Lecture 19: Numerical Solutions to IVPs for ODEs (One-Step Methods)

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Consider a first-order IVP of the form

$$y' = f(x,y), \quad y(x_0) = y_0, \tag{1}$$

where the function f may be linear or nonlinear. Recall the following basic existence and uniqueness theorem.

Theorem. If $f(x,y)$ is continuous in a rectangle

$$R = \{(x,y) \mid \ |x - x_0| \le a, \ |y - y_0| \le b\},$$

satisfies a Lipschitz condition in the second variable

$$|f(x,y_1) - f(x,y_2)| \le L|y_1 - y_2|, \ \ \forall\,(x,y_1),(x,y_2) \in R.$$

Then the IVP (1) has a unique solution in the interval $|x - x_0| < \alpha$, where $\alpha = \min\{a, \frac{b}{M}\}$ and $M = \max_R |f(x,y)|$.

Taylor-Series Method

Expanding $y(x)$ into a Taylor series about the point $x = x_0$, we have

$$\begin{aligned} y(x) &= y(x_0) + (x - x_0)y'(x_0) + \frac{(x - x_0)^2}{2!}y''(x_0) + \cdots \\ &= y_0 + hy'(x_0) + \frac{h^2}{2!}y''(x_0) + \cdots , \end{aligned}$$

where $x - x_0 = h$. If f is sufficiently differentiable with respect to both x and y, we can compute

$$\begin{aligned} y' &= f(x, y) \\ y'' &= f' = f_x + f_y y' = f_x + f_y f \\ y''' &= f'' = f_{xx} + f_{xy} f + f_{yx} f + f_{yy} f^2 + f_y f_x + f_y^2 f \\ &= f_{xx} + 2f_{xy} f + f_{yy} f^2 + f_x f_y + f_y^2 f \end{aligned}$$

and so on.

Let y_n be approximations to the true solution $y(x_n)$ at the points $x_n = x_0 + nh$, $n = 0, 1, 2 \ldots$. That is, $y_n \approx y(x_n)$.

Taylor's algorithm of order k. To find an approximate solution of the IVP

$$y' = f(x, y), \quad y(x_0) = y_0$$

over an interval $[a, b]$.
Step 1: Choose a step $h = (b - a)/N$. Set

$$x_n = x_0 + nh, \quad n = 0, 1, 2, \ldots, N.$$

Step 2: Generate approximations y_n to $y(x_n)$ from the recursion

$$y_{n+1} = y_n + hT_k(x_n, y_n), \quad n = 0, 1, \ldots, N - 1,$$

where $T_k(x, y)$ is defined by

$$T_k(x, y) = f(x, y) + \frac{h}{2!} f'(x, y) + \cdots + \frac{h^{k-1}}{k!} f^{(k-1)}(x, y), \quad k = 1, 2, \ldots$$

Here, $f^{(j)}$ denotes the jth total derivative of the function $f(x, y(x))$ with respect to x.
Note that we calculate y at $x = x_{n+1}$ by using only information about y and its derivatives at the previous step $x = x_n$, are frequently called one-step methods/single step methods.

Taylor's theorem with remainder shows that the local error of Taylor's algorithm of order k is

$$E = \frac{h^{k+1}}{(k+1)!} f^{(k)}(\xi, y(\xi)) = \frac{h^{k+1}}{(k+1)!} y^{(k+1)}(\xi), \; x_n < \xi < x_n + h.$$

The Taylor algorithm is said to be of order k if the local error E is $O(h^{(k+1)})$.

Example. Using Taylors series, find an approximate solution of the IVP

$$xy' = x - y, \quad y(2) = 2$$

at $x = 2.1$ correct to five decimal places.

The first few derivatives and their values at $x_0 = 2$, $y_0 = 2$ are

$$\begin{aligned} y'(x) &= 1 - \frac{y}{x}, & y_0' &= 0 \\ y''(x) &= \frac{-y'}{x} + \frac{y}{x^2}, & y_0'' &= \frac{1}{2} \end{aligned}$$

$$y'''(x) = \frac{-y''}{x} + \frac{2y'}{x^2} - \frac{2y}{x^3}, \qquad y_0''' = -\frac{3}{4}$$
$$y^{iv}(x) = \frac{-y'''}{x} + \frac{3y''}{x^2} - \frac{6y'}{x^3} + \frac{6y}{x^4}, \qquad y_0^{iv} = \frac{3}{2}$$

The Taylor series expansion about $x_0 = 2$ is

$$\begin{aligned} y(x) &= y_0 + (x-2)y_0' + \frac{1}{2}(x-2)^2 y_0'' + \frac{1}{6}(x-2)^3 y_0''' \\ &\quad + \frac{1}{24}(x-2)^4 y_0^{iv} + \cdots \\ &= 2 + (x-2)0 + \frac{1}{4}(x-2)^2 - \frac{1}{8}(x-2)^3 + \frac{1}{16}(x-2)^4 + \cdots \end{aligned}$$

At $x = 2.1$,

$$y(2.1) = 2 + 0.0025 - 0.000125 + 0.0000062 - \cdots \approx 2.00238$$

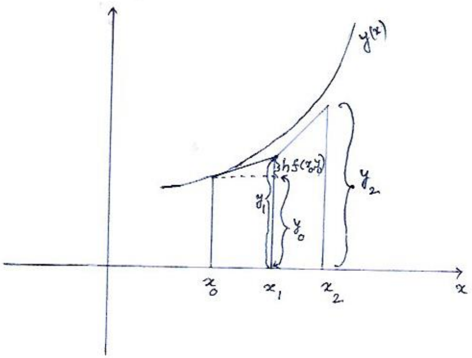
Euler's Method

Setting $k = 1$ (Taylor's algorithm of order 1), we obtain

$$y_{n+1} = y_n + h\,f(x_n, y_n) \tag{2}$$

which is known as Euler's method. The local error is

$$E = \frac{h^2}{2} y''(\xi).$$



Example. Consider the IVP:

$$y' = y, \quad y(0) = 1.$$

Here $f(x, y) = y$, $x_0 = 0$ and $y_0 = 1$. Apply Eulers method with $h = 0.01$ and retaining six decimal places, we obtain

$$\begin{aligned}
y(0.01) &\approx y_1 = y_0 + hf(x_0, y_0) = 1 + 0.01 = 1.01 \\
y(0.02) &\approx y_2 = y_1 + hf(x_1, y_1) = 1.01 + 0.01(1.01) = 1.0201 \\
y(0.03) &\approx y_3 = 1.0201 + 0.01(1.0201) = 1.030301 \\
y(0.04) &\approx y_4 = 1.030301 + 0.01(1.030301) = 1.040606
\end{aligned}$$

The exact solution of this equation is $y = e^x$ and $y(0.04) = 1.0408$.

Theorem (Error estimate). Let y_n be the approximate solution of

$$y' = f(x, y), \quad y(x_0) = y_0$$

generated by Eulers method. If $y'' \in C([x_0, b])$ and

$$|f_y(x, y)| < L, \qquad |y''(x)| < M, \;\; x \in [x_0, b],$$

for fixed positive constants L and M, then the error $e_n = y(x_n) - y_n$ of Eulers method at a point $x_n = x_0 + nh$ is bounded as follows:

$$|e_n| \leq \frac{hM}{2L}\left[e^{(x_n - x_0)L} - 1\right].$$

This theorem shows that the error is $O(h)$ and $e_n \to 0$ as $h \to 0$.
Example. Determine an upper bound for the discretization error of Eulers method in solving the equation $y' = y, \;\; y(0) = 1$ from $x = 0$ to $x = 1$. Here $f(x, y) = y$, hence we can take $L = 1$. Since $y = e^x$, then $y'' = e^x$ and $|y''(x)| < e$ for $0 < x < 1$. With $M = e$ and $x_n - x_0 = 1$, we have

$$|e(1)| \leq \frac{he}{2}(e - 1) < 2.4h.$$

*** Ends ***