Lecture 29: Stability Analysis Contd.. and Lax Equivalence Theorem

Rajen Kumar Sinha

Department of Mathematics
IIT Guwahati

Stability Criteria for Derivative BCs

Consider

$$\begin{aligned}
\frac{\partial U}{\partial t} &= \frac{\partial^2 U}{\partial x^2}, \quad 0 < x < 1, \\
\frac{\partial U}{\partial x} &= C_1(U - V_1) \text{ at } x = 0, \quad t > 0 \\
\frac{\partial U}{\partial x} &= -C_2(U - V_2) \text{ at } x = 1, \quad t > 0 \\
U(x, 0) &= f(x), \quad 0 \leq x \leq 1,
\end{aligned}$$

where C_1, C_2, V_1 and V_2 are constants with $C_1 \geq 0, C_2 \geq 0$.

Approximate the BCs by centeral difference scheme

$$\begin{aligned}
\frac{u_{1,j} - u_{-1,j}}{2h} &= C_1(u_{0,j} - V_1), \\
\frac{u_{N+1} - u_{N-1}}{2h} &= -C_2(u_{N,j} - V_2)
\end{aligned}$$

and the PDE by the explicit scheme

$$u_{i,j+1} = ru_{i-1,j} + (1 - 2r)u_{i,j} + ru_{i+1,j}.$$

Elimination of $u_{-1,j}$ and $u_{N+1,j}$, leads to the system

$$\begin{bmatrix} u_{0,j+1} \\ u_{1,j+1} \\ \vdots \\ u_{N-1,j+1} \\ u_{N,j+1} \end{bmatrix} = \begin{bmatrix} \{1-2r(1+C_1 h)\} & 2r & & & \\ r & (1-2r) & r & & \\ \ddots & \ddots & \ddots & & \\ & & r & (1-2r) & r \\ & & & 2r & \{1-2r(1+C_2 h))\} \end{bmatrix} \begin{bmatrix} u_{0,j} \\ u_{1,j} \\ \vdots \\ u_{N-1,j} \\ u_{N,j} \end{bmatrix} + \begin{bmatrix} 2rC_1 V_1 h \\ 0 \\ \vdots \\ 0 \\ 2rC_2 V_2 h \end{bmatrix}$$

The amplification matrix (A) determining the propagation error is

$$\begin{bmatrix} \{1-2r(1+C_1 h)\} & 2r & & & \\ r & (1-2r) & r & & \\ \ddots & \ddots & \ddots & & \\ & & r & (1-2r) & r \\ & & & 2r & \{1-2r(1+C_2 h))\} \end{bmatrix}$$

Note: Since the off-diagonal elements of this real matrix A are one-signed, all its eigenvalues are real.

Theorem(Gerschgorin's theorem)
Let P_s be the sum of the moduli of the elements along the sth row excluding the diagonal element $a_{s,s}$. Then each eigenvalue of the matrix M lies inside or on the boundary of at least one of the circles

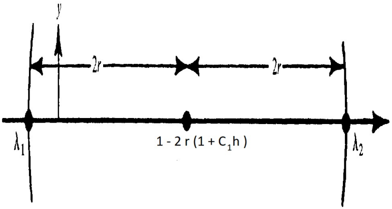
$$|\lambda - a_{s,s}| = P_s.$$

Application of the Gerschgorin's theorem to the amplification matrix A, with

$$a_{ss} = 1 - 2r(1 + C_1 h) \quad \text{and} \quad P_s = 2r$$

shows that some of its eigenvalues λ may lie on or within the circle

$$|\lambda - (1 - 2r(1 + C_1 h))| \leq 2r.$$



$$\lambda_1 = 1 - 2r(2 + C_1 h), \quad \lambda_2 = 1 - 2rC_1 h$$

For stability,

$$|\lambda_1 \leq 1, \quad |\lambda_2| \leq 1.$$

$$|\lambda_1| \leq 1 \quad \Longrightarrow \quad -1 \leq 1 - 2r(2 + C_1 h) \leq 1 \Longrightarrow r \leq \frac{1}{2 + C_1 h},$$

$$|\lambda_2| \leq 1 \quad \Longrightarrow \quad -1 \leq 1 - 2rC_1 h \leq 1 \Longrightarrow r \leq \frac{1}{C_1 h}.$$

The least of these is $r \leq \frac{1}{2 + C_1 h}$. Similarly, for the last row $i = N$, we have

$$r \leq \frac{1}{2 + C_2 h}.$$

For the row $i = 2, \ldots, N-1$, we have

$$a_{ss} = 1 - 2r, \quad P_s = 2r \Longrightarrow r \leq \frac{1}{2}.$$

Therefore, for overall stability,

$$r \leq \min\{\frac{1}{2 + C_1 h}; \ \frac{1}{2 + C_2 h}\}.$$

von Neumann's stability: This method applies to linear difference equation with constant coefficients. Strictly speaking only to initial value problems with periodic data.
Consider

$$\begin{aligned} U_t &= U_{xx}, \quad 0 \leq x \leq L, \ \ 0 < t \leq T, \\ U(x,0) &= f(x), \ \ 0 \leq x \leq L. \end{aligned}$$

Basic idea:

- Express initial data by finite fourier series along the initial mesh-point at $t = 0$.
- Study the growth of a function that reduced to this series at $t = 0$ by variable separable method.

The fourier series in complex exponential form can be expressed as

$$f(x) = \sum A_n e^{\frac{i\pi nx}{L}}, \quad i = \sqrt{-1}.$$

Notation: Let

$$\begin{aligned} x_p = ph, \ \ p = 0, 1, 2, \ldots, N \ \text{ with } Nh = L \\ t_q = qk, \ \ q = 0, 1, 2, \ldots, J \ \text{ with } Jk = T. \\ u(x_p, t_q) = u_{p,q} = u(ph, qk) \end{aligned}$$

Note that

$$A_n e^{\frac{i\pi n x_p}{L}} = A_n e^{\frac{i\pi nph}{Nh}} = A_n e^{i\beta_n ph}, \text{ where } \beta_n = \frac{\pi n}{Nh}.$$

At $q = 0$,

$$u_{p,0} = u(ph, 0) = \sum_{n=0}^{N} A_n e^{i\beta_n ph}.$$

Using the given $u_{p,0}$ (p=0,1,2,..., N), the constants A_n can be uniquely determined.
Since it is a linear problem, it is enough to investigate the propagation of only one initial value, such as $e^{i\beta ph}$, because separate solutions are additive. The coefficient A_n is a constant and can be neglected.

$$u_{p,q} = e^{i\beta x_p} e^{\alpha t_q} = e^{i\beta ph} \cdot e^{\alpha qk} = e^{i\beta ph} \cdot \xi^q,$$

where $\xi = e^{\alpha k}$, α is a constant.

By Lax-Richtmyer's definition of stability,

$$|u_{p,q}| \leq C \ \ \forall q \leq J,$$

as $h \to 0$ and $k \to 0$, and for all values of β needed to satisfy the initial conditions. Thus, $u_{p,q}$ remains bounded if

$$|\xi| \leq 1,$$

which is the necessary and sufficient condition for stability (von Neumann stability condition).

Example. Investigate the stability of Crank-Nicolson scheme

$$-ru_{p-1,q+1} + (2+2r)u_{p,q+1} - ru_{p+1,q+1} = ru_{p-1,q} + (2-2r)u_{p,q} + ru_{p+1,q}.$$

approximating $U_t = U_{xx}$.

Substituting $u_{p,q} = e^{i\beta ph}\xi^q$ in the difference equation, we obatin

$$\begin{aligned}
& e^{i\beta ph}\,\xi^{q+1}(-re^{-i\beta h} + (2+2r) - re^{i\beta h}) \\
& \qquad = e^{i\beta ph}\,\xi^{q}(re^{-i\beta h} + (2-2r) + re^{i\beta h}) \\
\implies \quad & \xi\{(2+2r) - 2r\cos(\beta h)\} = \{(2-2r) + 2r\cos(\beta h)\} \\
\implies \quad & \xi\{1 + r(1-\cos(\beta h))\} = \{1 - r(1-\cos(\beta h))\}
\end{aligned}$$

$$\Longrightarrow \qquad \xi = \frac{1 - 2r\sin^2(\frac{\beta h}{2})}{1 + 2r\sin^2(\frac{\beta h}{2})} \leq 1 \quad \forall r > 0, \text{ (always true)}$$

and for all β. Thus, von Neumann's stability condition is satisfied and hence the finite difference equations are unconditionally stable.

Exercise. Investigate the stability of the two-level explicit and Euler's implicit schemes approximating the heat equation $U_t = U_{xx}$ by von Neumann's stability method.

Remarks.

- Consistency is related to truncation error. We say a finite difference scheme is consistent with given PDE if the truncation error $T_{i,j} \to 0$ as $h \to 0$, $k \to 0$.
- Stability is related to the round-off error. We say that a finite difference scheme is stable if the exact solution of the finite difference equation does not grow with time.

Convergence. Let $U_{i,j}$ is the exact solution of the PDE at the mesh point (ih, jk), and let $u_{i,j}$ be its finite difference approximation. Then, we say $u_{i,j}$ converges to $U_{i,j}$ if $|u_{i,j} - U_{i,j}| \to 0$ as $h \to 0$ and $k \to 0$.

Lax Equivalence Theorem. Given a well-posed linear initial value problem and a consistent linear finite difference equation approximating it. Then stability is the necessary and sufficient condition for convergence. That is, for a consistent finite difference scheme,

$$\text{Stability} \iff \text{Convergence}$$

Remark.

$$\text{Consistency} + \text{Stability} \implies \text{Convergence}$$

*** Ends ***