1. Consider a $G/G/1$ queue with usual assumptions and notations. If the interarrival times (T) follow an $Exp(\lambda)$ distribution and the service times (S) are deterministic with value α, then the CDF $U(x)$ of the random variable $U = S - T$ is given by

   (A) $U(x) = \begin{cases} e^{-\lambda(\alpha - x)}, & x < \alpha \\ 1, & x \geq \alpha \end{cases}$

   (B) $U(x) = \begin{cases} e^{\lambda(\alpha - x)}, & x < \alpha \\ 1, & x \geq \alpha \end{cases}$

   (C) $U(x) = \begin{cases} e^{-\lambda(\alpha + x)}, & x < \alpha \\ 1, & x \geq \alpha \end{cases}$

Solution:

Answer: (A)

$$U(x) = P\left[S - T \leq x\right] = P\left[T \geq \alpha - x\right] = \begin{cases} e^{-\lambda(\alpha - x)}, & x < \alpha \\ 1, & x \geq \alpha \end{cases}$$

Alternatively, one can obtain the above from using $U(x) = \displaystyle\int_{\max(0,x)}^{\infty} B(y)dA(y - x)$ too.

2. Consider the $G/G/1$ system given in Question 1 above. Determine $E(U)$.

   (A) $E(U) = \rho - 1$.

   (B) $E(U) = (\rho - 1)/\lambda$.

   (C) $E(U) = (\rho + 1)/\lambda$.

   (D) $E(U) = \lambda(\rho - 1)$.

Solution:

Answer: (B)
$E(U) = E(S - T) = \alpha - E(T) = \alpha - 1/\lambda = (\rho - 1)/\lambda$ since $\rho = \lambda\alpha$.

3. Consider the $G/G/1$ system given in Question 1 above. Now, assume that $\alpha = 2$ and $\lambda = 1/4$. If $W_q \geq r_0$, where r_0 is the lower bound (as described in the lectures), then r_0 equals (approximately)

   (A) 2.23

   (B) 1.75

   (C) 1.33

   (D) 0.77

Solution:

Answer: (D)
Here, $\rho = \frac{\lambda}{\mu} = 1/2$, $\sigma_A^2 = \frac{1}{\lambda^2} = 16$, $\sigma_B^2 = 4 - 4 = 0$, (here $E(S) = \frac{1}{\mu} = 2, E(S^2) = 4$).
Form the previous solutions,

$$1 - U(x) = \begin{cases} 1 - e^{-\frac{1}{4}(2-x)}, & x < 2, \\ 0, & x \geq 2. \end{cases}$$

To get the lower bound, we solve $r_0 - \int_{-r_0}^{\infty}(1 - U(x))dx = 0$, which yields

$$\begin{aligned} r_0 - \int_{-r_0}^{2}(1 - e^{-\frac{1}{4}(2-x)})dx = 0 \\ \Rightarrow r_0 - r_0 + 2 - 4e^{-\frac{2+r_0}{4}} = 0 \\ \Rightarrow r_0 = -2 - 4\ln(1/2) = 0.7726. \end{aligned}$$

4. State whether the following statement is TRUE or FALSE.
In a G/G/1 queue in steady state, the distribution of line delay depends only on the distribution of the difference between the service time and interarrival time distributions, rather than on their individual distributions.

Solution:

Answer: TRUE, from the lectures.

5. State whether the following statement is TRUE or FALSE.
In a G/G/1 queue in steady state, the interarrival time is stochastically smaller than the time between a customer departure and the next start of service of a customer.

Solution:

Answer: FALSE, from the lectures. The reverse is true.

6. Cars arrive at a single station car wash at the rate of 15 per hour. The owner of the car wash goes away on vacation for a certain length of time, which is exponentially distributed with mean of 3.5 minutes and variance 2.5 minutes^2. The owner will go another vacation whenever he is not finding a car to wash on his return from vacation. Besides, the amount of time in washing a car has mean 3 minutes with a standard deviation of 1 minute. Determine the mean waiting time (rounded to the nearest minute) in the queue in the long run.

(A) 4
(B) 10
(C) 7
(D) 12

Solution:

Answer: (C)
The system is $M/G/1$ queue with multiple vacation. Here, $\lambda = 1/4$/minute, $E(S) = 3$ minutes, $E(S^2) = 1^2 + 3^2 = 10$ minutes^2, $E(V) = 3.5$ minutes, $E(V^2) = (3.5)^2 + 2.5 = 14.75$ minutes^2. Also $\rho = \lambda E(S) = 0.75$. From the lecture notes, $W_q = \frac{\lambda E(S^2)}{2(1-\rho)} + \frac{E(V^2)}{2E(V)} = 7.1071$ minutes.

7. State whether the following statement is TRUE or FALSE.
For an $M/G/1$ queue with vacations, if the number of customers present at the start of a busy period following a vacation or vacation period is always equal to 2, then the PGF of the number of arrivals during the limiting residual vacation period is equal to $(1 + z)/2$.

Solution:

Answer: TRUE, from the lectures. Here, $P\{N^* = 2\} = 1$ and hence $V(z) = \frac{1-R(z)}{(1-z)E(N^*)} = \frac{1-z^2}{(1-z)2} = (1 + z)/2$.

8. State whether the following statement is TRUE or FALSE.
In an $M/G/1$ queue with multiple vacations, if $V(z)$ is the PGF of the number in the system at a random point in time when the server is on vacation, then $V\left(\frac{s}{\lambda}\right)$ is the Laplace-Stieltjes transform of the the limiting residual vacation period.

Solution:

Answer: FALSE, from the lectures, since $V(1 - s/\lambda)$ is the required quantity.