Lecture 12: Hermite Interpolation Based on Divided Differences

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Recall

Given $(n+1)$ data points

$$(x_0, f(x_0)),\ (x_1, f(x_1)),\ \cdots,\ (x_n, f(x_n)).$$

Then, Newton's form of interpolating polynomial using divided difference is of the form

$$P_n(x) = f[x_0] + f[x_0, x_1](x - x_0) + f[x_0, x_1, x_2](x - x_0)(x - x_1)$$
$$+ \cdots + f[x_0, \cdots, x_n](x - x_0)(x - x_1)\cdots(x - x_{n-1})$$

AIM: To construct a polynomial $H(x)$ s.t.

$$\left.\begin{aligned} H(x_i) &= f(x_i), \\ H'(x_i) &= f'(x_i), \end{aligned}\right\} i = 0, 1, \cdots, n.$$

Hermite Interpolation Based on Divided Difference

Consider the polynomial interpolating $f(x)$ at nodes $z_1, z_2, \ldots, z_{2n}$. Wrting this polynomial in Newton divided difference form, we have

$$\begin{aligned} H(x) &= f(z_1) + (x - z_1)f[z_1, z_2] + (x - z_1)(x - z_2)f[z_1, z_2, z_3] \\ &\quad + \cdots + (x - z_1)\cdots(x - z_{2n-1})f[z_1, \ldots, z_{2n}]. \end{aligned} \tag{1}$$

The error in the approximation is (for $x \neq z_1, \ldots, z_{2n}$):

$$f(x) - H(x) = (x - z_1)\cdots(x - z_{2n})\, f[z_1, \ldots, z_{2n}, x]. \tag{2}$$

Letting nodes coincide, the formula (1) will still exist. In particular, let

$$z_1, z_2 \to x_1, \quad z_3, z_4 \to x_2, \ldots, z_{2n-1}, z_{2n} \to x_n$$

to obtain

$$\begin{aligned} H(x) &= f(x_1) + (x - x_1)f[x_1, x_1] + (x - x_1)^2 f[x_1, x_1, x_2] + \cdots \\ &+ (x - x_1)^2 \cdots (x - x_{n-1})^2 (x - x_n) f[x_1, x_1, \ldots, x_n, x_n]. \end{aligned} \tag{3}$$

$H(x)$ is a polynomial of degree $\leq 2n-1$. Again, taking limits

$$z_1, z_2 \to x_1,\ z_3, z_4 \to x_2,\ \ldots\ , z_{2n-1}, z_{2n} \to x_n.$$

in the error expression (2), using the continuity of the divided differences and assuming f is sufficiently differentiable, it follows that

$$f(x) - H(x) = (x - x_1)^2 \cdots (x - x_n)^2 f[x_1, x_1, \ldots, x_n, x_n, x]$$

It is easy to verify that

$$f(x_i) - H(x_i) = 0 \quad i = 1, 2, \ldots, n.$$ $\Rightarrow H(x_i) = f(x_i)$

$i = 1, \ldots, n$

Further,

$$f'(x) - H'(x) = (x - x_1)^2 \cdots (x - x_n)^2 \frac{d}{dx} f[x_1, x_1, \ldots, x_n, x_n, x]$$

$$+2f[x_1, x_1, \ldots, x_n, x_n, x] \sum_{i=1}^{n} \{(x - x_i) \prod_{\substack{j=1 \\ j \neq i}} (x - x_j)^2\}$$

$$\Longrightarrow \quad f'(x_i) - H'(x_i) = 0 \quad i = 1, \ldots, n$$ $\Rightarrow$ $H'(x_i) = f'(x_i)$, $i = 1, \ldots, n$

Thus degree of $H(x) \leq 2n-1$ and it satisfies

$$H(x_i) = f(x_i), \; H'(x_i) = f'(x_i).$$

By the uniqueness of the Hermite interpolating polynomial, $H(x)$ is unique.

The desired interpolating polynomial $H(x)$ is given by (3) and the associated error formula is

$$f(x) - H(x) = \prod_{j=1}^{n} (x - x_j)^2 f[x_1, x_1, \ldots, x_n, x_n, x].$$

Example. Find the interpolating polynomial from the following data:

$$p(a) = f(a), \; p(b) = f(b), \; p'(a) = f'(a), \; p'(b) = f'(b).$$

Using Hermite interpolation formula based on divided difference, we have

$$\begin{aligned} H(x) \; \cancel{p(x)} \quad &= \quad f(a) + (x-a)f[a,a] + (x-a)^2 f[a,a,b] \\ &\quad\; +(x-a)^2(x-b)f[a,a,b,b]. \end{aligned}$$

Just now, we have seen, if we have four sets of points z_1, z_2, z_3, & z_4, then

$$H(x) = f[z_1] + f[z_1, z_2](x - z_1) + f[z_1, z_2, z_3](x - z_1)(x - z_2)$$
$$+ f[z_1, z_2, z_3, z_4](x - z_1)(x - z_2)(x - z_3)$$

Letting $z_1, z_2 \to a$ & $z_3, z_4 \to b$, we obtain.

$$H(x) = f(a) + f[a, a](x - a) + f[a, a, b](x - a)(x - a)$$
$$+ f[a, a, b, b](x - a)(x - a)(x - b)$$

$$= f(a) + f[a, a](x - a) + f[a, a, b](x - a)^2$$
$$+ f[a, a, b, b](x - a)^2 (x - b).$$

$$f[a,a,b,b] = \frac{f(b)-f(a)}{b-a}$$

$$f[a,a,b] = \frac{f[a,b]-f[a,a]}{b-a}$$

$$= \frac{\frac{f(b)-f(a)}{b-a} - f'(a)}{b-a}$$

$$f[a,a,b] = \frac{f[a,b]-f'(a)}{b-a}$$

$$f[a,a,b,b] = \frac{f[a,b,b]-f[a,a,b]}{b-a}$$

$$= \frac{\frac{f[b,b]-f[a,b]}{b-a} - \frac{f[a,b]-f[a,a]}{b-a}}{b-a}$$

$$f[b,b] = f'(b)$$

$$= \frac{f'(b)-2f[a,b]+f'(a)}{(b-a)^2}.$$

The error in the interpolation is

$$f[x_0,\cdots,x_k] = \frac{f^{(k)}(\xi)}{k!}$$

$$f(x) - \overset{H}{\cancel{p}}(x) = (x-a)^2(x-b)^2 f[a,a,b,b,x]$$

$$x = \frac{a+b}{2}$$

$$= \frac{(x-a)^2(x-b)^2}{24} f^{(4)}(\xi_x), \quad \xi_x \in (a,b).$$

$$f[a,a,b,b,x]$$

$$\implies \max_{a\le x\le b} |f(x)-H(x)| \le \frac{(b-a)^4}{384} \max_{a\le t\le b} |f^{(4)}(t)|.$$

$$= \frac{f^{(iv)}(\xi_x)}{4!}$$

End