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Author(s): Mark B. Garman and Michael J. Klass
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Mark B. Garman and Michael J. Klass
University of California, Berkeley

On the Estimation of Security Price Volatilities from Historical Data*

Improved estimators of security price volatilities are formulated. These estimators employ data of the type commonly found in the financial pages of a newspaper: the high, low, opening, and closing prices and the transaction volume. The new estimators are seen to have relative efficiencies that are considerably higher than the standard estimators.

I. Introduction

This paper examines the problem of estimating capital asset price volatility parameters from the most available forms of public data. While many varieties of such data are possible, we shall consider here only those which are truly universal in their accessibility to investors, namely, data appearing in the financial pages of the newspaper. In particular, we shall consider volatility estimators which are based upon the historical opening, closing, high, and low prices and transaction volume. Alternative estimators of volatility may be constructed from such data as significant news events, "fundamental" information regarding a company's prospects, and other forms of publicly available data, but these will not be considered here.

Any parameter-estimation procedure must begin with a maintained hypothesis regarding the structural model within which estimation is to be made. Our structural model is given exposition in Section II. Section III discusses the "classical"

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estimation approach which forms the basis of current practice. In Section IV we introduce some more efficient estimators based upon the high and low prices. "Best" analytic estimators which simultaneously use the high, low, opening, and closing prices are formulated in Section V. Section VI considers the complications raised by trading volume, and Section VII provides a summary.

II. The Structural Model

The maintained model employed herein assumes that security prices are governed by a diffusion process of the form

$$P(t) = \phi(B(t)), \tag{1}$$

where P is the security price, t is time, ϕ is a monotonic time-independent[1] transformation, and $B(t)$ is a diffusion process with the differential representation

$$dB = \sigma dz, \tag{2}$$

where dz is the standard Gauss-Wiener process and σ is an unknown constant to be estimated. This formulation is sufficiently general to cover the usual hypothesis involving the geometric Brownian motion of stock prices, as well as some of the more recently proposed alternatives to the geometric hypothesis (Cox and Ross 1975). Throughout the remainder of this paper, it shall always be understood that we are dealing with the *transformed* price series $B = \phi^{-1}P$. Thus "price" would mean "logarithm of original price," and "volatility" would mean "variance of the logarithm of original prices," etc., in the case of the geometric Brownian motion hypothesis; the usage will be analogous for other hypotheses possessing other transformations.

Naturally, there are limitations to our maintained model. First, we are essentially considering each security in isolation, ignoring the covariation thought to exist among securities in various asset pricing models (e.g., Sharpe 1970). Second, only one parameter is to be estimated; simultaneous estimation of other unknown parameters, for example, the "drift," is not treated here. Third, the required form of ϕ rules out a significant number of alternative diffusion processes, including many having arbitrary nonzero drift, even when this drift is known. Fortunately, most of the foregoing difficulties tend to vanish as we shorten the interval over which estimation is made.[2] Finally, dividends and other discrete capital payouts are neglected, since these violate the continuous nature of the assumed diffusion sample paths.

1. Monotonicity and time independence are employed to assure that the same set of time points generates the maximum and minimum values of B and P.
2. See Thorpe (1976) for arguments and empirical evidence on this point.

Moreover, the current paper is not concerned with the question of whether the maintained model is the "correct" model of asset price fluctuations. Such a study has been an ongoing subject with many authors over many years, and we certainly could not aspire to settle this complex issue here. Rather, our purposes are to develop the estimation consequences of the model, given the data restrictions described earlier.

III. "Classical" Estimation

Under the maintained model, (transformed) price changes over any time interval are normally distributed with mean zero and variance proportional to the length of the interval. Moreover, the prices will always exhibit continuous sample paths. Yet we will not assume that these paths may be everywhere observed. There are at least two factors that interfere with our abilities to continuously observe prices: the first is the fact that transactions often occur only at discrete points in time;[3] the second is that stock exchanges are normally closed during certain periods of time. Our maintained model assumes that the continuous Brownian motion of (2) is followed during periods between transactions and during periods of exchange closure, even though prices cannot be observed in such intervals.

As a matter of choice, we shall concentrate herein on estimators of the variance parameter σ^2 of $B(t)$. Any such choice of estimation parameter will have disadvantages in some contexts.[4] Since such bias typically vanishes with increasing sample size and is usually small relative to the other sources of error, we shall ignore this issue to concentrate upon the estimation of σ^2 alone.

Moreover, it is convenient to think of the interval $t \in [0, 1]$ as representing 1 trading day, since this will prove to be a satisfactory paradigm for the problems of weekly and monthly data also. Our "day" will be divided into two portions, an initial period when the market is closed, followed immediately by a trading period. Figure 1 shows this diagrammatically.

In figure 1 trading is closed initially, starting with yesterday's closing with price C_0. The price sample path is then unobservable until trading opens, at time f and price O_1. In the interval $[f, 1]$ we shall assume (ignoring transaction volume for the moment) that the entire sample path is continuously monitored, having a high value of H_1, a low value of L_1, and a closing value of C_1. (The effects of monitoring at discrete transactions will be considered later.) We adopt notation as follows:

3. See Garman (1976) for a treatment of such a model.

4. As Boyle and Ananthanarayan (1977) have recently observed; any estimation procedure for σ^2 will produce bias in the estimation of any nonlinear function of σ^2, their example being its use in the "option pricing formula" (See Smith [1976] for a comprehensive survey of the option pricing model originated by Black and Scholes [1973].)

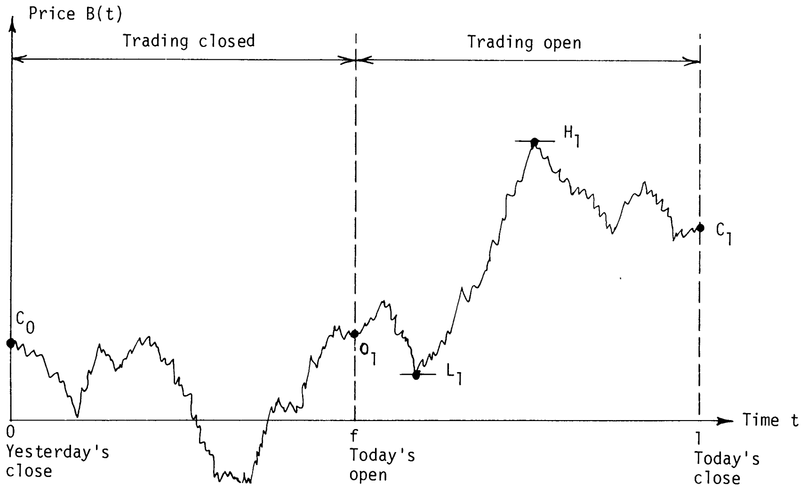

FIG. 1.—Price versus time

σ^2 = unknown variance (volatility) of price change;
f = fraction of the day (interval [0, 1]) that trading is closed;
$C_0 = B(0)$, previous closing price;
$O_1 = B(f)$, today's opening price;
$H_1 = \max_{f \leq t \leq 1} B(t)$, today's high;
$L_1 = \min_{f \leq t \leq 1} B(t)$, today's low;
$C_1 = B(1)$, today's close;
$u = H_1 - O_1$, the normalized high;
$d = L_1 - O_1$, the normalized low;
$c = C_1 - O_1$, the normalized close;
$g(u, d, c; \sigma^2)$ = the joint density of (u, d, c) given σ^2 and $f = 0$.

The classical estimation procedure employs

$$\hat{\sigma}_0^2 \equiv (C_1 - C_0)^2$$

as an unbiased estimator of σ^2. The advantages of the classical estimator $\hat{\sigma}_0^2$ are its simplicity of use and its freedom from obvious sources of error or bias. Closing prices are measured in a consistent fashion from period to period, and there is little question about the time interval being spanned by the estimator. Its principal disadvantage is the fact that it ignores other readily available information which may contribute to estimator efficiency. As we shall see, even minor additions to the utilized information can have remarkable impact.

For example, suppose that opening prices are available in addition to the closing prices. In this case, form the estimator

$$\hat{\sigma}_1^2 \equiv \frac{(O_1 - C_0)^2}{2f} + \frac{(C_1 - O_1)^2}{2(1 - f)}, \quad 0 < f < 1. \tag{3}$$

Then $\hat{\sigma}_1^2$ is a "better" estimator, in the sense discussed next.

The classical estimator $\hat{\sigma}_0^2$ will provide the benchmark by which we shall judge all other estimators. Therefore, define the relative efficiency of an arbitrary estimator $\hat{y}$ by the ratio

$$\text{Eff}(\hat{y}) \equiv \frac{\text{var}(\hat{\sigma}_0^2)}{\text{var}(\hat{y})}. \tag{4}$$

Since $\text{var}(\hat{\sigma}_0^2) = 2\sigma^4$ and $\text{var}(\hat{\sigma}_1^2) = \sigma^4$, it follows[5] that $\text{Eff}(\hat{\sigma}_1^2) = 2$, independent of f. Thus we see that, simply by including the opening price in our estimation procedure, we may halve the variance of our volatility estimates, given known f. This point argues strongly for the inclusion of opening prices in a correspondingly useful data base.

The importance of high relative efficiency is obvious, inasmuch as estimates of improved confidence may be contructed from our data bases. Alternatively, investigators may adopt the tactic of purposely restricting data usage to combat unforeseen nonstationarities. For example, suppose a researcher possesses a data base spanning 10 months. If he discovers an estimator having a high relative efficiency, say 10, he might choose to reduce his estimator confidence regions by a factor of $\sqrt{10}$. Alternatively, he might decide to use only 1 month's data and retain the old confidence regions; his reason for doing this would be to use the most recent month's data, since it has presumably more predictive content in the presence of unknown nonstationarities.

IV. High/Low Estimators

High and low prices during the trading interval require continuous monitoring to establish their values. The opening and closing prices, on the other hand, are merely "snapshots" of the process. Intuition would then tell us that high/low prices contain more information regarding volatility than do open/close prices. This intuition is correct, as Parkinson (1976) has recently shown.[6] He assumes $f = 0$ and constructs the estimator

$$\hat{\sigma}_2^2 \equiv \frac{(H_1 - L_1)^2}{4 \log_e 2} = \frac{(u - d)^2}{4 \log_e 2}. \tag{5}$$

Here, $\text{Eff}(\hat{\sigma}_2^2) \approx 5.2$. When the high, low, open, and close prices are simultaneously available, we can also form the composite estimator

$$\hat{\sigma}_3^2 \equiv a \frac{(O_1 - C_0)^2}{f} + (1 - a) \frac{(u - d)^2}{(1 - f)4 \log_e 2}, \quad 0 < f < 1, \tag{6}$$

5. Caveat: Note that we are dealing here with the variance of variances, so the fourth moments of the original quantities are involved.

6. Parkinson actually gives two estimators of volatility, the one described in formula (5) and another one which employs the square of the sum of the differences of high and low. However, this latter estimator is biased, and so we do not consider it here.

which has minimum variance when $a = 0.17$, independent of the value of f. In this case, $\mathrm{Eff}(\hat{\sigma}_3^2) \approx 6.2$.

One criticism of the estimators which are based solely on the quantity $(u - d)$ is that they ignore the joint effects between the quantities u, d, c, which may be utilized to further increase efficiency. In the following section we therefore characterize the best analytic estimators of σ^2.

V. "Best" Analytic Scale-invariant Estimators

For our purposes herein, an estimator is "best" when it has minimum variance and is unbiased. We shall also impose the requirements that the estimators be analytic and scale-invariant, as explained later.

To simplify the initial analysis, we suppose $f = 0$, that is, trading is open throughout the interval [0, 1]. Then consider estimators of form $D(u, d, c)$, that is, decision rules which are functions only of the quantities u, d, and c. We restrict attention to these normalized values because the process $B(t)$ renews itself everywhere, including $t = 0$, and so only the increments from the level O_1 ($= C_0$) are relevant. According to the lemma established in Appendix A, any minimum-squared-error estimator $D(u, d, c)$ should inherit the invariance properties of the joint density of (u, d, c). Two such invariance properties may be quickly enunciated: For all $\sigma^2 > 0$ and all $d \leqslant c \leqslant u$, $d \leqslant 0 \leqslant u$, we have

$$g(u, d, c; \sigma^2) = g(-d, -u, -c; \sigma^2) \tag{7}$$

and

$$g(u, d, c; \sigma^2) = g(u - c, d - c, -c; \sigma^2). \tag{8}$$

The first condition represents price symmetry: for Brownian motion of form (2), $B(t)$ and $-B(t)$ have the same distribution. Whenever $B(t)$ generates the realization (u, d, c), $-B(t)$ generates $(-d, -u, -c)$. The second condition represents time symmetry: $B(t)$ and $B(1 - t) - B(1)$ have identical distributions. Whenever $B(t)$ produces (u, d, c), $B(1 - t) - B(1)$ yields $(u - c, d - c, -c)$. By the lemma of Appendix A, then, any decision rule $\hat{\sigma}^2 \equiv D(u, d, c)$ may be replaced by an alternative decision rule which preserves the invariance properties (7) and (8) without increasing the expected (convex) loss associated with the estimator. Therefore, we seek decision rules which satisfy

$$D(u, d, c) = D(-d, -u, -c) \tag{9}$$

and

$$D(u, d, c) = D(u - c, d - c, -c). \tag{10}$$

Next, we observe that a scale-invariance property should hold in the parameter space: for any $\lambda > 0$,

$$g(u, d, c; \sigma^2) = g(\lambda u, \lambda d, \lambda c; \lambda^2\sigma^2). \tag{11}$$

In consequence of (11), we restrict our attention now to scale-invariant decision rules for which

$$D(\lambda u, \lambda d, \lambda c) = \lambda^2 D(u, d, c), \quad \lambda > 0. \tag{12}$$

If we now adopt the regularity condition that the decision rules considered must be analytic in a neighborhood of the origin, condition (12) implies that the decision rule $D(u, d, c)$ must be quadratic in its arguments. (Proof of this is given in Appendix B.) Thus we have

$$D(u, d, c) = a_{200}u^2 + a_{020}d^2 + a_{002}c^2 + a_{110}ud + a_{101}uc + a_{011}dc. \tag{13}$$

Scale invariance and analyticity have been combined to reduce the search for a method of estimating σ^2 from an infinite-dimensional problem to a six-dimensional affair. Applying the symmetry property (9) and (13), we have the implications $a_{200} = a_{020}$ and $a_{101} = a_{011}$. By virtue of property (10), we have the additional constraint $2a_{200} + a_{110} + 2a_{101} = 0$, hence we have

$$D(u, d, c) = a_{200}(u^2 + d^2) + a_{002}c^2 - 2(a_{200} + a_{101})ud + a_{101}c(u + d). \tag{14}$$

Insisting that $D(u, d, c)$ be unbiased, that is, $E[D(u, d, c)] = \sigma^2$, leads to one further reduction. Since[7] $E[u^2] = E[d^2] = E[c^2] = E[c(u + d)] = \sigma^2$ and $E[ud] = (1 - 2 \log_e 2)\sigma^2$, we may restrict attention further to the two-parameter family of decision rules $D(.)$ of the form

$$D(u, d, c) = a_1(u - d)^2 + a_2[c(u + d) - 2ud] + [1 - (a_1 + a_2)4 \log_e 2 + a_2]c^2. \tag{15}$$

To minimize this quantity, note that, for any random variables X, Y, and Z, the quantity $V(a_1, a_2) \equiv E[(a_1X + a_2Y + Z)^2]$ is minimized by a_1 and a_2, which satisfy

$$E[(a_1X + a_2Y + Z)X] = E[(a_1X + a_2Y + Z)Y] = 0. \tag{16}$$

Solving the above for a_1 and a_2, we have

$$a_1^* = \frac{E[XY]E[YZ] - E[Y^2]E[XZ]}{E[X^2]E[Y^2] - (E[XY])^2} \tag{17a}$$

and

$$a_2^* = \frac{E[XY]E[XZ] - E[X^2]E[YZ]}{E[X^2]E[Y^2] - (E[XY])^2}. \tag{17b}$$

In our problem,

$$\begin{aligned} X &= (u - d)^2 - (4 \log_e 2)c^2, \\ Y &= c(u + d) - 2ud + (1 - 4 \log_e 2)c^2, \\ Z &= c^2. \end{aligned} \tag{18}$$

7. See Appendix C for the calculation of moments.

Analysis via generating functions (Appendix C) reveals the following fourth moments:

$$E[u^4] = E[d^4] = E[c^4] = 3\sigma^4,$$

$$E[u^2c^2] = E[d^2c^2] = 2\sigma^4,$$

$$E[u^3c] = E[d^3c] = 2.25\sigma^4,$$

$$E[uc^3] = E[dc^3] = 1.5\sigma^4,$$

$$E[ud^2c] = E[u^2dc] = \left[\frac{9}{4} - 2\log_e 2 - \frac{7}{8}\zeta(3)\right]\sigma^4 = -0.1881\sigma^4,$$

$$E[u^2d^2] = [3 - 4\log_e 2]\sigma^4 = 0.2274\sigma^4,$$

$$E[udc^2] = \left[2 - 2\log_e 2 - \frac{7}{8}\zeta(3)\right]\sigma^4 = -0.4381\sigma^4,$$

$$E[ud^3] = E[u^3d] = \left[3 - 3\log_e 2 - \frac{9}{8}\zeta(3)\right]\sigma^4 = -0.4318\sigma^4,$$

where $\zeta(3) = \Sigma_{k=1}^{\infty} 1/k^3 = 1.2021$ is Riemann's zeta function. Substituting the above moments into (17a) and (17b) via (18), we find that $a_1^* = 0.511$ and $a_2^* = -0.019$. Employing these values in (15) yields the best analytic scale-invariant estimator

$$\hat{\sigma}_4^2 \equiv 0.511(u - d)^2 - 0.019[c(u + d) - 2ud] - 0.383c^2. \quad (19)$$

We find that $\text{Eff}(\hat{\sigma}_4^2) \approx 7.4$. (The more "practical" estimator $\hat{\sigma}_5^2 \equiv 0.5[u - d]^2 - [2\log_e 2 - 1]c^2$ has virtually the same efficiency but eliminates the cross-product terms.)

Now suppose that $0 < f < 1$, that is, trading is both open and closed in [0, 1]. Then the opening price O_1 may differ from the previous closing price C_0, and so we may form the composite estimator

$$\hat{\sigma}_6^2 \equiv a\,\frac{(O_1 - C_0)^2}{f} + (1 - a)\,\frac{\hat{\sigma}_4^2}{(1 - f)}. \quad (20)$$

The variance of $\hat{\sigma}_6^2$ is minimized when $a = 0.12$, and in this case $\text{Eff}(\hat{\sigma}_6^2) \approx 8.4$. Thus, there exists an estimator possessing an efficiency factor which is more than eight times better than the classical estimator $\hat{\sigma}_0^2$, given only high, low, open, and close prices.

VI. Volume Effects

The derivation of all of the high-low estimators of the previous sections depends critically upon the assumption of continuously monitored price paths. When the path can only be monitored at discrete transactions, all our statistics will be biased. Technically speaking, the knowledge that only a finite number of observations are available should lead us to commence a new search for the best estimator; however, we shall

TABLE 1 **Expected Values of Volatility Estimators ($\sigma^2 = 1$)**

No. Transactions	$\hat{\sigma}_0^2$	$\hat{c}^2$	$\hat{\sigma}_2^2$	$\hat{\sigma}_4^2$
5	1.03	.48	.55	.38
10	1.01	.67	.65	.51
20	1.00	.82	.74	.64
50	1.00	.92	.82	.73
100	1.00	.97	.86	.80
200	1.00	.99	.89	.85
500	1.00	1.00	.92	.89

defer this task to a later paper. We instead confine our considerations here to determining the extent of the bias in using the estimators already described when only a finite set of observations is available. Simulation studies[8] were employed to arrive at table 1. Note that the close-to-close estimator $\hat{\sigma}_0^2$ has only a slight positive bias.[9] On the other hand, the expected values of volatility estimators $\hat{c}^2 \equiv (C_1 - O_1)^2$, $\hat{\sigma}_2^2$, and $\hat{\sigma}_4^2$ are significantly less than σ^2 whenever a finite number of transactions take place. Moreover, there are two distinguishable reasons for the observed biases. The estimator $\hat{c}^2$ has downward bias because the effective time period over which it is estimated is shortened to the span of the first to last transaction, since it is an open-to-close estimator. The estimators $\hat{\sigma}_2^2$ and $\hat{\sigma}_4^2$ are also subject to this effect when trading is closed a portion of the day. (In the absence of other considerations, finite transaction volume will make the opening appear later and the closing appear earlier. However, many exchanges will collect orders during the night for execution at the opening; additionally, some exchanges have a "closing rotation" in which firm-offer prices are quoted at the closing. Each of these procedures would tend to diminish the effective-time bias.) The second reason affects only the latter high-low estimators, which take on a downward bias because the observed highs and lows will be less in absolute magnitude than the actual highs and lows. In addition, the high-low estimators are indirectly affected by the effective-time bias since highs and lows tend to occur at the first and last transactions.

As a practical procedure, one should divide the corresponding empirical estimators by the numbers in table 1. Since estimators $\hat{\sigma}_3^2$, $\hat{\sigma}_6^2$ are linear combinations of the given estimators, their bias and that of many others may be quickly computed therefrom.

Random transaction volume is one source of predictable bias somewhat within the scope of the current model. But there are other important sources of bias which can be made predictable only by significant

8. Our finite-volume simulations assumed that all transactions are scattered "randomly" (i.e., uniformly) throughout [0, 1].
9. Scholes and Williams (1977) also consider this source of bias.

extension of the current model. Some of these unpredictable bias sources are the following: (1) to the extent that transactions themselves may convey new information, daytime volatilities may be different from nighttime volatilities; (2) bid-ask spreads exist, within which the transactions process may be quite complicated (Garman 1976); and (3) volatilities may otherwise be nonstationary in a variety of fashions.

VII. Conclusions

We have examined a number of estimators of price volatility. Efficiency factors which are at least eight times better than the classical estimators have been demonstrated. These same estimators are also subject to more sources of predictable bias, one of the most evident of which is finite transaction volume. Unpredicted sources of bias await further empirical work.

Appendix A

Estimator Invariance Properties

Lemma: Let Θ be a parameter space. Let $\mathbf{X} = (X_1, X_2, \ldots, X_n)$ be a vector of (not necessarily independent) observations whose joint density $f_\theta(\mathbf{X})$ depends on an unknown parameter $\theta \in \Theta$, to be estimated. Let $T{:}\mathbf{R}^n \to \mathbf{R}^n$ be a fixed measure-preserving transformation. Suppose that, for all $\theta \in \Theta$ and all $\mathbf{X}$ in the support of f_θ,

$$f_\theta(T\mathbf{X}) = f_\theta(\mathbf{X}). \tag{A1}$$

Let $D(\mathbf{X})$ be any decision rule which estimates θ. Let $L(\theta, D(\mathbf{X}))$ be any loss function such that $L(\theta,.)$ is a convex function for each fixed $\theta \in \Theta$. Defining $T^j \equiv TT^{j-1}$ where T^0 is the identity operator, let A_k be an averaging operator which maps decision rules into decision rules according to the prescription

$$A_k(D(\mathbf{X})) = \frac{1}{k}\sum_{j=1}^{k} D(T^{j-1}\mathbf{X}). \tag{A2}$$

Then, for all θ,

$$E_\theta L(\theta, A_k(D(\mathbf{X}))) \leqslant E_\theta L(\theta, D(\mathbf{X})). \tag{A3}$$

Comment: In particular, if $L(\theta,.)$ is strictly convex, then every noninvariant rule is subject to improvement. (A rule is invariant with respect to θ_0 if and only if $\mathrm{Prob}_{\theta_0}[D(\mathbf{X}) = D(T\mathbf{X})] = 1$.) In many cases an invariant rule can be constructed from $D(.)$. If, for example, $T^k\mathbf{X} \equiv \mathbf{X}$, then

$$\begin{aligned} A_k(D(T\mathbf{X})) &= \frac{1}{k}\sum_{j=1}^{k} D(T^j\mathbf{X}) \\ &= \frac{1}{k}\sum_{j=1}^{k-1} D(T^j\mathbf{X}) + \frac{D(\mathbf{X})}{k} \\ &= A_k(D(\mathbf{X})). \end{aligned} \tag{A4}$$

Second, if T is measure preserving, then, according to the Mean Ergodic Theorem, there exists a measurable function $D^*(.)$ such that, for almost all $\mathbf{X}$,

$$\lim_{k\to\infty} A_k(D(\mathbf{X})) = D^*(\mathbf{X}), \tag{A5}$$

$$D^*(T\mathbf{X}) = D^*(\mathbf{X}), \tag{A6}$$

and

$$E_\theta L(\theta, D^*(\mathbf{X})) \leqslant E_\theta L(\theta, D(\mathbf{X})). \tag{A7}$$

Thus, in the interest of minimizing expected loss, attention may often be restricted to those rules for which $D(\mathbf{X}) = D(T\mathbf{X})$.

Proof: By convexity,

$$L(\theta, A_k(D(\mathbf{X}))) \leqslant \frac{1}{k}\sum_{j=1}^{k} L(\theta, D(T^{j-1}\mathbf{X})).$$

Taking expectations,

$$E_\theta L(\theta, A_k(D(\mathbf{X}))) \leqslant \frac{1}{k}\sum_{j=1}^{k} E_\theta L(\theta, D(T^{j-1}\mathbf{X})).$$

It suffices to verify that

$$E_\theta L(\theta, D(T^j\mathbf{X})) = E_\theta L(\theta, D(\mathbf{X}))$$

for all $j \geqslant 1$. Computing,

$$\begin{aligned} E_\theta L(\theta, D(T^j\mathbf{X})) &= \int_{R^n} L(\theta, D(T^j\mathbf{X}))f_\theta(\mathbf{X})d\mathbf{X} \\ &= \int_{R^n} L(\theta, D(T^j\mathbf{X}))f_\theta(T^j\mathbf{X})d\mathbf{X}, \end{aligned}$$

the latter by (A1). By change of variable, this equals

$$\begin{aligned} &= \int_{Rn} L(\theta, D(\mathbf{Y}))f_\theta(\mathbf{Y})\mathbf{J}_{T^{-j}}d\mathbf{Y} \\ &= E_\theta L(\theta, D(\mathbf{X})), \end{aligned}$$

where the Jacobian $\mathbf{J}_{T^{-j}} = (\mathbf{J}_T)^{-j} = 1^{-j} = 1$, since T is measure preserving.

Comment: Note that the transformations prescribed by formulas (7) and (8) of the text are both linear and satisfy the relation $T_i^2 = I$ (the identity). Hence, it follows that $\mathbf{J}_{T_i} = |\det(T_i)| = 1$.

Appendix B

Analytic Estimators Are Quadratic

Lemma: Estimators $D(u, d, c)$ of σ^2 which are analytic in the neighborhood of the origin are quadratic in form.

Proof: If $D(u, d, c)$ is analytic in the neighborhood of the origin, we may write its Taylor series expansion as

$$D(u, d, c) = \sum_{i,j,k\geqslant 0} a_{ijk}u^i d^j c^k. \tag{B1}$$

Next define

$$F_{u,d,c}(\lambda) \equiv \lambda^2 \sum_{i,j,k\geq 0} a_{ijk}u^i d^j c^k - \sum_{i,j,k\geq 0} a_{ijk}\lambda^{i+j+k}u^i d^j c^k. \tag{B2}$$

From the scale-invariance property (12), $F_{u,d,c}(\lambda)$ must be identically zero. It may also be extended to an analytic function in some open neighborhood of the origin. Thus, by uniqueness, all coefficients of powers of λ in (B2) must be identically zero. It follows that $a_{ijk}u^i d^j c^k = 0$ for $i + j + k \neq 2$, that is, D is quadratic.

Appendix C

Generating Function of High, Low, and Close[10]

The expectation of the moments of u, d, and c is given by

$$E(u^p d^q c^r) = (-1)^{p+r}\frac{\sigma^n}{2^{n/2}\Gamma\left(\frac{n}{2}+1\right)}\left[\frac{\partial^n \mathbf{H}(t,s,z)}{\partial t^p \partial s^q \partial z^r}\right]_{t=s=z=0}, \tag{C1}$$

where $n = p + q + r$ and our generating function $\mathbf{H}$ is

$$\begin{aligned}\mathbf{H}(t,s,z) = {} & \frac{st}{1-z^2}\sum_{k=1}^{\infty}\Bigg[-\frac{1}{(2k+t)(2k+s+1-z)} - \frac{1}{(2k+s)(2k+t+1+z)} \\ & + \frac{1}{(2k+t+2)(2k+s+1-z)} + \frac{1}{(2k+s+2)(2k+t+1+z)}\Bigg] \\ & + \frac{1}{1-z^2}\left[1 - \frac{s}{1+s-z} - \frac{t}{1+t+z} + \frac{ts}{(2+t)(1+s-z)} + \frac{ts}{(2+s)(1+t+z)}\right].\end{aligned} \tag{C2}$$

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10. This generating function was kindly supplied to us by Oldrich Vasicek.