Lecture 28: Stability Analysis by Matrix Method

Rajen Kumar Sinha

Department of Mathematics
IIT Guwahati

We shall investigate stability of finite difference equations by

- Matrix method
- von Neumann's method (Fourier series method)

Some elements from matrix theory. Let A be an $n \times n$ matrix. Define

$$\|A\|_1 = \max_{1 \le j \le n} \sum_{i=1}^{n} |a_{ij}|$$

$$\|A\|_\infty = \max_{1 \le i \le n} \sum_{j=1}^{n} |a_{ij}|$$

$$\|A\| = \max_{x \neq 0} \frac{\|Ax\|}{\|x\|} = \max_{\|x\|=1} \|Ax\|$$

Let $v_1, v_2, \ldots, v_n$ be the eigenvectors corresponding to the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ of A. Then

$$\begin{aligned}
& Av_i = \lambda_i v_i \implies \|Av_i\| = \|\lambda_i v_i\| = |\lambda_i| \|v_i\| \\
\text{or} \quad & |\lambda_i| \|v_i\| = \|Av_i\| \le \|A\| \|v_i\| \implies |\lambda_i| \le \|A\|, \quad i = 1(1)N. \\
\implies \quad & \max_i |\lambda_i| \le \|A\| \implies \rho(A) \le \|A\|, \quad \rho(A) = \max_i |\lambda_i|,
\end{aligned}$$

where $\rho(A)$ (spectral radius) is the largest eigenvalue of A.

The Lax-Richtmyer stability condition implies

$$\|A\| \leq 1 \implies \rho(A) \leq 1.$$

Note: The converse is not true. That is,

$$\rho(A) \leq 1 \not\Rightarrow \|A\| \leq 1.$$

Example. $A = \begin{bmatrix} -0.7 & 0 \\ 0.5 & 0.6 \end{bmatrix}$

$$\lambda_1 = -0.7,\ \lambda_2 = 0.6,\ \rho(A) = 0.7$$
$$\|A\|_1 = 1.2, \quad \|A\|_\infty = 1.1.$$

However, if A is real and symmetric, then

$$\|A\| = \sqrt{\rho(A^T A)} = \sqrt{\rho(A^2)} = \sqrt{\rho^2(A)} = \rho(A).$$

The eigenvalues of an $n \times n$ matrix

$$\begin{bmatrix} a & b & & & \\ c & a & b & & \\ \ddots & \ddots & \ddots & & \\ & & c & a & b \\ & & & c & a \end{bmatrix}$$

are given by

$$\lambda_s = a + 2\{\sqrt{(bc)}\} \cos\left(\frac{s\pi}{n+1}\right), \quad s = 1(1)n,$$

where $a, b, c \in \mathbb{R}$.

Example. Investigate the stability of the classical explicit scheme

$$u_{i,j+1} = ru_{i-1,j} + (1-2r)u_{i,j} + ru_{i+1,j}, \quad i = 1(1)N-1.$$

Assume that the boundary values $u_{0,j}$ and $u_{N,j}$ are known for $j = 1, 2, \ldots$. For $i = 1(1)N-1$, we have

$$\begin{bmatrix} u_{1,j+1} \\ u_{2,j+1} \\ \vdots \\ u_{N-2,j+1} \\ u_{N-1,j+1} \end{bmatrix} = \begin{bmatrix} (1-2r) & r & & & \\ r & (1-2r) & r & & \\ \ddots & \ddots & \ddots & & \\ & & r & (1-2r) & r \\ & & & r & (1-2r) \end{bmatrix} \begin{bmatrix} u_{1,j} \\ u_{2,j} \\ \vdots \\ u_{N-2,j} \\ u_{N-1,j} \end{bmatrix} + \begin{bmatrix} ru_{0,j} \\ \vdots \\ ru_{N,j} \end{bmatrix}$$

The amplification matrix A is

$$A = \begin{bmatrix} (1-2r) & r & & & \\ r & (1-2r) & r & & \\ \ddots & \ddots & \ddots & & \\ & & r & (1-2r) & r \\ & & & r & (1-2r) \end{bmatrix}$$

When $1-2r \geq 0$ then $0 < r \leq 1/2$ and

$$\|A\|_\infty = r + (1-2r) + r = 1.$$

When $1-2r < 0$, $r > 1/2$, $|1-2r| = 2r-1$, and

$$\|A\|_\infty = r + 2r - 1 + r = 4r - 1 > 1.$$

Therefore, the scheme is stable for $0 < r \leq 1/2$. Further, since A is real and symmetric

$$\|A\| = \rho(A) = \max_s |\mu_s|,$$

where μ_s is the s^{th} eigenvalue of A.

The matrix A can be rewritten as

$$A = I_{N-1} + rT_{N-1}, \quad \text{where}$$

$$I_{N-1} = \begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ & & & & 1 \end{bmatrix}, \quad T_{N-1} = \begin{bmatrix} -2 & 1 & & & \\ 1 & -2 & 1 & & \\ & & \ddots & & \\ & & 1 & -2 & 1 \\ & & & 1 & -2 \end{bmatrix}.$$

The eigenvalues of T_{N-1} are

$$\lambda_s = -2 + 2\cos(\frac{s\pi}{N}) = -4\sin^2(\frac{s\pi}{2N}), \;\; s = 1(1)N-1.$$

Thus, the eigenvalues of A are

$$\mu_s = 1 - 4r\sin^2(\frac{s\pi}{2N}), \;\; s = 1(1)N-1.$$

For stability, we must have $\|A\| \leq 1$. That is, $\max_s |\mu_s| \leq 1$

$$\text{i.e.,} \qquad \max_s |1 - 4r\sin^2(\frac{s\pi}{2N})| \leq 1$$
$$\text{i.e.,} \qquad -1 \leq 1 - 4r\sin^2(\frac{s\pi}{2N}) \leq 1, \quad s = 1(1)N-1.$$

The left-hand side inequality

$$-1 \leq 1 - 4r\sin^2(\frac{s\pi}{2N}) \implies r \leq \frac{1}{2\sin^2(\frac{(N-1)\pi}{2N})}$$

As $h \to 0$, $N \to \infty$ and $\sin^2(\frac{(N-1)\pi}{2N}) \to 1$. This implies $r \leq \frac{1}{2}$.

Therefore, the explicit scheme is stable for $0 < r \leq 1/2$. (i.e., Conditionally stable).

Example. Investigate the stability of Euler's implicit scheme:

$$-ru_{i-1,j+1} + (1+2r)u_{i,j} - ru_{i+1,j+1} = u_{i,j}.$$

For $i = 1(1)N-1$, we have

$$\begin{bmatrix} (1+2r) & -r & & & \\ -r & (1+2r) & -r & & \\ \ddots & \ddots & \ddots & & \\ & & -r & (1+2r) & -r \\ & & & -r & (1+2r) \end{bmatrix} \begin{bmatrix} u_{1,j+1} \\ u_{2,j+1} \\ \vdots \\ u_{N-2,j+1} \\ u_{N-1,j+1} \end{bmatrix} = \begin{bmatrix} u_{1,j} \\ u_{2,j} \\ \vdots \\ u_{N-2,j} \\ u_{N-1,j} \end{bmatrix} + \begin{bmatrix} ru_{0,j+1} \\ 0 \\ \vdots \\ 0 \\ ru_{N,j+1} \end{bmatrix}$$

Observe that

$$\begin{bmatrix} (1+2r) & -r & & & \\ -r & (1+2r) & -r & & \\ \ddots & \ddots & \ddots & & \\ & & -r & (1+2r) & -r \\ & & & -r & (1+2r) \end{bmatrix} = I_{N-1} - rT_{N-1}.$$

The amplification matrix is

$$A = (I_{N-1} - rT_{N-1})^{-1}.$$

The eigenvalues of A are

$$\mu_s = \frac{1}{1 + 4r\sin^2(\frac{s\pi}{2N})}, \quad s = 1(1)N-1$$

Since A is symmetric, we have

$$\begin{aligned} & \|A\| = \rho(A) = \max_s |\mu_s| \le 1 \\ \Longrightarrow \quad & \frac{1}{1 + 4r\sin^2(\frac{s\pi}{2N})} \le 1, \quad \forall r > 0, \end{aligned}$$

which proves the scheme is unconditionally stable.

Exercise. Investigate the stability of the Crank-Nicolson scheme.

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