Lecture 31: First-Order Wave Equations

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Consider a first-order linear wave equation

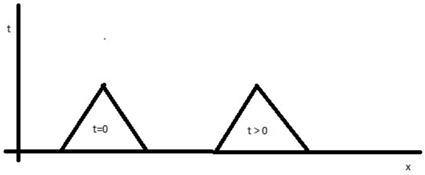
$$\begin{aligned} U_t + aU_x &= 0, \quad t > 0 \\ U(x,0) &= f(x), \quad 0 \le x \le 1, \end{aligned} \tag{1}$$

where a is a constant, t represents time and x represents the spatial variable. The solution to this IVP is given by

$$U(x,t) = f(x - at),$$

which can be easily verified. Set $\tau = x - at$. Then

$$U_x = f_\tau, \quad U_t = -af_\tau \implies U_t + aU_x = -af_\tau + af_\tau = 0.$$



The solution of (1) can be regarded as a wave that propagates with speed a without change of shape.

Condiser the first-order wave equation

$$\begin{aligned} aU_x + bU_t = c, &\quad t > 0 \\ \underbrace{U(x,0) = f(x)}_{initial\ condition}, & \end{aligned}$$

where a, b and c are functions of x, t and U. Such an equation is said to quasi-linear.

The charcteristics curve C and the solution U satisfy the following differential relationship.

$$\frac{dx}{a} = \frac{dt}{b} = \frac{dU}{c}$$

The solution U can be found either by solving

$$dU = (c/a)dx \quad \text{or} \quad dU = (c/b)dt.$$

Example.

$$tU_x + U_t = 2, \quad t > 0$$
$$U(x, 0) = U_0, \quad 0 \leq x \leq 1.$$

The differential equation for the family of charateristics curve C is

$$\frac{dx}{t} = \frac{dt}{1}.$$

The family of characteristics curve is

$$x = \frac{1}{2}t^2 + C_1,$$

where C_1 is a constant for each charateristics. If the characteristics curve passes through $(x_R, 0)$, then $C_1 = x_R$. The equation of this particular charateristics is

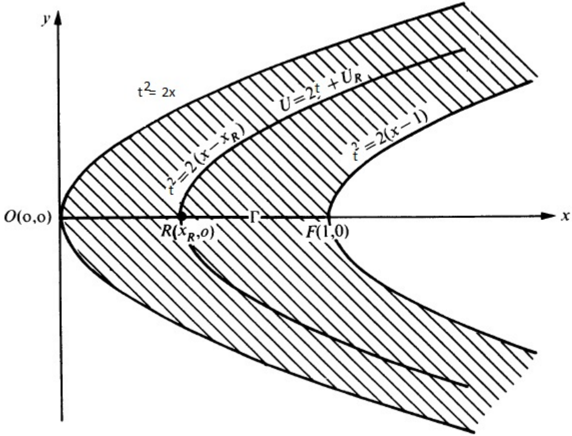
$$t^2 = 2(x - x_R).$$

The solution along the characteristics curve is

$$\frac{dU}{2} = \frac{dt}{1} \implies U = 2t + C_2,$$

where C_2 is constant along the particular charateristics.

$U = U_R$ at $R(x_R, 0) \implies C_2 = U_R$. The solution along the charateristics $t^2 = 2(x - x_R)$ is $U = 2t + U_R$.



Lax-Wendroff explicit method. Consider

$$aU_x + U_t = 0, \quad x \in \mathbb{R}, \quad t > 0,$$

where a is a positive constant. Introduce the mesh parameters h and k such that

$$\begin{aligned} x_i &= ih, \quad i = 0, \pm 1, \pm 2, \ldots \\ t_j &= jk, \quad j = 0, 1, 2, \ldots \end{aligned}$$

By Tayloy's expansion, we write

$$U_{i,j+1} = U(x_i, t_j + k) = U_{i,j} + k\left(\frac{\partial U}{\partial t}\right)_{i,j} + \frac{1}{2}k^2\left(\frac{\partial^2 U}{\partial t^2}\right)_{i,j} + \ldots$$

The differential equation can be used to eliminate the time derivatives by observing the fact

$$\frac{\partial}{\partial t} \equiv -a\frac{\partial}{\partial x}.$$

Therefore,

$$U_{i,j+1} = U_{i,j} - ka\left(\frac{\partial U}{\partial x}\right)_{i,j} + \frac{1}{2}k^2a^2\left(\frac{\partial^2 U}{\partial x^2}\right)_{i,j} + \ldots$$

Replacing the partial derivatives w.r.t x by central-difference approximation (up to terms k^2), we obtain an explicit finite difference equation

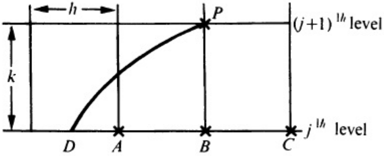
$$\begin{aligned} u_{i,j+1} &= u_{i,j} - \frac{ka}{2h}(u_{i+1,j} - u_{i-1,j}) + \frac{k^2 a^2}{2h^2}(u_{i-1,j} - 2u_{i,j} + u_{i+1,j}) \\ &= \frac{1}{2}a\mu(1 + a\mu)u_{i-1,j} + (1 - a^2\mu^2)u_{i,j} - \frac{1}{2}a\mu(1 - a\mu)u_{i+1,j} \quad (2) \end{aligned}$$

where $\mu = \frac{k}{h}$.
FACT.

- The local T.E. is $O(k^2) + O(h^2)$.
- The scheme is stable for $0 < a\mu \leq 1$.
- The above explicit scheme can be used for both initial-value problems as well as initial-boundary value problems.
- This scheme is often used to obtain numerical solutions to differential equations in fluid-flow problems when the solution changes rapidly with time. In such problems, k must be kept small.

Courant-Friedrich-Lewy (CFL) Condition. Consider an explicit scheme approximating a first-order hyperbolic PDE:

$$u_{i,j+1} = au_{i-1,j} + bu_{i,j} + cu_{i+1,j}$$



Let

- u_P be the finite difference solution at the mesh point P
- U_P be the exact solution at the mesh point P.

u_P depends on the values of u at the mesh points A, B and C. Let the characteristics curve through P meets the line AC at D. If the initial values along AC are altered then the solution at P of the finite difference equation (u_P) will change. However, these changes will not affect the exact solution U at P (i.e., U_P) which depends on the initial value at D.

In this case,

$$u_P \not\Rightarrow U_P \quad \text{as} \;\; h \to 0, \;\; k \to 0.$$

For convergence, D must lie between A and C (The CFL condition).

For example, consider Lax-Wendroff scheme (2) for the equation $U_t + aU_x = 0$. The slope of the characteristics $\frac{dt}{dx}$ is given by

$$\frac{dt}{1} = \frac{dx}{a} \implies \frac{dt}{dx} = \frac{1}{a}.$$

For convergence of the difference equation,

$$\text{Slope of PD} \geq \text{Slope of PA}$$
$$\implies \frac{1}{a} \geq \frac{k}{h} \implies a\mu \leq 1,$$

which coincides with the condition for stability, $i.e.$, $0 < a\mu \leq 1$ as $a > 0$ and $\mu > 0$.

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