Lecture 3: Fixed-Point Method

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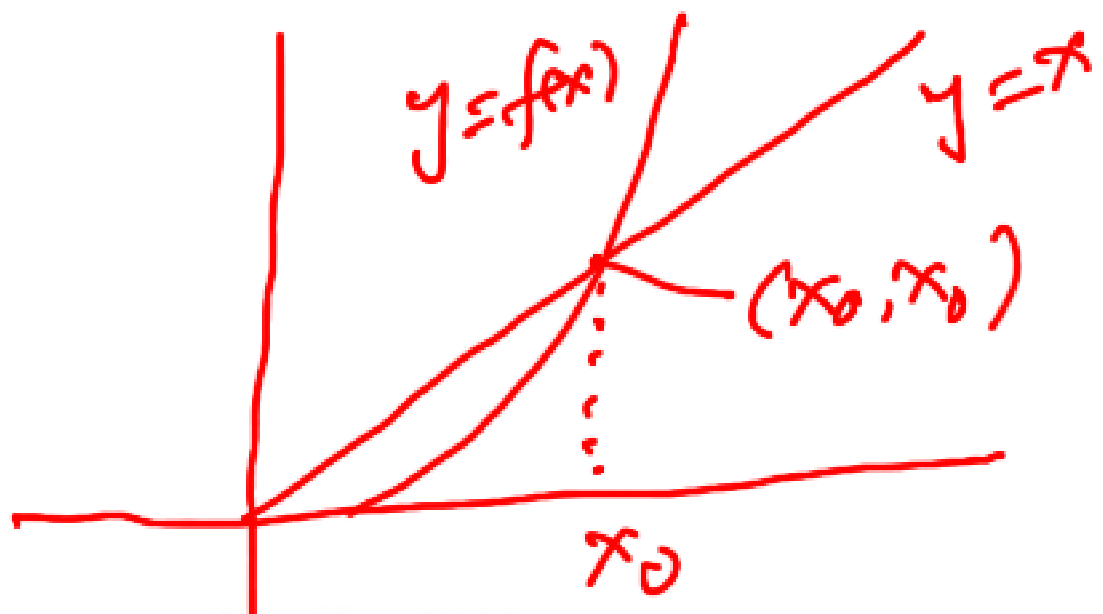
Fixed-Point Method



Definition. *Let $f : \mathbb{R} \to \mathbb{R}$. A point x_0 is called a fixed point of f if* $f(x_0) = x_0$.
The equation

$$f(x) = 0 \tag{1}$$

can be put in the form

$$x = g(x). \tag{2}$$

Any fixed-point of g is a solution of (1).
Consider $f(x) = x^2 - x - 2$. Then, possible choices for $g(x)$ are as follows.

$x^2 - x - 2 = 0$

$\Rightarrow \quad x = \underbrace{x^2 - 2}_{g(x)}$

$$g(x) = x^2 - 2; \quad g(x) = \sqrt{x+2};$$

$$g(x) = 1 + \frac{2}{x}; \quad g(x) = x - \frac{x^2 - x - 2}{m}, \quad m \neq 0.$$

$x^2 = x + 2$

or $\quad x = \underbrace{\sqrt{x+2}}_{g(x)}$

Each $g(x)$ is called an iteration function for solving (1).

Algorithm

$x_1 = g(x_0)$

$x_2 = g(x_1)$

$\vdots$

$x_{n+1} = g(x_n)$

Given an iteration function $g(x)$ and a starting point x_0.

$$\begin{cases} \text{Compute } x_{n+1} = g(x_n), \quad n = 0, 1, 2 \ldots . \\ \text{until } |x_n - \xi| < \epsilon \text{ (prescribed tolerance)} \end{cases}$$

ξ = fixed point of g.

To implement the above algorithm, one needs to ensure the following:

- For given x_0, we should be able to generate x_1, x_2, ...
- $\{x_n\} \to \xi$.
- ξ is a fixed point of $g(x)$, i.e., $\xi = g(\xi)$.

Eg: $g(x) = \sqrt{x}$

g is defined for $x \geq 0$.

$x_1 = g(x_0) = -\sqrt{x_0}$

Starting with x_0, we cannot compute x_2 as $x_1 < 0$

Theorem

Set $I = [a, b]$. Let $g(x)$ be an iteration function satisfying the following conditions:

1. $g : I \to I$
2. $g : I \to I$ *is continuous*
3. $\exists$ *a constant* $0 < K < 1$ *such that* $|g'(x)| \leq K \quad \forall x \in I$.

<u>Step 1:</u> claim: g has a fixed point in I. $I = [a,b]$

If $g(a) = a \Rightarrow a$ is a fixed pt. of g.

or $g(b) = b \Rightarrow b$ is a fixed pt. of g.

Suppose $g(a) \neq a$ and $g(b) \neq b$.

Note that $g(a), g(b) \in [a,b]$ $(\because g : [a,b] \to [a,b])$

Define $h(x) = g(x) - x$. Clearly, $h \in C([a,b])$

Further, notice that $h(a) = g(a) - a > 0$

$h(b) = g(b) - b < 0$

By IVT for cont's function, $\exists$ $\xi \in (a,b)$ s.t.

$h(\xi) = 0 \Rightarrow g(\xi) - \xi = 0$

$\Rightarrow g(\xi) = \xi$

$\Rightarrow g$ has a fixed point in I.

Then $g(x)$ has a unique fixed point $\xi \in I$ and the sequence of iterates $\{x_n\}$ generated by $x_{n+1} = g(x_n)$ converge to ξ.

Proof. Step 1: First, we need to show that g has a fixed point in $I = [a, b]$. If $g(a) = a$ or $g(b) = b$, then we are done. Otherwise, we have $g(a) \neq a$ and $g(b) \neq b$. Further, $g(a),\ g(b) \in I \implies g(a) > a$ and $g(b) < b$. Define

$$h(x) = g(x) - x.$$

Clearly, $h(x)$ is continuous and $h(a) > 0$, $h(b) < 0$. By IVT, $h(\xi) = 0$ for some ξ. This implies $g(\xi) = \xi$. Thus, $g(x)$ has a fixed point in I.

Step 2. Let $e_n = \xi - x_n$, $n = 0, 1, 2, \ldots$, denote the error in the nth iterate. Since $\xi = g(\xi)$ and $x_n = g(x_{n-1})$, we have

$$\begin{aligned} e_n = \xi - x_n &= g(\xi) - g(x_{n-1}) \\ &= g'(\eta_n)(\xi - x_{n-1}) = g'(\eta_n) e_{n-1}, \end{aligned}$$

for some η_n between ξ and x_{n-1}.

Hence

$$\begin{aligned} |e_n| &\leq K|e_{n-1}| \\ &\leq K^2|e_{n-2}| \\ &\leq \cdots \leq K^n|e_0| \\ &\qquad \Longrightarrow \lim_{n\to\infty} |e_n| \leq \lim_{n\to\infty} K^n|e_0| = 0 \\ &\qquad \Longrightarrow \{x_n\} \to \xi. \end{aligned}$$

$|g'(x)| \leq K.$

$|e_{n-1}| \leq K|e_{n-2}|$

$|e_{n-2}| \leq K|e_{n-3}|$

$\vdots$

$e_0 = \xi - x_0$

As $K < 1$

$\Rightarrow \lim_{n\to\infty} K^n \to 0$

Thus,

$$\lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} g(x_n) = g(\lim_{n\to\infty} x_n) \implies \xi = g(\xi).$$

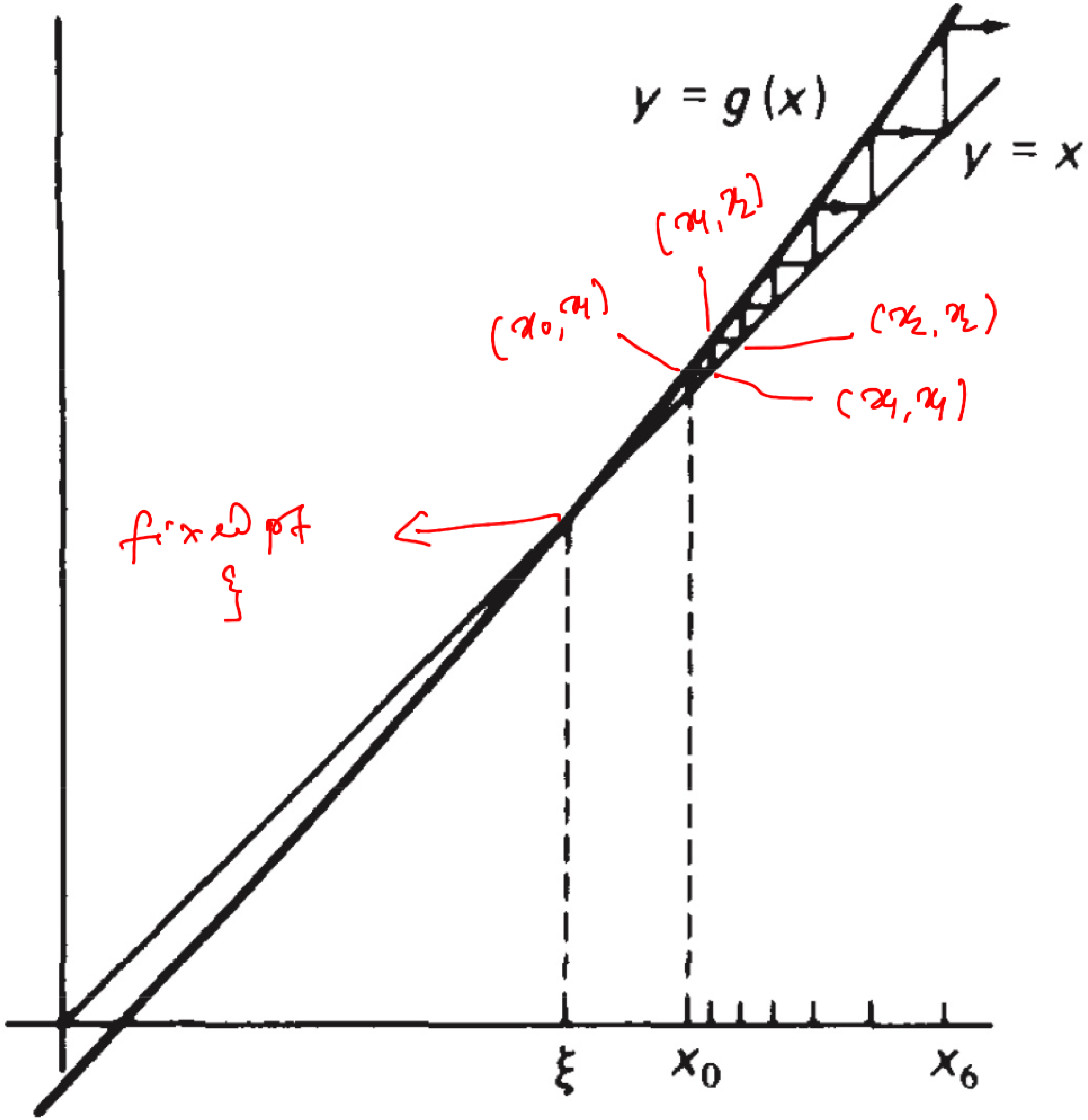
Step 3. **(Uniqueness of ξ)**

Let ξ_1 be also a fixed point of g, i.e., $\xi_1 = g(\xi_1)$. With $x_0 = \xi_1$, we have

$$\begin{aligned} & x_1 = g(x_0) = g(\xi_1) = \xi_1 \\ & \Longrightarrow |e_0| = |e_1| \leq K|e_0| \implies |e_0| = 0 \text{ as } K < 1. \\ & \Longrightarrow \xi = \xi_1. \text{ This completes the proof.} \end{aligned}$$

$e_0 = \xi - x_0$
$= \xi - \xi_1$

$e_1 = \xi - x_1$
$= \xi - \xi_1$



$x_1 = g(x_0)$

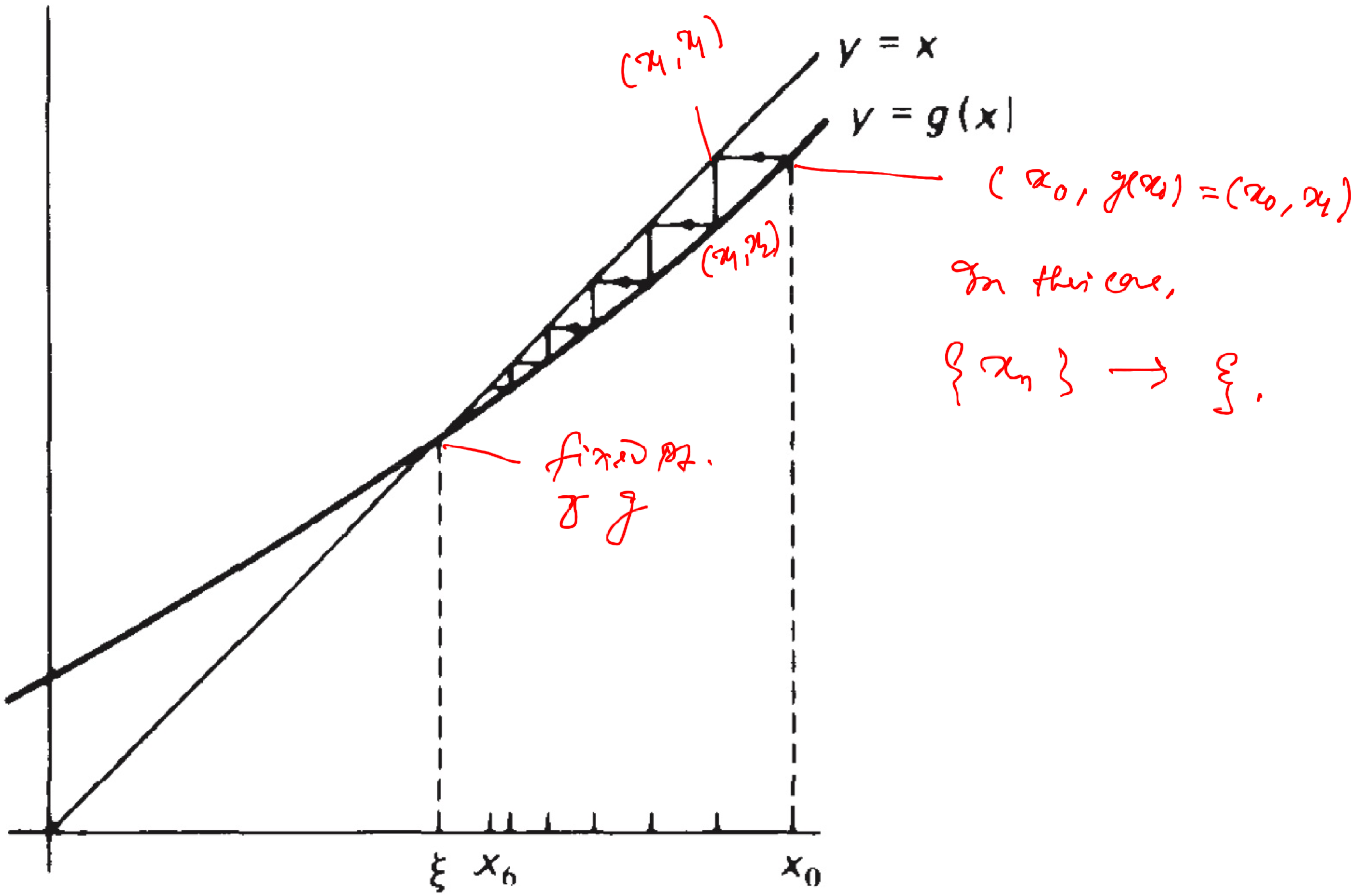
(x_0, x_1)

$\downarrow$

(x_1, x_2)

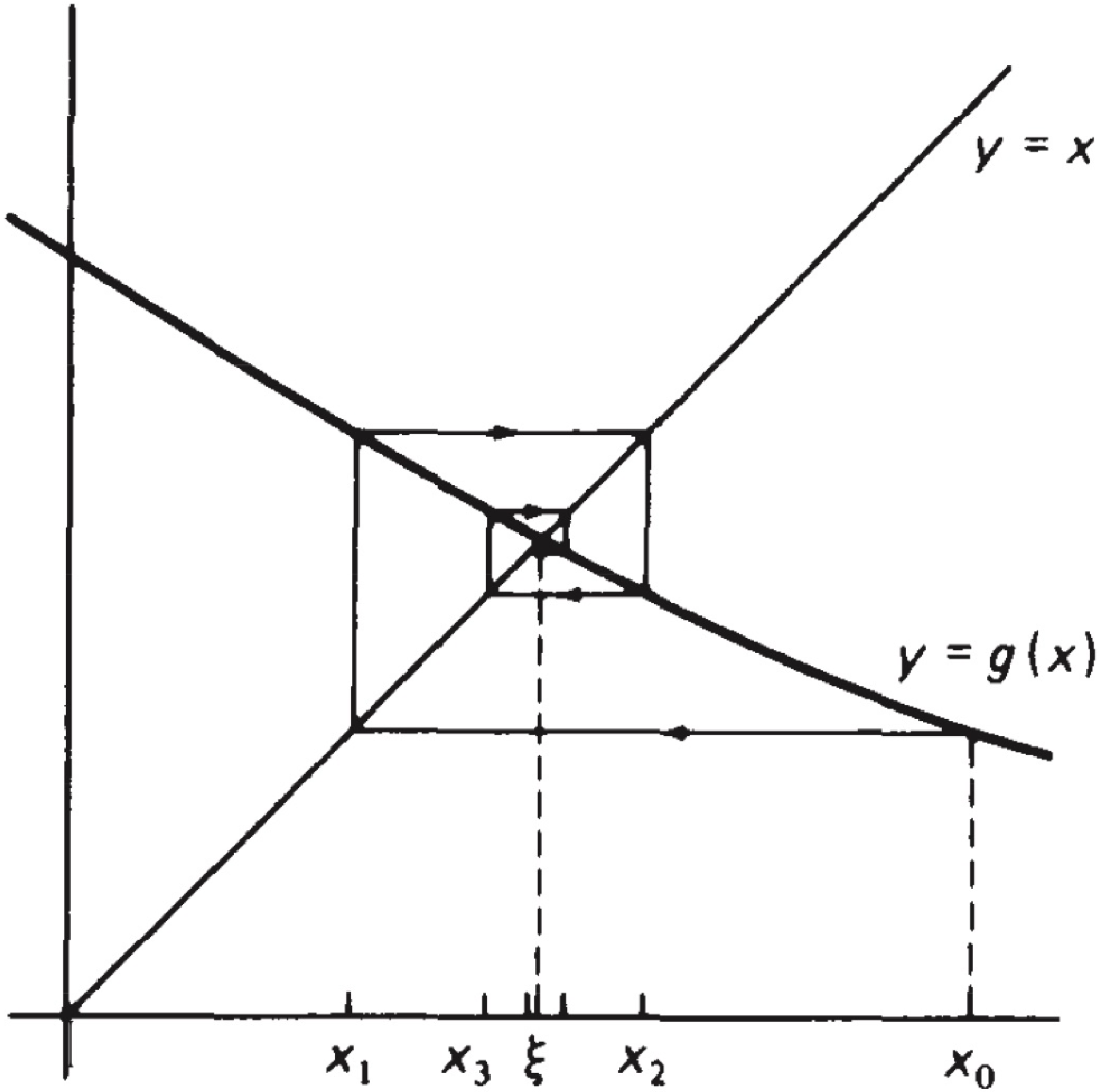
$\{x_n\} \not\to \xi$

Reason: $|g'(x)| > 1$



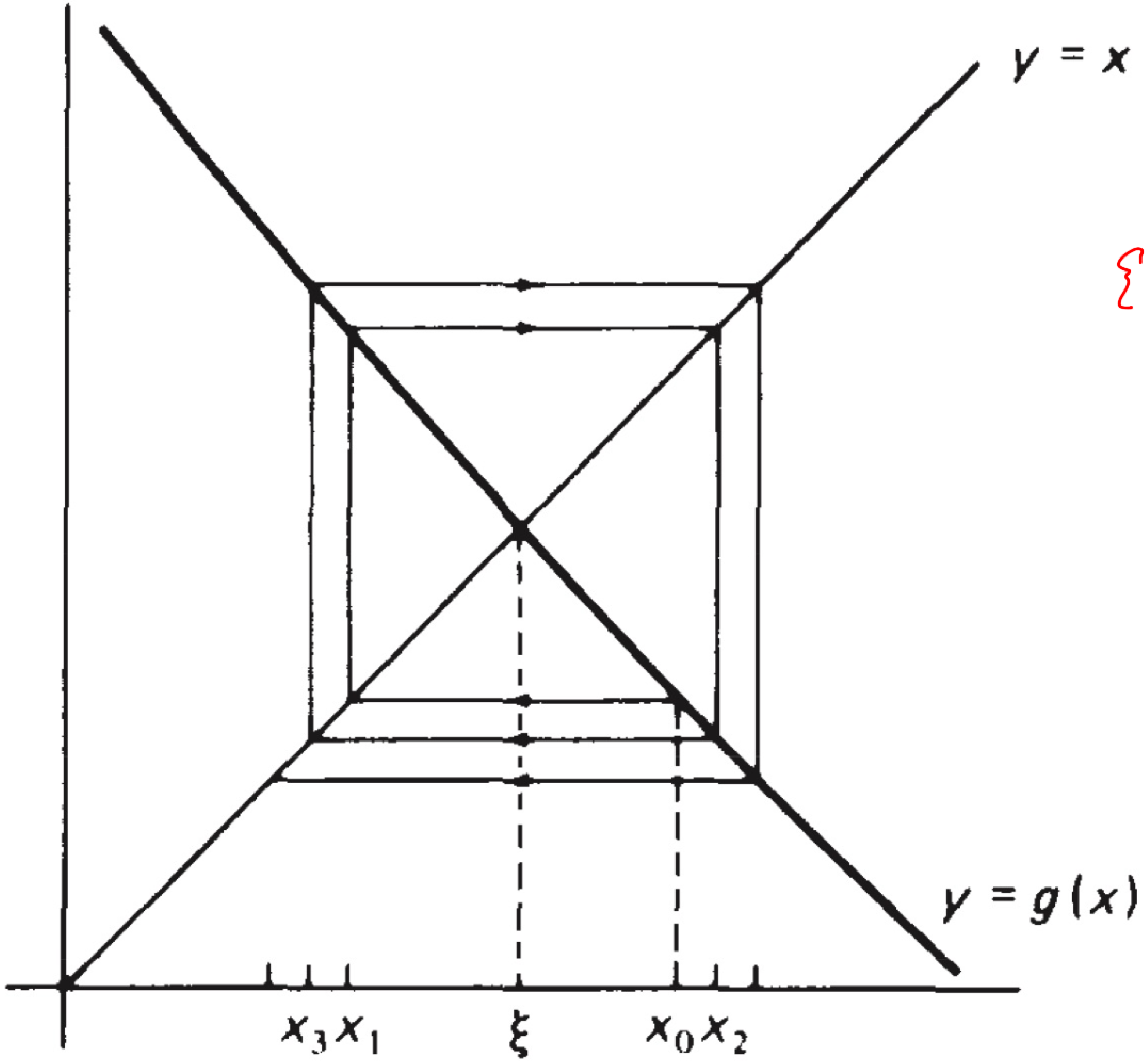
In this case,

$\{x_n\} \to \xi$.



$x_n = g(x_{n-1})$

$\{x_n\} \longrightarrow \xi.$



$\{x_n\} \not\to \xi.$

Order of Convergence: If $|e_{n+1}| \propto |e_n|^p$, then the order of convergence is p.

Theorem

Let ξ be the root of $x = g(x)$. Assume that $g \in C^m(N_\delta(\xi))$ for $m \geq 2$, where $N_\delta(\xi)$ denotes a neighbourhood of ξ. Assume that

$$g'(\xi) = \cdots = g^{(m-1)}(\xi) = 0.$$

Then, if the initial guess $x_0 \in N_\delta(\xi)$, the iteration given by

$$x_{n+1} = g(x_n), \quad n \geq 0$$

converges with order m and

$$\lim_{n\to\infty} \frac{(\xi - x_{n+1})}{(\xi - x_n)^m} = (-1)^{m-1} \cdot \frac{g^{(m)}(\xi)}{m!}.$$

Set $c = \left|(-1)^{m-1} \frac{g^{(m)}(\xi)}{m!}\right|$.

$|e_{n+1}| \propto |e_n|^m$.

Proof. By Taylor's theorem $= g(\xi + x_n - \xi)$

$$
\begin{aligned}
x_{n+1} &= g(x_n) = g(\xi) + (x_n - \xi)\underbrace{g'(\xi)}_{=0} + \cdots + \\
&\quad \frac{(x_n-\xi)^{m-1}}{(m-1)!}\underbrace{g^{(m-1)}(\xi)}_{=0} + \frac{(x_n-\xi)^m}{(m)!}g^{(m)}(\eta_n),
\end{aligned}
$$

for some η_n between x_n and ξ. Since

$$g'(\xi) = \cdots = g^{(m-1)}(\xi) = 0,$$

we have (using the fact $g(\xi) = \xi$)

$$\xi - x_{n+1} = -\frac{(x_n-\xi)^m}{(m)!}g^{(m)}(\eta_n) = (-1)^{m-1}\frac{(\xi - x_n)^m}{(m)!}g^{(m)}(\eta_n)$$

$$\implies \lim_{n\to\infty}\frac{\xi - x_{n+1}}{(\xi - x_n)^m} = (-1)^{m-1}\cdot\frac{g^{(m)}(\xi)}{m!}.$$

As $n \to \infty$
$x_n \to \xi$
η_n lies bet^n x_n & ξ
$\eta_n \to \xi$

This completes the proof.

End