$p_n^{(c)} = \sum_{m=(n-1)k+1}^{nk} p_m, \qquad n = 1, 2, \ldots.$$

- Special case: $k = 1$. ($M/M/1$ queue)
We have $P(z) = \dfrac{\mu(1-\rho)(1-z)}{\mu - (\lambda+\mu)z + \lambda z^2} = \dfrac{1-\rho}{1-\rho z}$ so that $p_n = (1-\rho)\rho^n, n \geq 0$ and $p_n^{(c)} = p_n$.

Example ($M/E_4/1$)

- Suppose that there is a service system to which the customers arrive according to a Poisson process with a mean rate of $16/h$.
- The mean service time is 2.5 minutes with a standard deviation of $\frac{5}{4}$ minutes. It is thought that the Erlang distribution would be a reasonable assumption for the service time distribution.
- How long a customer on an average must wait until getting into the service and how many customers are waiting for service?
- The appropriate model is an $M/E_k/1$ model.

$$\lambda = \frac{16}{60}, \quad \frac{1}{\mu} = 2.5, \quad \sigma^2 = \frac{1}{k\mu^2} = \frac{25}{16} \quad \Longrightarrow \quad k = 4.$$

We therefore have an $M/E_4/1$ model with $\rho = \frac{2}{3}$ for which

$$L_q = \frac{5}{8} \frac{\frac{4}{9}}{1 - \frac{2}{3}} = \frac{5}{6}, \qquad W_q = \frac{60}{16} \frac{5}{6} = \frac{25}{8} \text{ minutes.}$$

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Example ($M/M/2$ Vs. $M/D/1$)

- Components from a production line system that fail the quality control test arrive at a repair facility according to a Poisson process with a rate 18 per hour.
- The repair facility has two specialists and each can repair the component in an average of 5 minutes, with the repair time being exponentially distributed.
- The company is proposed with an alternative option of leasing one machine that can repair the components in exactly $2\frac{2}{3}$ minutes (i.e., no variation in repair time).
- Assume that the machine leasing cost is roughly equal to the salary and other benefit costs of the two staffs. Should the company lease the machine?

- We need to compare W and L under the two alternatives: $M/M/2$ Vs. $M/D/1$
- For $M/M/2$, $\lambda = 18/h, \mu = 12/h$ and this means that $W_q = \frac{3}{28}h = 6.4$ minutes, $W = 6.4 + 5 = 11.4$ minutes and $L = \lambda W = (18/60)(11.4) = 3.42$.
- For $M/D/1$, we use $\lim_{k\to\infty} M/E_k/1$. Given $\lambda = 18/h, \mu = \frac{3}{8}/min = 22.5/h$. Therefore,
$$W_q = \lim_{k\to\infty}\left(\frac{1+1/k}{2}\frac{\rho}{\mu(1-\rho)}\right) = \frac{\rho}{2\mu(1-\rho)} = \frac{4}{45}h = \frac{16}{3} \text{ minutes},$$
$W = \frac{16}{3} + \frac{8}{3} = 8$ minutes and $L = \lambda W = (18/60)(8) = 2.4$.
- Thus, leasing the machine is preferable.

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Erlang Arrival Model $(E_k/M/1)$

- We now consider a model where the interarrival times follow an Erlang type-k distribution with mean $\frac{1}{\lambda}$ (the rest being as usual).
- Like the previous model, here too an equivalence exists between $E_k/M/1$ and $M/M^{[k]}/1$.
- We can think as: An arrival is passing through k phases (each with mean $\frac{1}{k\lambda}$) before actually entering the system (again, a convenient device for analysis).
  ▶ The phases are numbered now frontward from 0 to $k-1$.
- Let $p_n^{(P)}$ denote the number of *arrival phases* in the system in steady state.
  ▶ The number of arrival phases in the system includes k phases for each customer who has already arrived (but not yet departed) as well as the completed phases corresponding to the next arrival (even though this customer has not officially "arrived").
- Then, the probability p_n (the number of customers being n in steady state) is given by

$$p_n = \sum_{j=nk}^{nk+k-1} p_j^{(P)}$$

- This system is identical in structure to the full-batch bulk-service model. The rate balance equations for $p_n^{(P)}$ in this model are identical to the rate balance equations for p_n in the bulk-service model, except with λ is now replaced by $k\lambda$ (and K by k). Therefore, we get

$$p_j^{(P)} = \rho(1 - r_0) r_0^{j-k}, \qquad j \geq k-1, \quad \rho = \frac{\lambda}{\mu},$$

where r_0 is the single root in $(0, 1)$ of the characteristic equation

$$\mu r^{k+1} - (k\lambda + \mu) r + k\lambda = 0.$$

- Thus, we obtain, for $n \geq 1$,

$$\begin{aligned} p_n = \sum_{j=nk}^{nk+k-1} p_j^{(P)} &= \rho(1 - r_0)(r_0^{nk-k} + r_0^{nk-k+1} + \cdots + r_0^{nk-1}) \\ &= \rho(1 - r_0) r_0^{nk-k}(1 + r_0 + \cdots + r_0^{k-1}) \\ &= \rho(1 - r_0^k)(r_0^k)^{n-1}. \end{aligned}$$

And, $p_0 = 1 - \rho$.

Performance Measures

- Note that $\overset{(c)}{p_n}$ has a geometric form (as with the M/M/1), but with r_0^k as the geometric parameter instead of ρ.
- The performance measures can be obtained as usual as

$$L = \rho(1 - r_0^k)\sum_{n=1}^{\infty} n(r_0^k)^{n-1} = \rho(1 - r_0^k)\frac{1}{(1 - r_0^k)^2} = \frac{\rho}{1 - r_0^k}.$$

$$L_q = L - \rho, \quad W = \frac{L}{\lambda}, \quad \text{and} \quad W_q = W - \frac{1}{\mu}.$$

Example ($E_2/M/1$)

- Arrivals occur to a single-server queueing system with an E_2 distributed interarrival times with a mean interarrival time of 30 minutes.
- The service times are exponentially distributed with a mean of 25 minutes.
- Determine the steady state system size probabilities and expected value measures of effectiveness of the system.

- Given $\lambda = 2/h, \mu = \frac{12}{5}/h$ and $k = 2$, the characteristic equation is $\frac{12}{5}r^3 - \frac{32}{5}r + 4 = \frac{4}{5}(3r^3 - 8r + 5) = (r-1)(3r^2 + 3r - 5) = 0$ and it has a positive root in $(0, 1)$ given by $r_0 = (-3 + \sqrt{69})/6 = 0.8844$.

  We then have

$$p_n = \rho(1 - r_0^k)(r_0^k)^{n-1} = (5/6)(0.2178)(0.7822)^{n-1} = (0.2320)(0.7822)^n, n \geq 1.$$

  and $p_0 = 1 - \rho = 1/6$.

  The mean system size is $L = \frac{\rho}{1-r_0^k} = \frac{5/6}{1-0.7822} = 3.8261$ and $W = L/\lambda = 1.9131\ h$.
  Similarly, $W_q = 1.9131 - 5/12 = 1.4964\ h$ and $L_q = 3.8261 - 5/6 = 2.9928$.