Lecture 6: Solutions of System of Nonlinear Equations

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System of Nonlinear Equations

Consider the system of two equations in two indendent variables x_1 and x_2:

$f_i : \mathbb{R}^2 \to \mathbb{R}$, $i = 1, 2$.

Assume that all partial derivatives up to second-order are continuous.

$$\begin{cases} f_1(x_1, x_2) = 0 \\ f_2(x_1, x_2) = 0. \end{cases} \tag{1}$$

Let (x_1^0, x_2^0) be an initial approximation to the exact solution, and let h_1^0 and h_2^0 be the corrections given to x_1^0 and x_2^0, respectively so that

$$\begin{aligned} f_1(x_1^0 + h_1^0, x_2^0 + h_2^0) &= 0, \\ f_2(x_1^0 + h_1^0, x_2^0 + h_2^0) &= 0. \end{aligned}$$

$x_1^0 + h_1^0$

$x_2^0 + h_2^0$

$f_i(x_1^0 + h_1^0, x_2^0 + h_2^0) = 0$ $i = 1, 2$.

Then, using Taylor's expansion upto linear term

$$0 = f_1(x_1^0 + h_1^0, x_2^0 + h_2^0) \approx f_1(x_1^0, x_2^0) + h_1^0 \frac{\partial f_1}{\partial x_1^0} + h_2^0 \frac{\partial f_1}{\partial x_2^0}$$

$$0 = f_2(x_1^0 + h_1^0, x_2^0 + h_2^0) \approx f_2(x_1^0, x_2^0) + h_1^0 \frac{\partial f_2}{\partial x_1^0} + h_2^0 \frac{\partial f_2}{\partial x_2^0},$$

$\frac{\partial f_i}{\partial x_i^0} = \frac{\partial f_i}{\partial x_i}\Big|_{(x_1^0, x_2^0)}$

where the partial derivatives are evaluated at (x_1^0, x_2^0).

$$f(a+h,\, b+k) = f(a,b) + h \left.\frac{\partial f}{\partial x}\right|_{(a,b)} + k \left.\left(\frac{\partial f}{\partial y}\right)\right|_{(a,b)} + \frac{1}{2}\left.\left[h^2 \frac{\partial^2 f}{\partial x^2} + 2hk \frac{\partial^2 f}{\partial x \partial y} + k^2 \frac{\partial^2 f}{\partial y^2}\right]\right|_{(a,b)} + \text{higher-order term.}$$

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we have

$$f_1(x_1^0, x_2^0) + h_1^0 \left.\frac{\partial f_1}{\partial x_1}\right|_{(x_1^0, x_2^0)} + h_2^0 \left.\frac{\partial f_1}{\partial x_2}\right|_{(x_1^0, x_2^0)} = 0$$

$$f_2(x_1^0, x_2^0) + h_1^0 \left.\frac{\partial f_2}{\partial x_1}\right|_{(x_1^0, x_2^0)} + h_2^0 \left.\frac{\partial f_2}{\partial x_2}\right|_{(x_1^0, x_2^0)} = 0$$

$$\Rightarrow \quad h_1^0 \frac{\partial f_1}{\partial x_1^0} + h_2^0 \frac{\partial f_1}{\partial x_2^0} = -f_1(x_1^0, x_2^0)$$

$$h_1^0 \frac{\partial f_2}{\partial x_1^0} + h_2^0 \frac{\partial f_2}{\partial x_2^0} = -f_2(x_1^0, x_2^0)$$

or

$$\underbrace{\begin{bmatrix} \frac{\partial f_1}{\partial x_1^0} & \frac{\partial f_1}{\partial x_2^0} \\ \frac{\partial f_2}{\partial x_1^0} & \frac{\partial f_2}{\partial x_2^0} \end{bmatrix}}_{\text{Jacobian matrix}} \underbrace{\begin{bmatrix} h_1^0 \\ h_2^0 \end{bmatrix}}_{\text{correction vector.}} = -\begin{bmatrix} f_1(x_1^0, x_2^0) \\ f_2(x_1^0, x_2^0) \end{bmatrix}$$

Rewrite the above equation as

$$\mathbf{J}\mathbf{h} = -\mathbf{f},$$

where the Jacobian matrix $\mathbf{J} = \mathbf{J}(f_1, f_2)$, the correction vector $\mathbf{h}$ and the vector $\mathbf{f}$ are given by

$$\mathbf{J} = \begin{bmatrix} \dfrac{\partial f_1}{\partial x_1^0} & \dfrac{\partial f_1}{\partial x_2^0} \\ \\ \dfrac{\partial f_2}{\partial x_1^0} & \dfrac{\partial f_2}{\partial x_2^0} \end{bmatrix}, \quad \mathbf{h} = \begin{bmatrix} h_1^0 \\ h_2^0 \end{bmatrix}, \quad \mathbf{f} = \begin{bmatrix} f_1(x_1^0, x_2^0) \\ \\ f_2(x_1^0, x_2^0) \end{bmatrix}.$$

If $\mathbf{J}$ is invertible, then the correction vector is dermined by

$$\mathbf{h} = -\mathbf{J}^{-1}\mathbf{f} \implies \begin{bmatrix} h_1^0 \\ h_2^0 \end{bmatrix} = -\mathbf{J}^{-1} \begin{bmatrix} f_1(x_1^0, x_2^0) \\ f_2(x_1^0, x_2^0) \end{bmatrix}.$$

first approximation to the root is
$x_1^1 = x_1^0 + h_1^0$
$x_2^1 = x_2^0 + h_2^0$

The first approximation to the solution is obtained as:

$$x_1^1 = x_1^0 + h_1^0, \quad x_2^1 = x_2^0 + h_2^0.$$

For a better approximation, set $x_1^2 = x_1^1 + h_1^1$ and $x_2^2 = x_2^1 + h_2^1$, and repeat the procedure to obtain

$J = J_{(x_1^1, x_2^1)}$

$$\begin{bmatrix} h_1^1 \\ h_2^1 \end{bmatrix} = -\mathbf{J}^{-1} \begin{bmatrix} f_1(x_1^1, x_2^1) \\ f_2(x_1^1, x_2^1) \end{bmatrix},$$

and so on. In general, Newton's iteration formula, in this case, reads

$$\begin{bmatrix} x_1^{(n+1)} \\ x_2^{(n+1)} \end{bmatrix} = \begin{bmatrix} x_1^{(n)} \\ x_2^{(n)} \end{bmatrix} + \begin{bmatrix} h_1^{(n)} \\ h_2^{(n)} \end{bmatrix}.$$

Remark. (i) At each stage of Newton's interation, one requires to solve the linear system

$$\mathbf{J} \begin{bmatrix} h_1^{(n)} \\ h_2^{(n)} \end{bmatrix} = - \begin{bmatrix} f_1(x_1^{(n)}, x_2^{(n)}) \\ f_2(x_1^{(n)}, x_2^{(n)}) \end{bmatrix}.$$

(ii) **Stopping Criteria.** If $|h_1^{(n)}| + |h_2^{(n)}| < tol$ then stop. Print the solution.

Consider the system

$$\mathbf{f}(\mathbf{x}) = 0, \tag{2}$$

where $\mathbf{f} = (f_1, f_2, \ldots, f_n)^T$, $\mathbf{x} = (x_1, x_2, \ldots, x_n)^T$. As before, a linearization of (2) leads to

$$0 = \mathbf{f}(\mathbf{x}^0 + \mathbf{h}^0) \approx \mathbf{f}(\mathbf{x}^0) + \mathbf{f}'(\mathbf{x}^0)\mathbf{h}^0,$$

where $\mathbf{h}^0 = (h_1^0, h_2^0, \ldots, h_n^0)^T$ is the correction vector and $\mathbf{f}'(\mathbf{x}^0)$ denotes the Jacobian matrix given by

$$\mathbf{f}'(\mathbf{x}^0) = \mathbf{J} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & & \vdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}_{(\mathbf{x}^0)} .$$

The correction vector $\mathbf{h}^0$ is determined as

$$\mathbf{h}^0 = -\mathbf{J}^{-1}\mathbf{f}(\mathbf{x}^0).$$

Newton's iteration reads

$$\mathbf{x}^{n+1} = \mathbf{x}^n + \mathbf{h}^n,$$

with

$$\mathbf{h}^n = -(\mathbf{J}^{-1})\mathbf{f}(\mathbf{x}^n) = -(\mathbf{f}'(\mathbf{x}^n))^{-1}\mathbf{f}(\mathbf{x}^n).$$

Example. Consider the system

$$\begin{aligned} f_1(x_1, x_2) &= 3x_1^2 x_2 - 10x_1 + 7 = 0 \\ f_2(x_1, x_2) &= x_2^2 - 5x_2 + 4 = 0. \end{aligned}$$

$$\frac{\partial f_1}{\partial x_1} = 6x_1x_2 - 10; \quad \frac{\partial f_1}{\partial x_2} = 3x_1^2,$$
$$\frac{\partial f_2}{\partial x_1} = 0; \quad \frac{\partial f_2}{\partial x_2} = 2x_2 - 5.$$

With starting value $x_1^0 = x_2^0 = 0.5$, the Jacobian matrix at (x_1^0, x_2^0) is

$$\mathbf{J} = \begin{bmatrix} -8.5 & 0.75 \\ 0 & -4 \end{bmatrix}; \quad f_1(x_1^0, x_2^0) = 2.375 \;\; f_2(x_1^0, x_2^0) = 1.75$$

Solving the system

$$\begin{bmatrix} -8.5 & 0.75 \\ 0 & -4 \end{bmatrix} \begin{bmatrix} h_1^0 \\ h_2^0 \end{bmatrix} = - \begin{bmatrix} 2.375 \\ 1.75 \end{bmatrix}$$

one gets $h_1^0 = 0.3180$ and $h_2^0 = 0.4375$.

The first approximation to the root is

$$x_1^1 = x_1^0 + h_1^0 = 0.8180, \quad x_2^1 = x_2^0 + h_2^0 = 0.9375.$$

For the second approximation, we have

$$\mathbf{J} = \begin{bmatrix} -5.3988 & 2.0074 \\ 0 & -3.125 \end{bmatrix}, f_1(x_1^1, x_2^1) = 0.7019, f_2(x_1^1, x_2^1) = 0.1914$$

$$h_1^1 = 0.1528, \quad h_2^1 = 0.0612,$$

$$x_1^2 = x_1^1 + h_1^1 = 0.9708, \quad x_2^2 = x_2^1 + h_2^1 = 0.9987.$$

After kth iteration,

$$\begin{bmatrix} x_1^{(k)} \\ x_2^{(k)} \end{bmatrix} = \begin{bmatrix} x_1^{(k-1)} \\ x_2^{(k-1)} \end{bmatrix} + \begin{bmatrix} h_1^{(k-1)} \\ h_2^{(k-1)} \end{bmatrix},$$

$|h_1^{(k)}| + |h_2^{(k)}|$

$< \text{TOL}$

print the

soln.

$(x_1^{(k+1)}, x_2^{(k+1)})$

where

$$\begin{bmatrix} h_1^{(k-1)} \\ h_2^{(k-1)} \end{bmatrix} = -(\mathbf{J}(x_1^{(k-1)}, x_2^{(k-1)}))^{-1}\mathbf{f}(x_1^{(k-1)}, x_2^{(k-1)}).$$

End

Recall

$$|e_{n+1}| = c\,|e_n|^p$$

$$|e_n| = C\,|e_{n-1}|^p$$

$e_n = x_n - \xi$

$$\begin{array}{l} x_0 \\ x_1 \\ x_2 \\ x_3 \end{array} \quad \begin{array}{l} \} \, e_{n-1} \\ \} \, e_n \\ \} \, e_{n+1} \end{array}$$

We need to find p?

$$\left|\frac{e_{n+1}}{e_n}\right| \approx \left|\frac{e_n}{e_{n-1}}\right|^p$$

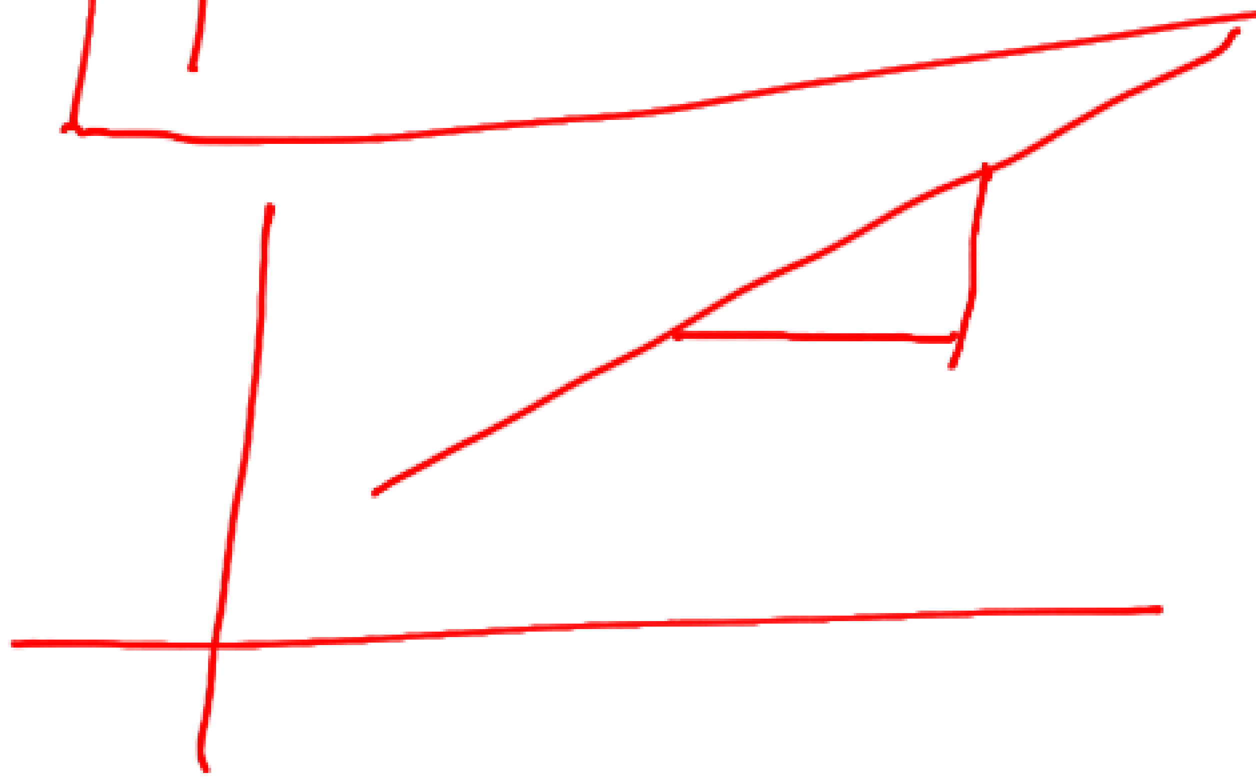
Taking log. both side.

$$p \approx \frac{\log\left(\left|\frac{e_{n+1}}{e_n}\right|\right)}{\log\left(\left|\frac{e_n}{e_{n-1}}\right|\right)}$$

Slope

$$\log|e_{n+1}| = \log c + p\,\underbrace{\log|e_n|}_{X}$$

$$\boxed{Y = C + mX}$$



$p \to 2.$ for Newton's method (in case of simple root)