Lecture 24: Finite Difference Approximations to Derivatives

Rajen Kumar Sinha

Department of Mathematics
IIT Guwahati

Consider the two-dimensional second-order linear PDE:

$$a\frac{\partial^2 U}{\partial x^2} + b\frac{\partial^2 U}{\partial x \partial y} + c\frac{\partial^2 U}{\partial y^2} + d\frac{\partial U}{\partial x} + e\frac{\partial U}{\partial y} + fU + g = 0, \tag{1}$$

where a, b, c, d, e, f, and g may be functions of the independent variables x and y. $U = U(x, y)$ is the dependent variable.

Classifications: The PDE (1) is said to be

- **Elliptic** when $b^2 - 4ac < 0$.
- **Parabolic** when $b^2 - 4ac = 0$.
- **Hyperbolic** when $b^2 - 4ac > 0$.

Elliptic equations: These problems are generally associated with equilibrium or steady-state problems. For example,

- $\dfrac{\partial^2 U}{\partial x^2} + \dfrac{\partial^2 U}{\partial y^2} = 0$ (Laplace's equation)

  The velocity potential V for the steady flow of incompressible non-viscous fluid satisfies Laplace's equation.

- $\dfrac{\partial^2 U}{\partial x^2} + \dfrac{\partial^2 U}{\partial y^2} = f(x, y)$ (Poisson's equation)

  The electric potential V associated with a two-dimensional electron distribution of charge density ρ satisfies Poisson's equation with $f = -\rho/\epsilon$, where ϵ is a dielectric constant.

Parabolic equations: The heat equation

$$\frac{\partial U}{\partial t} = \kappa \frac{\partial^2 U}{\partial x^2}$$

is the simplest example of parabolic equation, where U represents the temperature in a rod at a distance x unit of length after t seconds of heat conduction.

Hyperbolic equations: These equations generally originate from vibration problems, or from problems where the discontinuities can persist in time. The simplest hyperbolic equation is one-dimensional wave equation:

$$\frac{\partial^2 U}{\partial t^2} = c^2 \frac{\partial^2 U}{\partial x^2}.$$

Finte Difference Approximations to Derivatives

Functions of one-variable: Let $U : [a, b] \to \mathbb{R}$ be sufficinetly differentiable function. Let

$$a = x_0 < x_1 < x_2 < \cdots < x_n = b$$

be a partition of $[a, b]$ such that $x_n = x_0 + nh$, where $h = (x_n - x_0)/n$ is the discretization parameter. Set $x_i = x_0 + ih, \;\; i = 0, 1, \ldots, n$ and $U_i = U(x_i)$.

By Taylor's theorem

$$U(x+h) = u(x) + hU'(x) + \frac{h^2}{2}U''(x) + \frac{h^3}{6}U'''(x) + \cdots . \quad (2)$$

$$U(x-h) = u(x) - hU'(x) + \frac{h^2}{2}U''(x) - \frac{h^3}{6}U'''(x) + \cdots . \quad (3)$$

$$
\begin{aligned}
\left.\frac{dU}{dx}\right|_{x=x_i} &= \frac{U(x_i+h)-U(x_i)}{h} + O(h) \quad \text{(From (2))} \\
&\approx \frac{U_{i+1}-U_i}{h}, \quad \text{(Forward difference formula)} \\
&= \frac{U(x_i)-U(x_i-h)}{h} + O(h) \quad \text{(From (3))} \\
&\approx \frac{U_i-U_{i-1}}{h} \quad \text{(Backward difference formula)} \\
&= \frac{U(x_i+h)-U(x_i-h)}{2h} + O(h^2) \quad \text{(From (2)-(3))} \\
&\approx \frac{U_{i+1}-U_{i-1}}{2h} \quad \text{(Central difference formula)}
\end{aligned}
$$

$$
\begin{aligned}
\left.\frac{d^2U}{dx^2}\right|_{x=x_i} &= \frac{U(x_i+h)-2U(x_i)+U(x_i-h)}{h^2} + O(h^2) \\
&\approx \frac{U_{i+1}-2U_i+U_{i-1}}{h^2}.
\end{aligned}
$$

Functions of two-variables: Let $U : [0, a] \times [0, b] \to \mathbb{R}$ be a differentiable function of x and t. Introduce the mesh parameters h and k in the directions of x and t, respectively. Denote

$$x_i = ih, \quad i = 0, 1, 2, \ldots, N \text{ with } x_0 = 0, \ x_N = a.$$

$$t_j = jk, \quad j = 0, 1, 2, \ldots, J \text{ with } t_0 = 0, \ t_J = b.$$

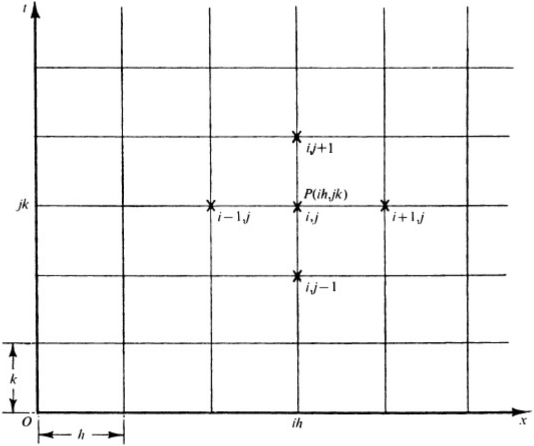
Notation: Set

$$U_{i,j} = U(x_i, t_j) = U(ih, jk), \ U_{i+1,j} = U(x_i + h, t_j) = U((i+1)h, jk),$$

$$U_{i-1,j} = U(x_i - h, t_j) = U((i-1)h, jk),$$

$$U_{i,j+1} = U(x_i, t_j + k) = U(ih, (j+1)k),$$

$$U_{i,j-1} = U(x_i, t_j - k) = U(ih, (j-1)k).$$



(Discretization of the domain)

$$
\begin{aligned}
\left.\frac{\partial U}{\partial x}\right|_{(x_i,t_j)} &= \frac{U_{i+1,j}-U_{i,j}}{h}+O(h)\\
&= \frac{U_{i,j}-U_{i-1,j}}{h}+O(h)\\
&= \frac{U_{i+1,j}-U_{i-1,j}}{2h}+O(h^2)\\
\left.\frac{\partial U}{\partial t}\right|_{(x_i,t_j)} &= \frac{U_{i,j+1}-U_{i,j}}{k}+O(k)\\
&= \frac{U_{i,j}-U_{i,j-1}}{k}+O(k)\\
&= \frac{U_{i,j+1}-U_{i,j-1}}{2k}+O(k^2)\\
\left.\frac{\partial^2 U}{\partial x^2}\right|_{(x_i,t_j)} &= \frac{U_{i+1,j}-2U_{i,j}+U_{i-1,j}}{h^2}+O(h^2)\\
\left.\frac{\partial^2 U}{\partial t^2}\right|_{(x_i,t_j)} &= \frac{U_{i,j+1}-2U_{i,j}+U_{i,j-1}}{k^2}+O(k^2)
\end{aligned}
$$

*** Ends ***