CONTINUOUS SYSTEMS

Lecture Outline
Transverse Vibration of Strings
Longitudinal Vibration of Rods
Torsional Vibration of Shafts
Variational Formulations
Bending Vibration of Beams
Membranes and Plates
Traveling Forces and Elastic Foundation

A continuous system has distributed mass, damping, and elasticity. To specify the position of every point in a continuous system, an infinite number of coordinates is needed. Hence a continuous system possesses an infinite DOF. In general, the vibration of a continuous system is governed by a partial differential equation with appropriate initial values and boundary conditions. There is not a prototype equation of motion for all continuous systems.

The equation of motion of a continuous system can be derived by either the Newtonian or the analytical formulation. In the Newtonian approach, a force balance is applied to an infinitesimal control element of the system. In the analytical formulation, Hamilton's principle is used, which generates both the equation of motion as well the associated boundary conditions. Lagrange's equations cannot be used because a continuous system has an infinite DOF. In general, continuous systems cannot be solved exactly and numerical solution is often necessary.

5.1 Transverse Vibration of Strings

Consider a uniform string with constant linear density (mass per unit length) ρ stretched to a length l under constant tension T. Let

$y(x,t)$ = transverse displacement at position x from equilibrium configuration
$f(x,t)$ = external transverse force per unit length

Assumptions
(a) An ideal string is perfectly flexible with negligible bending stiffness. The restoring forces during vibration are derived entirely from axial tension. (b) The tension T is great enough so that the equilibrium configuration of the stretched string is a straight line. (c) The transverse deflection is small enough so that T remains unchanged with deflection.

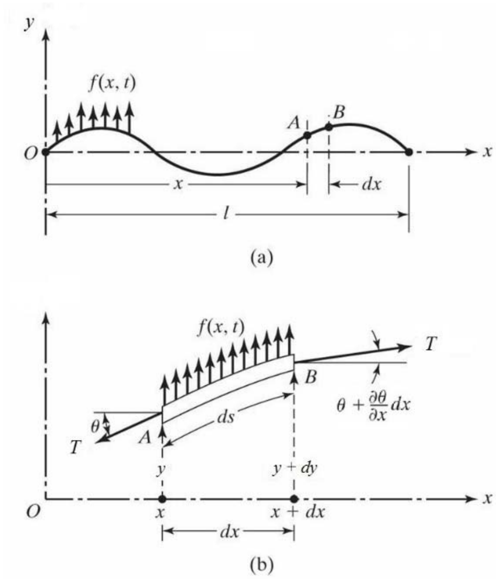
For an infinitesimal element of the string, the equation of motion in the transverse direction is

$$T\sin\left(\theta+\frac{\partial\theta}{\partial x}dx\right)-T\sin\theta+f(x,t)dx=\rho dx\frac{\partial^2 y}{\partial t^2}$$

In a small displacement, $\theta\cong\sin\theta\cong\tan\theta=\partial y/\partial x$. As a result,

$$T\frac{\partial\theta}{\partial x}dx + f(x,t)dx = \rho dx\frac{\partial^2 y}{\partial t^2}$$

$$\Rightarrow \qquad T\frac{\partial^2 y}{\partial x^2} + f(x,t) = \rho\frac{\partial^2 y}{\partial t^2}$$



The equation of free vibration is

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2}, \qquad c = \sqrt{\frac{T}{\rho}}$$

which is the one-dimensional **wave equation**, and c is a phase velocity associated with wave propagation along the string. The axial force in the string during transverse vibration is equal to

$$\frac{\partial}{\partial x}\left[T(x)\frac{\partial y}{\partial x}\right] + f(x,t) = \rho(x)\frac{\partial^2 y}{\partial t^2}$$

Some Definitions Involving Partial Differential Equations

The order of a PDE is that of the highest-order derivative present. The general solution of a PDE contains a number of arbitrary independent functions equal to the order of the equation (arbitrary constants for ODE). A boundary-value problem consists of a system of PDE together with values assigned on the physical boundary of the domain in which the problem is specified. A BVP may admit a unique solution if both initial values and boundary conditions are specified.

Linear Partial Differential Equations

The general linear PDE of order two in two independent variables has the form

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x\partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu = G$$

where A, B,…, G may depend on x and y but not on u. A second-order PDE that does not have the above form is called nonlinear. If $G = 0$ the equation is called homogeneous, while if $G \neq 0$ it is called nonhomogeneous. The general solution of a linear nonhomogeneous PDE is obtained by adding a particular solution of the nonhomogeneous equation to the general solution of the homogeneous equation (same property for linear ODE). If A, B,…, F are constants, then the general solution of the homogeneous equation can be found by assuming that $u = \exp(ax + by)$ where a and b are constants to be determined.

Because of the nature of the solutions, the equation is classified as elliptic if $B^2 - 4AC < 0$, parabolic if $B^2 - 4AC = 0$, and hyperbolic if $B^2 - 4AC > 0$. A similar terminology is used in the classification of conic sections. Any rotated ellipse, centered at the origin, is represented by a polynomial of the form

$$Ax^2 + Bxy + Cy^2 = 1$$

where $B^2 - 4AC < 0$ (there are linear terms in x or y if center of the ellipse is not at the origin). The conic section is a parabola if $B^2 - 4AC = 0$, it is a hyperbola if $B^2 - 4AC > 0$. For a uniform string, $B^2 - 4AC = T\rho > 0$ and the wave equation is hyperbolic.

Initial Values and Boundary Conditions

Initial values of a vibrating string may be specified by the initial profile $y(x, 0)$ and initial velocity $\dot{y}(x, 0)$. Boundary conditions are provided by the manner in which a string is held at its boundaries $x = 0$ and $x = l$. If a string is fixed at the end $x = 0$, then

$$y(0, t) = 0$$

This BC is called a **geometric** or essential BC since it imposes a geometric requirement of zero displacement at the boundary. Suppose a string has a free end at $x = l$. A transverse force is not exerted at the free end, therefore

$$(T \sin\theta)_{x=l} \cong \left(T\frac{\partial y}{\partial x}\right)_{x=l} = 0$$

$$\Rightarrow \qquad \frac{\partial y}{\partial x}(l, t) = 0$$

This BC is called a **dynamic** or natural BC since it imposes a force or moment balance at the boundary. The solution of a BVP contains arbitrary constants if only BC but not IV are specified.

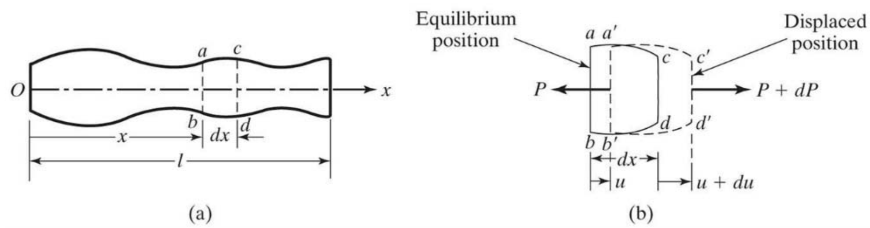
5.2 Longitudinal Vibration of Rods

A string can only store energy in tension. In comparison, a bar can store energy in compression, shear and bending. Thus a bar can sustain longitudinal, torsional and flexural vibration.

Consider a thin uniform rod with constant mass density ρ and constant cross-sectional area A in axial vibration. Let

$u(x,t)$ = longitudinal displacement at position x from equilibrium configuration
$f(x,t)$ = external longitudinal force per unit length

Assumptions
(a) Plane cross sections remain plane during longitudinal vibration. (b) The Poisson effect of lateral shrinkage and expansion can be ignored.



These assumptions are valid for a thin rod in small oscillations. The longitudinal force $P(x,t)$ at position x during deflection is given by

$$P(x,t) = EA\frac{\partial u(x,t)}{\partial x}$$

where E is the Young's modulus. The equation of free vibration is

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}, \qquad c = \sqrt{\frac{E}{\rho}}$$

where c is the velocity of longitudinal wave propagation along the rod. For a non-uniform rod with mass density $\rho(x)$ and varying cross-sectional area $A(x)$, the equation of motion is

$$\frac{\partial}{\partial x}\left[EA(x)\frac{\partial u}{\partial x}\right] + f(x,t) = \rho(x)A(x)\frac{\partial^2 u}{\partial t^2}$$

There are many applications of longitudinal vibration of rods. For example, the transmission of plane acoustic waves through air is analogous to the longitudinal vibration along a rod.

5.3 Torsional Vibration of Shafts

If a shaft is fixed at one end and the other end is twisted about the long axis and then released, a torsional wave will travel along the shaft. Consider a uniform shaft with constant mass density ρ and constant circular cross section. Let

$\theta(x,t)$ = angle of rotation of the cross section at position x from equilibrium
$f(x,t)$ = external torque per unit length

Assumption
Plane cross sections remain plane during torsional vibration.

This assumption is valid for a circular shaft in small oscillations. If the cross section of the shaft is not circular, warping is liable to occur and a plane cross section may distort out of plane. For a circular shaft, the torque $M_t(x,t)$ twisting the cross section at position x during rotation is given by

$$M_t(x,t) = GJ\frac{\partial \theta(x,t)}{\partial x}$$

where G is the shear modulus and J polar moment of inertia of the cross section in the case of a circular section. The equation of free vibration is

$$\frac{\partial^2 \theta}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 \theta}{\partial t^2}, \qquad c = \sqrt{\frac{G}{\rho}}$$

where c is the velocity of torsional wave propagation along the shaft. When the cross section is non-circular, a torsional shape constant γ is sometimes introduced so that the equation of free vibration becomes

$$\frac{\partial^2 \theta}{\partial x^2} = \frac{\rho J}{G\gamma}\frac{\partial^2 \theta}{\partial t^2}$$

For a non-uniform circular shaft, the equation of motion is

$$\frac{\partial}{\partial x}\left[GJ(x)\frac{\partial \theta}{\partial x}\right] + f(x,t) = I_0(x)\frac{\partial^2 \theta}{\partial t^2}$$

where $I_0(x)$ is the mass polar moment of inertia of the shaft per unit length.

The Golden Gate Bridge has a fundamental period of longitudinal vibration of 3.81 sec. It has a fundamental period of torsional vibration of 4.43 sec. The fundamental period for bending vibration of the Golden Gate Bridge is 18.2 sec in the horizontal direction and 10.9 sec in the vertical direction. The Tacoma Narrows Bridge in Washington collapsed on November 7, 1940 due to torsional vibration.

Remark. Strings in transverse vibration, rods in longitudinal vibration, and shafts in torsional vibration represent entirely analogous systems from a mathematical viewpoint.

5.4 Modal Analysis and Eigenfunction Expansions

There are many methods by which boundary-value problems involving linear partial differential equations can be solved. A common method is to assume that the solution is a product of unknown functions each of which depends on only one of the independent variables. If applicable, this separation of variables or product method reduces a PDE to a series of ODE.

5.4.1 Free Vibration of Uniform Strings

A uniform string in free vibration is governed by the wave equation

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2}$$

where $c = \sqrt{T/\rho}$. By inspection, the general solution of the wave equation is

$$y(x,t) = y_1(x - ct) + y_2(x + ct)$$

where y_1 and y_2 are arbitrary functions. To check this, let $w = x - ct$. Observe that

$$\begin{aligned}
& \frac{\partial y_1}{\partial x} = \frac{dy_1}{dw}\frac{\partial w}{\partial x} = \frac{dy_1}{dw} \\
\Rightarrow \quad & \frac{\partial^2 y_1}{\partial x^2} = \frac{d^2 y_1}{dw^2} \\
& \frac{\partial y_1}{\partial t} = \frac{dy_1}{dw}\frac{\partial w}{\partial t} = -c\frac{dy_1}{dw} \\
\Rightarrow \quad & \frac{\partial^2 y_1}{\partial t^2} = c^2\frac{d^2 y_1}{dw^2}
\end{aligned}$$

Thus $y_1(x - ct)$ is a solution. Similarly, with $w = x + ct$, it can be verified that $y_2(x + ct)$ is also a solution. Examples of particular solutions of the wave equation include $\log(x \pm ct)$, $\sin[\omega(t \pm x/c)]$, and $\cosh(x \pm ct)$.

Wave Nature of Homogeneous Solution

The solution $y_1(x - ct)$ represents a wave translating in the positive x-direction with speed c and a constant shape. For example, if t increases by a unit, x increases by a distance c so that

$$y_1(\{x + c\} - c\{t + 1\}) = y_1(x - ct)$$

remains constant. That means $y_1(x - ct)$ moves to the right as t increases. While the velocity of wave propagation is c, particles of the string move transversely about their equilibrium positions with particle speed $\partial y_1/\partial t$. In a similar manner, $y_2(x + ct)$ represents a wave traveling in the negative x-direction with speed c. The general solution of the wave equation consists of two traveling waves of arbitrary shapes moving in opposite directions.

Separation of Variables

In this method, solution of the wave equation is assumed to be separable in space and time such that

$$y(x,t) = Y(x)F(t)$$

This assumption implies that the spatial profile of vibration is the same at all times. It is known that undamped free vibration of SDOF systems is harmonic and that a solution of $\mathbf{M}\ddot{\mathbf{q}} + \mathbf{K}\mathbf{q} = \mathbf{0}$ is $\mathbf{q} = \mathbf{u}e^{i\omega t}$. Postulate that

$$y(x,t) = Y(x)e^{i\omega t}$$

Upon substitution, the wave equation is converted into

$$Y'' + \frac{\omega^2}{c^2}Y = 0$$

This is a differential eigenvalue problem with eigenvalue ω^2 and spatial eigenfunction $Y(x) \neq 0$.

Eigenvalues

In practical applications, the eigenvalues ω^2 are real and positive if any rigid-body motion is eliminated in the formulation. The parameter $\omega > 0$ is called a **natural frequency**. An eigenfunction associated with ω^2 is

$$Y(x) = A\cos\frac{\omega x}{c} + B\sin\frac{\omega x}{c}$$

When BC are imposed, the above equation generates a frequency equation from which infinitely many frequencies can be determined. In general, there are infinitely many natural frequencies. By convention, the natural frequencies are arranged in increasing order of magnitude such that $\omega_1 \leq \omega_2 \leq \omega_3 \leq \cdots$. The lowest frequency ω_1 is called the **fundamental frequency**.

To demonstrate that ω^2 are real and positive, consider a string fixed at both ends. Observe that

$$\begin{aligned} y(0,t) &= Y(0)e^{i\omega t} = 0 \qquad &\Rightarrow \qquad Y(0) = 0 \\ y(l,t) &= Y(l)e^{i\omega t} = 0 \qquad &\Rightarrow \qquad Y(l) = 0 \end{aligned}$$

As a consequence,

$$\begin{aligned} & Y'' + \frac{\omega^2}{c^2}Y = 0 \\ \Rightarrow \qquad & \frac{\omega^2}{c^2}\int_0^l Y^2 dx = -\int_0^l YY'' dx = -YY'|_0^l + \int_0^l (Y')^2 dx = \int_0^l (Y')^2 dx \\ \Rightarrow \qquad & \omega^2 > 0 \end{aligned}$$

if $Y'(x) \neq 0$. The eigenvalues are also positive for many other BC.

Eigenfunctions

Every eigenfunction can be chosen to be real. A real eigenfunction $Y(x)$ specifies a physical profile of vibration and is therefore called a **mode shape**. Because there are infinitely many natural frequencies, there are also infinitely many mode shapes.

Each eigenfunction can at most be determined up to a multiplicative constant. If $Y(x)$ is an eigenfunction, so is $cY(x)$ for any $c \neq 0$. The eigenfunctions $Y_n(x)$ may be normalized to have unit L_2 norm such that

$$\|Y_n\| = \left(\int_0^l Y_n^2 dx\right)^{1/2} = 1$$

This is analogous to requiring unit Euclidean norm for eigenvectors. Normalization has no physical significance and is just a matter of convenience. Each normalized eigenfunction is still not unique because its sign is arbitrary.

Normal Modes of Vibration

A normal mode associated with the natural frequency ω_n is the real part of the corresponding eigensolution so that

$$s_n(x) = \mathrm{Re}[A_n Y_n(x) e^{i\omega_n t}] = C_n Y_n(x)\cos(\omega_n t - \phi_n)$$

where $A_n = C_n e^{-i\phi_n}$ is an arbitrary complex multiplier and $Y_n(x)$ is a real eigenfunction. The profile of vibration in each mode does not change with time. A normal mode represents synchronous motion in which all points on the string perform simple harmonic motion with the same frequency ω_n and the amplitude ratio of any two points is independent of time. Any two points on the vibrating string are either in phase or 180° out of phase relative to each other. The general response in free vibration is the real part of

$$y(x,t) = \sum_{n=1}^{\infty} c_n Y_n(x) e^{i\omega_n t}$$

which is basically a linear combination of the normal modes. In general, there is no reason why the solution of a PDE can be factorized into a product of functions of independent variables. **The method of separation of variables is applicable to a continuous system only if time-independent synchronous spatial motion exists**. Many undamped continuous systems possess time-independent spatial profiles of vibration.

Eigenfunction Expansions of System Response

It will be shown that the real eigenfunctions of a differential eigenvalue problem constitute a basis and the system response can be expressed as a linear combination of the eigenfunctions with time-dependent coefficients.

Orthogonality of Eigenfunctions

Assume that all frequencies ω_n are distinct and the corresponding eigenfunctions $Y_n(x)$ are normalized to have unit L_2 norm. For all practical BC of a string, the eigenfunctions are orthogonal over the interval $0 < x < l$ in the sense that the inner product

$$(Y_m, Y_n) = \int_0^l Y_m Y_n dx = \delta_{mn}$$

To demonstrate the orthgonality of eigenfunctions, consider a uniform string fixed at both ends. Then $Y_n(0) = Y_n(l) = 0$. Observe that

$$\begin{aligned} & Y_m'' + \frac{\omega_m^2}{c^2} Y_m = 0 \\ \Rightarrow \quad & c^2 Y_m'' Y_n + \omega_m^2 Y_m Y_n = 0 \end{aligned}$$

On the other hand,

$$\begin{aligned} & c^2 Y_n'' + \omega_n^2 Y_n = 0 \\ \Rightarrow \quad & c^2 Y_n'' Y_m + \omega_n^2 Y_n Y_m = 0 \end{aligned}$$

By subtraction,

$$c^2 (Y_m'' Y_n - Y_n'' Y_m) + (\omega_m^2 - \omega_n^2) Y_m Y_n = 0$$

As a consequence,

$$\int_0^l Y_m Y_n dx = -\frac{c^2}{\omega_m^2 - \omega_n^2} \int_0^l (Y_m'' Y_n - Y_n'' Y_m) dx = -\frac{c^2}{\omega_m^2 - \omega_n^2} (Y_m' Y_n - Y_n' Y_m) \Big|_0^l = 0$$

It can be shown that the eigenfunctions are also orthogonal for many other BC.

A set of functions $f_1(x)$, $f_2(x)$,…, $f_n(x)$ is said to be linearly dependent over an interval $0 < x < l$ if, for some constants c_1, c_2,…, c_n not all zero,

$$c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x) = 0$$

identically over the interval. For example, e^x and xe^x are linearly independent. Observe that

$$\begin{aligned} & c_1 e^x + c_2 x e^x = 0 \\ \Rightarrow \quad & c_1 + c_2 x = 0 \\ \Rightarrow \quad & c_1 = c_2 = 0 \end{aligned}$$

Since the eigenfunctions $Y_n(x)$ are orthogonal, they are also linearly independent because

$$\begin{aligned} & \sum_{n=1}^{\infty} c_n Y_n(x) = 0 \\ \Rightarrow \quad & \textstyle\sum_{n=1}^{\infty} c_n \int_0^l Y_m Y_n dx = c_m = 0 \end{aligned}$$

for all m. The orthogonal eigenfunctions constitute a set of basis functions for the Hilbert space of all spatial functions satisfying the same BC.

Eigenfunction Expansion Theorem. A sufficiently continuous function $f(x)$ satisfying the same BC as the eigenfunctions of a differential eigenvalue problem can be expanded as a linear combination of the eigenfunctions, i.e.,

$$f(x) = \sum_{n=1}^{\infty} c_n Y_n(x)$$

As a consequence, the system response $y(x,t)$ can be written as

$$y(x,t) = \sum_{n=1}^{\infty} Y_n(x) p_n(t)$$

where the coefficients $p_n(t)$ are termed modal coordinates. Recall that $p_n(t) = e^{i\omega_n t}$ for a string in conservative free vibration. The above expression is referred to as modal expansion, mode superposition, or eigenfunction expansion of the system response $y(x,t)$. It is applicable to damped or undamped systems in free or forced vibration.

Theory of Eigenfunction Expansions
Consider a differential eigenvalue problem defined over the interval $0 < x < l$ with

$$L[w(x)] = \lambda \mu(x) w(x)$$

where L is a linear homogeneous differential operator and $\mu(x)$ is a function depicting the spatial distribution of mass. As an example, separation of variables of a vibrating string generates

$$\frac{d^2Y}{dx^2} + \frac{\omega^2}{c^2} Y = 0$$

so that

$$L = -\frac{d^2}{dx^2}, \qquad \mu(x) = \frac{1}{c^2}, \qquad \lambda = \omega^2$$

A **comparison function** is an arbitrary function that satisfies all BC but not necessarily the differential eigenvalue problem. An **admissible function** satisfies only the geometric BC. The class of comparison functions is considerably larger than the class of eigenfunctions and the class of admissible functions is even broader.

In the algebraic eigenvalue problem $\mathbf{Ku} = \omega^2\mathbf{Mu}$, the matrices are symmetric and positive definite. What is the concept of symmetry in a continuous system? What is positive definiteness? The operator L is said to be **self-adjoint** if

$$\int_0^l uL[v]dx = \int_0^l vL[u]dx$$

for any two comparison functions $u(x)$ and $v(x)$. For example, a string with fixed ends is self-adjoint because

$$\int_0^l uL[v]dx = -\int_0^l u\frac{d^2v}{dx^2}dx = -u\frac{dv}{dx}\bigg|_0^l + \int_0^l \frac{du}{dx}\frac{dv}{dx}dx = \int_0^l \frac{du}{dx}\frac{dv}{dx}dx$$
$$\int_0^l vL[u]dx = -\int_0^l v\frac{d^2u}{dx^2}dx = -v\frac{du}{dx}\bigg|_0^l + \int_0^l \frac{du}{dx}\frac{dv}{dx}dx = \int_0^l \frac{du}{dx}\frac{dv}{dx}dx$$

It can be shown that a string with many other BC is also self-adjoint. A self-adjoint operator L is **positive definite** if

$$\int_0^l uL[u]dx > 0$$

for any comparison function $u(x) \neq 0$. If L is positive definite, all eigenvalues λ are real and positive, and the corresponding eigenfunctions can be chosen real. If L is self-adjoint and the eigenvalues are distinct, then the eigenfunctions are orthogonal in the sense that

$$\int_0^l uL[v]dx = 0$$

for any two eigenfunctions $u(x)$ and $v(x)$. The orthogonal eigenfunctions a self-adjoint system constitute a basis in the space of functions satisfying the same BC. The eigenfunction expansion theorem states that any comparison function of a self-adjoint differential eigenvalue problem can be expanded as a linear combination of the corresponding eigenfunctions.

A Hilbert space is a function space which is complete in its natural norm (the norm derived from the inner product). As an example, the Hilbert space of a fixed-fixed string consists of all shapes of a string fixed at both ends. The mode shapes of a fixed-fixed string form a basis in this Hilbert space. The mode shapes of a fixed-fixed string are

$$Y_n(x) = \sin\frac{n\pi x}{l}$$

Any function $f(x)$ satisfying the condition $f(0) = f(l) = 0$ is a comparison function and it can be written as

$$f(x) = \sum_{n=1}^{\infty} c_n Y_n(x) = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{l}$$

As a consequence, the response $y(x, t)$ of a fixed-fixed string in free or forced vibration can be expressed as a modal expansion in the form

$$y(x,t) = \sum_{n=1}^{\infty} Y_n(x) p_n(t) = \sum_{n=1}^{\infty} \sin\frac{n\pi x}{l} p_n(t)$$

where $p_n(t)$ are the modal coordinates. Using orthogonality of the eigenfunctions, decoupled modal equations for $p_n(t)$ may be obtained and solved independently.

L. Meirovitch, *Principles and Techniques of Vibrations*, Prentice Hall, Upper Saddle River, New Jersey, 379-390 (1997).

Example. A uniform string with linear density ρ and length l is stretched to a tension T and set into transverse vibration. Suppose the string is fixed at both ends, determine the frequency equation and plot the first three mode shapes.

Solution

The equation of motion is

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2}$$

with geometric boundary conditions

$$y(0,t) = y(l,t) = 0$$

By separation of variables,

$$y(x,t) = Y(x)e^{i\omega t}$$

$$\Rightarrow \qquad Y'' + \frac{\omega^2}{c^2}Y = 0$$

$$\Rightarrow \qquad Y(x) = A\cos\frac{\omega x}{c} + B\sin\frac{\omega x}{c}$$

When BC are imposed, the spatial eigenfunction becomes independent of x and a frequency equation is generated. Invoke the BC,

$$y(0,t) = 0 \quad \Rightarrow \quad Y(0) = 0$$

$$\Rightarrow \quad A = 0$$

$$y(l,t) = 0 \quad \Rightarrow \quad Y(l) = 0$$

$$\Rightarrow \quad B\sin\frac{\omega l}{c} = 0$$

For a nontrivial eigenfunction,

$$\sin\frac{\omega l}{c} = 0$$

which is the frequency equation of a fixed-fixed string. The nth natural frequency is

$$\frac{\omega_n l}{c} = n\pi$$

$$\Rightarrow \qquad \omega_n = \frac{n\pi c}{l}, \qquad\qquad n \geq 1$$

The value $\omega_0 = 0$ is not allowed because $Y(x) \neq 0$. Within a multiplicative constant, the mode shape associated with frequency ω_n is given by

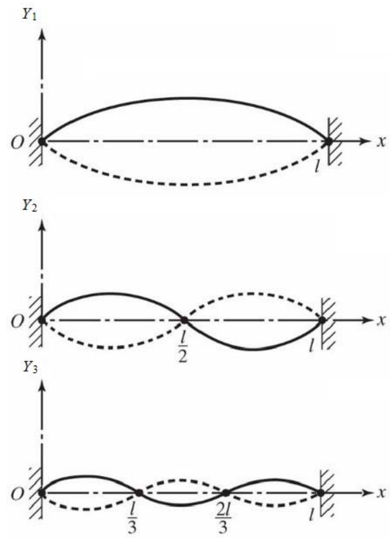
$$Y_n(x) = \sin\frac{\omega_n x}{c}$$

The eigensolution $Y_n(x)e^{i\omega_n t}$ is called the nth mode of vibration and the general solution is a linear combination given by

$$y(x,t) = \sum_{n=1}^{\infty} a_n Y_n(x) e^{i\omega_n t} = \sum_{n=1}^{\infty} a_n \sin\frac{\omega_n x}{c} e^{i\omega_n t}$$

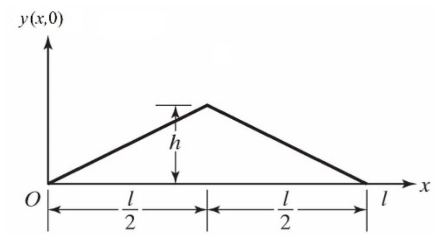
The complex constants $a_n = C_n - iD_n$ are determined by the initial values $y(x,0)$ and $\dot{y}(x,0)$. Taking the real part,

$$y(x,t) = \mathrm{Re}\left[\sum_{n=1}^{\infty} a_n \sin\frac{\omega_n x}{c} e^{i\omega_n t}\right] = \sum_{n=1}^{\infty} \sin\frac{\omega_n x}{c}\left(C_n\cos\omega_n t + D_n\sin\omega_n t\right)$$



If each end of a uniform string is either fixed or free, all higher frequencies are integral multiples of the fundamental frequency and all mode shapes are given by sinusoidal functions.

Example. A uniform string of length l, fixed at both ends and initially at rest, is plucked at its midpoint by producing a transverse displacement h and then released. (a) Determine the response in free vibration. (b) Plot the first three nonzero mode shapes. (c) Using the first three nonzero mode shapes, plot the deflection shapes of the string at times $t = 0$, $l/(4c)$, $l/(3c)$, $l/(2c)$, and l/c, where c is the transverse wave speed on the string.



Solution

(a) The initial values of a string plucked at the midpoint are

$$y(x,0) = \begin{cases} \dfrac{2h}{l}x, & 0 < x \le \dfrac{l}{2} \\ \dfrac{2h}{l}(l-x), & \dfrac{l}{2} < x \le l \end{cases}$$

$$\dot{y}(x,0) = 0$$

The free vibration of a fixed-fixed string is given by

$$y(x,t) = \sum_{n=1}^{\infty} \sin\frac{\omega_n x}{c}(C_n \cos\omega_n t + D_n \sin\omega_n t)$$

where $\omega_n = n\pi c/l$. Hence,

$$y(x,0) = \sum_{n=1}^{\infty} C_n \sin\frac{\omega_n x}{c} = \sum_{n=1}^{\infty} C_n \sin\frac{n\pi x}{l}$$

$$\dot{y}(x,0) = \sum_{n=1}^{\infty} \omega_n D_n \sin\frac{\omega_n x}{c} = \sum_{n=1}^{\infty} \omega_n D_n \sin\frac{n\pi x}{l}$$

which are Fourier series in the interval $0 \le x \le l$. The constants C_n, D_n can be calculated as Fourier coefficients from

$$C_n = \frac{2}{l}\int_0^l y(x,0)\sin\frac{n\pi x}{l}dx = \frac{8h}{(n\pi)^2}\sin\frac{n\pi}{2}$$

$$D_n = \frac{2}{\omega_n l}\int_0^l \dot{y}(x,0)\sin\frac{n\pi x}{l}dx = 0$$

It follows that for a string plucked at the center,

$$C_1 = \frac{8h}{\pi^2}, \qquad \frac{C_3}{C_1} = -\frac{1}{9}, \qquad \frac{C_5}{C_1} = \frac{1}{25}$$

$$C_2 = C_4 = C_6 = \cdots = 0$$

and so forth. All even-order modes are absent. This is not surprising because each of the absent modes has a node at the midpoint of the string, where it is initially pulled aside. Harmonics having a node at the point where a string is plucked cannot be excited. The complete response of the string is given by

$$y(x,t) = \frac{8h}{\pi^2}\left\{\sin\frac{\pi x}{l}\cos\frac{\pi ct}{l} - \frac{1}{9}\sin\frac{3\pi x}{l}\cos\frac{3\pi ct}{l} + \frac{1}{25}\sin\frac{5\pi x}{l}\cos\frac{5\pi ct}{l} - \cdots\right\}$$

(b) Within a multiplicative constant, the mode shape associated with frequency ω_n is given by

$$Y_n(x) = \sin\frac{\omega_n x}{c}, \qquad \omega_n = \frac{n\pi c}{l}$$

(c) Using only the first three nonzero modes,

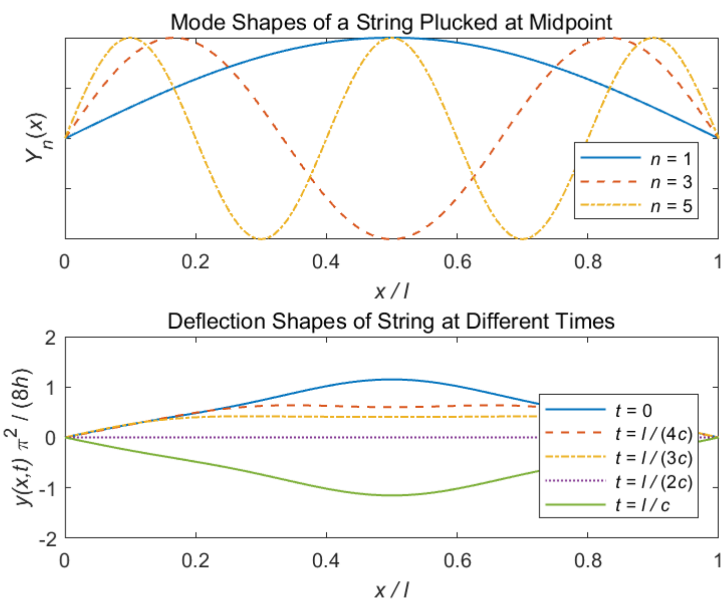
$$y(x,0) = \frac{8h}{\pi^2}\left\{\sin\frac{\pi x}{l} - \frac{1}{9}\sin\frac{3\pi x}{l} + \frac{1}{25}\sin\frac{5\pi x}{l}\right\}$$

$$y(x,l/(4c)) = \frac{8h}{\pi^2}\left\{\frac{1}{\sqrt{2}}\sin\frac{\pi x}{l} + \frac{1}{9\sqrt{2}}\sin\frac{3\pi x}{l} - \frac{1}{25\sqrt{2}}\sin\frac{5\pi x}{l}\right\}$$

$$y(x,l/(3c)) = \frac{8h}{\pi^2}\left\{\frac{1}{2}\sin\frac{\pi x}{l} + \frac{1}{9}\sin\frac{3\pi x}{l} + \frac{1}{50}\sin\frac{5\pi x}{l}\right\}$$

$$y(x,l/(2c)) = 0$$

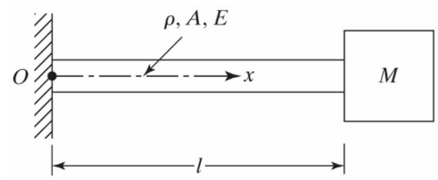
$$y(x,l/c) = \frac{8h}{\pi^2}\left\{-\sin\frac{\pi x}{l} + \frac{1}{9}\sin\frac{3\pi x}{l} - \frac{1}{25}\sin\frac{5\pi x}{l}\right\}$$



5.4.2 Systems with Discrete Elements

When a string is fixed at one end, what it means in practice is that the string is attached to a heavy mass or a large spring. At the appended boundary, the transverse forces on the string and on the attached element balance each other. In string instruments such as violins, guitars, or pianos, the vibrating strings are connected to a sounding board by bridges. Most of the sound radiation in string instruments comes from vibration of the top plate, which is driven by the vibrating strings through the flexible supports. It is the resonant frequencies of the top plate that determine the quality of the instrument.

Example. A uniform rod is fixed at one end with a mass M attached at the other end. (a) Determine the frequency equation of longitudinal vibration. (b) Find the first three natural frequencies and plot the corresponding mode shapes if $M = \rho Al$, i.e. the attached mass is equal to the mass of the rod.



Solution

(a) Suppose the rod is fixed at $x = 0$. Then

$$u(0, t) = 0$$

which is a geometric BC. At the end $x = l$, the longitudinal force in the rod must be equal to the inertia force of the vibrating mass M, and so

$$EA\frac{\partial u}{\partial x}(l, t) = -M\frac{\partial^2 u}{\partial t^2}(l, t)$$

which is a dynamic BC. Assume a solution of the form

$$u(x, t) = U(x)e^{i\omega t}$$

$$\Rightarrow \qquad U(x) = C_1 \cos\frac{\omega x}{c} + C_2 \sin\frac{\omega x}{c}$$

A frequency equation is generated when BC are imposed. Invoke the BC,

$$\begin{aligned}
u(0, t) = 0 \quad &\Rightarrow \quad U(0) = 0 \\
&\Rightarrow \quad C_1 = 0 \\
EA\frac{\partial u}{\partial x}(l, t) = -M\frac{\partial^2 u}{\partial t^2}(l, t) \quad &\Rightarrow \quad EAU'(l) = M\omega^2 U(l) \\
&\Rightarrow \quad C_2 EA\frac{\omega}{c}\cos\frac{\omega l}{c} = C_2 M\omega^2 \sin\frac{\omega l}{c} \\
&\Rightarrow \quad \cot\frac{\omega l}{c} = \frac{Mc\omega}{EA} = \frac{Mc^2}{EAl}\frac{\omega l}{c} = \frac{M}{\rho Al}\frac{\omega l}{c} \\
&\Rightarrow \quad \cot\beta = \frac{M}{m}\beta, \qquad \beta = \frac{\omega l}{c}
\end{aligned}$$

where $m = \rho Al$ is the mass of the rod.

(b) If $M = m = \rho Al$, the frequency equation becomes

$$\cot\beta = \beta$$

The frequency equation is a transcendental equation because it contains the transcendental function $\cot\beta$ (trigonometric functions are transcendental functions). In general, numerical or graphical methods are used for the solution of a transcendental equation. Upon solution,

$$\beta_1 = 0.8603 \quad \Rightarrow \quad \omega_1 = 0.8603\frac{c}{l}$$
$$\beta_2 = 3.4256 \quad \Rightarrow \quad \omega_2 = 3.4256\frac{c}{l}$$
$$\beta_3 = 6.4373 \quad \Rightarrow \quad \omega_3 = 6.4373\frac{c}{l}$$

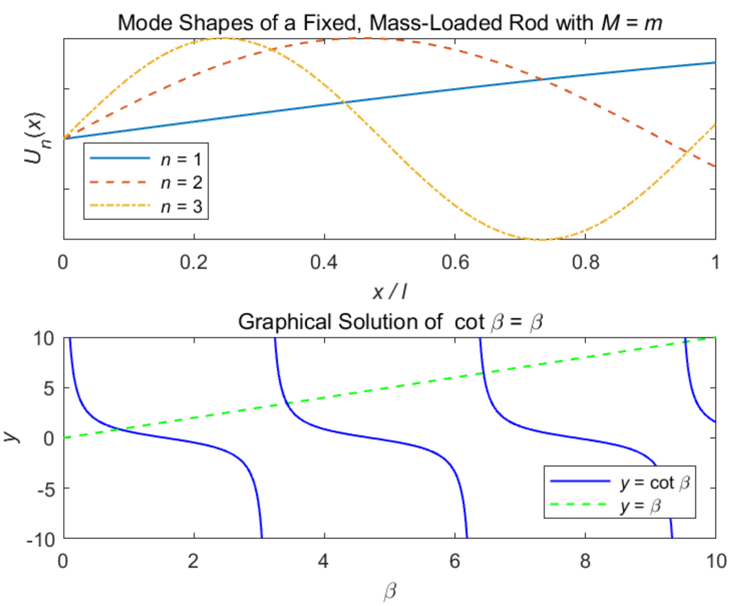
The higher frequencies are not integral multiples of the fundamental frequency. Within a multiplicative constant, the mode shape associated with frequency ω_n is given by

$$U_n(x) = \sin\frac{\omega_n x}{c} = \sin\frac{\beta_n x}{l}$$

In comparison, the fundamental frequency of a fixed-free rod ($M = 0$) is

$$\omega_1 = \frac{\pi c}{2l} = 1.5708\frac{c}{l}$$

With an attached mass $M = m$, the fundamental frequency is about 55% of the fixed-free value.



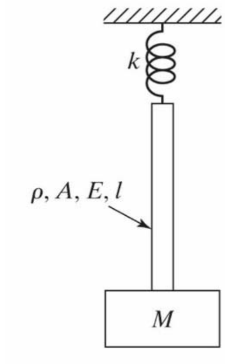
The MATLAB function **fzero** finds the roots of a nonlinear function. An initial estimate of the location of each root is required. The command

$$x = \text{fzero}(@fun, x0)$$

returns a root of fun near an initial point or within an initial interval $x0$. Using an initial point $x0 = 6.4$, **fzero** generates $\beta_3 = 6.2832$ up to four decimal digits. However, $\cot(6.2832) = 6.8 \times 10^4$ and therefore **fzero** does not converge. Using an initial interval $x0 = [6.4 \;\; 6.5]$, the correct value $\beta_3 = 6.4373$ is obtained.

The higher frequencies of a mass-loaded rod are not integral multiples of ω_1, and this is sometimes advantageous in practical applications. As an illustration, consider a mass-loaded nickel tube used to produce a pure tone at ω_1 when driven by an AC current in a coil mounted on the tube. Unless the AC current is well filtered, it has harmonic components at $2\omega_1$, $3\omega_1$, etc. However, higher frequencies of the mass-loaded tube will not be excited by these harmonic components.

Example. A uniform rod is loaded at one end with mass M and at the other end with spring k. Determine the frequency equation of longitudinal vibration of the rod. Write the equation in terms of $\beta = \omega l/c$, M, k, the mass of the rod $m = \rho A l$, and the axial stiffness EA/l. Check that the frequency equation reduces to that for a fixed, spring-loaded rod if M increases without bound.



Solution
Suppose spring k is attached to the boundary at $x = 0$ and mass M at $x = l$. The boundary conditions are

$$EA\frac{\partial u}{\partial x}(0,t) = ku(0,t)$$
$$EA\frac{\partial u}{\partial x}(l,t) = -M\frac{\partial^2 u}{\partial t^2}(l,t)$$

Assume a solution of the form

$$u(x,t) = U(x)e^{i\omega t}$$
$$\Rightarrow \quad U(x) = C_1\cos\frac{\omega x}{c} + C_2\sin\frac{\omega x}{c}$$

Invoke the boundary conditions,

$$EA\frac{\partial u}{\partial x}(0,t) = ku(0,t)$$
$$\Rightarrow \quad EAU'(0) = kU(0)$$
$$\Rightarrow \quad C_2 EA\frac{\omega}{c} = C_1 k$$
$$EA\frac{\partial u}{\partial x}(l,t) = -M\frac{\partial^2 u}{\partial t^2}(l,t)$$
$$\Rightarrow \quad EAU'(l) = M\omega^2 U(l)$$
$$\Rightarrow \quad C_1\left(-\frac{EA}{c}\sin\frac{\omega l}{c} - M\omega\cos\frac{\omega l}{c}\right) + C_2\left(\frac{EA}{c}\cos\frac{\omega l}{c} - M\omega\sin\frac{\omega l}{c}\right) = 0$$

These two equations can be cast in the vector form

$$\begin{bmatrix} k & -\dfrac{EA\omega}{c} \\ -\dfrac{EA}{c}\sin\dfrac{\omega l}{c} - M\omega\cos\dfrac{\omega l}{c} & \dfrac{EA}{c}\cos\dfrac{\omega l}{c} - M\omega\sin\dfrac{\omega l}{c} \end{bmatrix}\begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

If the determinant of coefficient matrix is zero, then

$$\left(1 + \frac{\rho EA^2}{Mk}\right)\tan\beta = \frac{\rho Al}{M\beta} - \frac{EA\beta}{kl}, \qquad \beta = \frac{\omega l}{c}$$

The frequency equation may be written as

$$\left(1+\frac{mEA}{Mkl}\right)\tan\beta=\frac{m}{M\beta}-\frac{EA\beta}{kl}$$

where $m=\rho Al$ is the mass of the rod. As $M\rightarrow\infty$, the above equation reduces to

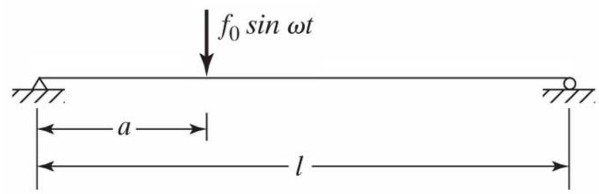
$$\tan\beta=-\frac{EA}{kl}\beta$$

which is the frequency equation for a fixed, spring-loaded rod.

5.4.3 Forced and Damped Vibration

The modal equations for p_n can be decoupled and solved independently when a system is driven by an external force and when damping is classical.

Example. A uniform string with linear density ρ and length l is stretched to a tension T between rigid supports. Find the steady-state response if the string is subjected to a harmonic force $f_0 \sin \omega t$ applied at $x = a$.



Solution

The boundary-value problem is given by

$$T\frac{\partial^2 y}{\partial x^2} + f(x,t) = \rho\frac{\partial^2 y}{\partial t^2}$$
$$y(0,t) = y(l,t) = 0$$

where $f(x,t) = -f_0 \sin \omega t\, \delta(x-a)$. The mode shapes of a fixed-fixed string are

$$Y_n(x) = \sin\tfrac{\omega_n x}{c}, \qquad \omega_n = \tfrac{n\pi c}{l}$$

Assume that the steady-state response is an eigenfunction expansion of the form

$$y(x,t) = \sum_{n=1}^{\infty} Y_n(x)p_n(t) = \sum_{n=1}^{\infty} \sin\frac{\omega_n x}{c}p_n(t)$$

where $p_n(t)$ are the modal coordinates. Substitute into the equation of motion to obtain

$$-T\frac{\omega_n^2}{c^2}\sum_{n=1}^{\infty}\sin\frac{\omega_n x}{c}p_n(t) - f_0 \sin\omega t\, \delta(x-a) = \rho\sum_{n=1}^{\infty}\sin\frac{\omega_n x}{c}\ddot{p}_n(t)$$

Upon simplification,

$$\sum_{n=1}^{\infty}[\ddot{p}_n(t) + \omega_n^2 p_n(t)]\sin\frac{\omega_n x}{c} = -\frac{f_0}{\rho}\sin\omega t\, \delta(x-a)$$

The mode shapes are orthogonal in such a way that

$$\int_0^l Y_m(x)Y_n(x)dx = \int_0^l \sin\frac{\omega_m x}{c}\sin\frac{\omega_n x}{c}dx = \int_0^l \sin\frac{m\pi x}{l}\sin\frac{n\pi x}{l}dx = \frac{l}{2}\delta_{mn}$$

Using the orthogonality relationship, the modal coordinates are decoupled and

$$[\ddot{p}_n(t) + \omega_n^2 p_n(t)]\frac{l}{2} = -\frac{f_0}{\rho}\sin\omega t\int_0^l \sin\frac{\omega_n x}{c}\delta(x-a)dx$$

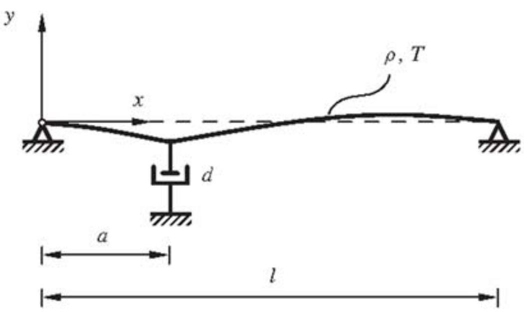
$$\Rightarrow \qquad \ddot{p}_n(t) + \omega_n^2 p_n(t) = -\tfrac{2f_0}{\rho l}\sin\omega t \sin\tfrac{\omega_n a}{c}$$

$$\Rightarrow \qquad p_n(t) = \tfrac{1}{\omega^2-\omega_n^2}\tfrac{2f_0}{\rho l}\sin\omega t\sin\tfrac{\omega_n a}{c}, \qquad \omega \neq \omega_n$$

$$\Rightarrow \qquad y(x,t) = \tfrac{2f_0 \sin\omega t}{\rho l}\textstyle\sum_{n=1}^{\infty}\tfrac{1}{\omega^2-\omega_n^2}\sin\tfrac{n\pi a}{l}\sin\tfrac{n\pi x}{l}$$

As n increases, $1/(\omega^2 - \omega_n^2) \rightarrow 0$ and only the lower-order modes contribute significantly to the overall response.

Example. A uniform string with linear density ρ and length l is stretched to a tension T between rigid supports. Indicate a method for finding the free response if a viscous damper d is attached to the string at position $x = a$.



Solution

The boundary-value problem is given by

$$T\frac{\partial^2 y}{\partial x^2} + f(x,t) = \rho\frac{\partial^2 y}{\partial t^2}$$
$$y(0,t) = y(l,t) = 0$$

where

$$f(x,t) = -d\frac{\partial y}{\partial t}\delta(x-a)$$

The mode shapes of a fixed-fixed string are

$$Y_n(x) = \sin\frac{\omega_n x}{c}, \qquad \omega_n = \frac{n\pi c}{l}$$

Assume that the damped free response is an eigenfunction expansion of the form

$$y(x,t) = \sum_{n=1}^{\infty} Y_n(x)p_n(t) = \sum_{n=1}^{\infty} \sin\frac{\omega_n x}{c} p_n(t)$$

where $p_n(t)$ are the modal coordinates. Substitute into the equation of motion to obtain

$$-T\frac{\omega_n^2}{c^2}\sum_{n=1}^{\infty} \sin\frac{\omega_n x}{c} p_n(t) - d\sum_{n=1}^{\infty} \sin\frac{\omega_n x}{c}\dot{p}_n(t)\delta(x-a) = \rho\sum_{n=1}^{\infty} \sin\frac{\omega_n x}{c}\ddot{p}_n(t)$$

Upon simplification,

$$\sum_{n=1}^{\infty} [\ddot{p}_n(t) + \omega_n^2 p_n(t)]\sin\frac{\omega_n x}{c} = -\frac{d}{\rho}\sum_{n=1}^{\infty} \sin\frac{\omega_n x}{c}\dot{p}_n(t)\delta(x-a)$$

The mode shapes are orthogonal in such a way that

$$\int_0^l Y_m(x)Y_n(x)dx = \int_0^l \sin\frac{\omega_m x}{c}\sin\frac{\omega_n x}{c}dx = \int_0^l \sin\frac{m\pi x}{l}\sin\frac{n\pi x}{l}dx = \frac{l}{2}\delta_{mn}$$

Using the orthogonality relationship, the modal coordinate $p_n(t)$ is governed by

$$\ddot{p}_n(t) + \omega_n^2 p_n(t) = -\frac{2d}{\rho l}\int_0^l \sum_{m=1}^{\infty} \sin\frac{\omega_m x}{c}\dot{p}_m(t)\delta(x-a)\sin\frac{\omega_n x}{c}dx$$

which is equivalent to

$$\ddot{p}_n(t) + \frac{2d}{\rho l}\sin\frac{\omega_n a}{c}\sum_{m=1}^{\infty}\sin\frac{\omega_m x}{c}\dot{p}_m(t) + \omega_n^2 p_n(t) = 0$$

It is necessary to calculate the modal coordinates, which are coupled by damping, to generate the damped free response. As a vector differential equation,

$$\begin{bmatrix}\ddot{p}_1\\ \ddot{p}_2\\ \vdots\end{bmatrix} + \frac{2d}{\rho l}\begin{bmatrix}\sin^2\frac{\omega_1 a}{c} & \sin\frac{\omega_1 a}{c}\sin\frac{\omega_2 a}{c} & \cdots\\ \sin\frac{\omega_2 a}{c}\sin\frac{\omega_1 a}{c} & \sin^2\frac{\omega_2 a}{c} & \cdots\\ \vdots & \vdots & \ddots\end{bmatrix}\begin{bmatrix}\dot{p}_1\\ \dot{p}_2\\ \vdots\end{bmatrix} + \begin{bmatrix}\omega_1^2 & 0 & \cdots\\ 0 & \omega_2^2 & \cdots\\ \vdots & \vdots & \ddots\end{bmatrix}\begin{bmatrix}p_1\\ p_2\\ \vdots\end{bmatrix} = \begin{bmatrix}0\\ 0\\ \vdots\end{bmatrix}$$

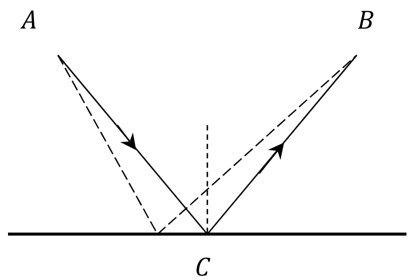
The damping matrix is symmetric, positive semidefinite, and with rank one. Further, it couples all the modes of the undamped system to produce a non-classically damped system.

P. Hagedorn and A. DasGupta, *Vibrations and Waves in Continuous Mechanical Systems*, Wiley, Hoboken, New Jersey, 55-56 (2007).

5.5 Variational Formulations

The evolution of physical systems is often governed by a **principle of least action**. There are many principles of least action in play. In optics, Fermat's principle states that light travels between two points along a path in such a way that the time taken by light has a stationary value. As a consequence, light travels in a straight line from point A to point B so that the time taken is a minimum.

The law of reflection can be easily deduced. Suppose light travels from A to B by reflection at an undetermined position C. Then the ray path $AC + CB$ is minimized if C reaches a position at which the angle of incidence is equal to the angle of reflection.



The evolution of a dynamical system is governed by a principle of least action called Hamilton's principle (1834).

Hamilton's Principle

A conservative holonomic system moves along a path between two observed configurations from time t_1 to time t_2 in such a way that the action integral

$$I = \int_{t_1}^{t_2} (T - V)dt$$

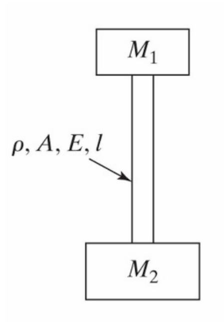
has a stationary value.

The stationary value is usually a minimum. Hamilton's principle can be extended to non-conservative systems by incorporating the virtual work of non-conservative forces. It is easier to apply Hamilton's principle to a conservative system. Afterwards, non-conservative forces can be added to the equation of motion. When applied to a continuous system, Hamilton's principle generates both the differential equations of motion and the associated boundary conditions.

Let δ be the variational operator. The action integral has a stationary value if

$$\delta I = \int_{t_1}^{t_2} (\delta T - \delta V)dt = 0$$

Example. A uniform rod is loaded at one end with mass M_1 and at the other end with mass M_2. Use Hamilton's principle to derive the equation of longitudinal vibration and the associated boundary conditions.



Solution

Suppose M_1 is attached to the upper boundary at $x = 0$ and M_2 to the lower boundary at $x = l$. For an infinitesimal element at position x of length dx, the kinetic energy is

$$dT = \frac{1}{2}\rho A dx \left(\frac{\partial u}{\partial t}\right)^2$$

The system kinetic energy is given by

$$T = \frac{1}{2}\int_0^l \rho A \left(\frac{\partial u}{\partial t}\right)^2 dx + \frac{1}{2}M_1\dot{u}^2(0,t) + \frac{1}{2}M_2\dot{u}^2(l,t)$$

The elongation of the element dx is

$$\delta(dx) = \frac{\partial u}{\partial x}dx$$

Thus the potential energy for the element is

$$dV = \frac{1}{2}P(x,t)\delta(dx) = \frac{1}{2}EA\left(\frac{\partial u}{\partial x}\right)^2 dx$$

The system potential energy is

$$V = \frac{1}{2}\int_0^l EA\left(\frac{\partial u}{\partial x}\right)^2 dx$$

From Hamilton's principle,

$$\delta\int_{t_1}^{t_2}(T - V)dt = 0$$

which implies that

$$\delta\int_{t_1}^{t_2}\left[\frac{1}{2}\int_0^l \rho A\left(\frac{\partial u}{\partial t}\right)^2 dx + \frac{1}{2}M_1\dot{u}^2(0,t) + \frac{1}{2}M_2\dot{u}^2(l,t) - \frac{1}{2}\int_0^l EA\left(\frac{\partial u}{\partial x}\right)^2 dx\right]dt = 0$$

Upon simplification,

$$\int_{t_1}^{t_2}\left[\int_0^l \rho A\frac{\partial u}{\partial t}\delta\left(\frac{\partial u}{\partial t}\right)dx + M_1\dot{u}(0,t)\delta\dot{u}(0,t) + M_2\dot{u}(l,t)\delta\dot{u}(l,t) - \int_0^l EA\frac{\partial u}{\partial x}\delta\left(\frac{\partial u}{\partial x}\right)dx\right]dt = 0$$

Observe that $\delta u = 0$ at $t = t_1$ and $t = t_2$ because the configuration of the rod is specified at those instants. Hence,

$$\int_{t_1}^{t_2} \rho A \frac{\partial u}{\partial t} \delta\left(\frac{\partial u}{\partial t}\right) dt = \int_{t_1}^{t_2} \rho A \frac{\partial u}{\partial t} \frac{\partial}{\partial t} (\delta u) dt$$

$$= \rho A \frac{\partial u}{\partial t} \delta u \bigg|_{t_1}^{t_2} - \int_{t_1}^{t_2} \frac{\partial}{\partial t}\left(\rho A \frac{\partial u}{\partial t}\right) \delta u dt = -\int_{t_1}^{t_2} \rho A \frac{\partial^2 u}{\partial t^2} \delta u dt$$

Similarly,

$$\int_{t_1}^{t_2} M_1 \dot{u}(0,t) \delta \dot{u}(0,t) dt = -\int_{t_1}^{t_2} M_1 \frac{\partial^2 u}{\partial t^2}(0,t) \delta u(0,t) dt$$

$$\int_{t_1}^{t_2} M_2 \dot{u}(l,t) \delta \dot{u}(l,t) dt = -\int_{t_1}^{t_2} M_2 \frac{\partial^2 u}{\partial t^2}(l,t) \delta u(l,t) dt$$

In addition,

$$\int_0^l EA \frac{\partial u}{\partial x} \delta\left(\frac{\partial u}{\partial x}\right) dx = \int_0^l EA \frac{\partial u}{\partial x} \frac{\partial}{\partial x} (\delta u) dx = EA \frac{\partial u}{\partial x} \delta u \bigg|_{x=0}^{l} - \int_0^l \frac{\partial}{\partial x}\left(EA \frac{\partial u}{\partial x}\right) \delta u dx$$

As a consequence,

$$\int_{t_1}^{t_2} \left[\int_0^l \left(EA \frac{\partial^2 u}{\partial x^2} - \rho A \frac{\partial^2 u}{\partial t^2} \right) \delta u dx - EA \frac{\partial u}{\partial x} \delta u \bigg|_{x=0}^{l} - M_1 \frac{\partial^2 u}{\partial t^2}(0,t) \delta u(0,t) \right.$$
$$\left. - M_2 \frac{\partial^2 u}{\partial t^2}(l,t) \delta u(l,t) \right] dt = 0$$

The virtual displacement δu is arbitrary except that it must vanish at $t = t_1$ and $t = t_2$. Invoke the subclass of δu with the additional constraint that $\delta u = 0$ at $x = 0$ and $x = l$,

$$EA \frac{\partial^2 u}{\partial x^2} - \rho A \frac{\partial^2 u}{\partial t^2} = 0$$

$$\Rightarrow \qquad \frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2}, \qquad c = \sqrt{\frac{E}{\rho}}$$

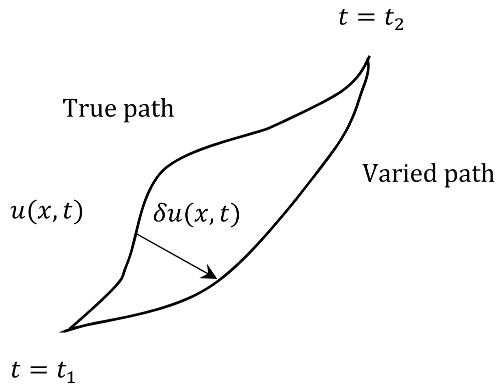
which is the equation of motion. Furthermore, the boundary conditions are revealed by

$$-\left(EA \frac{\partial u}{\partial x} + M_2 \frac{\partial^2 u}{\partial t^2}\right) \delta u \bigg|_{x=l} + \left(EA \frac{\partial u}{\partial x} - M_1 \frac{\partial^2 u}{\partial t^2}\right) \delta u \bigg|_{x=0} = 0$$

The boundary conditions are

$$EA \frac{\partial u}{\partial x}(0,t) = M_1 \frac{\partial^2 u}{\partial t^2}(0,t)$$

$$EA \frac{\partial u}{\partial x}(l,t) = -M_2 \frac{\partial^2 u}{\partial t^2}(l,t)$$


$t = t_2$
True path
Varied path
$u(x,t)$
$\delta u(x,t)$
$t = t_1$

5.6 Bending Vibration of Beams

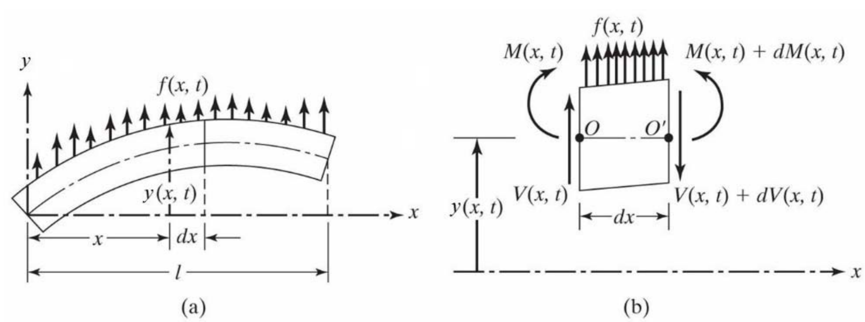
A bar is capable of vibrating longitudinally, torsionally, and transversely. In longitudinal vibration the bar derives its restoring forces through axial tension. In torsional vibration the restoring forces are generated by rotational stiffness. In transverse vibration the restoring forces are provided by bending stiffness. The three forms of vibration are coupled through internal strains in the bar, and one motion invariably produces the other. Plates are two-dimensional versions of beams.

Consider the transverse vibration of a uniform beam with constant mass density ρ, constant cross-sectional area A, and flexural rigidity (bending stiffness) EI. Let

$y(x,t)$ = transverse deflection at position x from equilibrium position
$M(x,t)$ = bending moment at position x during deflection
$V(x,t)$ = shear force at position x during deflection
$f(x,t)$ = external transverse force per unit length

Assumptions

(a) An ideal beam derives its restoring forces entirely from bending stiffness. (b) Plane cross sections remain plane during bending vibration.



If **deformation due to shear stress is negligible**, the moment-curvature relation is

$$M(x,t) = EI\frac{\partial^2 y}{\partial x^2}(x,t)$$

where I is the moment of inertia of the cross-sectional area about the z-axis (perpendicular to the (x,y) plane and passes through the centroid of cross section). If **rotational inertia is negligible**, rotational equilibrium of the element yields the standard relation

$$V(x,t) = \frac{\partial M}{\partial x}(x,t)$$

between shear force and bending moment. From mechanics of materials, the equation of motion of the element in the y-direction is

$$\rho A\frac{\partial^2 y}{\partial t^2} = f(x,t) - \frac{\partial V}{\partial x}$$

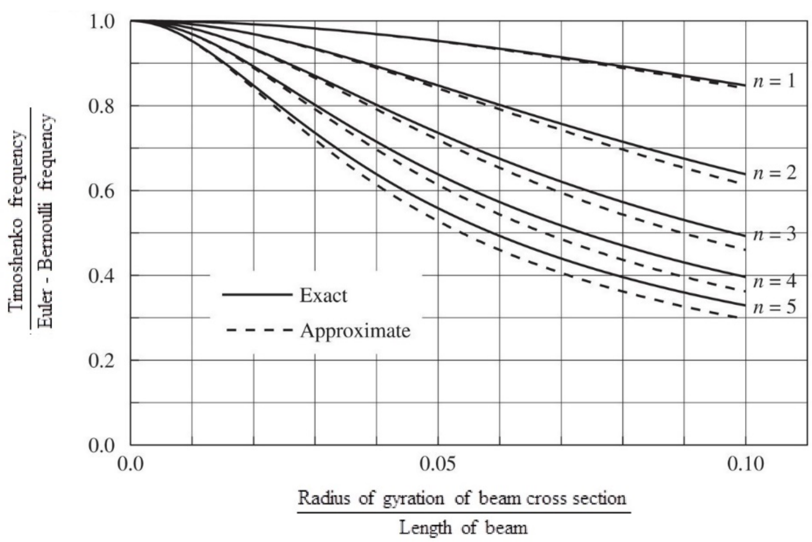
The equation of transverse vibration of a uniform beam simplifies to

$$EI\frac{\partial^4 y}{\partial x^4} + \rho A\frac{\partial^2 y}{\partial t^2} = f(x,t)$$

This is a fourth-order PDE called the Euler-Bernoulli equation or thin beam equation. For a non-uniform beam with varying cross-sectional area $A(x)$ and flexural rigidity $EI(x)$, the governing equation is

$$\frac{\partial^2}{\partial x^2}\left[EI(x)\frac{\partial^2 y}{\partial x^2}\right] + \rho A(x)\frac{\partial^2 y}{\partial t^2} = f(x,t)$$

Shear deformation and rotational inertia are incorporated in Timoshenko beam theory. They have the effect of lowering the natural frequencies. In general, the correction is less than 5% for the fundamental frequency. It may be significant for higher frequencies.



A. K. Chopra, *Dynamics of Structures: Theory and Applications to Earthquake Engineering*, 4*th* ed., Prentice Hall, Upper Saddle River, New Jersey, 705-707 (2012).

Free Vibration of Uniform Beams
For free vibration of a uniform beam,

$$c^2\frac{\partial^4 y}{\partial x^4} + \frac{\partial^2 y}{\partial t^2} = 0, \qquad c = \sqrt{\frac{EI}{\rho A}}$$

By direct substitution, functions of the form $y(x - ct)$ and $y(x + ct)$ are not solutions. Postulate that

$$y(x,t) = Y(x)e^{i\omega t}$$

Upon substitution, one obtains a differential eigenvalue problem

$$c^2 \frac{d^4Y}{dx^4} - \omega^2 Y = 0$$

$$\Rightarrow \quad \frac{d^4Y}{dx^4} - \beta^4 Y = 0, \qquad \beta^4 = \frac{\omega^2}{c^2} = \frac{\rho A \omega^2}{EI}$$

For all practical BC, the eigenvalues ω^2 are real and positive. That means β^4 are also positive. The roots of the characteristic equation

$$\lambda^4 - \beta^4 = 0$$

are $\pm\beta$ and $\pm i\beta$. As a consequence, the eigenfunction associated with β^4 is

$$Y(x) = B_1 e^{\beta x} + B_2 e^{-\beta x} + B_3 e^{i\beta x} + B_4 e^{-i\beta x}$$

$$\Rightarrow \quad Y(x) = C_1 \cos \beta x + C_2 \sin \beta x + C_3 \cosh \beta x + C_4 \sinh \beta x$$

using $e^{\pm i\theta} = \cos\theta \pm i \sin\theta$ and $e^{\pm\theta} = \cosh\theta \pm \sinh\theta$. The constants C_1, C_2, C_3 and C_4 can be determined from boundary conditions.

By convention, the natural frequencies are arranged in increasing order of magnitude such that $\omega_1 \le \omega_2 \le \omega_3 \le \cdots$. The lowest frequency ω_1 is called the fundamental frequency. The general response in free vibration is

$$y(x,t) = \sum_{n=1}^{\infty} Y_n(x) e^{i\omega_n t}$$

Boundary Conditions of Uniform Beams

The common boundary conditions are as follows.

1. Clamped (fixed) end: both the deflection and the angle of slope are zero.

$$y = 0$$

$$\frac{\partial y}{\partial x} = 0$$

2. Simply supported (pinned) end: rotation but not deflection is allowed. If the beam is supported by a hinge and rotation is not restricted, the bending moment is zero.

$$y = 0$$

$$M = EI \frac{\partial^2 y}{\partial x^2} = 0 \qquad \Rightarrow \qquad \frac{\partial^2 y}{\partial x^2} = 0$$

3. Free end: both bending moment and shear force are zero.

$$M = EI\frac{\partial^2 y}{\partial x^2} = 0$$

$$V = \frac{\partial}{\partial x}\left(EI\frac{\partial^2 y}{\partial x^2}\right) = 0 \quad \Rightarrow \qquad \frac{\partial^3 y}{\partial x^3} = 0$$

4. Sliding end: deflection but not rotation is allowed. If deflection is not restricted, the shear force is zero.

$$\frac{\partial y}{\partial x} = 0$$

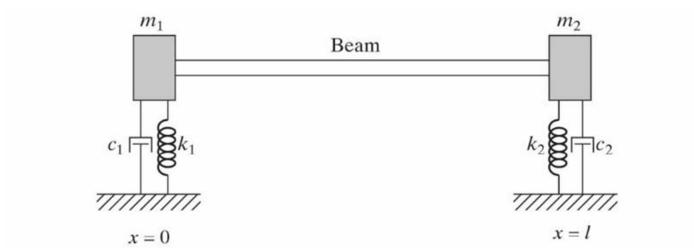
$$V = \frac{\partial}{\partial x}\left(EI\frac{\partial^2 y}{\partial x^2}\right) = 0 \quad \Rightarrow \qquad \frac{\partial^3 y}{\partial x^3} = 0$$

5. End connected to mass, damper, and linear spring: the shear force at the end balances the resisting force due to mass, damper, and spring. Thus

$$V = \frac{\partial}{\partial x}\left(EI\frac{\partial^2 y}{\partial x^2}\right) = a\left[m\frac{\partial^2 y}{\partial t^2} + c\frac{\partial y}{\partial t} + ky\right]$$

where $a = -1$ for left end and $a = 1$ for right end of the beam. In addition, the bending moment is zero because rotation is allowed so that

$$M = EI\frac{\partial^2 y}{\partial x^2} = 0 \qquad \Rightarrow \qquad \frac{\partial^2 y}{\partial x^2} = 0$$



Orthogonality of Eigenfunctions

Assume that the frequencies ω_n are distinct and the eigenfunctions $Y_n(x)$ are normalized to have unit L_2 norm so that

$$\|Y_n\| = \left(\int_0^l Y_n^2 dx\right)^{1/2} = 1$$

For many practical BC, the normalized eigenfunctions are orthogonal in the sense that the inner product

$$(Y_m, Y_n) = \int_0^l Y_m Y_n dx = \delta_{mn}$$

To demonstrate orthgonality of eigenfunctions, consider a beam with clamped, free, or simply supported boundaries. At a clamped end, $Y_n = 0$ and $Y_n' = 0$. For a free end, $Y_n'' = 0$ and $Y_n''' = 0$. At a simply supported end, $Y_n = 0$ and $Y_n'' = 0$. Observe that

$$c^2 \frac{d^4 Y_m}{dx^4} - \omega_m^2 Y_m = 0$$
$$c^2 \frac{d^4 Y_n}{dx^4} - \omega_n^2 Y_n = 0$$

Combine the two equations to obtain

$$c^2 \frac{d^4 Y_m}{dx^4} Y_n - \omega_i^2 Y_m Y_n - c^2 \frac{d^4 Y_n}{dx^4} Y_m + \omega_n^2 Y_n Y_m = 0$$

As a consequence,

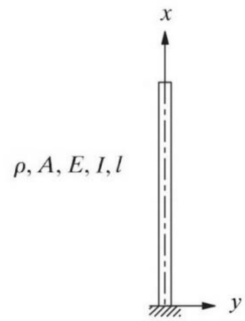
$$\int_0^l Y_m Y_n dx = \frac{c^2}{\omega_m^2 - \omega_n^2} \int_0^l \left(\frac{d^4 Y_m}{dx^4} Y_n - \frac{d^4 Y_n}{dx^4} Y_m \right) dx$$
$$= -\frac{c^2}{\omega_m^2 - \omega_n^2} \left(\frac{d^3 Y_m}{dx^3} Y_n - \frac{d^2 Y_m}{dx^2} \frac{dY_n}{dx} - \frac{d^3 Y_n}{dx^3} Y_m + \frac{d^2 Y_n}{dx^2} \frac{dY_m}{dx} \right) \Bigg|_0^l = 0$$

for any combination of clamped, free, or simply supported end conditions. It can be shown that the eigenfunctions are also orthogonal for many other BC. The orthogonal eigenfunctions are linearly independent over $0 < x < l$ and they constitute a basis in the space of functions satisfying the same BC. As a consequence, the system response $y(x,t)$ in free or forced vibration can be expressed as an eigenfunction expansion of the form

$$y(x,t) = \sum_{n=1}^{\infty} Y_n(x) p_n(t)$$

where $p_n(t)$ are the modal coordinates. Using orthogonality of the eigenfunctions, decoupled modal equations for $p_n(t)$ may be obtained and solved independently.

Example. Determine the frequency equation of a uniform beam clamped at one end and free at the other. Find the first three natural frequencies and plot the corresponding mode shapes.



Solution

A cantilever beam is a clamped-free beam. Let the beam be fixed at $x = 0$ and free at $x = l$. The boundary conditions are

$$y(0,t) = 0, \qquad \frac{\partial y}{\partial x}(0,t) = 0$$
$$\frac{\partial^2 y}{\partial x^2}(l,t) = 0, \qquad \frac{\partial^3 y}{\partial x^3}(l,t) = 0$$

Assume a solution of the form

$$y(x,t) = Y(x)e^{i\omega t}$$
$$\Rightarrow \qquad \frac{d^4Y}{dx^4} - \beta^4 Y = 0, \qquad \beta^4 = \frac{\rho A \omega^2}{EI}$$
$$\Rightarrow \qquad Y(x) = C_1 \cos\beta x + C_2 \sin\beta x + C_3 \cosh\beta x + C_4 \sinh\beta x$$

When BC are imposed, the spatial eigenfunction becomes independent of x and a frequency equation is generated. Observe that

$$Y'(x) = -C_1\beta \sin\beta x + C_2\beta \cos\beta x + C_3\beta \sinh\beta x + C_4\beta \cosh\beta x$$

using $d(\cosh x)/dx = \sinh x$ and $d(\sinh x)/dx = \cosh x$. Invoke the BC at $x = 0$,

$$Y(0) = 0 \qquad \Rightarrow \qquad C_1 + C_3 = 0$$
$$Y'(0) = 0 \qquad \Rightarrow \qquad C_2 + C_4 = 0$$

These two equations give

$$Y(x) = C_1(\cos\beta x - \cosh\beta x) + C_2(\sin\beta x - \sinh\beta x)$$

Invoke the BC at $x = l$,

$$Y''(l) = 0$$
$$\Rightarrow \qquad C_1(-\cos\beta l - \cosh\beta l) + C_2(-\sin\beta l - \sinh\beta l) = 0$$
$$Y'''(l) = 0$$
$$\Rightarrow \qquad C_1(\sin\beta l - \sinh\beta l) + C_2(-\cos\beta l - \cosh\beta l) = 0$$

The last two equations can be cast in the vector form

$$\begin{bmatrix} \cos\beta l + \cosh\beta l & \sin\beta l + \sinh\beta l \\ \sin\beta l - \sinh\beta l & -\cos\beta l - \cosh\beta l \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

For a nontrivial solution of $[C_1 \quad C_2]^T$, determinant of the coefficient matrix is zero. This yields the frequency equation, using $\cos^2 x + \sin^2 x = 1$ and $\cosh^2 x + \sinh^2 x = 1$,

$$1 + \cos\beta l \cosh\beta l = 0$$

By numerical iterations,

$$\beta_1 l = 1.8751 \qquad \Rightarrow \qquad \omega_1 = \frac{3.52}{l^2}\sqrt{\frac{EI}{\rho A}}$$

$$\beta_2 l = 4.6941 \qquad \Rightarrow \qquad \omega_2 = \frac{22.03}{l^2}\sqrt{\frac{EI}{\rho A}}$$

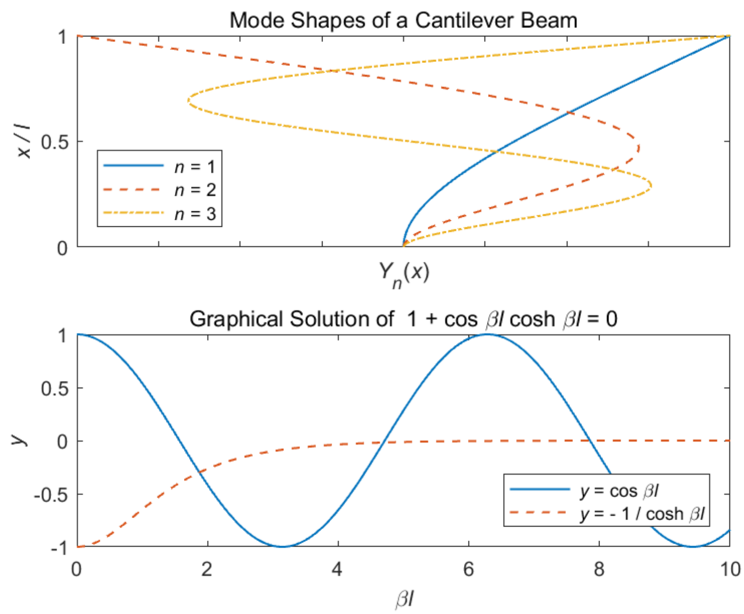
$$\beta_3 l = 7.8548 \qquad \Rightarrow \qquad \omega_3 = \frac{61.70}{l^2}\sqrt{\frac{EI}{\rho A}}$$

The higher frequencies are not integral multiples of the fundamental frequency. From the equation associated with $Y''(l) = 0$,

$$C_2 = -C_1 \frac{\cos\beta l + \cosh\beta l}{\sin\beta l + \sinh\beta l}$$

Within a multiplicative constant, the mode shape associated with frequency ω_n is given by

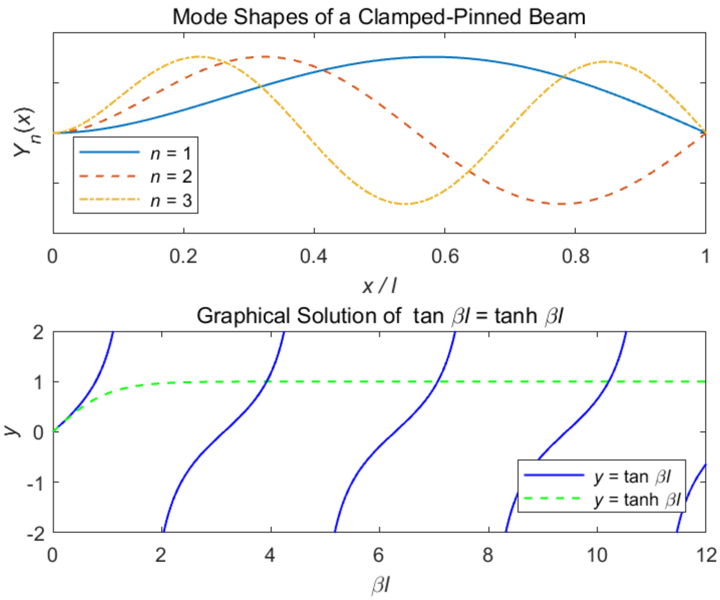
$$Y_n(x) = \cos\beta_n x - \cosh\beta_n x - \frac{\cos\beta_n l + \cosh\beta_n l}{\sin\beta_n l + \sinh\beta_n l}(\sin\beta_n x - \sinh\beta_n x)$$



A clamped-pinned beam and a pinned-free beam have the same frequency equation of

$$\tan\beta l = \tanh\beta l$$

but their mode shapes are different. The pinned-free beam has a rigid-body mode.



A free-free beam and a clamped-clamped beam have the same frequency equation of

$$\cos\beta l \cosh\beta l = 1$$

but their mode shapes are different. The free-free beam has a rigid-body mode.

The frequency equation and mode shapes of a uniform beam with both ends simply supported are

$$\sin\beta l = 0$$

$$Y_n(x) = \sin\beta_n x$$

The frequency equation and mode shapes of a simply supported beam and a fixed-fixed string have the same functional forms. Among common boundary conditions, the higher frequencies of a uniform beam are integral multiples of the fundamental frequency only when simply supported.

5.7 Membranes and Plates

A membrane is a two-dimensional version of string. Standard examples of membrane include drumheads and soap films. In contrast, a plate is a two-dimensional version of beam. Standard examples of thin plate include diaphragms of ordinary telephone microphones and receivers. In a membrane, the restoring force arises entirely from tension, whereas in a thin plate the restoring force results from bending stiffness. Choice of a coordinate system matching the boundary conditions will greatly simplify obtaining and interpreting solutions. For example, Cartesian coordinates should be used for a rectangular boundary and polar coordinates for a circular boundary.

Rectangular Membranes

Consider a uniform membrane with constant surface density (mass per unit area) ρ stretched over an area in the (x, y) plane under constant tension per unit length T. Let

$w(x, y, t)$ = transverse displacement at position (x, y) normal to equilibrium plane
$f(x, y, t)$ = external transverse force per unit area

The equation of motion of the membrane is

$$T\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) + f(x, y, t) = \rho \frac{\partial^2 w}{\partial t^2}$$

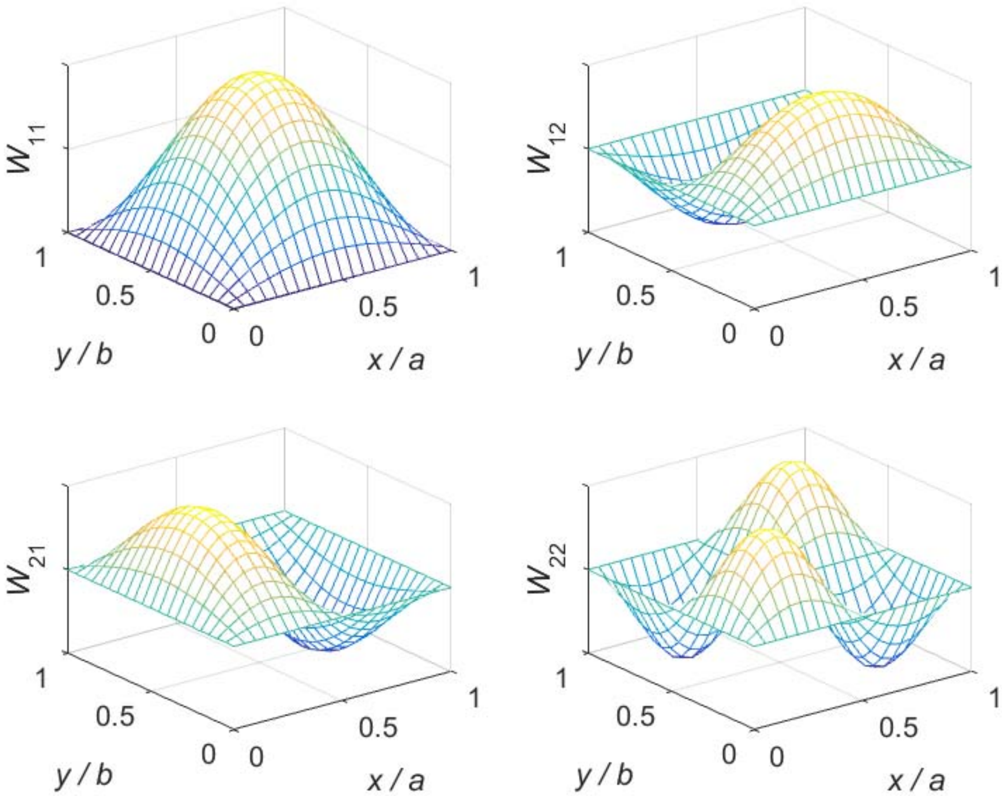
In free vibration,

$$\nabla^2 w = \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = \frac{1}{c^2}\frac{\partial^2 w}{\partial t^2}, \qquad c = \sqrt{\frac{T}{\rho}}$$

where ∇^2 is the two-dimensional Laplacian operator. This is the two-dimensional wave equation in rectangular coordinates. A solution is obtained by assuming that

$$w(x, y, t) = W(x, y)e^{i\omega t} = X(x)Y(y)e^{i\omega t}$$

For a rectangular membrane fixed at the boundary, the mode shape $W_{mn}(x, y)$ of the membrane is the product of $Y_m(x)$ and $Y_n(y)$, where $Y_n(x)$ is the mode shape of order n of a fixed-fixed string. In higher-order modes, there are nodal lines that separate regions of nonzero displacements. For example, on a rectangular membrane of side a along the x-axis, there is a nodal line at $x/a = 1/2$ in the mode shape $W_{21}(x, y)$. It is possible to insert rigid supports along the nodal lines without affecting the mode shape for the particular frequency involved. There may be two or more mode shapes of a membrane associated with the same natural frequency.



W_{11}
W_{12}
W_{21}
W_{22}
y / b
x / a
0
0.5
1

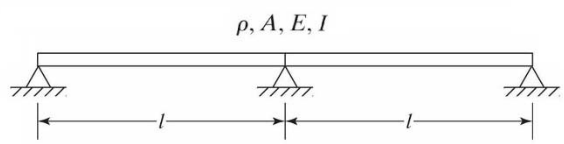
COMPOSITE AND HYBRID SYSTEMS

Important applications of continuous systems are considered herein.

6.1 Beams on Multiple Supports

For a beam on many supports, divide the beam into segments so that each segment is only supported at the boundaries. A boundary-value problem is set up for each segment, and the overall response is obtained by matching the interface conditions between adjacent segments.

Example. A uniform beam of length $2l$ rests on three simple supports as shown. Derive the frequency equation of transverse vibration of the beam.



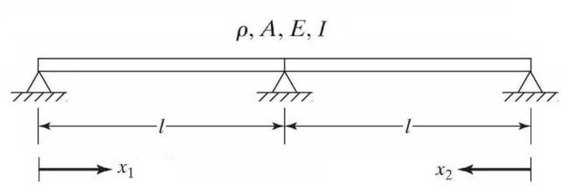
Solution

Divide the beam into two segments and set up two coordinates x_1 and x_2 as shown. The equations of motion of the uniform beam are

$$c^2\frac{\partial^4 y_1}{\partial x_1^4}+\frac{\partial^2 y_1}{\partial t^2}=0, \qquad 0<x_1<l$$
$$c^2\frac{\partial^4 y_2}{\partial x_2^4}+\frac{\partial^2 y_2}{\partial t^2}=0, \qquad 0<x_2<l$$

where

$$c=\sqrt{\frac{EI}{\rho A}}$$



For pinning at $x_1=0$ and $x_2=0$,

$$y_1(0,t)=0, \qquad \frac{\partial^2 y_1}{\partial x_1^2}(0,t)=0$$
$$y_2(0,t)=0, \qquad \frac{\partial^2 y_2}{\partial x_2^2}(0,t)=0$$

For pinning at the midpoint,

$$y_1(l,t)=y_2(l,t)=0$$

In addition, the slope and bending moment must be continuous across the interface at $x_1=l$ or $x_2=l$, which requires that (a positive slope in x_1 coordinate is a negative slope in x_2 coordinate)

$$\frac{\partial y_1}{\partial x_1}(l,t)=-\frac{\partial y_2}{\partial x_2}(l,t)$$

$$\frac{\partial^2 y_1}{\partial x_1^2}(l,t) = \frac{\partial^2 y_2}{\partial x_2^2}(l,t)$$

Altogether there are 8 boundary equations. Assume a solution of the form

$$y_1(x_1,t) = Y_1(x_1)e^{i\omega t}, \qquad 0 < x_1 < l$$
$$y_2(x_2,t) = Y_2(x_2)e^{i\omega t}, \qquad 0 < x_2 < l$$

The spatial eigenfunctions can be written as

$$Y_1(x_1) = B_1 \cos\beta x_1 + B_2 \sin\beta x_1 + B_3 \cosh\beta x_1 + B_4 \sinh\beta x_1$$
$$Y_2(x_2) = C_1 \cos\beta x_2 + C_2 \sin\beta x_2 + C_3 \cosh\beta x_2 + C_4 \sinh\beta x_2$$

where

$$\beta^4 = \frac{\omega^2}{c^2} = \frac{\rho A \omega^2}{EI}$$

There are 8 boundary conditions involving 9 unknowns B_1, B_2, B_3, B_4, C_1, C_2, C_3, C_4 and β. In principle, the first 8 unknowns can be eliminated by the 8 boundary equations, resulting in a frequency equation in β. The algebra can be streamlined because x_1, x_2 coordinates are mirror images of each other.

Consider the segment on the left. Invoke the boundary conditions at $x_1 = 0$,

$$Y_1(0) = 0 \qquad \Rightarrow \qquad B_1 + B_3 = 0$$
$$\frac{d^2Y_1}{dx_1^2}(0) = 0 \qquad \Rightarrow \qquad -B_1 + B_3 = 0$$

These two equations give $B_1 = B_3 = 0$ and so

$$Y_1(x_1) = B_2 \sin\beta x_1 + B_4 \sinh\beta x_1$$

Invoke the boundary condition at $x_1 = l$,

$$Y_1(l) = 0$$
$$\Rightarrow \qquad B_2 \sin\beta l + B_4 \sinh\beta l = 0$$
$$\Rightarrow \qquad Y_1(x_1) = B_2\left(\sin\beta x_1 - \frac{\sin\beta l}{\sinh\beta l}\sinh\beta x_1\right) \tag{1}$$

Consider the segment on the right. By symmetry with respect to the middle support,

$$Y_2(x_2) = C_2\left(\sin\beta x_2 - \frac{\sin\beta l}{\sinh\beta l}\sinh\beta x_2\right) \tag{2}$$

Invoke the interface conditions,

$$\frac{dY_1}{dx_1}(l) = -\frac{dY_2}{dx_2}(l)$$
$$\Rightarrow \qquad \left[\cos\beta l - \frac{\sin\beta l}{\sinh\beta l}\cosh\beta l\right](B_2 + C_2) = 0 \tag{3}$$
$$\frac{d^2Y_1}{dx_1^2}(l) = \frac{d^2Y_2}{dx_2^2}(l)$$
$$\Rightarrow \qquad \sin\beta l\,(B_2 - C_2) = 0 \tag{4}$$

The last two equations can be cast in the vector form

$$\begin{bmatrix} \cos\beta l - \dfrac{\sin\beta l}{\sinh\beta l}\cosh\beta l & \cos\beta l - \dfrac{\sin\beta l}{\sinh\beta l}\cosh\beta l \\ \sin\beta l & -\sin\beta l \end{bmatrix}\begin{bmatrix} B_2 \\ C_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

For a nontrivial solution of $[B_2 \quad C_2]^T$, determinant of the coefficient matrix is zero. This yields the frequency equation

$$\left[\cos\beta l - \frac{\sin\beta l}{\sinh\beta l}\cosh\beta l\right]\sin\beta l = 0 \tag{5}$$

The above frequency equation is satisfied if either

$$\cos \beta l - \frac{\sin \beta l}{\sinh \beta l} \cosh \beta l = 0$$

$$\Rightarrow \quad \tan \beta l = \tanh \beta l \tag{6}$$

or

$$\sin \beta l = 0 \tag{7}$$

The roots of these two component equations do not overlap and the two equations cannot be satisfied simultaneously. Observe that Eq. (6) is the frequency equation of a clamped-pinned beam. If this equation is valid,

$$\sin \beta l \neq 0$$

$$\Rightarrow \quad B_2 - C_2 = 0 \qquad \text{using Eq. (4)}$$

$$\Rightarrow \quad Y_1(x) = Y_2(x)$$

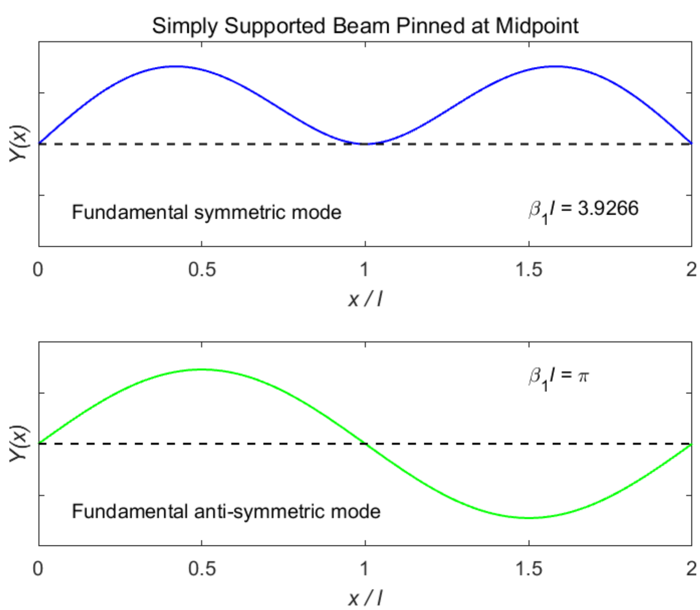
and the mode shape is symmetric with respect to the middle support. On the other hand, Eq. (7) is the frequency equation of a simply supported beam. If this equation is valid,

$$\cos \beta l - \frac{\sin \beta l}{\sinh \beta l} \cosh \beta l \neq 0$$

$$\Rightarrow \quad B_2 + C_2 = 0 \qquad \text{using Eq. (3)}$$

$$\Rightarrow \quad Y_1(x) = -Y_2(x)$$

and the mode shape is anti-symmetric with respect to the middle support. A symmetric mode and an anti-symmetric mode are shown.



S. S. Rao, *Vibration of Continuous Systems*, Wiley, Hoboken, New Jersey, 359-363 (2007).

6.2 Traveling Forces and Elastic Foundation

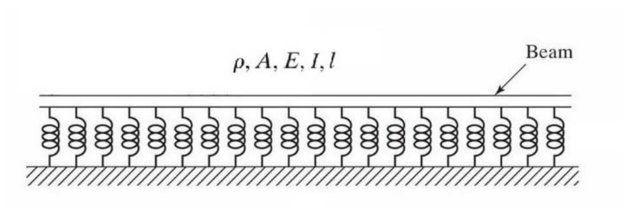
Important applications include automobile-bridge, jet-aircraft carrier, and train-track.

Beams on Elastic Foundation

Suppose a uniform beam rests on an elastic foundation such as a rail track on soil. Let the stiffness per unit length of the foundation be k_f. Thus the transverse force per unit length exerted by the foundation on the beam is $f(x,t) = -k_f y(x,t)$. The equation of transverse vibration is

$$EI\frac{\partial^4 y}{\partial x^4} + \rho A\frac{\partial^2 y}{\partial t^2} = f(x,t)$$

$$\Rightarrow \qquad EI\frac{\partial^4 y}{\partial x^4} + \rho A\frac{\partial^2 y}{\partial t^2} + k_f y = 0$$



Assume a solution of the form

$$y(x,t) = Y(x)e^{i\omega t}$$

$$\Rightarrow \qquad EI\frac{d^4Y}{dx^4} - (\rho A\omega^2 - k_f)Y = 0$$

$$\Rightarrow \qquad \frac{d^4Y}{dx^4} - \beta^4 Y = 0$$

where

$$\beta^4 = \frac{\rho A\omega^2}{EI} - \frac{k_f}{EI} = \frac{\omega^2}{c^2} - \frac{k_f}{EI}, \qquad c = \sqrt{\frac{EI}{\rho A}}$$

This has the same form as that of a beam without a foundation, but with a different β. The response of a beam on elastic foundation with various boundary conditions can be obtained by simply updating β. The eigenfunction can be written as

$$Y(x) = C_1\cos\beta x + C_2\sin\beta x + C_3\cosh\beta x + C_4\sinh\beta x$$

For a simply supported beam on an elastic foundation,

$$\beta_n l = n\pi$$

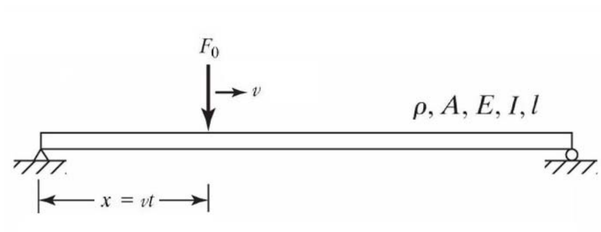
$$\Rightarrow \qquad \omega_n = c\beta_n^2\sqrt{1 + \frac{k_f}{\rho A\beta_n^4}} = c\left(\frac{n\pi}{l}\right)^2\sqrt{1 + \frac{k_f}{\rho A}\left(\frac{l}{n\pi}\right)^4}$$

Within a multiplicative constant, the mode shape associated with frequency ω_n is given by

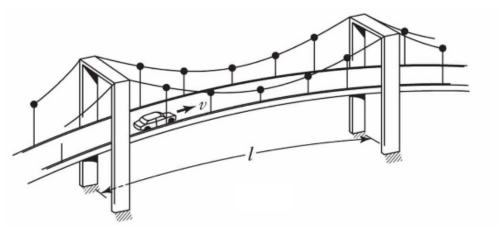
$$Y_n(x) = \sin \beta_n x = \sin \frac{n\pi x}{l}$$

An elastic foundation increases the frequencies but it does not change the mode shapes. The forced response can also be found from the corresponding result without a foundation.

Example. A vehicle of weight F_0 moving at a constant speed v on a bridge can be modeled as a concentrated load traveling on a simply supported beam. Find the transverse displacement of the bridge with initial conditions $y(x,0) = 0$ and $\partial y/\partial t(x,0) = 0$.



Solution
A real-life picture of the mathematical model is shown.



The equation of transverse vibration of a uniform beam is

$$EI\frac{\partial^4 y}{\partial x^4} + \rho A\frac{\partial^2 y}{\partial t^2} = f(x,t)$$

where $f(x,t) = -F_0\delta(x - vt)$. For a simply supported beam, the boundary conditions are

$$y(0,t) = 0, \qquad \frac{\partial^2 y}{\partial x^2}(0,t) = 0$$

$$y(l,t) = 0, \qquad \frac{\partial^2 y}{\partial x^2}(l,t) = 0$$

The mode shapes of a simply supported beam are

$$Y_n(x) = \sin\beta_n x = \sin\frac{n\pi x}{l}, \qquad \beta_n^4 = \frac{\rho A\omega_n^2}{EI} = \left(\frac{n\pi}{l}\right)^4$$

Assume that the response is an eigenfunction expansion of the form

$$y(x,t) = \sum_{n=1}^{\infty} Y_n(x)p_n(t) = \sum_{n=1}^{\infty} \sin\beta_n x\, p_n(t)$$

where $p_n(t)$ are the modal coordinates. Substitute into the equation of motion to obtain

$$\sum_{n=1}^{\infty}\left[EI\beta_n^4 \sin\beta_n x\, p_n(t) + \rho A\sin\beta_n x\, \ddot{p}_n(t)\right] = -F_0\delta(x - vt)$$

Upon simplification,

$$\sum_{n=1}^{\infty}\left[\ddot{p}_n(t) + \omega_n^2 p_n(t)\right]\sin\beta_n x = -\frac{F_0}{\rho A}\delta(x - vt)$$

The mode shapes are orthogonal in such a way that

$$\int_0^l Y_m(x)Y_n(x)dx = \int_0^l \sin\beta_m x\sin\beta_n x\, dx = \int_0^l \sin\frac{m\pi x}{l}\sin\frac{n\pi x}{l}dx = \frac{l}{2}\delta_{mn}$$

Using the orthogonality relationship, the modal coordinates are decoupled and

$$[\ddot{p}_n(t) + \omega_n^2 p_n(t)]\frac{l}{2} = -\frac{F_0}{\rho A}\int_0^l \sin\beta_n x\,\delta(x - vt)dx$$

$$\Rightarrow \qquad \ddot{p}_n(t) + \omega_n^2 p_n(t) = -\tfrac{2F_0}{\rho A l}\sin\beta_n vt, \qquad \beta_n v \neq \omega_n$$

$$\Rightarrow \qquad p_n(t) = C_n\cos\omega_n t + D_n\sin\omega_n t - \tfrac{1}{(i\beta_n v)^2 + \omega_n^2}\tfrac{2F_0}{\rho A l}\sin\beta_n vt$$

As a result,

$$y(x,t) = \sum_{n=1}^{\infty}\left[C_n\cos\omega_n t + D_n\sin\omega_n t + \frac{1}{(n\pi v/l)^2 - \omega_n^2}\frac{2F_0}{\rho A l}\sin\frac{n\pi vt}{l}\right]\sin\frac{n\pi x}{l}$$

Since the beam is initially at rest and is undeformed,

$$y(x,0) = \sum_{n=1}^{\infty} C_n\sin\frac{n\pi x}{l} = 0$$

$$\Rightarrow \qquad C_n = 0$$

$$\dot{y}(x,0) = \sum_{n=1}^{\infty}\left[\omega_n D_n + \frac{1}{(n\pi v/l)^2 - \omega_n^2}\frac{2n\pi vF_0}{\rho A l^2}\right]\sin\frac{n\pi x}{l} = 0$$

$$\Rightarrow \qquad D_n = -\tfrac{1}{(n\pi v/l)^2 - \omega_n^2}\tfrac{2n\pi vF_0}{\rho A l^2\omega_n}$$

A load F_0 traveling with velocity v takes an interval $t_d = l/v$ to move on the beam. The complete solution in the duration $0 \le t \le t_d$ is

$$y(x,t) = \frac{2F_0}{\rho A l}\sum_{n=1}^{\infty}\frac{1}{(n\pi v/l)^2 - \omega_n^2}\left[\sin\frac{n\pi vt}{l} - \frac{n\pi v}{\omega_n l}\sin\omega_n t\right]\sin\frac{n\pi x}{l}$$

Once the load F_0 leaves the beam, the beam undergoes free vibration. The shape of the beam and its velocity at the onset of free vibration can be obtained from the above solution at $t = t_d$. When $t > t_d$,

$$y(x,t) = \sum_{n=1}^{\infty}\sin\frac{\omega_n x}{c}(A_n\cos\omega_n t + B_n\sin\omega_n t)$$

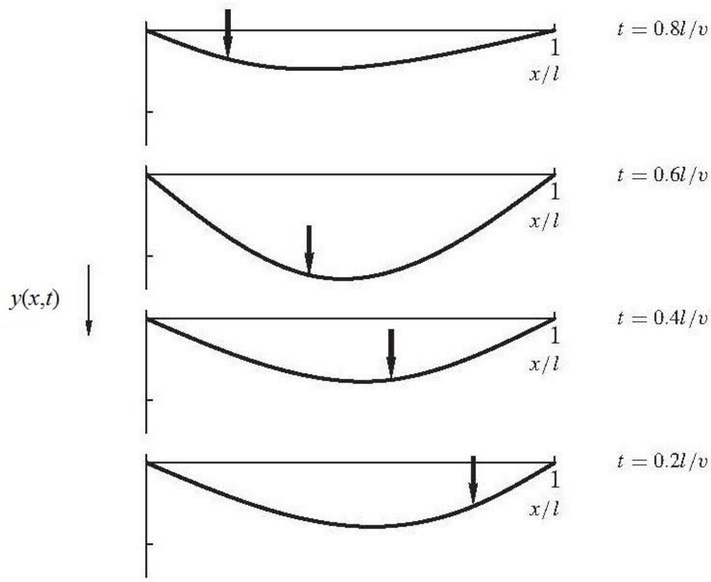
Matching the values of $y(x,t)$ and $\dot{y}(x,t)$ at $t = t_d$, A_n and B_n can be found. The solution for $t > t_d$ is

$$y(x,t) = \frac{2F_0}{\rho A l}\sum_{n=1}^{\infty}\frac{1}{(n\pi v/l)^2 - \omega_n^2}\frac{n\pi v}{\omega_n l}\left[(-1)^n\sin\omega_n\left(t - \frac{l}{v}\right) - \sin\omega_n t\right]\sin\frac{\omega_n x}{c}$$

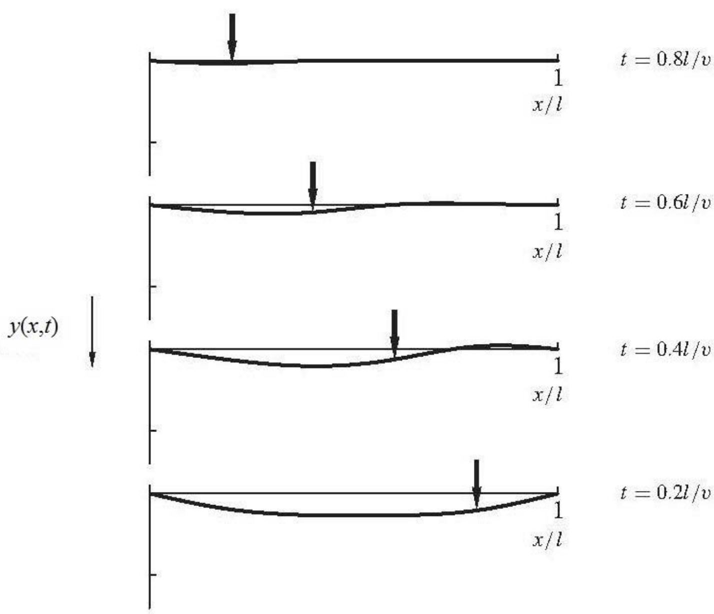
The shapes of the beam at selected time instants are shown for a load traveling at $v = \omega_1 l/(4\pi) < \omega_1/\beta_1 = \omega_1 l/\pi$ and at $v = \omega_1 l\pi/4$ where

$$\omega_2/\beta_2 = 2\omega_1 l/\pi < \omega_1 l\pi/4 = 2.47\omega_1 l/\pi < \omega_3/\beta_3 = 3\omega_1 l/\pi$$

In general, response contributions from higher modes are small.



Response of a beam with constant force with $v/l = \omega_1/4\pi$



Response of a beam with constant force with $v/l = \omega_1\pi/4$

A. K. Chopra, *Dynamics of Structures: Theory and Applications to Earthquake Engineering*, 4*th* ed., Prentice Hall, Upper Saddle River, New Jersey, 713-716 (2012).