FUNDAMENTAL CONCEPTS

Lecture Outline
Introduction
Analysis in Time Domain
Analysis in Frequency Domain

The subject of vibration is concerned with the dynamics of oscillatory motion.

1.1 Introduction

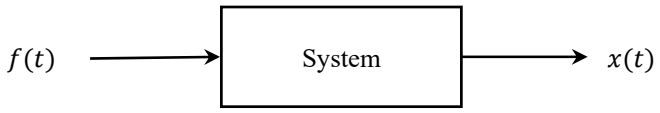
Any physical system that possesses mass and elasticity is capable of vibration. Mass refers to the ability of an element within the system to transfer momentum to adjacent elements. Elasticity refers to the ability of a displaced element to return to its equilibrium position. Vibration manifests itself in a great variety of ways in daily life.

1.2 Classification of Systems

A system is an assemblage of components acting as a whole. The motion of a system is described by a set of variables known as generalized coordinates. The external excitation $f(t)$ may be regarded as an input, and the resulting response $x(t)$ an output. From a theoretical standpoint, the input and output are connected by a differential equation

$$L[x(t)] = f(t)$$

The differential operator L provides a system model.



The number of degrees of freedom of a system is the minimum number of independent coordinates required to specify its motion completely.

Discrete and Continuous Systems

A discrete system has a finite DOF; its operator L generates a system of ODE the number of which is equal to the DOF. A continuous system has an infinite DOF; its operator L generates a system of PDE.

Linear and Nonlinear Systems

A system is linear if and only if the principle of superposition applies.

Principle of Superposition

If $L[x_1(t)] = f_1(t)$ and $L[x_2(t)] = f_2(t)$ under the same initial conditions, then

$$L[\alpha x_1(t) + \beta x_2(t)] = \alpha f_1(t) + \beta f_2(t)$$

When the input $f(t)$ is a Gaussian random process, then the system is linear if and only if the output $x(t)$ is a Gaussian random process.

Time-Varying and Time-Invariant Systems
A system is time-invariant if a delay in the excitation produces an identical delay in the response. For an arbitrary constant τ, a time-invariant system satisfies

$$L[x(t-\tau)] = f(t-\tau)$$

A linear system is time-invariant if and only if it possesses constant coefficients. When there is a time-dependent coefficient, the system is time-varying.

Characterization of Vibration Problems
Among $f(t)$, $x(t)$ and L, three different types of problems can be formulated.

	Excitation $f(t)$	System Model L	Response $x(t)$
Analysis	√	√	?
Calibration	?	√	√
System Identification	√	?	√

1.3 Prototype System
Any SDOF LTI system can be represented as a mass-spring-damper equation, governed by the equation

$$m\ddot{x} + c\dot{x} + kx = f(t)$$

Using equivalent viscous damping, this is true even for systems with nonlinear damping.

Equilibrium Position as Reference
In deriving the equation of motion of a linear system, it is useful to define the displacement variables to be zero at equilibrium rather than at the position of zero spring deflections. Such a choice allows balancing forces associated with equilibrium to be ignored. For nonlinear systems, all forces, including the static forces associated with equilibrium, should be included in analysis.

1.4 Viscously Damped Free Vibration

A system in damped free vibration is governed by the homogeneous equation

$$m\ddot{x} + c\dot{x} + kx = 0$$

Solution for Damped Free Vibration

Postulate that

$$x = Ae^{\lambda t}$$

Upon substitution, one obtains a characteristic equation

$$\begin{aligned} & m\lambda^2 + c\lambda + k = 0 \\ \Rightarrow \quad & \lambda = \frac{1}{2m}(-c \pm \sqrt{c^2 - 4mk}) \\ \Rightarrow \quad & \mathrm{Re}[\lambda] < 0 \end{aligned}$$

By superposition, the general solution is

$$x = A_1 e^{\lambda_1 t} + A_2 e^{\lambda_2 t}$$

where A_1 and A_2 are constants depending on the initial conditions. Define a non-dimensional damping ratio by

$$\zeta = \frac{c}{2m\omega_n} = \frac{c}{2\sqrt{mk}} = \frac{c}{c_c}$$

where $\omega_n = \sqrt{k/m}$ is the undamped natural frequency and c_c is called the critical damping. In terms of ζ,

$$\lambda_1, \lambda_2 = -\frac{c}{2m} \pm \frac{1}{2m}\sqrt{4mk\left(\frac{c^2}{4mk} - 1\right)} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$

Categories of Damped Motion

There are three types of damped free response.

Case 1. $\zeta > 1$: overdamping. The two roots λ_1, λ_2 are distinct and negative. The free response is non-oscillatory and exponentially decaying. Instruments used to measure steady-state values, for example a scale measuring stationary weight, are usually overdamped by design.

Case 2. $\zeta = 1$: critical damping. The two roots are real and equal, with $\lambda_1 = \lambda_2 = -\omega_n$. The free response for the special case of equal roots is given by

$$x = (A_1 + A_2 t)e^{-\omega_n t}$$

The motion is again non-oscillatory and exponentially decaying. In general, a critically damped

system will approach equilibrium faster than will an overdamped system.

Case 3. $\zeta < 1$: underdamping. The roots λ_1, λ_2 are complex conjugates. Define the frequency of damped vibration by

$$\omega_d = \omega_n\sqrt{1-\zeta^2}$$

The general solution can be expressed as

$$x = e^{-\zeta\omega_n t}\left(A_1 e^{i\omega_d t} + A_2 e^{-i\omega_d t}\right)$$

Since the physical response x is real, $A_2 = \bar{A}_1$ ($\bar{A}$ is the complex conjugate of A and A^* is the complex conjugate transpose) and, as a result,

$$x = e^{-\zeta\omega_n t}\mathrm{Re}[2A_1 e^{i\omega_d t}]$$

Write $2A_1$ in polar form as $2A_1 = Ce^{-i\phi}$. It follows that

$$x = Ce^{-\zeta\omega_n t}\cos(\omega_d t - \phi)$$

This can also be expressed as

$$x = Ce^{-\zeta\omega_n t}\sin(\omega_d t - \phi + \pi/2)$$

because $\sin(\pi/2 \pm A) = \cos A$. The free response consists of sinusoidal oscillations within an exponentially decaying envelope. Many real-life systems are underdamped with $\zeta < 5\%$. These include buildings, bridges, dams, nuclear power plants, and offshore structures.

Remark. Free vibration decays exponentially with any amount of viscous damping. It is usually ignored if the external excitation $f(t)$ persists over a sufficiently long duration.

1.5 Forced Vibration in Time Domain

A quadrature solution of the nonhomogeneous equation

$$m\ddot{x} + c\dot{x} + kx = f(t)$$

will be derived in the time domain. The general solution of a linear nonhomogeneous ODE is obtained by adding the homogeneous solution to a particular solution,

$$x(t) = x_h(t) + x_p(t)$$

Because of this fundamental theorem on linear ODE, it is feasible to consider separately the tasks of finding general solutions of homogeneous equations and particular solutions of nonhomogeneous equations. The homogeneous solution has already been determined. It decays exponentially with any amount of viscous damping and is often ignored. The particular solution is maintained by the driving force $f(t)$ and is independent of initial conditions. Sometimes it is useful to classify the excitation as harmonic, periodic, or non-periodic.

There are two things that should be kept in mind. First, the overall solution must be used to satisfy the initial conditions in order to establish the two arbitrary constants contained in the homogeneous solution. Second, the general solution of a nonlinear nonhomogeneous ODE is not a superposition of the homogeneous solution and a particular solution.

Laplace Transforms

A linear differential equation with constant coefficients is converted into an algebraic equation by Laplace transform. The Laplace transform of a function $x(t)$ is defined as

$$X(s) = \mathcal{L}[x(t)] = \int_0^\infty e^{-st}x(t)dt$$

where s is, in general, a complex quantity called the subsidiary variable. The inverse transform is provided by the Bromwich's integral formula:

$$x(t) = \mathcal{L}^{-1}[X(s)] = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{st}X(s)ds, \qquad t > 0$$

and $x(t) = 0$ for $t < 0$. The real number γ is chosen so that the vertical line $\gamma + iy$ lies to the right of all singularities of $X(s)$ in the complex plane but is otherwise arbitrary. Tables of transform pairs of elementary functions are generally available.

Using integration by parts,

$$\mathcal{L}[\dot{x}(t)] = \int_0^\infty e^{-st}\dot{x}(t)dt = \int_0^\infty e^{-st}dx(t) = e^{-st}x(t)|_0^\infty - \int_0^\infty x(t)de^{-st}$$
$$= -x(0) + s\int_0^\infty e^{-st}x(t)dt = s\mathcal{L}[x(t)] - x(0)$$

In a similar fashion,

$$\mathcal{L}[\ddot{x}(t)] = s^2\mathcal{L}[x(t)] - sx(0) - \dot{x}(0)$$

Taking Laplace transform of the prototype equation of motion,

$$\begin{aligned} & m\mathcal{L}[\ddot{x}] + c\mathcal{L}[\dot{x}] + k\mathcal{L}[x] = \mathcal{L}[f(t)] \\ \Rightarrow \quad & (ms^2 + cs + k)X(s) - (ms + c)x(0) - m\dot{x}(0) = F(s) \\ \Rightarrow \quad & x(t) = \mathcal{L}^{-1}[X(s)] = \mathcal{L}^{-1}\left[\frac{(ms+c)x(0)+m\dot{x}(0)}{ms^2+cs+k}\right] + \mathcal{L}^{-1}\left[\frac{F(s)}{ms^2+cs+k}\right] \end{aligned}$$

The first term on the right side is the homogeneous solution because it contains the initial conditions and is independent of the driving force. Since the homogeneous solution has been determined earlier,

$$x_h(t) = \mathcal{L}^{-1}\left[\frac{(ms + c)x(0) + m\dot{x}(0)}{ms^2 + cs + k}\right] = A_1e^{\lambda_1 t} + A_2e^{\lambda_2 t}$$

If $x(0) = \dot{x}(0) = 0$, the homogeneous solution vanishes and the general solution reduces to

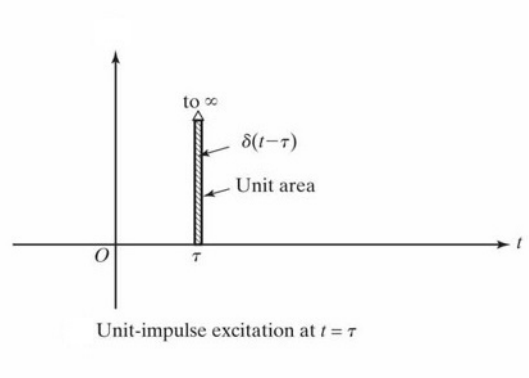
$$\begin{aligned} & x(t) = \mathcal{L}^{-1}\left[\frac{F(s)}{ms^2 + cs + k}\right] \\ \Rightarrow \quad & \frac{\mathcal{L}[x(t)]}{\mathcal{L}[f(t)]} = \frac{X(s)}{F(s)} = \frac{1}{ms^2+cs+k} = T(s) \end{aligned}$$

The **transfer function** is the ratio of output to input in the Laplace domain assuming zero initial conditions. What temporal function does $T(s)$ represent?

Impulse Response

An impulse is a very large force that acts for a very short time such that its time integral is finite. A unit impulse applied at $t = \tau$ is denoted by the delta function $\delta(t - \tau)$, which has the properties that

$$\begin{aligned} & \delta(t - \tau) = 0, \qquad t \neq \tau \\ & \int_{-\infty}^{\infty} \delta(t - \tau)dt = 1 \\ & \int_{-\infty}^{\infty} f(t)\delta(t - \tau)dt = f(\tau) \end{aligned}$$



Unit-impulse excitation at $t = \tau$

The impulse response $h(t)$ is the response of a system with zero initial conditions to the unit impulse $\delta(t)$.

$$\begin{aligned} & m\ddot{h} + c\dot{h} + kh = \delta(t), && h(0) = \dot{h}(0) = 0 \\ \Rightarrow \quad & m\mathcal{L}[\ddot{h}] + c\mathcal{L}[\dot{h}] + k\mathcal{L}[h] = \mathcal{L}[\delta(t)] \\ \Rightarrow \quad & H(s) = \mathcal{L}[h(t)] = \frac{1}{ms^2+cs+k} = T(s) \end{aligned}$$

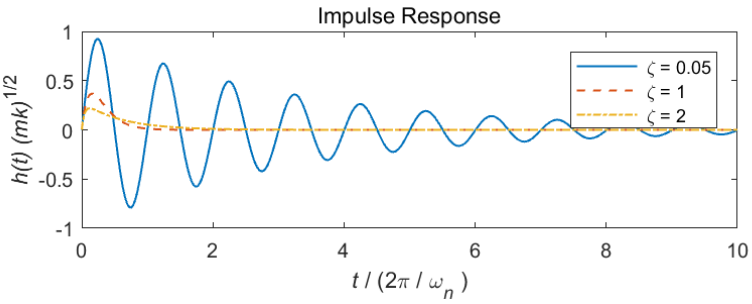
Remark. The Laplace transform of the impulse response $h(t)$ is the transfer function.

The impulse response is either oscillatory with frequency $\omega_d = \omega_n\sqrt{1-\zeta^2}$ or non-oscillatory.

$$h(t) = \begin{cases} \dfrac{1}{m\omega_d} e^{-\zeta\omega_n t} \sin \omega_d t\,, & \zeta < 1 \\ \dfrac{1}{m\omega_n\sqrt{\zeta^2-1}} e^{-\zeta\omega_n t} \sinh(\omega_n\sqrt{\zeta^2-1}\, t)\,, & \zeta > 1 \\ \dfrac{t}{m} e^{-\zeta\omega_n t}, & \zeta = 1 \end{cases}$$

$$\Rightarrow \quad \sqrt{mk}h(t) = \begin{cases} \dfrac{1}{\sqrt{1-\zeta^2}} e^{-\zeta 2\pi t/T} \sin(\sqrt{1-\zeta^2}\,\dfrac{2\pi t}{T})\,, & \zeta < 1 \\ \dfrac{1}{\sqrt{\zeta^2-1}} e^{-\zeta 2\pi t/T} \sinh(\sqrt{\zeta^2-1}\,\dfrac{2\pi t}{T})\,, & \zeta > 1 \\ \dfrac{2\pi t}{T} e^{-\zeta 2\pi t/T}, & \zeta = 1 \end{cases}$$

where $t > 0$ and $T = 2\pi/\omega_n$ is the natural period.



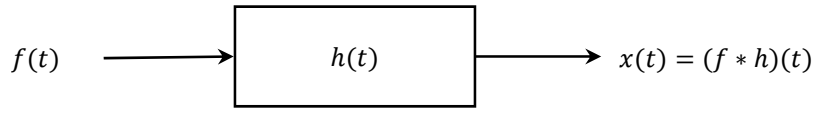
Response to Arbitrary Input
When driven by an excitation $f(t)$, the particular solution is

$$x(t) = \mathcal{L}^{-1}\left[\frac{F(s)}{ms^2 + cs + k}\right] = \mathcal{L}^{-1}[F(s)H(s)] = \mathcal{L}^{-1}\{\mathcal{L}[f(t)]\mathcal{L}[h(t)]\}$$

By the convolution theorem for Laplace transforms,

$$x(t) = \int_0^t f(\tau)h(t-\tau)d\tau = (f * h)(t)$$

Remark. In the time domain, the particular solution is the Laplace convolution of the impulse response and excitation.



Convolution is commutative because

$$(f * h)(t) = \int_0^t f(\tau)h(t-\tau)d\tau = \int_0^t h(\tau)f(t-\tau)d\tau = (h * f)(t)$$

FREQUENCY RESPONSE METHODS

Any temporal function $f(t)$ can be specified uniquely in terms of its frequencies. The recent popularity of frequency-domain analysis is due to the availability of FFT.

2.1 Harmonically Excited Vibration

When a system is subjected to harmonic excitation, it is forced to vibrate at the same frequency as that of the excitation. Common sources of harmonic excitation are unbalance in rotating machines, forces produced by reciprocating engines, and the pure tone produced by a tuning fork.

Complex Exponential Method of Solution

A harmonic function is a sinusoidal function which can be expressed as

$$f(t) = f_0 \cos(\omega t - \phi) = f_0 \mathrm{Re}[e^{i(\omega t - \phi)}] = \mathrm{Re}[f_0 e^{-i\phi} e^{i\omega t}] = \mathrm{Re}[F_0 e^{i\omega t}]$$

where $F_0 = f_0 e^{-i\phi}$ is a complex amplitude. The use of complex exponential representation provides a very effective way to manipulate with harmonic functions. Unless stated otherwise, a harmonic force is written as $F_0 e^{i\omega t}$ for which the **real part** is intended.

The homogeneous solution of

$$m\ddot{x} + c\dot{x} + kx = f(t)$$

has been represented by the exponential form $Ae^{\lambda t}$, where A and λ are often complex. The particular solution is also expressible in complex exponential form if $f(t)$ is harmonic. Consider the real differential equation

$$m\ddot{x} + c\dot{x} + kx = f_0 \cos \omega t$$

To obtain a solution of the above equation, introduce the complex differential equation

$$m\ddot{x} + c\dot{x} + kx = f_0 e^{i\omega t}$$

Since the excitation is complex, the solution is also complex. It can be proved that **the real part of the complex solution is a solution of the original real differential equation**. Suppose the homogeneous solution can be neglected. Postulate that the steady-state solution is

$$x(t) = Xe^{i\omega t}$$

Upon substitution,

$$m(i\omega)^2 Xe^{i\omega t} + c(i\omega)Xe^{i\omega t} + kXe^{i\omega t} = f_0 e^{i\omega t}$$

$$\Rightarrow \quad X = \frac{f_0}{m(i\omega)^2 + c(i\omega) + k}$$

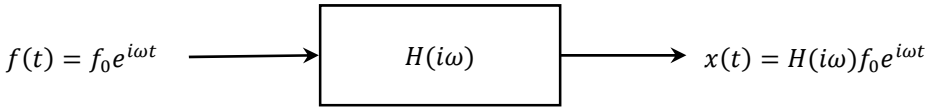
$$\Rightarrow \qquad H(i\omega) = \frac{Xe^{i\omega t}}{f_0 e^{i\omega t}} = \frac{1}{m(i\omega)^2 + c(i\omega) + k} = |H(i\omega)|e^{-i\phi}$$

The **complex frequency response** $H(i\omega)$, or frequency-response function, is the ratio of output to harmonic input represented by complex exponential form. It can be checked that

$$H(i\omega) = \left.\frac{1}{ms^2 + cs + k}\right|_{s=i\omega} = T(s)|_{s=i\omega}$$

where $T(s)$ is the transfer function in the Laplace domain.

Remark. The steady-state response to harmonic excitation is also harmonic with the same frequency as the excitation.



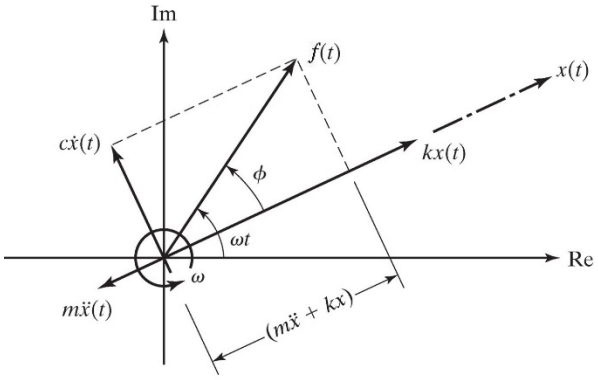
The physical response is

$$\mathrm{Re}[x(t)] = \mathrm{Re}[H(i\omega) f_0 e^{i\omega t}] = |H(i\omega)| f_0 \cos(\omega t - \phi)$$

Theoretically speaking, a harmonic function can also be taken as the imaginary part of a complex exponential representation. It can be readily shown that

$$f(t) = \mathrm{Im}[f_0 e^{i\omega t}] = f_0 \sin \omega t$$
$$\Rightarrow \qquad x(t) = \mathrm{Im}[H(i\omega) f_0 e^{i\omega t}] = |H(i\omega)| f_0 \sin(\omega t - \phi)$$

When driven by $f_0 e^{i\omega t}$, the response is magnified in amplitude by a factor $|H(i\omega)|$ and it lags the excitation by a phase angle ϕ. In the complex plane, the physical response and excitation are projections of the respective phasors on the real axis.



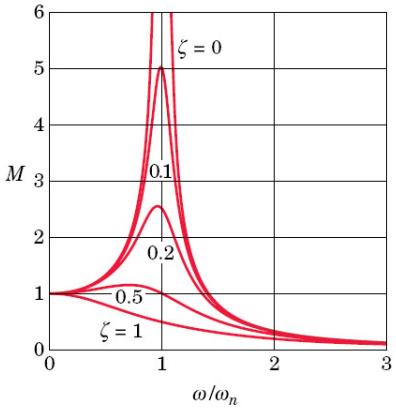
A non-dimensional magnification factor M is defined by

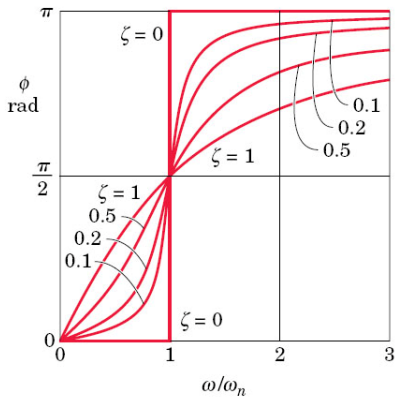
$$M = k\left|\frac{x(t)}{f(t)}\right| = k|H(i\omega)| = \frac{1}{\{[1-(\omega/\omega_n)^2]^2 + [2\zeta\omega/\omega_n]^2\}^{1/2}}$$

In addition,

$$\phi = \tan^{-1}\frac{2\zeta\omega/\omega_n}{1-(\omega/\omega_n)^2}$$

A plot of $M = k|H(i\omega)|$ against the excitation frequency is called a **frequency response curve** or a resonance curve. A resonant frequency is a driving frequency at which the largest response amplitude occurs. At resonance, the driving force supplies maximum power to the system and the phase angle $\phi = 90°$ for any ζ.





In general, a plot of the amplitude of any response quantity against the excitation frequency is called a frequency response curve. For example, one may produce a frequency response curve for $x^2 + \dot{x}^2$. When frequency response is plotted using a logarithmic scale, the curve constitutes a Bode plot of the system.

2.2 Response to Periodic Excitation

Many forces are periodic or nearly periodic. The torque generated by a reciprocating engine, for example, often deviates from a pure sinusoidal form due to slight imperfections.

2.2.1 Fourier Series

Let $f(t)$ be periodic with minimum period T so that $f(t) = f(t+T)$. Then $\omega = 2\pi/T$ is called the fundamental frequency. Around 1822, Fourier expounded a theory to synthesize periodic functions.

Fourier Theorem. Any periodic function may be expressed as a sum of harmonic functions whose frequencies are integral multiples of the fundamental frequency.

In trigonometric form, the Fourier series is given by

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos n\omega t + b_n \sin n\omega t)$$

where the real Fourier coefficients a_n and b_n are

$$a_n = \frac{2}{T}\int_0^T f(t) \cos n\omega t \, dt$$
$$b_n = \frac{2}{T}\int_0^T f(t) \sin n\omega t \, dt$$

The constant term $a_0/2$ is the mean of $f(t)$ over a period. The trigonometric form of Fourier series can be cast in a complex form:

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{in\omega t}$$

where the complex Fourier coefficients are

$$c_n = \frac{1}{T}\int_0^T f(t) e^{-in\omega t} dt$$

A complex Fourier series represents a real function because $c_{-n}e^{-in\omega t} = (c_n e^{in\omega t})^*$. The negative frequencies $n\omega$ for $n < 0$ arise from complex conjugation and they do not have any physical meaning. The trigonometric and complex forms of Fourier series are equivalent. It can be shown that

$$c_0 = \tfrac{1}{2}a_0, \qquad c_n = \tfrac{1}{2}(a_n - ib_n), \qquad c_{-n} = c_n^* = \tfrac{1}{2}(a_n + ib_n)$$

Convergence

There are at present no known necessary and sufficient conditions for the convergence of Fourier series. The continuity of $f(t)$ alone does not ensure convergence of its Fourier series. A sufficient condition for the convergence of Fourier series is the **Dirichlet condition**, which requires that $f(t)$ and $\dot{f}(t)$ be piecewise continuous. However, this is not a necessary condition.

Practical Significance of Periodic Excitation
In most applications, it is not necessary to go beyond periodic functions. If a non-periodic signal $f(t)$ is transient, it is quiescent outside a finite window. Then $f(t)$ is represented as a periodic function whose period is the finite time window. Even if a non-periodic signal $f(t)$ persists over a very long duration, the signal can only be sampled in a finite window. Again $f(t)$ is taken as a periodic function with the sampling window as its period.

2.2.2 Forced Periodic Response

Consider a system driven by a periodic force so that

$$m\ddot{x} + c\dot{x} + kx = \sum_{n=-\infty}^{\infty} c_n\, e^{in\omega t}$$

Each term of the Fourier series is a harmonic function.

$$\begin{aligned} & f(t) = c_n e^{in\omega t} \\ \Rightarrow \quad & x(t) = H(in\omega) c_n e^{in\omega t} \end{aligned}$$

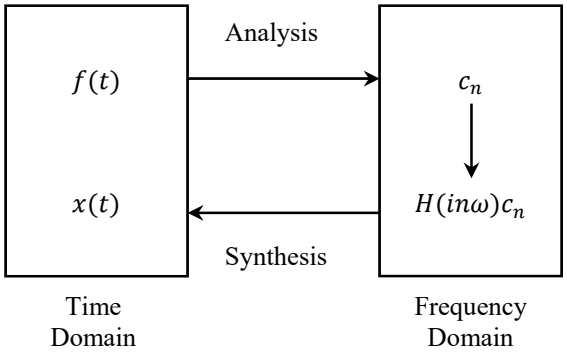
By superposition,

$$\begin{aligned} & f(t) = \sum_{n=-\infty}^{\infty} c_n\, e^{in\omega t} \\ \Rightarrow \quad & x(t) = \textstyle\sum_{n=-\infty}^{\infty} H(in\omega) c_n\, e^{in\omega t} \end{aligned}$$

The response $x(t)$ is itself a Fourier series whose Fourier coefficients are those of $f(t)$ modulated by the complex frequency response. The system may be driven into resonance if $\omega_n = m\omega$ for some integer m.

Remark. The steady-state response to periodic excitation is also periodic with the same period as the excitation.

The determination of the harmonic components of a periodic function is called **Fourier analysis**. The inverse process of combining harmonics to form a periodic function is called **Fourier synthesis**. Computation of the steady-state response to periodic excitation is carried out by transformation to the frequency domain.



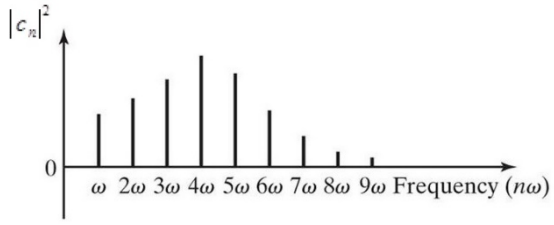
A. K. Chopra, *Dynamics of Structures: Theory and Applications to Earthquake Engineering*, 5*th* ed., Pearson, Hoboken, New Jersey, 901-920 (2017).

Visualization of Frequency Contents

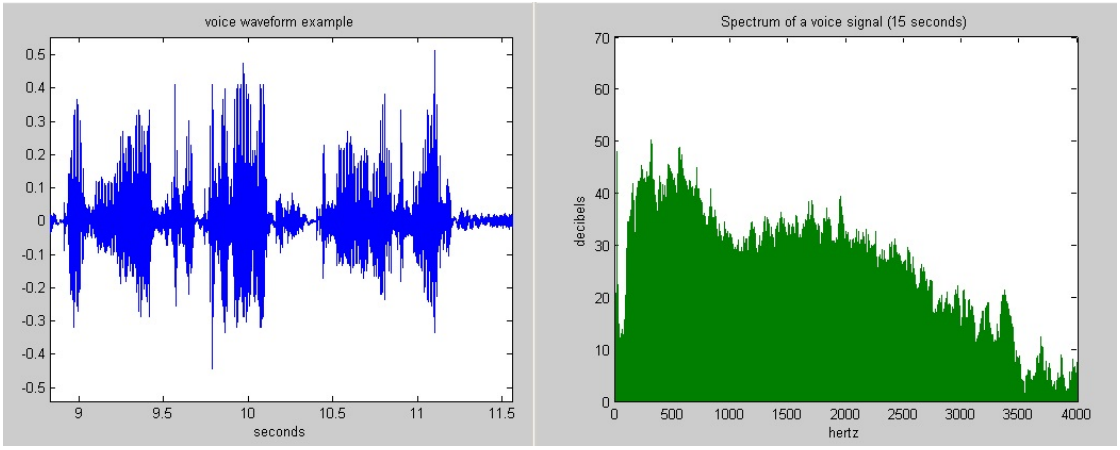
The **power spectrum** of a time function $f(t)$ is a frequency response curve of its energy density. For a periodic function,

$$\frac{1}{T}\int_0^T f^2(t)dt = \sum_{n=-\infty}^{\infty} |c_n|^2$$

by Parseval's identity. Since the integral on the left-hand side is the average energy of the signal, the power spectrum may be taken as a plot of $|c_n|^2$ versus the frequency $n\omega$. It is sufficient to plot the power spectrum for $n\omega \geq 0$ only because $|c_n| = |c_{-n}|$. A power spectrum shows the relative significance of the frequencies characterizing $f(t)$.



The **amplitude spectrum** is the square root of the power spectrum. A spectrum analyzer is an electronic device that generates the power spectrum of a waveform and displays the output as a bar graph. The MATLAB function **fft** computes, among other things, the amplitude spectrum of a function using a FFT algorithm.



The voice waveform over time (left) has a broad audio power spectrum (right)

2.3 Response to Non-Periodic Excitation

Conceptually, a non-periodic function can be regarded as a periodic function with a very large period. As the period T of a periodic function increases indefinitely, its Fourier series becomes a Fourier integral associated with Fourier transforms.

Fourier Transforms

Let $f(t)$ be periodic with fundamental frequency $\omega_0 = 2\pi/T$. As $T \to \infty$, the Fourier series of $f(t)$ converges to a Fourier integral such that

$$f(t) = \sum_{n=-\infty}^{\infty} c_n \, e^{in\omega_0 t} \to \int_{-\infty}^{\infty} F(i\omega) e^{i\omega t} d\omega$$

where the temporal function $f(t)$ and spectral function $F(i\omega)$ constitute a Fourier transform pair.

$$F(i\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt = \mathcal{F}[f(t)]$$
$$f(t) = \int_{-\infty}^{\infty} F(i\omega) e^{i\omega t} d\omega = \mathcal{F}^{-1}[F(i\omega)]$$

The constants pre-multiplying the forward and inverse Fourier transforms can be chosen arbitrarily as long as their product is $1/(2\pi)$. There is one-to-one correspondence between functions in the time and frequency domains by means of Fourier transforms.

Numerical Processing and Amplitude Spectrum

The MATLAB function **fft** computes the Fourier transform of a function by using a FFT algorithm. As a by-product, the amplitude spectrum is produced. The function $f(t)$ is sampled within a chosen time window to generate a vector x of length n. Then the command

$$Y = \text{fft}(x, n)$$

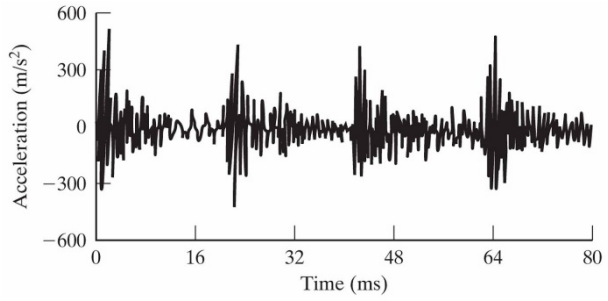
returns the n-point discrete Fourier transform of vector x, which is complex and $|Y| = |\mathcal{F}[f(t)]|$ gives the amplitude spectrum. The speed of **fft** is the highest if n is a power of 2 and it is several times slower if n is a large prime number. The command

$$y = \text{ifft}(X, n)$$

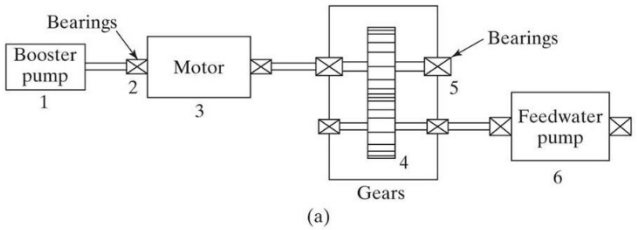
returns the n-point inverse discrete Fourier transform of vector X, computed with a FFT algorithm.

2.4 Machine-Condition Monitoring

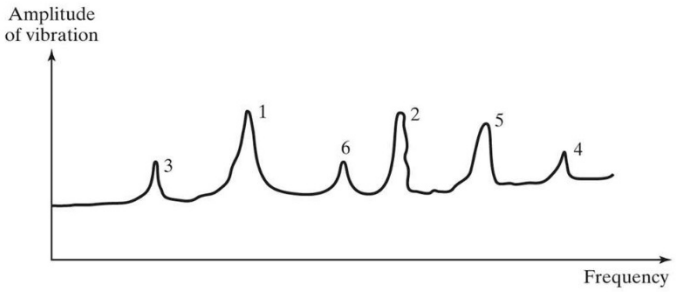
Vibration analysis is most commonly used for machine-condition monitoring. Time-domain analysis can be used to study vibration waveforms. As an example, the time-domain waveform of a faulty gearbox is shown.



In frequency-domain analysis, **changes in the amplitude spectrum of a machine are used as an indicator of degradation** or damage. In many cases, each rotating element in a machine generates an identifiable frequency in the amplitude spectrum. Thus changes in the amplitude spectrum at a specified frequency can even be attributed directly to a specific machine component.



(a)



(b)

ANALYSIS OF DISCRETE SYSTEMS

Lecture Outline
Form of Equations of Motion
Configuration Space Analysis
Modal Analysis
State Space Analysis
Estimation and Control

Analytical mechanics is an approach to mechanics in which the notion of systems is emphasized. In analytical mechanics, the treatment of a system is based upon three scalar quantities: the system kinetic energy T, the system potential energy V, and the work function W. There are two principal methods, one due to Lagrange (1788) and a second one due to Hamilton (1834). All formulations of mechanics are equivalent.

3.1 Lagrange's Equations

For a holonomic system specified by n independent generalized coordinates q_1, q_2,…, q_n, the equations of motion are

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} + \frac{\partial V}{\partial q_i} = Q_i$$

where $i = 1, 2, \ldots, n$ and Q_i is the non-conservative generalized force associated with the coordinate q_i. Conservative forces are already incorporated in the potential energy V. The Lagrangian formulation utilizes only the q's, $\dot{q}$'s, and t as independent variables. In general, $T = T(\mathbf{q}, \dot{\mathbf{q}}, t)$, $V = V(\mathbf{q}, t)$, and $Q_i = Q_i(\mathbf{q}, \dot{\mathbf{q}}, t)$.

The generalized force Q_i is defined so that, if q_i changes by δq_i while all other q_j $(j \neq i)$ are held constant, the work done is given by

$$\delta W = Q_i \delta q_i$$

Lagrange's equations do not involve accelerations explicitly. The Lagrangian formulation is only applicable to systems with a finite DOF.

D. T. Greenwood, *Principles of Dynamics*, 2*nd* ed., Prentice Hall, Englewood Cliffs, New Jersey, 262-269 (1988).

Form of Equations of Motion
The following important developments can be made.

1. The Lagrange's equations of a system can be conveniently written as a vector differential equation in the form

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\mathbf{q}}}\right) - \frac{\partial T}{\partial \mathbf{q}} + \frac{\partial V}{\partial \mathbf{q}} = \mathbf{Q}$$

Note that the derivative of a multivariate scalar function f with respect to an n-dimensional vector such as $\mathbf{q}$ is defined by

$$\frac{\partial f}{\partial \mathbf{q}} = \nabla f = \left[\frac{\partial f}{\partial q_1} \quad \frac{\partial f}{\partial q_2} \quad \cdots \quad \frac{\partial f}{\partial q_n}\right]^T$$

2. The Lagrange's equations of an n-degree-of-freedom nonlinear system can be expressed as

$$\mathbf{M}(\mathbf{q}, t)\ddot{\mathbf{q}} + \mathbf{f}(\mathbf{q}, \dot{\mathbf{q}}, t) = \mathbf{0}$$

where $\mathbf{M}(\mathbf{q}, t)$ is a symmetric matrix function and $\mathbf{f}(\mathbf{q}, \dot{\mathbf{q}}, t)$ is a nonlinear vector function. Lagrange's equations are linear in the $\ddot{q}$'s.

3. Upon linearization about the equilibrium position at $\mathbf{q} = \mathbf{0}$, a second-order linear time-varying equation

$$\mathbf{M}(t)\ddot{\mathbf{q}} + \mathbf{A}(t)\dot{\mathbf{q}} + \mathbf{B}(t)\mathbf{q} = \mathbf{f}(t)$$

is obtained, where $\mathbf{M}(t)$ is a symmetric matrix function.

4. Assume that the system is time-invariant, the equation of motion becomes

$$\mathbf{M}\ddot{\mathbf{q}} + (\mathbf{C} + \mathbf{G})\dot{\mathbf{q}} + (\mathbf{K} + \mathbf{H})\mathbf{q} = \mathbf{f}(t)$$

where $\mathbf{M} = \mathbf{M}^T$ is the mass matrix, $\mathbf{C} = \mathbf{C}^T$ is the damping matrix, $\mathbf{G} = -\mathbf{G}^T$ is the gyroscopic matrix, $\mathbf{K} = \mathbf{K}^T$ is the stiffness matrix, and $\mathbf{H} = -\mathbf{H}^T$ is the circulatory matrix.

5. For non-gyroscopic and non-circulatory LTI systems, the equation of motion is

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{C}\dot{\mathbf{q}} + \mathbf{K}\mathbf{q} = \mathbf{f}(t)$$

D. T. Greenwood, *Principles of Dynamics*, 2*nd* ed., Prentice Hall, Englewood Cliffs, New Jersey, 482-486, 522-524 (1988).

L. Meirovitch, *Principles and Techniques of Vibrations*, Prentice Hall, Upper Saddle River, New Jersey, 162-167 (1997).

Example. Consider a conservative system specified by n independent generalized coordinates q_1, q_2,…, q_n. Suppose (a) the kinetic energy is a quadratic form of the velocities so that

$$T = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} m_{ij}\dot{q}_i\dot{q}_j = \frac{1}{2}\dot{\mathbf{q}}^T\mathbf{M}\dot{\mathbf{q}}$$

and (b) the potential energy is a function of the displacements so that

$$V = V(q_1, q_2, \dots, q_n) = V(\mathbf{q})$$

Show that the equation of free vibration is

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{K}\mathbf{q} = \mathbf{0}$$

Solution

A quadratic form is a homogenous polynomial of degree two. There is no loss of generality in assuming that the matrix **M** associated with a quadratic form is symmetric. This is so because any real square matrix can be expressed as the sum of a real symmetric matrix and a real skew-symmetric matrix. If **A** is a real square matrix, then

$$\mathbf{A} = \frac{1}{2}(\mathbf{A} + \mathbf{A}^T) + \frac{1}{2}(\mathbf{A} - \mathbf{A}^T)$$

where

$$(\mathbf{A} + \mathbf{A}^T)^T = \mathbf{A}^T + \mathbf{A} = \mathbf{A} + \mathbf{A}^T$$
$$(\mathbf{A} - \mathbf{A}^T)^T = \mathbf{A}^T - \mathbf{A} = -(\mathbf{A} - \mathbf{A}^T)$$

As a consequence,

$$\mathbf{x}^T\mathbf{A}\mathbf{x} = \frac{1}{2}\mathbf{x}^T(\mathbf{A} + \mathbf{A}^T)\mathbf{x}$$

The quadratic forms corresponding to **A** and its symmetric component are the same.

Expand V in a Taylor series about the equilibrium position $\mathbf{q} = \mathbf{0}$,

$$V(q_1, q_2, \dots, q_n) = V(\mathbf{0}) + \sum_{i=1}^{n}\frac{\partial V}{\partial q_i}\, q_i + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\partial^2 V}{\partial q_i\, \partial q_j}\, q_i q_j + \cdots$$

where all partial derivatives of V are evaluated at $\mathbf{q} = \mathbf{0}$. Choose the reference level of potential energy so that $V(\mathbf{0}) = 0$. Define the stiffness coefficients by

$$k_{ij} = \frac{\partial^2 V}{\partial q_i\, \partial q_j} = \frac{\partial^2 V}{\partial q_j\, \partial q_i} = k_{ji}$$

Then

$$V \cong \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} k_{ij}q_i q_j \; = \frac{1}{2}\mathbf{q}^T\mathbf{K}\mathbf{q}$$

The Lagrange's equations are

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} + \frac{\partial V}{\partial q_i} = 0, \qquad i = 1, 2, \dots, n$$
$$\Rightarrow \quad \sum_{j=1}^{n} m_{ij}\ddot{q}_j + \sum_{j=1}^{n} k_{ij}q_j \; = 0, \qquad i = 1, 2, \dots, n$$
$$\Rightarrow \quad \mathbf{M}\ddot{\mathbf{q}} + \mathbf{K}\mathbf{q} = \mathbf{0}$$

where $\mathbf{M} = \mathbf{M}^T$ and $\mathbf{K} = \mathbf{K}^T$.

The mass matrix $\mathbf{M} = \mathbf{M}^T$ is positive definite because, if the system is in motion,

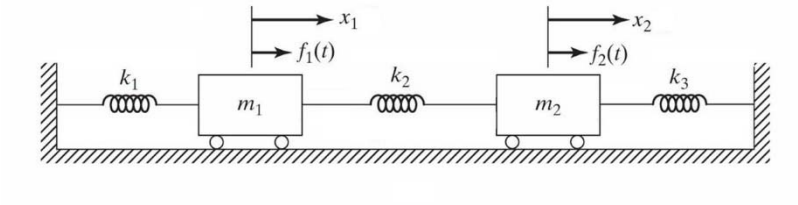
$$\dot{\mathbf{q}} \neq \mathbf{0} \qquad \Rightarrow \qquad T = \tfrac{1}{2}\dot{\mathbf{q}}^T\mathbf{M}\dot{\mathbf{q}} > 0$$

Suppose $\mathbf{q} = \mathbf{0}$ is a stable equilibrium position. The stiffness matrix $\mathbf{K} = \mathbf{K}^T$ is positive semidefinite because

$$\mathbf{q} \neq \mathbf{0} \qquad \Rightarrow \qquad V = \frac{1}{2}\mathbf{q}^T\mathbf{K}\mathbf{q} \geq 0$$

If $\mathbf{q} = \mathbf{0}$ is the only equilibrium position and the system is asymptotically stable, then the stiffness matrix $\mathbf{K}$ is positive definite.

Example. Derive the equations of motion of the elastic system shown.



Solution

Let x_1 and x_2 be the displacements of m_1 and m_2 from static equilibrium. Then

$$T = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2$$

$$V = \frac{1}{2}k_1x_1^2 + \frac{1}{2}k_2(x_2 - x_1)^2 + \frac{1}{2}k_3x_2^2$$

where $V = 0$ at the equilibrium configuration. The non-conservative forces are $f_1(t)$ and $f_2(t)$. When x_1 is incremented by δx_1 with x_2 held constant, the work input is

$$f_1(t)\delta x_1 = Q_1\delta x_1$$

$$\Rightarrow \quad Q_1 = f_1(t)$$

Similarly, it is found that $Q_2 = f_2(t)$. The Lagrange's equations are

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_1}\right) - \frac{\partial T}{\partial x_1} + \frac{\partial V}{\partial x_1} = Q_1$$

$$\Rightarrow \quad m_1\ddot{x}_1 + k_1x_1 - k_2(x_2 - x_1) = f_1(t)$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_2}\right) - \frac{\partial T}{\partial x_2} + \frac{\partial V}{\partial x_2} = Q_2$$

$$\Rightarrow \quad m_2\ddot{x}_2 + k_2(x_2 - x_1) + k_3x_2 = f_2(t)$$

These equations can be written as a vector differential equation with matrix coefficients such that

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} f_1(t) \\ f_2(t) \end{bmatrix}$$

By defining the displacement variables to be zero at equilibrium rather than at the position of zero spring deflection, static forces associated with equilibrium can be ignored. Suppose the springs k_1, k_2, k_3 are stretched by δ_1, δ_2 and δ_3 respectively at equilibrium. Balance of forces in the springs at equilibrium requires

$$k_1\delta_1 = k_2\delta_2 = k_3\delta_3$$

Clearly, T remains unchanged and

$$\begin{aligned} V &= \frac{1}{2}k_1(x_1 + \delta_1)^2 - \frac{1}{2}k_1\delta_1^2 + \frac{1}{2}k_2(x_2 - x_1 + \delta_2)^2 - \frac{1}{2}k_2\delta_2^2 + \frac{1}{2}k_3(x_2 - \delta_3)^2 - \frac{1}{2}k_3\delta_3^2 \\ &= \frac{1}{2}k_1x_1^2 + \frac{1}{2}k_2(x_2 - x_1)^2 + \frac{1}{2}k_3x_2^2 + k_1x_1\delta_1 + k_2(x_2 - x_1)\delta_2 - k_3x_2\delta_3 \\ &= \frac{1}{2}k_1x_1^2 + \frac{1}{2}k_2(x_2 - x_1)^2 + \frac{1}{2}k_3x_2^2 \end{aligned}$$

3.2 Configuration Space Analysis

The equation of motion of an n-degree-of-freedom damped linear system can be written as

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{C}\dot{\mathbf{q}} + \mathbf{K}\mathbf{q} = \mathbf{f}(t)$$

Properties of solution of this equation are similar to those of SDOF equation. The general solution is obtained by adding the homogeneous solution to a particular solution (a solution without arbitrary constants),

$$\mathbf{q}(t) = \mathbf{q}_h(t) + \mathbf{q}_p(t)$$

3.2.1 Steady State Response

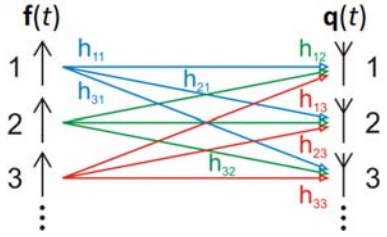
Under viscous damping, the homogeneous solution $\mathbf{q}_h(t)$ decays exponentially and it may be ignored. Theoretically speaking, the particular solution $\mathbf{q}_p(t)$ can be given in an exact form.

In control engineering, an MDOF system with excitation vector $\mathbf{f}(t)$ and response vector $\mathbf{q}(t)$ is regarded as an multi-input multi-output system. When a single input, say $f_1(t)$, is applied to channel 1 (other elements of $\mathbf{f}(t)$ being zero), an output $q_i(t)$ is generated for all i.

Impulse Response Method

Suppose a unit impulse $\delta(t)$ is applied to channel j of an MIMO system with zero initial conditions (the excitation being zero in other channels). Denote the impulse response in channel i by $h_{ij}(t)$. When an arbitrary force $f_j(t)$ is applied to channel j, the response in channel i is, by convolution, equal to

$$q_i(t) = \int_0^t h_{ij}(\tau) f_j(t-\tau) d\tau$$



When driven by an excitation $\mathbf{f}(t)$, the total response in channel i is, by superposition, equal to

$$q_i(t) = \sum_{j=1}^{n} \int_0^t h_{ij}(\tau) f_j(t-\tau) d\tau$$

Define an impulse response matrix by $\mathbf{H}(t) = [h_{ij}(t)]$, which is of order $n \times n$. In the time domain, the particular solution is

$$\mathbf{q}(t) = \int_0^t \mathbf{H}(\tau)\mathbf{f}(t-\tau)d\tau$$

This convolution is not commutative because $\mathbf{f}(\tau)\mathbf{H}(t-\tau)$ is not defined due to conformability requirements. It is also not efficient to use this formula to evaluate the output of an MDOF system.

Frequency Response Method

The complex frequency-response matrix, of order $n \times n$, is defined by

$$\mathbf{H}(i\omega) = [\mathbf{M}(i\omega)^2 + \mathbf{C}(i\omega) + \mathbf{K}]^{-1}$$

For a sinusoidal excitation $\mathbf{f}(t) = \mathbf{f}_0 e^{i\omega t}$, the steady-state response is

$$\mathbf{q}(t) = \mathbf{H}(i\omega)\mathbf{f}_0 e^{i\omega t}$$

While this formula is exact, it is not efficient for the evaluation of output because of matrix inversion.

Numerical Solution

The MATLAB function **ode45** solves systems of ordinary differential equations. It consists of an automatic step-size Runge-Kutta integration method which is a combination of a fourth and fifth-order method. In order to use this function, the differential equation is first written as a first-order equation

$$\dot{\mathbf{y}} = \mathbf{f}(\mathbf{y}, t)$$

Before calling the solver, the equation

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{C}\dot{\mathbf{q}} + \mathbf{K}\mathbf{q} = \mathbf{f}(t)$$

should be cast in the form

$$\dot{\mathbf{y}} = \begin{bmatrix}\dot{\mathbf{q}} \\ \ddot{\mathbf{q}}\end{bmatrix} = \begin{bmatrix}\mathbf{0} & \mathbf{I} \\ -\mathbf{M}^{-1}\mathbf{K} & -\mathbf{M}^{-1}\mathbf{C}\end{bmatrix}\begin{bmatrix}\mathbf{q} \\ \dot{\mathbf{q}}\end{bmatrix} + \begin{bmatrix}\mathbf{0} \\ \mathbf{M}^{-1}\mathbf{f}(t)\end{bmatrix} = \mathbf{A}\mathbf{y} + \mathbf{g}(t)$$

The first-order equation is then specified by a function file

$$\text{function } dydt = odefun(t, y)$$

whose handle is $@odefun$, and with t, $\mathbf{y}$ as input and $\dot{\mathbf{y}}$ as output. The command

$$[t, y] = \text{ode45}(@odefun, tspan, y0)$$

solves the differential equation and returns a solution array y associated with time points in the interval $tspan = [t_0, t_f]$, where t_0 and t_f are respectively the beginning and final values of t. The vector $y0 = \mathbf{y}(t_0)$ contains the initial values. If $\mathbf{y}$ is n-dimensional, the solution array y has n

columns. Each row in the array y gives $\mathbf{y}(t)$ at a value returned in the column vector t.

The solver **ode45** is a general-purpose solver suitable for both linear and nonlinear differential equations. If the equation is stiff, as indicated by a long-running solution process or by a warning or error message, then use **ode15s**.

3.2.2 Stability of Homogeneous System

In free vibration, the equation of motion is

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{C}\dot{\mathbf{q}} + \mathbf{K}\mathbf{q} = \mathbf{0}$$

Postulate that

$$\mathbf{q} = \mathbf{v}e^{\lambda t}$$

where $\mathbf{v}$ is an n-dimensional column vector of unspecified constants and λ is a scalar parameter. Upon substitution, one obtains a quadratic (nonlinear) eigenvalue problem

$$(\mathbf{M}\lambda^2 + \mathbf{C}\lambda + \mathbf{K})\mathbf{v} = \mathbf{0}$$

The eigenvalues are the roots of the polynomial equation

$$\det(\mathbf{M}\lambda^2 + \mathbf{C}\lambda + \mathbf{K}) = 0$$

There are $2n$ eigenvalues λ_j but there cannot be more than n linearly independent eigenvectors $\mathbf{v}_j$. If the eigenvalues λ_j are distinct, the homogeneous solution is a linear combination of the form

$$\mathbf{q} = \sum_{j=1}^{2n} c_j \mathbf{v}_j e^{\lambda_j t}$$

If the quadratic eigenvalue problem is defective, powers of t will appear. For example, if $\lambda_1 = \lambda_2$ and there is only one eigenvector $\mathbf{v}_1$, the term $(c_1 + c_2 t)\mathbf{v}_1 e^{\lambda_1 t}$ appears in the homogeneous solution. An LTI system is stable if its solution is bounded or decays over time.

Remark. The homogeneous equation is stable if $\mathrm{Re}[\lambda_j] \leq 0$ for simple eigenvalues and $\mathrm{Re}[\lambda_j] < 0$ for repeated eigenvalues. The homogeneous system is asymptotically stable if $\mathrm{Re}[\lambda_j] < 0$ for every eigenvalue. The system is unstable if $\mathrm{Re}[\lambda_j] > 0$ for any eigenvalue.

If $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$ are symmetric and positive definite, energy is continuously dissipated by damping and the homogeneous system decays exponentially. As time progresses, the asymptotically stable homogeneous solution may be ignored. Let $\mathbf{v}_j e^{\lambda_j t}$ be an eigensolution, where λ_j and $\mathbf{v}_j$ may be complex. Pre-multiply

$$(\mathbf{M}\lambda_j^2 + \mathbf{C}\lambda_j + \mathbf{K})\mathbf{v}_j = \mathbf{0}$$

by the complex conjugate transpose $\mathbf{v}_j^*$ to obtain

$$\mathbf{v}_j^*\mathbf{M}\mathbf{v}_j\lambda_j^2 + \mathbf{v}_j^*\mathbf{C}\mathbf{v}_j\lambda_j + \mathbf{v}_j^*\mathbf{K}\mathbf{v}_j = 0$$

The roots of the above quadratic equation are given by

$$\lambda_j = \frac{-\mathbf{v}_j^*\mathbf{C}\mathbf{v}_j \pm \sqrt{(\mathbf{v}_j^*\mathbf{C}\mathbf{v}_j)^2 - 4(\mathbf{v}_j^*\mathbf{M}\mathbf{v}_j)(\mathbf{v}_j^*\mathbf{K}\mathbf{v}_j)}}{2\mathbf{v}_j^*\mathbf{M}\mathbf{v}_j}$$

$$\Rightarrow \quad \mathrm{Re}[\lambda_j] < 0$$

Numerical Solution

The MATLAB function **polyeig** solves a polynomial eigenvalue problem. For $(\mathbf{M}\lambda^2 + \mathbf{C}\lambda + \mathbf{K})\mathbf{v} = \mathbf{0}$, the command

$$[\mathrm{X}, \mathrm{e}] = \mathrm{polyeig}(\mathrm{K}, \mathrm{C}, \mathrm{M})$$

returns the eigenvalues and eigenvectors in the form

$$\mathrm{X} = [\mathbf{v}_1 \quad \mathbf{v}_2 \quad \cdots \quad \mathbf{v}_{2n}]$$

$$\mathrm{e} = \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_{2n} \end{bmatrix}$$

The Euclidean norm of any computed eigenvector $\mathbf{v}_j$ is one.

3.3 Modal Analysis

Modal analysis is a method for decoupling the equation of motion by means of a coordinate transformation. If an MDOF system can be decoupled, it may be regarded as composing of independent SDOF systems. Classical modal analysis utilizes a real coordinate transformation in the configuration space. Complex modal analysis utilizes a complex transformation in the state space.

Coordinate Coupling

In general, the vector differential equation for an MDOF system is coupled. The ith component equation involves not only q_i and its derivatives but also other coordinates and their derivatives as well. For example,

$$\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{bmatrix} + \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} + \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} f_1(t) \\ f_2(t) \end{bmatrix}$$

If **M** is diagonal, the system is said to be inertially decoupled. If **K** is diagonal, the system is elastically decoupled. Coupling is not an inherent property of a system but depends on the generalized coordinates used.

3.3.1 Conservative Free Vibration

A non-gyroscopic system in **conservative free vibration** is governed by

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{K}\mathbf{q} = \mathbf{0}$$

where the mass matrix **M** and the stiffness matrix **K** are symmetric and positive definite. By the complex exponential method, postulate that

$$\mathbf{q} = \mathbf{u}e^{\lambda t}$$

The physical response is the real part of the assumed solution if it is complex. Upon substitution, one obtains an eigenvalue problem

$$(\mathbf{M}\lambda^2 + \mathbf{K})\mathbf{u} = \mathbf{0}$$

The above quadratic eigenvalue problem in λ and $\mathbf{u}$ may be regarded as a linear eigenvalue problem in λ^2 and $\mathbf{u}$.

Eigenvalues

The eigenvalues are the roots of a polynomial equation

$$\det(\mathbf{M}\lambda^2 + \mathbf{K}) = 0$$

The fundamental theorem of algebra states that a polynomial equation of degree n has n roots, counting multiplicities of repeated roots. An explicit formula for the roots of a polynomial equation of degree $n > 4$ has not been found. Hence determination of eigenvalues is iterative in nature for large n.

All n eigenvalues λ^2 are real and negative if $\mathbf{M}$, $\mathbf{K}$ are symmetric and positive definite. Pre-multiply $(\mathbf{M}\lambda^2 + \mathbf{K})\mathbf{u} = \mathbf{0}$ by $\mathbf{u}^*$ to obtain

$$\begin{aligned} & \mathbf{u}^*\mathbf{M}\mathbf{u}\lambda^2 + \mathbf{u}^*\mathbf{K}\mathbf{u} = 0 \\ \Rightarrow \quad & \lambda^2 = -\frac{\mathbf{u}^*\mathbf{K}\mathbf{u}}{\mathbf{u}^*\mathbf{M}\mathbf{u}} < 0 \\ \Rightarrow \quad & \lambda = \pm i\omega, \qquad\qquad \omega > 0 \end{aligned}$$

Eigenvectors
Each eigenvector can at most be determined **up to a multiplicative constant**. If $\mathbf{u} \neq \mathbf{0}$ is an eigenvector, so is $c\mathbf{u}$ for any $c \neq 0$.

Every eigenvector can be chosen to be real. Let $\mathbf{u}$ be an eigenvector associated with the eigenvalue λ^2. Consider two mutually exclusive possibilities. First, suppose $\mathbf{u}$ is purely imaginary. In this case, $i\mathbf{u}$ is a real eigenvector associated with λ^2. Second, suppose $\mathbf{u}$ is not purely imaginary. Then $\mathbf{u} + \bar{\mathbf{u}}$ is real, and

$$\mathbf{K}(\mathbf{u} + \bar{\mathbf{u}}) = \mathbf{K}\mathbf{u} + \bar{\mathbf{K}}\bar{\mathbf{u}} = \lambda^2\mathbf{M}\mathbf{u} + \overline{\lambda^2}\bar{\mathbf{M}}\bar{\mathbf{u}} = \lambda^2\mathbf{M}(\mathbf{u} + \bar{\mathbf{u}})$$

Thus $\mathbf{u} + \bar{\mathbf{u}}$ is a real eigenvector associated with λ^2. It will be assumed that $\mathbf{u}$ is always taken as real.

Natural Frequencies and Mode Shapes
The postulation that $\mathbf{q} = \mathbf{u}e^{\lambda t}$ is equivalent to $\mathbf{q} = \mathbf{u}e^{\pm i\omega t}$ where $\omega > 0$ and $\mathbf{u}$ is real. When real parts are taken, $\mathbf{u}e^{i\omega t}$ and $\mathbf{u}e^{-i\omega t}$ generate the same physical response. There is no loss of generality in assuming that

$$\mathbf{q} = \mathbf{u}e^{i\omega t}$$

The parameter $\omega > 0$ is called a **natural frequency** and the corresponding eigenvector $\mathbf{u} \neq \mathbf{0}$ is termed a **mode shape**. The eigenvalue problem that generates the natural frequencies and mode shapes may be rewritten as

$$\mathbf{K}\mathbf{u} = \omega^2\mathbf{M}\mathbf{u}$$

By convention, the natural frequencies are arranged in order of increasing magnitude such that $\omega_1 \leq \omega_2 \leq \cdots \leq \omega_n$. The lowest frequency ω_1 is called the **fundamental frequency**.

Remark. There are n natural frequencies and all mode shapes are real.

Normal Modes of Vibration
It is well known that an undamped linear dynamical system possesses normal or natural modes, and that in each mode different parts of the system vibrate with the same frequency, passing through their equilibrium configuration at the same instant of time. Theoretically speaking, a normal mode of vibration associated with natural frequency ω_j is the real part of the corresponding

eigensolution so that

$$\mathbf{s}_j(t) = \mathrm{Re}[A_j \mathbf{u}_j e^{i\omega_j t}] = C_j \mathbf{u}_j \cos(\omega_j t - \phi_j)$$

where $A_j = C_j e^{-i\phi_j}$ is an arbitrary complex multiplier and $\mathbf{u}_j$ is a real eigenvector. A normal mode represents synchronous motion in which all system components perform simple harmonic motion with the same frequency ω_j and the **amplitude ratio of any two components is independent of time**. Any two system components are either in phase or 180° out of phase with each other.

There are n normal modes of vibration. Each normal mode is independently excitable. For example, $\mathbf{s}_j(t)$ can be excited with the initial conditions $\mathbf{q}(0) = \mathbf{u}_j$, $\dot{\mathbf{q}}(0) = 0$. The general response in free vibration is the real part of $\mathbf{q}(t)$ as obtained from

$$\mathbf{q}(t) = \sum_{j=1}^{n} c_j \mathbf{u}_j e^{i\omega_j t}$$

which is basically a linear combination of the normal modes.

Numerical Solution

The MATLAB function **eig** solves an eigenvalue problem. For $\mathbf{Ax} = \lambda\mathbf{x}$, the command

$$[\mathrm{V}, \mathrm{D}] = \mathrm{eig}(\mathrm{A})$$

returns the eigenvalues and eigenvectors of **A** in the form

$$\mathrm{V} = [\mathbf{x}_1 \quad \mathbf{x}_2 \quad \cdots \quad \mathbf{x}_n]$$
$$\mathrm{D} = \mathrm{diag}[\lambda_1, \lambda_2, \ldots, \lambda_n]$$

For $\mathbf{Ku} = \omega^2\mathbf{Mu}$, the command

$$[\mathrm{V}, \mathrm{D}] = \mathrm{eig}(\mathrm{K}, \mathrm{M})$$

returns the eigenvalues and eigenvectors in the form

$$\mathrm{V} = [\mathbf{u}_1 \quad \mathbf{u}_2 \quad \cdots \quad \mathbf{u}_n]$$
$$\mathrm{D} = \mathrm{diag}[\omega_1^2, \omega_2^2, \ldots, \omega_n^2]$$

Normalization of Eigenvectors

Each eigenvector of an eigenvalue problem can at most be determined up to a multiplicative constant. Scale factors are often used to standardize the eigenvectors. This process is called normalization. In general, output vectors are normalized by MATLAB to have unit Euclidean norm. For example, each eigenvector of $\mathbf{Ax} = \lambda\mathbf{x}$ satisfies

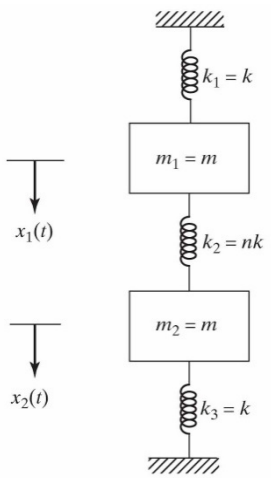
$$(\mathbf{x}^*\mathbf{x})^{1/2} = (\bar{\mathbf{x}}^T\mathbf{x})^{1/2} = \left(\sum_{i=1}^{n}|x_i|^2\right)^{1/2} = 1$$

In theoretical mechanics, it is customary to normalize the mode shapes $\mathbf{u}_j$ generated by $\mathbf{Ku} = \omega^2\mathbf{Mu}$ in accordance with

$$\mathbf{u}_j^T\mathbf{M}\mathbf{u}_j = 1, \qquad j = 1, 2, \ldots, n$$

All real eigenvectors of $\mathbf{Ku} = \omega^2\mathbf{Mu}$ are mass-normalized by MATLAB to conform to popular use. Normalization has no physical significance and is just a matter of convenience. A normalized eigenvector $\mathbf{u}_j$ is called a normal mode. Each normalized eigenvector is still not unique because its sign is arbitrary.

Example. Find the natural frequencies and mode shapes of the mass-spring system shown, which is constrained to move in a vertical direction only. Take $n = 1$.



Solution

The equation of motion is given by

$$\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} 2k & -k \\ -k & 2k \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

In modal vibration, $\mathbf{x} = \mathbf{u}e^{i\omega t}$ where $\mathbf{Ku} = \omega^2\mathbf{Mu}$. The frequency equation is

$$\begin{aligned} & \det(\mathbf{K} - \omega^2\mathbf{M}) = 0 \\ \Rightarrow \quad & \begin{vmatrix} 2k - \omega^2 m & -k \\ -k & 2k - \omega^2 m \end{vmatrix} = 0 \\ \Rightarrow \quad & \begin{vmatrix} 2 - \lambda & -1 \\ -1 & 2 - \lambda \end{vmatrix} = 0, \qquad \lambda = \frac{\omega^2 m}{k} \\ \Rightarrow \quad & \lambda^2 - 4\lambda + 3 = 0 \\ \Rightarrow \quad & \omega_1, \omega_2 = \sqrt{\frac{k}{m}}, \sqrt{\frac{3k}{m}} \end{aligned}$$

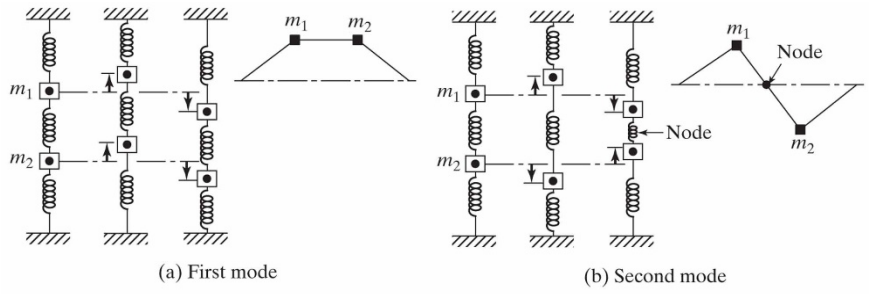
To determine the first mode shape $\mathbf{u}_1$, observe that

$$\begin{aligned} & (\mathbf{K} - \omega_1^2\mathbf{M})\mathbf{u}_1 = \mathbf{0} \\ \Rightarrow \quad & \left\{ \begin{bmatrix} 2k & -k \\ -k & 2k \end{bmatrix} - \frac{k}{m} \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \right\} \begin{bmatrix} u_{11} \\ u_{21} \end{bmatrix} = k \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} u_{11} \\ u_{21} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \Rightarrow \quad & \mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \end{aligned}$$

In a similar fashion,

$$\mathbf{u}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

A mode shape is shown by plotting its elements against the indices. A node is a non-clamped position in a vibrating system at which the displacement is zero at all times. The second mode shape $\mathbf{u}_2$ has a node.



The general response in free vibration is a linear combination of two natural modes of vibration such that

$$\mathbf{x}(t) = c_1\mathbf{u}_1 e^{i\omega_1 t} + c_2\mathbf{u}_2 e^{i\omega_2 t} = c_1\begin{bmatrix}1\\1\end{bmatrix}\exp\left\{i\sqrt{\frac{k}{m}}t\right\} + c_2\begin{bmatrix}1\\-1\end{bmatrix}\exp\left\{i\sqrt{\frac{3k}{m}}t\right\}$$

3.3.2 Eigenvector Expansions of System Response

It will be shown that the real eigenvectors of a symmetric eigenvalue problem constitute a basis and the system response can be expressed as a linear combination of the eigenvectors with time-dependent coefficients.

Orthogonality of Modes

Assume that all frequencies ω_i are distinct and the corresponding eigenvectors $\mathbf{u}_i$ are mass-normalized. Then

$$\begin{aligned} & \mathbf{K}\mathbf{u}_i = \omega_i^2 \mathbf{M}\mathbf{u}_i \\ \Rightarrow \quad & \mathbf{u}_j^T \mathbf{K}\mathbf{u}_i = \omega_i^2 \mathbf{u}_j^T \mathbf{M}\mathbf{u}_i \\ \Rightarrow \quad & \mathbf{u}_i^T \mathbf{K}\mathbf{u}_j = \omega_i^2 \mathbf{u}_i^T \mathbf{M}\mathbf{u}_j \end{aligned}$$

On the other hand,

$$\begin{aligned} & \mathbf{K}\mathbf{u}_j = \omega_j^2 \mathbf{M}\mathbf{u}_j \\ \Rightarrow \quad & \mathbf{u}_i^T \mathbf{K}\mathbf{u}_j = \omega_j^2 \mathbf{u}_i^T \mathbf{M}\mathbf{u}_j \end{aligned}$$

By subtraction,

$$\begin{aligned} & (\omega_i^2 - \omega_j^2)\mathbf{u}_i^T \mathbf{M}\mathbf{u}_j = 0 \\ \Rightarrow \quad & \mathbf{u}_i^T \mathbf{M}\mathbf{u}_j = \delta_{ij} \end{aligned}$$

because $\omega_i \neq \omega_j$ if $i \neq j$. It also follows that

$$\mathbf{u}_i^T \mathbf{K}\mathbf{u}_j = \omega_i^2 \delta_{ij}$$

The case of repeated eigenvalues can be treated by orthogonalization. In general, $\mathbf{u}_i^T \mathbf{u}_j \neq 0$ and the eigenvectors are not orthogonal in an ordinary sense.

Remark. The eigenvectors of $\mathbf{K}\mathbf{u} = \omega^2 \mathbf{M}\mathbf{u}$ are orthogonal with respect to either **M** or **K**.

A set of vectors is called linearly independent if no vector in the set can be expressed as a linear combination of the remaining ones. Any set of n linearly independent vectors constitutes a basis in n-dimensional Euclidean space. For example, let $\mathbf{e}_i$ be the Cartesian unit vector with 1 as the ith element and 0 elsewhere. Then the Cartesian unit vectors form a basis. If $\mathbf{x}$ is an n-dimensional vector,

$$\mathbf{x} = [x_1 \quad x_2 \quad \cdots \quad x_n]^T = \sum_{i=1}^{n} x_i \mathbf{e}_i$$

Since the eigenvectors of $\mathbf{K}\mathbf{u} = \omega^2 \mathbf{M}\mathbf{u}$ are orthogonal, they are also linearly independent because

$$\sum_{i=1}^{n} c_i \mathbf{u}_i = 0$$

$$\Rightarrow \quad \mathbf{u}_j^T \mathbf{M} \sum_{i=1}^{n} c_i \mathbf{u}_i = \sum_{i=1}^{n} c_i \mathbf{u}_j^T \mathbf{M} \mathbf{u}_i = \sum_{i=1}^{n} c_i \delta_{ji} = c_j = 0$$

Eigenvector Expansion Theorem. Any real n-dimensional vector can be expressed as a linear combination of the eigenvectors of $\mathbf{Ku} = \omega^2 \mathbf{Mu}$.

As a consequence, the response $\mathbf{q}(t)$ of any MDOF system can be written as

$$\mathbf{q}(t) = \sum_{j=1}^{n} p_j(t) \mathbf{u}_j$$

where the coefficients $p_j(t)$ are termed principal or modal coordinates. Recall that $p_j(t) = c_j e^{i\omega_j t}$ in conservative free vibration. The above expression is referred to as modal expansion, mode superposition, or eigenvector expansion of the system response $\mathbf{q}(t)$. It is applicable to damped or undamped systems in free or forced vibration.

Modal and Spectral Matrices
Define the modal and spectral matrices, respectively, by

$$\mathbf{U} = [\mathbf{u}_1 \quad \mathbf{u}_2 \quad \cdots \quad \mathbf{u}_n]$$
$$\mathbf{\Omega} = \text{diag}[\omega_1^2, \omega_2^2, \ldots, \omega_n^2]$$

Upon normalization of the natural modes with respect to the mass matrix, the generalized orthogonality of the modes can be expressed in a compact form:

$$\mathbf{U}^T \mathbf{M} \mathbf{U} = \mathbf{I}$$
$$\mathbf{U}^T \mathbf{K} \mathbf{U} = \mathbf{\Omega}$$

The modal matrix defines a linear transformation that diagonalizes **M**, **K** simultaneously.

Modal Equations for Undamped Systems
An undamped system is governed by

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{K}\mathbf{q} = \mathbf{f}(t)$$

By mode superposition,

$$\mathbf{q} = \sum_{j=1}^{n} p_j \mathbf{u}_j = [\mathbf{u}_1 \quad \mathbf{u}_2 \quad \cdots \quad \mathbf{u}_n] \begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_n \end{bmatrix} = \mathbf{U}\mathbf{p}$$

This can be regarded as coordinate transformation from $\mathbf{q}$ to $\mathbf{p}$. If q_1, q_2,…,q_n are linearly

independent, the modal coordinates p_1, p_2,…,p_n are also independent because

$$\begin{aligned} & \sum_{i=1}^{n} c_i p_i = 0 \\ \Rightarrow \quad & \mathbf{c}^T\mathbf{p} = \mathbf{c}^T\mathbf{U}^T\mathbf{M}\mathbf{q} = \mathbf{0} \\ \Rightarrow \quad & \mathbf{c}^T\mathbf{U}^T\mathbf{M} = \mathbf{0} \\ \Rightarrow \quad & c_i = 0 \end{aligned}$$

In terms of $\mathbf{p}$, the equation of motion becomes

$$\begin{aligned} & \mathbf{M}\mathbf{U}\ddot{\mathbf{p}} + \mathbf{K}\mathbf{U}\mathbf{p} = \mathbf{f}(t) \\ \Rightarrow \quad & \mathbf{U}^T\mathbf{M}\mathbf{U}\ddot{\mathbf{p}} + \mathbf{U}^T\mathbf{K}\mathbf{U}\mathbf{p} = \mathbf{U}^T\mathbf{f}(t) \\ \Rightarrow \quad & \ddot{\mathbf{p}} + \boldsymbol{\Omega}\mathbf{p} = \mathbf{U}^T\mathbf{f}(t) \end{aligned}$$

Remark. An undamped system can always be decoupled in real space by modal analysis.

Initial Conditions
The initial values are transformed in accordance with

$$\begin{aligned} \mathbf{p}(0) &= \mathbf{U}^T\mathbf{M}\mathbf{q}(0) \\ \dot{\mathbf{p}}(0) &= \mathbf{U}^T\mathbf{M}\dot{\mathbf{q}}(0) \end{aligned}$$

Driving Forces
The excitation $\mathbf{f}(t)$ is transformed into

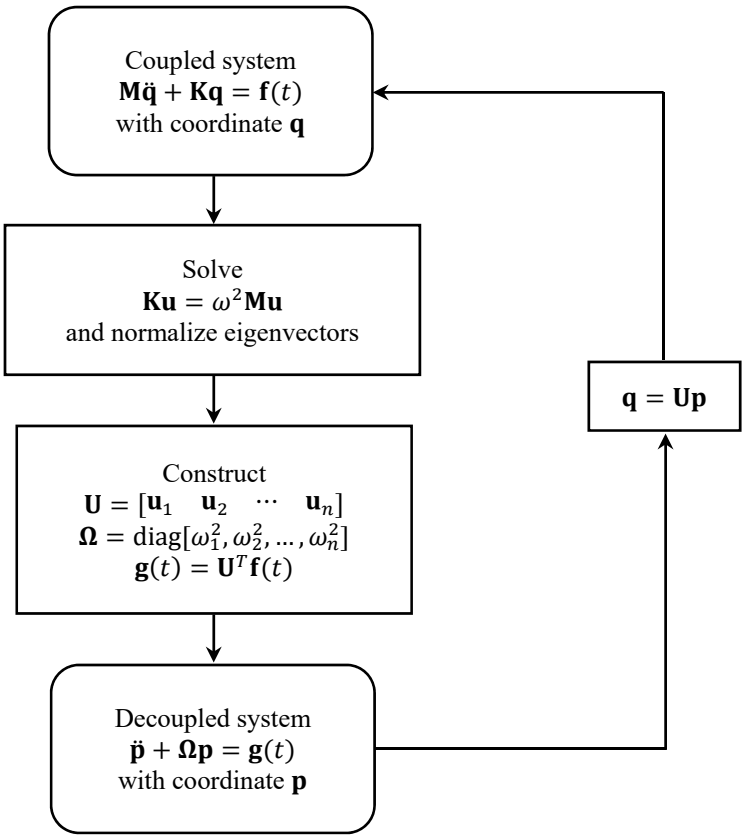
$$\mathbf{U}^T\mathbf{f}(t) = \begin{bmatrix} \mathbf{u}_1^T \\ \mathbf{u}_2^T \\ \vdots \\ \mathbf{u}_n^T \end{bmatrix} \mathbf{f}(t) = \begin{bmatrix} \mathbf{u}_1^T\mathbf{f}(t) \\ \mathbf{u}_2^T\mathbf{f}(t) \\ \vdots \\ \mathbf{u}_n^T\mathbf{f}(t) \end{bmatrix}$$

The modal equation associated with p_i is

$$\ddot{p}_i + \omega_i^2 p_i = \mathbf{u}_i^T\mathbf{f}(t), \qquad i = 1, 2, \dots, n$$

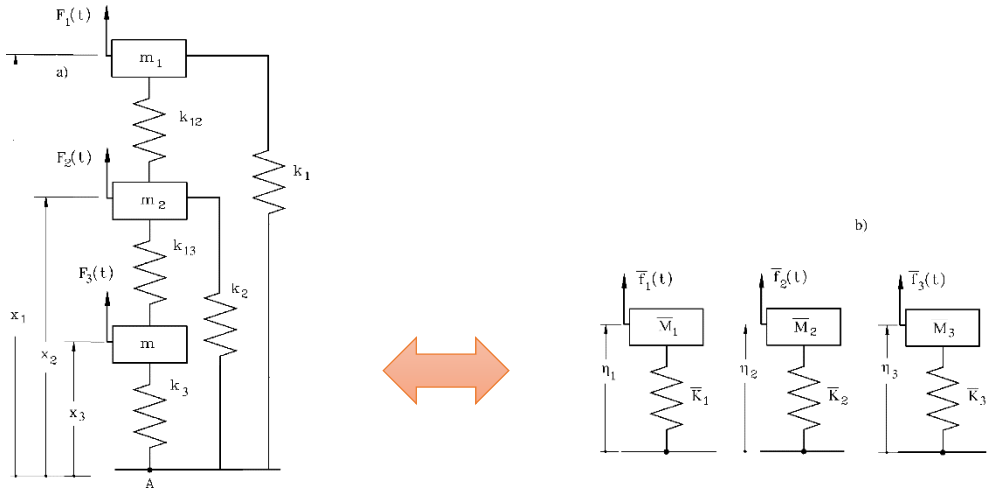
where the modal force $\mathbf{u}_i^T\mathbf{f}(t)$ is a linear combination of the elements of $\mathbf{f}(t)$ weighted by $\mathbf{u}_i$. Each decoupled modal equation is solved independently, and solution of the original equation is readily recovered from $\mathbf{q} = \mathbf{U}\mathbf{p}$.

A flowchart for modal analysis is given in the following diagram.



Visualization of System Decoupling

In modal analysis, a coupled system is decoupled into real and independent systems by a real and invertible coordinate transformation in the configuration space. In the accompanying diagram, the coupled system (a) and the decoupled modal systems (b) are equivalent.



G. Genta, *Vibration Dynamics and Control*, Springer, New York, 103 (2009).

3.3.3 Damped Systems and Decoupling Approximations

A linear system can always be decoupled inertially and elastically, and any coupling occurs ultimately through damping.

Modal Equations for Damped Systems

Let $\mathbf{q} = \mathbf{U}\mathbf{p}$ be the real modal transformation. Then

$$\begin{aligned} & \mathbf{M}\ddot{\mathbf{q}} + \mathbf{C}\dot{\mathbf{q}} + \mathbf{K}\mathbf{q} = \mathbf{f}(t) \\ \Rightarrow \quad & \mathbf{M}\mathbf{U}\ddot{\mathbf{p}} + \mathbf{C}\mathbf{U}\dot{\mathbf{p}} + \mathbf{K}\mathbf{U}\mathbf{p} = \mathbf{f}(t) \\ \Rightarrow \quad & \mathbf{U}^T\mathbf{M}\mathbf{U}\ddot{\mathbf{p}} + \mathbf{U}^T\mathbf{C}\mathbf{U}\dot{\mathbf{p}} + \mathbf{U}^T\mathbf{K}\mathbf{U}\mathbf{p} = \mathbf{U}^T\mathbf{f}(t) \\ \Rightarrow \quad & \ddot{\mathbf{p}} + \mathbf{D}\dot{\mathbf{p}} + \boldsymbol{\Omega}\mathbf{p} = \mathbf{U}^T\mathbf{f}(t) \end{aligned}$$

The matrix $\mathbf{D} = \mathbf{U}^T\mathbf{C}\mathbf{U}$ is called the modal damping matrix. A damped system is decoupled if and only if $\mathbf{D}$ is diagonal.

Proportional Damping

In Section 97 of "The Theory of Sound" in 1894, Rayleigh asserted that a damped system can be decoupled by modal analysis if

$$\mathbf{C} = \alpha\mathbf{M} + \beta\mathbf{K}$$

for some scalar constants α and β. This requirement, referred to as proportional damping, is sufficient but not necessary for decoupling by modal analysis. Under proportional damping, the modal damping matrix $\mathbf{D}$ is diagonal because

$$\mathbf{D} = \mathbf{U}^T\mathbf{C}\mathbf{U} = \mathbf{U}^T(\alpha\mathbf{M} + \beta\mathbf{K})\mathbf{U} = \alpha\mathbf{I} + \beta\boldsymbol{\Omega}$$

Classical Damping

A system is classically damped if it can be decoupled by classical modal analysis, whereby the modal damping matrix $\mathbf{D} = \mathbf{U}^T\mathbf{C}\mathbf{U}$ is diagonal. A necessary and sufficient condition under which a system is classically damped is

$$\mathbf{C}\mathbf{M}^{-1}\mathbf{K} = \mathbf{K}\mathbf{M}^{-1}\mathbf{C}$$

A proportionally damped system is classically damped but the converse is not true. If $\mathbf{C} = \alpha\mathbf{M} + \beta\mathbf{K}$, then

$$\begin{aligned} \mathbf{C}\mathbf{M}^{-1}\mathbf{K} &= (\alpha\mathbf{M} + \beta\mathbf{K})\mathbf{M}^{-1}\mathbf{K} = \alpha\mathbf{K} + \beta\mathbf{K}\mathbf{M}^{-1}\mathbf{K} \\ \mathbf{K}\mathbf{M}^{-1}\mathbf{C} &= \mathbf{K}\mathbf{M}^{-1}(\alpha\mathbf{M} + \beta\mathbf{K}) = \alpha\mathbf{K} + \beta\mathbf{K}\mathbf{M}^{-1}\mathbf{K} \end{aligned}$$

T. K. Caughey and M. E. J. O'Kelly, Classical normal modes in damped linear dynamic systems, *ASME Journal of Applied Mechanics* **32**, 583-588 (1965).

Non-Classical Damping

A system is non-classically damped if it cannot be decoupled by classical modal analysis. There is no reason why damping in a linear system should be classical. Practically speaking, **classical**

damping means that energy dissipation is almost uniformly distributed throughout the system. This condition is violated for systems consisting of two or more parts with significantly different levels of damping. Examples of such systems include soil-structure systems, base-isolated structures, and systems in which coupled vibration of structures and fluids occurs. In fact, experimental modal testing suggests that no physical system is strictly classically damped.

Mathematically, the "classical decoupling problem" is equivalent to the problem of simultaneous conversion of **M**, **C** and **K** into diagonal forms. It has been shown that no time-invariant linear transformations in the configuration space can decouple all damped systems. Even partial decoupling, i.e. simultaneous transformation of the coefficient matrices of the equation of motion to upper triangular forms, cannot be ensured with time-invariant linear transformations. As a consequence, any universal decoupling transformation in the configuration space, if it exists, must be at least time-varying or even nonlinear.

Decoupling Approximations
It is routine for engineers to invoke a whole array of approximations to continue to base the analysis of MDOF systems upon real modal analysis.

Neglect of Off-Diagonal Elements in Modal Damping Matrix
The off-diagonal elements of the modal damping matrix **D** may be ignored since they are usually small and have different signs.

$$\mathbf{D} \cong \text{diag}[d_{11}, d_{22}, \ldots, d_{nn}]$$

This is a common procedure called the decoupling approximation. Physically, it means that coupling is a second-order effect when damping is sufficiently small. This approximation is intuitively appealing if **D** is diagonally dominant or if the natural frequencies are well separated. However, the errors introduced by the decoupling approximation have not been rigorously analyzed.

Over a finite range, it is possible for errors in the decoupling approximation to continuously increase while the modal damping matrix becomes more and more diagonal with its off-diagonal elements decreasing in magnitude continuously. Within a practical range of applications, diagonal dominance of damping would not be sufficient for neglecting modal coupling.

M. Morzfeld, N. Ajavakom and F. Ma, Diagonal dominance of damping and the decoupling approximation in linear vibratory systems, *Journal of Sound and Vibration* **320**(1-2), 406-420 (2009).

Proportional Damping Approximation
Assume that

$$\mathbf{C} \cong \alpha\mathbf{M} + \beta\mathbf{K}$$

The optimal values of α and β are determined by minimizing the mean-square error such that

$$\min_{\alpha,\beta}\|\mathbf{C}-\alpha\mathbf{M}-\beta\mathbf{K}\|^2=\min_{\alpha,\beta}\sum_{i=1}^{n}\sum_{j=1}^{n}(c_{ij}-\alpha m_{ij}-\beta k_{ij})^2$$

where the Euclidean norm of matrices is used.

Power Series Approximation
This is an extension of proportional damping approximation. A necessary and sufficient condition under which a system is classically damped is

$$\begin{aligned} & \mathbf{C}\mathbf{M}^{-1}\mathbf{K}=\mathbf{K}\mathbf{M}^{-1}\mathbf{C} \\ \Leftrightarrow\quad & \mathbf{M}^{-1}\mathbf{C}\mathbf{M}^{-1}\mathbf{K}=\mathbf{M}^{-1}\mathbf{K}\mathbf{M}^{-1}\mathbf{C} \end{aligned}$$

The commutativity condition is satisfied if $\mathbf{M}^{-1}\mathbf{C}$ is a power series in $\mathbf{M}^{-1}\mathbf{K}$. Assume that

$$\mathbf{M}^{-1}\mathbf{C}\cong\sum_{j=0}^{\infty}a_j(\mathbf{M}^{-1}\mathbf{K})^j$$

Cayley-Hamilton Theorem. A matrix satisfies its own characteristic equation.

For a square matrix $\mathbf{A}$ of order n, its characteristic equation is

$$\begin{aligned} & \det(\mathbf{A}-\lambda\mathbf{I})=c_n\lambda^n+c_{n-1}\lambda^{n-1}+\cdots+c_1\lambda+c_0=0 \\ \Rightarrow\quad & c_n\mathbf{A}^n+c_{n-1}\mathbf{A}^{n-1}+\cdots+c_1\mathbf{A}+c_0\mathbf{I}=\mathbf{0} \end{aligned}$$

As a result, any power of $\mathbf{A}$ higher than $n-1$ can be expressed as a finite series in powers of $\mathbf{A}$ up to $n-1$. Thus the power series of $\mathbf{M}^{-1}\mathbf{C}$ in terms of $\mathbf{M}^{-1}\mathbf{K}$ need not contain powers of $\mathbf{M}^{-1}\mathbf{K}$ higher than $n-1$.

$$\begin{aligned} & \mathbf{M}^{-1}\mathbf{C}=\sum_{j=0}^{n-1}a_j(\mathbf{M}^{-1}\mathbf{K})^j \\ \Rightarrow\quad & \mathbf{C}=\mathbf{M}\textstyle\sum_{j=0}^{n-1}a_j(\mathbf{M}^{-1}\mathbf{K})^j \end{aligned}$$

The coefficients of the power series are chosen by

$$\min_{a_0,a_1,\ldots,a_{n-1}}\left\|\mathbf{C}-\mathbf{M}\sum_{j=0}^{n-1}a_j(\mathbf{M}^{-1}\mathbf{K})^j\right\|$$

The first three terms of the power series are

$$\begin{gathered} a_0\mathbf{M}(\mathbf{M}^{-1}\mathbf{K})^0=a_0\mathbf{M} \\ a_1\mathbf{M}(\mathbf{M}^{-1}\mathbf{K})^1=a_1\mathbf{K} \\ a_2\mathbf{M}(\mathbf{M}^{-1}\mathbf{K})^2=a_2\mathbf{K}\mathbf{M}^{-1}\mathbf{K} \end{gathered}$$

If only the first two terms of the power series are used,

$$\mathbf{C} = \mathbf{M}\sum_{j=0}^{1} \mathrm{a_j}(\mathbf{M}^{-1}\mathbf{K})^j = a_0\mathbf{M} + a_1\mathbf{K}$$

which reduces to proportional damping. If only the first three terms of the power series are used,

$$\mathbf{C} = \mathbf{M}\sum_{j=0}^{2} a_j(\mathbf{M}^{-1}\mathbf{K})^j = a_0\mathbf{M} + a_1\mathbf{K} + a_2\mathbf{K}\mathbf{M}^{-1}\mathbf{K}$$

A system with a damping matrix $\mathbf{C}$ as defined above is classically but non-proportionally damped. It is obvious that

$$\mathbf{C}\mathbf{M}^{-1}\mathbf{K} = a_0\mathbf{K} + a_1\mathbf{K}\mathbf{M}^{-1}\mathbf{K} + a_2\mathbf{K}\mathbf{M}^{-1}\mathbf{K}\mathbf{M}^{-1}\mathbf{K} = \mathbf{K}\mathbf{M}^{-1}\mathbf{C}$$

A. K. Chopra, *Dynamics of Structures: Theory and Applications to Earthquake Engineering*, 5*th* ed., Pearson, Hoboken, New Jersey, 442-445 (2017).

3.4 State Space Analysis

Classical modal analysis is a process of real analysis in the configuration space. It can be extended to a process of complex modal analysis in the state space to treat non-classically damped systems. However, the state-space approach has never appealed to practicing engineers. There are several reasons for this situation. A common excuse is that the state-space approach is computationally more involved because the dimension of the state space is twice the DOF. Another reason is that complex modal analysis still cannot decouple all non-classically damped systems. The state companion matrix must be non-defective in order for complex modal analysis to achieve complete decoupling. More importantly, there is little physical insight associated with different elements of complex modal analysis. Classical modal analysis is amenable to physical interpretation. For example, a modal vector represents a physical profile of vibration. Even the symmetric eigenvalue problem $\mathbf{Ku} = \omega^2\mathbf{Mu}$ can be interpreted geometrically as the problem of finding the principal axes of an n-dimensional ellipsoid.

Standard State Equation

The equation of motion of an n-degree-of-freedom linear system

$$\mathbf{M\ddot{q}} + \mathbf{C\dot{q}} + \mathbf{Kq} = \mathbf{f}(t)$$

can be cast in $2n$-dimensional state space as a first-order equation

$$\dot{\mathbf{x}} = \begin{bmatrix}\dot{\mathbf{q}} \\ \ddot{\mathbf{q}}\end{bmatrix} = \begin{bmatrix}\mathbf{0} & \mathbf{I} \\ -\mathbf{M}^{-1}\mathbf{K} & -\mathbf{M}^{-1}\mathbf{C}\end{bmatrix}\begin{bmatrix}\mathbf{q} \\ \dot{\mathbf{q}}\end{bmatrix} + \begin{bmatrix}\mathbf{0} \\ \mathbf{M}^{-1}\mathbf{f}(t)\end{bmatrix} = \mathbf{Ax} + \mathbf{g}(t)$$

Solutions obtained by time-domain or frequency-domain analysis are not efficient for applications unless **A** is a diagonal matrix.

Complex Modal Analysis

This is a method for decoupling the state equation by means of a state transformation. The resulting independent equations are generally complex. The state companion matrix **A** is diagonalizable if and only if it has $2n$ linearly independent eigenvectors (matrix is non-defective). A diagonalizable matrix can be represented in terms of its eigenvalues and eigenvectors.

Method of Eigen Decomposition

Assume that **A** is diagonalizable. Consider the eigenvalue problem

$$\mathbf{Au} = \lambda\mathbf{u}$$

If the $2n$ eigenvalues λ_j are distinct, the corresponding $2n$ eigenvectors $\mathbf{u}_j$ are linearly independent. The eigenvectors $\mathbf{u}_j$ are known as complex modes. Define the modal and spectral matrices, respectively, by

$$\mathbf{U} = [\mathbf{u}_1 \quad \mathbf{u}_2 \quad \cdots \quad \mathbf{u}_{2n}]$$
$$\mathbf{\Lambda} = \text{diag}[\lambda_1, \lambda_2, \ldots, \lambda_{2n}]$$

In general, **U** and **Λ** are complex. Clearly,

$$\mathbf{AU} = [\mathbf{Au}_1 \quad \mathbf{Au}_2 \quad \cdots \quad \mathbf{Au}_{2n}] = [\lambda_1\mathbf{u}_1 \quad \lambda_2\mathbf{u}_2 \quad \cdots \quad \lambda_{2n}\mathbf{u}_{2n}] = \mathbf{U\Lambda}$$

It follows that

$$\mathbf{A} = \mathbf{U\Lambda U}^{-1}$$

The above factorization of a matrix $\mathbf{A}$ into its eigenvalues and eigenvectors is termed the eigen decomposition of $\mathbf{A}$. Define a state transformation by $\mathbf{x} = \mathbf{Uy}$. Then

$$\begin{aligned} & \mathbf{U}\dot{\mathbf{y}}(t) = \mathbf{AUy} + \mathbf{g}(t) \\ \Rightarrow \quad & \dot{\mathbf{y}}(t) = \mathbf{\Lambda y} + \mathbf{U}^{-1}\mathbf{g}(t) \end{aligned}$$

In terms of $\mathbf{y}$, $2n$ independent equations are obtained. Since the decoupled equations are generally complex, physical insight is greatly diminished.

R. A. Horn and C. R. Johnson, *Matrix Analysis*, Cambridge University Press, Cambridge, United Kingdom, 46-47 (1985).

State Equations with Symmetric Coefficients
The equation of motion can be cast in the state space in a symmetric form

$$\begin{bmatrix} \mathbf{C} & \mathbf{M} \\ \mathbf{M} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \dot{\mathbf{q}} \\ \ddot{\mathbf{q}} \end{bmatrix} + \begin{bmatrix} \mathbf{K} & \mathbf{0} \\ \mathbf{0} & -\mathbf{M} \end{bmatrix} \begin{bmatrix} \mathbf{q} \\ \dot{\mathbf{q}} \end{bmatrix} = \begin{bmatrix} \mathbf{f}(t) \\ \mathbf{0} \end{bmatrix}$$

or in the form

$$\begin{bmatrix} \mathbf{0} & \mathbf{M} \\ \mathbf{M} & \mathbf{C} \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{q}} \\ \dot{\mathbf{q}} \end{bmatrix} + \begin{bmatrix} -\mathbf{M} & \mathbf{0} \\ \mathbf{0} & \mathbf{K} \end{bmatrix} \begin{bmatrix} \dot{\mathbf{q}} \\ \mathbf{q} \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{f}(t) \end{bmatrix}$$

These state equations possess symmetric and nonsingular coefficients and are analogous to $\mathbf{M\ddot{q}} + \mathbf{Kq} = \mathbf{f}(t)$. However, the eigenvalues and eigenvectors of

$$\lambda \begin{bmatrix} \mathbf{C} & \mathbf{M} \\ \mathbf{M} & \mathbf{0} \end{bmatrix} \mathbf{w} + \begin{bmatrix} \mathbf{K} & \mathbf{0} \\ \mathbf{0} & -\mathbf{M} \end{bmatrix} \mathbf{w} = \mathbf{0}$$

are generally complex. This is also true for

$$\lambda \begin{bmatrix} \mathbf{0} & \mathbf{M} \\ \mathbf{M} & \mathbf{C} \end{bmatrix} \mathbf{w} + \begin{bmatrix} -\mathbf{M} & \mathbf{0} \\ \mathbf{0} & \mathbf{K} \end{bmatrix} \mathbf{w} = \mathbf{0}$$

If non-defective, each state equation can be decoupled by utilizing the associated state eigenvectors but the decoupled equations are complex. Real modal analysis is not applicable to state equations with symmetric coefficients.

In classical modal analysis, symmetry and positive definiteness of $\mathbf{M}$ and $\mathbf{K}$ are required in $\mathbf{Ku} = \omega^2\mathbf{Mu}$ to ensure that the eigenvalue ω^2 and eigenvector $\mathbf{u}$ are real. However, at least one

symmetric coefficient in each state equation is not definite. In the first state equation,

$$\begin{bmatrix}\mathbf{0}\\ \mathbf{1}\end{bmatrix}^T \begin{bmatrix}\mathbf{K} & \mathbf{0}\\ \mathbf{0} & -\mathbf{M}\end{bmatrix}\begin{bmatrix}\mathbf{0}\\ \mathbf{1}\end{bmatrix} = -\mathbf{1}^T\mathbf{M}\mathbf{1} < 0$$
$$\begin{bmatrix}\mathbf{1}\\ \mathbf{0}\end{bmatrix}^T \begin{bmatrix}\mathbf{K} & \mathbf{0}\\ \mathbf{0} & -\mathbf{M}\end{bmatrix}\begin{bmatrix}\mathbf{1}\\ \mathbf{0}\end{bmatrix} = \mathbf{1}^T\mathbf{K}\mathbf{1} > 0$$

In the second state equation,

$$\begin{bmatrix}\mathbf{1}\\ \mathbf{0}\end{bmatrix}^T \begin{bmatrix}-\mathbf{M} & \mathbf{0}\\ \mathbf{0} & \mathbf{K}\end{bmatrix}\begin{bmatrix}\mathbf{1}\\ \mathbf{0}\end{bmatrix} = -\mathbf{1}^T\mathbf{M}\mathbf{1} < 0$$
$$\begin{bmatrix}\mathbf{0}\\ \mathbf{1}\end{bmatrix}^T \begin{bmatrix}-\mathbf{M} & \mathbf{0}\\ \mathbf{0} & \mathbf{K}\end{bmatrix}\begin{bmatrix}\mathbf{0}\\ \mathbf{1}\end{bmatrix} = \mathbf{1}^T\mathbf{K}\mathbf{1} > 0$$

The second-order equation $\mathbf{M}\ddot{\mathbf{q}} + \mathbf{C}\dot{\mathbf{q}} + \mathbf{K}\mathbf{q} = \mathbf{f}(t)$ can be cast in different state equations but all state-space representations are equivalent. The corresponding state eigenvalue problems are isospectral, i.e., they have the same eigenvalues with the same algebraic multiplicities.

K. A. Foss, Co-ordinates which uncouple the equations of motion of damped linear dynamic systems, *ASME Journal of Applied Mechanics* **25**, 361-364 (1958).

Example. Let $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$ be symmetric and positive definite of the same order. Show that the following state equations are equivalent:

$$\begin{bmatrix}\dot{\mathbf{q}}\\ \ddot{\mathbf{q}}\end{bmatrix} = \begin{bmatrix}\mathbf{0} & \mathbf{I}\\ -\mathbf{M}^{-1}\mathbf{K} & -\mathbf{M}^{-1}\mathbf{C}\end{bmatrix}\begin{bmatrix}\mathbf{q}\\ \dot{\mathbf{q}}\end{bmatrix} + \begin{bmatrix}\mathbf{0}\\ \mathbf{M}^{-1}\mathbf{f}(t)\end{bmatrix}$$

$$\begin{bmatrix}\mathbf{C} & \mathbf{M}\\ \mathbf{M} & \mathbf{0}\end{bmatrix}\begin{bmatrix}\dot{\mathbf{q}}\\ \ddot{\mathbf{q}}\end{bmatrix} + \begin{bmatrix}\mathbf{K} & \mathbf{0}\\ \mathbf{0} & -\mathbf{M}\end{bmatrix}\begin{bmatrix}\mathbf{q}\\ \dot{\mathbf{q}}\end{bmatrix} = \begin{bmatrix}\mathbf{f}(t)\\ \mathbf{0}\end{bmatrix}$$

$$\begin{bmatrix}\mathbf{0} & \mathbf{M}\\ \mathbf{M} & \mathbf{C}\end{bmatrix}\begin{bmatrix}\ddot{\mathbf{q}}\\ \dot{\mathbf{q}}\end{bmatrix} + \begin{bmatrix}-\mathbf{M} & \mathbf{0}\\ \mathbf{0} & \mathbf{K}\end{bmatrix}\begin{bmatrix}\dot{\mathbf{q}}\\ \mathbf{q}\end{bmatrix} = \begin{bmatrix}\mathbf{0}\\ \mathbf{f}(t)\end{bmatrix}$$

Solution

If conformable matrices are partitioned in a compatible fashion, the submatrices can be treated just as scalars when performing the operations of addition and multiplication. By direct calculations,

$$\begin{bmatrix}\mathbf{C} & \mathbf{M}\\ \mathbf{M} & \mathbf{0}\end{bmatrix}\begin{bmatrix}\mathbf{A}_1 & \mathbf{A}_2\\ \mathbf{A}_3 & \mathbf{A}_4\end{bmatrix} = \begin{bmatrix}\mathbf{I} & \mathbf{0}\\ \mathbf{0} & \mathbf{I}\end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix}\mathbf{A}_1 & \mathbf{A}_2\\ \mathbf{A}_3 & \mathbf{A}_4\end{bmatrix} = \begin{bmatrix}\mathbf{C} & \mathbf{M}\\ \mathbf{M} & \mathbf{0}\end{bmatrix}^{-1} = \begin{bmatrix}\mathbf{0} & \mathbf{M}^{-1}\\ \mathbf{M}^{-1} & -\mathbf{M}^{-1}\mathbf{C}\mathbf{M}^{-1}\end{bmatrix}$$

Observe that

$$\begin{bmatrix}\mathbf{C} & \mathbf{M}\\ \mathbf{M} & \mathbf{0}\end{bmatrix}^{-1}\begin{bmatrix}\mathbf{K} & \mathbf{0}\\ \mathbf{0} & -\mathbf{M}\end{bmatrix} = \begin{bmatrix}\mathbf{0} & \mathbf{M}^{-1}\\ \mathbf{M}^{-1} & -\mathbf{M}^{-1}\mathbf{C}\mathbf{M}^{-1}\end{bmatrix}\begin{bmatrix}\mathbf{K} & \mathbf{0}\\ \mathbf{0} & -\mathbf{M}\end{bmatrix} = \begin{bmatrix}\mathbf{0} & -\mathbf{I}\\ \mathbf{M}^{-1}\mathbf{K} & \mathbf{M}^{-1}\mathbf{C}\end{bmatrix}$$

$$\begin{bmatrix}\mathbf{C} & \mathbf{M}\\ \mathbf{M} & \mathbf{0}\end{bmatrix}^{-1}\begin{bmatrix}\mathbf{f}(t)\\ \mathbf{0}\end{bmatrix} = \begin{bmatrix}\mathbf{0} & \mathbf{M}^{-1}\\ \mathbf{M}^{-1} & -\mathbf{M}^{-1}\mathbf{C}\mathbf{M}^{-1}\end{bmatrix}\begin{bmatrix}\mathbf{f}(t)\\ \mathbf{0}\end{bmatrix} = \begin{bmatrix}\mathbf{0}\\ \mathbf{M}^{-1}\mathbf{f}(t)\end{bmatrix}$$

As a result,

$$\begin{bmatrix}\mathbf{C} & \mathbf{M}\\ \mathbf{M} & \mathbf{0}\end{bmatrix}\begin{bmatrix}\dot{\mathbf{q}}\\ \ddot{\mathbf{q}}\end{bmatrix} + \begin{bmatrix}\mathbf{K} & \mathbf{0}\\ \mathbf{0} & -\mathbf{M}\end{bmatrix}\begin{bmatrix}\mathbf{q}\\ \dot{\mathbf{q}}\end{bmatrix} = \begin{bmatrix}\mathbf{f}(t)\\ \mathbf{0}\end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix}\dot{\mathbf{q}}\\ \ddot{\mathbf{q}}\end{bmatrix} + \begin{bmatrix}\mathbf{C} & \mathbf{M}\\ \mathbf{M} & \mathbf{0}\end{bmatrix}^{-1}\begin{bmatrix}\mathbf{K} & \mathbf{0}\\ \mathbf{0} & -\mathbf{M}\end{bmatrix}\begin{bmatrix}\mathbf{q}\\ \dot{\mathbf{q}}\end{bmatrix} = \begin{bmatrix}\mathbf{C} & \mathbf{M}\\ \mathbf{M} & \mathbf{0}\end{bmatrix}^{-1}\begin{bmatrix}\mathbf{f}(t)\\ \mathbf{0}\end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix}\dot{\mathbf{q}}\\ \ddot{\mathbf{q}}\end{bmatrix} = \begin{bmatrix}\mathbf{0} & \mathbf{I}\\ -\mathbf{M}^{-1}\mathbf{K} & -\mathbf{M}^{-1}\mathbf{C}\end{bmatrix}\begin{bmatrix}\mathbf{q}\\ \dot{\mathbf{q}}\end{bmatrix} + \begin{bmatrix}\mathbf{0}\\ \mathbf{M}^{-1}\mathbf{f}(t)\end{bmatrix}$$

This shows that the first two state equations are equivalent. Similarly,

$$\begin{bmatrix}\mathbf{0} & \mathbf{M}\\ \mathbf{M} & \mathbf{C}\end{bmatrix}^{-1}\begin{bmatrix}-\mathbf{M} & \mathbf{0}\\ \mathbf{0} & -\mathbf{K}\end{bmatrix} = \begin{bmatrix}-\mathbf{M}^{-1}\mathbf{C}\mathbf{M}^{-1} & \mathbf{M}^{-1}\\ \mathbf{M}^{-1} & \mathbf{0}\end{bmatrix}\begin{bmatrix}-\mathbf{M} & \mathbf{0}\\ \mathbf{0} & -\mathbf{K}\end{bmatrix} = \begin{bmatrix}\mathbf{M}^{-1}\mathbf{C} & \mathbf{M}^{-1}\mathbf{K}\\ -\mathbf{I} & \mathbf{0}\end{bmatrix}$$

$$\begin{bmatrix}\mathbf{0} & \mathbf{M}\\ \mathbf{M} & \mathbf{C}\end{bmatrix}^{-1}\begin{bmatrix}\mathbf{0}\\ \mathbf{f}(t)\end{bmatrix} = \begin{bmatrix}-\mathbf{M}^{-1}\mathbf{C}\mathbf{M}^{-1} & \mathbf{M}^{-1}\\ \mathbf{M}^{-1} & \mathbf{0}\end{bmatrix}\begin{bmatrix}\mathbf{0}\\ \mathbf{f}(t)\end{bmatrix} = \begin{bmatrix}\mathbf{M}^{-1}\mathbf{f}(t)\\ \mathbf{0}\end{bmatrix}$$

It follows that

$$\begin{bmatrix}\mathbf{0} & \mathbf{M}\\ \mathbf{M} & \mathbf{C}\end{bmatrix}\begin{bmatrix}\ddot{\mathbf{q}}\\ \dot{\mathbf{q}}\end{bmatrix} + \begin{bmatrix}-\mathbf{M} & \mathbf{0}\\ \mathbf{0} & \mathbf{K}\end{bmatrix}\begin{bmatrix}\dot{\mathbf{q}}\\ \mathbf{q}\end{bmatrix} = \begin{bmatrix}\mathbf{0}\\ \mathbf{f}(t)\end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix}\ddot{\mathbf{q}}\\ \dot{\mathbf{q}}\end{bmatrix} + \begin{bmatrix}\mathbf{0} & \mathbf{M}\\ \mathbf{M} & \mathbf{C}\end{bmatrix}^{-1}\begin{bmatrix}-\mathbf{M} & \mathbf{0}\\ \mathbf{0} & -\mathbf{K}\end{bmatrix}\begin{bmatrix}\dot{\mathbf{q}}\\ \mathbf{q}\end{bmatrix} = \begin{bmatrix}\mathbf{0} & \mathbf{M}\\ \mathbf{M} & \mathbf{C}\end{bmatrix}^{-1}\begin{bmatrix}\mathbf{0}\\ \mathbf{f}(t)\end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix}\ddot{\mathbf{q}}\\ \dot{\mathbf{q}}\end{bmatrix} + \begin{bmatrix}\mathbf{M}^{-1}\mathbf{C} & \mathbf{M}^{-1}\mathbf{K}\\ -\mathbf{I} & \mathbf{0}\end{bmatrix}\begin{bmatrix}\dot{\mathbf{q}}\\ \mathbf{q}\end{bmatrix} = \begin{bmatrix}\mathbf{M}^{-1}\mathbf{f}(t)\\ \mathbf{0}\end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix}\dot{\mathbf{q}}\\ \ddot{\mathbf{q}}\end{bmatrix} = \begin{bmatrix}\mathbf{0} & \mathbf{I}\\ -\mathbf{M}^{-1}\mathbf{K} & -\mathbf{M}^{-1}\mathbf{C}\end{bmatrix}\begin{bmatrix}\mathbf{q}\\ \dot{\mathbf{q}}\end{bmatrix} + \begin{bmatrix}\mathbf{0}\\ \mathbf{M}^{-1}\mathbf{f}(t)\end{bmatrix}$$

Thus the first and third state equations are equivalent.

3.5 Estimation and Control

The fundamental frequency of an MDOF system is of greater interest than higher frequencies because the amplitude of the fundamental mode is often the largest. By mode superposition, the response of a system in free or forced vibration can be written as

$$\mathbf{q}(t) = \sum_{j=1}^{n} p_j(t)\mathbf{u}_j$$

where $p_j(t)$ are the modal coordinates. If n is large, this may be regarded as an infinite series. For the series to converge, $p_j(t)\mathbf{u}_j$ must necessarily approach zero as j increases. In general, only the lower-order modes contribute significantly to the overall response.

Estimation of Fundamental Frequencies

The Rayleigh's quotient is the ratio

$$R(\mathbf{x}) = \frac{\mathbf{x}^T\mathbf{K}\mathbf{x}}{\mathbf{x}^T\mathbf{M}\mathbf{x}}$$

Since $\mathbf{M}$ is positive definite, $R(\mathbf{x})$ exists for any $\mathbf{x} \neq \mathbf{0}$. Clearly,

$$\mathbf{K}\mathbf{u}_i = \omega_i^2\mathbf{M}\mathbf{u}_i$$
$$\Rightarrow \quad R(\mathbf{u}_i) = \frac{\mathbf{u}_i^T\mathbf{K}\mathbf{u}_i}{\mathbf{u}_i^T\mathbf{M}\mathbf{u}_i} = \omega_i^2$$

Rayleigh's Principle. The Rayleigh's quotient is bounded by the eigenvalues ω_1^2 and ω_n^2 and it attains a stationary value at each eigenvector.

If $\mathbf{x}$ is an arbitrary vector, then by modal expansion,

$$\mathbf{x} = \sum_{i=1}^{n} c_i\mathbf{u}_i$$
$$\Rightarrow \quad R(\mathbf{x}) = \frac{\sum_{i=1}^{n}\sum_{j=1}^{n} c_i c_j \mathbf{u}_i^T\mathbf{K}\mathbf{u}_j}{\sum_{i=1}^{n}\sum_{j=1}^{n} c_i c_j \mathbf{u}_i^T\mathbf{M}\mathbf{u}_j} = \frac{\sum_{i=1}^{n} c_i^2\omega_i^2}{\sum_{i=1}^{n} c_i^2}$$
$$\Rightarrow \quad \omega_1^2 \leq R(\mathbf{x}) \leq \omega_n^2$$

The multivariate scalar function $R(\mathbf{x})$ has a stationary value if

$$\frac{\partial R}{\partial x_k} = 0, \qquad k = 1, 2, \ldots, n$$
$$\Rightarrow \quad \frac{d}{d\mathbf{x}}R(\mathbf{x}) = \nabla R(\mathbf{x}) = \left[\frac{\partial}{\partial x_1} \quad \frac{\partial}{\partial x_2} \quad \cdots \quad \frac{\partial}{\partial x_n}\right]^T R(\mathbf{x}) = \mathbf{0}$$

Gradient of a Quadratic Form

If $\mathbf{A}$ is a symmetric matrix,

$$\frac{\partial}{\partial x_k}(\mathbf{x}^T\mathbf{A}\mathbf{x}) = \frac{\partial}{\partial x_k}\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}x_i x_j = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}\left(\frac{\partial x_i}{\partial x_k}x_j + x_i\frac{\partial x_j}{\partial x_k}\right)$$
$$= \sum_{j=1}^{n} a_{kj}x_j + \sum_{i=1}^{n} a_{ik}x_i = 2\sum_{j=1}^{n} a_{kj}x_j$$

It follows that

$$\frac{d}{d\mathbf{x}}(\mathbf{x}^T\mathbf{A}\mathbf{x}) = \begin{bmatrix}\frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \cdots & \frac{\partial}{\partial x_n}\end{bmatrix}^T (\mathbf{x}^T\mathbf{A}\mathbf{x}) = 2\mathbf{A}\mathbf{x}$$

As a consequence,

$$\frac{d}{d\mathbf{x}}R(\mathbf{x}) = \frac{(\mathbf{x}^T\mathbf{M}\mathbf{x})\frac{d}{d\mathbf{x}}(\mathbf{x}^T\mathbf{K}\mathbf{x}) - (\mathbf{x}^T\mathbf{K}\mathbf{x})\frac{d}{d\mathbf{x}}(\mathbf{x}^T\mathbf{M}\mathbf{x})}{(\mathbf{x}^T\mathbf{M}\mathbf{x})^2}$$
$$= \frac{2(\mathbf{x}^T\mathbf{M}\mathbf{x})\mathbf{K}\mathbf{x} - 2(\mathbf{x}^T\mathbf{K}\mathbf{x})\mathbf{M}\mathbf{x}}{(\mathbf{x}^T\mathbf{M}\mathbf{x})^2} = \frac{2}{\mathbf{x}^T\mathbf{M}\mathbf{x}}[\mathbf{K} - R(\mathbf{x})\mathbf{M}]\mathbf{x}$$

Hence,

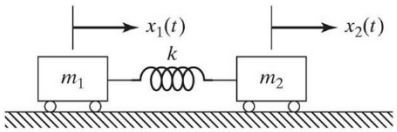
$$\frac{d}{d\mathbf{x}}R(\mathbf{u}_i) = \frac{2}{\mathbf{u}_i^T\mathbf{M}\mathbf{u}_i}[\mathbf{K} - R(\mathbf{u}_i)\mathbf{M}]\mathbf{u}_i = \frac{2}{\mathbf{u}_i^T\mathbf{M}\mathbf{u}_i}[\mathbf{K}\mathbf{u}_i - \omega_i^2\mathbf{M}\mathbf{u}_i] = \mathbf{0}$$

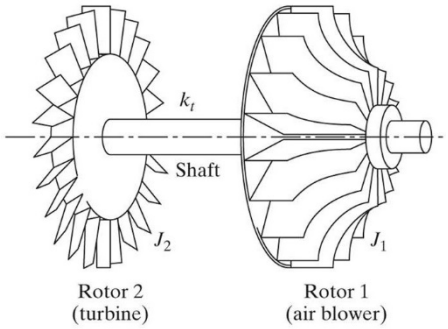
Thus $R(\mathbf{x})$ has a stationary value at each eigenvector $\mathbf{u}_i$. The stationary value may be a relative maximum, a relative minimum, or a saddle point. The stationary value is a minimum at the fundamental mode $\mathbf{u}_1$.

The essence of Rayleigh's method for the estimation of fundamental frequency is to assume a mode shape for $\mathbf{u}_1$. At a stationary value, $R(\mathbf{x})$ does not change and there is a loss of sensitivity to changes in $\mathbf{x}$. If $\mathbf{x} \cong \mathbf{u}_1$ is accurate to order ϵ, then $R(\mathbf{x}) \cong R(\mathbf{u}_1)$ is accurate to ϵ^2. In general, **the static deflection curve of an elastic body provides a fairly accurate value of the fundamental frequency**. The Rayleigh-Ritz method provides an improved estimate using a linear combination of trial functions with unknown coefficients.

Unrestrained Systems

An unrestrained system is one that has no restraints or supports and that can move as a rigid body. It is also known as a semidefinite or degenerate system. A common example is the motion of two railway cars with masses m_1 and m_2 and a coupling spring k. The stiffness matrix $\mathbf{K}$ of an unrestrained system is positive semidefinite.

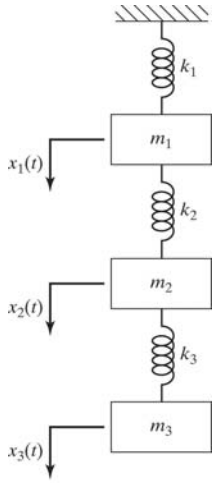




If **M** is positive definite but **K** is only positive semidefinite, there is a zero natural frequency. The corresponding eigenvector is a rigid-body mode. There are at most six rigid-body modes. Rigid-body modes can be eliminated by reducing the DOF in the formulation.

If **M** is only positive semidefinite and **K** is positive definite, there is an infinite natural frequency. The corresponding eigenvector is a static mode. There can be as many static modes as the DOF. Static modes can be eliminated by reducing the DOF in the formulation by static condensation.

Example. Using Rayleigh's principle, estimate the fundamental frequency of vibration of the system shown, where $m_1 = m_2 = m_3 = m$ and $k_1 = k_2 = k_3 = k$. Compare the estimate with the exact fundamental frequency.



Solution

Let x_1, x_2, x_3 be the displacements of m_1, m_2, m_3 from static equilibrium. The equation of motion system is

$$\begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 & 0 \\ -k_2 & k_2 + k_3 & -k_3 \\ 0 & -k_3 & k_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

With the specified values of masses and spring constants,

$$m \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{bmatrix} + k \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Apply a constant vertical force to m_3 to produce nonzero deflections in the masses. At equilibrium, the extensions in individual springs k_1, k_2, k_3 are equal. Thus the static displacements x_1, x_2, x_3 are in the ratio $1:2:3$. Equivalently, the static deflection pattern of the system is given by

$$\mathbf{x} = \mathbf{K}^{-1} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$\Rightarrow \quad R(\mathbf{x}) = \frac{\mathbf{x}^T \mathbf{K} \mathbf{x}}{\mathbf{x}^T \mathbf{M} \mathbf{x}} = 0.2143 \frac{k}{m}$$

$$\Rightarrow \quad \omega_1 = 0.4629 \sqrt{\frac{k}{m}}$$

The exact fundamental frequency and mode shape are

$$\omega_1 = 0.4450 \sqrt{\frac{k}{m}}\,, \qquad \mathbf{u}_1 = \begin{bmatrix} 1 \\ 1.8019 \\ 2.2470 \end{bmatrix}$$

Thus the estimate is about 4% larger than the exact fundamental frequency.

If the assumed fundamental mode is

$$\mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\Rightarrow \qquad R(\mathbf{x}) = \frac{\mathbf{x}^T\mathbf{K}\mathbf{x}}{\mathbf{x}^T\mathbf{M}\mathbf{x}} = 0.3333\frac{k}{m}$$

$$\Rightarrow \qquad \omega_1 = 0.5774\sqrt{\frac{k}{m}}$$

The estimate is about 30% larger than the exact fundamental frequency. The estimate is not accurate because the assumed mode is quite different from the fundamental mode. If the system is loaded with forces proportional to the masses, the static deflection pattern is

$$\mathbf{x} = \mathbf{K}^{-1}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \\ 6 \end{bmatrix}$$

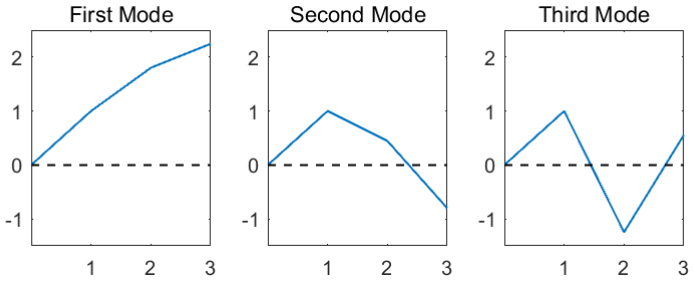
$$\Rightarrow \qquad R(\mathbf{x}) = \frac{\mathbf{x}^T\mathbf{K}\mathbf{x}}{\mathbf{x}^T\mathbf{M}\mathbf{x}} = 0.2\frac{k}{m}$$

$$\Rightarrow \qquad \omega_1 = 0.4472\sqrt{\frac{k}{m}}$$

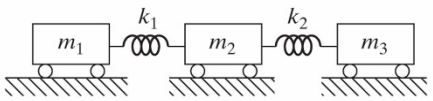
The estimate is about 0.5% larger than the exact fundamental frequency. When applicable, the fundamental mode is closely approximated by a static deflection pattern obtained by loading the system with forces proportional to the masses. For this elastic system,

$$\omega_2 = 1.2470\sqrt{\frac{k}{m}}, \qquad \mathbf{u}_2 = \begin{bmatrix} 1 \\ 0.4450 \\ -0.8019 \end{bmatrix}$$

$$\omega_3 = 1.8019\sqrt{\frac{k}{m}}, \qquad \mathbf{u}_3 = \begin{bmatrix} 1 \\ -1.2470 \\ 0.5550 \end{bmatrix}$$



Example. (a) Three freight cars are coupled by two springs as shown. Find the natural frequencies and plot the mode shapes for $m_1 = m$, $m_2 = 2m$, $m_3 = 3m$ and $k_1 = k_2 = k$. (b) If the stiffness matrix **K** of a system is only positive semidefinite, show that at least one natural frequency is zero.



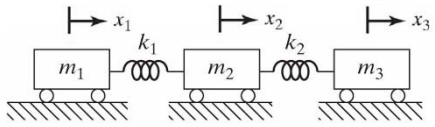
Solution

(a) Let x_1, x_2, x_3 be the displacements of m_1, m_2, m_3 from static equilibrium. The equation of motion system is

$$\begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{bmatrix} + \begin{bmatrix} k_1 & -k_1 & 0 \\ -k_1 & k_1 + k_2 & -k_2 \\ 0 & -k_2 & k_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

It can be checked that

$$\det(\mathbf{K}) = k_1[(k_1 + k_2)k_2 - k_2^2] + k_1(-k_1 k_2) = 0$$



With the specified values of masses and spring constants,

$$m \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{bmatrix} + k \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

As a consequence,

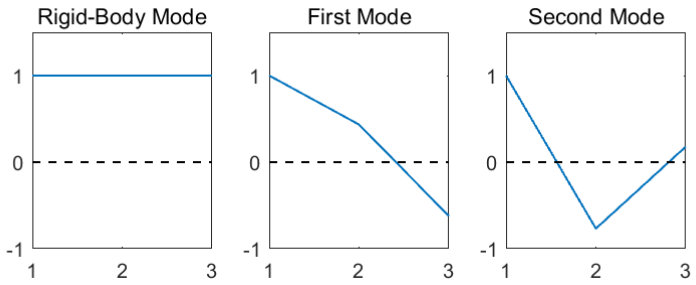
$$\det(\mathbf{K} - \omega^2 \mathbf{M}) = 0$$

$$\Rightarrow \quad \begin{vmatrix} 1-\lambda & -1 & 0 \\ -1 & 2-2\lambda & -1 \\ 0 & -1 & 1-3\lambda \end{vmatrix} = 0, \qquad \lambda = \frac{m\omega^2}{k}$$

$$\Rightarrow \quad \omega_0, \omega_1, \omega_2 = 0,\ 0.7522\sqrt{\frac{k}{m}},\ 1.3295\sqrt{\frac{k}{m}}$$

The mode shapes are

$$\mathbf{u}_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \qquad \mathbf{u}_1 = \begin{bmatrix} 1 \\ 0.4343 \\ -0.6228 \end{bmatrix}, \qquad \mathbf{u}_2 = \begin{bmatrix} 1 \\ -0.7676 \\ 0.1784 \end{bmatrix}$$



(b) If $\mathbf{K}$ is positive semidefinite, there exists a vector $\mathbf{x} \neq 0$ such that $\mathbf{x}^T\mathbf{K}\mathbf{x} = 0$. This implies

$$R(\mathbf{x}) = \frac{\mathbf{x}^T\mathbf{K}\mathbf{x}}{\mathbf{x}^T\mathbf{M}\mathbf{x}} = 0$$

By Rayleigh's principle, $R(\mathbf{x}) \geq \omega_1^2$. Therefore, $\omega_1 = 0$.