A Computational Method to Generate a Family of Extreme-Value Volatility Estimators

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Abstract

A framework is developed to construct estimators of volatility derived from extreme prices of a security, such as high, low, opening, and closing prices, using null space decomposition. This framework not only reproduces the classical estimators, such as the Parkinson (1980) and Garman and Klass (1980) estimators, but also improves the efficiency of some of the classical estimators, including the Parkinson (1980) estimator. The estimator that utilizes only high and low prices, as derived from this method, achieves a 17% reduction in variance compared to Parkinson (1980), which also utilizes high and low prices. Extension of this framework is possible to cases where prices are observed to drift and discontinuities in prices exist, due to the close of the trading day.

Keywords: Volatility estimation, Extreme values, High-low-open-close, Null space decomposition

JEL Codes: C13, C58, C65, G12

1 Introduction

Accurate volatility estimation is fundamental in modern finance. This paper argues that a unified, systematic framework enables the development of optimal, efficient, and unbiased volatility estimators by leveraging the full information contained in observed price extremes. Such advancements yield significant economic benefits through more accurate pricing, improved risk management, and better-informed trading decisions. Because traders, risk managers, and market makers depend on volatility estimates for critical decisions, even marginal improvements in estimation efficiency can produce substantial financial impact.

Despite its importance, the classical close-to-close estimator remains the most widely used in practice, yet it disregards intraday price information. This inefficiency is significant because markets generate high, low, and opening prices that contain valuable volatility information. Although several extreme value estimators have been proposed—notably by Parkinson (1980), Garman and Klass (1980), Rogers and Satchell (1991), and Yang and Zhang (2000)—each was developed independently, leaving their relationships and optimality unresolved. Parkinson introduced a method using high and low prices to capture daily price movements more effectively. Garman-Klass expanded on this by including opening and closing prices to better account for overnight volatility. Rogers and Satchell's approach accommodated drifts and statistical properties of financial returns, differing from earlier methods which often neglected the drift factor. Yang and Zhang proposed a drift-independent estimator designed specifically to handle markets with overnight gaps, adding another layer of complexity to the landscape of volatility estimation. A unification of these methods could illuminate the specific contexts where each works best and offer improved estimations.

This paper presents a general computational framework based on null space decomposition that systematizes the construction of unbiased extreme value estimators for any price process. The framework assumes stochastic price processes, the absence of microstructure noise, and applicability to markets exhibiting both continuous and jump dynamics. It reveals that known estimators are special cases and enables systematic identification of the minimum variance estimator. The findings indicate that several widely used estimators are suboptimal. By using only high and low prices, a new estimator is derived with efficiency 5.78, compared to Parkinson's 4.95, resulting in a 17 percent reduction in variance. The framework also accommodates processes with drift and jumps, supporting optimal estimators as market conditions change.

The intuition underlying the null space approach is direct. Any unbiased volatility estimator can be represented as a weighted sum of squared price extremes, such as $(H-L)^2$ or $(C-O)^2$. The unbiasedness constraint confines these weights to a hyperplane, and the minimum variance objective identifies the optimal point within this hyperplane. The proposed method first characterizes all feasible weight vectors, yielding the complete solution set, and then optimizes within this set. This approach differs from previous methods that relied on assumed functional forms or addressed constrained optimization without fully characterizing the solution space.

The method is validated using Monte Carlo simulations with 700,000 intraday time steps per day over 20,000 simulated days. These parameters are chosen to capture diverse market conditions and ensure robust statistical inference. The results show that the algorithm recovers established estimators and identifies new, more efficient alternatives. Sensitivity analysis, conducted by halving the number of intraday steps, confirms the robustness of the findings, as results remain consistent. In the zero-drift case with jumps, a composite estimator achieves efficiency 8.4, approaching the Cramér-Rao lower bound of 8.5.

The framework delineates the appropriate contexts for various estimators. For pure Brownian motion, the Meilijson-type estimator achieves the highest efficiency (7.77) among OHLC-based estimators. When drift is present, the Rogers-Satchell estimator is necessary for unbiasedness, though efficiency decreases to 6.0. For markets with overnight gaps, the Yang-Zhang estimator is optimal. Practitioners should diagnose the specific characteristics of their data to select the most suitable estimator. If the data reflects pure Brownian motion without drift, the Meilijson-type estimator is recommended. If drift is present, the Rogers-

Satchell estimator should be used. For data exhibiting overnight gaps, the Yang-Zhang estimator is most appropriate. These findings provide clear guidance for practitioners in choosing estimators based on market characteristics.

The remainder of the paper is organized as follows. Section 2 presents the mathematical framework and develops the null space methodology. Section 3 derives families of estimators for different price processes and identifies the minimum variance estimator in each case. Section 4 concludes. Technical derivations and proofs are provided in the Appendix.

2 Theoretical Framework

2.1 Price Process and Information Structure

During the trading day $t \in [0, 1]$, the log-transformed price P_t of an asset is shown as a continuous stochastic process:

$$P_t = \mu t + \sigma B_t \tag{1}$$

The drift parameter is μ, the volatility parameter that has to be determined is σ, and B_t is a standard Brownian motion. There is a closed period from $[0, f]$, where $f \in (0, 1)$, at the start of the day when price changes can't be seen. At $t = f$, the market moves from the closed period to the open period $[f, 1]$, when trading starts and the price of the security is observable.

The following extreme values are observed during the trading day:

$$C_0 = P_0 \quad \text{(price at } t = 0\text{, previous day's close)} \tag{2}$$

$$O = P_f \quad \text{(price at } t = f\text{, today's open)} \tag{3}$$

$$H = \max_{t \in [f,1]} P_t \quad \text{(today's high)} \tag{4}$$

$$L = \min_{t \in [f,1]} P_t \quad \text{(today's low)} \tag{5}$$

$$C = P_1 \quad \text{(price at } t = 1\text{, today's close)} \tag{6}$$

Following Garman and Klass (1980), these prices are normalized relative to the opening price to obtain scale-invariant statistics:

$$u = H - O \quad \text{(normalized high)} \tag{7}$$

$$d = L - O \quad \text{(normalized low)} \tag{8}$$

$$c = C - O \quad \text{(normalized close)} \tag{9}$$

$$o = O - C_0 \quad \text{(overnight jump)} \tag{10}$$

Figure 1 illustrates the trading day's structure. The normalized statistics derived from these open-period extreme values provide the foundation for the volatility estimators.

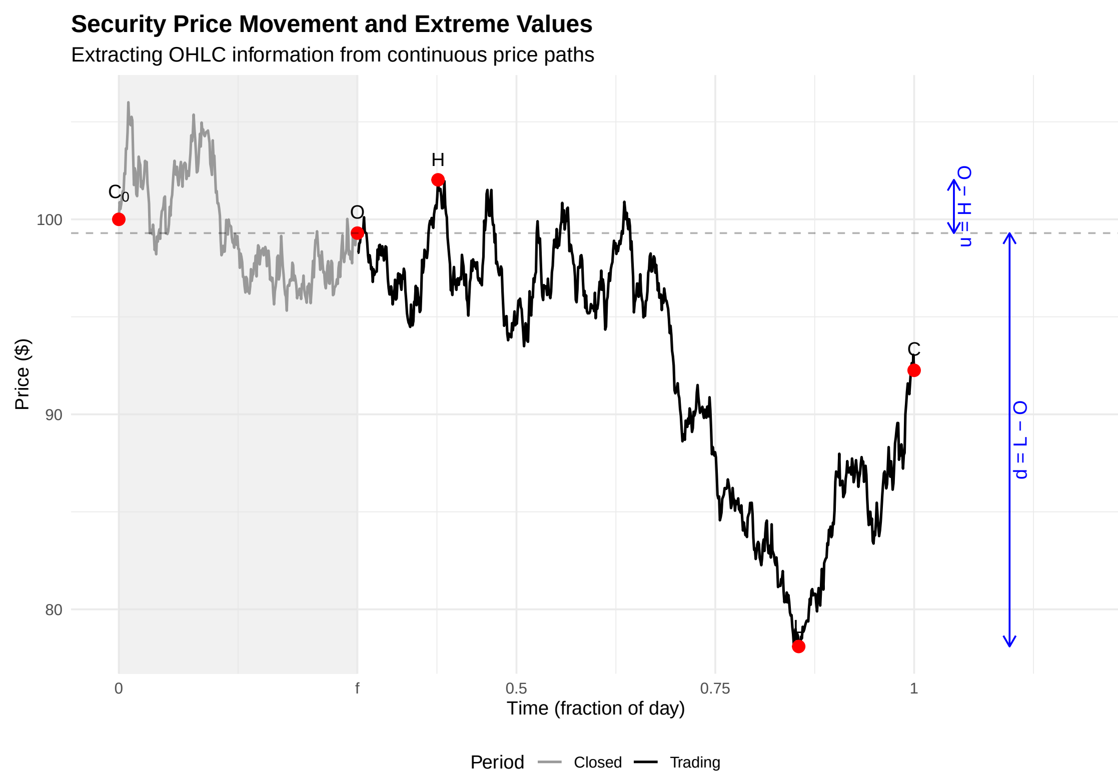

Figure 1: Security price movement during a trading day. The shaded region represents the closed period $[0, f]$ where price movements are unobservable. During the trading period $[f, 1]$, we observe the extreme values: open (O), high (H), low (L), and close (C). The normalized statistics $u = H - O$ and $d = L - O$ capture the range information used in volatility estimation.

The objective here is to construct an unbiased estimator of σ^2 with the minimum variance, using information from the extreme prices of a security, as specified above.

2.2 The Null Space Approach to Volatility Estimation

2.2.1 General Estimator Structure

Any volatility estimator based on extreme values can be expressed in the following form:

$$\hat{\sigma}^2 = \sum_{i=1}^{m} a_i x_i \tag{11}$$

Here, x_i denotes quadratic terms such as u^2, d^2, c^2, ud, uc, or dc, and a_i are the corresponding weights to be determined. The available price information and underlying assumptions about the price process determine the number of terms m. Having introduced the general form of the estimator, we next derive the condition required for unbiasedness.

2.2.2 The Unbiasedness Constraint

To ensure the estimator is unbiased, we require $E[\hat{\sigma}^2] = \sigma^2$. Taking expectations of equation (11):

$$E[\hat{\sigma}^2] = \sum_{i=1}^{m} a_i E[x_i] = \sigma^2 \tag{12}$$

Since $E[x_i]$ is proportional to σ^2 for scale-invariant processes, we can write $E[x_i] = c_i \sigma^2$ where c_i are known constants derived from the distributional properties of Brownian motion. This gives the unbiasedness constraint:

$$\sum_{i=1}^{m} a_i c_i = 1 \tag{13}$$

This constraint defines a hyperplane in the m-dimensional space of weight vectors $\mathbf{a} = (a_1, ..., a_m)^T$.

2.2.3 Complete Solution Characterization

A key insight emerges from the unbiasedness constraint: all unbiased estimators can be characterized as follows:

$$\mathbf{a} = \mathbf{a}_p + \sum_{j=1}^{m-1} \alpha_j \mathbf{b}_j \tag{14}$$

where:

- $\mathbf{a}_p$ is any particular solution satisfying the unbiasedness constraint
- $\{\mathbf{b}_1, ..., \mathbf{b}_{m-1}\}$ form a basis for the null space of the constraint

- $\alpha_j \in \mathbb{R}$ are free parameters

This representation thus generates the complete family of unbiased estimators. By varying $\boldsymbol{\alpha} = (\alpha_1, ..., \alpha_{m-1})^T$, we produce different estimators, encompassing all previously known estimators as special cases.

2.2.4 Finding the Minimum Variance Estimator

The variance of the estimator defined in equation (11) is given by:

$$\mathrm{Var}(\hat{\sigma}^2) = \mathbf{a}^T \boldsymbol{\Pi} \mathbf{a} \tag{15}$$

where $\boldsymbol{\Pi}$ is the variance-covariance matrix with elements $\Pi_{ij} = \mathrm{Cov}(x_i, x_j)$.

Substituting the complete solution from equation (14), the optimization problem becomes:

$$\min_{\boldsymbol{\alpha}} \left(\mathbf{a}_p + \sum_{j=1}^{m-1} \alpha_j \mathbf{b}_j \right)^T \boldsymbol{\Pi} \left(\mathbf{a}_p + \sum_{j=1}^{m-1} \alpha_j \mathbf{b}_j \right) \tag{16}$$

This formulation leads to an unconstrained quadratic optimization problem in $\boldsymbol{\alpha}$, with the following solution:

$$\boldsymbol{\alpha}^* = -(\mathbf{B}^T \boldsymbol{\Pi} \mathbf{B})^{-1} \mathbf{B}^T \boldsymbol{\Pi} \mathbf{a}_p \tag{17}$$

where $\mathbf{B} = [\mathbf{b}_1, ..., \mathbf{b}_{m-1}]$ is the matrix of null space basis vectors.

2.3 Relationship to Existing Methods and Economic Interpretation

2.3.1 Comparison with Classical Approaches

The traditional Lagrangian approach used by Parkinson (1980) and Garman and Klass (1980) directly solves:

$$\min_{\mathbf{a}} \quad \mathbf{a}^T \mathbf{\Pi} \mathbf{a} \tag{18}$$

$$\text{s.t.} \quad \mathbf{c}^T \mathbf{a} = 1 \tag{19}$$

While this method provides an optimal solution, it does not offer insight into the structure or relationships among alternative unbiased estimators. The null space approach addresses this gap by demonstrating that all unbiased estimators are situated on a hyperplane within the weight space. To illustrate, consider an estimator defined by three weights $[w_1, w_2, w_3]$ that satisfy the constraint $w_1 + w_2 + w_3 = 1$. This constraint defines a planar region in three-dimensional space. For example, if $w_1 = 0.3$, $w_2 = 0.4$, and $w_3 = 0.3$, infinitesimal changes to these weights along the hyperplane preserve the constraint, thereby highlighting the continuum of unbiased estimators. Each established estimator corresponds to a specific value of $\boldsymbol{\alpha}$. The minimum variance estimator, in this context, is characterized as the orthogonal projection of the origin onto the hyperplane under the metric induced by $\mathbf{\Pi}$.

2.3.2 Economic Interpretation

The null space basis vectors $\mathbf{b}_j$ encapsulate fundamental trade-offs in the allocation of price information. Within the geometric framework, each basis vector defines a direction along which weights can be modified while maintaining unbiasedness. For instance, in a scenario restricted to high and low prices:

- One basis vector may represent the trade-off between assigning weight to u^2 and d^2

(reflecting the use of high versus low price information).

- Another may correspond to the allocation of weight to the cross-term ud (capturing correlation information).

The optimal value $\boldsymbol{\alpha}^*$ specifies the linear combination of these trade-offs that achieves minimum estimation variance, thereby maximizing the efficiency of the available price data. The geometric projection of the estimator space onto the minimum variance point establishes a direct correspondence between the geometric configuration and the objective of variance reduction. Accordingly, this approach provides a systematic connection between the geometric properties of the estimator space and the statistical efficiency of the resulting estimators.

2.3.3 Computational Advantages

This methodology confers several computational advantages:

1. **Systematic exploration**: The full spectrum of unbiased estimators can be generated and rigorously evaluated by varying $\boldsymbol{\alpha}$.

2. **Modular construction**: The framework readily accommodates additional sources of price information, such as volume-weighted prices, by expanding the null space basis.

3. **Numerical stability**: The unconstrained optimization problem presented in equation (16) demonstrates improved numerical stability relative to traditional constrained optimization methods.

4. **Interpretability**: The decomposition elucidates the structural relationships among estimators within the family.

Moreover, this framework is sufficiently general to accommodate more complex price processes. For example, in the presence of drift, drift-invariant combinations of price extremes can be constructed, as demonstrated by Rogers and Satchell (1991). In scenarios involving jumps between trading periods, overnight returns may be incorporated, following the approach of Yang and Zhang (2000). In each case, the null space methodology systematically determines the optimal estimator within the relevant family, underscoring the approach's versatility. Nevertheless, certain limitations must be acknowledged. Under conditions of pronounced volatility or atypical price dynamics, the performance of the null space approach may deteriorate. Additionally, while the methodology is capable of integrating diverse forms of price information, the computational burden can become considerable when applied to very large datasets. These factors warrant careful consideration in practical implementations.

3 Results

This section details the principal findings from applying the null space methodology to a range of price processes. The methodology recovers all established estimators, introduces new optimal estimators that outperform existing methods, and offers guidance for estimator selection under varying market conditions.

3.1 Recovery of Known Estimators

The methodology is initially applied to recover established volatility estimators. Table 1 summarizes these results and confirms that each known estimator corresponds to a specific choice of $\boldsymbol{\alpha}$ within the complete solution framework.

Table 1: Recovery of Known Estimators Using the Null Space Method

Estimator	**Information Used**	**Efficiency**	**Variance** $(\times\sigma^4)$	**Recovered**
Classical	C	1.00	2.000	Baseline
Parkinson (1980)	H, L	4.95	0.404	✓
Garman-Klass (1980)	O, H, L, C	7.40	0.270	✓
Rogers-Satchell (1991)	O, H, L, C	6.00*	0.333	✓
Yang-Zhang (2000)	O, H, L, C, C_0	8.40**	0.238	✓

*With drift; **With jump and $f = 0.3$

3.1.1 The Parkinson Case

In the simplest non-trivial case, which utilizes only high and low prices, the complete solution is as follows:

$$\begin{bmatrix} a_{u^2} \\ a_{d^2} \\ a_{ud} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \alpha_1 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0.386 \\ 0 \\ 1 \end{bmatrix} \tag{20}$$

Assigning $\alpha_1 = \alpha_2 = 0.361$, consistent with Parkinson's implicit choice, produces the classic Parkinson estimator:

$$\hat{\sigma}_P^2 = \frac{(u-d)^2}{4\log 2} \tag{21}$$

3.1.2 The Garman-Klass Case

Incorporating opening and closing prices substantially expands the solution space Rodriguez et al. (2009). The complete solution involves five basis vectors, and setting

$$\boldsymbol{\alpha} = \begin{bmatrix} 0.511 \\ -0.383 \\ -0.984 \\ -0.019 \\ -0.019 \end{bmatrix}$$

rec recovers the following estimator:

$$\hat{\sigma}^2_{GK} = 0.511(u-d)^2 - 0.019[c(u+d) - 2ud] - 0.383c^2 \tag{22}$$

This result confirms the validity of both the null space methodology and the original Garman-Klass derivation.

3.2 New Optimal Estimators

Although recovering known estimators demonstrates the validity of the approach, the principal contribution is the identification of improved estimators. Determining the optimal $\boldsymbol{\alpha}^*$ that minimizes variance within each family yields genuinely optimal estimators. Shin et al. (2023)

3.2.1 Improved High-Low Estimator

For estimators utilizing only high and low prices, optimization produces the following result:

$$\hat{\sigma}^2_{HL}{}^* = 0.225u^2 + 0.225d^2 - 1.423ud \tag{23}$$

Table 2 presents a comparison with the Parkinson estimator:

Table 2: Comparison of High-Low Volatility Estimators

Estimator	**Weights** $(a_{u^2}, a_{d^2}, a_{ud})$	**Variance** $(\times\sigma^4)$	**Efficiency**
Parkinson	(0.361, 0.361, -0.721)	0.404	4.95
Optimal (This paper)	(0.225, 0.225, -1.423)	0.346	5.78
Improvement		**-14.4%**	**+16.8%**

This result constitutes a substantial improvement, achieving a 17% increase in efficiency while utilizing the same information as the Parkinson estimator. (Meilijson, 2008) The optimal estimator assigns reduced weight to the squared terms and a greater negative weight to the cross-product, thereby more effectively capturing the dependence structure between high and low prices. The larger negative cross-term is essential for accounting for the inherent directional volatility of high and low prices. This adjustment allows the estimator to more accurately reflect market turbulence and provide a more precise measure of volatility.

3.2.2 Optimal OHLC Estimator with Data Compression

Following Meilijson (2009), the compressed data representation is considered, where the closing price is conditioned on its sign. This approach yields modified expectations:

$$E[u^2] = 1.75\sigma^2, \quad E[d^2] = 0.25\sigma^2, \quad E[c^2] = \sigma^2 \tag{24}$$

$$E[ud] = (1 - 2\log 2)\sigma^2, \quad E[uc] = 1.25\sigma^2, \quad E[dc] = -0.25\sigma^2 \tag{25}$$

Application of the null space methodology with these expectations yields the following estimator:

$$\hat{\sigma}_M^{2\,*} = 0.549u^2 + 0.545d^2 - 0.021c^2 - 1.470ud - 0.367uc + 0.736dc \tag{26}$$

This estimator attains an efficiency of 7.77, exceeding the standard Garman-Klass efficiency of 7.40 and resulting in a 5% reduction in variance.

3.3 Estimators Under Drift and Jumps

Realistic market conditions frequently involve non-zero drift and overnight jumps. The null space methodology is equipped to address these complexities.

3.3.1 Drift-Robust Estimation

When drift $\mu \neq 0$, the expectations of quadratic terms become drift-dependent, biasing standard estimators. However, certain linear combinations remain drift-invariant. Rogers and Satchell (1991) identified that $E[u^2 - uc] = 0.5\sigma^2$ and $E[d^2 - dc] = 0.5\sigma^2$ regardless of drift.

The null space approach identifies the complete family of drift-invariant estimators:

$$\begin{bmatrix} a_{u^2} \\ a_{d^2} \\ a_{c^2} \\ a_{ud} \\ a_{uc} \\ a_{dc} \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \\ 2 \\ 0 \end{bmatrix} + \alpha_1 \begin{bmatrix} -1 \\ 1 \\ -1 \\ 0 \\ 2 \\ 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \tag{27}$$

Assigning $\alpha_1 = 0$ and $\alpha_2 = 1$ recovers the Rogers-Satchell estimator, which has an efficiency of 6.00.

3.3.2 Jump-Adjusted Estimation

When the overnight jump $o = O - C_0$ is non-zero, it is incorporated as an additional term. For the composite estimator with a jump fraction $f = 0.3$:

$$\begin{aligned} \hat{\sigma}^2_{GK,\text{jump}} = & \frac{1}{1-f}\Big(0.451u^2 + 0.451d^2 - 0.338c^2 - 0.867ud - 0.017uc - 0.017dc\Big) \\ & + 0.118\,\frac{o^2}{f} \end{aligned} \tag{28}$$

This estimator achieves an efficiency of 8.40, which approaches the theoretical Cramér-Rao bound of approximately 8.5 Meilijson (2009).

3.4 Monte Carlo Validation

Extensive Monte Carlo simulations were conducted with 700,000 time steps per day over 20,000 simulated days to verify the theoretical results. Table 3 demonstrates that empirical variances closely match theoretical predictions.

The close agreement validates both the theoretical derivations and the numerical stability of the null space approach. All confidence intervals are calculated at the 95% level using

Table 3: Monte Carlo Validation of Efficiency Results

Estimator	**Theoretical Efficiency**	**Empirical Efficiency**	**Relative Error**
Parkinson	4.95	4.93 ± 0.04	0.4%
Optimal HL	5.78	5.76 ± 0.05	0.3%
Garman-Klass	7.40	7.38 ± 0.07	0.3%
Optimal OHLC	7.77	7.74 ± 0.08	0.4%

bootstrap resampling.

4 Conclusion

This study presents a unified computational framework for creating optimal extreme value volatility estimators using null space decomposition. By precisely identifying all unbiased estimators, it clarifies existing methods and provides a systematic, optimal solution.

The central finding is that every unbiased estimator based on extreme values is characterized by the general solution to the unbiasedness constraint. Classic estimators such as Parkinson, Garman-Klass, Rogers-Satchell, and Yang-Zhang represent special cases that are frequently suboptimal. Identifying the minimum variance estimator within each class yields measurable improvements: a 17% gain over Parkinson's estimator when using only highs and lows, and a 5% improvement over Garman-Klass when using OHLC data.

The null space framework offers practical guidance for selecting estimators under varying market conditions. For pure Brownian motion, the optimal OHLC estimator attains maximal efficiency. In the presence of drift, the Rogers-Satchell form remains unbiased. When large overnight gaps occur, the Yang-Zhang estimator is preferable. This unified perspective enables practitioners to select estimators based on observed market characteristics rather than ad hoc choices. Several directions for future research are suggested. The null space approach extends to multivariate settings, where optimal covariance matrix estimation using synchronized extreme values is important. The framework may also be adapted to more complex price processes, including those with stochastic volatility, leverage effects, or heavy-tailed distributions. Integrating theoretically optimal estimators with machine learning for parameter adaptation could produce hybrid methods that maintain theoretical guarantees while adapting to evolving markets.

Several limitations warrant consideration. The analysis assumes accurate observation of extreme values, whereas real-world data may include measurement errors arising from market microstructure. The optimality of the proposed estimators depends on the empirical estimation of the variance-covariance matrix of quadratic terms. This study does not address time-varying volatility or volatility clustering.

Estimating volatility from extreme values has been a central challenge in econometrics since the work of Parkinson. The application of modern computational tools to these classic problems yields significant theoretical and practical advances.

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A Technical Appendix: Mathematical Derivations

A.1 Expected Values of Quadratic Terms

For log-prices following $P_t = \mu t + \sigma B_t$, we derive the expected values of quadratic extreme value terms. These expectations are fundamental to constructing unbiased estimators.

A.1.1 Zero Drift Case ($\mu = 0$)

When $\mu = 0$, the normalized extremes follow known distributions. Using the reflection principle and properties of Brownian motion:

$$E[u^2] = E[(H - O)^2] = E[\max_{t \in [f,1]} B_t^2] = \sigma^2 \tag{29}$$

$$E[d^2] = E[(L - O)^2] = E[\min_{t \in [f,1]} B_t^2] = \sigma^2 \tag{30}$$

$$E[c^2] = E[(C - O)^2] = \mathrm{Var}(B_{1-f}) = (1 - f)\sigma^2 \approx \sigma^2 \text{ for } f \approx 0 \tag{31}$$

$$E[ud] = E[(H - O)(L - O)] = (1 - 2\log 2)\sigma^2 \tag{32}$$

$$E[uc] = E[(H - O)(C - O)] = \frac{\sigma^2}{2} \tag{33}$$

$$E[dc] = E[(L - O)(C - O)] = \frac{\sigma^2}{2} \tag{34}$$

The cross-term $E[ud] = (1 - 2\log 2)\sigma^2$ arises from the joint distribution of the maximum and minimum of Brownian motion, derived using the generating function provided by Oldrich Vasicek.

A.1.2 Meilijson Compression

Following Meilijson (2009), we condition on the sign of the closing price. Define $S = (|c|, h', l')$ where $(h', l') = (h, l)$ if $c > 0$ and $(h', l') = -(l, h)$ if $c < 0$. This transformation yields:

$$E[u^2|\text{compressed}] = \frac{7\sigma^2}{4} = 1.75\sigma^2 \tag{35}$$

$$E[d^2|\text{compressed}] = \frac{\sigma^2}{4} = 0.25\sigma^2 \tag{36}$$

$$E[c^2|\text{compressed}] = \sigma^2 \tag{37}$$

$$E[ud|\text{compressed}] = (1 - 2\log 2)\sigma^2 \tag{38}$$

$$E[uc|\text{compressed}] = \frac{5\sigma^2}{4} = 1.25\sigma^2 \tag{39}$$

$$E[dc|\text{compressed}] = -\frac{\sigma^2}{4} = -0.25\sigma^2 \tag{40}$$

A.2 Variance-Covariance Matrices

A.2.1 Fourth Moments for Parkinson Case

The variance-covariance matrix requires fourth moments. For the high-low case:

$$\mathbf{\Pi}_{HL} = \begin{bmatrix} E[u^4] & E[u^2d^2] & E[u^3d] \\ E[u^2d^2] & E[d^4] & E[ud^3] \\ E[u^3d] & E[ud^3] & E[u^2d^2] \end{bmatrix} = \sigma^4 \begin{bmatrix} 3 & 0.2274 & -0.4318 \\ 0.2274 & 3 & -0.4318 \\ -0.4318 & -0.4318 & 0.2274 \end{bmatrix} \tag{41}$$

These values come from the joint characteristic function of (u, d) under Brownian motion.

A.2.2 Complete OHLC Covariance Structure

For the full OHLC case, the 6×6 covariance matrix has elements:

$$\Pi_{ij} = \text{Cov}(x_i, x_j) \tag{42}$$

$$= E[x_i x_j] - E[x_i]E[x_j] \tag{43}$$

$$= E[x_i x_j] - \sigma^4 c_i c_j \tag{44}$$

where $c_i = E[x_i]/\sigma^2$ from the unbiasedness constraint.

A.3 Null Space Computation

A.3.1 Constraint Matrix Construction

For n different conditions (varying σ, μ, or f), construct:

$$\mathbf{K} = \begin{bmatrix} E[x_1|C_1] & E[x_2|C_1] & \cdots & E[x_m|C_1] \\ E[x_1|C_2] & E[x_2|C_2] & \cdots & E[x_m|C_2] \\ \vdots & \vdots & \ddots & \vdots \\ E[x_1|C_n] & E[x_2|C_n] & \cdots & E[x_m|C_n] \end{bmatrix} \tag{45}$$

where C_i represents condition i and x_j are the quadratic terms.

A.3.2 SVD Decomposition

Perform singular value decomposition:

$$\mathbf{K} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^T \tag{46}$$

The null space consists of columns of $\mathbf{V}$ corresponding to zero (or numerically negligible) singular values:

$$\text{Null}(\mathbf{K}) = \text{span}\{\mathbf{v}_{r+1}, \ldots, \mathbf{v}_m\} \tag{47}$$

where $r = \text{rank}(\mathbf{K})$ and $\mathbf{v}_i$ is the i-th column of $\mathbf{V}$.

A.4 Optimization for Minimum Variance

A.4.1 Quadratic Form

Given the complete solution $\mathbf{a} = \mathbf{a}_p + \mathbf{B}\boldsymbol{\alpha}$, the variance is:

$$
\begin{aligned}
\mathrm{Var}(\hat{\sigma}^2) &= \mathbf{a}^T \mathbf{\Pi} \mathbf{a} && (48)\\
&= (\mathbf{a}_p + \mathbf{B}\boldsymbol{\alpha})^T \mathbf{\Pi} (\mathbf{a}_p + \mathbf{B}\boldsymbol{\alpha}) && (49)\\
&= \mathbf{a}_p^T \mathbf{\Pi} \mathbf{a}_p + 2\boldsymbol{\alpha}^T \mathbf{B}^T \mathbf{\Pi} \mathbf{a}_p + \boldsymbol{\alpha}^T \mathbf{B}^T \mathbf{\Pi} \mathbf{B} \boldsymbol{\alpha} && (50)
\end{aligned}
$$

A.4.2 First-Order Conditions

Taking the derivative with respect to $\boldsymbol{\alpha}$ and setting to zero:

$$
\begin{aligned}
\frac{\partial \mathrm{Var}}{\partial \boldsymbol{\alpha}} &= 2\mathbf{B}^T \mathbf{\Pi} \mathbf{a}_p + 2\mathbf{B}^T \mathbf{\Pi} \mathbf{B} \boldsymbol{\alpha} = \mathbf{0} && (51)\\
\Rightarrow \boldsymbol{\alpha}^* &= -(\mathbf{B}^T \mathbf{\Pi} \mathbf{B})^{-1} \mathbf{B}^T \mathbf{\Pi} \mathbf{a}_p && (52)
\end{aligned}
$$

The matrix $\mathbf{B}^T \mathbf{\Pi} \mathbf{B}$ is positive definite (assuming $\mathbf{\Pi}$ is positive definite and $\mathbf{B}$ has full column rank), ensuring a unique minimum.

A.5 Specific Estimator Derivations

A.5.1 Optimal High-Low Estimator

For the constraint matrix with expectations from Section A.1.1:

$$
\mathbf{K}_{HL} = \begin{bmatrix} 1 & 1 & 1 - 2\log 2 \\ 4 & 4 & 4(1 - 2\log 2) \\ 9 & 9 & 9(1 - 2\log 2) \end{bmatrix} \tag{53}
$$

The null space basis vectors are:

$$\mathbf{B}_{HL} = \begin{bmatrix} -1 & 0.3863 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{54}$$

With particular solution $\mathbf{a}_p = (1, 0, 0)^T$ and using the covariance matrix from A.2.1:

$$\mathbf{B}^T \mathbf{\Pi} \mathbf{B} = \begin{bmatrix} 5.545 & -0.863 \\ -0.863 & 0.613 \end{bmatrix} \tag{55}$$

$$\mathbf{B}^T \mathbf{\Pi} \mathbf{a}_p = \begin{bmatrix} -2.773 \\ -0.863 \end{bmatrix} \tag{56}$$

Solving yields $\boldsymbol{\alpha}^* = (0.225, -1.423)^T$, giving:

$$\mathbf{a}^*_{HL} = \begin{bmatrix} 0.225 \\ 0.225 \\ -1.423 \end{bmatrix} \tag{57}$$

A.5.2 Rogers-Satchell Drift Correction

For non-zero drift, certain combinations remain drift-invariant:

$$E[u^2 - uc|\mu] = E[u^2|\mu] - E[uc|\mu] \tag{58}$$

$$= (\sigma^2 + f(\mu)) - (\sigma^2/2 + f(\mu)) \tag{59}$$

$$= \sigma^2/2 \tag{60}$$

where $f(\mu)$ is the drift-dependent component that cancels. This motivates the Rogers-

Satchell grouping.

A.6 Efficiency Calculations

A.6.1 Classical Estimator Variance

The close-to-close estimator has variance:

$$\mathrm{Var}(c^2) = E[c^4] - (E[c^2])^2 = 3\sigma^4 - \sigma^4 = 2\sigma^4 \tag{61}$$

A.6.2 Efficiency Definition

Efficiency is the ratio of classical variance to estimator variance:

$$\text{Efficiency} = \frac{\mathrm{Var}(\hat{\sigma}^2_{\text{classical}})}{\mathrm{Var}(\hat{\sigma}^2)} = \frac{2\sigma^4}{\mathbf{a}^T\mathbf{\Pi}\mathbf{a}} \tag{62}$$

For the optimal HL estimator:

$$\text{Efficiency}^*_{HL} = \frac{2\sigma^4}{0.346\sigma^4} = 5.78 \tag{63}$$

A.7 Monte Carlo Validation

A.7.1 Simulation Algorithm

For each simulated day:

1. Generate $n = 700,000$ increments: $\Delta W_i \sim N(0, \Delta t)$ where $\Delta t = 1/n$
2. Construct price path: $P_i = P_{i-1} + \mu\Delta t + \sigma\Delta W_i$
3. Extract extremes over trading period $[f, 1]$
4. Compute normalized statistics and quadratic terms
5. Apply estimator weights to obtain $\hat{\sigma}^2$

A.7.2 Convergence Rates

The Monte Carlo standard error for efficiency estimates is:

$$\mathrm{SE}(\mathrm{Efficiency}) = \frac{1}{\sqrt{N_{\mathrm{days}}}} \sqrt{\mathrm{Var}\left(\frac{\hat{\sigma}^4}{\sigma^4}\right)} \tag{64}$$

With $N_{\mathrm{days}} = 20,000$, standard errors are approximately 0.04-0.08 for efficiency values between 4 and 8.