MODULE 11: Semi-Markovian Queueing Systems (contd...)

LECTURE 41

M/G/1/K Queues, Additional Insights on M/G/1 Queues

M/G/1/K Queues

- $M/G/1/K$ analysis is very simlar to that of $M/G/1$. So, let us look at the main ideas/results.
- The PK mean formulas will not hold now, since the expected number of (joined) arrivals during a service period must be conditioned on the system size.
- Since $M/G/1/K$ has only finite number of states, we can find steady state probabilities and then obtain the mean value results.
- The single-step transition matrix must here be truncated at $K-1$ (and not K, as we are observing just after a departure), so that (we assume $K > 1$)

$$P = ((p_{ij})) = \begin{bmatrix} k_0 & k_1 & k_2 & \dots & 1 - \sum_{n=0}^{K-2} k_n \\ k_0 & k_1 & k_2 & \dots & 1 - \sum_{n=0}^{K-2} k_n \\ 0 & k_0 & k_1 & \dots & 1 - \sum_{n=0}^{K-3} k_n \\ 0 & 0 & k_0 & \dots & 1 - \sum_{n=0}^{K-4} k_n \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & 1 - k_0 \end{bmatrix}$$

- The stationary equations now become

$$\pi_i = \begin{cases} \pi_0 k_i + \sum_{j=1}^{i+1} \pi_j k_{i-j+1}, & i = 0, 1, 2, \ldots, K-2 \\ 1 - \sum_{j=0}^{K-2} \pi_j, & i = K-1 \end{cases}$$

- These K (consistent) equations in K unknowns can then be solved for all the probabilities.

- The average system size at points of departure is thus given by $L = \sum_{i=0}^{k-1} i\pi_i$.

- The first portion of the stationary equation is identical to that of the unlimited $M/G/1$.
  - ▶ The respective stationary probabilities $\{\pi_i\}$ for $M/G/1/K$ and $\{\pi_i^*\}$ for $M/G/1/\infty$ must be at worst proportional for $i \leq K-1$; that is, $\pi_i = C\pi_i^*$, $i = 0, 1, \ldots, K-1$.
  - ▶ The usual condition that the probabilities sum to one implies that $C = 1/\sum_{i=0}^{K-1} \pi_i^*$.

- Also, the probability distribution for the system size encountered by an arrival will be different from $\{\pi_i\}$, since now the state space must be enlarged to include K.
- Let a'_n denote the probability that an arriving customer finds a system with n customers.
  - ▶ Here, we are talking about the distribution of arriving customers whether or not they join the queue, as opposed to only those arrivals who join, denoted by a_n.
  - ▶ The distribution $\{a'_n\}$ also has its own significance.
- Recall that in the proof that $\pi_n = p_n$, the equality holds as longs as arrivals occur singly and service is not in bulk.
- Similar is the case with a'_n , except that the state spaces are different. This difference is taken care of by first noting that

$$\pi_n = P\left\{\text{arrivals finds } n | \text{customer does in fact join}\right\}$$

$$= a_n = \frac{a'_n}{1 - a'_K}, \quad 0 \leq n \leq K - 1.$$

Therefore, $a'_n = (1 - a'_K)\pi_n, \quad 0 \leq n \leq K - 1.$

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14/ 18

- To get a'_K, we use an approach similar to Markovian models where we equate the effective arrival rate with the effective departure rate, i.e.,

$$\lambda\left(1 - a'_K\right) = \mu\left(1 - p_0\right).$$

Therefore,

$$a'_n = \frac{\left(1 - p_0\right)\pi_n}{\rho}, \quad 0 \leq n \leq K - 1$$

$$a'_K = \frac{\rho - 1 + p_0}{\rho}.$$

- But, since the original arrival process is Poisson, $a'_n = p_n$ for all n. Thus,

$$a'_0 = p_0 = \frac{\left(1 - p_0\right)\pi_0}{\rho} \Rightarrow p_0 = \frac{\pi_0}{\pi_0 + \rho}.$$

Finally,

$$a'_n = \frac{\pi_n}{\pi_0 + \rho}.$$

202%

M/G/1: Some Additional Results/Insights

- Now we study some additional results for $M/G/1$ queues, including impatience, output, transience, finite source, and batching.
- For impatience, we can easily introduce balking into the $M/G/1$ queue by prescribing a probability b that an arrival decides to actually join the system. Then the true input process becomes a (filtered) Poisson process with mean $b\lambda t$, and k_n's thus has to be written as

$$k_n = \int_0^\infty \frac{e^{-b\lambda t}(b\lambda t)^n}{n!} dB(t).$$

- The rest of the analysis goes through parallel to that for the regular $M/G/1$ queue, with the probability of idleness, p_0, now equal to $1 - b\lambda/\mu$.

- As far as output is concerned, we have that the steady-state $M/M/1$ has Poisson output, but $M/G/1$ queues do not posses this property. This is because such queues are never reversible, so that their output processes cannot probabilistically match their inputs.
- What is the distribution of an arbitrary $M/G/1$ interdeparture time in the steady state? Define $C(t)$ as the CDF of the interdeparture times, then (with $B(t)$ as servicetime CDF)

$$
\begin{aligned}
C(t) =& P\{\text{interdeparture time} \leq t\} \\
=& P\{\text{system experienced no idleness during interdeparture period}\} \\
& \times P\{\text{interdeparture time} \leq t | \text{no idleness}\} \\
& + P\{\text{system experienced some idleness during interdeparture period}\} \\
& \times P\{\text{interdeparture time} \leq t | \text{some idleness}\} \\
=& \rho B(t) + (1-\rho) \int_0^t B(t-u) \lambda e^{-\lambda u} du,
\end{aligned}
$$

since the length of an interdeparture period with idleness is the sum of the idle time and service time.

Exercise: Exponentiality of $C(t)$ implies exponentiality of $B(t)$.

- The fact that $M/M/1$ is the only $M/G/1$ with exponential output has serious negative implications for the solution of series models. The output of a first stage will be exponential, which we would like it to be, only if it is $M/M/1$.
  - ▶ Small $M/G/1$ series problems can be handled numerically with the help $C(t)$.
- By putting a capacity restriction on the $M/G/1$ queue at $K = 1$, it can be seen that such queues also have IID interdeparture times. This is because the successive departure epochs are identical to the busy cycles, which are found as the sums of each idle time paired with an adjacent service time.

- To get the transient results for the $M/G/1$ queue, we appeal directly to the theory of Markov chains and the Chapman-Kolmogorov equation

$$p_j^{(m)} = \sum_k p_k^{(0)} p_{kj}^{(m)},$$

where $p_j^{(m)}$ is then the probability that the system state isin state j just after the mth customer has departed.
  - ▶ In practice, a careful truncation of the transition matrix is needed for computations.

- The finite-source $M/G/1$ is essentially the machine-repairmen problem with arbitrarily distributed repair times and has been solved in the literature, again using an embedded Markov chain approach.
- The bulk-input $M/G/1$, denoted by $M^{[X]}/G/1$, and the bulk-service $M/G/1$, denoted by $M/G^{[Y]}/1$, can also be solved with the use of Markov chains.
  - ▶ While the bulk-input model is relatively easy (in a manner similar to $M/G/1$ but with certain differences), the bulk-service problem is messy (but doable).
- With respect to priorities, we can obtain expected value measures in an easier manner for certain models.