Angular Momentum

For Advanced Nuclear Physics

Syllabus

Addition of angular momentum – Clebsch-Gordan coefficients – symmetry properties – evaluation of Clebsch-Gordan coefficients in simple cases – Rotational matrices for spinors – Spherical tensors – rotation matrices – Wigner Eckart theorem – simple applications.

Contents

1. Angular momentum operators **1**
1.1. Definition of angular momentum operator in quantum mechanics 1
1.2. Physical interpretation of $\hat{J}$ 1
1.3. Raising and lowering operators 2
1.4. Spectrum of eigenvalues for $\hat{J}^2$ and $\hat{J}_z$ 3

2. Coupling of Two Angular Momenta **4**
2.1. The Clebsch-Gordan coefficients 4
2.2. Some simple properties 5
2.3. General expressions for C.G. coefficients 7
2.4. Fortran program to evaluate the C.G. coefficients 7
2.5. Symmetry properties 8
2.6. Notations 9
2.7. Evaluation of C.G. Coefficients in simple cases 10

3. Rotation Matrices **11**
3.1. Definition of rotation matrix 11
3.2. Rotation in terms of Euler angles 11
3.3. Transformation of a vector under rotation of coordinate system 12
3.4. The rotation matrix $D^1(\alpha,\beta,\gamma)$ 13
3.5. The rotation operator 13
3.6. The $d^j_{m'm}(\beta)$ matrix 15
3.7. The rotation matrix for spinors 16
3.8. The Clebsch-Gordan series 19
3.9. The inverse Clebsch-Gordan series 19
3.10. Application of rotational operator – Cranking Model 20

4. The Wigner-Eckart theorem **21**
4.1. Theorem 21
4.2. Proof of the theorem 21
4.3. Matrix element of spherical harmonics 23

1. Angular momentum operators

1.1. Definition of angular momentum operator in quantum mechanics

In classical mechanics, the angular momentum vector is defined as the cross product of the position vector $\hat{r}$ and the momentum vector $\hat{p}$

$$\hat{L} = \hat{r} \times \hat{p} \qquad \ldots(1)$$

Both $\hat{r}$ and $\hat{p}$ change sign under inversion of coordinate system and so they are called polar vectors. It is easy to see that $\hat{L}$ behaves differently and will not change sign under inversion of coordinate system and it is known as a *Pseudo-vector* or *axial vector*.

The transition to quantum mechanics can be made by incorporating the uncertainty principle expressed in the form of commutators

$$[\hat{x}, \hat{p}_x] = i\hbar,\ [\hat{y}, \hat{p}_y] = i\hbar,\ [\hat{z}, \hat{p}_z] = i\hbar \qquad \ldots(2)$$

We obtain the commutation relations for the components of angular momentum operator

$$[\hat{L}_x, \hat{L}_y] = i\hbar\hat{L}_z$$
$$[\hat{L}_y, \hat{L}_z] = i\hbar\hat{L}_x \qquad \ldots(3)$$
$$[\hat{L}_y, \hat{L}_x] = i\hbar\hat{L}_y$$

These commutators define the angular momentum in quantum mechanics and this definition is more general and admits half-integral quantum numbers. For this purpose, let us denote the quantum mechanical angular momentum operator by $\hat{J}$ and also use the convention that the angular momentum is expressed in units of $\hbar$.

$$[\hat{J}_x, \hat{J}_y] = i\hat{J}_z$$
$$[\hat{J}_y, \hat{J}_z] = i\hat{J}_x \qquad \ldots(4)$$
$$[\hat{J}_y, \hat{J}_x] = i\hat{J}_y$$

In a compact notation, eqs(4) become

$$[\hat{J} \times \hat{J}] = i\hat{J} \qquad \ldots(5)$$

Eqn.(5) is the starting point of an investigation and our aim is to draw as much information as possible from this definition.

1.2. Physical interpretation of $\hat{J}$

Although the components of the angular momentum operator do not commute among themselves, it is easy to show that the square of the angular momentum operator

$$\hat{J}^2 = \hat{J}_x^2 + \hat{J}_y^2 + \hat{J}_z^2 \qquad \ldots(6)$$

commutes with each of its component.

$$[\hat{J}^2, \hat{J}_x] = 0 , \quad [\hat{J}^2, \hat{J}_y] = 0 , \quad [\hat{J}^2, \hat{J}_z] = 0 \qquad \ldots(7)$$

Eqs (4) and (7) are amenable to simple physical interpretation. It is possible to find the simultaneous eigenvalues of $\hat{J}^2$ and one of its components, say $\hat{J}_z$, alone, but it is impossible to find the eigenvalues of $\hat{J}_x$ and $\hat{J}_y$ at the same time. Representing the operators by matrices, one can say that $\hat{J}^2$ and $\hat{J}_z$ can be diagonalized in the same representation but not the other components $\hat{J}_x$ and $\hat{J}_y$. Physically this means that one can know at the most, the magnitude of the angular momentum vector and its projection on one of the axis. The projections on the other two axes cannot be determined.

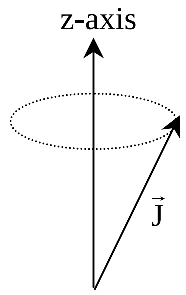

Fig.1

This is illustrated in Fig.(1) in which the angular momentum vector is depicted to lie anywhere on the core.

If ψ_{jm} is the eigenfunction of the operators $\hat{J}^2$ and $\hat{J}_z$, then

$$\hat{J}^2 \psi_{jm} = \eta_j \psi_{jm} \qquad \ldots(8)$$

$$\hat{J}_z \psi_{jm} = m \psi_{jm} \qquad \ldots(9)$$

In the above, j and m are the quantum numbers used to define the eigenfunction and the corresponding eigenvalues are η_j and m. We are interested in finding the spectrum of values that j and m can take and also the eigenvalue η_j.

1.3. Raising and lowering operators

Let us define two more operators $\hat{J}_+$ and $\hat{J}_-$ which we shall call the raising and lowering operators.

$$\hat{J}_\pm = \hat{J}_x \pm i \hat{J}_y \qquad \ldots(10)$$

This nomenclature will become obvious once their roles are understood. The following commutation relations can be easily obtained.

$$[\hat{J}^2, \hat{J}_\pm] = 0 ; \quad [\hat{J}_z, \hat{J}_\pm] = \pm \hat{J}_\pm \qquad \ldots(11)$$

Let us generate a new function $\phi_\pm$ by allowing $\hat{J}_\pm$ to operate on ψ_{jm} and examine whether this new function is an eigenfunction of $\hat{J}^2$ and $\hat{J}_z$ operators. If so, what are their eigenvalues?

$$\hat{J}_\pm \psi_{jm} = \phi_\pm \qquad \ldots(12)$$

$$\hat{J}^2 \phi_\pm = \hat{J}^2 \hat{J}_\pm \psi_{jm}$$

$$= \hat{J}_\pm \hat{J}^2 \psi_{jm} \qquad \{ \text{ using eq.(11)}$$

$$= \eta_j \, \phi_\pm \qquad \ldots(13)$$

$$\hat{J}_z \, \phi_\pm = \hat{J}_z \, \hat{J}_\pm \psi_{jm}$$

$$= (\hat{J}_\pm \, \hat{J}_z \pm \hat{J}_\pm)\psi_{jm} \qquad \{ \text{ using eq.(11)}$$

$$= \hat{J}_\pm \, (\hat{J}_z \pm 1) \, \psi_{jm}$$

$$= (m \pm 1) \, \phi_\pm \qquad \ldots(14)$$

Thus we find that $\phi_\pm$ is an eigenfunction of $\hat{J}^2$ and $\hat{J}_z$ operators. The eigenvalue of the operator $\hat{J}^2$ remains unchanged but the eigenvalue of the operator $\hat{J}_z$ is stepped up or stepped down by unity. It is precisely for this reason, the operator $\hat{J}_\pm$ is called the raising or lowering operator.

1.4. Spectrum of eigenvalues for $\hat{J}^2$ and $\hat{J}_z$

Starting from the quantum mechanical definition of angular momentum given by eq.(5), we can show that the eigenvalues of $\hat{J}^2$ operator is j(j+1) where j can take integral or half-integral values and for a given j, the eigenvalue of $\hat{J}_z$ operator, viz., m can take a spectrum of values from –j to +j in steps of unity.

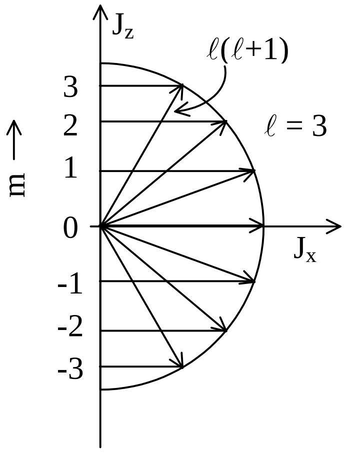

Fig. 2.

Writing the angular momentum operators in spherical coordinates and doing some simplifications using their commutation relations, we can write

$$\hat{J}^2 = - \, \Delta_{\theta,\varphi}$$

where $\Delta_{\theta,\varphi}$ is the part of the Laplacian operator acting on the variables θ and φ only. In this context we also write down the eigenfunctions of $\hat{J}^2$ and $\hat{J}_z$:

$$\hat{J}^2 \, Y_{\ell m}(\theta,\varphi) = \ell(\ell+1) \, Y_{\ell m}(\theta,\varphi)$$

$$\hat{J}_z \, Y_{\ell m}(\theta,\varphi) = m \, Y_{\ell m}(\theta,\varphi),$$

$$m = -\ell, -\ell+1, \ldots, 0, \ldots, \ell$$

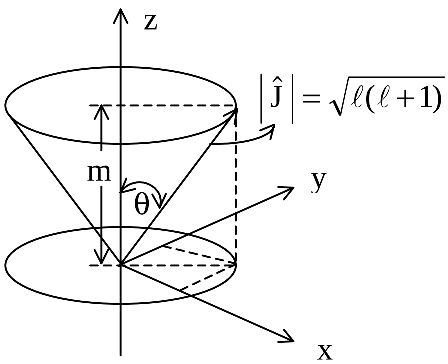

Fig. 3.

Obviously the spectrum of $\hat{J}^2$ and $\hat{J}_z$ is always discrete. And from the above equations we infer that the z projection of an angular momentum $\hat{J}$ with absolute value $\sqrt{\ell(\ell+1)}$ takes $2\ell+1$ different m values. This is illustrated in Fig. 2.

The angle between the angular momentum and the direction of quantization can have only certain values:

$$\cos\theta = \frac{m}{\sqrt{\ell(\ell+1}}$$

This is sometimes called *quantization of direction* and means nothing more than the quantization of the z component of angular momentum, i.e. $\hat{J}_z$. Thus the obtained results can be interpreted in a pictorial way (see Fig. 3); the angular momentum vector $\hat{J}_z$ precesses on a cone around the direction of quantization (z-axis). As a result, the x and y components of angular momentum are not constant in time. This illustrates the uncertainty relations between $\hat{J}_z$ and $\hat{J}_x$ and between $\hat{J}_z$ and $\hat{J}_y$.

2. Coupling of Two Angular Momenta

2.1. The Clebsch-Gordan coefficients

Problems involving the addition of two angular momenta arouse in Physics. They may be the angular momenta of the two particles in a system or the orbital and spin angular momenta of a single particle.

If $\hat{J}_1$ and $\hat{J}_2$ are the operators corresponding to the two angular momenta, then the resultant angular momentum operator $\hat{J}$ is obtained by the vector addition

$$\hat{J} = \hat{J}_1 + \hat{J}_2 \qquad \ldots(1)$$

It follows that

$$\hat{J}_x = \hat{J}_{1x} + \hat{J}_{2x}$$

$$\hat{J}_y = \hat{J}_{1y} + \hat{J}_{2y} \qquad \ldots(2)$$

$$\hat{J}_z = \hat{J}_{1z} + \hat{J}_{2z}$$

Squaring (1) we obtain,

$$\hat{J}^2 = \hat{J}_1^2 + \hat{J}_2^2 + 2\ \hat{J}_1 \cdot \hat{J}_2 \qquad \ldots(3)$$

Since by our construction $\hat{J}$ is and angular momentum operator, it should obey the same commutation relation as $\hat{J}_1$ and $\hat{J}_2$. That is,

$$[\hat{J}^2, \hat{J}_z] = 0$$

$$[\hat{J}_{z,} \hat{J}_y] = i\ \hat{J}_z \qquad \ldots(4)$$

and the other cyclic relations. The commutation relations (4) can be deduced from the commutation relations obeyed by $\hat{J}_1$ and $\hat{J}_2$. It is to be noted that $\hat{J}_1$ and $\hat{J}_2$ are two independent operators and hence they should mutually commute. However, it is found that

$$[\hat{J}^2, \hat{J}_{1z}] = -[\hat{J}^2, \hat{J}_{2z}] \neq 0 \qquad \ldots(5)$$

Thus we have two sets of numerically mutually commuting operators

$$\text{Set I} \quad : \quad \hat{J}_1^2, \hat{J}_2^2, \hat{J}_{1z}, \hat{J}_{2z}$$
$$\text{Set II} \quad : \quad \hat{J}_1^2, \hat{J}_2^2, \hat{J}^2, \hat{J}_z \qquad \ldots(6)$$

So it is possible to find the simultaneous eigenvalues of either the first set of operators or the second set but not both. The eigenfunctions, denoted by their quantum numbers $|j_1\ j_2\ m_1\ m_2\rangle$ corresponding to the first set of operators are said to be in the uncoupled representation and the eigenfunctions $|j_1\ j_2\ j\ m\rangle$ corresponding to the second set belong to the coupled representation. The functions $|j_1\ j_2\ j\ m\rangle$ can be expanded in terms of functions $|j_1\ j_2\ m_1\ m_2\rangle$ and vice versa.

$$|j_1\ j_2\ j\ m\rangle = \sum_{m_1, m_2} \begin{pmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m \end{pmatrix} |j_1\ j_2\ m_1\ m_2\rangle \qquad \ldots(7)$$

The quantity $\begin{pmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m \end{pmatrix}$ is the expansion coefficient whose dependence on the quantum numbers is explicitly denoted. This coefficient is known as the *Clebsch-Gordan coefficient* (C.G.coefficient) or the vector addition coefficient and it is the unitary transformation coefficient that occurs when one goes from the uncoupled to the coupled representation. From (7) we get

$$\langle j_1\ j_2\ m_1\ m_2 | j_1\ j_2\ j\ m\rangle = \begin{pmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m \end{pmatrix} \qquad \ldots(8)$$

This C.G. coefficient can be determined without the phase factor and the standard phase convention is such as to make the C.G. coefficient real. Then taking the complex conjugate of (8)

$$\langle j_1\ j_2\ j\ m | j_1\ j_2\ m_1\ m_2\rangle = \begin{pmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m \end{pmatrix} \qquad \ldots(9)$$

Which is equivalent to the inverse relation of (7), viz.,

$$|j_1\ j_2\ m_1\ m_2\rangle = \sum_{j} \begin{pmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m \end{pmatrix} |j_1\ j_2\ j\ m\rangle \qquad \ldots(10)$$

2.2. Some simple properties

(a) If $m_1 + m_2 \neq m$ then C.G. coefficients will vanish

It is easy to show that $m_1 + m_2 = m$. Otherwise the C.G. coefficients will vanish. Operating J_z on the LHS and $J_{1z} + J_{2z}$ on the RHS of eq.(7) we obtain

$$m\,|j_1\ j_2\ j\ m\rangle = \sum_{m_1, m_2} (m_1 + m_2) \begin{pmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m \end{pmatrix} |j_1\ j_2\ m_1\ m_2\rangle \qquad \ldots(11)$$

Expanding $|j_1\ j_2\ j\ m\rangle$ once again in terms of $|j_1\ j_2\ m_1\ m_2\rangle$ we get

$$\sum_{m_1, m_2} (m - m_1 - m_2) \begin{pmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m \end{pmatrix} \; |j_1\ j_2\ m_1\ m_2\rangle = 0 \qquad \ldots(12)$$

Since the function $|j_1\ j_2\ m_1\ m_2\rangle$ are linearly independent, it follows that each of the coefficients in the summation should be identically zero.

$$(m - m_1 - m_2) \begin{pmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m \end{pmatrix} = 0 \qquad \ldots(13)$$

Thus it is evident that unless $m = m_1 + m_2$, the C.G. coefficient should vanish.

(b) C.G. coefficients will exist only when the triangular condition is satisfied

There are in total $(2j_1+1)(2j_2+1)$ linearly independent functions $|j_1\ j_2\ m_1\ m_2\rangle$. Since the total number of linearly independent functions is preserved in any unitary transformation, the number of independent functions $|j_1\ j_2\ j\ m\rangle$ in the coupled representation should be the same. Hence

$$\sum_{j} (2j+1) = (2j_1+1)(2j_2+1) \qquad \ldots(14)$$

j_{max} should be obviously $(j_1 + j_2)$ since the maximum value of $m = j_1 + j_2$, j_1 and j_2 being the maximum value of m_1 and m_2. By simple enumeration, one can find $j_{min} = |j_1 - j_2|$. Thus j can assume a spectrum of values from $|j_1 - j_2|$ to $j_1 + j_2$ in steps of unity. Thereby j_1, j_2 and j obey the triangular condition $\Delta(j_1\ j_2\ j)$. Otherwise the C.G. coefficients will vanish.

(c) The distinction between the uncoupled and the coupled representations vanishes if one of the two angular momenta were to vanish

Hence the C.G. coefficient which is the element of the unitary transformation becomes unity.

$$\begin{pmatrix} j & 0 & j \\ m & 0 & m \end{pmatrix} = \begin{pmatrix} 0 & j & j \\ 0 & m & m \end{pmatrix} = 1 \qquad \ldots(15)$$

(d) The functions $|j_1\ j_2\ m_1\ m_2\rangle$ *and* $|j_1\ j_2\ j\ m\rangle$ *are orthonormal*

$$\langle j_1\ j_2\ m_1'\ m_2' | j_2\ j_2\ m_1\ m_2 \rangle = \delta_{m_1 m_1'} \delta_{m_2 m_2'} \qquad \ldots(16)$$

$$\langle j_1\ j_2\ j'\ m' \,|\, j_2\ j_2\ j\ m \rangle = \delta_{j\,j'}\ \delta_{m\,m'} \qquad \ldots(17)$$

Using the expansion (7) we obtain

$$\langle j_1\ j_2\ j'\ m' | j_1\ j_2\ j\ m \rangle = \sum_{\substack{m_1, m_2 \\ m_1', m_2'}} \begin{pmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m \end{pmatrix} \begin{pmatrix} j_1 & j_2 & j' \\ m_1' & m_2' & m' \end{pmatrix} \langle j_1\ j_2\ m_1'\ m_2' | j_1\ j_2\ m_1\ m_2 \rangle \qquad \ldots(18)$$

Application of eqs (16) and (17) yields

$$\delta_{j\,j'}\,\delta_{m\,m'} = \sum_{m_1,\,m_2} \begin{pmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m \end{pmatrix} \begin{pmatrix} j_1 & j_2 & j' \\ m_1 & m_2 & m' \end{pmatrix} \qquad \ldots(19)$$

In a similar way, starting from eq (16) and applying the expression (10) twice we get one more relation

$$\delta_{m_1\,m_1'}\,\delta_{m_2\,m_2'} = \sum_{j} \begin{pmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m \end{pmatrix} \begin{pmatrix} j_1 & j_2 & j \\ m_1' & m_2' & m \end{pmatrix} \qquad \ldots(20)$$

The vectors (19) and (20) are known as orthonormality relations of the C.G. coefficients.

It may be observed that in eqs(7) and (19) although there are two summations m_1 and m_2, one is redundant because of the constraint $m_1 + m_2 = m$.

2.3. General expressions for C.G. coefficients

General expressions for C.G. coefficients have been derived by E.P.Wigner using group theory and by G.Racah using the algebraic methods. Here we give only Racah's closed expression for C.G. coefficients since it is more convenient for writing a computer program for numerical evaluation of C.G. coefficients

$$\begin{pmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m \end{pmatrix} = \delta_{m\,m_1+m_2} \left\{ (2j+1)\frac{(j_1+j_2-j)!\,(j+j_1-j_2)!\,(j+j_2-j_1)!}{(j_1+j_2+j+1)!} \right\}^{1/2}$$

$$\times \quad \{(j_1+m_1)!\,(j_1-m_1)!\,(j_2+m_2)!\,(j_2-m_2)!\,(j+m)!\,(j-m)!\}^{1/2}$$

$$\times \quad \sum_{\nu} \frac{(-1)^{\nu}}{\nu!} \{(j_1+j_2-j-\nu)!\,(j_1-m_1-\nu)!(j_2+m_2-\nu)!$$

$$\times (j-j_2+m_1+\nu)!\,(j-j_1-m_2+\nu)!\}^{-1} \qquad \ldots(21)$$

The summation index ν assumes all integer values for which the factorial arguments are not negative.

2.4. Fortran program to evaluate the C.G. coefficients

```
**********************************************************************
************Numerical Evaluation of C.G.Coefficients***************
********************using Racah expansion**************************
**********************************************************************
      Real j1,j2,j,m1,m2,m
      Read(*,*)j1,j2,j
      Read(*,*)m1,m2,m
      Write(*,*)CGC(j1,j2,j,m1,m2,m),CGC(j1,j2,j,m1,m2,m)**2
      End

      Function CGC(j1,j2,j,m1,m2,m)
      Real j1,j2,j,m1,m2,m
```

```
      Real*8 Fact

      Term1=(2*j+1)*Fact(Int(j1+j2-j))*Fact(Int(j+j1-j2))
     1  *Fact(Int(j+j2-j1))/Fact(Int(j1+j2+j+1))

      Term2=Fact(Int(j1+m1))*Fact(Int(j1-m1))*Fact(Int(j2+m2))
     1  *Fact(Int(j2-m2))*Fact(Int(j+m))*Fact(Int(j-m))

      Term3=0

      Do nu = 0,10
        ia=Int(j1+j2-j-nu)
        ib=Int(j1-m1-nu)
        ic=Int(j2+m2-nu)
        id=Int(j-j2+m1+nu)
        ie=Int(j-j1-m2+nu)
        If ((ia.GE.0.0).AND.(ib.GE.0.0).AND.(ic.GE.0.0).AND.
     1   (id.GE.0.0).AND.(ie.GE.0.0)) Then
           Term3=Term3+(-1)**nu/(Fact(Int(nu))*Fact(ia)*Fact(ib)
     1       *Fact(ic)*Fact(id)*Fact(ie))
        Endif
      End Do

      CGC= Kdel(m,m1+m2)*Sqrt(Term1*Term2)*Term3
      End

      Function Kdel(delK1,delK2)
      If(delK1.EQ.delK2) then
        Kdel=1
      Else
        Kdel=0
      Endif
      End

      Real*8 Function Fact(JFact)
      I=JFact
      Fact = 1.0
      Do While (I.GT.0)
        Fact = Fact*I
        I=I-1
      End Do
      End
```

Note: *In the above program, some steps that may give arise to a faster program have been ignored for the sake of readability.*

2.5. Symmetry properties

A study of the general expressions for the C.G. coefficients will reveal the following symmetry properties.

$$\begin{pmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m \end{pmatrix} = (-1)^{j_1+j_2-j} \begin{pmatrix} j_1 & j_2 & j \\ -m_1 & -m_2 & -m \end{pmatrix} \qquad \ldots(22a)$$

$$= (-1)^{j_1+j_2-j} \begin{pmatrix} j_2 & j_1 & j \\ m_2 & m_1 & m \end{pmatrix} \qquad \ldots(22b)$$

$$\begin{pmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m \end{pmatrix} = (-1)^{j_1-m_1} \frac{[j]}{[j_2]} \begin{pmatrix} j_1 & j & j_2 \\ m_1 & -m & -m_2 \end{pmatrix} \qquad \ldots(22c)$$

$$= (-1)^{j_2+m_2} \frac{[j]}{[j_1]} \begin{pmatrix} j & j_2 & j_1 \\ -m & m_2 & -m_1 \end{pmatrix} \qquad \ldots(22d)$$

where $\quad [j] = \sqrt{2j+1}$

Relations 22(a),(b),(c) and (d) bring out the symmetry properties of the C.G. coefficients under the permutation of any two columns or the reversal of the sign of the projection quantum numbers. Note that when the third column is permuted with the first or the second, there is a reversal of the sign of the projection quantum numbers of the permuted columns. This is essential to preserve the relation $m_1+m_2=m$.

By using the symmetry relation (22a) one finds

$$\begin{pmatrix} j_1 & j_2 & j \\ 0 & 0 & 0 \end{pmatrix} = (-1)^{j_1+j_2-j} \begin{pmatrix} j_1 & j_2 & j \\ 0 & 0 & 0 \end{pmatrix} \qquad \ldots(23)$$

$$\begin{pmatrix} j_1 & j_2 & j \\ 0 & 0 & 0 \end{pmatrix} = 0 \qquad \ldots(24)$$

if $j_1 + j_2 - j$ is odd. Moreover, the quantum numbers j_1, j_2 & j should all be integers, otherwise the projection quantum numbers cannot be zero. This special C.G. coefficient is known as the parity C.G. coefficient since in physical problems such as a coefficient contains the parity selections rule.

The C.G. Coefficients are the expansion coefficients and the sum of their squares should be unity since the eigenfunctions are normalized. i.e.,

$$\sum_{m_1} \begin{pmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m \end{pmatrix}^2 = 1 \qquad \ldots(25)$$

This argument can be verified directly from the orthonormality relations [Eqns. (19) and (20)].

2.6. Notations

Different notations have come into vague for the C.G. coefficients. $\langle j_1\ m_1\ j_2\ m_2 |\ j\ m\rangle$, $C(j_1\ j_2\ j\ m_1\ m_2\ m)$ are some of the notations commonly used in literature.

The Wigner 3-j symbol is related to the C.G. coefficient by the relation,

$$\langle j_1\ j_2\ m_1\ m_2 \mid j\ m\rangle = (-1)^{j_1-j_2+m}\sqrt{2j+1}\ \underbrace{\begin{pmatrix} j_1 & j_2 & j \\ m_1 & m_2 & -m \end{pmatrix}}_{\text{3-j symbol}} \qquad \ldots(26)$$

2.7. Evaluation of C.G. Coefficients in simple cases

(a) $\begin{pmatrix} 2 & 2 & 0 \\ 1 & -1 & 0 \end{pmatrix} = -\frac{1}{[2]} \begin{pmatrix} 2 & 0 & 2 \\ 1 & 0 & 1 \end{pmatrix} = -\frac{1}{\sqrt{5}}$

The symmetry property (22b) is used.

(b) $\begin{pmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \end{pmatrix} = 1$

It is a stretched case.

(c) $\begin{pmatrix} 1/2 & 1/2 & 1 \\ 1/2 & -1/2 & 0 \end{pmatrix} = \begin{pmatrix} 1/2 & 1/2 & 1 \\ -1/2 & 1/2 & 0 \end{pmatrix} = -\frac{1}{\sqrt{2}}$

Only the above two C.G. Coefficients occur in the expansion of the eigenfunction $|\tfrac{1}{2}\ \tfrac{1}{2}\ 1\ 0\rangle$ and hence the sum of their squares should be unity.

(d) $\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} = 0$

This is a parity C.G. Coefficient and it is zero since $j_1 + j_2 - j$ is odd.

(e) To find $\begin{pmatrix} 2 & 1 & 2 \\ 1 & -1 & 0 \end{pmatrix}$

$$\sum_{m_1} \begin{pmatrix} 2 & 1 & 2 \\ m_1 & m_2 & 0 \end{pmatrix}^2 = 1$$

In the expansion there are three C.G. Coefficients, of which one is the parity C.G. Coefficient $\begin{pmatrix} 2 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$ which is zero. The other two C.G. Coefficients are determined using the above property and the symmetry relation.

$$\begin{pmatrix} 2 & 1 & 2 \\ 1 & -1 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 1 & 2 \\ -1 & 1 & 0 \end{pmatrix} = \frac{1}{\sqrt{2}}$$

(f) $\begin{pmatrix} 1 & 1 & 3 \\ 1 & -1 & 0 \end{pmatrix} = 0, \quad$ since the triangular condition $\Delta(j_1, j_2, j)$ is not obeyed.

(g) $\begin{pmatrix} 1 & 1 & 2 \\ 2 & 0 & 2 \end{pmatrix} = 0, \quad$ since $m_1 > j_1$ is not allowed.

(h) $\begin{pmatrix} 1 & 1 & 2 \\ 1 & -1 & 2 \end{pmatrix} = 0, \quad \text{since } m \neq m_1 + m_2.$

(i) $\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} = 1, \quad \text{since it is a stretched case as (b).}$

(j) $\begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \end{pmatrix} = \frac{1}{[1]}\begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix} = \frac{1}{\sqrt{3}}.$

3. Rotation Matrices

3.1.Definition of rotation matrix

The rotation matrices define the transformation properties of angular momentum eigenfunctions ψ_{jm} under rotation of coordinate system.

$$\psi_{jm}(\hat{r}') = \sum_{m'} D^j_{m'm}(\alpha\beta\gamma)\ \psi_{jm'}(\hat{r}), \qquad \ldots(1)$$

where $D^j_{m'm}(\alpha\beta\gamma)$ denotes an element of the Rotation matrix, the rotation being described by a set of three Euler angles α,β,γ. The angular momentum eigenfunctions $\psi_{jm}(\hat{r}')$ are in the rotated coordinate system S′, whereas the functions $\psi_{jm}(\hat{r})$ denote the eigenfunctions in the original coordinate system S. Hence these functions should be related by a unitary transformation. In this section we shall obtain the rotation matrices from a consideration of the transformations properties of a vector (spherical tensor of rank 1).

3.2. Rotation in terms of Euler angles

Consider a right-handed coordinate system. Any general rotation R in three dimensional space can be conveniently described in terms of the three Euler angles α, β and γ ($0 < \alpha < 2\pi$, $0 < \beta < \pi$, $0 < \gamma < 2\pi$).

$$R = R_{Z_2}(\gamma)\ R_{Y_1}(\beta)\ R_Z(\alpha) \qquad \ldots(2)$$

$R_Z(\alpha)$ denotes a rotation through an angle α about the Z axis. This result in the change of the reference frame $XYZ \rightarrow X_1Y_1Z_1$, Z_1 axis coinciding with the Z axis. This is followed by a rotation through an angle β about the Y_1 axis and then through an angle γ about the Z axis. The complete rotation R can be denoted explicitly in the following sequence.

$$XYZ \longrightarrow X_1Y_1Z_1 \longrightarrow X_2Y_2Z_2 \longrightarrow X'Y'Z' \qquad \ldots(3)$$

3.3. Transformation of a vector under rotation of coordinate system

This subsection just gives a flavor about how rotation can be represented effectively using the matrix formalism. Eventhough the exact transformation of the spherical components has been skipped, the forthcoming arguments provide reasonably good understandings that how a rotational matrix would look like.

Let us now consider the transformation of the cartesian components of a vector $\vec{A}$ under a general rotation R of the coordinate system and obtain the transformation matrix. This is done in three steps. First let us make a rotation through an angle α about the Z axis. The cartesian components of $\vec{A}$ transform as follows:

$$\begin{aligned} A_{X_1} &= A_X \cos\alpha + A_Y \sin\alpha, \\ A_{Y_1} &= A_Y \cos\alpha - A_X \sin\alpha, \\ A_{Z_1} &= A_Z. \end{aligned} \qquad \ldots(4)$$

This transformation can be expressed more elegantly in the matrix form as follows.

$$\begin{bmatrix} A_{X_1} \\ A_{Y_1} \\ A_{Z_1} \end{bmatrix} = \begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_X \\ A_Y \\ A_Z \end{bmatrix} \qquad \ldots(5)$$

In a concise notation,

$$\vec{A}_1 = M_Z(\alpha)\,\vec{A} \quad , \qquad \ldots(6)$$

where $M_Z(\alpha)$ is the transformation matrix for rotation about the Z axis through an angle α.

Next let us consider a rotation through an angle β about the Y axis. The cartesian components A_{X_1}, A_{Y_1}, A_{Z_1} transform into A_{X_2}, A_{Y_2}, A_{Z_2} and the equations of transformation are given below.

$$\begin{aligned} A_{X_2} &= A_{X_1} \cos\beta - A_{Z_1} \sin\beta, \\ A_{Y_2} &= A_{Y_1}, \\ A_{Z_2} &= A_{Z_1} \cos\beta + A_{X_1} \sin\beta. \end{aligned} \qquad \ldots(7)$$

In matrix notation,

$$\begin{bmatrix} A_{X_2} \\ A_{Y_2} \\ A_{Z_2} \end{bmatrix} = \begin{bmatrix} \cos\beta & 0 & -\sin\beta \\ 0 & 1 & 0 \\ \sin\beta & 0 & \cos\beta \end{bmatrix} \begin{bmatrix} A_{X_1} \\ A_{Y_1} \\ A_{Z_1} \end{bmatrix} . \qquad \ldots(8)$$

Denoting the transformation matrix by $M_Y(\beta)$,

$$\vec{A}_2 = M(\beta)\,\vec{A}_1 \quad , \qquad \ldots(9)$$

$$= M_{Y_1}(\beta)\, M_Z(\alpha)\ \vec{A} \qquad \ldots(10)$$

Lastly, we have to perform a rotation through an angle γ about the Z_2 axis. The resulting transformation matrix is the product of the three transformation matrices obtained for rotations through the three Euler angles.

$$M(\alpha,\beta,\gamma) = M_{Z_2}(\gamma)\, M_{Y_1}(\beta)\, M_Z(\alpha), \qquad \ldots(11)$$

and the transformed vector $\vec{A}'$ is given by

$$\vec{A}' = M(\alpha,\beta,\gamma)\, \vec{A}\,. \qquad \ldots(12)$$

3.4.The rotation matrix $D^1(\alpha,\beta,\gamma)$

It is to be pointed out that the transformation matrix M is not the rotation matrix defined in this material. According to the law of matrix multiplication, any component of the transformed vector $\vec{A}'$ is given by

$$A'_\mu = \sum_\nu M_{\mu\nu}(\alpha,\beta,\gamma) A_\nu, \qquad \ldots(13)$$

whereas the rotation matrix $D^1(\alpha,\beta,\gamma)$ is defined such that

$$A'_\mu = \sum_\nu D^1_{\nu\mu}(\alpha,\beta,\gamma) A_\nu, \qquad \ldots(14)$$

Hence the rotation matrix D is the transpose of the transformation matrix M defined in eq.(9) & (10).

3.5.The rotation operator

Let us consider an infinitesimal rotation $\delta\alpha$ about the Z-axis of a right-handed coordinate system and investigate how the wavefunction transforms.

$$\Psi'(\hat{r}) = R_Z(\delta\alpha)\, \Psi(\hat{r}) = \Psi(\hat{r}') \qquad \ldots(15)$$

where $R_Z(\delta\alpha)$ is the rotation operator which causes a rotation of the coordinate system $S \to S'$ through an infinitesimal angle $\delta\alpha$ about the Z-axis. Under rotation,

$$\hat{r} \to \hat{r}', \qquad \ldots(16)$$

$$\Psi(\hat{r}) \to \Psi(\hat{r}') = \Psi'(\hat{r})\,. \qquad \ldots(17)$$

Under the rotation of coordinate system $S \to S'$, the coordinates of a physical point changes from $\hat{r}$ to $\hat{r}'$ and the function $\Psi(\hat{r})$ transforms to $\Psi(\hat{r}')$, which, in turn, becomes a new function $\Psi'(\hat{r})$ when expressed in terms of the old coordinate $\hat{r}$.

$$\begin{aligned} \Psi'(\hat{r}) &= \Psi(\hat{r}'), \\ &= \Psi(x + y\delta\alpha,\, y - x\delta\alpha,\, z), \\ &= \Psi(x,y,z) + \delta\alpha\left(y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}\right)\Psi(x,y,z)\,. \end{aligned} \qquad \ldots(18)$$

The last step is obtained by applying the Taylor series expansion and neglecting terms involving higher powers of $\delta\alpha$. Since the Z-component of the orbital angular momentum operator is given by

$$L_Z = -i\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x} \right), \qquad \ldots(19)$$

we have

$$\Psi'(\hat{r}) = (1 - i\,\delta\alpha\,\hat{L}_Z)\,\Psi(\hat{r})\,. \qquad \ldots(20)$$

Let us now generalize the relation (20) and replace the operator $\hat{L}$ by $\hat{J}$.

$$\Psi'(\hat{r}) = (1 - i\,\delta\alpha\,\hat{J}_Z)\,\Psi(\hat{r})\,. \qquad \ldots(21)$$

Equation (21) gives the transformation of the function due to an infinitesimal rotation through an angle $\delta\alpha$ about the Z-axis. Making a large number (n) of such infinitesimal rotations, one can obtain a finite rotation α about the Z-axis.

$$R_Z(\alpha)\Psi(\hat{r}) = (1 - i\,\delta\alpha\,\hat{J}_Z)^n\,\Psi(\hat{r}) = e^{-i\alpha\hat{J}_Z}\Psi(\hat{r})\,. \qquad \ldots(22)$$

where $\alpha = n\,\delta\alpha$. In a similar way, we can find the rotation operator corresponding to a rotation about the Y-axis.

$$R_Y(\beta)\Psi(\hat{r}) = e^{-i\alpha\hat{J}_Y}\Psi(\hat{r})\,. \qquad \ldots(23)$$

It is to be noted that $\hat{J}^2$ commutes with the rotation operators and hence j is a good quantum number under rotation.

Any general rotation can be described in terms of three parameters. They may be three Euler angles α, β, γ or they may correspond to a rotation ψ about an axis $\hat{n}$ in which is fixed by the two parameters θ and ϕ.

$$R_{\hat{n}}(\psi) = R(\alpha, \beta, \gamma)\,, \qquad \ldots(24)$$

where

$$R_{\hat{n}}(\psi) = e^{-i\psi\,\hat{n}\cdot\hat{J}}\,, \qquad \ldots(25)$$

and

$$\begin{aligned} R(\alpha, \beta, \gamma) &= R_{Z_2}(\gamma) + R_{Y_1}(\beta) + R_Z(\alpha)\,, \\ &= e^{-i\gamma\hat{J}_{Z_2}}\; e^{-i\beta\hat{J}_{Y_1}}\; e^{-i\gamma\hat{J}_Z}\,. \end{aligned} \qquad \ldots(26)$$

We have the following relation between the parameters specifying the single rotation and the Euler angles.

$$\begin{aligned} &\cos\frac{\psi}{2} = \cos\frac{\beta}{2}\,\cos\frac{\alpha+\gamma}{2}\,, \\ &\sin\frac{\psi}{2}\,\sin\theta = \sin\frac{\beta}{2}\,, \\ &\phi = \frac{\gamma-\alpha}{2} + \frac{\pi}{2}\,. \end{aligned} \qquad \ldots(27)$$

In the expansion for R(α, β, γ) given by eq.(26), only the rotation through an angle α is carried out about the Z-axis of the original coordinate system but the rotations β and γ are carried out about the axes Y_1 and Z_2 of the new coordinate systems obtained in successive rotations.

$$XYZ \xrightarrow[R_Z(\alpha)]{} X_1Y_1Z_1 \xrightarrow[R_{Y_1}(\beta)]{} X_1Y_1Z_1 \xrightarrow[R_{Z_2}(\gamma)]{} X'Y'Z' .$$

Since the rotations are unitary transformations, we can subject the operators to unitary transformations successively in order to denote all the rotations with respective to the original coordinate system. For instance,

$$e^{-i\gamma \hat{J}_{Z_2}} = R_{Y_1}(\beta)\, e^{-i\gamma \hat{J}_{Z_1}} [R_{Y_1}(\beta)]^{-1} ,$$

$$= e^{-i\beta \hat{J}_{Y_1}}\, e^{-i\gamma \hat{J}_{Z_1}}\, e^{-i\beta \hat{J}_{Y_1}} . \qquad \ldots(28)$$

Substituting Eq.(28) in Eq.(26), we get

$$R(\alpha, \beta, \gamma) = e^{-i\beta \hat{J}_{Y_1}}\, e^{-i\gamma \hat{J}_{Z_1}}\, e^{-i\alpha \hat{J}_Z} . \qquad \ldots(29)$$

Once again, we can subject the operators in the coordinate system $X_1Y_1Z_1$ to a unitary transformation and obtain the corresponding operators in the coordinate system XYZ.

$$e^{-i\beta \hat{J}_{Y_1}}\, e^{-i\gamma \hat{J}_{Z_1}} = e^{-i\alpha \hat{J}_Z}\, e^{-i\beta \hat{J}_Y}\, e^{-i\gamma \hat{J}_Z}\, e^{i\alpha \hat{J}_Z} . \qquad \ldots(30)$$

Substituting (30) into (29), we get finally,

$$R(\alpha, \beta, \gamma) = e^{-i\alpha \hat{J}_Z}\, e^{-i\beta \hat{J}_Y}\, e^{-i\gamma \hat{J}_Z} \qquad \ldots(31)$$

In the expression (31) for R(α, β, γ), all the rotations are carried out in the original coordinate system and its usefulness will be seen in the next section. The rotation operator R is unitary, that is

$$R^\dagger R = RR^\dagger = 1 ; \qquad R^{-1} = R^\dagger \qquad \ldots(32)$$

3.6. The $d^j_{m'm}(\beta)$ matrix

The rotations matrix $D^j_{m'm}(\alpha, \beta, \gamma)$ has been defined in Eq.(1) and now we can express its elements as the matrix elements of the rotation operator R(α, β, γ).

$$\Psi_{jm}(\hat{r}') = R(\alpha, \beta, \gamma)\, \Psi_{jm}(\hat{r}) ,$$

$$= \sum_m D^j_{m'm}(\alpha, \beta, \gamma)\, \Psi_{jm'}(\hat{r}) . \qquad \ldots(33)$$

or

$$D^j_{m'm}(\alpha, \beta, \gamma) = \langle \Psi_{jm'}(\hat{r}) \,|\, R(\alpha, \beta, \gamma) \,|\, \Psi_{jm}(\hat{r}) \rangle ,$$

$$= e^{-i\alpha m'} \langle jm' \,|\, R(\alpha, \beta, \gamma) \,|\, jm \rangle . \qquad \ldots(34)$$

Using explicit form (31) for $R(\alpha, \beta, \gamma)$ and remembering that the angular momentum functions are eigenfunctions of J_Z operator, we obtain

$$D^j_{m'm}(\alpha, \beta, \gamma) = \langle\, jm' \,|\, e^{-i\,\alpha\,\hat{J}_Z}\ e^{-i\,\beta\,\hat{J}_Y}\ e^{-i\,\gamma\,\hat{J}_Z} \,|\, jm\rangle\ ,$$

$$= e^{-i\,\alpha\,m'} \langle\, jm' \,|\, e^{-i\,\beta\,\hat{J}_Y} \,|\, jm\rangle\, e^{-i\,\gamma\,m}\ . \qquad \ldots(35)$$

The last step was obtained by allowing the operator $e^{-i\,\alpha\,\hat{J}_Z}$ to operate on the left state and the operator $e^{-i\,\gamma\,\hat{J}_Z}$ on the right state. This was possible only because both the operators and the states correspond to the same coordinate system.

In our representation, J_Y is purely imaginary and hence the matrix element $\langle\, jm' \,|\, e^{-i\,\beta\,\hat{J}_Y} \,|\, jm\rangle$ is real. Denoting this matrix element by , we have

$$D^j_{m'm}(\alpha, \beta, \gamma) = e^{-i\,\alpha\,m'}\, d^j_{m'm}(\beta)\, e^{-i\,\gamma\,m} \qquad \ldots(36)$$

Since is unitary and real, the following symmetry relations are satisfied.

$$\begin{aligned} d^j_{m'm}(\beta) &= d^j_{m\,m'}(-\beta)\ , \\ &= (-1)^{m'-m}\, d^j_{m\,m'}(\beta)\ , \\ &= d^j_{-m,-m'}(\beta)\ . \end{aligned} \qquad \ldots(37)$$

Once we obtain the matrix $d^j_{m'm}(\beta)$, the construction of the full rotation matrix $D^j_{m'm}(\alpha, \beta, \gamma)$ is simple because of Eq.(36). Also the construction of $d^j_{m'm}(\beta)$ for higher j-values can be done starting from the lower j-values using the coupling rule for rotation matrices (inverse C.G. series) to be discussed in Sec.3.9.

3.7. The rotation matrix for spinors

We shall now obtain the rotation matrix for j = ½. For a rotation about the Y-axis, the rotation operator is given by

$$R_Y(\beta) = e^{-i\,\beta\,S_Y}\ , \qquad \ldots(38)$$

where S_Y is the Y-component of the spin operator. Expressing it in terms of the Pauli spin operator σ_y, we have

$$R_Y(\beta) = e^{-i\frac{\beta}{2}\sigma_y}\ . \qquad \ldots(39)$$

Recalling the following series expansions

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\ , \qquad \ldots(40)$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots\ , \qquad \ldots(41)$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots\ , \qquad \ldots(42)$$

and the property of the Pauli matrices,

$$\sigma_x^2 = \sigma_y^2 = \sigma_z^2 = 1, \qquad \ldots(43)$$

we obtain a simple form for the rotation matrix.

$$R_Y(\beta) = e^{-i\frac{\beta}{2}\sigma_y} \qquad \ldots(44)$$

$$= 1 - i\frac{\beta}{2}\sigma_y - \frac{\left(\frac{\beta}{2}\sigma_y\right)^2}{2!} + i\,\frac{\left(\frac{\beta}{2}\sigma_y\right)^3}{3!} \ldots ,$$

$$= 1 - i\frac{\beta}{2}\sigma_y - \frac{\left(\frac{\beta}{2}\right)^2}{2!} + i\,\frac{\left(\frac{\beta}{2}\right)^3}{3!}\sigma_y \ldots ,$$

$$= \left(1 - \frac{\left(\frac{\beta}{2}\right)^2}{2!} + \frac{\left(\frac{\beta}{2}\right)^4}{4!} - \ldots\right) - i\,\sigma_y\left(\frac{\beta}{2} - \frac{\left(\frac{\beta}{2}\right)^3}{3!} + \ldots\right),$$

$$= \cos\frac{\beta}{2} - i\,\sigma_y \sin\frac{\beta}{2}. \qquad \ldots(45)$$

Substituting the matrix elements of σ_y, we obtain the matrix representation for the operator and it is denoted by.

$$d^{1/2}(\beta) = R_Y(\beta) = \cos\frac{\beta}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - i\sin\frac{\beta}{2}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix},$$

$$= \begin{bmatrix} \cos\frac{\beta}{2} & -\sin\frac{\beta}{2} \\ \sin\frac{\beta}{2} & \cos\frac{\beta}{2} \end{bmatrix}. \qquad \ldots(46)$$

In a similar way, we can obtain the rotation matrices for rotations about the X or Z axis

$$R_X(\theta) = \cos\frac{\theta}{2} - i\,\sigma_x \sin\frac{\theta}{2},$$

$$= \begin{bmatrix} \cos\frac{\theta}{2} & -i\sin\frac{\theta}{2} \\ -\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{bmatrix} \qquad \ldots(47)$$

$$R_X(\theta) = \cos\frac{\theta}{2} - i\,\sigma_x \sin\frac{\theta}{2},$$

$$= \begin{bmatrix} e^{-i\frac{\theta}{2}} & 0 \\ 0 & e^{i\frac{\theta}{2}} \end{bmatrix} \qquad \ldots(48)$$

Let us now investigate the effect of rotation of the coordinate system on the eigenfunction Ψ_m. A rotation through an angle β about the Y-axis yields

$$\chi_m = \sum_{m'} d_{m'm}(\beta)\,\Psi_{m'}. \qquad \ldots(49)$$

In Eq.(), an explicit mention of the quantum number j is omitted but it is understood that j = ½ in the following discussion. If we wish to express the eigenfunctions Ψ and χ as column vectors and d as a matrix, and use the usual rule of matrix multiplication, then we find the matrix d^T which is the transpose of the d matrix to be more convenient.

$$\chi_m = \sum_{m'} (d^T(\beta))_{mm'} \Psi_{m'} . \qquad \ldots(50)$$

Writing explicitly, we have

$$\begin{bmatrix} \chi_{1/2} \\ \chi_{-1/2} \end{bmatrix} = \begin{bmatrix} \cos\frac{\beta}{2} & \sin\frac{\beta}{2} \\ -\sin\frac{\beta}{2} & \cos\frac{\beta}{2} \end{bmatrix} \begin{bmatrix} \Psi_{1/2} \\ \Psi_{-1/2} \end{bmatrix} \qquad \ldots(51)$$

If we start with a pure state $\Psi_{1/2}$ which is a spinor with spin up $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, then a rotation through an angle 2π about the Y-axis yields

$$\begin{bmatrix} \chi_{1/2} \\ \chi_{-1/2} \end{bmatrix} = \begin{bmatrix} \cos\pi & \sin\pi \\ -\sin\pi & \cos\pi \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

$$= -\begin{bmatrix} 1 \\ 0 \end{bmatrix}. \qquad \ldots(52)$$

This is in contradiction to the case of a vector for which the rotation through an angle 2π leaves the vector undisturbed. In the case of spinor, a rotation through an angle 4π is necessary to get the same spinor. That is why the spinors are sometimes referred to as 'half-vectors'.

Also there is an interesting feature that a spinor exhibits. For a spinor located at the origin of the coordinate system, a rotation through an angle π about the X-axis is not equivalent to a rotation through an angle π about the Y-axis.

$$R_X(\pi)\, \Psi_{1/2} = -\, i\, \Psi_{-1/2} = \varphi \qquad \ldots(53)$$

$$R_Y(\pi)\, \Psi_{1/2} = -\, i\, \Psi_{-1/2} = \varphi \qquad \ldots(54)$$

For a vector located at the origin, these two rotations will invert the vector. But it is not so in the case of spinors. However, it can be shown that the two spinors φ and φ' differ by a rotation through an angle π about the Z-axis.

$$R_Z(\pi)\, \varphi = -\Psi_{1/2} = \varphi' \qquad \ldots(55)$$

$$R_Z(-\pi)\, \varphi' = -\, i\, \Psi_{1/2} = \varphi \qquad \ldots(56)$$

That is why a spinor can be considered a vector with thickness.

3.8. The Clebsch-Gordan series

In this section, we shall obtain a coupling rule for rotation matrices and it is deduced from the coupling scheme of two angular momenta.

$$|j_1m_1\rangle\,|j_2m_2\rangle = \sum_j \begin{bmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m \end{bmatrix} |j\,m\rangle \qquad \ldots(57)$$

Rotating the coordinate system through Euler angles (α, β, γ), we obtain

$$\sum_{\nu_1\nu_2} D^{j_1}_{\nu_1 m_1}(\omega)\, D^{j_2}_{\nu_2 m_2}(\omega)\,|j_1\nu_1\rangle\,|j_2\nu_2\rangle = \sum_{j\mu} \begin{bmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m \end{bmatrix} D^{j}_{\nu m}(\omega)\ |j\mu\rangle\,, \qquad \ldots(58)$$

Where the argument ω of the D matrix stands for the set of Euler{angles α, β, γ). The state $|j\mu\rangle$ on the right hand side can be expanded as

$$|j\mu\rangle = \sum_{\mu'} \begin{bmatrix} j_1 & j_2 & j \\ \mu_1 & \mu_2 & \mu \end{bmatrix} \ |j_1\mu'_1\rangle\,|j_2\mu'_{21}\rangle\,, \qquad \ldots(59)$$

Inserting this into (58) and taking the scalar product with $|j_1\,\mu_1\rangle\,|j_2\,\mu_2\rangle$

$$\sum_{\nu_1\nu_2} D^{j_1}_{\nu_1 m_1}(\omega)\, D^{j_2}_{\nu_2 m_2}(\omega)\, \delta_{\mu_1\nu_1}\, \delta_{\mu_2\nu_2} =$$

$$\sum_{j\mu\mu'_1} \begin{bmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m \end{bmatrix} \begin{bmatrix} j_1 & j_2 & j \\ \mu_1 & \mu_2 & \mu \end{bmatrix} D^{j}_{\mu m}(\omega)\ \delta_{\mu_1\mu'_1}\, \delta_{\mu_2\mu'_2}\ , \qquad \ldots(60)$$

The sum over μ on the right hand side of Eqn.(60) can be replaced by $\mu'_2 = \mu - \mu'_1$, Now performing the summation over the projection quantum numbers, we obtain

$$D^{j_1}_{\mu_1 m_1}(\omega)\, D^{j_2}_{\mu_2 m_2}(\omega) = \sum_j \begin{bmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m \end{bmatrix} \begin{bmatrix} j_1 & j_2 & j \\ \mu_1 & \mu_2 & \mu \end{bmatrix} D^{j}_{\mu m}(\omega)\ . \qquad \ldots(61)$$

This is known as the Clebsch-Gordan (G.G. Series).

3.9. The inverse Clebsch-Gordan series

Starting from the C.G. series (Eq.61), an inverse series can be obtained using the orthogonality of the C.G. coefficients. Multiplying both sides of Eq.(61) by $\begin{bmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m \end{bmatrix}$ and summing over m_1, we obtain

$$\sum_{m_1} \begin{bmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m \end{bmatrix} D^{j_1}_{\nu_1 m_1}(\omega)\, D^{j_2}_{\nu_2 m_2}(\omega)$$

$$= \sum_j \delta_{j\,j'} \begin{bmatrix} j_1 & j_2 & j \\ \mu_1 & \mu_2 & \mu \end{bmatrix} D^{j}_{\mu m}(\omega)\ \ , \begin{bmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m \end{bmatrix} = D^{j'}_{\mu m}(\omega) \qquad \ldots(62)$$

Eq.(62) was obtained by applying the orthonormality condition of C.G. coefficients.

Once again, multiplying both sides by $\begin{bmatrix} j_1 & j_2 & j' \\ \mu_1 & \mu_2 & \mu \end{bmatrix}$ and summing over μ_1, we obtain

$$\sum_{m_1 \mu_1} \begin{bmatrix} j_1 & j_2 & j' \\ m_1 & m_2 & m \end{bmatrix} \begin{bmatrix} j_1 & j_2 & j' \\ \mu_1 & \mu_2 & \mu \end{bmatrix} D^{j_1}_{\mu_1 m_1}(\omega)\, D^{j_2}_{\mu_2 m_2}(\omega) = D^{j'}_{\mu m}(\omega) \quad \ldots(63)$$

This is known as the inverse C.G. series. This can be used to generate the elements of all the matrices $D^j(\omega)$, ($j > ½$), if the rotation matrix $D^{½}(\omega)$ is given.

3.10. Application of rotation operator – Cranking Model

The time dependent Schrodinger equation can in the laboratory system be written as

$$i\hbar \frac{\partial}{\partial t} \psi = H\psi \quad \ldots(64)$$

The wavefunction ψ and the Hamiltonian H can be expressed in terms of the body-fixed coordinate by means of the rotation operator.

$$R(\omega t) = e^{-i\omega t J'_Z/\hbar} \quad \ldots(65)$$

as

$$\psi = R(\omega t)\, \psi' \quad \ldots(66)$$

and

$$H = R(\omega t)\, H_0\, R(\omega t)^{-1} \quad \ldots(67)$$

where ψ' and H_0 relates to the body-fixed system. The Schrodinger equation can then be rewritten as

$$i\hbar \frac{\partial}{\partial t} (R(\omega t)\, \psi') = R(\omega t)\, H_0\, \psi' \quad \ldots(68)$$

By inserting the expression (65) for $R(\omega t)$ and computing the time derivative yields

$$i\hbar \frac{\partial}{\partial t} \psi' = (H_0 - \omega J'_Z)\, \psi' = H_\omega\, \psi' \quad \ldots(69)$$

i.e., $H_\omega = H_0 - \omega J'_Z$

4. The Wigner-Eckart theorem

4.1. Theorem

The Wigner-Eckart theorem states that the matrix element of an irreducible tensor operator taken between any two well-defined angular momentum states can be factored out into two parts, one part depending on the magnetic quantum numbers and the other part completely independent of them.

The first part contains the entire geometry or the symmetry properties of the system and the second part is concerned with the dynamics of the physical process. The theorem states that the entire dependence of the matrix element on the magnetic quantum numbers can be factored out as a C.G. coefficient and the other factor, which is independent of the projection quantum numbers, is known as the reduced matrix element or the double-bar matrix element.

$$\left\langle j_f\ m_f \,\middle|\, T_k^{\mu} \,\middle|\, j_i\ m_i \right\rangle = \begin{pmatrix} j_i & k & j_f \\ m_i & \mu & m_f \end{pmatrix} \left\langle j_f \,\middle\|\, T_k^{\mu} \,\middle\|\, j_i \right\rangle \qquad \ldots(1)$$

Eq.(1) is the mathematical statement of the Wigner-Eckart theorem. Unfortunately there is no uniformity in the precise statement of the Wigner-Eckart theorem and consequently the reduced matrix element differs from one author to another.

It can be easily seen that the first factor, viz., the C.G. coefficient depends on the coordinate system that is used to evaluate the matrix element and it also implies the law of conservation of angular momentum. If this factorization is possible in one coordinate system, then it is easy to show that it is possible in every other coordinate system obtained by rotation.

The foregoing discussion cannot be considered as the proof of the Wigner-Eckart theorem although it serves as a consistency check. There are atleast three different proofs of the Wigner-Eckart theorem, one due to Wigner, another due to Schrodinger and the third due to Racah.

4.2. Proof of the theorem

We shall first write down explicitly the matrix element Q of an irreducible tensor operator of rank k.

$$Q = \left\langle j_f\ m_f \,\middle|\, T_k^{\mu} \,\middle|\, j_i\ m_i \right\rangle$$

$$= \int \psi^{*}_{j_f\ m_f}(\hat{r}) \quad T_k^{\mu}(\hat{r}) \quad \psi_{j_i\ m_i}(\hat{r})\, d\Omega \qquad \ldots(2)$$

The angular integration in eq(2) can be carried out either by rotating the functions in fixed coordinate system or by rotating the coordinate system keeping the functions fixed. We shall opt for the latter method. Let us consider a rotation of the coordinate system through the Euler angles such that the angular coordinate ř goes from (0, 0) to (θ, φ).

$$Q = \sum_{m'_f\,\mu'\,m'_i} D^{j_f\,*}_{m'_f\,m_f}(\Omega) \quad D^{k}_{\mu'\,\mu}(\Omega) \quad D^{j_i}_{m'_i\,m_i}(\Omega)$$

$$\psi^{*}_{j_f\,m'_f}(0,0) \quad T^{\mu'}_{k}(0,0) \quad \psi_{j_i\,m'_i}(0,0) \quad d\Omega \qquad \ldots(3)$$

We shall first couple the two D-matrices using the C.G.series.

$$D^{j_i}_{m'_i\,m_i}(\Omega) \quad D^{k}_{\mu'\,\mu}(\Omega) = \sum_{J} \begin{pmatrix} j_i & k & J \\ m'_i & \mu' & M' \end{pmatrix} \begin{pmatrix} j_i & k & J \\ m_i & \mu & M \end{pmatrix} \quad D^{J}_{M'\,M}(\Omega) \qquad \ldots(4)$$

Substituting this into eq.(3) we obtain

$$Q = \sum_{m'_f\,\mu'\,m'_i} \sum_{J} \begin{pmatrix} j_i & k & J \\ m'_i & \mu' & M' \end{pmatrix} \begin{pmatrix} j_i & k & J \\ m_i & \mu & M \end{pmatrix} \psi^{*}_{j_f\,m'_f}(0,0) \quad T^{\mu'}_{k}(0,0) \quad \psi_{j_i\,m'_i}(0,0)$$

$$\int D^{j_f\,*}_{m'_f\,m_f}(\Omega) \quad D^{J}_{M'\,M}(\Omega) \quad d\Omega \qquad \ldots(5)$$

where $\quad d\Omega = \int_0^{\pi} \sin\theta d\theta \int_0^{2\pi} d\phi \qquad \ldots(6)$

The integration over $d\Omega$ can be carried out easily.

$$\int D^{j_f\,*}_{m'_f\,m_f}(\Omega) \quad D^{J}_{M'\,M}(\Omega) \quad d\Omega = \frac{4\pi^2}{2j_f+1} \; \delta_{j_f\,J} \; \delta_{m'_f\,M'} \; \delta_{m_f\,M} \qquad \ldots(7)$$

Inserting (7) into (5) and summing over J and m'_f, we obtain

$$Q = \begin{pmatrix} j_i & k & J \\ m_i & \mu & M \end{pmatrix} \delta_{m_f\,M} \left\{ \begin{matrix} \dfrac{4\pi^2}{2j_f+1} \displaystyle\sum_{m'_i\,\mu'} \begin{pmatrix} j_i & k & J \\ m'_i & \mu' & M' \end{pmatrix} \\ \psi^{*}_{j_f\,m'_f}(0,0) \quad T^{\mu'}_{k}(0,0) \quad \psi_{j_i\,m'_i}(0,0) \end{matrix} \right\} \qquad \ldots(8)$$

The quantity within the curly brackets in eq(8) is independent of the projection quantum numbers because of the summation over m_i' and μ'. Thus the matrix element Q depends on the projection quantum numbers only through the C.G. coefficients. The reduced matrix element is the quantity within the curly bracket and as we have shown it is independent of the projection quantum numbers.

$$\left\langle j_f \left\| T^{\mu}_{k} \right\| j_i \right\rangle = \frac{4\pi^2}{2j_f+1} \sum_{m'_i\,\mu'} \begin{pmatrix} j_i & k & J \\ m'_i & \mu' & M' \end{pmatrix} \psi^{*}_{j_f\,m'_f}(0,0) \quad T^{\mu'}_{k}(0,0) \quad \psi_{j_i\,m'_i}(0,0) \qquad \ldots(9)$$

From (8) & (9) we can write

$$Q = \left\langle j_f \; m_f \left| T^{\mu}_{k} \right| j_i \; m_i \right\rangle = \begin{pmatrix} j_i & k & j_f \\ m_i & \mu & m_f \end{pmatrix} \left\langle j_f \left\| T^{\mu}_{k} \right\| j_i \right\rangle \qquad \ldots(10)$$

Hence the proof.

4.3. Matrix element of spherical harmonics

It will be instructive to calculate the reduced matrix element in the special case of spherical harmonics.

From eq(9) we have

$$\langle \ell_f \| Y_\ell \| \ell_i \rangle = \frac{4\pi^2}{2\ell_f+1} \sum_{m_i' \, m'} \begin{pmatrix} \ell_i & \ell & \ell_f \\ m_i' & m' & m_f' \end{pmatrix} Y^*_{\ell_f \, m_f}(0,0) \;\; Y_{\ell \, m'}(0,0) \;\; Y_{\ell_i \, m_i'}(0,0) \qquad \ldots(11)$$

$$\text{since} \quad Y_{\ell \, m}(0,0) = \sqrt{\frac{2\ell+1}{4\pi}} \, \delta_{m\,0} \qquad \ldots(12)$$

eq(11) simplifies to

$$\langle \ell_f \| Y_\ell \| \ell_i \rangle = \left\{ \frac{(2\ell_i+1)(2\ell+1)}{4\pi \, (2\ell_f+1)} \right\}^{1/2} \begin{pmatrix} \ell_i & \ell & \ell_f \\ 0 & 0 & 0 \end{pmatrix} \qquad \ldots(13)$$

Thus, according to the Wigner-Eckart theorem, the matrix element of $Y_{\ell \, m}$ is

$$\langle \ell_f \; m_f | Y_{\ell m} | \ell_i \; m_i \rangle = \begin{pmatrix} \ell_i & \ell & \ell_f \\ m_i & m & m_f \end{pmatrix} \langle \ell_f \| Y_\ell \| \ell_i \rangle \qquad \ldots(14)$$

where the reduced matrix element is given by eq(13). This result is identical with the result obtained earlier using the coupling rule of the spherical harmonics.