Lecture 33: Second-Order Wave Equations

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Consider a second order quasilinear PDE:

$$aU_{xx} + bU_{xy} + cU_{yy} + e = 0, \tag{1}$$

where

$$a = a(x, y, U, U_x, U_y); \qquad b = b(x, y, U, U_x, U_y),$$
$$c = c(x, y, U, U_x, U_y); \qquad e = e(x, y, U, U_x, U_y),$$

Put $U_x = p,\ U_y = q,\ U_{xx} = r,\ U_{xy} = s,\ U_{yy} = t$. Then Eq. (1) takes the form

$$ar + bs + ct + e = 0. \tag{2}$$

Let C be a curve in the xy-plane of the solution domain. Assume that $U, U_x = p, U_y = q$ are known along C. Further, along C we have

$$dU = \frac{\partial U}{\partial x} dx + \frac{\partial U}{\partial y} dy = p\,dx + q\,dy, \tag{3}$$

where $\frac{dy}{dx}$ is the slope of the tangent to C. Note that, r, s, and t on C must satisfy

$$dp = \frac{\partial p}{\partial x} dx + \frac{\partial p}{\partial y} dy = r\,dx + s\,dy \tag{4}$$

$$dq = \frac{\partial q}{\partial x} dx + \frac{\partial q}{\partial y} dy = s\, dx + t\, dy \tag{5}$$

Elimination of r and t from Eqs. (2) by utilizing (4) and (5) leads to

$$\begin{aligned}
& a\Big(\frac{dp}{dx} - s\frac{dy}{dx}\Big) + bs + c\Big(\frac{dq}{dy} - s\frac{dx}{dy}\Big) + e = 0 \\
\Longrightarrow \quad & a\frac{dp}{dx} + c\frac{dq}{dy} + e - s\left(a\frac{dy}{dx} - b + c\frac{dx}{dy}\right) = 0 \\
\Longrightarrow \quad & a\frac{dp}{dx}\,\frac{dy}{dx} - as\left(\frac{dy}{dx}\right)^2 + bs\frac{dy}{dx} + c\frac{dq}{dy}\,\frac{dy}{dx} - cs + e\frac{dy}{dx} = 0 \\
\Longrightarrow \quad & s\left[a\left(\frac{dy}{dx}\right)^2 - b\frac{dy}{dx} + c\right] - \left[a\frac{dp}{dx}\,\frac{dy}{dx} + c\frac{dq}{dy}\,\frac{dy}{dx} + e\frac{dy}{dx}\right] = 0. \; (6)
\end{aligned}$$

Eqn. (6) can be made independent of s by choosing the curve C such a way that the slope $\frac{dy}{dx}$ satisfies

$$a\left(\frac{dy}{dx}\right)^2 - b\left(\frac{dy}{dx}\right) + c = 0. \tag{7}$$

Along these directions, we have

$$a\frac{dp}{dx}\,\frac{dy}{dx} + c\frac{dq}{dy}\,\frac{dy}{dx} + e\frac{dy}{dx} = 0. \tag{8}$$

- Depending on the nature of the roots of (7), we classify Eqn. (1) as follows.

  $b^2 - 4ac > 0 \implies$ the roots are real and distinct $\implies$ (1) is of hyperbolic type.

  $b^2 - 4ac = 0 \implies$ the roots are equal $\implies$ (1) is of parabolic type.

  $b^2 - 4ac < 0 \implies$ the roots are complex $\implies$ (1) is of elliptic type.

- Let $\frac{dy}{dx} = f$ and $\frac{dy}{dx} = g$. Along these directions, we have

$$a\,\frac{dp}{dx}\,\frac{dy}{dx} + c\,\frac{dq}{dy}\,\frac{dy}{dx} + e\,\frac{dy}{dx} = 0$$

or

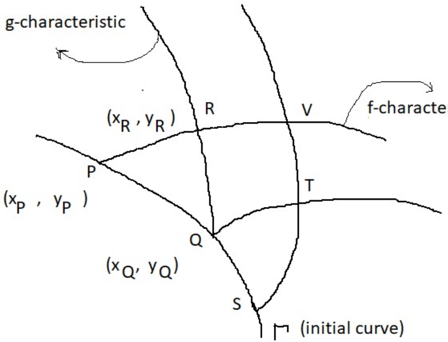
$$a\frac{dy}{dx}\,dp + c\,dq + e\,dy = 0.$$

The method of characteristics

Let

$$\frac{dy}{dx} = f, \quad \frac{dy}{dx} = g$$

be two charateristics directions associated a second order quasi-linear hyperbolic equation. Let Γ be the initial curve which does not coincide with C. Let P and Q be two points on Γ that are close together. Assume that the values of U, $U_x = p$, $U_y = q$ are known along Γ.



The numerical integration along the charateristic curves proceed as follows.
Step 1:

- (a) Computing the approximate values $(x_R^{(1)}, y_R^{(1)})$ of (x_R, y_R) from the equations:

$$\begin{aligned} (y_R^{(1)} - y_P) &= f_P\,(x_R^{(1)} - x_P), \\ (y_R^{(1)} - y_Q) &= g_Q\,(x_R^{(1)} - x_Q), \end{aligned}$$

  where $f_P = f(P)$ and $g_Q = g(Q)$.

- (b) Determine the approximate values of $p = U_x$, $q = U_y$ as follows.
  Along PR:

$$a_P\, f_P\,(p_R^{(1)} - p_P) + c_P\,(q_R^{(1)} - q_P) + e_P\,(y_R^{(1)} - y_P) = 0$$

  Along QR:

$$a_Q\, g_Q\,(p_R^{(1)} - p_Q) + c_Q\,(q_R^{(1)} - q_Q) + e_Q\,(y_R^{(1)} - y_Q) = 0$$

- (c) The values of U can be computed from the relation $dU = p\,dx + q\,dy$ as

$$(U_R^{(1)} - U_P) = \frac{1}{2}(p_P + p_R^{(1)})(x_P^{(1)} - x_P) + \frac{1}{2}(q_R^{(1)} + q_P)(y_R^{(1)} - y_P).$$

Step 2: Improve the values of $(x_R^{(1)}, y_R^{(1)})$, $(p_R^{(1)}, q_R^{(1)})$ and $U_R^{(1)}$ as

- (a) Compute $(x_R^{(2)}, y_R^{(2)})$ by solving

$$
\begin{aligned}
(y_R^{(2)} - y_P) &= \frac{1}{2}(f_P + f_R)\,(x_R^{(2)} - x_P),\\
(y_R^{(2)} - y_Q) &= \frac{1}{2}(g_Q + g_R)\,(x_R^{(2)} - x_Q),
\end{aligned}
$$

- (b) Determine the values of $(p_R^{(2)}, q_R^{(2)})$ as

$$
\begin{aligned}
&\frac{1}{2}(a_P + a_R)\,\frac{1}{2}(f_P + f_R)\,(p_R^{(2)} - p_P) + \frac{1}{2}(c_P + c_R)\,(q_R^{(2)} - q_P)\\
&+\frac{1}{2}(e_P + e_R)\,(y_R^{(2)} - y_P) = 0\\
&\frac{1}{2}(a_Q + a_R)\,\frac{1}{2}(g_Q + g_R)\,(p_R^{(2)} - p_Q) + \frac{1}{2}(c_Q + c_R)\,(q_R^{(2)} - q_Q)\\
&+\frac{1}{2}(e_Q + c_R)\,(y_R^{(2)} - y_Q) = 0
\end{aligned}
$$

- (c) Improved values of $U^{(2)}$ can be computed as

$$
(U_R^{(2)} - U_P) = \frac{1}{2}(p_P + p_R^{(2)})(x_R^{(2)} - x_P) + \frac{1}{2}(q_P + q_R^{(2)})(y_R^{(2)} - y_P).
$$

Repeat step 2 until (x_R, y_R), (p_R, q_R) and U_R attain the desired accuray.

Consider the second-order wave equation in one space dimension:

$$U_{tt} = U_{xx}, \quad t > 0 \tag{9}$$
$$\underbrace{U(x,0) = f(x)}_{initial\ displacement}, \quad \underbrace{U_t(x,0) = g(x)}_{initial\ velocity}$$

The slopes $\frac{dt}{dx}$ of the characteristics curves are given by

$$\left(\frac{dt}{dx}\right)^2 = 1.$$

Thus, the characteristics through the point (x_p, t_p) are given by

$$t - t_p = \pm(x - x_p) \tag{10}$$

The above straight lines meeting the x-axis at $D(x_p - t_p, 0)$ and $E(x_p + t_p, 0)$. The solution to IVP (9) (D'Alembert's solution) is given by

$$U(x,t) = \frac{1}{2}\left[f(x+t) + f(x-t) + \int_{x-t}^{x+t} g(\tau)\, d\tau\right].$$

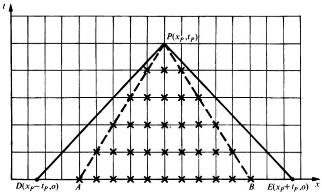
The solution at $P(x_p, t_p)$ is

$$U(x_p, t_p) = \frac{1}{2}\left[f(x_p + t_p) + f(x_p - t_p) + \int_{x_p - t_p}^{x_p + t_p} g(\tau)\, d\tau\right],$$

which depends upon

- the value of $f(x)$ at D and E, and
- the values of $g(x)$ at every point of the closed interval DE.

That is, the solution U depends on the initial data (f and g) along the interval of dependence DE. The area PDE is called the domain of dependence of the point P.



Finite Difference Approximation: A CTCS (central-time and central-space) scheme for the wave equation (9) at the mesh points $(x_i, t_j) = (ih, jk)$ leads to

$$\frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{k^2} = \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2}$$

$$\Longrightarrow \qquad u_{i,j+1} = \mu^2 u_{i-1,j} + 2(1-\mu^2)u_{i,j} + \mu^2 u_{i+1,j} - u_{i,j-1}, \quad (11)$$

where $\mu = \frac{k}{h}$. This is an explicit formula giving approximation values at mesh points along $t = 2k, 3k, \ldots$ as soon as the mesh values along $t = k$ have been determined.

Putting $j = 0$ in equation (11) yields

$$\begin{aligned} u_{i,1} &= \mu^2 u_{i-1,0} + 2(1-\mu^2)u_{i,0} + \mu^2 u_{i+1,0} - u_{i,-1} \\ &= \mu^2 f_{i-1} + 2(1-\mu^2)f_i + \mu^2 f_{i+1} - u_{i,-1} \qquad (12) \end{aligned}$$

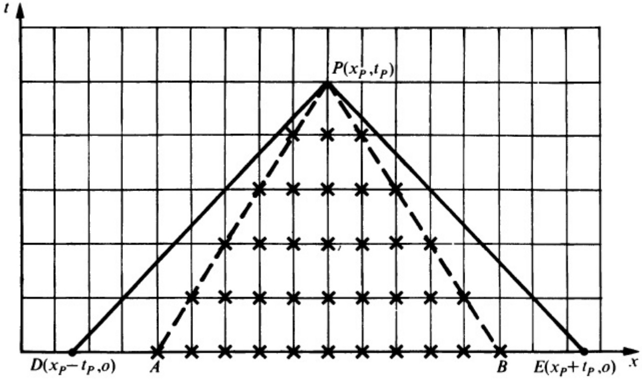
A central difference approximation to initial derivative condition $U_t(x, 0) = g$ gives

$$\frac{u_{i,1} - u_{i,-1}}{2k} = g_{i,0} \Longrightarrow u_{i,-1} = u_{i,1} - 2kg_{i,0} \qquad (13)$$

Substituting (13) in (12), we get

$$u_{i,1} = \frac{1}{2}\left[\mu^2 f_{i-1} + 2(1-\mu^2)f_i + \mu^2 f_{i+1} + 2kg_{i,0}\right]. \tag{14}$$

Courant Friedrich and Lewy (CFL) Condition:



Let u_P be the finite difference solution at the mesh point P, and the exact solution U at P be denoted U_P. Observe that u_P at the mesh point depends on the values of $u_{i,j}$ at the mesh points marked with crosses. This set of mesh points is called the numerical domain of dependence of the point P. The lines PA and PB are ofter called the numerical characteristics. Suppose initial conditions along DA and BE are changed, these changes will alter the analytical solution U of PDE at P but not the numerical solution given by (11) and (14). In this case,

$$u_P \not\Rightarrow U_P \quad \text{as } h \to 0, \quad k \to 0.$$

The CFL conditions states that the numerical domain of dependence of the difference equation must include the domain of dependence of the differential equation. Thus, for convergence of numerical solutions, we must have

$$0 < \mu \leq 1.$$

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