Module 8: Queueing Networks

Lecture 29

Burke's Theorem, General Setup, Tandem Networks

Two-Node Queueing Network

- Consider the following simple network of two queues (or nodes):
  - ▶ Customers arrive at the first node according to a Poisson process with rate λ.
  - ▶ The fist node is a single-server system with exponentially distributed service times (with parameter μ_1) having infinite queueing capacity.
  - ▶ Once the service is completed in the first node, the customer moves to the second node which is also a single-server system with exponentially distributed service times (with parameter μ_2) having infinite queueing capacity.
  - ▶ Once the service is completed in the second node, the customer departs the system.
  - ▶ No feedback or departure at the first node. No arrival at the second node.
  - ▶ The system can be represented as $M/M/1 \to \bullet/M/1$

- The system state can be represented as a two-dimensional CTMC with state space $S = \{(n_1, n_2) : n_1, n_2 = 0, 1, 2, \ldots\}$.
- Denote the probability of n_1 customers in the first node and n_2 customers in the second node in steady state by p_{n_1,n_2}.
- Then, the steady state solution for this system exists under the condition that $\rho_1 = \lambda/\mu_1 < 1$ and $\rho_2 = \lambda/\mu_2 < 1$ and can be obtained from the balance equations given by

$$\begin{aligned}
(\lambda + \mu_1 + \mu_2)p_{n_1,n_2} &= \lambda p_{n_1-1,n_2} + \mu_1 p_{n_1+1,n_2-1} + \mu_2 p_{n_1,n_2+1}, \quad n_1 \geq 1, n_2 \geq 1 \\
(\lambda + \mu_1)p_{n_1,0} &= \lambda p_{n_1-1,0} + \mu_2 p_{n_1,1}, \quad n_1 \geq 1 \\
(\lambda + \mu_2)p_{0,n_2} &= \mu_1 p_{1,n_2-1} + \mu_2 p_{0,n_2+1}, \quad n_2 \geq 1 \\
\lambda p_{0,0} &= \mu_2 p_{0,1}
\end{aligned}$$

It can be shown that

$$p_{n_1,n_2} = \rho_1^{n_1}\rho_2^{n_2} p_{0,0} \quad \text{and} \quad p_{0,0} = (1-\rho_1)(1-\rho_2).$$

Thus,

$$p_{n_1,n_2} = \left[(1-\rho_1)\rho_1^{n_1}\right]\left[(1-\rho_2)\rho_2^{n_2}\right], \; n_1, n_2 \geq 0.$$

- A queueing network of this type where the joint distribution of the number of customers in each node can be written as a product of terms involving the number in individual nodes is referred to as a **product-form network**.
  - In many situations and under some conditions, this type of solution is observed to hold.
- The form of the solution indicates that in steady state each node behaves independently of the other (the joint distribution factors into product of marginals).
- The second stage behaves like a system with an input process that is Poisson with rate λ. That is, it behaves as an $M/M/1$ queue independent of the behaviour of the first stage.
- This can be proved if we can characterize the output process of the first node.
- The output process (distribution of times between successive departures) of the first node is the input process to the second node.
- Burke's theorem determines the output process of $M/M/c$ queues.
  - Of all the systems with FCFS, $M/M/c$ is the only system with the stated property.

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The Output Process

Theorem (Burke's Theorem)

In an $M/M/c$ queueing system in steady state, the interdeparture times are IID exponential random variables with parameter λ. In other words, the output process is Poisson with the same parameter as the input process.

Proof:- Let $N(t)$ be the number in the system at time t and $t'_1, t'_2, \ldots, t'_n, t'_{n+1}, \ldots$ denote the successive departure instants so that $L = t'_{n+1} - t'_n$ is the n^{th} interdeparture interval.
Let $F_k(t) = P\{N(t'_n + t) = k, t'_{n+1} - t'_n > t\}, \quad t > 0, k = 0, 1, 2, \ldots$ be the joint probability that there are k in the system at time t after the last departure and that t is less than the interdeparture time.
Then, the CDF of L is

$$C(t) = P\{L \leq t\} = 1 - \sum_{k=0}^{\infty} F_k(t)$$

Since the input is a Poisson process, the probability that a departing customer leaves k in the system is equal to the probability that the number in the system is k and we therefore have

$$F_k(0) = p_k \quad \text{(of } M/M/c \text{ queue)}, \quad k = 0, 1, 2, \ldots.$$

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Now, for an infinitesimal interval of length Δt,

$$F_0(t+\Delta t) = (1-\lambda\Delta t)F_0(t) + o(\Delta t)$$
$$F_k(t+\Delta t) = (1-\lambda\Delta t)(1-k\mu\Delta t)F_k(t) + \lambda\Delta t(1-k\mu\Delta t)F_{k-1}(t) + o(\Delta t), \qquad 1 \le k \le c$$
$$F_k(t+\Delta t) = (1-\lambda\Delta t)(1-c\mu\Delta t)F_k(t) + \lambda\Delta t(1-c\mu\Delta t)F_{k-1}(t) + o(\Delta t), \qquad k \ge c$$

Moving $F_n(t)$ from the right-hand-side of each of the above equations to the LHS, dividing by Δt, and taking the limit as $\Delta t \to 0$, we obtain the differential-difference equations as

$$F_0'(t) = -\lambda F_0(t)$$
$$F_k'(t) = -(\lambda + k\mu)F_k(t) + \lambda F_{k-1}(t) \qquad 1 \le k \le c$$
$$F_k'(t) = -(\lambda + c\mu)F_k(t) + \lambda F_{k-1}(t), \qquad k \ge c$$

Using the boundary condition $F_k(0) = p_k$ and solving the above (in a similar manner as we did for the Poisson process), we obtain

$$F_k(t) = p_k e^{-\lambda t}, \qquad \text{where} \quad p_{k+1} = \begin{cases} \frac{\lambda}{(k+1)\mu} p_k, & 1 \le k \le c \\ \frac{\lambda}{c\mu} p_k, & k \ge c \end{cases}$$

Thus, we obtain the CDF of L as

$$C(t) = 1 - \sum_{k=0}^{\infty} p_k e^{-\lambda t} = 1 - e^{-\lambda t} \quad \text{and this implies that} \quad L \sim Exp(\lambda).$$

Now, again consider

$$P\{N(t'_{n+1} + 0) = k, t \leq t'_{n+1} - t'_n < t + \Delta t\} = F_{k+1}(t)\ P\{\text{one service completion in } (t, t + \Delta t)\}.$$

For $k + 1 \leq c$, $k + 1$ servers are busy and rate of service is $(k + 1)\mu$, while the rate of service is $c\mu$ when $k + 1 > c$.
Thus, the right-hand-side expression for the above reduces to $p_{k+1}e^{-\lambda t}(k + 1)\mu\Delta t + o(\Delta t)$ for $k + 1 \leq c$ and to $p_{k+1}e^{-\lambda t}c\mu\Delta t + o(\Delta t)$ for $k + 1 > c$. Substituting the appropriate expressions for p_{k+1}, both of them reduces to $p_k\lambda e^{-\lambda t}\Delta t + o(\Delta t)$.
Thus,

$$P\{N(t'_{n+1} + 0) = k, t \leq t'_{n+1} - t'_n < t + \Delta t\} = p_k\lambda e^{-\lambda t}\Delta t + o(\Delta t),$$

proving that $N(t'_{n+1} + 0)$ and $t'_{n+1} - t'_n$ are independent. i.e., $N(L)$ and L are independent.

Now we look at the independence of interdeparture intervals.
Let Λ represent the set of lengths of an arbitrary number of interdeparture intervals subsequent to the interval of length L. The Markov property implies that

$$P(\Lambda|N(L)) = P(\Lambda|N(L), L).$$

Since $N(L)$ and L are independent, we have

$$P(N(L), L) = P(N(L))\,P(L).$$

The joint probability function of the initial interval length, the state at the end of the interval, and the set of subsequent interval lengths may be written as

$$P(L, N(L), \Lambda) = P(\Lambda|N(L), L)P(N(L), L) = P(\Lambda|N(L))P(N(L))P(L) = P(\Lambda, N(L))P(L)$$

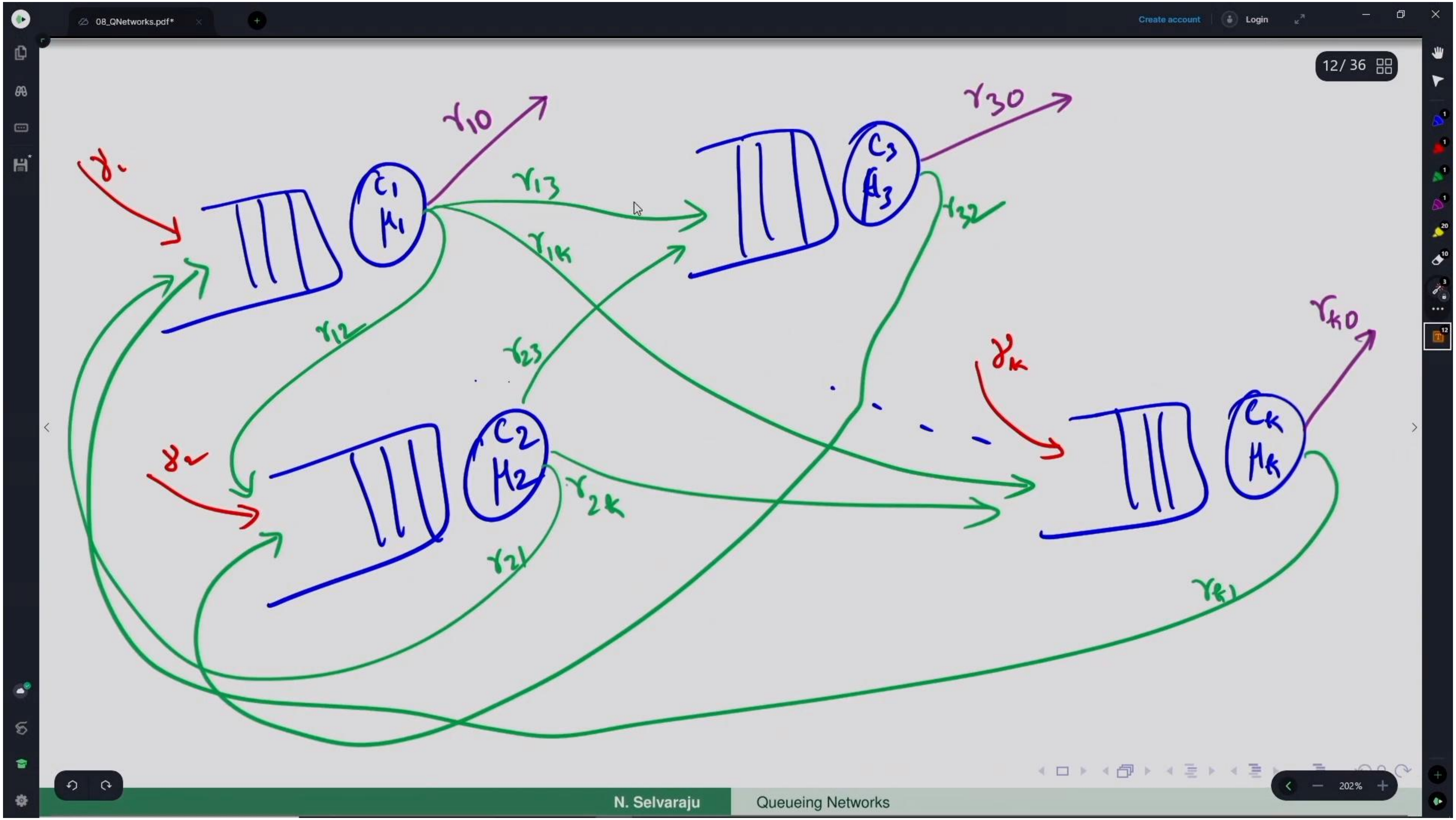
$$\Longrightarrow \quad P(L, \Lambda) = \sum_{N(L)=0}^{\infty} P(\Lambda, N(L))P(L) = P(L)P(\Lambda),$$

thus proving the mutual independence of all the intervals.

Queueing Networks - The General Setup

- The system is a network of queues consisting of a group of k nodes.
- Each node represents a service facility with c_i servers at node $i, i = 1, 2, 3, \ldots, k$ and are assumed to have infinite buffers.
- In general, customers enter the system at any node, traverse from node to node and depart from any node (all customers need not have the same path).
- There may be feedback or may be only feed-forward, and need not visit all nodes too.
- The main characteristics of our networks are:
  1. Arrivals from the 'outside' to node i follow a Poisson process with rate γ_i.
  2. Service (holding) times at each channel at node i are IID $Exp(\mu_i)$ (may depend on the queue length).
  3. The probability that a customer who has completed service at node i will go next to node j (routing probability) is $r_{ij}, 1 \leq i \leq k, 0 \leq j \leq k$ (independent of the state of the system) and r_{i0} indicates the probability that a customer will leave the system from node i.
- The networks having the above properties are called **Jackson networks** (Jackson, 1957, 1963).

  ♦ They have product-form solutions.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Queueing Networks

- Networks with $\gamma_i = 0, \ \forall i$ and $r_{i0} = 0, \ \forall i$ are referred to as 'closed' Jackson networks (the general case described above are 'open' Jackson networks).
- The finite-source queue (machine repairmen problem) is a closed network, with two nodes (one representing operating machines and the other the repair facility).
  - Here, $i = 1, 2; j = 0, 1, 2; r_{12} = r_{21} = 1$ and all other r_{ij}'s are zero.
- We will start with open networks where

$$\gamma_i = \begin{cases} \lambda, & i = 1 \\ 0, & \text{otherwise} \end{cases} \quad \text{and} \quad \gamma_{ij} = \begin{cases} 1, & j = i+1, 1 \leq i \leq k-1 \\ 1, & i = k, j = 0 \\ 0, & \text{otherwise} \end{cases}$$

  - These networks are called 'series' or 'tandem' queues where customers may enter from the outside only at node 1 and depart only from node k.
- We then generalize to a general open Jackson networks and then to closed Jackson networks.
- We consider only Markovian systems, wherein all the exogenous inputs are Poisson, the holding times are exponential, and the routing probabilities are known and state independent.

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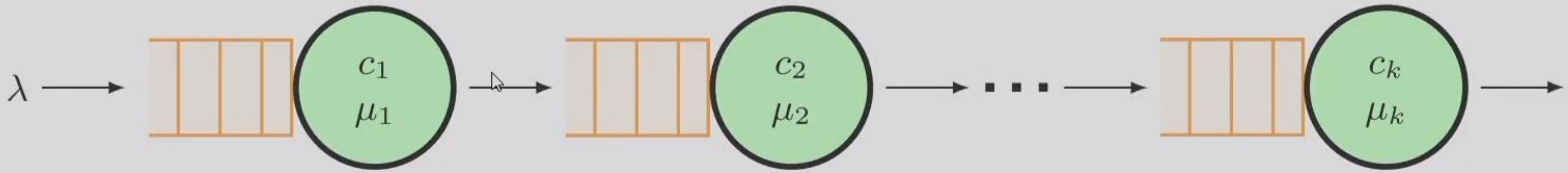
Series or Tandem Queueing Networks

- There are series of service stations through which each calling unit must progress prior to leaving the system.
  - Typical in manufacturing or assembly-line processes, clinical systems.



- The calling units arrive according to PP(λ), no restriction on the capacity of the waiting room between stations, and the service times of each server at station i are $Exp(\mu_i)$, $i = 1, 2, \ldots, k$.
- From our knowledge gained from the consideration of the two-stage series network earlier, we see that each station can be analyzed separately as a single-stage queueing model.
- By Burke's theorem, all stations are independent $M/M/c_i/\infty$ models, as long as there are ample holding spaces in front of each node.
- Thuus, the results of $M/M/c$ can be used on node individually and a complete analysis of this series or tandem queueing network is possible.

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- The steady state probability $p_{n_1,n_2,\dots,n_k}$ that there are n_i customers in the ith node is therefore given by

$$p_{n_1,n_2,\dots,n_k} = p_{n_1} p_{n_2} \cdots p_{n_k}, \qquad n_i \geq 0, i = 1, 2, \dots, k,$$

  where p_{n_i} is the probability that there are n_i customers in the system in an $M/M/c_i$ queue in steady state (which exists under the condition $\rho_i = \dfrac{\lambda}{c_i \mu_i} < 1$).
- The product-form result obtained above holds good in more general cases of Jackson networks too.
- It is possible to carry out analysis of a series network wherein only the last node can have capacity limitations (with blocked customers dropped from the system), using the ideas as above.
- Analysis of feedforward networks (i.e., networks in which customers are not allowed to revisit previously visited nodes) is quite similar to that of the series networks.

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Example (A Supermarket)

- In a supermarket, as customers complete their shopping, they enter the lounge and wait if all the checkout counters are busy.
- Customers arrive according to a PP(40/h) and shopping times and checkout times are exponentially distributed with an average of 45 minutes and 4 minutes, respectively.
  - $a)$ Minimum number of checkout counters needed?
  - $b)$ If we add one more counter to the minimum, what is the average waiting time in lounge? How many customers, on an average, will be in the lounge? How many people, on an average, will be in the entire supermarket?
- This situation can be modelled by a two-node tandem queue.
  - First Stage: $M/M/\infty$ with $\lambda = 40, \mu = \frac{4}{3}$.
  - Second Stage: $M/M/c$ with $\lambda = 40, \mu = 15$.
- For steady-state convergence, we need $c\mu > \lambda \implies c > \frac{\lambda}{\mu} = 2.67 \implies c = 3$. That is, the minimum number of checkout counters needed is 3.

Example

- If we add one more to the minimum number of counters, then the new $c = 4$ and node-2 now is an $M/M/4$ system.
- From the results of $M/M/c$, we have that $p_0 = 0.06$ and

$$W_q = 1.14 \text{ minutes} \qquad \text{and} \qquad L_q = \lambda W_q = 0.76.$$

The total number of customers in the supermarket is

$$\begin{aligned} L &= L\ (\text{of } M/M/\infty) + L\ (\text{of } M/M/4) \\ &= \frac{\lambda}{\mu} + \lambda\left(W_q + \frac{\lambda}{\mu}\right) = \frac{40}{(4/3)} + 40\left(0.019 + \frac{4}{60}\right) = 30 + 3.44 = 33.44 \end{aligned}$$

- Now, one can do similar calculations with the minimum number of checkout counters (i.e., 3) to see how much these numbers increases.