Question 01

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Given q = Langrange poly of deg. n for $(x_0, y_0) \cdots (x_n, y_n)$

r = Langrange poly of deg n for $(x_1, y_1) \cdots (x_{n+1}, y_{n+1})$.

$q(x) = y_0 l_0(x) + \cdots y_n l_n(x)$
$r(x) = y_1 l_1'(x) + \cdots y_{n+1} l_{n+1}'(x)$.
} by def^n. with l and l' being basis.

Now consider, $\dfrac{(x-x_0) r(x) - (x - x_{n+1}) q(x)}{(x_{n+1} - x_0)}$. $(= p(x))$

We take 3 different cases —

i) for first term

$y_0 l_0(x)$ is multiplied by $\dfrac{-(x - x_{n+1})}{(x_{n+1} - x_0)}$.

It becomes

$$= \boxed{y_0 \frac{(x-x_1)(x-x_2) \cdots (x-x_n)(x-x_{n+1})}{(x_0-x_1) \cdots (x_0-x_n)(x_0-x_{n+1})}} \quad \text{— (1)}$$

→ taking -ve sign here.

ii) Last term

$y_{n+1} l'_{n+1}(x)$ is multiplied by $\dfrac{(x-x_0)}{(x_{n+1} - x_0)}$

It becomes

$$= \boxed{y_{n+1} \frac{(x-x_0)(x-x_1) \cdots (x-x_n)}{(x_{n+1}-x_0)(x_{n+1}-x_1) \cdots (x_{n+1}-x_n)}} \quad \text{— (2)}$$

(iii) Any general kth term (for k in 1, 2, 3 ... n)

$$\left(\frac{x-x_0}{x_{n+1}-x_0}\right) y_k l_k' - \frac{(x-x_{n+1})}{(x_{n+1}-x_0)} y_k l_k .$$

Solving this by taking LCM,

$$\frac{y_k (x-x_0)(x-x_1) \cdots (x-x_{k-1})(x-x_{k+1}) \cdots (x-x_{n+1}) \left[x_k - x_0 - x_k + x_{n+1}\right]}{(x_k-x_0)(x_k-x_1) \cdots (x_k-x_n)(x_{n+1}-x_0)}$$

$$= y_k \frac{(x-x_0) \cdots (x-x_{n+1})}{(x_k-x_0)(x_k-x_1) \cdots (x_k-x_n)} \frac{\cancel{(x_{n+1}-x_0)}}{\cancel{(x_{n+1}-x_0)}}$$

$$= \boxed{y_k \prod_{\substack{j \neq k \\ \in \{1,2,\ldots n+1\}}} \frac{(x-x_j)}{(x_k-x_j)}} \text{ ——— (3)}$$

From (1), (2) and (3), we have

$$\boxed{\text{p(x) is a Langrange polynomial of degree n+1 for points, } (x_0, y_0) \cdots (x_{n+1}, y_{n+1}).}$$

Thus proved!

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Part 2

$$E(h) = \frac{f(h) - f(-h)}{2h} + \left(\frac{\varepsilon_+ + \varepsilon_-}{2h}\right) - f'(0).$$

Using taylor's thm (since f is triple differentiable).

$$f(h) = f(0) + f'(0)h + f''(0)\frac{h^2}{2!} + \frac{f'''(\xi)}{3!}h^3 \text{ for some } \xi \in (-h, h)$$

Hence, $E(h)$

$$= \frac{\left(f(0) + f'(0)h + \frac{f''(0)h^2}{2} + \frac{f'''(\xi)h^3}{6}\right) - \left(f(0) - f'(0)h + \frac{f''(0)h^2}{2} - \frac{f'''(\xi)h^3}{6}\right)}{2h}$$

$$+ \frac{(\varepsilon_+ + \varepsilon_-)}{2h} - f'(0)$$

$$= \left(\frac{\varepsilon_+ + \varepsilon_-}{2h}\right) + \frac{h^3}{3}\frac{f'''(\xi)}{2h} + \cancel{f'(0)} - \cancel{f'(0)}$$

$$\Rightarrow \boxed{E(h) = \frac{1}{6}h^2 f'''(\xi) + \frac{\varepsilon_+ - \varepsilon_-}{2h}}$$

Part (3)

$$E(h) = \frac{1}{6}h^2 f'''(\xi) + \frac{\varepsilon_+ - \varepsilon_-}{2h}$$ (from last part)

$$|E(h)| \leq \left|\frac{1}{6}h^2 f'''(\xi)\right| + \left|\frac{\varepsilon_+ - \varepsilon_-}{2h}\right|$$

(since $|x+y| \leq |x| + |y|$)

$$\Rightarrow |E(h)| \leq \frac{1}{6}h^2 |f'''(\xi)| + \left|\frac{\varepsilon_+}{2h}\right| + \left|\frac{-\varepsilon_-}{2h}\right|$$

Let $|\varepsilon_+| = |\varepsilon_-| = \varepsilon$.

$$\Rightarrow |E(h)| \leq \frac{h^2}{6}|f'''(\xi)| + \frac{\varepsilon}{h}$$

Since $\xi \in (-h, h)$ and $f \in C^3[-h, h]$, so f''' is bounded,

$$\Rightarrow \boxed{|E(h)| \leq \frac{h^2}{6}M + \frac{\varepsilon}{h}}$$ for a constant $M > 0$.

RHL $= \frac{1}{6}h^2 M + \frac{\varepsilon}{h} = g(h)$

On differentiating, we get $h = \left(\frac{\varepsilon}{3M}\right)^{1/3}$ (we see $g''(h) > 0$ so, its local minima)

Minimum value of RHL

$$= \frac{1}{6}h^2 M + \frac{\varepsilon}{h} = \boxed{\frac{1}{6}\left(\frac{\varepsilon}{3}\right)^{2/3} M^{1/3} + \varepsilon^{2/3}(3M)^{1/3}}$$

Question 2

$Q(f) = C_{-1} f(-1) + C_0 f(0) + C_1 f(1) + C_2 f(2)$

Exact integral value $\int_0^1 f(x)\, dx$.

As f is a polynomial of degree 3,

$f(x) = x^0 \Rightarrow \int_0^1 1\, dx = 1 = C_{-1} + C_0 + C_1 + C_2$

$f(x) = x \Rightarrow \int_0^1 x\, dx = \frac{1}{2} = C_{-1}(-1) + C_0(0) + C_1(1) + C_2(2)$

$f(x) = x^2 \Rightarrow \int_0^1 x^2 dx = \frac{1}{3} = C_{-1}(1) + C_0(0) + C_1(1) + C_2(4)$

$f(x) = x^3 \Rightarrow \int_0^1 x^3 dx = \frac{1}{4} = C_{-1}(-1) + C_0(0) + C_1(1) + C_2(8)$

On solving the 4 eq^ns we get,

$$\boxed{C_{-1} = \frac{-1}{24},\quad C_0 = \frac{13}{24},\quad C_1 = \frac{13}{24},\quad C_2 = \frac{-1}{24}}$$