Lecture 2: The Solution of Nonlinear Equations

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The Solution of Nonlinear Equations

To find a real root of the equation

$$f(x) = 0, \tag{1}$$

where $f : [a, b] \to \mathbb{R}$. Here, the function $f(x)$ may be

- a polynomial in x or
- a transcendental function or
- a combination of the above.

Consider the following examples.

Example 1. $e^{-x} - \sin x = 0$.

Example 2. $x - a \sin x = b$ for various values of a and b.

Note. In rare cases it may be possible to obtain the exact roots of (1). In general, one can hope to obtain only approximate solutions. That is, to find a point $c \in [a, b]$ for which $|f(c)|$ is close to 0.

The following theorem ensures the existence of atleast one root of $f(x) = 0$.

Theorem

Assume that $f \in C[a, b]$ and $f(a)f(b) < 0$. Then there exists atleast one number $\xi \in (a, b)$ such that $f(\xi) = 0$.

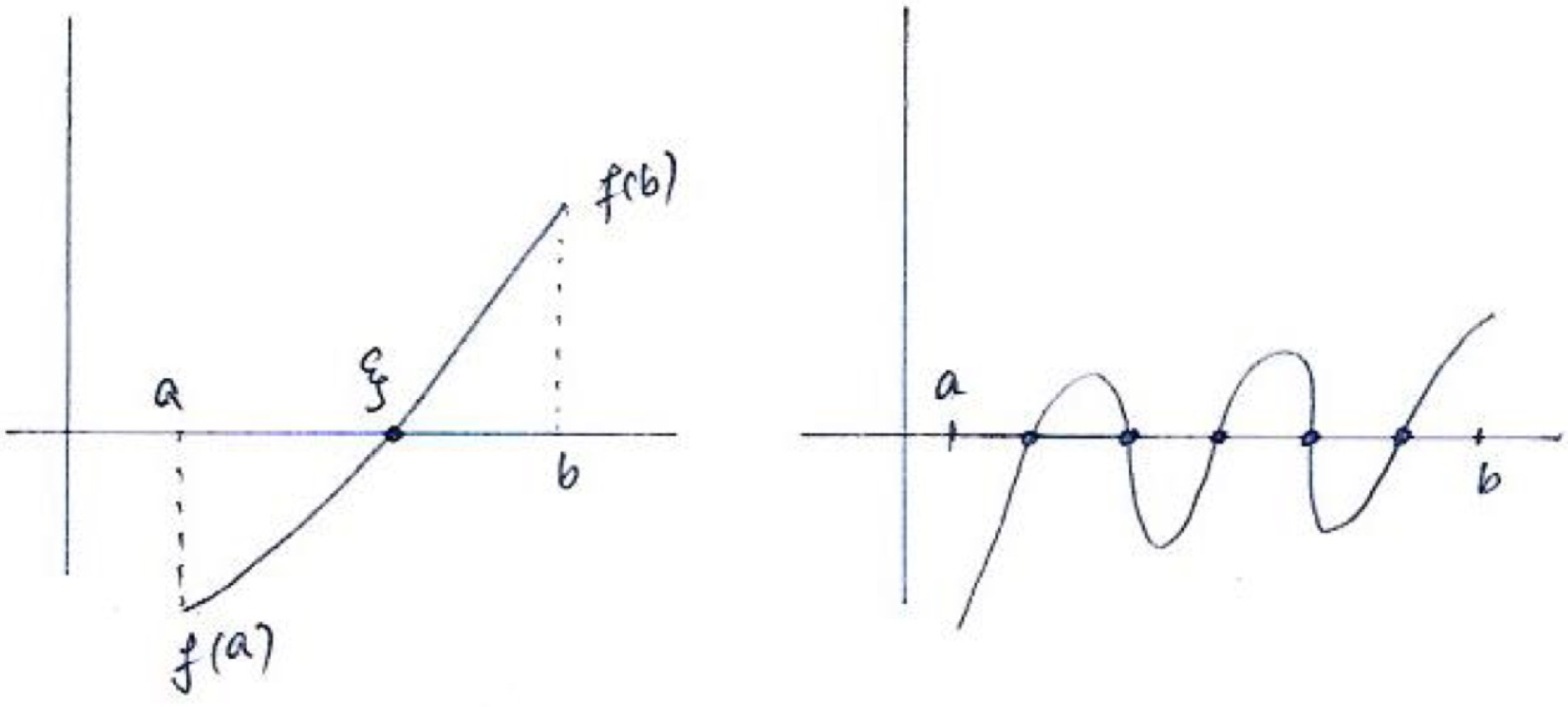
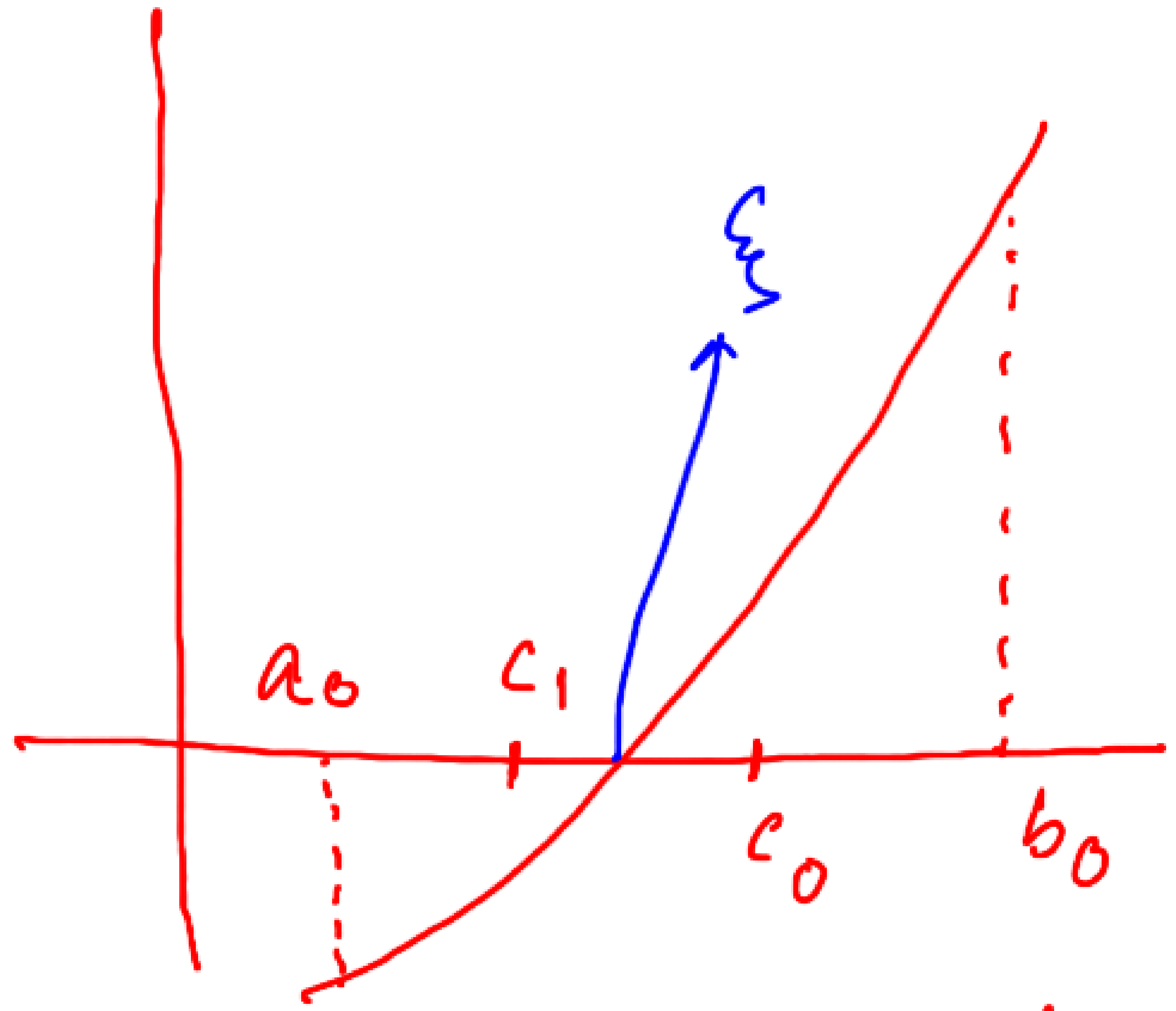


Figure : 1

The Bisection Method:



Suppose $f(a_0) f(b_0) < 0$

$\Rightarrow \exists$ a number $\xi \in (a_0, b_0)$ s.t. $f(\xi) = 0$

Set $c_0 = \dfrac{a_0 + b_0}{2}$

if $|f(c_0)| < \epsilon$ (TOL) then

accept c_0, stop.

$c_0 = \dfrac{a_0 + b_0}{2}$

$c_1 = \dfrac{a_0 + c_0}{2}$

Otherwise, check whether $\xi \in [a_0, c_0]$ or $\xi \in [c_0, b_0]$.

Suppose $f(a_0) f(c_0) < 0 \Rightarrow \xi \in (a_0, c_0)$

Set $c_1 = \dfrac{a_0 + c_0}{2}$

Again, if $|f(c_1)| < \epsilon$ (TOL) then accept c_1, stop.

Otherwise, repeat the above process.

$$f(a_0)f(b_0) < 0; \quad c_0 = (a_0 + b_0)/2.$$
$$f(a_0)f(c_0) < 0; \quad c_1 = (a_0 + c_0)/2.$$

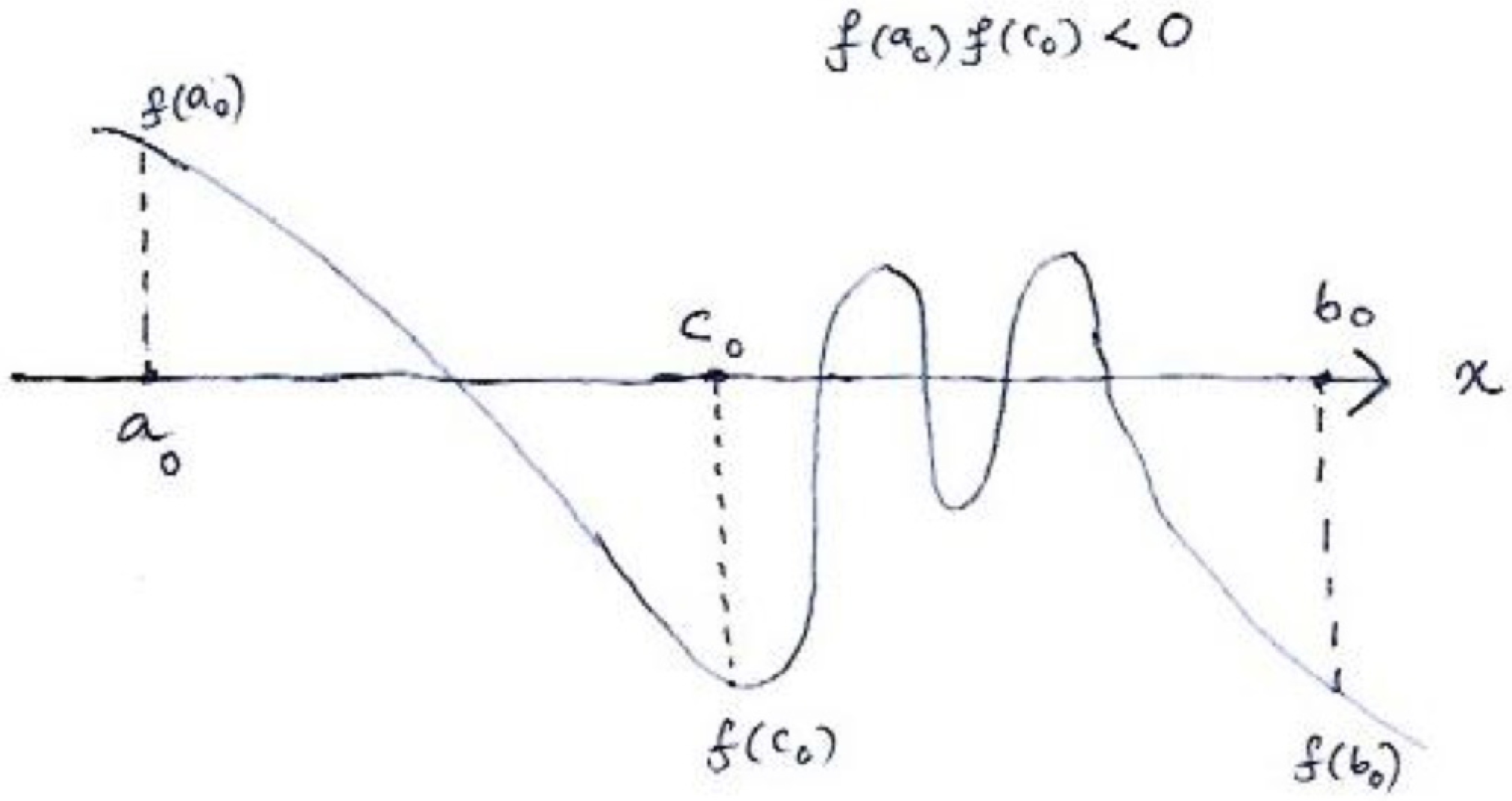


Figure : The bisection method determines the left interval

$$f(a_0)f(b_0) < 0; \quad c_0 = (a_0 + b_0)/2.$$
$$f(c_0)f(b_0) < 0; \quad c_1 = (c_0 + b_0)/2.$$

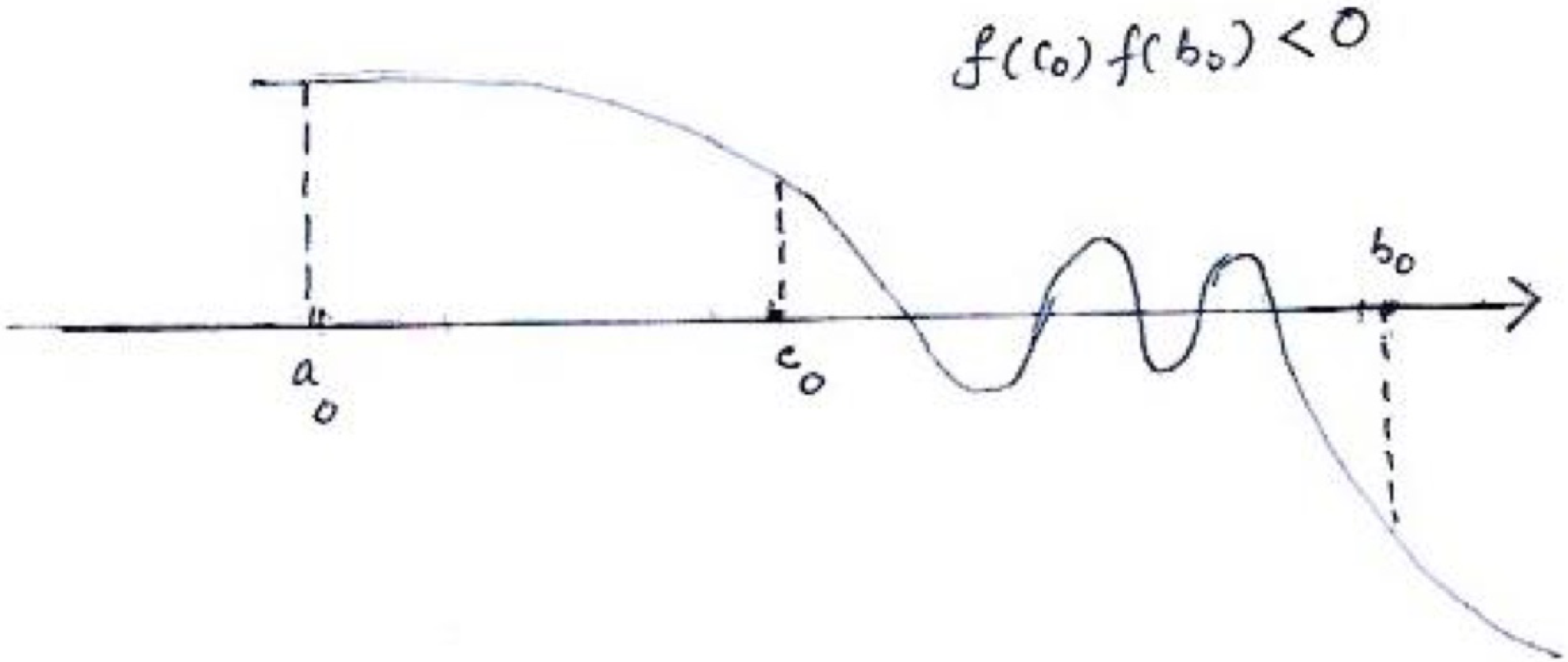


Figure : The bisection method determines the right interval

Algorithm

Given $f \in C[a_0, b_0]$ be such that $f(a_0)f(b_0) < 0$.

For $n = 0, 1, 2, \ldots,$ do:

1. Set $c_n = (a_n + b_n)/2$.
2. If $|f(c_n)| < \epsilon$ (prescribed tolerance), then accept c_n, stop.
3. If $f(a_n)f(c_n) < 0$, then set $a_{n+1} = a_n,\ b_{n+1} = c_n$; go to step 1;
4. else set $a_{n+1} = c_n,\ b_{n+1} = b_n$; go to step 1.

Remark. To avaid cancellation error, compute the mid-point c_n as

$$c_n = a_n + (b_n - a_n)/2.$$

instead of $c_n = (a_n + b_n)/2$.

Example. Find a real root of $x^3 - 2x - 5 = 0$.
Here $f(x) = x^3 - 2x - 5$. Note that $f(2) = -1$ and $f(3) = 16$. Thus, the root $\xi \in (2, 3)$.

n	c_n
0	2.5
1	2.25
2	2.125
3	2.0625
$\vdots$	$\vdots$
10	2.09473
11	2.09424

The absolute error is $|x_{11} - x_{10}| = 0.0005$ which is correct up to three decimal places. The percentage error is

$$\left|\frac{x_{11} - x_{10}}{x_{11}}\right| \times 100 = \frac{0.0005}{2.09424} \times 100 = 0.02\%.$$

Error Analysis

Let $[a_0, b_0]$, $[a_1, b_1], \ldots,$ be successive intervals arise in the process of the Bisection method with

$$a_0 \leq a_1 \leq a_2 \leq \cdots \leq b_0, \quad \text{and} \quad b_0 \geq b_1 \geq b_2 \cdots \geq a_0.$$

We have

$$b_{n+1} - a_{n+1} = \frac{1}{2}(b_n - a_n).$$

Note that the sequence $\{a_n\}$ is $\uparrow$ and bounded above, and hence converges. Similarly, $\{b_n\}$ converges. Since

$$b_n - a_n = \frac{1}{2}(b_{n-1} - a_{n-1}) = \cdots = \frac{1}{2^n}(b_0 - a_0),$$

$b_n - a_n = \frac{1}{2}(b_{n-1} - a_{n-1})$

$= \frac{1}{2}\left[\frac{1}{2}(b_{n-2} - a_{n-2})\right]$

$= \frac{1}{2^2}\left[b_{n-2} - a_{n-2}\right]$

$= \cdots$

$= \frac{1}{2^n}\left[b_0 - a_0\right]$

it follows that

$$\lim_{n\to\infty} b_n - \lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{2^n}(b_0 - a_0) = 0$$

$$\implies \lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = \xi \text{ (say)}.$$

$= f(\lim_{n\to\infty} a_n)\, f(\lim_{n\to\infty} b_n)$ [∵ f is conts.]

$$\lim_{n\to\infty} f(a_n)f(b_n) \le 0 \implies \{f(\xi)\}^2 \le 0 \implies f(\xi) = 0.$$

Suppose we want to stop the process in $[a_n, b_n]$ then the best estimate of the root is $c_n = (a_n + b_n)/2$. The error e_n at the nth step is

→ an approximation to ξ

$$e_n = |\xi - c_n| \le \frac{1}{2}(b_n - a_n) = \frac{1}{2^{n+1}}(b_0 - a_0).$$

Theorem

Assume that $f \in C[a_0, b_0]$. Let $[a_0, b_0], \cdots, [a_n, b_n] \cdots$ denotes the intervals in the bisection method. Then $\lim_{n\to\infty} a_n$ and $\lim_{n\to\infty} b_n$ exist, are equal and represent a zero of f. If $\xi = \lim_{n\to\infty} c_n$, where $c_n = (a_n + b_n)/2$, then

$$|\xi - c_n| \le 2^{-(n+1)}(b_0 - a_0).$$

NB: The bisection method converges to the root linearly.

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