Statistical Modelling 2: Homework #1

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Problem 1

Problem 2

Part A

$$\begin{aligned}
cov(x) &= E\{(x-\mu)(x-\mu)^T\} \\
&= E\{(x-\mu)(x^T-\mu^T)\}, \text{using property of transpose} \\
&= E\{xx^T - x\mu^T - \mu x^T + \mu\mu^T\}, \text{using distributive property of matrix multiplication} \\
&= E(xx^T) - E(x\mu^T) - E(\mu x^T) + E(\mu\mu^T), \text{using Linearity of Expectation} \\
&= E(xx^T) - E(x)\mu^T - \mu E(x^T) + \mu\mu^T, \mu \text{ being a constant can be pushed out of the expectation} \\
&= E(xx^T) - \mu\mu^T - \mu\mu^T + \mu\mu^T \\
&= E(xx^T) - \mu\mu^T
\end{aligned}$$

$$\begin{aligned}
cov(Ax+b) &= E\{(Ax+b)(Ax+b)^T\} - E(Ax+b)E(Ax+b)^T, \text{using previous result} \\
&= E\{(Ax+b)(x^TA^T+b^T)\} - \{E(Ax)+b\}\{E(Ax)+b\}^T, \text{using property of transpose} \\
&\stackrel{1}{=} E(Axx^TA^T) + E(Axb^T) + E(bx^TA^T) + E(bb^T) - (A\mu+b)(A\mu+b)^T \\
&\stackrel{2}{=} AE(xx^T)A^T + AE(x)b^T + bE(x^T)A^T + bb^T - (A\mu+b)(\mu^TA^T+b^T) \\
&\stackrel{3}{=} AE(xx^T)A^T + A\mu b^T + b\mu^TA^T + bb^T - A\mu\mu^TA^T - A\mu b^T - b\mu^TA^T - bb^T \\
&= AE(xx^T)A^T - A\mu\mu^TA^T \\
&= Acov(x)A^T, \text{using previous result}
\end{aligned}$$

1 follows from linearity of expectation and distributive property of matrix multiplication. 2 follows from the fact that A and μ being constants can be pushed out of the expectation. 3 follows from the definition of μ

Part D

Let us try to find the moment generating function for the random variable z.

$$\begin{aligned}
MGF_x(t) &= E\{\exp(t^Tx)\}, \text{using definition of MGF} \\
&= E\{\exp(t^T(Lz+\mu))\}, \text{substituting for } x \text{ in terms of } z \\
&= E\{\exp(t^TLz)\}E\{\exp(t^T\mu)\}, \text{distributing the powers} \\
&= E\{\exp(t^TLz)\}\exp(t^T\mu), t^T\mu \text{ is constant for a given } t \\
&= E\{\exp((L^Tt)^Tz)\}\exp(t^T\mu), \text{using basic property of transpose} \\
&= \exp((L^Tt)^TI(L^Tt))\exp(t^T\mu), \text{using MGF for standard multivariate normal} \\
&= \exp(t^TLL^Tt)\exp(t^T\mu) \\
&= \exp(t^T\mu + t^T(LL^T)t), \text{combining the powers}
\end{aligned}$$

Now using the if part of the result in part C, we can easily see that $x \sim N(u, LL^T)$

Part E

Suppose $x \sim N(\mu, \Sigma)$. We know that the matrix Σ is symmetric since for any i, j we have that $cov(x_i, x_j) = cov(x_j, x_i)$. From the spectral theorem, there exists an orthonormal matrix U and a diagonal matrix D such that $\Sigma = UDU^T$. We can rewrite this decomposition as $\Sigma = UD^{\frac{1}{2}}(UD^{\frac{1}{2}})^T$ where $D^{\frac{1}{2}}$ is the matrix obtained by taking the square roots of the the diagonal entries of D.

Problem 3

Problem 4

Problem 5

Problem 18

Problem 19

Problem 6