# Lecture - 5

## Newton's Method via Fixed point method and Multiple roots.

Newton's method can be analyzed by the fixed point method.

Let $\xi$ be a simple root of $f(x) = 0$.

Recall Newton's formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n \geq 0$$

Considering this method as a fixed point method

$$x_{n+1} = g(x_n), \quad \text{where}$$

$$g(x) = x - \frac{f(x)}{f'(x)}.$$

Note that

$$g(\xi) = \xi - \frac{f(\xi)}{f'(\xi)}$$

$$= \xi \quad (\because f(\xi) = 0)$$

$\Rightarrow \xi$ is a fixed point of $g$.

Compute

$$g'(x) = 1 - \frac{(f'(x))^2 - f(x) f''(x)}{(f'(x))^2}$$

$$= \frac{(f'(x))^2 - (f'(x))^2 + f(x) f''(x)}{(f'(x))^2} = \frac{f(x) f''(x)}{(f'(x))^2}$$

$$g'(\xi) = \frac{f(\xi) f''(\xi)}{(f'(\xi))^2} = 0 \quad [\because f(\xi) = 0]$$

$$g''(x) = \frac{\left(f'(x) f''(x) + f'''(x) f(x)\right)(f'(x))^2 - 2 f'(x) (f''(x))^2 f(x)}{(f'(x))^4}$$

$$g''(\xi) = \frac{f'(\xi) f''(\xi) (f'(\xi))^2}{(f'(\xi))^4} = \frac{f''(\xi)}{f'(\xi)} \neq 0$$

Note that

$$g(\xi) = \xi, \quad g'(\xi) = 0 \quad \& \quad g''(\xi) \neq 0.$$

Recall the following result from fixed-point method:

$\underline{\underline{Th^m}}$: Let $\xi$ be a fixed point of $g(x)$. Let $g \in C^m(N_\delta(\xi))$, $m \geq 2$, and be s.t.

$$g'(\xi) = g''(\xi) = \cdots = g^{(m-1)}(\xi) = 0.$$

If the starting value $x_0 \in N_\delta(\xi)$, then iteration $x_{n+1} = g(x_n)$ converges to $\xi$ with order $m$ & $\lim_{n\to\infty} \frac{|e_{n+1}|}{|e_n|^m} = c$, $c = \left|\frac{g^{(m)}(\xi)}{m!}\right|$

By the above result, Newton's method yields the second-order convergence.

## Newton's method and Multiple roots:

Def$^n$: If $\xi \in I$ is a root of $f(x)=0$ with multiplicity $m$ then

$$f(x) = (x-\xi)^m h(x) , \qquad h(\xi) \neq 0$$

& $h(x)$ is conts at $x=\xi$.

If $h \in C^m(I)$, then

$$f(\xi) = f'(\xi) = \cdots = f^{(m-1)}(\xi) = 0 , \quad f^{m}(\xi) \neq 0.$$

For example: if $\xi$ is a double root of $f(x)=0$.

$$f(x) = (x-\xi)^2 h(x)$$

Clearly, $f(\xi)=0$, $\quad f'(x) = 2(x-\xi)\,h(x) + (x-\xi)^2 h'(x)$

$f'(\xi) = 0$. But, $f''(\xi) \neq 0$.

We will see how the presence of multiple root affects the rates of convergence in Newton's method.

Let $\xi$ be a multiple root of $f(x)=0$ with multiplicity $m$.

$$f(x) = (x-\xi)^m h(x)$$

$$f'(x) = (x-\xi)^m h'(x) + m(x-\xi)^{m-1} h(x).$$

In this case,

$$g(x) = x - \frac{f(x)}{f'(x)}$$

$$= x - \frac{(x-\xi)^m h(x)}{(x-\xi)^m h'(x) + m(x-\xi)^{m-1} h(x)}$$

$$= x - \frac{(x-\xi) h(x)}{(x-\xi) h'(x) + m\, h(x)}.$$

Clearly, $g(\xi) = \xi$

$$g'(x) = 1 - \frac{h(x)}{(x-\xi)h'(x) + m\,h(x)} - \underbrace{(x-\xi)\frac{d}{dx}\left[\frac{h(x)}{(x-\xi)h'(x) + m\,h(x)}\right]}_{\substack{\shortparallel \\ 0}}$$

$$g'(\xi) = 1 - \frac{1}{m} \neq 0$$

as $m > 1$

$\Rightarrow$ Any fixed-point $th^m$, the order of convergence is linear

**Remark:** The presence of multiple root slows down the order of convergence.

We can recover the second-order convergence by using the modified Newton's formula

$$x_{n+1} = x_n - m\frac{f(x_n)}{f'(x_n)}.$$

In this case, the iteration function $g(x) = x - m\dfrac{f(x)}{f'(x)}$

It is easy to verify that $g(\xi) = \xi$, $g'(\xi) = 0$

Then, we can regain the second-order convergence.