# Lecture 11: Hermite Interpolation

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# Hermite Interpolation

Given $(n+1)$ distinct points (nodes) $x_0, x_1, \ldots, x_n$ and the values $f(x_i) = y_i$ and $f'(x_i) = y_i'$ for $i = 0, 1, \ldots, n$.

**Objective.** To look for a polynomial $H(x)$ of degree at most $2n+1$ such that

$$H(x_i) = f(x_i) = y_i \text{ and } H'(x_i) = f'(x_i) = y_i', \quad i = 0, 1, \ldots, n.$$

Here, $y_i, y_i' \in \mathbb{R}$

In analogy with Lagrange's formula, we write

$$H(x) = \sum_{i=0}^{n} y_i\, h_i(x) + \sum_{i=0}^{n} y_i'\, \tilde{h}_i(x),$$

where $h_i(x)$ and $\tilde{h}_i(x)$ are polynomials of degree $\leq 2n+1$ and satisfy the following properties:

$$h_i'(x_j) = \tilde{h}_i(x_j) = 0, \quad i, j = 1, \ldots, n.$$

$$h_i(x_j) = \tilde{h}_i'(x_j) = \begin{cases} 0 & i \neq j, \\ 1 & i = j. \end{cases}$$

Recall Langrange's interpolating polynomial:

$$P_n(x) = \sum_{i=0}^{n} f(x_i) L_i(x) ,$$

$$L_i(x) = \prod_{\substack{j=0 \\ j \neq i}}^{n} \frac{(x - x_j)}{(x_i - x_j)}$$

$$L_i(x_j) = \delta_{ij} = \begin{cases} 1 , & j = i \\ 0 , & j \neq i \end{cases}$$

$$P_n(x_j) = \sum_{i=0}^{n} f(x_i) L_i(x_j) = f(x_j)$$

$$H(x) = \sum_{i=0}^{n} y_i h_i(x) + \sum_{i=0}^{n} y_i' \tilde{h}_i(x)$$

Basic Idea: To construct $h_i(x)$ & $\tilde{h}_i(x)$ such that

$$\underbrace{H(x_i) = y_i,}_{(1)} \qquad \underbrace{H'(x_i) = y_i',}_{(2).} \qquad i = 0, 1, \dots, n.$$

In order to satisfy the cond$^n$ (1), we must have

$$\tilde{h}_i(x_j) = 0 \quad \forall\ i, j$$

$$h_i(x_j) = \delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$$

$$H(x_j) = \sum_{i=0}^{n} y_i h_i(x_j) + \underbrace{\sum_{i=0}^{n} y_i' \tilde{h}_i(x_j)}_{= 0} \qquad \left(\because \tilde{h}_i(x_j) = 0 \ \forall\ i, j\right)$$

$$= \sum_{i=0}^{n} y_i h_i(x_j) = y_j \qquad \left[\because h_i(x_j) = \delta_{ij}\right].$$

For $H(x)$ to satisfy the second condⁿ (2), we must have

$$h_i'(x_j) = 0 \quad \forall \; i, j$$

and

$$\tilde{h}_i'(x_j) = \delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$$

Note that

$$H'(x) = \sum_{i=0}^{n} y_i \, h_i'(x) + \sum_{i=0}^{n} y_i' \, \tilde{h}_i'(x)$$

$$H'(x_j) = \underbrace{\sum_{i=1}^{n} y_i \, h_i'(x_j)}_{=0} + \sum_{i=0}^{n} y_i' \, \tilde{h}_i'(x_j) \qquad \left[\because h_i'(x_j) = 0 \;\; \forall \; i, j\right]$$

$$= \sum_{i=0}^{n} y_i' \, \tilde{h}_i'(x_j) = y_j' \qquad \left[\because \tilde{h}_i'(x_j) = \delta_{ij}\right].$$

It is easy to verify that

$$H(x_j) = y_j, \quad H'(x_j) = \sum_{i=0}^{n} h_i'(x_j)\, y_i + \sum_{i=0}^{n} \tilde{h}_i'(x_j)\, y_i' = y_j'.$$

It remains to construct $h_i(x)$ and $\tilde{h}_i(x)$. Recall the basis for the Lagrange interpolating polynomial:

$$L_i(x) = \prod_{\substack{j=0 \\ j \neq i}} \frac{(x - x_j)}{(x_i - x_j)}, \quad 0 \leq i \leq n.$$

Define

$$\begin{aligned} h_i(x) &= [1 - 2\, L_i'(x_i)\,(x - x_i)]\,(L_i(x))^2, \quad 0 \leq i \leq n. \\ \tilde{h}_i(x) &= (x - x_i)\,(L_i(x))^2. \end{aligned}$$

The Hermite ploynomial (Lagrange form) is

$$H(x) = \sum_{i=0}^{n} h_i(x)\, y_i + \sum_{i=0}^{n} \tilde{h}_i(x)\, y_i'.$$

We have

$$h_i(x) = \left[1 - 2L_i'(x_i)(x - x_i)\right](L_i(x))^2$$

$$\tilde{h}_i(x) = (x - x_i)(L_i(x))^2$$

Both $h_i(x)$ & $\tilde{h}_i(x)$ are polys. of deg $\leq 2n+1$.

$$\tilde{h}_i(x_j) = 0 \quad \forall\, i,j \quad \left[\begin{aligned}\tilde{h}_i(x_j) &= (x_j - x_i)(L_i(x_j))^2 \\ &= 0 \quad \forall\, i,j\end{aligned}\right]$$

$$h_i(x_j) = \left[1 - 2L_i'(x_i)(x_j - x_i)\right](L_i(x_j))^2$$

$$= (L_i(x_j))^2 - \underbrace{2L_i'(x_i)(x_j - x_i)(L_i(x_j))^2}_{=0}$$

$$= (L_i(x_j))^2 = \delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases} \qquad \left(\because L_i(x_j) = \delta_{ij}\right)$$

$$\tilde{h}_i(x) = (x - x_i)\left(L_i(x)\right)^2$$

$$\tilde{h}_i'(x) = \left(L_i(x)\right)^2 + (x - x_i)\, 2 L_i(x) L_i'(x)$$

$$\tilde{h}_i'(x_j) = \left(L_i(x_j)\right)^2 + \underbrace{(x_j - x_i)\, 2 L_i(x_j) L_i'(x_j)}_{=0 \ \forall i,j}$$

$$= \left(L_i(x_j)\right)^2$$

$$= \delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$$

<u>Check</u>: $h_i'(x_j) = 0 \quad \forall i, j$

All conditions are now verified.

Let us take $n = 1$. Then the given data are:

$$f(x_0) = y_0,\ f(x_1) = y_1, \text{ and } f'(x_0) = y_0',\ f'(x_1) = y_1'.$$

Then

$$H(x) = h_0(x)y_0 + h_1(x)y_1 + \tilde{h}_0(x)y_0' + \tilde{h}_1(x)y_1',$$

where

$$\begin{aligned}
h_0(x) &= [1 - 2\,L_0'(x_0)\,(x - x_0)]\,(L_0(x))^2, \\
h_1(x) &= [1 - 2\,L_1'(x_1)\,(x - x_1)]\,(L_1(x))^2, \\
\tilde{h}_0(x) &= (x - x_0)\,(L_0(x))^2 \\
\tilde{h}_1(x) &= (x - x_1)\,(L_1(x))^2, \quad \text{with}
\end{aligned}$$

$$L_0(x) = \frac{x - x_1}{x_0 - x_1},\quad L_1(x) = \frac{x - x_0}{x_1 - x_0}$$

$$L_0'(x) = \frac{1}{x_0 - x_1},\quad L_1'(x) = \frac{1}{x_1 - x_0}.$$

## To show the uniqueness of $H(x)$.

Suppose there exists a second polynomial $G(x)$ with degree $\leq 2n+1$ with $G(x_i) = y_i$ and $G'(x_i) = y_i'$, $i = 0, 1, \ldots, n$. Define

$$R(x) = H(x) - G(x).$$

R(x) is a poly. of deg $\leq 2n+1$

Then

$R(x_i) = H(x_i) - G(x_i) = y_i - y_i = 0$

$R'(x_i) = H'(x_i) - G'(x_i) = y_i' - y_i' = 0$

$i = 0, 1, \ldots, n$

$$R(x_i) = R'(x_i) = 0, \quad i = 0, 1, \ldots, n.$$

$$\implies R(x) \text{ has } (n+1) \text{ double roots at } x_0, x_1, \ldots, x_n.$$

$$\implies R(x) = q(x)(x - x_0)^2(x - x_1)^2 \cdots (x - x_n)^2,$$

for some polynomial $q(x)$.

If $q(x) \neq 0$, then degree of $R(x) \geq 2n+2$, which is a contradiction to the fact that degree of $R(x) \leq 2n+1$.

Therefore, $R(x) = 0 \implies H(x) = G(x)$. This proves the uniqueness.

***End***