Lecture 30: Two-Dimensional Heat Equations

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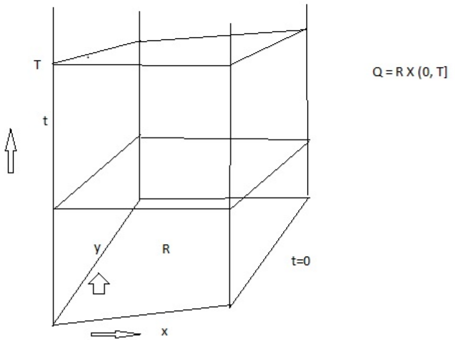
Let $R\ :\ \{(x,y)\mid 0<x<a,\ \ 0<y<b\}\subset\mathbb{R}^2,$ and $Q=R\times(0,T].$

Consider the following model equation heat equation:

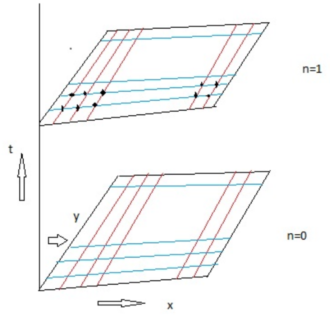
$$
\begin{aligned}
U_t &= U_{xx}+U_{yy}\ \text{ in } Q\\
\text{IC:}\ \ U(x,y,0) &= f(x,y),\ \ 0\le x\le a,\ 0\le y\le b,\\
\text{BCs:}\ \ U(0,y,t) &= g_1(y,t),\ \ U(a,y,t)=g_2(y,t),\ \ 0<y<b,\ \ t>0,\\
U(x,0,t) &= g_3(x,t),\ \ U(x,b,t)=g_4(x,t),\ \ 0<x<a,\ \ t>0.
\end{aligned}
$$



Introduce the mesh parameters Δx, Δy and Δt along x, y and t directions, respectively such that

$$\begin{aligned} x_i &= i\Delta x, \quad i = 0, 1, 2, \ldots, M \ \text{ with } \ M\Delta x = a, \\ y_j &= j\Delta y, \quad j = 0, 1, 2, \ldots, N \ \text{ with } \ N\Delta y = b, \\ t_n &= n\Delta t, \quad n = 0, 1, 2, \ldots, J \ \text{ with } \ J\Delta t = T. \end{aligned}$$

Denote $U(x_i, y_j, t_n) = U(i\Delta x, j\Delta y, n\Delta t) = U^n_{i,j}$.



Define the following central difference operator:

$$\begin{aligned}\delta_t\phi^n_{i,j} &= \phi^{n+\frac{1}{2}}_{i,j} - \phi^{n-\frac{1}{2}}_{i,j}\\ \delta_x\phi^n_{i,j} &= \phi^n_{i+\frac{1}{2},j} - \phi^n_{i-\frac{1}{2},j}\\ \delta_y\phi^n_{i,j} &= \phi^n_{i,j+\frac{1}{2}} - \phi^n_{i,j-\frac{1}{2}}.\end{aligned}$$

Explicit scheme: Approximate the heat equation $U_t = U_{xx} + U_{yy}$ at the grid point $(i\Delta x, j\Delta y, n\Delta t)$ using FTCS (forward-time and central-space) scheme to have

$$(U_t)^n_{i,j} \approx \frac{u^{n+1}_{i,j} - u^n_{i,j}}{\Delta t} = \frac{1}{\Delta t}\delta_t u^{n+\frac{1}{2}}_{i,j}$$

$$(U_{xx})^n_{i,j} \approx \frac{u^n_{i+1,j} - 2u^n_{i,j} + u^n_{i-1,j}}{(\Delta x)^2} = \frac{1}{(\Delta x)^2}\delta_x^2 u^n_{i,j}$$

$$(U_{yy})^n_{i,j} \approx \frac{u^n_{i,j+1} - 2u^n_{i,j} + u^n_{i,j-1}}{(\Delta y)^2} = \frac{1}{(\Delta y)^2}\delta_y^2 u^n_{i,j}$$

$$\frac{1}{\Delta t}\delta_t u^{n+\frac{1}{2}}_{i,j} = \frac{1}{(\Delta x)^2}\delta_x^2 u^n_{i,j} + \frac{1}{(\Delta y)^2}\delta_y^2 u^n_{i,j}.$$

$$\Longrightarrow \qquad u^{n+1}_{i,j} = u^n_{i,j} + \Delta t\left(\frac{1}{(\Delta x)^2}\delta_x^2 + \frac{1}{(\Delta y)^2}\delta_y^2\right)u^n_{i,j}. \qquad (1)$$

Remarks.

- It is an explicit scheme
- T.E $= O(\Delta t) + O((\Delta x)^2) + O((\Delta y)^2)$
- This scheme is conditionally stable. The condition for stability is

$$\Delta t \left(\frac{1}{(\Delta x)^2} + \frac{1}{(\Delta y)^2} \right) \leq \frac{1}{2}.$$

In particular, for $\Delta x = \Delta y = h$, the scheme (1) becomes

$$\begin{aligned} u_{i,j}^{n+1} &= u_{i,j}^n + \frac{\Delta t}{h^2} \left(\delta_x^2 u_{i,j}^n + \delta_y^2 u_{i,j}^n \right) \\ &= u_{i,j}^n + r \left(\delta_x^2 + \delta_y^2 \right) u_{i,j}^n, \quad r = \frac{\Delta t}{h^2}. \end{aligned}$$

In this case, the stabilty criteria reads

$$r = \frac{\Delta t}{h^2} \leq \frac{1}{4}.$$

Crank-Nicolson Scheme: Apply CTCS (central-time and central-space) schemes at the mesh point $(x_i, y_j, t_{n+1/2})$ to obtain

$$
\begin{aligned}
(U_t)_{i,j}^{n+\frac{1}{2}} &\approx \frac{u_{i,j}^{n+1} - u_{i,j}^{n}}{\Delta t} \\
(U_{xx})_{i,j}^{n+\frac{1}{2}} &\approx \frac{1}{2(\Delta x)^2}\, \delta_x^2 (u_{i,j}^{n+1} + u_{i,j}^{n}) \\
(U_{yy})_{i,j}^{n+\frac{1}{2}} &\approx \frac{1}{2(\Delta y)^2}\, \delta_y^2 (u_{i,j}^{n+1} + u_{i,j}^{n})
\end{aligned}
$$

With $\Delta x = \Delta y = h$, we have

$$
\begin{aligned}
& \frac{u_{i,j}^{n+1} - u_{i,j}^{n}}{\Delta t} = \frac{1}{2h^2}[\delta_x^2 (u_{i,j}^{n+1} + u_{i,j}^{n}) + \delta_y^2 (u_{i,j}^{n+1} + u_{i,j}^{n})] \\
\Longrightarrow \quad & u_{i,j}^{n+1} - u_{i,j}^{n} = \frac{r}{2}\left[(\delta_x^2 + \delta_y^2) u_{i,j}^{n} + (\delta_x^2 + \delta_y^2) u_{i,j}^{n+1}\right], \quad r = \frac{k}{h^2} \\
\Longrightarrow \quad & \left[1 - \frac{r}{2}(\delta_x^2 + \delta_y^2)\right] u_{i,j}^{n+1} = \left[1 + \frac{r}{2}(\delta_x^2 + \delta_y^2)\right] u_{i,j}^{n}.
\end{aligned}
$$

Remarks.

- It is an implicit scheme which is unconditionally stable.
- At each time level, we need to solve a linear system involving $(N-1)\times(N-1)$ unknowns.
- The local T.E. $= O(h^2) + O(k^2)$. $(\Delta t = k, \Delta x = \Delta y = h)$

Aternating Direction Implicit (ADI) Scheme: (Due to Peaceman and Rachford)

In this method, the computations from n^{th} time-level to $(n+1)^{th}$ time-level proceed as follows.

- **Step 1:** In the first step, we advance from n^{th} level to $(n+\frac{1}{2})^{th}$ level by approximating the term $\frac{\partial^2 U}{\partial x^2}$ by an implicit scheme, and for the term $\frac{\partial^2 U}{\partial y^2}$ use an explicit scheme.

  With $\Delta x = \Delta y = h$ and $r = \frac{\Delta t}{h^2}$, we have

$$\frac{u_{i,j}^{n+\frac{1}{2}} - u_{i,j}^{n}}{\frac{\Delta t}{2}} = \frac{1}{h^2}[\delta_x^2 u_{i,j}^{n+\frac{1}{2}} + \delta_y^2 u_{i,j}^{n})]$$

$$\Longrightarrow \qquad \left(1 - \frac{r}{2}\delta_x^2\right)u_{i,j}^{n+\frac{1}{2}} = \left(1 + \frac{r}{2}\delta_y^2\right)u_{i,j}^{n}. \tag{2}$$

- **Step 2:** While advancing from $(n+\frac{1}{2})^{th}$ level to $(n+1)^{th}$ level, approximate $\frac{\partial^2 U}{\partial x^2}$ by an explicit scheme, and use an implicit scheme for the term $\frac{\partial^2 U}{\partial y^2}$. In this case, we get

$$\frac{u_{i,j}^{n+1} - u_{i,j}^{n+\frac{1}{2}}}{\frac{\Delta t}{2}} = \frac{1}{h^2}[\delta_x^2 u_{i,j}^{n+\frac{1}{2}} + \delta_y^2 u_{i,j}^{n+1}]$$

$$\implies \quad \left(1 - \frac{r}{2}\delta_y^2\right) u_{i,j}^{n+1} = \left(1 + \frac{r}{2}\delta_x^2\right) u_{i,j}^{n+\frac{1}{2}}. \qquad (3)$$

The intermediate boundary values for $u_{i,j}^{n+\frac{1}{2}}$ can be obtained from (2) and (3) as follows:

$$u_{i,j}^{n+\frac{1}{2}} = \frac{1}{2}\left[(1 + \frac{r}{2}\delta_y^2) u_{i,j}^{n} + (1 - \frac{r}{2}\delta_y^2) u_{i,j}^{n+1}\right]$$

Remark:

- ADI scheme is unconditionally stable. Unlike Crank-Nicolson scheme, in this case we solve a linear system involving only $N-1$ unknowns.
- The local T.E. $= O(h^2) + O(k^2)$.

*** Ends ***