Lecture 9: Interpolation Contd..

Department of Mathematics
IIT Guwahati

Rajen Kumar Sinha

Drawback in Lagrange interpolation formula

Recall the Lagrange interpolation formula: We know

$$p_n(x) = \sum_{i=0}^{n} f(x_i) L_i(x)$$

where

$$L_i(x) = \prod_{\substack{j=0 \\ j \neq i}}^{n} \frac{(x - x_j)}{(x_i - x_j)}, \quad i = 0, 1, \cdots, n.$$

The main disadvantage in the Lagrange's form is as follows:

- In practice, one is uncertain about how many interpolation points to use. If one increases interpolation points then the functions $L_i(x)$ have to be recomputed and the previous calculation of $L_i(x)$ of little use. In other words, in calculating $p_k(x)$, no obvious advantage can be taken of the fact that one already has $p_{k-1}(x)$ available.

Newton's interpolating polynomial

Let $(x_i, f(x_i))$, $i = 0, 1, \ldots n$ be the set of $(n+1)$ data points, where x_i's are distinct. We know that $\exists$ a unique polynomial $p_n(x)$ of degree $\leq n$ such that

$$p(x_i) = f(x_i), \quad i = 0, 1, \ldots, n. \tag{1}$$

Consider the interpolating polynomial $p_n(x)$ in Newton form

$$\begin{aligned} p_n(x) &= c_0 + c_1(x - x_0) + c_2(x - x_0)(x - x_1) \\ &\quad + \cdots + c_n(x - x_0)(x - x_1)\cdots(x - x_{n-1}) \\ &= \sum_{j=0}^{n} c_j q_j(x), \ \text{ where } q_j(x) = \prod_{k=0}^{j-1}(x - x_k), \end{aligned}$$

$$q_0(x) = 1, \ q_1(x) = (x - x_0), \ q_2(x) = (x - x_0)(x - x_1),$$

$$\ldots, \ q_n(x) = (x - x_0)(x - x_1)\cdots(x - x_{n-1}).$$

The interpolation conditions (1) lead to

$$p_n(x_i) = \sum_{j=0}^{n} c_j q_j(x_i) = f(x_i), \quad i = 0, 1, \ldots, n$$
$$\implies Ac = F, \text{ where}$$

$$A = (a_{ij}) \text{ with } a_{ij} = q_j(x_i),\ 0 \leq i, j \leq n,$$
$$c = [c_0, c_1, \ldots, c_n]^T, \quad F = [f(x_0), f(x_1), \ldots, f(x_n)]^T.$$

The matrix A is lower triangular because

$$q_j(x) = \prod_{k=0}^{j-1}(x - x_k), \ \ q_j(x_i) = \prod_{k=0}^{j-1}(x_i - x_k) = 0 \text{ if } i \leq j - 1.$$

For $n = 2$, we note that

$$p_2(x) = c_0 + c_1(x - x_0) + c_2(x - x_0)(x - x_1)$$

$$p_2(x_0) = c_0, \ \ p_2(x_1) = c_0 + c_1(x_1 - x_0),$$
$$p_2(x_2) = c_0 + c_1(x_2 - x_0) + c_2(x_2 - x_0)(x_2 - x_1).$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & (x_1 - x_0) & 0 \\ 1 & (x_2 - x_0) & (x_2 - x_0)(x_2 - x_1) \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} f(x_0) \\ f(x_1) \\ f(x_2) \end{bmatrix}$$

Observe that c_0 depends on $f(x_0)$, c_1 depends on $f(x_0)$ and $f(x_1)$ and so on. In general, c_n depends on $f(x_0), f(x_1), \ldots, f(x_n)$. Thus, we write $c_n = f[x_0, x_1, \ldots, x_n]$.

Define the symbol $f[x_0, x_1, \ldots, x_n]$ to be the coefficients of q_n when $p(x) = \sum_{k=0}^{n} c_k q_k$ interpolates f at $x_0, x_1, \ldots, x_n$. Thus, we arrive at the **Newton formula for the interpolating polynomial**

$$\begin{aligned} p_n(x) &= f[x_0] + f[x_0, x_1](x - x_0) + f[x_0, x_1, x_2](x - x_0)(x - x_1) \\ &\quad + \cdots + f[x_0, x_1, \ldots, x_n](x - x_0)(x - x_1) \cdots (x - x_{n-1}) \\ &= \sum_{k=0}^{n} f[x_0, x_1, \ldots, x_k] \prod_{j=0}^{k-1} (x - x_j). \end{aligned}$$

Explicit formula for computing divided differences

$$
\begin{aligned}
&P_n(x_0) = f(x_0) \implies f[x_0] = f(x_0). \\
&p_n(x_1) = f(x_1) \implies f(x_0) + f[x_0, x_1](x_1 - x_0) = f(x_1) \\
&\implies f[x_0, x_1] = \frac{f(x_1) - f(x_0)}{x_1 - x_0}.
\end{aligned}
$$

In general, we have the following formula for computing higher-order divided differences.

$$
f[x_0, x_1, \ldots, x_n] = \frac{f[x_1, x_2, \ldots, x_n] - f[x_0, x_1, \ldots, x_{n-1}]}{x_n - x_0}. \tag{2}
$$

Let's prove the identity (2). Let $p_k(x)$ be the polynomial of degree $\leq k$ such that

$$
p_k(x_i) = f(x_i), \quad i = 0, 1, \ldots, k.
$$

Let $q(x)$ be the polynomial of degree $\leq n-1$ such that

$$
q(x_i) = f(x_i), \quad i = 1, 2, \ldots, n.
$$

Then

$$p_n(x) = q(x) + \frac{x - x_n}{x_n - x_0}\{q(x) - p_{n-1}(x)\} \qquad (3)$$

is a polynomial of degree at most n. Further, note that, for $i = 1, 2, \ldots, n-1$,

$$\begin{aligned} p_n(x_i) &= q(x_i) + \frac{x_i - x_n}{x_n - x_0}\{q(x_i) - p_{n-1}(x_i)\} \\ &= f(x_i) + \frac{x_i - x_n}{x_n - x_0}\{f(x_i) - f(x_i)\} = f(x_i). \end{aligned}$$

For $i = 0$ and $i = n$, we have

$$p_n(x_0) = q(x_0) - \{q(x_0) - p_{n-1}(x_0)\} = p_{n-1}(x_0) = f(x_0)$$
$$p_n(x_n) = q(x_n) = f(x_n).$$

The LHS and RHS polynomials in (3) are identical. Equating the coeffcients of x^n, we have

$$f[x_0, x_1, \ldots, x_n] = \frac{f[x_1, x_2, \ldots, x_n] - f[x_0, x_1, \ldots, x_{n-1}]}{x_n - x_0}$$

Using the identity (2), we compute

$$f[x_0, x_1] = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

$$f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0} = \frac{\frac{f(x_2)-f(x_1)}{x_2-x_1} - \frac{f(x_1)-f(x_0)}{x_1-x_0}}{x_2 - x_0}$$

Divided differences of order 0, 1, 2 and 3 are given below.

x_0	$f(x_0)$	$f[x_0, x_1]$	$f[x_0, x_1, x_2]$	$f[x_0, x_1, x_2, x_3]$
x_1	$f(x_1)$	$f[x_1, x_2]$	$f[x_1, x_2, x_3]$	
x_2	$f(x_2)$	$f[x_2, x_3]$		
x_3	$f(x_3)$			

Example. Find the Newton interpolating polynomial for the following values.

x	0	1	2	4
$f(x)$	3	4	7	19

$$
\begin{array}{llllll}
0 & 3 & f[x_0,x_1]=1 & f[x_0,x_1,x_2]=1 & f[x_0,x_1,x_2,x_3]=0 \\
1 & 4 & f[x_1,x_2]=3 & f[x_1,x_2,x_3]=1 & \\
2 & 7 & f[x_2,x_3]=6 & & \\
4 & 19 & & &
\end{array}
$$

The desired interpolating polynomial is

$$
\begin{aligned}
p(x) &= 3+1(x-0)+1(x-0)(x-1) \\
&\quad +0(x-0)(x-1)(x-2) \\
&= 3+x^2.
\end{aligned}
$$

Theorem. *Let $p_n(x)$ be the polynomial of degree $\leq n$ such that $p_n(x_i) = f(x_i)$, $i = 0, 1, \ldots, n$. If t is a point such that $t \neq x_0, x_1, \ldots, x_n$, then*

$$f(t) - p(t) = f[x_0, x_1, \ldots, x_n, t] \prod_{j=0}^{n} (t - x_j).$$

Proof. Let $q(x)$ be the polynomial of degree $\leq n + 1$ such that

$$q(x_i) = f(x_i), \quad i = 0, 1, \ldots, n \text{ and } q(t) = f(t).$$

Then $q(x)$ is obtained from $p(x)$ by adding one term. That is,

$$q(x) = p(x) + f[x_0, x_1, \ldots, x_n, t] \prod_{j=0}^{n} (x - x_j).$$

Since $q(t) = f(t)$, we immediately get

$$f(t) = p(t) + f[x_0, x_1, \ldots, x_n, t] \prod_{j=0}^{n} (t - x_j). \qquad \square \text{ *End*}$$