Lecture 32: First-Order Wave Equations Contd..

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Recall the following example:

$$
\begin{aligned}
&tU_x + U_t = 2, \quad t > 0 \\
&U(x,0) = U_0 \ \text{ on the initial curve } \Gamma : \ t = 0, \ 0 \leq x \leq 1.
\end{aligned}
$$

The differential equation for the family of charateristics curve C is

$$\frac{dx}{t} = \frac{dt}{1}.$$

The family of characteristics curve is $x = \frac{1}{2}t^2 + C_1$, where C_1 is a constant. If the characteristics curve passes through $(x_R, 0)$, then $C_1 = x_R$. The equation of this particular charateristics is

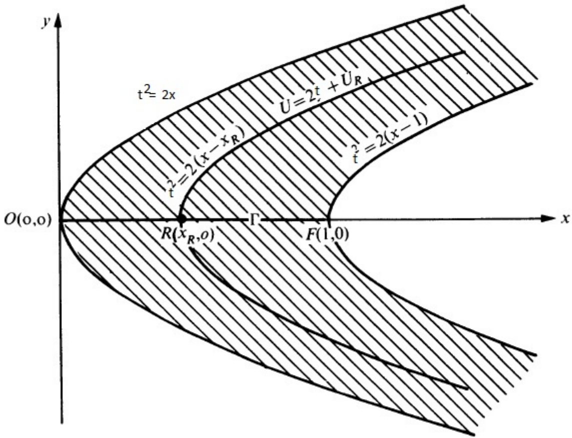
$$t^2 = 2(x - x_R).$$

The solution along the characteristics curve is

$$\frac{dU}{2} = \frac{dt}{1} \implies U = 2t + C_2,$$

where C_2 is constant along the particular charateristics.

If $U = U_R$ at $R(x_R, 0) \implies C_2 = U_R$. The solution along the charateristics $t^2 = 2(x - x_R)$ is $U = 2t + U_R$.



If the charactreistics curve C coincides with initial curve Γ. The solution is no longer unique.

Take $\Gamma = C: \; t^2 = 2x$. The solution is given by

$$U(x,t) = 2t + U_0 + K_1(t^2 - 2x), \quad K_1 \text{ is an arbitrary constant.}$$

Note that, along the characteristics curve $t^2 = 2x$, the solution is $U(x,t) = 2t + U_0$. Away from the characteristic curve i.e., $t^2 \neq 2x$, we have

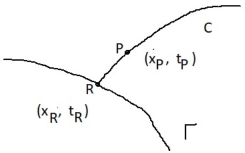
$$U_t = 2 + 2K_1 t; \quad U_x = -2K_1,$$

which implies $U(x,t)$ satisfies $tU_x + U_t = 2$ and $U(x,0) = U_0$. Since K_1 is arbitrary, we have infinitely many solutions.

Method of Charateristics: Consider

$$\begin{aligned} &aU_x + bU_t = c, \quad t > 0 \\ &U(x,0) \text{ is prescribed on the initial curve } \Gamma, \end{aligned}$$

where the coefficients a, b and c may be functions of x, t and U.



Let C be the charateristics curve passes through the point $R(x_R, t_R)$. Let $P(x_P, t_P)$ be a point on C such that $x_P - x_R$ is very small. Assume that x_P is known. The method of charateristics (numerical approximation along the charateristics) proceeds as follows.

Step 1: Given x_P, compute the first approximation $t_P^{(1)}$ by solving

$$a_R(t_P^{(1)} - t_R) = b_R(x_P - x_R),$$

Next, compute the first approximation to the solution $u_P^{(1)}$ from

$$a_R(u_P^{(1)} - u_R) = c_R(x_P - x_R),$$

where $a_R = a(R), \;\; b_R = b(R), \;\; c_R = c(R).$

Step 2: Improve the values of $t_P^{(1)}$ and $u_P^{(1)}$ by considering the mean values of the coefficients a, b and c along RP:

$$\begin{aligned} \frac{1}{2}(a_R + a_P^{(1)})(t_P^{(2)} - t_R) &= \frac{1}{2}(b_R + b_P^{(1)})(x_P - x_R) \\ \frac{1}{2}(a_R + a_P^{(1)})(u_P^{(2)} - u_R) &= \frac{1}{2}(c_R + c_P^{(1)})(x_P - x_R) \end{aligned}$$

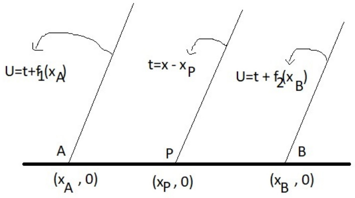
Determine $t_P^{(2)}$ and $u_P^{(2)}$ from the above equations and repeat the Step 2 until the successive iterates agree to a specified number of decimal places.

Propagation of Discontinuity: Consider the IVP:

$$U_x + U_t = 1, \quad t > 0$$
$$U(x,0) = \begin{cases} f_1(x), & -\infty < x < x_P, \\ f_2(x), & x_P < x < \infty. \end{cases}$$

The characteristic curves through the points $(x_P, 0)$, $(x_A, 0)$ and $(x_B, 0)$ are $t = x - x_P$, $t = x - x_A$ and $t = x - x_B$, respectively.

The solution along the charateriestic curve $t = x - x_B$ is $U(x,t) = t + f_2(x_B)$. Similarly, the solution along the charateriestic curve $t = x - x_A$ is $U(x,t) = t + f_1(x_A)$.



Let $U_L(x,t)$ be the solution to the left of the characteristic curve $t = x - x_P$, and let $U_R(x,t)$ be the solution to the right of the characteristic curve $t = x - x_P$.

For fixed t, $U_L(x,t) - U_R(x,t) = f_1(x_A) - f_2(x_B)$. Now, as $x_A, x_B \to x_P$, $U_L(x,t) - U_R(x,t) \neq 0$. This implies U is discontinuous along the curve $t = x - x_P$.

Thus, if the initial data is given to be discontinuous at a point $(x_P, 0)$, then the solution remains discontinuous along the charateristic curve $t = x - x_P$ through this point.

Exercise. For the IVP:

$$U_x + U_t = 1, \quad t > 0$$
$$U(x,0) = \begin{cases} 0, & -\infty < x \leq 0, \\ x, & 0 < x < \infty. \end{cases}$$

Show that the solution remains continuous along the charateristic through the point $(0,0)$ but their partial derivatives are discontinuous for all $t \geq 0$.

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