Lecture - 8

Error in Lagrange's Interpolation

Recap of Lecture-7

Given $(x_0, f(x_0)), (x_1, f(x_1)), \ldots, (x_n, f(x_n))$ a set of $(n+1)$ data points, where x_i's are distinct point in $[a, b]$.

The Lagrange's interpolating polynomial is given by

$$P_n(x) = \sum_{i=0}^{n} f(x_i) L_i(x), \quad \text{where}$$

$$L_i(x) = \prod_{j \neq i} \frac{(x - x_j)}{(x_i - x_j)}, \quad i = 0, 1, \ldots, n.$$

Lagrange's basis functions

$$L_i(x_j) = \delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$$

Theorem: (Error estimate)

Assume that $f \in C^{n+1}([a,b])$. If $p_n(x)$ is a polynomial of degree $\le n$ s.t.

$$p_n(x_i) = f(x_i), \quad i = 0, 1, \dots, n,$$

where x_i's are distinct points in $[a,b]$. Then, $\forall\, t \in [a,b]$, $\exists\, \xi = \xi(t) \in (a,b)$ s.t.

$$f(t) - p_n(t) = f(t) - \sum_{j=0}^{n} f(x_j)\, L_j(t)$$

$$= \underbrace{\frac{f^{(n+1)}(\xi)}{(n+1)!} \prod_{j=0}^{n} (t - x_j)}_{\text{Error.}}$$

Proof. From the error expression, we observe that

$$f(t) - P_n(t) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(t-x_0)(t-x_1)\cdots(t-x_n)$$

If $t = x_i$, $i = 0, 1, \ldots, n$, then

$$f(x_i) - P_n(x_i) = 0, \quad i = 0, 1, \ldots, n.$$

Then, $L.H.S = R.H.S.$

Take $t \neq x_i$, $\forall\, i = 0, 1, \ldots, n.$

Define $E(x) = f(x) - P_n(x)$, where $P_n(x) = \sum_{j=0}^{n} f(x_j) L_j(x).$

Set $G(x) = E(x) - \dfrac{\psi(x)}{\psi(t)} E(t)$, $\psi(x) = \prod_{j=0}^{n} (x - x_j)$

Notice that $G(x) \in C^{n+1}([a,b])$ as $\psi(x), E(x) \in C^{n+1}([a,b])$

$$G(x_i) = E(x_i) - \frac{\psi(x_i)}{\psi(t)} E(t) = 0, \quad i = 0, 1, 2, \ldots, n.$$

$$G(t) = E(t) - E(t) = 0$$

$G(x)$ has $(n+2)$ distinct zeros in $[a, b]$

$\Rightarrow$ By MVT, $G'(x)$ has $(n+1)$ distinct zeros in $[a,b]$.

$\Rightarrow$ $G''(x)$ has n distinct zeros in $[a,b]$.

Likewise, $G^{(n+1)}(x)$ has only one zero in (a,b).

$\Rightarrow$ $\exists$ a number $\xi \in (a,b)$ s.t.

$$G^{(n+1)}(\xi) = 0$$

Observe that

$$G^{(n+1)}(x) = E^{(n+1)}(x) - \frac{\psi^{(n+1)}(x)}{\psi(t)} E(t)$$

$$= f^{(n+1)}(x) - \frac{(n+1)!}{\psi(t)} E(t)$$

$$G^{(n+1)}(\xi) = 0$$

$$\Rightarrow \quad f^{(n+1)}(\xi) - \frac{(n+1)!}{\psi(t)} E(t) = 0$$

$$\Rightarrow \quad E(t) = \frac{f^{(n+1)}(\xi)}{(n+1)!} \psi(t)$$

$$= \frac{f^{(n+1)}(\xi)}{(n+1)!} \prod_{j=0}^{n} (t - x_j)$$

i.e

$$\boxed{f(t) - P_n(t) = \frac{f^{(n+1)}(\xi)}{(n+1)!} \prod_{j=0}^{n} (t - x_j)}$$

Example: To compute the error in the linear interpolation

i.e $f(t) - P_1(t)$.

Using the error formula, we write

$$E(t) = f(t) - P_1(t)$$

$$= \frac{f''(\xi)}{2!} (t - x_0)(t - x_1),$$

where ξ lies bet^n x_0 and x_1.

Suppose $|f''(x)| \le M$ on $[x_0, x_1]$. Then

$$|E(t)| = |f(t) - P_1(t)| \le \frac{M}{2} |(t - x_0)(t - x_1)|$$

Observe that $\max\limits_{t \in [x_0, x_1]} |(t - x_0)(t - x_1)| = \frac{(x_1 - x_0)^2}{4}$

($\max\limits_{t \in [x_0, x_1]} |(t - x_0)(t - x_1)|$ occurs at $t = \frac{x_0 + x_1}{2}$)

$$|E(t)| = |f(t) - P_1(t)| \le \underbrace{\frac{M}{8}(x_1 - x_0)^2}_{\text{Error in the linear interpolation.}}$$

If $x_1 - x_0 = h$,

$$\boxed{|f(t) - P_1(t)| \le \frac{M}{8} h^2}$$

Rounding error analysis for linear interpolation :

Let

$$\left.\begin{aligned} f(x_0) &= f_0 + \epsilon_0 \\ f(x_1) &= f_1 + \epsilon_1 \end{aligned}\right\}$$ Here, ϵ_0, ϵ_1 are rounding errors.

Define the error

$$\mathcal{E}(x) = f(x) - \frac{(x_1 - x_0) f_0 + (x - x_0) f_1}{(x_1 - x_0)}$$

Using $f_i = f(x_i) - \epsilon_i$, $(i = 0, 1)$, we obtain

$$\mathcal{E}(x) = \underbrace{f(x) - \frac{(x_1 - x) f(x_0) + (x - x_0) f(x_1)}{(x_1 - x_0)}}_{E(x)} + \underbrace{\frac{(x_1 - x)\epsilon_0 + (x - x_0)\epsilon_1}{(x_1 - x_0)}}_{R(x)}$$

interpolation error

$R(x)$ - Round off error

$$= E(x) + R(x)$$

interpolation error

round off error

$$|\mathcal{E}(x)| \le \frac{(x_1 - x_0)^2}{8} M + \max_{x \in [x_0, x_1]} |R(x)|.$$

$$\max |R(x)| \le \max \{|\epsilon_0|, |\epsilon_1|\}$$

$$\Rightarrow \boxed{|\mathcal{E}(x)| \le \frac{M}{8}(x_1 - x_0)^2 + \max\{|\epsilon_0|, |\epsilon_1|\}.}$$