# Module 10: Semi-Markovian Queueing Systems

LECTURE 37

Regenerative Processes, Semi-Markov Processes

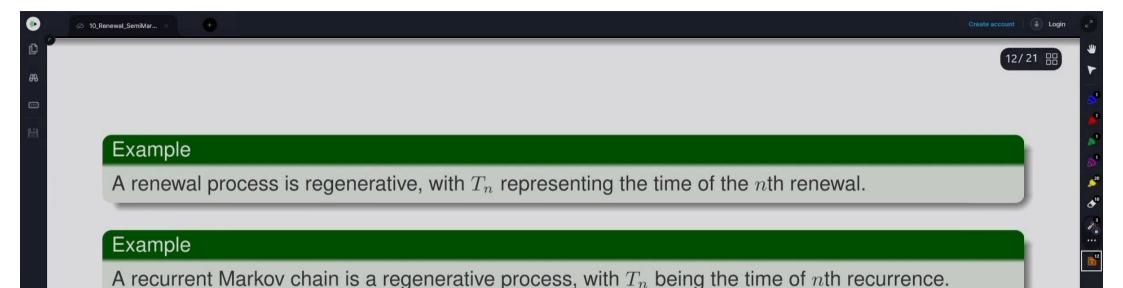
### Definition

A stochastic process  $\{X(t), t \geq 0\}$  is called a regenerative process if there exists time points  $0 = T_0 < T_1 < T_2 < \dots$  such that, for n > 1,

- the process  $\{X(T_n+t), t \geq 0\}$  is independent of the process  $\{X(t), 0 \leq t < T_n\}$ .
- the processes  $\{X(T_n+t), t \geq 0\}$  and  $\{X(t), t \geq 0\}$  have the same joint distribution.

The  $T_n$ 's are called regeneration epochs (times or points) and the lengths  $T_1 - T_0, T_2 - T_1, \ldots$  are called regeneration cycles.  $T_n$  's are IID random variables and  $\{T_n, n \geq 0\}$  defines a renewal process. The renewal process is said to be embedded in  $\{X(t)\}$  at the epochs  $T_1, T_2, \ldots$  Every time a renewal occurs a cycle is said to be completed.

Thus, a regenerative process is a stochastic process with time points starting from which the process is a probabilistic replica if the whole process starting at 0.



### Example

In an M/G/c queue, whenever the queue is empty, all servers are idle and only the arival process has an effect on the future. Thus, the system process regenerates at the points  $T_n$  of the system becoming idle for the nth time. The durations  $T_{n+1} - T_n$  are IID. Hence, the M/G/c system process is a regenerative process.

### Example (Alternating Renewal Process)

Another example of a regenerative process is an alternating renewal process. Such a process can be envisaged by considering that a system can be in one of two possible states - say, 0 and 1. Initially, it is at state 0 and remains at that state for a time  $Y_1$ , and then a change of state to state 1 occurs in which it remains for a time  $Z_1$ , after which it again goes to state 0 for a time  $Y_2$  and then goes to state 1 for a time  $Z_2$  and so on. That is, its movement could be denoted by  $0 \to 1 \to 0 \to 1 \dots$  (For the initial state of 1, the movement sequence is  $1 \to 0 \to 1 \to 0 \to 1$ ...)

• Suppose that  $\{Y_n\}$ ,  $\{Z_n\}$  are two sequences of IID random variables and that  $Y_n$  and  $Z_n$  need not be independent, Let  $T_n = T_{n-1} + Y_n + Z_n \quad n = 1, 2, \dots$ 

Then at time  $T_1$  the process restarts itself, and so also at times  $T_2, T_3 \dots$  The interval  $T_n - T_{n-1}$  denotes a complete cycle, and the process restarts itself after each complete cycle.

• Let  $E[Y_n] = E[Y]$ ,  $E(Z_n) = E(Z)$ . Then the long-run proportions of time that the system is at states 0 and 1 are, respectively, are

$$p_0 = \lim_{t \to \infty} P\{X(t) = 0\} = \frac{E(Y)}{E(Y) + E(Z)}$$
 and 
$$p_1 = \lim_{t \to \infty} P\{X(t) = 1\} = \frac{E(Z)}{E(Y) + E(Z)} = 1 - p_0.$$

# Use in Queueing



- The results of the alternating renewal process example given above have an important application in queueing theory.
- Consider a single-server queueing system such that an arriving customer is immediately taken for service if the server is ree, but joins a waiting line if the server is busy.
- The system can be considered to be in two states (idle or busy) according to whether the server is idle or busy.
- The idle and busy states alternate and together constitute a cycle of an alternating renewal process. A busy period starts as soon as a customer arrives before an idle server and ends at the instant when the server becomes free for the first time.
- The epochs of commencement of busy periods are regeneration points. Let  $I_n$  and  $B_n$  denote the lengths of nth idle and busy periods, respectively, and let

$$E(I_n) = E(I)$$
 and  $E(B_n) = E(B)$ .



• Then the long-run proportion of time that the server is idle equals

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$$p_0 = \frac{E(I)}{E(I) + E(B)} \tag{1}$$

and the long-run proportion of time that the server is busy equals

$$p_1 = \frac{E(B)}{E(I) + E(B)}.$$
 (2)

• Remark: If the arrival process is Poisson with mean  $\lambda t$ , then it follows (from its lack of memory property) that an idle period is exponentially distributed with mean  $1/\lambda$ , i.e,  $E(I) = 1/\lambda$ . Then when  $p_0$  or  $p_1$  is known, E(B) can be found.

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### Markov Renewal Processes and Semi-Markov Processes

We consider a special class of regenerative processes that is important for the analysis of many queueing systems. And, this class generalizes Markov processes and renewal processes at the same time.

#### Definition

Let S denote a countable state space. For every  $n=0,1,2,\ldots$ , let  $X_n$  denote a RV on S and  $T_n$  a nonnegative RV such that  $0=T_0< T_1< T_2<\ldots$  and  $\sup_{n\to\infty} T_n=\infty$  almost surely. Define the process  $\{Y(t), t\geq 0\}$  by

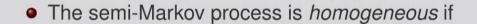
$$Y(t) = X_n$$
 for  $T_n \le t < T_{n+1}$ 

for all  $t \geq 0$ . If

$$P\{X_{n+1} = j, T_{n+1} - T_n \le u | X_0, \dots, X_n, T_0, \dots, T_n\} = P\{X_{n+1} = j, T_{n+1} - T_n \le u | X_n\}$$

hods for all  $n=0,1,2,\ldots,j\in S$ , and  $u\geq 0$ , then  $\{Y(t),t\geq 0\}$  is called a semi-Markov process on S.

The sequence of random variables  $\{(X_n, T_n), n \ge 0\}$  is called the embedded Markov renewal chain (or simply the Markov renewal process).



$$Q_{ij}(t) = P\{X_{n+1} = j, T_{n+1} - T_n \le t \mid X_n = i\}$$

is independent of n. We consider only this case.

- By definition, a semi-Markov process is a pure jump process and hence the sample paths are step functions.
- By construction, the semi-Markov process is determined by the embedded Markov renewal chain and vice versa.
- Let  $\{Y(t), t \geq 0\}$  is a homogeneous Markov process with S and parameters  $\lambda_i, i \in S$  for the exponential holding times. The embedded Markov chain  $\{X_n\}$  has the transition matrix  $P = (p_{ij})$ . Then  $\{Y(t)\}$  is a semi-Markov process with  $Q_{ij}(t) = p_{ij} \left(1 e^{-\lambda_i t}\right)$  for all  $i, j \in S$ . Thus, for a Markov process, the distribution of  $T_{n+1} T_n$  is exponential and independent of the state entered at time  $T_{n+1}$ . These are the two features for which the semi-Markov process is a generalization of the Markov process on a discrete state space (i.e., CTMC).

• It can be shown easily that, for a semi-Markov process  $\{Y(t), t \geq 0\}$  with embedded Markov renewal chain  $\{(X_n, T_n), n \geq 0\}$ , the chain  $\{X_n, n \geq 0\}$  is a Markov chain (called embedded Markov chain). And, we denote its TPM by  $P = (p_{ij})_{i,j \in S}$ . Then the following relation holds for all  $i, j \in S$ :

$$p_{ij} = P\{X_{n+1} = j | X_n = i\} = \lim_{t \to \infty} Q_{ij}(t).$$

- According to its embedded Markov chain  $\{X_n\}$  we call a semi-Markov process irreducible, recurrent or transient.
  - An irreducible recurrent semi-Markov process is regenerative, as one can fix any initial state  $i \in S$  and the times of visiting this state to be a renewal process.
- Define  $F_{ij}(t) = Q_{ij}(t)/p_{ij}$  for all  $t \ge 0$  and  $i, j \in S$  if  $p_{ij} > 0$ , while  $F_{ij}(t) = 0$  otherwise. Then, this can be interpreted as

$$F_{ij}(t) = P\{T_{ij} \le t\} = P\{T_{n+1} - T_n \le t | X_n = i, X_{n+1} = j\},\$$

i.e, this is the distribution function of  $T_{ij}$ , the conditional sojourn time at state i given that the next transition is to state j.

Then, the unconditional sojourn time at state i equals  $\tau_i = \sum_j p_{ij} T_{ij}$ .

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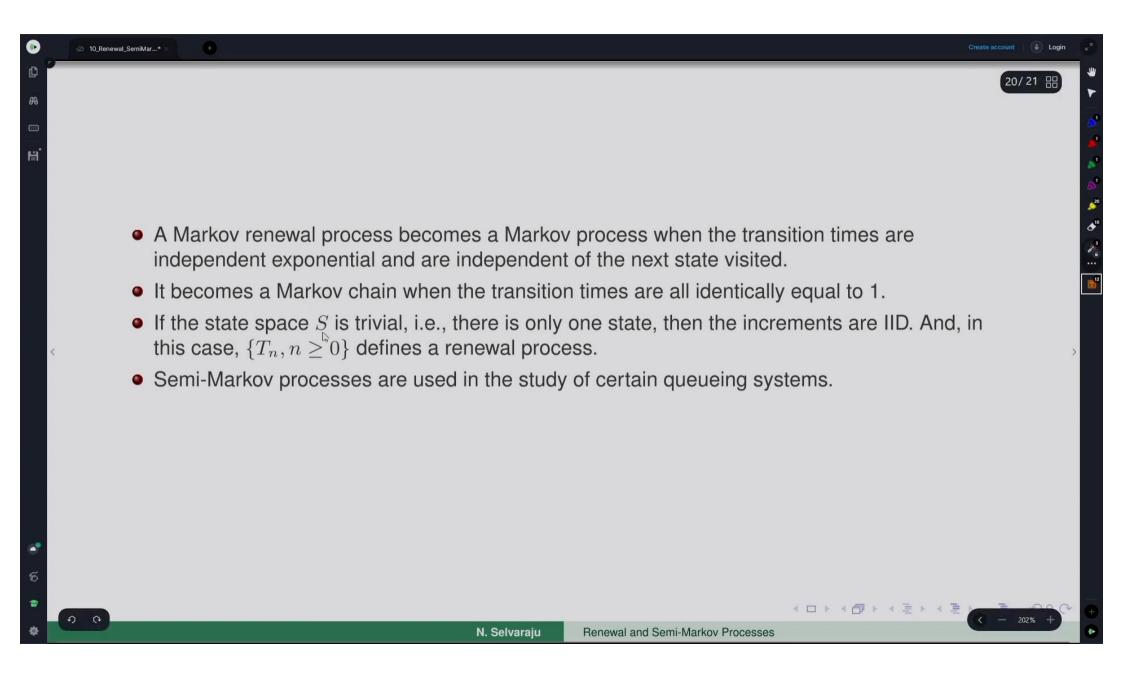
## Example

A pure-birth process is a special type of Markov renewal process with

$$Q_{ij}(t) = 1 - e^{a_i t}, \quad j = i + 1,$$
  
= 0, otherwise

Then

$$p_{ij}=1,\quad j=i+1,$$
  $=0$  otherwise  $F_{ij}(t)=Q_{ij}(t),\quad au_i=T_{ij},\quad j=i+1$ 



- Let  $p_k = \lim_{t \to \infty} P\{Y(t) = k\}$  for  $k \in S$ .
- Suppose that the embedded Markov chain  $\{X_n\}$  is irreducible and positive recurrent with stationary distribution  $\{\nu_j, j \in S\}$ . That is,  $\nu_j = \lim_{n \to \infty} p_{ij}^{(n)}$  exists and are given as the unique nonnegative solution of

$$\nu_j = \sum_{k \in S} \nu_k p_{kj}, \quad j \in S, \quad \sum_j \nu_j = 1.$$

Then, we would expect that  $p_k$  to be proportional to  $\nu_k \mu_k$ , i.e.,

$$p_k = \frac{\nu_k \mu_k}{\sum_{j \in S} \nu_j \mu_j},$$

where  $\mu_k = E(\tau_k)$  is the expected sojourn time in state k until the next transition happens at time  $T_{n+1}$ .

(Refer to any standard text for a proof)