# AMS 231: Nonlinear Control Theory: Winter 2018

## Homework #1

Name: ...........................................................................................................

Due: January 23, 2018

NOTE: Please show all the steps in your solution. Turn in a hard copy of your HW stapled with this as cover sheet with your name written in the above field. Please submit your HW in class on the due date.

## Problem 1

**Modeling of Nonlinear Control Systems** $(2+2+2+(3\times 3)+5 = 20$ **points)**

A simple model of car or car-like mobile robot is given by

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{pmatrix} = \begin{pmatrix} v\cos\theta \\ v\sin\theta \\ \dfrac{v}{\ell}\tan\phi \end{pmatrix},$$

where the control variables are: the speed of the car $v$, and the steering angle $\phi$. Assume limited range for steering angle $\phi$, say $\phi \in [0, \frac{\pi}{2})$. Also assume $|v| \leq v_{\max}$. The parameter $\ell$ denotes the distance between the front and the rear axles.

(a) What is the state space $\mathscr{X}$ for this system?

(b) What is the control space $\mathscr{U}$ for this system?

(c) Is this an autonomous or non-autonomous control system? Give reason in one sentence.

(d) Most commercially available indoor mobile robots do not move like a car. Instead, they have differential drive consisting of a single axle which connects two independently controlled wheels. Each wheel is driven by its own motor, and is free to rotate without affecting the other wheel. The model is given by

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{pmatrix} = \begin{pmatrix} \frac{r}{2}(\omega_{\text{right}} + \omega_{\text{left}})\cos\theta \\ \frac{r}{2}(\omega_{\text{right}} + \omega_{\text{left}})\sin\theta \\ \frac{r}{\ell}(\omega_{\text{right}} - \omega_{\text{left}}) \end{pmatrix},$$

where the control variables are the angular velocities of the right and left wheels, denoted by $\omega_{\text{right}}$ and $\omega_{\text{left}}$, respectively. The parameter $r$ is the wheel radius. What type of motion occurs if we set: (i) $\omega_{\text{right}} = \omega_{\text{left}} > 0$? (ii) $\omega_{\text{right}} = \omega_{\text{left}} < 0$? (iii) $\omega_{\text{right}} = -\omega_{\text{left}} \neq 0$?

(e) The fundamental difference between the model in part (d) with the one considered in parts (a–c) is that the rotation rate ($\dot{\theta}$) of a differential drive robot can be set independent of its translational velocity, whereas the same cannot be done for a car. To show this explicitly, consider transforming the control variables in part (d) as $\omega^+ := \frac{1}{2}(\omega_{\text{right}} + \omega_{\text{left}})$, $\omega^- := \omega_{\text{right}} - \omega_{\text{left}}$. Further let $\widetilde{v} := r\omega^+$ and $\widetilde{\phi} := \tan^{-1}(\omega^-)$. Using these new control variables $\widetilde{v}$ and $\widetilde{\phi}$, show that the $\theta$ dynamics in part (d) does not depend on $\widetilde{v}$.

## Solution

(a) $\mathscr{X} = \mathbb{R}^2 \times \mathbb{S}^1$.

(b) $\mathscr{U} = [-v_{\max}, v_{\max}] \times [0, \frac{\pi}{2})$.

(c) This is an autonomous control system since the vector field has no explicit time dependence.

(d) The motions are: (i) move forward while keeping the orientation fixed, (ii) move backward while keeping the orientation fixed, (iii) pure rotation.

(e) We get

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{pmatrix} = \begin{pmatrix} r\omega^+ \cos\theta \\ r\omega^+ \sin\theta \\ \frac{r}{\ell}\omega^- \end{pmatrix} = \begin{pmatrix} \widetilde{v}\cos\theta \\ \widetilde{v}\sin\theta \\ \frac{r}{\ell}\tan\widetilde{\phi} \end{pmatrix}.$$

The $\dot{\theta}$ dynamics is independent of $\widetilde{v}$.

## Problem 2

**Planar Nonlinear Systems** **(**5+10+15+10 = 40 **points)**

Consider the planar nonlinear autonomous system

$$\dot{x}_1 = -x_1 - \frac{x_2}{\ln\sqrt{x_1^2 + x_2^2}}, \qquad \dot{x}_2 = -x_2 + \frac{x_1}{\ln\sqrt{x_1^2 + x_2^2}}.$$

(a) Prove that origin is the unique fixed point of this dynamical system.

(b) Linearize the system about the fixed point and show that it is a stable node of the linearized system. (Hint: compute the Jacobian evaluated at the fixed point)

(c) By computing the phase portrait curves by hand, show that the global behavior of the nonlinear system resembles stable focus. (Hint: rewrite the dynamics in polar co-ordinates)

(d) Explain the conclusions in part (b) from the results obtained in part (c).

## Solution

(a) Any fixed point $\begin{pmatrix} x_1^* \\ x_2^* \end{pmatrix}$ of the given dynamics satisfies

$$\dot{x}_1^* = 0 \;\Rightarrow\; \ln\sqrt{(x_1^*)^2 + (x_2^*)^2} = -\frac{x_2^*}{x_1^*}, \qquad \dot{x}_2^* = 0 \;\Rightarrow\; \ln\sqrt{(x_1^*)^2 + (x_2^*)^2} = \frac{x_1^*}{x_2^*},$$

and hence $-\frac{x_2^*}{x_1^*} = \frac{x_1^*}{x_2^*} \Rightarrow (x_1^*)^2 + (x_2^*)^2 = 0$, which is possible iff $x_1^* = x_2^* = 0$. Therefore, origin is the unique fixed point.

(b) By direct calculation, we get

$$\left.\frac{\partial f_1}{\partial x_1}\right|_{(0,0)} = \left.\left(-1 + \frac{x_1 x_2}{(x_1^2 + x_2^2)\left(\ln\sqrt{x_1^2 + x_2^2}\right)^2}\right)\right|_{(0,0)} = -1 + \lim_{r\to 0} \frac{\cos\theta\sin\theta}{(\ln r)^2} = -1 + 0 = -1,$$

$$\left.\frac{\partial f_1}{\partial x_2}\right|_{(0,0)} = \left.\left(0 - \frac{1}{\ln\sqrt{x_1^2 + x_2^2}} + \frac{x_2^2}{(x_1^2 + x_2^2)\left(\ln\sqrt{x_1^2 + x_2^2}\right)^2}\right)\right|_{(0,0)} = 0 + \lim_{r\to 0}\left(-\frac{1}{\ln r} + \frac{\sin^2\theta}{(\ln r)^2}\right) = 0,$$

$$\left.\frac{\partial f_2}{\partial x_1}\right|_{(0,0)} = \left.\left(0 + \frac{1}{\ln\sqrt{x_1^2 + x_2^2}} - \frac{x_1^2}{(x_1^2 + x_2^2)\left(\ln\sqrt{x_1^2 + x_2^2}\right)^2}\right)\right|_{(0,0)} = 0 + \lim_{r\to 0}\left(\frac{1}{\ln r} - \frac{\cos^2\theta}{(\ln r)^2}\right) = 0,$$

$$\left.\frac{\partial f_2}{\partial x_2}\right|_{(0,0)} = \left.\left(-1 - \frac{x_1 x_2}{(x_1^2 + x_2^2)\left(\ln\sqrt{x_1^2 + x_2^2}\right)^2}\right)\right|_{(0,0)} = -1 - \lim_{r\to 0} \frac{\sin\theta\cos\theta}{(\ln r)^2} = -1 - 0 = -1.$$

Hence, the Jacobian evaluated at the unique fixed point $(x_1^*, x_2^*) = (0, 0)$ is

$$\left.\begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{pmatrix}\right|_{(0,0)} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Therefore, origin is a hyperbolic fixed point and by Hartman-Grobman Theorem, locally behaves like an improper stable node.

(c) Recall that the unit vectors in polar coordinates are related to the same in Cartesian coordinates as:

$$\begin{pmatrix} \widehat{e}_r \\ \widehat{e}_\theta \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \widehat{e}_x \\ \widehat{e}_y \end{pmatrix} \Rightarrow \begin{pmatrix} \widehat{e}_x \\ \widehat{e}_y \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \widehat{e}_r \\ \widehat{e}_\theta \end{pmatrix}.$$

Thus, the velocity vector

$$\underline{v} = f_1\widehat{e}_x + f_2\widehat{e}_y = (f_1\cos\theta + f_2\sin\theta)\widehat{e}_r + (-f_1\sin\theta + f_2\cos\theta)\widehat{e}_\theta = \dot{r}\widehat{e}_r + r\dot{\theta}\widehat{e}_\theta,$$

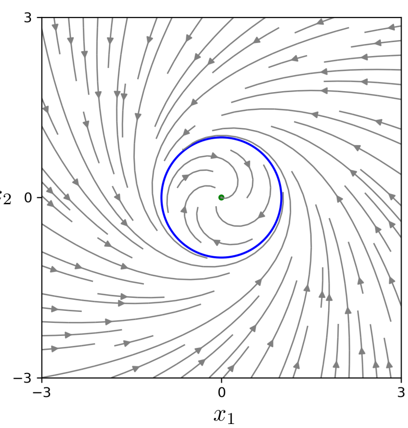
where the last equality comes from transport theorem applied on position vector $\underline{r} = r\widehat{e}_r$, since the angular velocity is $\underline{\omega} = \dot{\theta}\widehat{e}_z$ and hence $\underline{v} = \left(\frac{\mathrm{d}}{\mathrm{d}t}r\right)\widehat{e}_r + \underline{\omega}\times\underline{r} = \dot{r}\widehat{e}_r + r\dot{\theta}\widehat{e}_\theta$. Using the relations $x_1 = r\cos\theta$, $x_2 = r\sin\theta$, we get

$$\begin{aligned}
\dot{r} &= f_r(r,\theta) = f_1(r\cos\theta, r\sin\theta)\cos\theta + f_2(r\cos\theta, r\sin\theta)\sin\theta = -r,\\
\dot{\theta} &= f_\theta(r,\theta) = \frac{1}{r}\left(-f_1(r\cos\theta, r\sin\theta)\sin\theta + f_2(r\cos\theta, r\sin\theta)\cos\theta\right) = \frac{1}{\ln r},
\end{aligned}$$

which gives the vector field in polar coordinates as $\dot{r} = -r, \dot{\theta} = \frac{1}{\ln r}$, and therefore

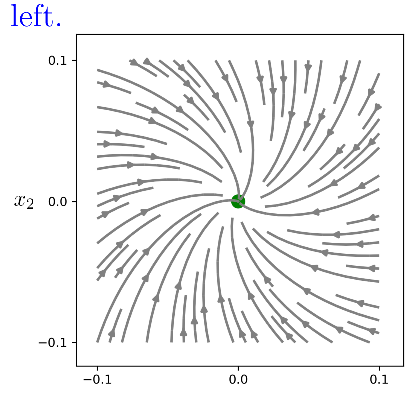
$$\frac{\mathrm{d}r}{\mathrm{d}\theta} = \frac{\dot{r}}{\dot{\theta}} = -r\ln r \Rightarrow \frac{\mathrm{d}r}{r\ln r} + \mathrm{d}\theta = 0 \Rightarrow \ln(\ln(r)) + \theta = \text{constant}.$$

Initial conditions determine the value of the constant. In particular, different values of the constant gives different curves in the phase portrait of the nonlinear system.




From the above expression of the polar curve, it is clear that globally the trajectories spiral in toward the origin (notice that $r(t) = r_0 e^{-t}$, $\theta(t) = \theta_0 - \ln(\ln(r_0) - t))$ resembling stable focus, with the exception of the unit circle since $\ln(\ln(1)) = -\infty$. Explicitly, the phase portrait curve for $r = 1$ is $\theta = +\infty$, i.e., the unit circle is a limit cycle on which the vector field rotates counter-clockwise. Because trajectories spiral in for both $r > 1$ (counter-clockwise, since $\frac{1}{\ln r}$ is positive) and $r < 1$ (clockwise, , since $\frac{1}{\ln r}$ is negative), the unit circle is a semi-stable limit cycle (attracting from outside, repelling from inside); see Fig. on the left.

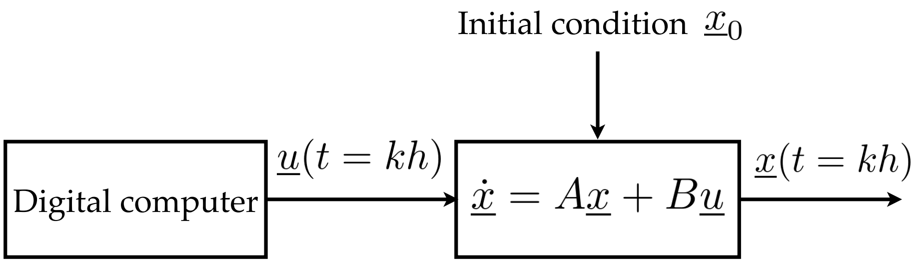
(d) Linearizing the dynamics in polar coordinates about $r = 0$ (origin), yields $\dot{r} = -r$, $\dot{\theta} =$ indeterminate. This tells us that near the origin, $r$ approaches the origin radially as time progresses, i.e., the phase portrait locally resembles a node near the origin, as already established in part (b). The phase portrait near origin is depicted on the right; the fixed point is shown in green.



## Problem 3

**Computer Controlled Continuous-time LTI System** **(40 points)**

Consider the continuous-time LTI system being controlled by a digital computer, as shown.




The sampling period is $h$. The control signal $\underline{u}(t) = \underline{u}(k) := \underline{u}(t = kh)$, $\forall\, t \in [kh, (k+1)h)$. This makes $\underline{u}$ a piecewise constant function of time $t$.

Prove that the above system is equivalent to a discrete-time LTI system

$$\underline{x}(k+1) = A_{\mathrm{d}}\,\underline{x}(k) + B_{\mathrm{d}}\,\underline{u}(k), \qquad k = 0, 1, 2, \ldots$$

by finding the matrices $A_{\mathrm{d}}$ and $B_{\mathrm{d}}$ as functions of $A, B$ and $h$.
(Hint: The solution of LTI system $\dot{\underline{x}} = A\underline{x} + B\underline{u}$ with initial condition $\underline{x}_0$ is given by

$$\underline{x}(t) = e^{At}\underline{x}_0 \; + \int_0^t e^{A(t-\tau)}B\underline{u}(\tau)\mathrm{d}\tau.$$

Then use the fact that $\underline{x}(k) := \underline{x}(t = kh)$, etc.)

## Solution

We have

$$\begin{aligned}
\underline{x}(k+1) := \underline{x}(t = (k+1)h) &= e^{A(k+1)h}\underline{x}_0 \; + \int_0^{(k+1)h} e^{A((k+1)h-\tau)}B\underline{u}(\tau)\mathrm{d}\tau \\
&= e^{Ah}e^{Akh}\underline{x}_0 \; + \; e^{Ah}\int_0^{kh} e^{A(kh-\tau)}B\underline{u}(\tau)\mathrm{d}\tau \; + \int_{kh}^{(k+1)h} e^{A((k+1)h-\tau)}B\underline{u}(\tau)\mathrm{d}\tau \\
&= e^{Ah}\underline{x}(k) \; + \int_{kh}^{(k+1)h} e^{A((k+1)h-\tau)}B\underline{u}(k)\mathrm{d}\tau.
\end{aligned}$$

Let $\sigma := (k+1)h - \tau$. Then $\mathrm{d}\sigma = -\mathrm{d}\tau$. Furthermore, $\tau = kh \Rightarrow \sigma = h$, and $\tau = (k+1)h \Rightarrow \sigma = 0$. Thus the integral

$$\int_{kh}^{(k+1)h} e^{A((k+1)h-\tau)}B\underline{u}(k)\mathrm{d}\tau = \left(-\int_h^0 e^{A\sigma}\mathrm{d}\sigma\right)B\underline{u}(k) = \left(\int_0^h e^{A\sigma}\mathrm{d}\sigma\right)B\underline{u}(k).$$

Letting $A_{\mathrm{d}} := e^{Ah}$ and $B_{\mathrm{d}} := \left(\int_0^h e^{A\sigma}\mathrm{d}\sigma\right)B$, we arrive at $\underline{x}(k+1) = A_{\mathrm{d}}\,\underline{x}(k) + B_{\mathrm{d}}\,\underline{u}(k)$, as desired.