A Computational Method to Generate Family of Extreme Value Volatility Estimators

Rakesh Nigam, Prachi Srivastava, Parush Arora, Abhishek Chand

November 21, 2019

Abstract

An algorithm to construct a family of unbiased extreme value volatility estimators is developed when the log prices of assets follow simple Brownian motion with drift and jump. From this family of estimators, the minimum variance estimator can be computed and is found to be competitive to the estimators of Parkinson (1980) and Yang and Zhang (2000) (no drift and positive jump case). Other estimators like the Garman and Klass (1980) and the Rogers and Satchell (1991) are recovered. The novelty of this approach is that it is constructive and, moreover it can be extended to compute estimators when prices follow more realistic models.

1 Introduction

This paper estimates the volatility of a financial security, whose log-price process (for short it will be called the price) at time t is $P_t = \mu t + \sigma B_t$, with two constant unknown parameters, the drift and volatility, μ and σ^2 respectively. The stochastic nature of P_t is inherited from the standard Brownian motion B_t. The volatility is estimated using extreme value information of the stochastic process P_t such as the open, high, low and close prices of trading days. The jump between the closing price of the previous day $(t-1)$ and the opening price of the current day t is also incorporated into the estimation procedure. It is well known that volatility plays an important role in pricing options in the Black-Scholes framework and also for modelling portfolio risk.

The classical variance estimator used in the literature is based on close-to-close prices. The main drawback of this approach is that it ignores extreme value information present in the log-price process. By including the extreme values in the estimation process, it is found that the variance of the volatility estimator is reduced further.
Parkinson (1980) and Garman and Klass (1980), constructed extreme value estimators for the case with no jumps and zero drift. Parkinson's estimator is based only on high and low prices. His estimator is unbiased and 2.5-5 times better than the classical estimator. Later, Garman and Klass proposed a minimum variance unbiased estimator which incorporated the opening and closing prices in addition to the high and low prices that were used in the Parkinson's estimator. By including additional information, the Garman and Klass estimator had lower variance (efficiency= 7.4) compared to Parkinson. As an extension of Garman and Klass, Meilijson (2011) provided a maximum likelihood unbiased volatility estimator using a different approach. He separated the security price data into two data sets depending upon whether the normalized closing price of a day is above zero or below. For example, in 100 days of data, 40 days have normalized closing price as positive and 60 days have normalized closing price as negative. So, the 40 days will form one data set and 60 days will form another data set. The probability of any day to fall in any of the two data sets is 0.5. Then he calculated the expectations of the normalized high, low, opening and closing values separately for the two datasets and derived a new estimator. The estimator yielded an efficiency of 7.73 which is higher than Garman-Klass. In the case of non-zero drift, Parkinson, Meilijson and Garman and Klass become biased. Rogers and Satchell (1991) proposed a drift independent unbiased estimator that generalized both Parkinson's and Garman and Klass's estimators for the case of non-zero drift. The efficiency of the estimator is 6. Yang and Zhang (2000) generalized Rogers and Satchell's estimator by incorporating jumps between trading days. It is found that adding jumps increase the efficiency of the estimator as efficiency is a function of jump.
The contribution of this paper is to provide a coherent framework under which all the previous estimators can be rederived. A methodology (which uses null space solution) is developed through which a family of volatility estimators is calculated. The methodology is applicable on price movement modelled as Brownian motions with different assumptions (drift and jump) and can be generalized to more realistic movements. Through this, we have validated and rederived the minimum variance estimator forvdifferent Brownian motions

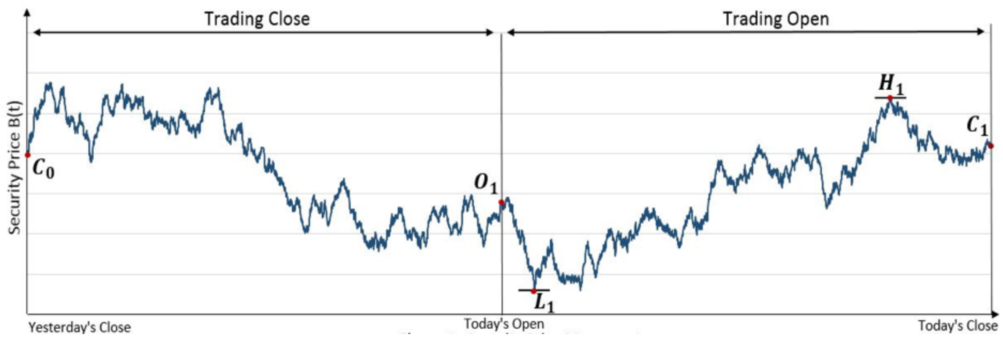

Figure 1: Security Price Movement

assumptions (drift and jump) and improved upon the two case.

1. Best minimum variance estimator for security price modelled as Brownian motion with no drift and no jumps which uses only high and low price data. (Case of Parkinson)

2. Best minimum variance estimator for security price modelled as Brownian motion with no drift but positive jump using high, low, opening and closing prices.

Data has been generated with 700,000 time steps (representing a single day) to make it closer to the continuous case. Increasing the number of time steps taken between the interval [0, 1], better is the approximation recovered for volatility as discussed by Rogers and Satchell. Such 20,000 days are simulated to calculate the expectations of high, low, opening and closing values. These extreme values are available on websites and newspapers which makes the improved estimator very much applicable in real life scenarios.

This paper is divided in four sections. Section II explains and outlines the mathematical modelling and hypothesis for our model along with new approach that we recommend. Section III sums upon the findings from the simulations. Section IV is the conclusion.

2 Mathematical Framework for calculating Volatility Estimator

2.1 Graphical Representation

We will start with explaining the structural model for price movement. For convenience, we are assuming the time interval of price as $t \in [0, 1]$ representing a single day. The day is divided into two parts, first one being where the trading is closed (the movement of price is not observable) and the later with the trading being opened. Figure 1 shows the price movement graphically.

The trading is closed initially in the interval $[0, f]$ and C_0 (yesterday's closing price) is observed at time 0. The trading starts when price is O_1 (today's opening price) at time f and its movement is observable in the interval $[f, 1]$. The following are the notations which are taken from Garman Klass and used in the whole paper.

- σ^2 = unknown constant volatility of price change
- μ =drift
- f = fraction of the day where trading is closed
- $C_0 = B(0)$, yesterday's closing price
- $O_1 = B(f)$, today's opening price
- $H_1 = maxB(t), f < t < 1$, today's high
- $L_1 = minB(t), f < t < 1$, today's low
- $C_1 = B(1)$, today's close
- $u = H_1 - O_1$, the normalized high
- $d = L_1 - O_1$, the normalized low
- $c = C_1 - O_1$, the normalized close
- $o = O_1 - C_0$, the normalized open

The price movement modelled as Brownian motion is generated using simulations.

2.2 Estimator Derivation using Constraint Optimization (Classical Approach)

We will demonstrate the classical procedure to derive the minimum variance unbiased estimator. We have tried to write the procedure step wise so that it is easy to understand.

2.2.1 Defining the Unbiased Constraint

The estimator for volatility of price movement has a standard form which is represented as a function of extreme values.

$$\hat{\sigma}^2 = \sum_{i=1}^{n} a_i x_i \tag{1}$$

a_i's are the weights and x_i's are the quadratic extreme value terms like u^2, d^2, c^2 etc. The number of terms (n) used to construct the estimator will differ depending upon the Brownian motion assumptions. The unbiased constraint can be derived by taking the expectations of (1).

$$E(\hat{\sigma}^2) = \sum_{i=1}^{n} a_i E(x_i) \tag{2}$$

$$1 = \sum_{i=1}^{n} a_i (E(x_i)/\sigma^2) \tag{3}$$

$$1 = \sum_{i=1}^{n} a_i c_i \tag{4}$$

$$1 = \vec{A}\vec{C} \tag{5}$$

The expected values of quadratic extreme values $E(x_i)$ are provided in the past.

2.2.2 Minimize the variance covariance matrix ($\sum cov$)with respect to the unbiased constraint

The optimal a_i's will come from minimizing the variance-covariance matrix of the volatility estimator with respect to the unbiased constraint. The optimization problem will have the following form.

Min:

$$\begin{bmatrix} a_1 & a_2 \dots a_n \end{bmatrix} \begin{bmatrix} E(x_1^4) & E(x_1^2 x_2^2) & \dots & \dots & E(x_1^2 x_n^2) \\ E(x_2^2 x_1^2) & E(x_2^4) & \dots & \dots & E(x_2^2 x_n^2) \\ \vdots & \vdots & \ddots & \dots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ E(x_n^2 x_1^2) & E(x_n^2 x_2^2) & \dots & \dots & E(x_n^4) \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ \vdots \\ a_n \end{bmatrix} = \vec{A}^T \Pi \vec{A}$$

$$s.t \quad 1 = \vec{A}\vec{C} \tag{6}$$

The values of the expectation terms in Π for various cases of Brownian motion were provided by the authors and are reconfirmed through simulations. We can use the Lagrange multiplier and its first order conditions to solve for a_i's.

$$L(\vec{A}, \lambda) = \vec{A}^T \Pi \vec{A} + \lambda(\vec{A}\vec{C} - 1) \tag{7}$$

$$\frac{\partial L(\vec{A}, \lambda)}{\partial \vec{A}} = 2\Pi\vec{A} + \lambda\vec{C} = \vec{0} \tag{8}$$

$$\frac{\partial L(\vec{A}, \lambda)}{\partial \vec{A}} = \vec{A}\vec{C} = 1 \tag{9}$$

Solving (8) and (9) will give the vector $\vec{A}$ which will minimize the variance of the estimator.

2.3 New Approach to derive the family of estimators and the minimum variance estimator

We suggest an alternate approach which will give all the possible unbiased estimators (we will call it the family of estimator) for the given Brownian motion case and the minimum variance estimator among it. The approach uses the complete solution which comprise of a particular solution and a homogeneous solution (null space) to find the family of estimators.

2.3.1 Derive the basis vectors of the null space (Homogeneous solution)

We will start with the $E(x_i)$ from the unbiased constraint (2) and construct a general matrix that will have different values of $E(x_i)$'s based on different assumptions of Brownian motion considered. The following matrix form will be used to derive the basis vector of the null space.

$$\vec{K} = \begin{bmatrix} E(x_{C_1,1}) & E(x_{C_1,2}) & \dots & \dots & E(x_{C_1,m}) \\ E(x_{C_2,1}) & E(x_{C_2,2}) & \dots & \dots & E(x_{C_2,m}) \\ \vdots & \vdots & \ddots & \dots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ E(x_{C_n,1}) & E(x_{C_n,2}) & \dots & \dots & E(x_{C_n,m}) \end{bmatrix} \tag{10}$$

Each row has expected value of quadratic extreme values with respect to given assumptions C_i (combination of drift, jump and volatility). The matrix will have n rows as n distinct values of volatility is considered and m columns which are the total number of quadratic extreme values considered to form the estimator. Each column has different expected values of the same quadratic extreme value as each row correspond to different volatility, drift or jump. Solving $\vec{K}\vec{x} = 0$ will give the basis vectors $\vec{x}$ which will form the null space or the homogeneous solution which will form our family of estimator.

2.3.2 Calculate the Particular Solution

The particular solution is easy to calculate by using the unbiased constraint (2) and $\vec{K}$. Solving $\vec{K}\vec{x} = \vec{\sigma}^2$ will give the particular solution.

$$\begin{bmatrix} E(x_{C_1,1}) & E(x_{C_1,2}) & \dots & \dots & E(x_{C_1,m}) \\ E(x_{C_2,1}) & E(x_{C_2,2}) & \dots & \dots & E(x_{C_2,m}) \\ \vdots & \vdots & \ddots & \dots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ E(x_{C_n,1}) & E(x_{C_n,2}) & \dots & \dots & E(x_{C_n,m}) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ \vdots \\ x_m \end{bmatrix} = \begin{bmatrix} \sigma^2_{C_1} \\ \sigma^2_{C_2} \\ \vdots \\ \vdots \\ \sigma^2_{C_n} \end{bmatrix} \tag{11}$$

$\vec{\sigma}^2$ is the vector containing values of σ^2 for various assumptions or conditions C_1. As the quadratic extreme values are a linear function of σ^2, we can have many vectors $\vec{x}$ which will solve the problem. Thus, we will choose and stick to one such vector which will solve the matrix equation problem and will be termed as our particular solution. The values in basis vectors will change accordingly with the particular solution chosen. We have provided the particular solutions for all the cases in the result section that we have considered corresponding to the homogeneous solution.

2.3.3 Constructing the family of unbiased estimators

After calculating the particular solution and the homogeneous solution, we can represent our findings as the complete solution for calculating a_i's.

$$\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ \vdots \\ a_n \end{bmatrix} = \alpha_0 \begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ \vdots \\ p_n \end{bmatrix} + \alpha_1 \begin{bmatrix} b_{11} \\ b_{21} \\ \vdots \\ \vdots \\ b_{n1} \end{bmatrix} + \alpha_2 \begin{bmatrix} b_{12} \\ b_{22} \\ \vdots \\ \vdots \\ b_{n2} \end{bmatrix} + \alpha_3 \begin{bmatrix} b_{13} \\ b_{23} \\ \vdots \\ \vdots \\ b_{n3} \end{bmatrix} + \ldots\ldots\ldots\ldots + \alpha_m \begin{bmatrix} b_{1m} \\ b_{2m} \\ \vdots \\ \vdots \\ b_{nm} \end{bmatrix} \tag{12}$$

OR

$$\vec{a_l} = \alpha_0 \vec{p} + \alpha_1 \vec{B_1} + \alpha_2 \vec{B_2} + \alpha_3 \vec{B_3} + \ldots\ldots\ldots + \alpha_m \vec{B_m} \tag{13}$$

$\vec{p}$ is the particular solution vector. b_{ij}'s are the i^{th} term of the j^{th} basis vector $\vec{B_i}$ of the null space corresponding to each a_i's respectively. α_j's are the coefficients which take values in the interval $(-\infty, \infty)$. Different values of α_j's will give different estimators. We have considered $\alpha_0 = 0$ for simplicity. Hence, the family of estimator is formed which has infinite unbiased estimators based on different values of α_j's. One of the combinations of α_j's will give the minimum variance unbiased estimator.

2.3.4 Derive the best estimator among the family of estimator

To find that combination of α_j's which gives the estimator with the minimum variance among the family of estimators, we will form a quadratic programming problem. For this, we will construct a matrix which has the particular solution and all the basis vectors in it.

$$\begin{bmatrix} p_1 & b_{11} & \ldots & \ldots & b_{1m} \\ p_2 & b_{21} & \ldots & \ldots & b_{2m} \\ \vdots & \vdots & \ddots & \ldots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ p_n & b_{n1} & \ldots & \ldots & b_{nm} \end{bmatrix} = \vec{N} \tag{14}$$

We will use $\vec{N}$ to find the minimum variance estimator. The minimization problem looks like the following

$$\begin{aligned} Min: \quad & \vec{\alpha}\vec{N^T}\Pi\vec{N}\vec{\alpha^T} \\ s.t \quad & 1 = \alpha_0 \end{aligned} \tag{15}$$

Solving the constraint optimization problem will give the α_j's which will give the minimum variance volatility estimator

3 Results: Derivation of the family of estimator using the new approach

In section 2, we explained the derivation of the family of estimators for a given Brownian motion using the new approach. In this section, we will show how the complete solution looks like for Brownian motion with varying assumptions and also show the best estimators for two distinct cases.

3.1 Brownian Motion with zero drift and no jump

3.1.1 Parkinson Case

1. **Unbiased Constraint:**

$$E(u^2) = \sigma^2 \quad E(d^2) = \sigma^2 \quad E(ud) = (1 - 2\log 2)\sigma^2$$

2. **Variance-Covariance Matrix $\sum_P$:**

$$\begin{bmatrix} E(u^4) & E(u^2d^2) & E(u^3d) \\ E(u^2d^2) & E(d^4) & E(ud^3) \\ E(u^3d) & E(ud^3) & E(u^2d^2) \end{bmatrix} = \begin{bmatrix} 3\sigma^4 & 0.2274\sigma^4 & -0.4318\sigma^4 \\ 0.2274\sigma^4 & 3\sigma^4 & -0.4318\sigma^4 \\ -0.4318\sigma^4 & -0.4318\sigma^4 & 0.2274\sigma^4 \end{bmatrix}$$

3. **$\vec{K}$ to calculate the family of estimators:**

$$\vec{K_P} = \begin{bmatrix} E(u^2_{\sigma_1}) & E(d^2_{\sigma_1}) & E(ud_{\sigma_1}) \\ E(u^2_{\sigma_2}) & E(d^2_{\sigma_2}) & E(ud_{\sigma_2}) \\ E(u^2_{\sigma_3}) & E(d^2_{\sigma_3}) & E(ud_{\sigma_3}) \\ E(u^2_{\sigma_4}) & E(d^2_{\sigma_4}) & E(ud_{\sigma_4}) \\ E(u^2_{\sigma_5}) & E(d^2_{\sigma_5}) & E(ud_{\sigma_5}) \\ E(u^2_{\sigma_6}) & E(d^2_{\sigma_6}) & E(ud_{\sigma_6}) \end{bmatrix} = \begin{bmatrix} {\sigma_1}^2 & {\sigma_1}^2 & (1-2\log 2){\sigma_1}^2 \\ {\sigma_2}^2 & {\sigma_2}^2 & (1-2\log 2){\sigma_2}^2 \\ {\sigma_3}^2 & {\sigma_3}^2 & (1-2\log 2){\sigma_3}^2 \\ {\sigma_4}^2 & {\sigma_4}^2 & (1-2\log 2){\sigma_4}^2 \\ {\sigma_5}^2 & {\sigma_5}^2 & (1-2\log 2){\sigma_5}^2 \\ {\sigma_6}^2 & {\sigma_6}^2 & (1-2\log 2){\sigma_6}^2 \end{bmatrix}$$

4. **Complete Case:**
To calculate the Parkinson's estimator, we can show a similar form of family of estimator with a smaller number of quadratic extreme values involved.

$$\begin{bmatrix} {a_P}^{u^2} \\ {a_P}^{d^2} \\ {a_P}^{ud} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \alpha_1 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0.3862944 \\ 0 \\ 1 \end{bmatrix}$$

Parkinson: $\alpha_P : \alpha_1 = 0.2251 \quad \alpha_2 = -1.4232$

5. **Minimum Variance Solution:**
Putting values of alpha in the complete case above gives the following solution:

$${a_P}^{u^2} = 0.225108 \quad {a_P}^{d^2} = 0.225108 \quad {a_P}^{ud} = -1.423226$$

Parkinson showed that his estimator was superior than the classical estimator, but he didn't show whether it is a minimum variance estimator if we consider only the high and low values. Thus, using new null space technique it can be shown that the minimum variance estimator is different from Parkinson's estimator.

3.1.2 Garman and Klass Case

1. **Unbiased Constraint:**

$$E(u^2) = \sigma^2 \quad E(d^2) = \sigma^2 \quad E(c^2) = \sigma^2 \quad E(ud) = (1 - 2\log 2)\sigma^2$$
$$E(uc) = 0.5\sigma^2 \quad E(dc) = 0.5\sigma^2$$

2. **Variance-Covariance Matrix $\sum_{GK}$:**

$$\begin{bmatrix} E(u^4) & E(u^2d^2) & E(u^2c^2) & E(u^3d^2) & E(u^3c) & E(u^2dc) \\ E(u^2d^2) & E(d^4) & E(d^2c^2) & E(ud^3) & E(u^2d^2) & E(u^3d) \\ E(u^2c^2) & E(d^2c^2) & E(c^4) & E(udc^2) & E(uc^3) & E(dc^3) \\ E(u^3d) & E(ud^3) & E(udc^2) & E(u^2d^2) & E(u^2dc) & E(ud^2c) \\ E(u^3c) & E(ud^2c) & E(uc^3) & E(u^2dc) & E(u^2c^2) & E(udc^2) \\ E(u^2dc) & E(d^3c) & E(dc^3) & E(ud^2c) & E(udc^2) & E(d^2c^2) \end{bmatrix}$$

$$= \begin{bmatrix} 3\sigma^4 & 0.2274\sigma^4 & 2\sigma^4 & -0.4318\sigma^4 & 2.25\sigma^4 & -0.1881\sigma^4 \\ 0.2274\sigma^4 & 3\sigma^4 & 2\sigma^4 & -0.4318\sigma^4 & -0.1881\sigma^4 & 2.25\sigma^4 \\ 2\sigma^4 & 2\sigma^4 & 3\sigma^4 & -0.4381\sigma^4 & 1.5\sigma^4 & 1.5\sigma^4 \\ -0.4318\sigma^4 & -0.4318\sigma^4 & -0.4381\sigma^4 & 0.2274\sigma^4 & -0.1881\sigma^4 & -0.1881\sigma^4 \\ 2.25\sigma^4 & -0.1881\sigma^4 & 1.5\sigma^4 & -0.1881\sigma^4 & 2\sigma^4 & -0.4381\sigma^4 \\ -0.1881\sigma^4 & 2.25\sigma^4 & 1.5\sigma^4 & -0.1881\sigma^4 & -0.4381\sigma^4 & 2\sigma^4 \end{bmatrix}$$

3. **$\vec{K}$ to calculate the family of estimators:**

$$\vec{K}_{GK} = \begin{bmatrix} E(u^2_{\sigma_1}) & E(d^2_{\sigma_1}) & E(c^2_{\sigma_1}) & E(ud_{\sigma_1}) & E(uc_{\sigma_1}) & E(dc_{\sigma_1}) \\ E(u^2_{\sigma_2}) & E(d^2_{\sigma_2}) & E(c^2_{\sigma_2}) & E(ud_{\sigma_2}) & E(uc_{\sigma_2}) & E(dc_{\sigma_2}) \\ E(u^2_{\sigma_3}) & E(d^2_{\sigma_3}) & E(c^2_{\sigma_3}) & E(ud_{\sigma_3}) & E(uc_{\sigma_3}) & E(dc_{\sigma_3}) \\ E(u^2_{\sigma_4}) & E(d^2_{\sigma_4}) & E(c^2_{\sigma_4}) & E(ud_{\sigma_4}) & E(uc_{\sigma_4}) & E(dc_{\sigma_4}) \\ E(u^2_{\sigma_5}) & E(d^2_{\sigma_5}) & E(c^2_{\sigma_5}) & E(ud_{\sigma_5}) & E(uc_{\sigma_5}) & E(dc_{\sigma_5}) \\ E(u^2_{\sigma_6}) & E(d^2_{\sigma_6}) & E(c^2_{\sigma_6}) & E(ud_{\sigma_6}) & E(uc_{\sigma_6}) & E(dc_{\sigma_6}) \end{bmatrix}$$

$$= \begin{bmatrix} {\sigma_1}^2 & {\sigma_1}^2 & {\sigma_1}^2 & (1-2\log 2){\sigma_1}^2 & 0.5{\sigma_1}^2 & 0.5{\sigma_1}^2 \\ {\sigma_2}^2 & {\sigma_2}^2 & {\sigma_2}^2 & (1-2\log 2){\sigma_2}^2 & 0.5{\sigma_2}^2 & 0.5{\sigma_2}^2 \\ {\sigma_3}^2 & {\sigma_3}^2 & {\sigma_3}^2 & (1-2\log 2){\sigma_3}^2 & 0.5{\sigma_3}^2 & 0.5{\sigma_3}^2 \\ {\sigma_4}^2 & {\sigma_4}^2 & {\sigma_4}^2 & (1-2\log 2){\sigma_4}^2 & 0.5{\sigma_4}^2 & 0.5{\sigma_4}^2 \\ {\sigma_5}^2 & {\sigma_5}^2 & {\sigma_5}^2 & (1-2\log 2){\sigma_5}^2 & 0.5{\sigma_5}^2 & 0.5{\sigma_5}^2 \\ {\sigma_6}^2 & {\sigma_6}^2 & {\sigma_6}^2 & (1-2\log 2){\sigma_6}^2 & 0.5{\sigma_6}^2 & 0.5{\sigma_6}^2 \end{bmatrix} \quad (16)$$

4. **Complete Case:**
The family of estimators for volatility of security price modelled as Brownian motion without drift

$$\begin{bmatrix} {a_{GK}}^{u^2} \\ {a_{GK}}^{d^2} \\ {a_{GK}}^{c^2} \\ {a_{GK}}^{ud} \\ {a_{GK}}^{uc} \\ {a_{GK}}^{dc} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \alpha_1 \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \alpha_3 \begin{bmatrix} 0.3862944 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \alpha_4 \begin{bmatrix} -0.5 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + \alpha_5 \begin{bmatrix} -0.5 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

The solution has the first vector as the particular solution and the rest of the 5 vectors are the basis vectors which comprise the null space for the estimator. Different values of $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ and $\alpha_5$will give different estimators

Garman and Klass: α_{GK} : $\alpha_1 = 0.5111 \quad \alpha_2 = -0.3831 \quad \alpha_3 = -0.9838 \quad \alpha_4 = -0.0192 \quad \alpha_5 = -0.0192$

5. **Minimum Variance Solution:**
Putting values of alpha in the complete case above gives the following solution:

$$\begin{gathered} {a_{GK}}^{u^2} = 0.511119 \quad {a_{GK}}^{d^2} = 0.511119 \quad {a_{GK}}^{c^2} = -0.383063 \quad {a_{GK}}^{ud} = -0.983808 \\ {a_{GK}}^{uc} = -0.019215 \quad {a_{GK}}^{dc} = -0.019215 \end{gathered}$$

3.1.3 Meilijson Case

1. **Unbiased Constraint:**

$$\begin{gathered} E(u^2) = 1.75\sigma^2 \quad E(d^2) = 0.25\sigma^2 \quad E(c^2) = \sigma^2 \quad E(ud) = (1-2\log 2)\sigma^2 \\ E(uc) = 1.25\sigma^2 \quad E(dc) = -0.25\sigma^2 \end{gathered}$$

2. **Variance-Covariance Matrix $\sum_M$:**

$$\begin{bmatrix} E(u^4) & E(u^2d^2) & E(u^2c^2) & E(u^3d^2) & E(u^3c) & E(u^2dc) \\ E(u^2d^2) & E(d^4) & E(d^2c^2) & E(ud^3) & E(u^2d^2) & E(u^3d) \\ E(u^2c^2) & E(d^2c^2) & E(c^4) & E(udc^2) & E(uc^3) & E(dc^3) \\ E(u^3d) & E(ud^3) & E(udc^2) & E(u^2d^2) & E(u^2dc) & E(ud^2c) \\ E(u^3c) & E(ud^2c) & E(uc^3) & E(u^2dc) & E(u^2c^2) & E(udc^2) \\ E(u^2dc) & E(d^3c) & E(dc^3) & E(ud^2c) & E(udc^2) & E(d^2c^2) \end{bmatrix}$$

$$= \begin{bmatrix} 5.8125\sigma^4 & 0.22741\sigma^4 & 3.875\sigma^4 & -0.7159\sigma^4 & 4.59375\sigma^4 & -0.53376\sigma^4 \\ 0.22741\sigma^4 & 0.1875\sigma^4 & 0.125\sigma^4 & -0.1478\sigma^4 & 0.15758\sigma^4 & -0.09375\sigma^4 \\ 3.875\sigma^4 & 0.125\sigma^4 & 3\sigma^4 & -0.43808\sigma^4 & 3.375\sigma^4 & -0.375\sigma^4 \\ -0.7159\sigma^4 & -0.1478\sigma^4 & -0.43808\sigma^4 & 0.22741\sigma^4 & -0.53376\sigma^4 & 0.15758\sigma^4 \\ 4.59375\sigma^4 & 0.15758\sigma^4 & 3.375\sigma^4 & -0.53376\sigma^4 & 3.875\sigma^4 & -0.43808\sigma^4 \\ -0.53376\sigma^4 & -0.09375\sigma^4 & -0.375\sigma^4 & 0.15758\sigma^4 & -0.43808\sigma^4 & 0.125\sigma^4 \end{bmatrix}$$

3. **$\vec{K}$ to calculate the family of estimators:**

$$\vec{K}_M = \begin{bmatrix} E(u_{\sigma_1}^2) & E(d_{\sigma_1}^2) & E(c_{\sigma_1}^2) & E(ud_{\sigma_1}) & E(uc_{\sigma_1}) & E(dc_{\sigma_1}) \\ E(u_{\sigma_2}^2) & E(d_{\sigma_2}^2) & E(c_{\sigma_2}^2) & E(ud_{\sigma_2}) & E(uc_{\sigma_2}) & E(dc_{\sigma_2}) \\ E(u_{\sigma_3}^2) & E(d_{\sigma_3}^2) & E(c_{\sigma_3}^2) & E(ud_{\sigma_3}) & E(uc_{\sigma_3}) & E(dc_{\sigma_3}) \\ E(u_{\sigma_4}^2) & E(d_{\sigma_4}^2) & E(c_{\sigma_4}^2) & E(ud_{\sigma_4}) & E(uc_{\sigma_4}) & E(dc_{\sigma_4}) \\ E(u_{\sigma_5}^2) & E(d_{\sigma_5}^2) & E(c_{\sigma_5}^2) & E(ud_{\sigma_5}) & E(uc_{\sigma_5}) & E(dc_{\sigma_5}) \\ E(u_{\sigma_6}^2) & E(d_{\sigma_6}^2) & E(c_{\sigma_6}^2) & E(ud_{\sigma_6}) & E(uc_{\sigma_6}) & E(dc_{\sigma_6}) \end{bmatrix}$$

$$= \begin{bmatrix} 1.75{\sigma_1}^2 & 0.25{\sigma_1}^2 & {\sigma_1}^2 & (1-2\log 2){\sigma_1}^2 & 1.25{\sigma_1}^2 & -0.25{\sigma_1}^2 \\ 1.75{\sigma_2}^2 & 0.25{\sigma_2}^2 & {\sigma_2}^2 & (1-2\log 2){\sigma_2}^2 & 1.25{\sigma_2}^2 & -0.25{\sigma_2}^2 \\ 1.75{\sigma_3}^2 & 0.25{\sigma_3}^2 & {\sigma_3}^2 & (1-2\log 2){\sigma_3}^2 & 1.25{\sigma_3}^2 & -0.25{\sigma_3}^2 \\ 1.75{\sigma_4}^2 & 0.25{\sigma_4}^2 & {\sigma_4}^2 & (1-2\log 2){\sigma_4}^2 & 1.25{\sigma_4}^2 & -0.25{\sigma_4}^2 \\ 1.75{\sigma_5}^2 & 0.25{\sigma_5}^2 & {\sigma_5}^2 & (1-2\log 2){\sigma_5}^2 & 1.25{\sigma_5}^2 & -0.25{\sigma_5}^2 \\ 1.75{\sigma_6}^2 & 0.25{\sigma_6}^2 & {\sigma_6}^2 & (1-2\log 2){\sigma_6}^2 & 1.25{\sigma_6}^2 & -0.25{\sigma_6}^2 \end{bmatrix} \tag{17}$$

4. **Complete Case:**
The family of estimators for volatility of security price modelled as Brownian motion without drift

$$\begin{bmatrix} {a_M}^{u^2} \\ {a_M}^{d^2} \\ {a_M}^{c^2} \\ {a_M}^{ud} \\ {a_M}^{uc} \\ {a_M}^{dc} \end{bmatrix} = \begin{bmatrix} 0.5 \\ 0.5 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \alpha_1 \begin{bmatrix} -0.14 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} -0.57 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \alpha_3 \begin{bmatrix} 0.22 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \alpha_4 \begin{bmatrix} -0.71 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + \alpha_5 \begin{bmatrix} 0.14 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

The solution has the first vector as the particular solution and the rest of the 5 vectors are the basis vectors which comprise the null space for the estimator. Different values of $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ and $\alpha_5$will give different estimators

Meilijson: $\alpha_M : \alpha_1 = 0.0453 \quad \alpha_2 = --0.0219 \quad \alpha_3 = -1.4707 \quad \alpha_4 = -0.3662 \quad \alpha_5 = -0.7374$

5. **Minimum Variance Solution:**
Putting values of alpha in the complete case above gives the following solution:

$$a_M{}^{u^2} = 0.549 \quad a_M{}^{d^2} = 0.545 \quad a_M{}^{c^2} = -0.0214 \quad a_M{}^{ud} = -1.470$$
$$a_M{}^{uc} = -0.367 \quad a_M{}^{dc} = 0.736$$

3.1.4 Using Meilijson Technique on Parkinson estimator

We can further improve the Minimum variance estimator by using the approach suggested in Meilijson (2011) to calculate volatility estimator. This estimator is 1.23 times better than the Parkinson's estimator.

1. **Unbiased Constraint:**

$$E(u^2) = 1.75\sigma^2 \quad E(d^2) = 0.25\sigma^2 \quad E(ud) = (1 - 2\log 2)\sigma^2$$

2. **Variance-Covariance Matrix $\sum_{MP}$:**

$$\begin{bmatrix} E(u^4) & E(u^2d^2) & E(u^3d) \\ E(u^2d^2) & E(d^4) & E(ud^3) \\ E(u^3d) & E(ud^3) & E(u^2d^2) \end{bmatrix} = \begin{bmatrix} 5.8125\sigma^4 & 0.22741\sigma^4 & -0.7159\sigma^4 \\ 0.22741\sigma^4 & 0.1875\sigma^4 & -0.1478\sigma^4 \\ -0.7159\sigma^4 & -0.1478\sigma^4 & 0.22741\sigma^4 \end{bmatrix}$$

3. $\vec{K}$ **to calculate the family of estimators:**

$$\vec{K}_{MP} = \begin{bmatrix} E(u^2_{\sigma_1}) & E(d^2_{\sigma_1}) & E(ud_{\sigma_1}) \\ E(u^2_{\sigma_2}) & E(d^2_{\sigma_2}) & E(ud_{\sigma_2}) \\ E(u^2_{\sigma_3}) & E(d^2_{\sigma_3}) & E(ud_{\sigma_3}) \\ E(u^2_{\sigma_4}) & E(d^2_{\sigma_4}) & E(ud_{\sigma_4}) \\ E(u^2_{\sigma_5}) & E(d^2_{\sigma_5}) & E(ud_{\sigma_5}) \\ E(u^2_{\sigma_6}) & E(d^2_{\sigma_6}) & E(ud_{\sigma_6}) \end{bmatrix} = \begin{bmatrix} 1.75\sigma_1{}^2 & 0.25\sigma_1{}^2 & (1-2\log 2)\sigma_1{}^2 \\ 1.75\sigma_2{}^2 & 0.25\sigma_2{}^2 & (1-2\log 2)\sigma_2{}^2 \\ 1.75\sigma_3{}^2 & 0.25\sigma_3{}^2 & (1-2\log 2)\sigma_3{}^2 \\ 1.75\sigma_4{}^2 & 0.25\sigma_4{}^2 & (1-2\log 2)\sigma_4{}^2 \\ 1.75\sigma_5{}^2 & 0.25\sigma_5{}^2 & (1-2\log 2)\sigma_5{}^2 \\ 1.75\sigma_6{}^2 & 0.25\sigma_6{}^2 & (1-2\log 2)\sigma_6{}^2 \end{bmatrix}$$

4. **Complete Case:**
To calculate the new estimator, we can show a similar form of family of estimator with a smaller number of quadratic extreme values involved.

$$\begin{bmatrix} a_{MP}{}^{u^2} \\ a_{MP}{}^{d^2} \\ a_{MP}{}^{ud} \end{bmatrix} = \begin{bmatrix} 0.5 \\ 0.5 \\ 0 \end{bmatrix} + \alpha_1 \begin{bmatrix} -0.1428 \\ 1 \\ 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0.221 \\ 0 \\ 1 \end{bmatrix}$$

Meilijison/Parkinson: $\alpha_{MP} : \alpha_1 = 0.109354668 \quad \alpha_2 = -1.1062027834$check

5. **Minimum Variance Solution:**
Putting values of alpha in the complete case above gives the following solution:

$$a_{MP}{}^{u^2} = 0.2401951 \quad a_{MP}{}^{d^2} = 0.60935466 \quad a_{MP}{}^{ud} = -1.1062027834$$

3.2 Brownian Motion with zero drift and positive jump

3.2.1 Garman and Klass

1. **Unbiased Constraint:**

$$E(u^2) = (1-f)\sigma^2 \quad E(d^2) = (1-f)\sigma^2 \quad E(c^2) = (1-f)\sigma^2 \quad E(ud) = (1-f)(1-2\log 2)\sigma^2$$
$$E(uc) = 0.5(1-f)\sigma^2 \quad E(dc) = 0.5(1-f)\sigma^2 \quad E(o^2) = f\sigma^2$$

2. **Variance-Covariance Matrix $\sum_{GK}$:**
In order to make the estimator independent of f, the coefficient assigned to o^2 should be equal to 1. As the coefficient for o^2 is fixed, we can exclude the term from the variance covariance matrix. Therefore, the variance covariance matrix will look similar to the case where jump is zero and the best estimator for positive jump case will have the same coefficients for u^2, d^2, c^2, ud, uc and dc as were in the no jump case.

$$\begin{bmatrix} E(u^4) & E(u^2d^2) & E(u^2c^2) & E(u^3d^2) & E(u^3c) & E(u^2dc) \\ E(u^2d^2) & E(d^4) & E(d^2c^2) & E(ud^3) & E(u^2d^2) & E(u^3d) \\ E(u^2c^2) & E(d^2c^2) & E(c^4) & E(udc^2) & E(uc^3) & E(dc^3) \\ E(u^3d) & E(ud^3) & E(udc^2) & E(u^2d^2) & E(u^2dc) & E(ud^2c) \\ E(u^3c) & E(ud^2c) & E(uc^3) & E(u^2dc) & E(u^2c^2) & E(udc^2) \\ E(u^2dc) & E(d^3c) & E(dc^3) & E(ud^2c) & E(udc^2) & E(d^2c^2) \end{bmatrix}$$

$$= (1-f)^2 \begin{bmatrix} 3\sigma^4 & 0.2274\sigma^4 & 2\sigma^4 & -0.4318\sigma^4 & 2.25\sigma^4 & -0.1881\sigma^4 \\ 0.2274\sigma^4 & 3\sigma^4 & 2\sigma^4 & -0.4318\sigma^4 & -0.1881\sigma^4 & 2.25\sigma^4 \\ 2\sigma^4 & 2\sigma^4 & 3\sigma^4 & -0.4381\sigma^4 & 1.5\sigma^4 & 1.5\sigma^4 \\ -0.4318\sigma^4 & -0.4318\sigma^4 & -0.4381\sigma^4 & 0.2274\sigma^4 & -0.1881\sigma^4 & -0.1881\sigma^4 \\ 2.25\sigma^4 & -0.1881\sigma^4 & 1.5\sigma^4 & -0.1881\sigma^4 & 2\sigma^4 & -0.4381\sigma^4 \\ -0.1881\sigma^4 & 2.25\sigma^4 & 1.5\sigma^4 & -0.1881\sigma^4 & -0.4381\sigma^4 & 2\sigma^4 \end{bmatrix}$$

3. **$\vec{K}$ to calculate the family of estimators:**

$$\vec{K}_{GK} = \begin{bmatrix} (1-f_1)\sigma_1{}^2 & \dots & (1-f_1)(1-2\log 2)\sigma_1{}^2 & \dots & (1-f_1)0.5\sigma_1{}^2 \\ (1-f_1)\sigma_2{}^2 & \dots & (1-f_1)(1-2\log 2)\sigma_2{}^2 & \dots & (1-f_1)0.5\sigma_2{}^2 \\ (1-f_2)\sigma_1{}^2 & \dots & (1-f_2)(1-2\log 2)\sigma_1{}^2 & \dots & (1-f_2)0.5\sigma_1{}^2 \\ (1-f_2)\sigma_2{}^2 & \dots & (1-f_2)(1-2\log 2)\sigma_2{}^2 & \dots & (1-f_2)0.5\sigma_2{}^2 \\ \vdots & \dots & \vdots & \dots & \vdots \\ \vdots & \dots & \vdots & \dots & \vdots \end{bmatrix} \qquad (18)$$

4. **Complete Case:(Missing)**
   The family of estimators for volatility of security price modelled as Brownian motion without drift

$$\begin{bmatrix} {a_{GK}}^{u^2} \\ {a_{GK}}^{d^2} \\ {a_{GK}}^{c^2} \\ {a_{GK}}^{ud} \\ {a_{GK}}^{uc} \\ {a_{GK}}^{dc} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \alpha_1 \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \alpha_3 \begin{bmatrix} 0.3862944 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \alpha_4 \begin{bmatrix} -0.5 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + \alpha_5 \begin{bmatrix} -0.5 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

   The solution has the first vector as the particular solution and the rest of the 5 vectors are the basis vectors which comprise the null space for the estimator. Different values of $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ and $\alpha_5$will give different estimators

   Garman and Klass: $\alpha_{GK} : \alpha_1 = 0 \quad \alpha_2 = -0. \quad \alpha_3 = -0. \quad \alpha_4 = -0.0 \quad \alpha_5 = -0.0$

5. **Minimum Variance Solution:(Missing)**
   Putting values of alpha in the complete case above gives the following solution:

$${a_{GK}}^{u^2} = 0. \quad {a_{GK}}^{d^2} = 0.5 \quad {a_{GK}}^{c^2} = -0. \quad {a_{GK}}^{ud} = -0.$$
$${a_{GK}}^{uc} = -0.0 \quad {a_{GK}}^{dc} = -0.0$$

3.2.2 Meilijson Case

1. **Unbiased Constraint:**

$$E(u^2) = 1.75(1-f)\sigma^2 \quad E(d^2) = 0.25(1-f)\sigma^2 \quad E(c^2) = (1-f)\sigma^2 \quad E(ud) = (1-f)(1-2\log 2)\sigma^2$$
$$E(uc) = 1.25(1-f)\sigma^2 \quad E(dc) = -0.25(1-f)\sigma^2$$

2. **Variance-Covariance Matrix $\sum_M$:**

$$\begin{bmatrix} E(u^4) & E(u^2d^2) & E(u^2c^2) & E(u^3d^2) & E(u^3c) & E(u^2dc) \\ E(u^2d^2) & E(d^4) & E(d^2c^2) & E(ud^3) & E(u^2d^2) & E(u^3d) \\ E(u^2c^2) & E(d^2c^2) & E(c^4) & E(udc^2) & E(uc^3) & E(dc^3) \\ E(u^3d) & E(ud^3) & E(udc^2) & E(u^2d^2) & E(u^2dc) & E(ud^2c) \\ E(u^3c) & E(ud^2c) & E(uc^3) & E(u^2dc) & E(u^2c^2) & E(udc^2) \\ E(u^2dc) & E(d^3c) & E(dc^3) & E(ud^2c) & E(udc^2) & E(d^2c^2) \end{bmatrix}$$

$$= (1-f)^2 \begin{bmatrix} 5.8125\sigma^4 & 0.22741\sigma^4 & 3.875\sigma^4 & -0.7159\sigma^4 & 4.59375\sigma^4 & -0.53376\sigma^4 \\ 0.22741\sigma^4 & 0.1875\sigma^4 & 0.125\sigma^4 & -0.1478\sigma^4 & 0.15758\sigma^4 & -0.09375\sigma^4 \\ 3.875\sigma^4 & 0.125\sigma^4 & 3\sigma^4 & -0.43808\sigma^4 & 3.375\sigma^4 & -0.375\sigma^4 \\ -0.7159\sigma^4 & -0.1478\sigma^4 & -0.43808\sigma^4 & 0.22741\sigma^4 & -0.53376\sigma^4 & 0.15758\sigma^4 \\ 4.59375\sigma^4 & 0.15758\sigma^4 & 3.375\sigma^4 & -0.53376\sigma^4 & 3.875\sigma^4 & -0.43808\sigma^4 \\ -0.53376\sigma^4 & -0.09375\sigma^4 & -0.375\sigma^4 & 0.15758\sigma^4 & -0.43808\sigma^4 & 0.125\sigma^4 \end{bmatrix}$$

3. $\vec{K}$ **to calculate the family of estimators:**

$$\vec{K}_M = \begin{bmatrix} 1.75(1-f_1){\sigma_1}^2 & \dots & (1-f_1)(1-2\log 2){\sigma_1}^2 & \dots & -0.25(1-f_1){\sigma_1}^2 \\ 1.75(1-f_1){\sigma_2}^2 & \dots & (1-f_1)(1-2\log 2){\sigma_2}^2 & \dots & -0.25(1-f_1){\sigma_1}^2 \\ 1.75(1-f_2){\sigma_1}^2 & \dots & (1-f_2)(1-2\log 2){\sigma_1}^2 & \dots & -0.25(1-f_2){\sigma_1}^2 \\ 1.75(1-f_2){\sigma_2}^2 & \dots & (1-f_2)(1-2\log 2){\sigma_2}^2 & \dots & -0.25(1-f_2){\sigma_1}^2 \\ \vdots & \dots & \vdots & \dots & \vdots \\ \vdots & \dots & \vdots & \dots & \vdots \end{bmatrix} \tag{19}$$

4. **Complete Case:(to check)**
The family of estimators for volatility of security price modelled as Brownian motion without drift

$$\begin{bmatrix} {a_M}^{u^2} \\ {a_M}^{d^2} \\ {a_M}^{c^2} \\ {a_M}^{ud} \\ {a_M}^{uc} \\ {a_M}^{dc} \\ {a_M}^{o^2} \end{bmatrix} = \begin{bmatrix} 0.5 \\ 0.5 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} + \alpha_1 \begin{bmatrix} -0.14 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} -0.57 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \alpha_3 \begin{bmatrix} 0.22 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \alpha_4 \begin{bmatrix} -0.71 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \alpha_5 \begin{bmatrix} 0.14 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

The solution has the first vector as the particular solution and the rest of the 5 vectors are the basis vectors which comprise the null space for the estimator. Different values of $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ and $\alpha_5$will give different estimators

Meilijson: $\alpha_M : \alpha_1 = \quad \alpha_2 = \quad \alpha_3 = \quad \alpha_4 = \quad \alpha_5 =$

5. **Minimum Variance Solution:(Missing)**
Putting values of alpha in the complete case above gives the following solution:

$${a_M}^{u^2} = \quad {a_M}^{d^2} = \quad {a_M}^{c^2} = \quad {a_M}^{ud} =$$
$${a_M}^{uc} = \quad {a_M}^{dc} = \quad {a_M}^{o^2} = 1$$

3.2.3 Yang and Zhang Case

1. **Unbiased Constraint:(Missing)**

$$E(u^2) = \quad E(d^2) = \quad E(c^2) = \quad E(ud) =$$
$$E(uc) = \quad E(dc) =$$

2. **Variance-Covariance Matrix $\sum_M$(Missing):**

$$\begin{bmatrix} E(u^4) & E(u^2d^2) & E(u^2c^2) & E(u^3d^2) & E(u^3c) & E(u^2dc) \\ E(u^2d^2) & E(d^4) & E(d^2c^2) & E(ud^3) & E(u^2d^2) & E(u^3d) \\ E(u^2c^2) & E(d^2c^2) & E(c^4) & E(udc^2) & E(uc^3) & E(dc^3) \\ E(u^3d) & E(ud^3) & E(udc^2) & E(u^2d^2) & E(u^2dc) & E(ud^2c) \\ E(u^3c) & E(ud^2c) & E(uc^3) & E(u^2dc) & E(u^2c^2) & E(udc^2) \\ E(u^2dc) & E(d^3c) & E(dc^3) & E(ud^2c) & E(udc^2) & E(d^2c^2) \end{bmatrix}$$

$$= (1-f)^2 \begin{bmatrix} 5.8125\sigma^4 & 0.22741\sigma^4 & 3.875\sigma^4 & -0.7159\sigma^4 & 4.59375\sigma^4 & -0.53376\sigma^4 \\ 0.22741\sigma^4 & 0.1875\sigma^4 & 0.125\sigma^4 & -0.1478\sigma^4 & 0.15758\sigma^4 & -0.09375\sigma^4 \\ 3.875\sigma^4 & 0.125\sigma^4 & 3\sigma^4 & -0.43808\sigma^4 & 3.375\sigma^4 & -0.375\sigma^4 \\ -0.7159\sigma^4 & -0.1478\sigma^4 & -0.43808\sigma^4 & 0.22741\sigma^4 & -0.53376\sigma^4 & 0.15758\sigma^4 \\ 4.59375\sigma^4 & 0.15758\sigma^4 & 3.375\sigma^4 & -0.53376\sigma^4 & 3.875\sigma^4 & -0.43808\sigma^4 \\ -0.53376\sigma^4 & -0.09375\sigma^4 & -0.375\sigma^4 & 0.15758\sigma^4 & -0.43808\sigma^4 & 0.125\sigma^4 \end{bmatrix}$$

3. **$\vec{K}$ to calculate the family of estimators(Missing):**

$$\vec{K}_{YZ} = \begin{bmatrix} (1-f_1){\sigma_1}^2 & \dots & (1-f_1)(1-2\log 2){\sigma_1}^2 & \dots & 0.5(1-f_1){\sigma_1}^2 \\ (1-f_1){\sigma_2}^2 & \dots & (1-f_1)(1-2\log 2){\sigma_2}^2 & \dots & 0.5(1-f_1){\sigma_1}^2 \\ (1-f_2){\sigma_1}^2 & \dots & (1-f_2)(1-2\log 2){\sigma_1}^2 & \dots & 0.5(1-f_2){\sigma_1}^2 \\ (1-f_2){\sigma_2}^2 & \dots & (1-f_2)(1-2\log 2){\sigma_2}^2 & \dots & 0.5(1-f_2){\sigma_1}^2 \\ \vdots & \dots & \vdots & \dots & \vdots \\ \vdots & \dots & \vdots & \dots & \vdots \end{bmatrix} \tag{20}$$

4. **Complete Case:(to check)**
The family of estimators for volatility of security price modelled as Brownian motion without drift

$$\begin{bmatrix} {a_{YZ}}^{u^2} \\ {a_{YZ}}^{d^2} \\ {a_{YZ}}^{c^2} \\ {a_{YZ}}^{ud} \\ {a_{YZ}}^{uc} \\ {a_{YZ}}^{dc} \\ {a_{YZ}}^{o^2} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} + \alpha_1 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \alpha_3 \begin{bmatrix} 0.3862944 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \alpha_4 \begin{bmatrix} -0.5 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \alpha_5 \begin{bmatrix} -0.5 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

The solution has the first vector as the particular solution and the rest of the 5 vectors are the basis vectors which comprise the null space for the estimator. Different values of $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ and $\alpha_5$will give different estimators

Yang and Zhang: $\alpha_{YZ} : \alpha_1 = \quad \alpha_2 = \quad \alpha_3 = \quad \alpha_4 = \quad \alpha_5 =$

5. **Minimum Variance Solution:(Missing)**
   Putting values of alpha in the complete case above gives the following solution:

$$
\begin{gathered}
{a_{YZ}}^{u^2} = \quad {a_{YZ}}^{d^2} = \quad {a_{YZ}}^{c^2} = \quad {a_{YZ}}^{ud} = \\
{a_{YZ}}^{uc} = \quad {a_{YZ}}^{dc} = \quad {a_{YZ}}^{o^2} = 1
\end{gathered}
$$

3.2.4 Best Analytical Estimator

1. **Unbiased Constraint:(Missing)**

$$
\begin{gathered}
E(u^2) = \quad E(d^2) = \quad E(c^2) = \quad E(ud) = \\
E(uc) = \quad E(dc) =
\end{gathered}
$$

2. **Variance-Covariance Matrix $\sum_M$(Missing):**

$$
\begin{bmatrix}
E(u^4) & E(u^2d^2) & E(u^2c^2) & E(u^3d^2) & E(u^3c) & E(u^2dc) \\
E(u^2d^2) & E(d^4) & E(d^2c^2) & E(ud^3) & E(u^2d^2) & E(u^3d) \\
E(u^2c^2) & E(d^2c^2) & E(c^4) & E(udc^2) & E(uc^3) & E(dc^3) \\
E(u^3d) & E(ud^3) & E(udc^2) & E(u^2d^2) & E(u^2dc) & E(ud^2c) \\
E(u^3c) & E(ud^2c) & E(uc^3) & E(u^2dc) & E(u^2c^2) & E(udc^2) \\
E(u^2dc) & E(d^3c) & E(dc^3) & E(ud^2c) & E(udc^2) & E(d^2c^2)
\end{bmatrix}
$$
$$
= (1-f)^2 \left[\ldots\right]
$$

3. **$\vec{K}$ to calculate the family of estimators(Missing):**

$$
\vec{K}_{BE} = [\ldots] \tag{21}
$$

4. **Complete Case:(Missing)**
   The family of estimators for volatility of security price modelled as Brownian motion without drift

$$
\begin{bmatrix}
{a_{BE}}^{u^2} \\ {a_{BE}}^{d^2} \\ {a_{BE}}^{c^2} \\ {a_{BE}}^{ud} \\ {a_{BE}}^{uc} \\ {a_{BE}}^{dc} \\ {a_{BE}}^{o^2}
\end{bmatrix}
=
\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}
+ \alpha_1 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}
+ \alpha_2 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}
+ \alpha_3 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}
+ \alpha_4 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}
+ \alpha_5 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}
$$

   The solution has the first vector as the particular solution and the rest of the 5 vectors are the basis vectors which comprise the null space for the estimator. Different values of $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ and $\alpha_5$will give different estimators

   Best Analytical Estimator: $\alpha_{BE} : \alpha_1 = \quad \alpha_2 = \quad \alpha_3 = \quad \alpha_4 = \quad \alpha_5 =$

5. **Minimum Variance Solution:**
(Yang & Zhang, 2000) provided an estimator of volatility for Brownian motion with jumps and zero drift. The estimator resembled very closely to Garman-Klass estimator (Let's call it GK Jump) and is till now considered to be the best estimator. Although, by using (Meilijson, 2008) approach, an estimator can be generated which is superior than the GK Jump estimator. For this, we have considered a two-day case for our simulations as considered by Yang and Zhang, but the following can be generalized to n case as well.
Estimator form:

$$a^{u^2}\sum u^2 + a^{d^2}\sum d^2 + a^{c^2}\sum c^2 + a^{u^2}\sum ud + a^{uc}\sum uc + a^{dc}\sum dc + a^{o^2}\sum o^2$$

GK Jump Estimator:

$$a_{GK}{}^{u^2} = 0.5111 \quad a_{GK}{}^{d^2} = 0.5111 \quad a_{GK}{}^{c^2} = -0.383 \quad a_{GK}{}^{ud} = -0.9838$$

$$a_{GK}{}^{uc} = -0.0192 \quad a_{GK}{}^{dc} = -0.0192 \quad a_{GK}{}^{o^2} = 1$$

Best Estimator:

$$a_{BE}{}^{u^2} = 0.54704 \quad a_{BE}{}^{d^2} = 0.54704 \quad a_{BE}{}^{c^2} = -0.02303 \quad a_{BE}{}^{ud} = -1.47413$$

$$a_{BE}{}^{uc} = -0.363654 \quad a_{BE}{}^{dc} = 0.743706 \quad a_{BE}{}^{o^2} = 1$$

3.3 Brownian Motion with drift and no jump

3.3.1 Rogers and Satchell Case

As discussed earlier, the expectation of quadratic extreme values is a complex function when drift is introduced. Though, by analysing the expectations through graphical representation, it is easy to infer that the pattern followed by these expectations can be exploited to form groups whose expectations are drift independent.

There are three patterns which can be noted.

1. The distance between u^2 and uc is $\sigma^2/2$ irrespective of the drift value.
2. The distance between d^2 and dc is $sigma^2/2$ irrespective of the drift value.
3. The distance between $(u^2 + d^2)$ and c^2 is σ^2 irrespective of the drift value.

Let us out look at the solutions in case of Roger and Satchell.

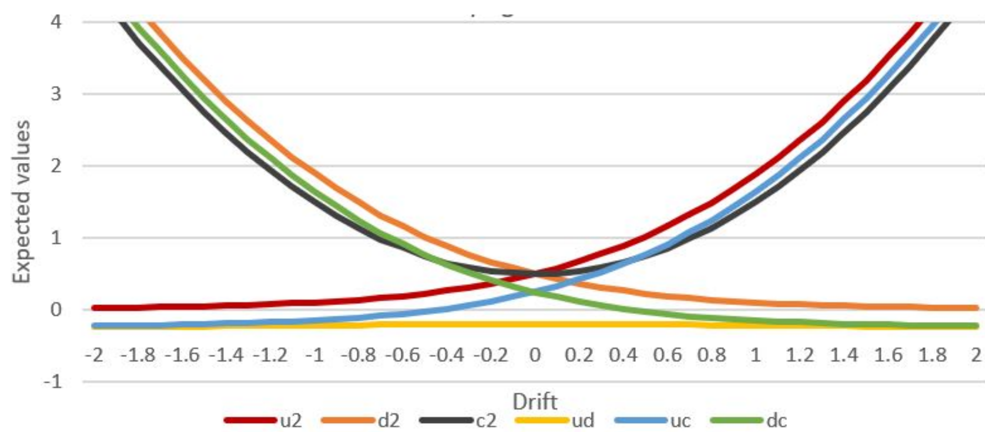

Figure 2: Expectation of quadratic extreme values with drift 0.5

1. **Unbiased Constraint:**
   The function of the expected values of quadratic extreme terms becomes non linear and complex when drift is non zero. Rogers and Satchell grouped the terms in such a way that the expectation of the group is independent of the drift.

$$E(u^2 - uc) = 0.5\sigma^2 \quad E(d^2 - dc) = 0.5\sigma^2$$

2. **Variance-Covariance Matrix $\sum_{RS}$:(to check)**
   The cross moments are not independent of the drift. Thus, we have put the values of cross moments corresponding to zero drift case as done by the Rogers and Satchell as well.

$$\begin{bmatrix} E[(u^2 - uc)^2] & E[(u^2 - uc)(d^2 - dc)] \\ E[(u^2 - uc)(d^2 - dc)] & E[(d^2 - dc)^2] \end{bmatrix} = \begin{bmatrix} \sigma^4/2 & 0.331\sigma^4 \\ 0.331\sigma^4 & 2\sigma^4/2 \end{bmatrix}$$

3. **$\vec{K}$ to calculate the family of estimators:(to check)**
   The groups formed by Rogers and Satchell which were drift independent came from the first two patterns. We will try to explore the third pattern in our results. Using these patterns, we have provided the matrix form to calculate the null space.

$$\vec{K}_{RS} = \begin{bmatrix} E({u^2}_{\sigma_1\mu_1}) & E({d^2}_{\sigma_1\mu_1}) & E({c^2}_{\sigma_1\mu_1}) & E(ud_{\sigma_1\mu_1}) & E(uc_{\sigma_1\mu_1}) & E(dc_{\sigma_1\mu_1}) \\ E({u^2}_{\sigma_2\mu_1}) & E({d^2}_{\sigma_2\mu_1}) & E({c^2}_{\sigma_2\mu_1}) & E(ud_{\sigma_2\mu_1}) & E(uc_{\sigma_2\mu_1}) & E(dc_{\sigma_2\mu_1}) \\ E({u^2}_{\sigma_1\mu_2}) & E({d^2}_{\sigma_1\mu_2}) & E({c^2}_{\sigma_1\mu_2}) & E(ud_{\sigma_1\mu_2}) & E(uc_{\sigma_1\mu_2}) & E(dc_{\sigma_1\mu_2}) \\ E({u^2}_{\sigma_2\mu_2}) & E({d^2}_{\sigma_2\mu_2}) & E({c^2}_{\sigma_2\mu_2}) & E(ud_{\sigma_2\mu_2}) & E(uc_{\sigma_2\mu_2}) & E(dc_{\sigma_2\mu_2}) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

$$= \begin{bmatrix} \sigma_1/2 + {\theta^{ud}}_{\sigma_1\mu_1} & \sigma_1/2 + {\theta^{dc}}_{\sigma_1\mu_1} & {\theta^{ud}}_{\sigma_1\mu_1} + {\theta^{dc}}_{\sigma_1\mu_1} & ? & {\theta^{uc}}_{\sigma_1\mu_1} & {\theta^{dc}}_{\sigma_1\mu_1} \\ \sigma_2/2 + {\theta^{ud}}_{\sigma_2\mu_1} & \sigma_2/2 + {\theta^{dc}}_{\sigma_2\mu_1} & {\theta^{ud}}_{\sigma_2\mu_1} + {\theta^{dc}}_{\sigma_2\mu_1} & ? & {\theta^{uc}}_{\sigma_2\mu_1} & {\theta^{dc}}_{\sigma_2\mu_1} \\ \sigma_1/2 + {\theta^{ud}}_{\sigma_1\mu_2} & \sigma_1/2 + {\theta^{dc}}_{\sigma_1\mu_2} & {\theta^{ud}}_{\sigma_1\mu_2} + {\theta^{dc}}_{\sigma_1\mu_2} & ? & {\theta^{uc}}_{\sigma_1\mu_2} & {\theta^{dc}}_{\sigma_1\mu_2} \\ \sigma_2/2 + {\theta^{ud}}_{\sigma_2\mu_2} & \sigma_2/2 + {\theta^{dc}}_{\sigma_2\mu_2} & {\theta^{ud}}_{\sigma_2\mu_2} + {\theta^{dc}}_{\sigma_2\mu_2} & ? & {\theta^{uc}}_{\sigma_2\mu_2} & {\theta^{dc}}_{\sigma_2\mu_2} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix} \tag{22}$$

To explain the expected values, we will start with the 5th column ${\theta^{uc}}_{\sigma_1\mu_1}$ is the expected value of uc for a given volatility as σ_1 and drift as mu_1. From pattern 1, the expected value of u^2 (for the first row) is the sum of the expected value of uc (which is equal to ${\theta^{uc}}_{\sigma_1\mu_1}$ and $\sigma_2/2$. Similarly, the expected value of d^2 (for the first row) can be derived from the expected value of dc (which is equal to ${\theta^{dc}}_{\sigma_1\mu_1}$ using pattern 2. We used pattern 3 to derive the expected value of c^2 (for the first row) which is the sum of expected values of ${\theta^{uc}}_{\sigma_1\mu_1}$ and ${\theta^{dc}}_{\sigma_1\mu_1}$. The expected value of ud does not form any pattern with any other quadratic extreme value and thus is omitted from the matrix.

4. **Complete Case:**
The family of estimators for volatility of security price modelled as Brownian motion without drift

$$\begin{bmatrix} {a_{RS}}^{u^2} \\ {a_{RS}}^{d^2} \\ {a_{RS}}^{c^2} \\ {a_{RS}}^{ud} \\ {a_{RS}}^{uc} \\ {a_{RS}}^{dc} \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \\ 2 \\ 0 \end{bmatrix} + \alpha_1 \begin{bmatrix} -1 \\ 1 \\ -1 \\ 0 \\ 2 \\ 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

Rogers and Satchell: $\alpha_{RS} : \alpha_1 = 0 \quad \alpha_2 = 1$

5. **Minimum Variance Solution:(to check)**
Putting values of alpha in the complete case above gives the following solution:

$$ {a_{RS}}^{u^2} = 1 \quad {a_{RS}}^{d^2} = 1 \quad {a_{RS}}^{c^2} = 0 \quad {a_{RS}}^{ud} = 0 $$
$$ {a_{RS}}^{uc} = 3 \quad {a_{RS}}^{dc} = 1 $$

3.3.2 Prachi Estimator

1. **Unbiased Constraint:**

$$E(u^2 - uc) = 0.5\sigma^2 \quad E(d^2 - dc) = 0.5\sigma^2$$

2. **Variance-Covariance Matrix $\sum_{RS}$:(to check)**

$$\begin{bmatrix} E[(u^2-uc)^2] & E[(u^2-uc)(d^2-dc)] \\ E[(u^2-uc)(d^2-dc)] & E[(d^2-dc)^2] \end{bmatrix} = \begin{bmatrix} \sigma^4/2 & 0.331\sigma^4 \\ 0.331\sigma^4 & 2\sigma^4/2 \end{bmatrix}$$

3. **$\vec{K}$ to calculate the family of estimators:(to check)**

$$\vec{K}_{PC} = \begin{bmatrix} E(u^2{}_{\sigma_1\mu_1}) & E(d^2{}_{\sigma_1\mu_1}) & E(c^2{}_{\sigma_1\mu_1}) & E(ud_{\sigma_1\mu_1}) & E(uc_{\sigma_1\mu_1}) & E(dc_{\sigma_1\mu_1}) \\ E(u^2{}_{\sigma_2\mu_1}) & E(d^2{}_{\sigma_2\mu_1}) & E(c^2{}_{\sigma_2\mu_1}) & E(ud_{\sigma_2\mu_1}) & E(uc_{\sigma_2\mu_1}) & E(dc_{\sigma_2\mu_1}) \\ E(u^2{}_{\sigma_1\mu_2}) & E(d^2{}_{\sigma_1\mu_2}) & E(c^2{}_{\sigma_1\mu_2}) & E(ud_{\sigma_1\mu_2}) & E(uc_{\sigma_1\mu_2}) & E(dc_{\sigma_1\mu_2}) \\ E(u^2{}_{\sigma_2\mu_2}) & E(d^2{}_{\sigma_2\mu_2}) & E(c^2{}_{\sigma_2\mu_2}) & E(ud_{\sigma_2\mu_2}) & E(uc_{\sigma_2\mu_2}) & E(dc_{\sigma_2\mu_2}) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

$$= \begin{bmatrix} \sigma_1/2+\theta^{ud}{}_{\sigma_1\mu_1} & \sigma_1/2+\theta^{dc}{}_{\sigma_1\mu_1} & \theta^{ud}{}_{\sigma_1\mu_1}+\theta^{dc}{}_{\sigma_1\mu_1} & ? & \theta^{uc}{}_{\sigma_1\mu_1} & \theta^{dc}{}_{\sigma_1\mu_1} \\ \sigma_2/2+\theta^{ud}{}_{\sigma_2\mu_1} & \sigma_2/2+\theta^{dc}{}_{\sigma_2\mu_1} & \theta^{ud}{}_{\sigma_2\mu_1}+\theta^{dc}{}_{\sigma_2\mu_1} & ? & \theta^{uc}{}_{\sigma_2\mu_1} & \theta^{dc}{}_{\sigma_2\mu_1} \\ \sigma_1/2+\theta^{ud}{}_{\sigma_1\mu_2} & \sigma_1/2+\theta^{dc}{}_{\sigma_1\mu_2} & \theta^{ud}{}_{\sigma_1\mu_2}+\theta^{dc}{}_{\sigma_1\mu_2} & ? & \theta^{uc}{}_{\sigma_1\mu_2} & \theta^{dc}{}_{\sigma_1\mu_2} \\ \sigma_2/2+\theta^{ud}{}_{\sigma_2\mu_2} & \sigma_2/2+\theta^{dc}{}_{\sigma_2\mu_2} & \theta^{ud}{}_{\sigma_2\mu_2}+\theta^{dc}{}_{\sigma_2\mu_2} & ? & \theta^{uc}{}_{\sigma_2\mu_2} & \theta^{dc}{}_{\sigma_2\mu_2} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix} \tag{23}$$

4. **Complete Case:**
The family of estimators for volatility of security price modelled as Brownian motion without drift

$$\begin{bmatrix} a_{PC}{}^{u^2} \\ a_{PC}{}^{d^2} \\ a_{PC}{}^{c^2} \\ a_{PC}{}^{ud} \\ a_{PC}{}^{uc} \\ a_{PC}{}^{dc} \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \\ 2 \\ 0 \end{bmatrix} + \alpha_1 \begin{bmatrix} -1 \\ 1 \\ -1 \\ 0 \\ 2 \\ 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

Prachi Estimator: $\alpha_{PC} : \alpha_1 = 1 \quad \alpha_2 = 0$

5. **Minimum Variance Solution:(to check)**
Putting values of alpha in the complete case above gives the following solution:

$$a_{PC}{}^{u^2} = 1 \quad a_{PC}{}^{d^2} = 1 \quad a_{PC}{}^{c^2} = -1 \quad a_{PC}{}^{ud} = 0$$
$$a_{PC}{}^{uc} = 4 \quad a_{PC}{}^{dc} = 0$$

3.4 Brownian Motion with non-zero drift and positive jump

3.4.1 Yang and Zhang

1. **Unbiased Constraint:**
Yang and Zhang showed that it is impossible to construct a single period estimator

which is independent of both drift and jump(Yang & Zhang, 2000). Thus, they constructed a multi period estimator which was independent of both drift and jump. The terms were grouped together to make them drift independent.

$$E(o_i - \bar{o})^2 = f\sigma^2 \quad E(c_i - \bar{c})^2 = (1-f)\sigma^2 \quad E({u_i}^2 - uc_i + {d_i}^2 - dc_i) = (1-f)\sigma^2$$

2. **Variance-Covariance Matrix $\sum_{YZ}$:(to check)**
We can ignore the o^2 term from the variance covariance matrix as its coefficient is fixed as 1.

$$\begin{bmatrix} E(c_i - \bar{c})^4 & E[(c_i - \bar{c})^2({u_i}^2 - uc_i + {d_i}^2 - dc_i)] \\ E[(c_i - \bar{c})^2({u_i}^2 - uc_i + {d_i}^2 - dc_i)] & E({u_i}^2 - uc_i + {d_i}^2 - dc_i)^2 \end{bmatrix}$$

$$= \begin{bmatrix} [(n+1)/(n-1)]\sigma^4(1-f)^2 & \sigma^4(1-f)^2 \\ \sigma^4(1-f)^2 & [\partial + n - 1)/n]\sigma^4(1-f)^2 \end{bmatrix}$$

Where $\partial = E({u_i}^2 - uc_i + {d_i}^2 - dc_i)^2\sigma^4(1-f)^2$ and n =number of periods.

3. **$\vec{K}$ to calculate the family of estimators:(to check)**
As an unbiased estimator can be obtained by only using the multi period data and grouping, we have directly considered groups in the matrix so that it is convenient to calculate the family of estimator rather than putting individual quadratic extreme values.

$$\vec{K}_{YZ} = \begin{bmatrix} E(c_i - \bar{c})^2{}_{{\sigma_1}^2\mu_1 f_1} & E({u_i}^2 - uc_i + {d_i}^2 - dc_i)_{{\sigma_1}^2\mu_1 f_1} \\ E(c_i - \bar{c})^2{}_{{\sigma_1}^2\mu_1 f_2} & E({u_i}^2 - uc_i + {d_i}^2 - dc_i)_{{\sigma_1}^2\mu_1 f_2} \\ E(c_i - \bar{c})^2{}_{{\sigma_1}^2\mu_2 f_1} & E({u_i}^2 - uc_i + {d_i}^2 - dc_i)_{{\sigma_1}^2\mu_2 f_1} \\ E(c_i - \bar{c})^2{}_{{\sigma_1}^2\mu_2 f_2} & E({u_i}^2 - uc_i + {d_i}^2 - dc_i)_{{\sigma_1}^2\mu_2 f_2} \\ E(c_i - \bar{c})^2{}_{{\sigma_2}^2\mu_1 f_1} & E({u_i}^2 - uc_i + {d_i}^2 - dc_i)_{{\sigma_2}^2\mu_1 f_1} \\ E(c_i - \bar{c})^2{}_{{\sigma_2}^2\mu_1 f_2} & E({u_i}^2 - uc_i + {d_i}^2 - dc_i)_{{\sigma_2}^2\mu_1 f_2} \\ E(c_i - \bar{c})^2{}_{{\sigma_2}^2\mu_2 f_1} & E({u_i}^2 - uc_i + {d_i}^2 - dc_i)_{{\sigma_2}^2\mu_2 f_1} \\ E(c_i - \bar{c})^2{}_{{\sigma_2}^2\mu_2 f_2} & E({u_i}^2 - uc_i + {d_i}^2 - dc_i)_{{\sigma_2}^2\mu_2 f_2} \\ \vdots & \vdots \\ \vdots & \vdots \end{bmatrix}$$

$$= \begin{bmatrix} (1-f)^2{\sigma_1}^2{}_{{\sigma_1}^2\mu_1 f_1} & (1-f)^2{\sigma_1}^2{}_{{\sigma_1}^2\mu_1 f_1} \\ (1-f)^2{\sigma_1}^2{}_{{\sigma_1}^2\mu_1 f_2} & (1-f)^2{\sigma_1}^2{}_{{\sigma_1}^2\mu_1 f_2} \\ (1-f)^2{\sigma_1}^2{}_{{\sigma_1}^2\mu_2 f_1} & (1-f)^2{\sigma_1}^2{}_{{\sigma_1}^2\mu_2 f_1} \\ (1-f)^2{\sigma_1}^2{}_{{\sigma_1}^2\mu_2 f_2} & (1-f)^2{\sigma_1}^2{}_{{\sigma_1}^2\mu_2 f_2} \\ (1-f)^2{\sigma_2}^2{}_{{\sigma_2}^2\mu_1 f_1} & (1-f)^2{\sigma_2}^2{}_{{\sigma_2}^2\mu_1 f_1} \\ (1-f)^2{\sigma_2}^2{}_{{\sigma_2}^2\mu_1 f_2} & (1-f)^2{\sigma_2}^2{}_{{\sigma_2}^2\mu_1 f_2} \\ (1-f)^2{\sigma_2}^2{}_{{\sigma_2}^2\mu_2 f_1} & (1-f)^2{\sigma_2}^2{}_{{\sigma_2}^2\mu_2 f_1} \\ (1-f)^2{\sigma_2}^2{}_{{\sigma_2}^2\mu_2 f_2} & (1-f)^2{\sigma_2}^2{}_{{\sigma_2}^2\mu_2 f_2} \\ \vdots & \vdots \\ \vdots & \vdots \end{bmatrix} \quad (24)$$

4. **Complete Case:**
The family of estimators for volatility of security price modelled as Brownian motion without drift
$$\begin{bmatrix} {a_{YZ}}^{(c_i-\bar{c})^2} \\ {a_{YZ}}^{({u_i}^2-uc_i+{d_i}^2-dc_i)} \\ {a_{YZ}}^{(o_i-\bar{o})^2} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \alpha_1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$
The solution has the first vector as the particular solution and the second vector as the basis vector which comprise the null space for the estimator. Different values of α_1 will give different estimators. The form was provided by Yang and Zhang as well in their paper where they used k_0 instead of α_1. Yang and Zhang estimator is derived with the following values.
Yang and Zhang Estimator(2 period case): $\alpha_{YZ} : \alpha_1 = 0.0783$

5. **Minimum Variance Solution:(to check)**
Putting values of alpha in the complete case above gives the following solution:
$${a_{YZ}}^{(c_i-\bar{c})^2} = 1.0783 \quad {a_{YZ}}^{({u_i}^2-uc_i+{d_i}^2-dc_i)} = 1.0783 \quad {a_{YZ}}^{(o_i-\bar{o})^2} = 0$$

4 Testing real market data to evaluate the performance of different estimators(to do)

To do

5 Conclusion

In conclusion, this paper has tried to develop a methodology using which a family of estimators can be obtained and the family can be used to rederive the minimum variance estimators for varying assumptions of Brownian motion as showcased by various authors in the past. Using the methodology, we were able to improve upon the two cases: Best minimum variance estimator for security price modelled as Brownian motion with no drift and no jumps which uses only high and low-price data. (Case of Parkinson) Result:

$${a_{PMV}}^{u^2} = 0.2401973 \quad {a_{PMV}}^{d^2} = 0.6093938 \quad {a_{PMV}}^{ud} = -1.1061674$$

Best minimum variance estimator for security price modelled as Brownian motion with no drift but non-zero jumps using high, low, opening and closing prices. Result:

$${a_{BE}}^{u^2} = 0.54704 \quad {a_{BE}}^{d^2} = 0.54704 \quad {a_{BE}}^{c^2} = -0.02303 \quad {a_{BE}}^{ud} = -1.47413$$

$${a_{BE}}^{uc} = -0.363654 \quad {a_{BE}}^{dc} = 0.743706 \quad {a_{BE}}^{o^2} = 1$$

This methodology for calculating minimum variance estimator can be extended for ”n assets case” as well. Also, the methodology can be tested on security price following more realistic

models which can become another research question.
(Yang & Zhang, 2000) (Garman & Klass, 1980)(Meilijson, 2008)(Parkinson, 1980)(Rogers & Satchell, 1991)

References

Garman, M. B., & Klass, M. J. (1980). On the estimation of security price volatilities from historical data. *Journal of business*, 67–78.

Meilijson, I. (2008). The garman-klass volatility estimator revisited. *arXiv preprint arXiv:0807.3492*.

Parkinson, M. (1980). The extreme value method for estimating the variance of the rate of return. *Journal of business*, 61–65.

Rogers, L. C. G., & Satchell, S. E. (1991). Estimating variance from high, low and closing prices. *The Annals of Applied Probability*, 504–512.

Yang, D., & Zhang, Q. (2000). Drift-independent volatility estimation based on high, low, open, and close prices. *The Journal of Business*, *73*(3), 477–492.