1. Customers arrive in groups at a single-server queue. These groups arrive according to a Poisson process with a rate of 3 groups per hour. With probability 1/3, a group consists of two customers, and with probability 2/3, it consists of only one customer. All customers have an exponential service time with a mean of 6 minutes. Groups are served in order of arrival and within a group, customers are served in random order. Then, the steady state average number of customers waiting in the queue equals

   (A) $\frac{5}{6}$

   (B) $\frac{8}{15}$

   (C) $\frac{13}{30}$

   (D) $\frac{2}{5}$

Solution:

Answer: (C)
The given queueing model is $M^{[X]}/M/1$ with $E(X) = 4/3$, and $E(X^2) = 2$, $\lambda = 3/\text{hr.}$, $\mu = 10/\text{hr.}$ Also, $r = 3/10$, $\rho = 4/10$.

$$L = \frac{\rho + rE(X^2)}{2(1-\rho)} = 5/6 = 0.8333, \quad L_q = L - \rho = 13/30 = 0.4333.$$

2. Consider an $M^{[X]}/M/1$ queueing system with parameters λ and μ. At each of the arrival instants, one new customer will enter the system with probability 0.5 or two new customers will enter with probability 0.5. Then, with ρ defined in the usual manner, which of the following expressions represent the expected number of customers in the system at equilibrium?

   (A) $\frac{2r}{1-\rho}$.

   (B) $\frac{\rho}{1-\rho}$.

   (C) $\frac{4\lambda}{2\mu - 3\lambda}$.

   (D) $\frac{4\lambda}{\mu - \lambda}$.

Solution:

Answer: (A) and (C) [Hint: There may be more than one correct options.]
From the given data, $E(X) = 1 * 1/2 + 2 * 1/2 = 3/2, E(X^2) = 1^2 * 1/2 + 2^2 * 1/2 = 5/2$ and hence $\rho = \frac{\lambda}{\mu}E(X) = \frac{3\lambda}{2\mu}$. Then, we have that

$$L = \frac{r[E(X) + E(X^2)]}{2(1-\rho)} = \frac{r * 8/2}{2(1-\rho)} = \frac{2r}{1-\rho} = \frac{4\lambda}{2\mu - 3\lambda}.$$

3. State whether the following statement is TRUE or FALSE:
In an $M/M^{[K]}/1$ partial-batch bulk service model with $K = 1$ and with parameters λ and μ, we have that $L_q = (\lambda/\mu)L$.

Solution:

Answer: TRUE

With $K = 1$, the system is actually an $M/M/1$ system for which the given relationship holds. Alternatively, you can obtain the single root r_0 from the characteristics equation for the bulk service and see that this holds.

4. Consider a two-stage facility and suppose that all the jobs go through an $Exp(\mu_1)$ distributed processing time at stage-1. After completing service at stage-1, a job either exits the facility with probability $1 - p$ or go through a further processing for an $Exp(\mu_2)$ distributed duration at stage-2 with probability p $(0 < p < 1)$ and then exits the facility. The processing times in the two stages are independent of each other. If the total processing time of a job is modelled as a phase-type distribution $PH(\alpha, \tilde{Q})$, then $\tilde{Q}$ is given by

(A) $\begin{bmatrix} -\mu_1 & \mu_1 \\ 0 & -(1-p)\mu_2 \end{bmatrix}$

(B) $\begin{bmatrix} -\mu_1 & (1-p)\mu_1 \\ 0 & -\mu_2 \end{bmatrix}$

(C) $\begin{bmatrix} -\mu_1 & \mu_1 \\ 0 & -p\mu_2 \end{bmatrix}$

(D) $\begin{bmatrix} -\mu_1 & p\mu_1 \\ 0 & -\mu_2 \end{bmatrix}$

Solution:

Answer: (D)

Referrring to the lecture, the total processing time of the job can be modelled as the time to absorption of a CTMC on the state space $\{1, 2, 3\}$, with 3 as an absorbing state and with holding times in state i as $Exp(\mu_i), i = 1, 2$. The rate of transitions from state 1 to 2 is $p\mu_1$ (job moves to second stage) and from 1 to 3 it is $(1 - p)\mu_1$ (job exits). Similarly, the transition rate from state 2 to 3 is μ_2. This gives the matrix $\tilde{Q}$ corresponding to the non-absorbing states as $Q = \begin{bmatrix} -\mu_1 & p\mu_1 \\ 0 & -\mu_2 \end{bmatrix}$. This distribution is known as a two-phase Coxian distribution.

5. Consider the two-stage service facility as given in the Question 4 above. Then, the probability density function of the total processing time is given by

(A) $p\mu_1 e^{-\mu_1 t} + (1-p)\left(\frac{\mu_1\mu_2}{\mu_2 - \mu_1} e^{-\mu_1 t} - \frac{\mu_1\mu_2}{\mu_2 - \mu_1} e^{-\mu_2 t}\right)$

(B) $(1-p)\mu_1 e^{-\mu_1 t} + p\left(\frac{\mu_1\mu_2}{\mu_2 - \mu_1} e^{-\mu_1 t} - \frac{\mu_1\mu_2}{\mu_2 - \mu_1} e^{-\mu_2 t}\right)$

(C) $\frac{\mu_1\mu_2}{\mu_2 - \mu_1} e^{-\mu_1 t} - \frac{\mu_1\mu_2}{\mu_2 - \mu_1} e^{-\mu_2 t}$

(D) $(1-p)\mu_1 e^{-\mu_1 t} + p(\mu_1 + \mu_2)e^{-(\mu_1+\mu_2)t}$

Solution:

Answer: (B)

Again, referring to the lecture, one can solve the forward Kolmogorov equations to get the answer (as done in the case of hypoexponential). Alternatively, one can proceed as follows. If $X_1 \sim Exp(\mu_1)$ and $X_2 \sim Exp(\mu_2)$, then the total processing time X equals X_1 with probability $1-p$ and equals $X_1 + X_2$ with probability p, i.e, it is a mixture of these two distributions (one exponential and one hypoexponential). And, this gives the PDF as $(1-p)\mu_1 e^{-\mu_1 t} + p\left(\frac{\mu_1\mu_2}{\mu_2-\mu_1}e^{-\mu_1 t} - \frac{\mu_1\mu_2}{\mu_2-\mu_1}e^{-\mu_2 t}\right)$.

6. In a certain factory, machines break down according to a Poisson process at an average rate of two per hour. The repair of each machine consists of five basic steps that must be performed sequentially. The time taken to perform each of the five steps is found to have an exponential distribution with mean 5 minutes and is independent of the other steps. If there is only one repairman, what is the average idle time (in minutes) for each machine that has broken down?

   (A) 50
   (B) 10
   (C) 100
   (D) 40

Solution:

Answer: (C)

This is an $M/E_k/1$ queue with $k=5$, $\lambda = 1/30/$ min., $\mu = 1/25/\text{min}$. Hence $\rho = 5/6$, and

$$W_q = \frac{1+1/k}{2}\frac{\rho}{\mu(1-\rho)} = 75 \text{ min}, \quad W = W_q + \frac{1}{\mu} = 100 \text{ min}.$$

7. In a single server queueing system, the interarrival times are IID Erlang type-2 distribution with mean interarrival time of 15 minutes. The mean service time is 10 minutes and the service times are IID exponentially distributed. Then, the probability that there are two customers in the system in steady state equals _____.

Solution:

Answer: Range: 0.155 **to** 0.170 [Hint: Enter the answer in three decimals.]

From the given data for the $E_2/M/1$ model, $\lambda = \frac{1}{15}$ min $= 4$/h, $\mu = \frac{1}{10}$ min $= 6$/h, $k = 2$. The characteristic equation is

$$\mu r^{k+1} - (k\lambda+\mu)r + k\lambda = 0 \quad \Leftrightarrow \quad 6r^3 - 14r + 8 = 0$$
$$3r^3 - 7r + 4 = 0 \quad \Leftrightarrow \quad (r-1)(3r^2+3r-4) = 0$$

Here we have the single positive root in $(0,1)$ equal to $r_0 = (-3+\sqrt{57})/6 = 0.7583$. Then

$$p_n^{(c)} = \rho(1-r_0^k)(r_0^k)^{n-1} = \frac{4}{6}(1-0.7583^2)(0.7583^2)^{n-1} = 0.2833(0.5750)^{n-1}$$

and therefore the required probability is $p_2^{(c)} = 0.1629$.

8. State whether the following statement is TRUE or FALSE:
The usual performance measures of an $M/D/1$ queue can be obtained from the performance measures of a sequence of $M/E_k/1$ queues.

Solution:

Answer: TRUE

We know that as $k \to \infty$, the E_k distribution approaches a degenerate distribution (degenerate at the mean of E_k). This property can be utilized to obtain the performance measures of an $M/D/1$ queues from a sequence of $M/E_k/1$ queues.