Combinatorics Basics & Prime Numbers

Agenda:

- Introduction + Addition & Multiplication Principle
- Permutation & Combination Basics - nPr & nCr formula
- Properties of nCr
- Check if the given number is prime number.
- Find all prime numbers from 1 to N [Sieve].

< Question > : Given 10 Girls and & 7 Boys. How many different pairs can we form?

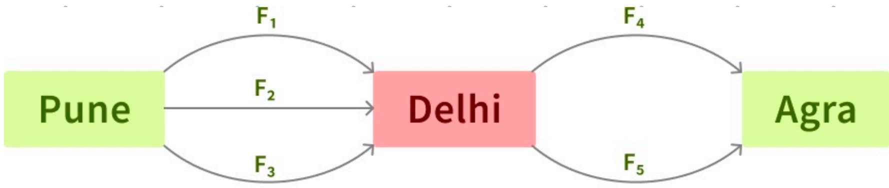
Note : Pair —> { 1boy , 1 girl }

Boys	Girls
B_1	G_1
B_2	G_2
B_3	G_3
B_4	G_4
B_5	G_5
B_6	G_6
B_7	G_7
	G_8
	G_9
	G_{10}

Total Pairs : No of Boys * No of Girls

$= 7 * 10 = 70$

Example-1 :



Number of ways to reach Agra from Pune via Delhi

Pune → Delhi x Delhi → Agra

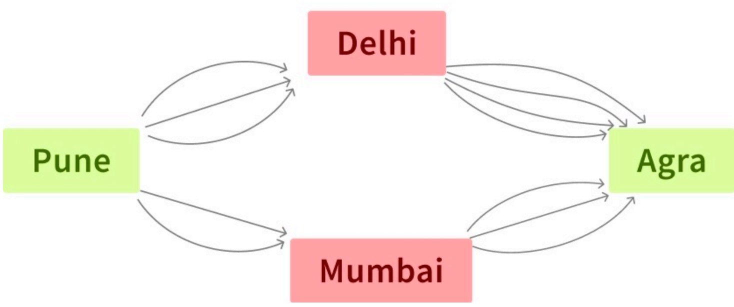
3 * 2 = 6

$F_1 F_4$ $F_2 F_4$ $F_3 F_4$

$F_1 F_5$ $F_2 F_5$ $F_3 F_5$

QUIZ

Number of ways of reaching Agra from Pune ?



via Delhi OR via Mumbai

Pune → Delhi & Delhi → Agra Pune → Mumbai & Mumbai → Agra

3 * 4 + 2 * 3

12 + 6 = 18

AND / MULTIPLICATION : Count possibilities that occur together in Sequence.

OR / ADDITION : Count possibilities that occur in Separate ways.

Permutations

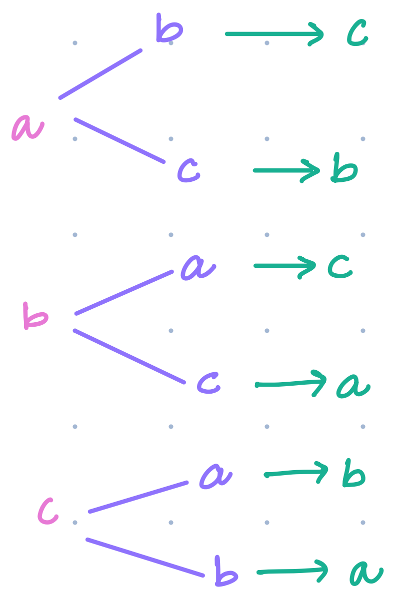
- Arrangement of objects.
- Order matters

$(a, b) \ != (b, a)$

< Question > : Given 3 distinct characters. In how many ways, we can arrange them?

S = " a b c "

$\underline{\ 3\ } * \underline{\ 2\ } * \underline{\ 1\ } = 6$ ways $= 3!$



QUIZ

< Question > : In how many ways n distinct characters can be arranged?

___ ___ ___ . . . . ___

$N * (N-1) * (N-2) \ \_ \ \_ \ \_ \ \ldots \ * 1$

$= N!$

< Question > : Given N distinct characters, in how many ways can we arrange r characters?

r places

$r=2$ $\rightarrow$ $N-1$ $= N-(2-1)$

$r=4$ $\rightarrow$ $N-3$ $N-(4-1)$

$N-(r-1)$

$$= N * (N-1) * (N-2) * (N-3) \cdots (N-(r-1))$$

$$= \frac{N * (N-1) * (N-2) \cdots (N-r+1) * (N-r) * (N-r-1) \cdots 1}{(N-r) * (N-r-1) \cdots 1}$$

$$\boxed{= \frac{N!}{(N-r)!} = {}^{N}P_{r}}$$

No of ways to arrange "r" characters from "n" distinct characters.

Combinations

- No of ways to select something.
- Order of selection doesn't matter.

$(a, b) = - (b, a)$

< Question > : In how many ways can we select 3 players from a pool of 4 players?

[P1 P2 P3 P4]

$$\left.\begin{array}{lll} P_1 & P_2 & P_3 \\ P_1 & P_3 & P_4 \\ P_1 & P_2 & P_4 \\ P_2 & P_3 & P_4 \end{array}\right\} \text{4 ways.}$$

< Question > : Number of ways to arrange the players in 3 slots

Given 4 players → [P1 P2 P3 P4]

Total no of ways to arrange 3 players out of 4 players = Select 3 players out of 4 players & Arrange the selected 3 players in 3 slots.

Arrangements

P_1	P_2	P_3	P_1	P_3	P_4	P_1	P_2	P_4	P_2	P_3	P_4
P_1	P_3	P_2	P_1	P_4	P_3	P_1	P_4	P_2	P_2	P_4	P_3
P_2	P_1	P_3	P_3	P_1	P_4	P_2	P_1	P_4	P_3	P_2	P_4
P_2	P_3	P_1	P_3	P_4	P_1	P_2	P_4	P_1	P_3	P_4	P_2
P_3	P_1	P_2	P_4	P_1	P_3	P_4	P_1	P_2	P_4	P_2	P_3
P_3	P_2	P_1	P_4	P_3	P_1	P_4	P_2	P_1	P_4	P_3	P_2

$P_1\ P_2\ P_3$ $\qquad$ $P_1\ P_3\ P_4$ $\qquad$ $P_1\ P_2\ P_4$ $\qquad$ $P_2\ P_3\ P_4$

Selection

For every selection = 6 arrangements

Total No of arrangements to arrange 3 players in 3 slots = 4 * 6 = 24 ways.

< Question > : Given N distinct elements, in how many ways can we select r distinct elements.

Total no of ways to arrange r players out of N players = Select r players out of N players & Arrange the selected r players in r slots.

Total no of ways to arrange r players out of N players $\rightarrow {}^{N}P_{r}$

Arrange the selected r players in r slots $\rightarrow r! = {}^{r}P_{r}$

${}^{N}P_{r}$ = No of selection of r players out of N players * $r!$

No of selection of r players out of N players $= \dfrac{{}^{N}P_{r}}{r!} = \dfrac{N!}{(N-r)!\, r!}$

No of selection of r players out of N players $\rightarrow {}^{N}C_{r}$

$$\boxed{{}^{N}C_{r} = \frac{{}^{N}P_{r}}{r!} = \frac{N!}{(N-r)!\, r!}}$$

Properties

1. $^{N}C_0 = \frac{N!}{(N-0)!\,0!} = \frac{\cancel{N!}}{\cancel{N!} \times 1} = 1$

No of ways to select NOTHING

2. $^{N}C_N = \frac{N!}{(N-N)!\,N!} = \frac{\cancel{N!}}{0!\,\cancel{N!}} = 1$

No of ways to select EVERYTHING.

3. $^{N}C_r = \frac{N!}{(N-r)!\,r!}$

4. $^{N}C_{N-r} = \frac{N!}{(N-(N-r))!\,(N-r)!} = \frac{N!}{(\cancel{N}-\cancel{N}+r)!\,(N-r)!}$

$$= \frac{N!}{r!\,(N-r)!} = {}^{N}C_r$$

$\downarrow$

No of ways to select $(N-r)$ things out of N things.

||

Rejecting r things.

$^{N}C_{N-r} = {}^{N}C_r$

Special Property : Given N distinct elements, select r elements.

$^{N}C_{r}$

_ _ _ _ _ _ _ _ _ _ _ n^{th} element

select not select

N^{th} element

Select OR Not select

Select (r-1) elements from (N-1) elements

Select r elements from (N-1) elements

$^{N-1}C_{r-1} + {}^{N-1}C_{r}$

$$^{N}C_{r} = {}^{N-1}C_{r-1} + {}^{N-1}C_{r}$$

Prime Numbers

No of factors = 2
(1 & the no itself)

Eg: 2, 3, 5, 7, 11 . . . .

< Question > : Given a number N. Check if it is prime or not.

N = 4 False

N = 13 True.

[Approach] :

```
boolean CheckPrime (int N) {
    int count = 0;
    for (i = 1; i*i <= N; i++) {
        if (N%i == 0) {
            if (i == N/i)
                count++;
            else
                count = count + 2;
        }
    }
    if (count == 2)
        return true;
    return false;
}
```

T.C : $O(\sqrt{N})$
S.C : $O(1)$

< Question > : Given an integer N. Check every number from 1 to N if it is a prime number or not and print it.

$1 \leq N \leq 10^6$

N = 10   2, 3, 5, 7

N = 20   2, 3, 5, 7, 11, 13, 17, 19

BF Idea For all no's from 1 → N call checkPrime()

```
void PrintallPrime (int N) {
    for (i = 1 ; i ≤ N ; i++) {
        if ( CheckPrime(i) == true)
            print (i);
    }
}
```

T.C : $O(N\sqrt{N})$

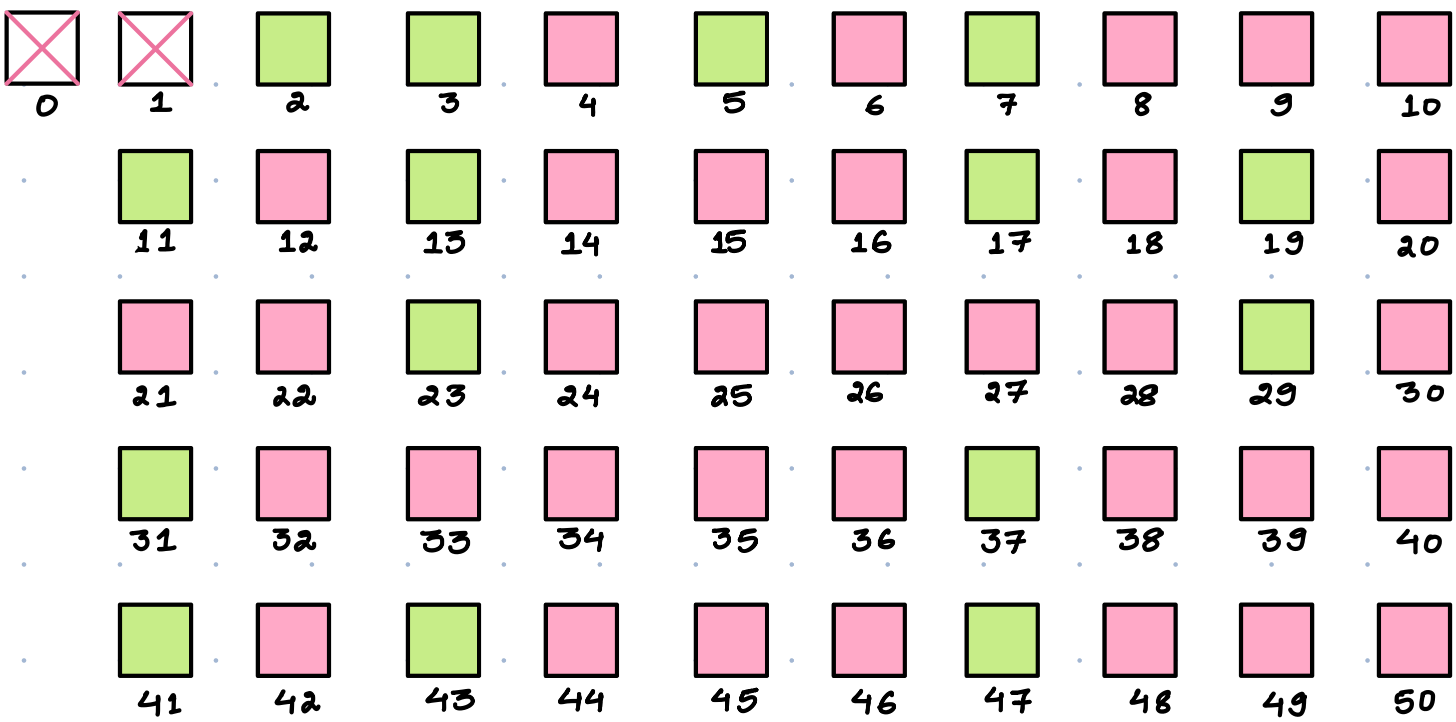
S.C : $O(1)$

Sieve of Eratosthenes / Prime Sieve.

Idea -2

Assumption : Initially assuming all no's from 2 to N to be PRIME.

N = 50



Observation :

i	Multiples of i are Marked as false
2	2*2 , 2*3 , 2*4 , 2*5 , 2*6 - - - - - -
3	3*2 , 3*3 , 3*4 , 3*5 , 3*6 - - - - - - -
4	________
5	5*2 , 5*3 , 5*4 , 5*5 , 5*6 - - - - - - - -

Observation 1 : Instead of marking multiples from 2*i , we can start marking from i*i .

Observation 2 : Stop when $i^2 > N$.

Approach:

- Assume initially all no's as PRIME.
- For each prime no, mark its multiples starting from i*i as False.
- Go to the next unmarked no & repeat the process till $i^2 < N$.

```
void PrintallPrimes(int N){
    // Mark all no's as PRIME
    bool ans[N+1] = {TRUE};
    ans[0] = false;
    ans[1] = false;

    for(i=2; i*i <= N; i++){
        // For every Prime No, mark its multiples as False.
        if(ans[i] == TRUE){
            for(j=i*i; j<=N; j=j+i){
                ans[j] = FALSE;
            }
        }
    }
    // Print all the unmarked no/Prime No.
    for(i=2; i<=N; i++){
        if(ans[i] == TRUE)
            print(i);
    }
}
```

Dry Run

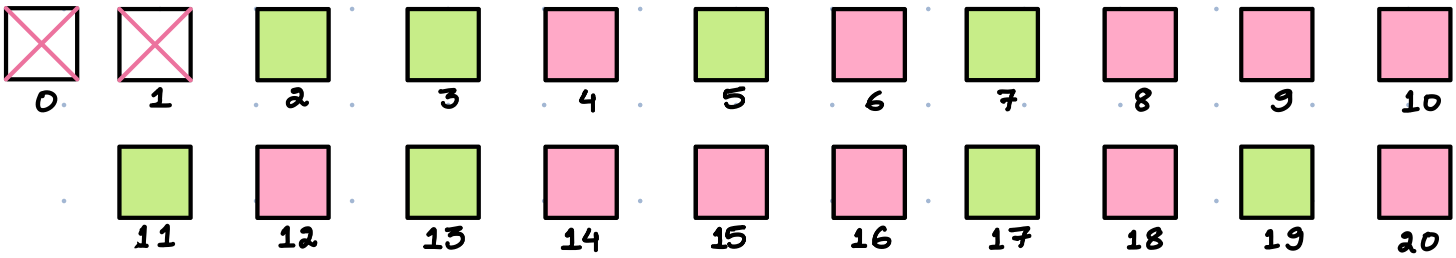
$i = \cancel{2}$
$\quad = 3$
$\quad\ \ 4$
$\quad\ \ 5$

$j = \cancel{2}\ \cancel{4}\ \cancel{6}\ \cancel{8}\ \cancel{10}\ \cancel{12}\ \cancel{14}\ \cancel{16}\ 18$

$j = \cancel{9}\ \cancel{12}\ \cancel{15}\ \cancel{18}\ 21$

$5^2 > 20 \rightarrow$ STOP

$N = 20$



T.C Analysis

$i\ [2 \rightarrow \sqrt{n}]$	$j\ [i*i \rightarrow n]$
2	$\simeq n/2$ iterations.
3	$\simeq n/3$ iterations
4	____
5	$\simeq n/5$ iterations.
$\vdots$	$\vdots$
$\sqrt{n}$	$[(\sqrt{n})^2 \rightarrow n]$ 1 iteration $n \rightarrow n$

Total No of Iterations : $\frac{n}{2} + \frac{n}{3} + \frac{n}{5} - \cdots 1$

$$= n\underbrace{\left[\frac{1}{2} + \frac{1}{3} + \frac{1}{5} - - - \cdot \frac{1}{n}\right]}_{\text{Sum of Reciprocal of all Prime No.}}$$

log(log N)

$$= n \log(\log n)$$

T.C : $O(N \log(\log N))$

S.C : $O(N)$