Module 10: Semi-Markovian Queueing Systems

Lecture 39

M/G/1 Queues, The Pollaczek-Khinchin Transform Formula

Example

- Assume that a system is currently working as an $M/M/1$ system with $\lambda = 10$ and $\mu = 12$, per hour.
- The server undergoes a training session at the end of which it is expected that while the mean service time would increase slightly, the variance would see an improvement.
  - The mean service time now is 5.5 minutes and the standard deviation is 4 minutes.
  - The system is now an $M/G/1$ system.
- Management is interested to know the impact of the training and whether they should have the server undergo further training.
- Let us compare L and W:
  For $M/M/1$: $L = 5$ and $W = 30$ minutes.
  For $M/G/1$: $L = 8.625$ and $W = 51.75$ minutes.
- Hence, it is not profitable to have the server better trained.
  - Here, with training, while the mean increased by 10%, the standard deviation decreased by 20% (from 5 to 4).
  - The performance is more sensitive to mean than to standard deviation.

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Example

- It may be of interest to calculate the reduction in variance required to make up for the increase of 0.5 in the mean.
- We can do this by solving for σ_B^2 in the PK formula for L:

$$L = 5 = \rho + \frac{\rho^2 + \lambda^2 \sigma_B^2}{2(1-\rho)},$$

  where $\rho = 11/12$. This yields $\sigma_B^2 < 0$, which is not possible.
  - ▶ This means that $L > 5$ always (even with $\sigma_B^2 = 0$).
  - ▶ The minimum value of L, achieved in the $M/D/1$ system, turns out to be $L = 6$.
- *Exercise: Determine the value of* σ_B^2 *required to yield the same* L *if the mean service time were increased to only* 5.2 *minutes after training.*

Derivation Using Departure Times

- We can derive the PK mean formulas given in table earlier considering the queue at times when customers depart from the queue.
- Considering the number of customers remaining in the system immediately after a customer has departed from the system, we can first derive a formula for the expected system size L at departure points.
- This is then seen to be equal to the expected steady-state system size at an arbitrary point in time.
- Instead of doing this, we will treat the steady state system size probabilities at departure points, from which too the PK mean formulas can be obtained.

Departure-Point System Size Probabilities: The Pollaczek-Khinchin (PK) Formula

- Let π_n denote the steady state probability of n in the system at a departure point.
  - In general, it need not be the case that $\pi_n = p_n$, but it is true here for the $M/G/1$ model.
- The $M/G/1$ queue, viewed only at departure times, leads to an embedded discrete-time Markov chain.
- The number in the system process $\{N(t), t \geq 0\}$ is not a Markov process here, because the state of the system after a transition depends not only on the state of $N(t)$, but also on the amount of elapsed service time of the person receiving service, if any. [Together, they form a Markov process]
  - If we consider the system only at those points when a customer completes his service, there will be no elapsed service time.
  - The evolution of $N(t)$ at those departure points can be captured nicely.

- Let $t_1, t_2, \ldots$ be the sequence of departure times from the system.
- Let $X_n = N(t_n+)$ be the number of customers left in the system immediately after the departure at time t_n.
- If $Y(t)$ denotes the number of customers left-behind in the system by the most recent departure. That is, $Y(t) = X_n, \quad t_n \leq t < t_{n+1}$.
  - $\{Y(t)\}$ is a semi-Markov process having $\{X_n, n = 0, 1, ...\}$ as its embedded Markov chain.
  - $\{(X_n, t_n), n = 0, 1, 2, \ldots\}$ is a Markov renewal process.
  - The sequence of intervals $\{t_{n+1} - t_n, n = 0, 1, 2, \ldots\}$ being the inter-departure times of successive units (or equivalently $\{t_n, n = 0, 1, 2, \ldots\}$) defines a renewal process.
- Let A_n be the number of customers who arrive during the service time of the nth customer. Then, for all $n \geq 1$.

$$X_{n+1} = \begin{cases} X_n - 1 + A_{n+1}, & X_n \geq 1, \\ A_{n+1}, & X_n = 0. \end{cases}$$

- We see that $\{X_n, n \geq 1\}$ is a Markov chain.
  ♦ Need to show that future states of the chain depend only on the present state – more specifically, we must show that given the present state X_n, the future state X_{n+1} is independent of previous states $X_{n-1}, X_{n-2}, \ldots$.
- First observe that X_{n+1} depends only on X_n and A_{n+1}. If A_{n+1} is independent of $X_{n-1}, X_{n-2}, \ldots$, then $\{X_n\}$ is a Markov chain.
- A_{n+1} is the number of customers arrive during the service time of the $(n+1)$th customer and depends on the length of this service time, but does not depend on events that occurred earlier (namely, the queue sizes at earlier departure points).
  ► Thus, A_{n+1} independent of $X_{n-1}, X_{n-2}, \ldots$ and hence $\{X_n\}$ is a MC.

- We now derive the transition probabilities for this Markov chain

$$p_{ij} = P\{X_{n+1} = j | X_n = i\}.$$

- The transition probabilities depend on the distribution of the number of customer who arrive during a service time.
- Let S denote a random service time (with CDF $B(\cdot)$) and A denote the random number of customers who arrive during this time (we drop the subscript as the distribution does not depend on the index of the customer). Define, for $i = 0, 1, 2, \ldots,$

$$k_i = P\{i \text{ arrivals during a service time}\} = P\left\{A = i\right\} = \int_0^\infty P\left\{A = i | S = t\right\} dB(t).$$

- Note that $A|S = t$ is a Poisson random variable with mean λt, and hence $P\left\{A = i | S = t\right\} = \dfrac{e^{-\lambda t}(\lambda t)^i}{i!}$ giving us

$$k_i = \int_0^\infty \frac{e^{-\lambda t}(\lambda t)^i}{i!} dB(t).$$

- Then from the relationship between X_n's, we get

$$p_{ij} = P\{X_{n+1} = j | X_n = i\} = \begin{cases} P\{A = j - i + 1\}, \; = k_{j-i+1} & i \geq 1 \\ P\{A = j\}, \; = k_j & i = 0 \end{cases}$$

- We the have the following transition probability matrix

$$P = ((p_{ij})) = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ \vdots \end{array} \begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & \cdots \end{array} \\ \begin{pmatrix} k_0 & k_1 & k_2 & k_3 & \ldots \\ k_0 & k_1 & k_2 & k_3 & \ldots \\ 0 & k_0 & k_1 & k_2 & \ldots \\ 0 & 0 & k_0 & k_1 & \ldots \\ 0 & 0 & 0 & k_0 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \end{pmatrix} \end{array}$$

- Assuming that steady state is achievable, the steady state probability vector $\boldsymbol{\pi} = \{\pi_n\}$ is found in the usual manner as the solution of the stationary equations:

$$\boldsymbol{\pi} = \boldsymbol{\pi} P, \quad \boldsymbol{\pi}\boldsymbol{e} = 1.$$

Writing down explicitly, these equations are

$$\begin{aligned}\pi_i &= \pi_0 k_i + \pi_1 k_i + + \pi_2 k_{i-1} + \cdots + \pi_{i+1} k_0 \\ &= \pi_0 k_i + \sum_{j=1}^{i+1} \pi_j k_{i-j+1}, \qquad i = 0, 1, 2, \ldots\end{aligned}$$

- Now define the generating functions

$$\Pi(z) = \sum_{i=0}^{\infty} \pi_i z^i \quad \text{and} \quad K(z) = \sum_{i=0}^{\infty} k_i z^i \quad (|z| \le 1)$$

- Multiplying the steady state equations by z^i, summing, and solving (Exercise!) for $\Pi(z)$ yields

$$\Pi(z) = \frac{\pi_0 (1-z) K(z)}{K(z) - z}$$

- Using the fact that $\Pi(1) = 1$, along with L'Hospital rule, and realizing that $K(1) = 1$ and $K'(1) = \lambda(1/\mu)$, we find that

$$\pi_0 = 1 - \rho \quad (\rho = \lambda E[S] < 1 \text{ is the condition for ergodicity})$$

and therefore we obtain finally

$$\boxed{\Pi(z) = \frac{(1-\rho)(1-z)K(z)}{K(z) - z}.}$$

This is known as Pollaczek-Khinchin (PK) Formula or Pollaczek-Khinchin (PK) Transform Formula.

- From $\Pi(z)$, we can obtain the PK mean formula for L and hence the other measures too.
- Given the service time distribution B, we can obtain k_i's and hence $K(z)$. Subsituting this, we obtain $\Pi(z)$, the PGF of the distribution of the departure epoch system size. $\{\pi_n\}$ can then be obtained from its PGF.
  - ▶ It is the case here that $\pi_n = p_n$.

$\pi_n = p_n$

- To prove that π_n, the steady-state probability of n in the system at a departure point, is equal to p_n, the steady-state probability of n in the system at an arbitrary point in time.
- We begin by considering a specific realization of the actual process over a long interval $(0, T)$.
- Let $N(t)$ be the system size at time t. Let $A_n(t)$ be the number of unit upward jumps or crossings (arrivals) from state n occurring in $(0, t)$. Let $D_n(t)$ be the number of unit downwards jumps (departures) to state n in $(0, t)$.
- Since arrivals occur singly and customers are served singly, we must have

$$|A_n(T) - D_n(T)| \leq 1. \tag{1}$$

- Furthermore, the total number of departures, $D(T)$, relates to the total number of arrivals, $A(T)$, by

$$D(T) = A(T) + N(0) - N(T). \tag{2}$$

- The departure-point probabilities are

$$\pi_n = \lim_{T \to \infty} \frac{D_n(T)}{D(T)}. \tag{3}$$

- By adding and subtracting $A_n(T)$ from the numerator of (3) and using (2) in its denominator,

$$\frac{D_n(T)}{D(T)} = \frac{A_n(T) + D_n(T) - A_n(T)}{A(T) + N(0) - N(T)} \tag{4}$$

- Since $N(0)$ is finite and $N(T)$ must be too because of the assumption of stationarity, it follows from (1), (4), and the fact that $A(T) \to \infty$

$$\lim_{T\to\infty} \frac{D_n(T)}{D(T)} = \lim_{T\to\infty} \frac{A_n(T)}{A(T)} \tag{5}$$

with probability one. Since the arrivals occur at the points of a Poisson process operating independently of the state of the process,

- Since the arrivals occur at the points of a Poisson process operating independently of the state of the process, we invoke the PASTA property that Poisson arrivals find time averages.
- Therefore the general-time probability p_n is identical to the arrival-point probability $a_n = \lim_{T\to\infty} \frac{A_n(T)}{A(T)}$, which is in turn, equal to departure-point probability from (5).
- Thus, all three sets of probabilities are equal for the $M/G/1$ problem.

$a_n = p_n = \pi_n$.

Example

If we set the service time distribution as exponential, then $M/G/1$ should reduce to $M/M/1$. Take $B(t) = 1 - e^{-\mu t}, t \geq 0$ (and 0 otherwise). Then

$$
\begin{aligned}
k_i &= \int_0^\infty \frac{e^{-\lambda t}(\lambda t)^i}{i!} dB(t) = \int_0^\infty \frac{e^{-\lambda t}(\lambda t)^i}{i!} \mu e^{-\mu t} dt \\
&= \frac{\lambda^i \mu}{i!} \int_0^\infty t^i e^{-(\lambda+\mu)t} dt = \frac{\lambda^i \mu}{i!} \, \frac{(i!)}{(\lambda+\mu)^{i+1}} \\
&= \left(\frac{\mu}{\lambda+\mu}\right)\left(\frac{\lambda}{\lambda+\mu}\right)^i, \quad i = 0, 1, 2, \ldots
\end{aligned}
$$

Therefore, $K(z) = \dfrac{1}{1+\rho-\rho z}$, where $\rho = \lambda/\mu$. Using $K(z)$, we can obtain the PGF $\Pi(z)$ as

$$
\Pi(z) = \frac{(1-\rho)(1-z)\frac{1}{1+\rho-\rho z}}{\frac{1}{1+\rho-\rho z} - z} = \frac{(1-\rho)(1-z)}{(1-z)(1-\rho z)} = \frac{1-\rho}{1-\rho z},
$$

which is the PGF in the $M/M/1$ model (equal to $P(z)$), as required.