Lecture 23: Stability of Multistep Methods

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Consider the sequence of real numbers defined by

$$
\begin{aligned}
y_0 = 1 \qquad\qquad y_1 &= \frac{1}{3} \\
y_{n+1} &= \frac{13}{3} y_n - \frac{4}{3} y_{n-1} \quad (n \geq 1)
\end{aligned}
$$

This recurrence relation generates the sequence

$$y_n = \left(\frac{1}{3}\right)^n .$$

Note that, for $n = m + 1$, we have

$$
\begin{aligned}
y_{m+1} &= \frac{13}{3} y_m - \frac{4}{3} y_{m-1} = \frac{13}{3}\left(\frac{1}{3}\right)^m - \frac{4}{3}\left(\frac{1}{3}\right)^{m-1} \\
&= \left(\frac{1}{3}\right)^{m-1}\left[\frac{13}{9} - \frac{4}{3}\right] = \left(\frac{1}{3}\right)^{m+1} .
\end{aligned}
$$

$$
\begin{aligned}
y_0 &= 1.0000000 \\
y_1 &= 0.3333333 \\
y_2 &= 0.1111112 \\
y_3 &= 0.0370373 \\
y_4 &= 0.0123466 \\
y_5 &= 0.0041187 \\
y_6 &= 0.0013857 \\
y_7 &= 0.0005131 \\
y_8 &= 0.0003757 \\
y_9 &= 0.0009437 \\
y_{10} &= 0.0035887 \\
y_{11} &= 0.0142927 \\
y_{12} &= 0.0571502 \\
y_{13} &= 0.2285939 \\
y_{14} &= 0.9143735 \\
y_{15} &= 3.657493 \quad \text{(incorrect with relative error of } 10^8\text{)}
\end{aligned}
$$

Stability of Multistep Methods

Consider the IVP: $y' = f(x, y), \;\; y(x_0) = y_0$. Integrating from x_{n-1} to x_{n+1} yields

$$\int_{x_{n-1}}^{x_{n+1}} y' dx = \int_{x_{n-1}}^{x_{n+1}} f(x, y)\, dx.$$

A mid-point rule for the right hand side integral leads to a multistep method

$$y_{n+1} = y_{n-1} + 2hf_n.$$

To understand the stability of this multistep method, let us consider the following problem:

$$y' = -2y + 1, \quad y(0) = 1.$$

The exact solution is $y(x) = \frac{1}{2}e^{-2x} + \frac{1}{2}$. Apply the above multistep method with $f_n = f(x_n, y_n) = -2y_n + 1$ to obtain

$$y_{n+1} + 4hy_n - y_{n-1} = 2h, \quad y_0 = 1.$$

The characteristics equation is $\beta^2 + 4h\beta - 1 = 0$, whose roots are

$$\beta_1 = -2h + \sqrt{1 + 4h^2}, \quad \beta_2 = -2h - \sqrt{1 + 4h^2}.$$

Expanding $\sqrt{1 + 4h^2}$ in a Taylor's series through linear terms, we obtain

$$\beta_1 = 1 - 2h + O(h^2), \quad \beta_2 = -(1 + 2h) + O(h^2).$$

$$y_n = C_1(1 - 2h + O(h^2))^n + C_2(-1)^n((1 + 2h) + O(h^2))^n + \frac{1}{2}. \qquad (1)$$

Note that $n = x_n/h$. Thus, for x_n fixed, we have

$$\lim_{h\to 0}(1 + 2h)^n = \lim_{h\to 0}(1 + 2h)^{(1/2h)(2x_n)} = e^{2x_n}.$$

$$\lim_{h\to 0}(1 - 2h)^n = e^{-2x_n}.$$

In the limit $h \to 0$, the solution (1) approaches

$$y_n = (C_1 e^{-2x_n} + \frac{1}{2}) + C_2(-1)^n e^{2x_n}.$$

- Note that the first term tends to the true solution of the differential equation. The second term is extraneous and arises only because we have replaced a first-order differential equation by a second-order difference equation.
- The error introduced from the extraneous solution will eventually dominate the true solution and lead to completely incorrect results.

Suppose a multistep method leads to a difference equation of order k whose characteristics equation is

$$\beta^k + a_{k-1}\,\beta^{k-1} + \cdots + a_0 = 0. \tag{2}$$

If (2) has k distinct zeros, say $\beta_1,\ \beta_2,\ \ldots,\ \beta_k$. The GS of the corresponding homogeneous difference equation is

$$y_n = c_1\beta_1^n + c_2\beta_2^n + \cdots + c_k\beta_k^n, \quad \forall n$$

One of these solutions, say β_1^n, will tend to the exact solution of the differential equation as $h \to 0$. All the other solutions $(\beta_2^n, \beta_3^n, \ldots, \beta_k^n)$ are extraneous.

A multistep method is said to be strongly stable if the extraneous roots satisfy the condition

$$|\beta_i| < 1, \quad i = 2, 3, \ldots, k.$$

If $|\beta_i| > 1$ for any $i = 2, 3, \ldots k$, then the errors will grow exponentially.

Example. For the IVP $y' = \lambda y, \;\; y(0) = 1$, Adams-Bashforth method leads to

$$\begin{aligned} y_{n+1} &= y_n + \frac{h}{24}(55f_n - 59f_{n-1} + 37f_{n-2} - 9f_{n-3}) \\ &= y_n + \frac{h\lambda}{24}(55y_n - 59y_{n-1} + 37y_{n-2} - 9y_{n-3}) \end{aligned} \tag{3}$$

The characteristics equation for this equation is

$$\beta^4 - \beta^3 - \frac{h\lambda}{24}(55\beta^3 - 59\beta^2 + 37\beta - 9) = 0,$$

which can be put in the form

$$p(\beta) + h\lambda q(\beta) = 0, \quad \text{where} \tag{4}$$

$$\begin{aligned} p(\beta) &= \beta^4 - \beta^3 \\ q(\beta) &= -\frac{1}{24}(55\beta^3 - 59\beta^2 + 37\beta - 9) \end{aligned}$$

As $h \to 0$, Eqn. (4) $\implies p(\beta) = 0$, whose roots are $\beta_1 = 1$, $\beta_2 = \beta_3 = \beta_4 = 0$. For $h \neq 0$, the GS of (3) will have the form

$$y_n = c_1\beta_1^n + c_2\beta_2^n + c_3\beta_3^n + c_4\beta_4^n, \text{ where}$$

β_i are the solutions of (4). One can show that β_1^n approaches the desired solution of $y' = \lambda y$, while the other roots correspond to extraneous solutions.

Since the roots of (4) are continuous functions of h, it follows that for h small enough,

$$|\beta_i| < 1 \quad \text{for } i = 2, 3, 4.$$

By the definition of stability that the Adams-Bashforth method is strongly stable.

Example. Let us investigate the stability properties of Milne's method for the same IVP:

$$y' = \lambda y, \quad y(0) = 1.$$

The Milne's method is given by

$$y_{n+1} = y_{n-1} + \frac{h}{3}(f_{n+1} + 4f_n + f_{n-1}).$$

Since $f(x, y) = \lambda y$, it follows that

$$y_{n+1} - y_{n-1} - \frac{h\lambda}{3}(y_{n+1} + 4y_n + y_{n-1}) = 0. \tag{5}$$

Its characteristics equation is given by

$$p(\beta) + h\lambda q(\beta) = 0, \quad \text{where} \tag{6}$$

$$\begin{aligned} p(\beta) &= \beta^2 - 1, \\ q(\beta) &= -\frac{1}{3}(\beta^2 + 4\beta + 1). \end{aligned}$$

As $h \to 0$, we have $p(\beta) = 0$, which has two roots $\beta_1 = 1$, $\beta_2 = -1$. By the definition, Milne's method is **not strongly stable**.

For h small, the roots of the stability polynomial (6) is

$$\beta_1 = 1 + \lambda h + O(h^2), \qquad \beta_2 = -(1 - \frac{\lambda h}{3}) + O(h^2).$$

The GS of (5) is

$$y_n = c_1(1 + \lambda h + O(h^2))^n + c_2(-1)^n(1 - \frac{\lambda h}{3}h + O(h^2))^n.$$

Set $n = x_n/h$ and let $h \to 0$, this solution approaches

$$\begin{aligned} y_n &= c_1 e^{\lambda x_n} + c_2(-1)^n e^{-\lambda x_n/3} \\ &=: y_{d,n}(x) + y_{e,n}(x) \end{aligned}$$

Observations

- If $\lambda > 0$, the desired solution $y_{d,n}(x)$ tends to the true soluion, and the extraneous solution $y_{e,n}(x)$ will be exponentially decreasing. Thus, Milne's method will be **stable**.
- On the other hand if $\lambda < 0$, then Milne's method will be **unstable** because the extraneous solution $y_{e,n}(x)$ will be exponentially increasing.
- Methods of this type whose stability depends upon the sign of λ for the test equation $y' = \lambda y$ are said to be weakly stable.

*** Ends ***