Lecture 10: Interpolation Contd..

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$f[x_0, x_1] = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$

Recall the interpolation error in Newton interpolating polynomial:

If t is a point such that $t \neq x_0, x_1, \ldots, x_n$, then

$$\boxed{f(t) - p_n(t) = f[x_0, x_1, \ldots, x_n, t] \prod_{j=0}^{n} (t - x_j).}$$

Recall, the error in Lagrange's interpolating poly:

$f(t) - p_n(t) = \frac{f^{(n+1)}(\xi)}{(n+1)!} \times \prod_{j=0}^{n} (t - x_j)$

Remark. Note that, we cannot evaluate the term $f[x_0, x_1, \ldots, x_n, t]$ without knowing the number $f(t)$. We now prove that the number $f[x_0, x_1, \ldots, x_n, t]$ is closely related to the $(n+1)$th derivative of $f(x)$.

Theorem. Let $f \in C^k([a, b])$. If $x_0, x_1, \ldots, x_k$ are $k+1$ distinct points in $[a, b]$, then there exists $\xi \in (a, b)$ such that

$$\boxed{f[x_0, x_1, \ldots, x_k] = \frac{f^{(k)}(\xi)}{k!}.}$$

$f^{(k)}(\xi) = \left.\frac{d^k f}{dx^k}\right|_{x=\xi}$

Proof. Take $k = 1$. Then by MVT,

$$f[x_0, x_1] = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = f'(\xi),$$

for some $\xi \in (a, b)$. For the general case, observe that

$e_k(x_i) = f(x_i) - p_k(x_i)$
$= f(x_i) - f(x_i)$
$= 0, \quad i = 0, 1, \ldots k.$

$$e_k(x) = f(x) - p_k(x)$$

has (at least) $k+1$ distinct zeros $x_0, x_1, \ldots, x_k$ in $[a, b]$. By Rolle's theorem, $e_k'(x)$ has at least k zeros in (a, b) $\implies$ $e_k''(x)$ has at least $k-1$ zeros in (a, b). Likewise, one can show that $e_k^{(k)}(x)$ has at least one zero in (a, b). That is, there exists $\xi \in (a, b)$ such that

$e_k(x) = f(x) - p_k(x)$
$e_k^{(k)}(x) = f^{(k)}(x) - p_k^{(k)}(x)$

$$e_k^{(k)}(\xi) = 0 \implies f^{(k)}(\xi) - p_k^{(k)}(\xi) = 0.$$

Since, for any x, $p_k^{(k)}(x) = f[x_0, x_1, \ldots, x_k]\, k!$, we have at once

$p_k^{(k)}(\xi) = f[x_0, \ldots x_k] \times k!$

$$f[x_0, x_1, \ldots, x_k] = \frac{f^{(k)}(\xi)}{k!}. \qquad \square$$

In view of the above result, we have the following expression for the interpolation error.

Theorem. Let $f \in C^{n+1}([a,b])$. If $p_n(x)$ is a polynomial of degree $\leq n$ such that

$$p_n(x_i) = f(x_i), \quad i = 0, 1, \ldots, n.$$

Then, for all $t \in [a,b]$, there exists $\xi = \xi(t) \in (a,b)$ such that

$$\boxed{f(t) - p_n(t) = \frac{f^{(n+1)}(\xi)}{(n+1)!} \prod_{j=0}^{n} (t - x_j).}$$

Proof. We know

$$f(t) - p_n(t) = f[x_0, x_1, \ldots, x_n, t] \prod_{j=0}^{n} (t - x_j).$$

Since $f[x_0, x_1, \ldots, x_n, t] = \frac{f^{(n+1)}(\xi)}{(n+1)!}$, we immediately obtain

$$f(t) - p_n(t) = \frac{f^{(n+1)}(\xi)}{(n+1)!} \prod_{j=0}^{n} (t - x_j). \qquad \square$$

Note that

$$\lim_{x \to x_0} f[x_0, x] = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = f'(x_0).$$

This suggest that the first divided difference at two identical points be defined as

$$f[x_0, x_0] = f'(x_0).$$

For any $n+1$ distinct points $x_0, x_1, \ldots, x_n$ in $[a; b]$ and any $f \in C^n[a, b]$, we have

$$f[x_0, x_1, \ldots, x_n] = \frac{f^{(n)}(\xi)}{n!}.$$

Letting all $x_i \to x_0$, $i \geq 1$, and ξ being trapped between them, $\xi \to x_0$,

we obtain $\boxed{f\,\underbrace{[x_0, x_0, \ldots, x_0]}_{\text{(n+1) times}} = \frac{f^{(n)}(x_0)}{n!}.}$

Inverse Interpolation

x	y
x_0	y_0
x_1	y_1
$\vdots$	$\vdots$
x_n	y_n

$P_n(x)$

Inv. Interpolation

y	x
y_0	x_0
y_1	x_1
$\vdots$	$\vdots$
y_n	x_n

An interesting application of interpolation, (in particular, of Newton's interpolation formula) is to approximate the solution of nonlinear equation

$$f(x) = 0. \tag{1}$$

$f(\xi) = 0.$ → root.

Let $x_0 \approx \xi$, $x_1 \approx \xi$ be two approximations of a root ξ of (1). Assume that f is monotone near ξ. Then

$$y = f(x) \implies x = f^{-1}(y).$$

Denote $g(y) = f^{-1}(y)$. Since $\xi = f^{-1}(0) = g(0)$, our aim is now to evaluate $g(0)$. Compute

$x_0 = f^{-1}(y_0) = g(y_0)$

$x_1 = g(y_1)$

$$y_0 = f(x_0),\ y_1 = f(x_1) \implies x_0 = g(y_0),\ x_1 = g(y_1).$$

The divided difference for inverse function g:

$$\begin{array}{lll} y_0 & x_0 & g[y_0, y_1] \\ y_1 & x_1 & \end{array}$$

$x_2 = g[y_0] + g[y_0, y_1](y - y_0)$

$\because y = 0,\ x_0 = g(y_0),\ x_2 = x_0 + (0 - y_0)\, g[y_0, y_1]$
$= x_0 - g[y_0, y_1]\, y_0$

To compute $g(0)$, an improved approximation by linear interpolation

$$x_2 = x_0 + (0 - y_0)g[y_0, y_1] = x_0 - y_0\, g[y_0, y_1].$$

Now evaluating $y_2 = f(x_2)$, we get $x_2 = g(y_2)$. The divided difference table can be updated as

$$\begin{array}{llll} y_0 & x_0 & g[y_0, y_1] & g[y_0, y_1, y_2] \\ y_1 & x_1 & g[y_1, y_2] & \\ y_2 & x_2 & & \end{array}$$

$x_3 = x_2 + g[y_0, y_1, y_2] \times (0 - y_0)(0 - y_1)$

This allows us to use quadratic interpolation to get

$$x_3 = x_2 + (0 - y_0)(0 - y_1)g[y_0, y_1, y_2] = x_2 + y_0 y_1\, g[y_0, y_1, y_2],$$

and then

$$y_3 = f(x_3) \ \text{ and } \ x_3 = g(y_3).$$

If necessary, we can continue updating the difference table,

$$\begin{array}{lllll}
y_0 & x_0 & g[y_0, y_1] & g[y_0, y_1, y_2] & g[y_0, y_1, y_2, y_3] \\
y_1 & x_1 & g[y_1, y_2] & g[y_1, y_2, y_3] & \\
y_2 & x_2 & g[y_2, y_3] & & \\
y_3 & x_3 & & &
\end{array}$$

$x_4 = x_3 - g[y_0, y_1, y_2, y_3] \times (0-y_0)(0-y_1)(0-y_2)$

and compute

$$x_4 = x_3 - y_0 y_1 y_2\, g[y_0, y_1, y_2, y_3], \quad y_4 = f(x_4), \quad x_4 = g(y_4)$$

so on. In general, the process will converge: $x_k \to \xi$ as $k \to \infty$.

End

Exercise: Calculate the cube root of 10 correct to three decimal places using inverse interpolation. Use the following data:

x:	2	3	4
$f(x) = y$:	8	27	64