Q1 ~~$E_\lambda[h$~~ $T = (X_1, X_2)$

$$E_\lambda[h(T)] = \int_{t \in T} h(t) \frac{e^{-\lambda}\lambda^{x_1}}{x_1!} \frac{e^{-\lambda}\lambda^{x_2}}{x_2!} dt$$

If we define
$$h(t) = \begin{cases} 1 & \text{for } x_1 > x_2 \\ -1 & \text{for } x_1 < x_2 \\ 0 & \text{for } x_1 = x_2 \end{cases}$$

this integral
$$E_\lambda[h(T)] = \left( e^{-2\lambda} \int_{t \in T} \frac{h(t)\,\lambda^{x_1+x_2}}{x_1!\,x_2!} dt \right)$$ is 0

but $h(T)$ is not equal to 0 with propability 1.

Therefore family induced by $T$ is not complete.

② $\mu_1 = E[X] = \int_{\theta_1}^{\theta_2} x \frac{1}{\theta_2-\theta_1}\, dx = \frac{\theta_2^2-\theta_1^2}{2(\theta_2-\theta_1)}$

$= \frac{\theta_1+\theta_2}{2}$

$\mu_2 = E[x^2] = \int_{\theta_1}^{\theta_2} x^2 \times \frac{1}{(\theta_2-\theta_1)}\, dx = \frac{\theta_2^3-\theta_1^3}{3(\theta_2-\theta_1)}$

$= \frac{\theta_1^2+\theta_1\theta_2+\theta_2^2}{3}$

By method of moments –

$\therefore \frac{\theta_1+\theta_2}{2} = \bar{x}$

$\frac{\theta_1^2+\theta_1\theta_2+\theta_2^2}{3} = \frac{\sum x_i^2}{n}$

$\theta_1 = 2\bar{x} - \theta_2$

$\frac{4\bar{x}^2+\theta_2^2-4\bar{x}\,\theta_2+(2\bar{x}-\theta_2)\theta_2+\theta_2^2}{3} = \frac{\sum x_i^2}{9}$

$3\left(4\bar{x}^2+\theta_2^2-2\bar{x}\,\theta_2\right) = \sum x_i^2$

$3(\theta_2-\bar{x})^2 = \sum x_i^2 - 9\bar{x}^2$

$\theta_2 = \bar{x} + \sqrt{\frac{\sum x_i^2}{3} - 3\bar{x}^2}$

$\theta_1 = \bar{x} - \sqrt{\frac{\sum x_i^2}{3} - 3\bar{x}^2}$

(3) (a) $f(x, \theta) = \frac{2^n}{\theta^n} x^n \exp\left[-\frac{\sum x_i^2}{\theta}\right]$

$$\ln f(x, \theta) = n \ln 2 + n \ln(x) - n \ln(\theta) - \frac{\sum x_i^2}{\theta}$$

$$\frac{\partial \ln f(x, \theta)}{\partial \theta} = 0 \Rightarrow -\frac{n}{\theta} + \frac{\sum x_i^2}{\theta^2} = 0$$

$$\frac{n}{\theta} = \frac{\sum x_i^2}{\theta^2}$$

$$\theta = \frac{\sum x_i^2}{n}$$

MLE for $T(\theta) = \frac{1}{\sqrt{\theta}} = \frac{1}{\sqrt{\frac{\sum x_i^2}{n}}} = \sqrt{\frac{n}{\sum x_i^2}}$

(3) (c)

$$f(x,\theta) = \frac{2}{\theta} x \exp\left(\frac{-x^2}{\theta}\right)$$

$$\ln f(x,\theta) = \ln\left(\frac{2}{\theta}\right) + \ln(x) - \frac{x^2}{\theta}$$

$$\frac{\partial}{\partial\theta} \ln f(x,\theta) = \frac{1}{(2/\theta)}\left(\frac{-2}{\theta^2}\right) + \frac{x^2}{\theta^2}$$

$$= \frac{-1}{\theta} + \frac{x^2}{\theta^2}$$

$$\frac{\partial^2}{\partial\theta^2} \ln f(x,\theta) = \frac{1}{\theta^2} - \frac{2x^2}{\theta^3}$$

$$E\left[\frac{\partial^2}{\partial\theta^2} \ln f(x,\theta)\right] = \frac{1}{\theta^2} - \frac{2}{\theta^3} E[X^2]$$

$$\boxed{E[X^2] = \int_0^\infty \frac{2}{\theta} x^2 \, x \exp\left(\frac{-x^2}{\theta}\right) dx = \theta}$$

$$\therefore \quad E\left[\frac{\partial^2}{\partial\theta^2} \ln f(x,\theta)\right] = \frac{1}{\theta^2} - \frac{2}{\theta^2} = \frac{-1}{\theta^2}$$

$$I_{X_1}(\theta) = -E\left[\frac{\partial^2}{\partial\theta^2} \ln(f(x,\theta))\right] = \frac{1}{\theta^2}$$

Or $\tau(\theta) = \frac{1}{\sqrt{\theta}}$

$\tau'(\theta) = \frac{-1}{2\theta\sqrt{\theta}}$

Cramer Rao lower bound $= \frac{(\tau'(\theta))^2}{n I_{X_1}}$

$= \frac{\frac{1}{4\theta^3}}{n\frac{1}{\theta^2}} = \frac{1}{4n\theta}$.

4. Likelihood function for Berouilli RV is

$$L = \prod_{i=1}^{n} f(x,p) = p^{\sum x_i} (1-p)^{n-\sum x_i}$$

$$\log L = \sum x_i \ln p + (n - \sum x_i) \ln(1-p)$$

$$\frac{\partial}{\partial p} \log L = 0 \Rightarrow \frac{\sum x_i}{p} + \frac{(n-\sum x_i)(-1)}{(1-p)} = 0$$

$$\Rightarrow p = \frac{\sum x_i}{n}$$

$$p = \frac{1}{1+e^{\theta}}$$

$$1+e^{\theta} = \frac{1}{p}$$

$$e^{\theta} = \frac{1}{p} - 1$$

$$\theta = \ln\left(\frac{1}{p} - 1\right)$$

$\therefore$ MLE for $\theta = \ln\left(\frac{n}{\sum x_i} - 1\right)$

by invariance property of MLE

MLE exist if $\frac{n}{\sum x_i} - 1 > 0$

$$\frac{1}{\bar{X}} > 1$$

$$\underline{\bar{X} < 1}$$

(5) Likelihood function

$$L(\theta, n) = \frac{1}{\theta^n} e^{-\frac{\sum x_i}{\theta}} \quad \text{for } \min_i x_i > 0$$

$$\Theta_0 = \{\theta_0\}$$

$$\Theta_0 \cup \Theta_1 = \mathbb{R}^+$$

$$\sup_{\theta \in \Theta_0} L(\theta) = L(\theta_0) = \frac{1}{\theta_0^n} e^{-\frac{\sum x_i}{\theta_0}}$$

$$\sup_{\theta \in \Theta_0 \cup \Theta_1} L(\theta) = L(\bar{x}) = \frac{1}{(\bar{x})^n} e^{-\left(\frac{\cancel{\sum x_i}\, n}{\cancel{\sum x_i}}\right)}$$

$$= \frac{e^{-n}}{(\bar{x})^n}$$

$$\log(L(\theta)) = -n \ln \theta - \frac{\sum x_i}{\theta}$$

for MLE –

$$\frac{\partial \log(L(\theta))}{\partial \theta} = 0 = -\frac{n}{\theta} + \frac{\sum x_i}{\theta^2} = 0$$

$$\theta = \frac{\sum x_i}{n} = \bar{x}$$

$$\frac{\partial^2 \log(L(\theta))}{\partial \theta^2} = \frac{n}{\theta^2} - \frac{2 \sum x_i}{\theta^3} = \frac{1}{\theta^2}\left(n - \frac{2n\bar{x}}{\theta}\right)$$

$$\frac{\partial^2}{\partial \theta^2} \log(L(\theta)) \text{ at } \theta = \bar{x} = \frac{1}{\bar{x}^2}(n - 2n) = \frac{-n}{\bar{x}^2} < 0$$

$\therefore$ MLE for $\theta = \hat{\theta} = \bar{x}$

$$\Lambda(x) = \frac{\sup_{\theta \in \Theta_0} L(\theta)}{\sup_{\theta \in \Theta_0 \cup \Theta_1}} = \frac{\frac{1}{\theta_0^n} e^{-\frac{\sum x_i}{\theta_0}}}{\frac{e^{-n}}{(\bar{X})^n}}$$

$$\Lambda = \left(\frac{\bar{x}}{\theta_0}\right)^n e^{n - \frac{\sum x_i}{\theta_0}} = \left(\frac{\bar{x}}{\theta_0}\right)^n e^{n - \frac{n\bar{x}}{\theta_0}}$$

We would reject the null hypothesis iff

$$\Lambda(x) < K \iff \left(\frac{\bar{x}}{\theta_0}\right)^n e^{n - \frac{n\bar{x}}{\theta_0}} < K$$

Taking log both side as it as an inc. $f^n$

$$\iff n \ln\left(\frac{\bar{x}}{\theta_0}\right) + n\left(1 - \frac{\bar{x}}{\theta_0}\right) < \ln K$$

$$\iff \ln\left(\frac{\bar{X}}{\theta}\right) - \frac{\bar{x}}{\theta_0} < \frac{\ln K}{n} - 1$$

let $\frac{\ln K}{n} - 1 = K_1$

and $\hat{x} = \frac{\bar{x}}{\theta_0}$

$\therefore$ ~~the~~ likelihood ratio level $\alpha$ test is -

$$\psi(x) = \begin{cases} 1 & \text{if } \ln \hat{x} - \hat{x} < k_1 \\ 0 & \text{o.w} \end{cases}$$

~~For k~~ For $k_1$ —

$$E_{\theta_0}(\psi(x)) = \alpha \iff P_{\theta_0}(\ln \hat{x} - \hat{x} < k_1) = \alpha$$