Lec. 10 (04/30/2020)

Implication of small gain Theorem for LTI:

Suppose S_1 and S_2 are both LTI with transfer matrices $G_1(s)$, $G_2(s)$.

Let $\|G_1\|_\infty \leq \gamma_1$, $\|G_2\|_\infty \leq \gamma_2$.

If $\gamma_1 \gamma_2 < 1$ then feedback interconnection is $\mathcal{L}_2$ stable.

Passivity

$\dot{x} = f(x) + g(x)u$

$y = h(x)$

$$\underline{u} \longrightarrow \boxed{S} \longrightarrow \underline{y}$$

$\underline{u}, \underline{y} \in \mathbb{R}^m$

$$\langle \underline{u}(t), \underline{y}(t) \rangle = \underline{u}^T(t)\,\underline{y}(t) = \underline{y}^T(t)\,\underline{u}(t)$$

$$\int_0^T \underbrace{\langle \underline{u}(t), \underline{y}(t) \rangle}_{P(t)} dt = \text{"Energy" of a control system}$$

$P(t)$ instantaneous power

$$\int_0^T \underbrace{F\,dx}_{W(t)}\,dt$$

Example:

- $u(t)$ could be current
  $y(t)$ " " voltage

- $u(t)$ could be force
  $y(t)$ " " displacement

Passive System:

Proposition: The following sentences are equivalent:
(If and only if)

① The system S is "passive".

② Rate inequality : $\exists\ C^1$ function $V(\underline{x}, t) \geq 0$ called "storage function" such that $\forall\, t > 0$, we have

2.1 $\dot{V} < 0$

2.2 $\underbrace{u^T(t)\, y(t)}_{\text{supply rate of energy}} \geq \underbrace{\dot{V} =: \frac{d}{dt} V}_{\text{Storage rate}} = \langle \nabla V, f \rangle + \left(\frac{\partial V}{\partial t}\right)$

~~supply rate of energy~~ Instantaneous Power

(if equality), we say S is "lossless".

③ Dissipativity inequality : The system satisfies :

$$\int_0^t \left(\underline{u}(\tau)\right)^T \left(\underline{y}(\tau)\right) d\tau \geq V(x(t), t) - V(\underline{x}(0), 0)$$

④ Dissipation equality :

$$\underbrace{\int_0^t (\underline{u}(\tau))^T \underline{y}(\tau)\, d\tau}_{\text{Energy supplied up until time } t} = \overbrace{V(\underline{x}(t), t) - V(\underline{x}(0), 0)}^{\text{Energy stored}} + \underbrace{d(\underline{x}(t))}_{\text{Energy dissipated}}$$

In short : $\boxed{\text{Supply} = \text{storage} + \text{dissipation}}$

(If dissipation $= 0$, $\Leftrightarrow$ System is Lossless)

Related notions :

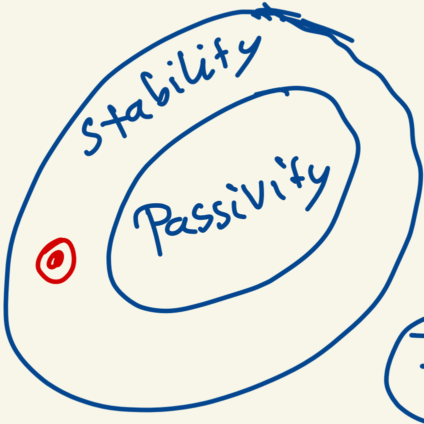
- Strictly Passive (SP) if $\underline{u}^T \underline{y} \geq \dot{V} + \psi(\underline{x})$ for some pos. def. function $\psi(.)$

- Input - feedforward passive (IFP) if $\underline{u}^T \underline{y} \geq \dot{V} + \underline{u}^T \underline{\xi}(\underline{u})$ for some $\underline{\xi}(\underline{u})$

- Input Strictly passive (ISP) if IFP + Extra condition ($\underline{u}^T \underline{\xi}(\underline{u}) > 0 \ \forall \underline{u} \neq 0$)

- Output Feedback passive (OFP) if $\underline{u}^T \underline{y} \geq \dot{V} + \underline{y}^T \underline{\eta}(\underline{y})$ for some $\underline{\eta}(\cdot)$

- Output strictly passive (OSP) if OFP + Extra condition ($\underline{y}^T \underline{\eta}(\underline{u}) > 0 \ \forall \underline{y} \neq 0$)



Theorem # 1

$$S \begin{cases} \dot{\underline{x}} = \underline{f}(\underline{x}, \underline{u}), & \underline{x} \in \mathbb{R}^n, \\ \underline{y} = \underline{h}(\underline{x}, \underline{u}), & \underline{y}, \underline{u} \in \mathbb{R}^m \end{cases}$$

If S is passive with storage function $V(\cdot)$, then origin of the unforced system $\dot{\underline{x}} = \underline{f}(\underline{x}, \underline{0})$ is stable.

Theorem # 2:

If S is OSP, with

$$u^T y \geq \dot{V} + \delta \underbrace{y^T y}_{= \|y(t)\|_2^2} \quad \text{for some } \delta > 0,$$

Then S is finite gain $\mathcal{L}_2$ stable with gain $\leq \left(\frac{1}{\delta}\right)$.

Definition: (Zero state Observability):

We say S is **zero-state observable** (ZSO) if no solution of $\dot{\underline{x}} = f(\underline{x}, \underline{0})$ can stay identically in the set

$$\{\underline{x} \in \mathbb{R}^n \mid \underline{h}(\underline{x}, \underline{0}) = \underline{0}\}$$

other than the trivial solution $\underline{x}(t) \equiv \underline{0} \;\; \forall t$.

Theorem #3:

If S is either SP or OSP + ZSO

then origin of $\dot{\underline{x}} = f(\underline{x}, \underline{0})$ is A.S.

If in addition, the storage function is radially unbounded then GAS.

Compositional property:

Passivity is preserved under
1. series connection
2. parallel connection
3. feedback connection.

Passivity for LTI

$$\underbrace{(A, B, C, D)}_{} \longleftrightarrow \underbrace{G(s)}_{\text{Transfer matrix}} = C(sI - A)^{-1}B + D$$

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

$$G(s) = \frac{N(s)}{D(s)} = \frac{\prod_i (s - z_i)}{\prod_i (s - p_i)}$$

zeros

poles

Theorem: The LTI minimal realization
((A, B) is controllable pair,
(A, C) is observable ")

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

$$G(s) = C(sI - A)^{-1}B + D$$

is

① passive if $G(s)$ is positive real (PR)

② strictly passive if $G(s)$ is strictly positive real (SPR).

We say a transfer function is <u>proper</u> if

- $G(j\infty)$ is finite (elementwise) $\iff$ (deg. denominator polynomial $\geq$ " numerator ")
- <u>Strictly proper</u> if $G(j\infty) = 0$ (elementwise) $\iff$ (deg. numerator polynomial $>$ deg. denominator polynomial)
- <u>Biproper</u> if both $G(\cdot)$ and $G^{-1}(\cdot)$ are proper (deg. numerator polynomial $=$ deg. denominator polynomial)

Positive real transfer Function / Matrix :

A $p \times p$ (square) proper rational transfer matrix $G(s)$ is positive real if

1. poles of all elements of $G(s)$ are in $Re(s) \leq 0$

2. the matrix $(G + G^*)(j\omega)$

$$= G(j\omega) + G^T(-j\omega) \succcurlyeq 0$$

$\forall \omega \in \mathbb{R}$ s.t. $j\omega$ is NOT a pole of any element of $G(s)$.

(If $p = 1$ (scalar/ transfer function case) then 2 becomes
$Re(G(j\omega)) \geq 0 \quad \forall \omega \in \mathbb{R}$)

③ Any pure imaginary pole $j\omega$ of any element of $G(s)$ is a simple pole AND the matrix $\lim_{s \to j\omega} (s - j\omega) G(s) \geq 0$.

We say $G(s)$ is SPR if $G(s-\epsilon)$ is PR for some $\epsilon > 0$.

---

① Positive real Lemma

② $\underbrace{\text{KYP}}$ Lemma

(Kalman - Yakubovich - Popov)