Theory Quiz 1: MA 322

Date: 10/04/2021

Examination Time: 11 am -12 noon &

Submission Time: by 12:10 pm on 10/04/2021

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1. **(2+2+2 points)** Given the distinct points $x_i,\ i = 0,\ 1, \ldots, n+1$, and the points $y_i,\ i = 0, 1, \ldots, n+1$, let q be the Lagrange polynomial of degree n for the set of points

$$\{(x_i, y_i) :\ i = 0, 1, \ldots, n\}$$

and let r be the Lagrange polynomial of degree n for the points

$$\{(x_i, y_i) :\ i = 1, 2, \ldots, n+1\}.$$

Define

$$p(x) = \frac{(x - x_0)r(x) - (x - x_{n+1})q(x)}{x_{n+1} - x_0}.$$

Show that p is the Lagrange polynomial of degree $n+1$ for the points

$$\{(x_i, y_i) :\ i = 0, 1, 2, \ldots, n+1\}.$$

Let

$$E(h) = \frac{(f(h) + \epsilon_+) - (f(-h) + \epsilon_-)}{2h} - f'(0).$$

Suppose that $f \in C^3[-h, h]$, then show that there exists $\xi \in (-h, h)$ such that

$$E(h) = \frac{1}{6}h^2 f'''(\xi) + \frac{\epsilon_+ - \epsilon_-}{2h}.$$

Hence, deduce that there exists constants $M,\ \epsilon > 0$ such that

$$\boxed{|E(h) \leq \frac{1}{6}h^2 M + \frac{\epsilon}{h}.}$$

Further, determine the value of h such that the right-hand side of the last inequality achieves its minimum.

2. **(4 points)** Determine the values of $c_j,\ j = -1, 0, 1, 2$, such that the quadrature rule

$$Q(f) = c_{-1}f(-1) + c_0 f(0) + c_1 f(1) + c_2 f(2)$$

gives the exact value for the integral

$$\int_0^1 f(x)dx$$

when f is any polynomial of degree 3. Show that, with these values of the weights c_j, and under appropriate conditions on the function f,

$$\left| \int_0^1 f(x)dx - Q(f) \right| \leq \frac{11}{720} M.$$

Give suitable conditions for the validity of this bound, and a definition of the quantity M.

End