Lec. 3 (04/07/2020)

Example:

(Pendulum with air damping)

$$\boxed{\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\alpha \sin x_1 - \beta x_2 , \end{aligned}}$$

$\alpha > 0$.
$\beta > 0$.

$$V(x_1, x_2) = \underbrace{\alpha(1 - \cos x_1)}_{\text{Potential energy}} + \underbrace{\frac{1}{2} x_2^2}_{\text{kinetic energy}}$$

$$V(0,0) = 0$$

$$V(x_1, x_2) > 0 \quad \forall\, (x_1, x_2) \in (S^1 \times \mathbb{R}) \setminus \{0,0\}$$

$$\dot{V} = \langle \nabla V, \underline{f} \rangle$$

$$= \frac{\partial V}{\partial x_1} f_1 + \frac{\partial V}{\partial x_2} f_2$$

$$= (\alpha \sin x_1)\, x_2 + (x_2)(-\alpha \sin x_1 - \beta x_2)$$

$$= \cancel{\alpha x_2 \sin x_1} - \cancel{\alpha x_2 \sin x_1} - \beta x_2^2$$

$$= -\beta x_2^2 \le 0$$

$\Rightarrow$ (0, 0) is STABLE but not sure if it is A.S.

LaSalle Invariance Principle:

(Handling the case $\dot{V} \leq 0$)

Theorem: Let $\Omega \subset \mathcal{D}$ be a compact set such that Ω is positively invariant in time, w.r.t. dynamics $\dot{\underline{x}} = \underline{f}(\underline{x})$.

Let $V : \mathcal{D} \mapsto \mathbb{R}$ be $C^1(\mathcal{D})$ function such that

① $\dot{V} \leq 0 \quad \forall \; \underline{x} \in \Omega$

② Let $\underline{E \subset \Omega}$ such that $\boxed{\dot{V} = 0}$ $\forall \; \underline{x} \in E$

③ Let M be the largest invariant set in E.

Then, every sol^{n} starting in Ω, approaches M as $t \to \infty$ ($\Leftrightarrow$ M is A.S.)

Special case: (Lasalle Invariance Principle for fixed pt.s)

Let $\underline{x}^* = \underline{0}$ be a fixed point.

Let $V : \mathcal{D} \mapsto \mathbb{R}$

① V is positive definite ($V(\underline{0}) = 0$, $V(\underline{x} \neq \underline{0}) > 0$)

② $\dot{V} \leq 0$ in $\mathcal{D}$

③ $\mathcal{S} := \{\underline{x} \in \mathcal{D} \mid \dot{V}(\underline{x}) = 0\}$ and suppose

No solution can stay identically in $\mathcal{S}$

Other than the trivial solution $\underline{x}(t) = \underline{0}$.

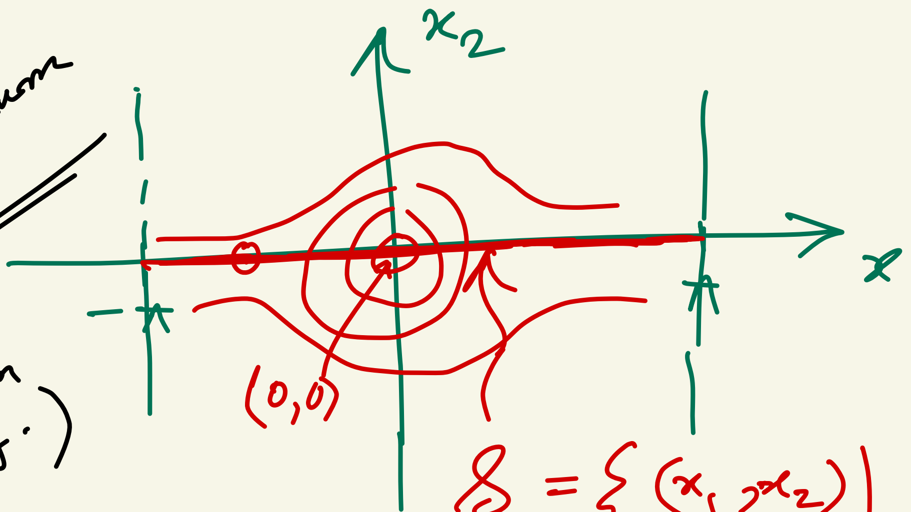
Then $\underline{x}^* = \underline{0}$ is A.S.

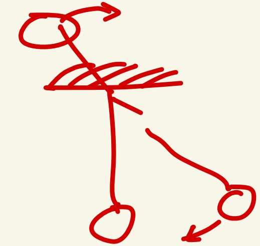
In our "Pendulum with damping" Example: $\dot{V} = -\beta x_2^2$

$$\mathcal{S} = \{(x_1, x_2) \in S^1 \times \mathbb{R} \mid x_2 = 0\}$$

The set S for pendulum in air

(see phase portrait in next pg.)

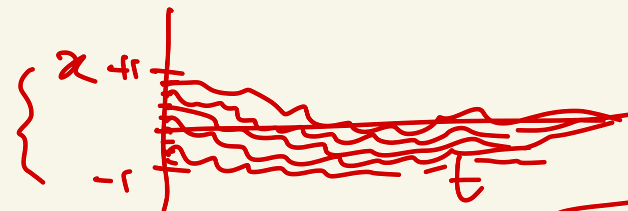




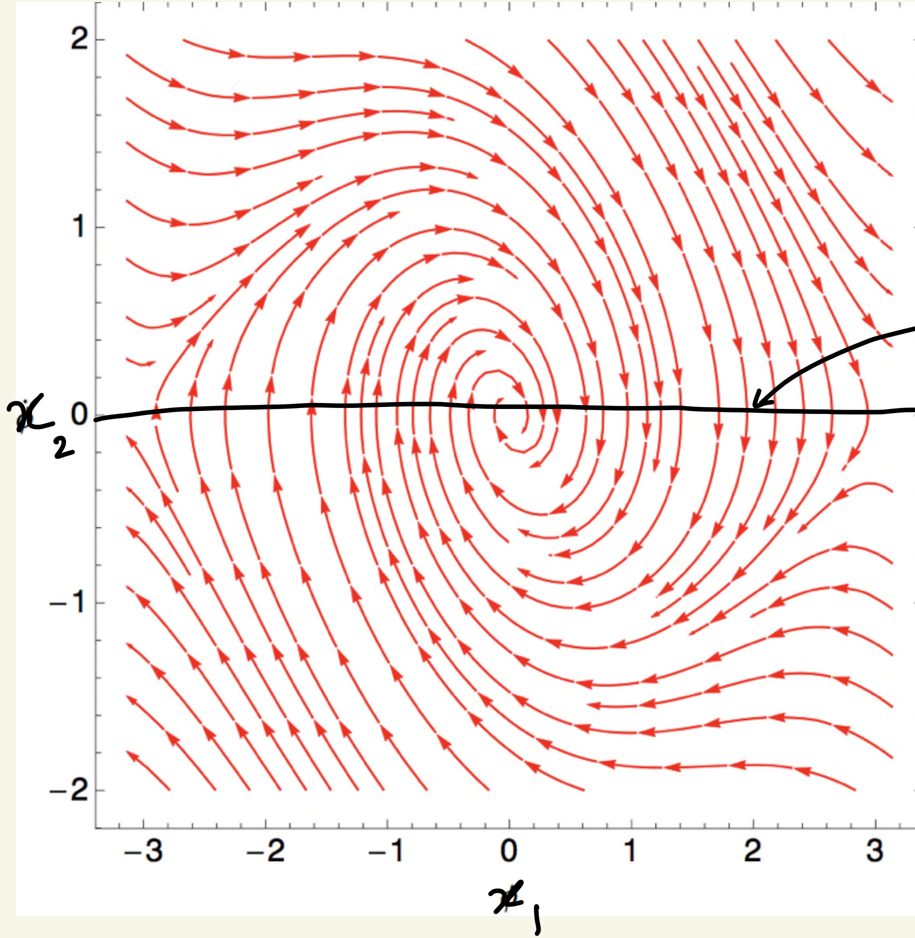
$$S = \{(x_1, x_2) \mid x_2 = 0\}$$

$\therefore$ $(0,0)$ is A.S.

$x_0 \in [-1, +1]$



Lyapunov function $\iff$ Lyapunov certificate



The set

$S := \{(x_1, x_2) \mid x_2 = 0\}$

superimposed with "pendulum in air" vector field

Other than the trivial sol$^{\underline{n}}$ $(0,0)$ all other points on S <u>cannot</u> stay identically on S.

__Example:__ (LaSalle Invariance for Limit Cycle A.S.)

$$\dot{x}_1 = x_2 - x_1(x_1^4 + 2x_2^2 - 10)$$
$$\dot{x}_2 = -x_1^3 - 3x_2^5(x_1^4 + 2x_2^2 - 10)$$

Prove that the set $x_1^4 + 2x_2^2 = 10$ is invariant

(stable limit cycle)

To show invariance:

$$\frac{d}{dt}(x_1^4 + 2x_2^2 - 10)$$
$$= -(4x_1^4 + 12x_2^6)\underbrace{(x_1^4 + 2x_2^2 - 10)}_{= 0 \text{ (on the curve)}}$$
$$= 0 \text{ (on the set)}$$

$\therefore$ Motion on the invariant set:

$$\left.\begin{aligned} \dot{x}_1 &= x_2 - \cancel{x_1(0)} \to 0 &&= x_2 \\ \dot{x}_2 &= -x_1^3 - \cancel{3x_2^5(0)} \to 0 &&= -x_1^3 \end{aligned}\right\}$$

Why A. S. ?

$$\underbrace{V := \left(x_1^4 + 2x_2^2 - 10\right)^2}_{\text{measures distance to the limit cycle}},$$

$$\dot{V} = \frac{\partial V}{\partial x_1} f_1 + \frac{\partial V}{\partial x_2} f_2$$

$$= -8 \left(x_1^4 + 2x_2^2 - 10\right)^2 \left(x_1^4 + 3x_2^6\right)$$

≤ 0 (neg. semi-definite) $\rightsquigarrow$ Stable but how to show A. S. ?

use LaSalle

$$\mathcal{E} := \left\{ \underline{x} \in \mathbb{R}^2 \mid \dot{V} = 0 \right\}$$

$$= \left\{ \underline{x} \in \mathbb{R}^2 \mid \text{either } x_1^4 + 2x_2^2 - 10 = 0 \text{ or } x_1^4 + 3x_2^6 = 0 \right\}$$

$$= \mathcal{E}_1 \cup \mathcal{E}_2$$

$$= \underbrace{\{(x_1, x_2) \in \mathbb{R}^2 \mid x_1^4 + 2x_2^2 - 10 = 0\}}_{\mathcal{E}_1} \quad \cup$$

(limit cycle itself)

$$\underbrace{\{(x_1, x_2) \in \mathbb{R}^2 \mid x_1^4 + 3x_2^6 = 0\}}_{\mathcal{E}_2}$$

(iff $x_1 = 0, x_2 = 0$
origin)

$$\therefore \quad M \equiv \mathcal{E} \subset \Omega_c := \{ \underline{x} \in \mathbb{R}^2 \mid V(\underline{x}) \leq c \}$$

For any c such that Ω_c contains both the limit cycle and the origin

Then, for any $\underline{x}_0 \in \Omega_c$, $\underline{x}(t)$ will converge <u>either</u> to the limit cycle

<u>or</u> to origin.

But if we choose : $\Omega_c := \{\underline{x} \in \mathbb{R}^2 \mid V(\underline{x}) \le c\}$

where $c = 100 - \varepsilon$, for some $\varepsilon > 0$, then the

$\boxed{\text{origin} \notin \Omega_c}$ but $\boxed{\text{limit cycle} \in \Omega_c}$

Then for any $\underline{x}_0 \in \Omega_c := \{\underline{x} \in \mathbb{R}^2 \mid V(\underline{x}) \le 100 - \varepsilon, \varepsilon > 0\}$

$\lim_{t \to \infty} \underline{x}(t) \to$ limit cycle $\Leftrightarrow$ limit cycle is A.S.

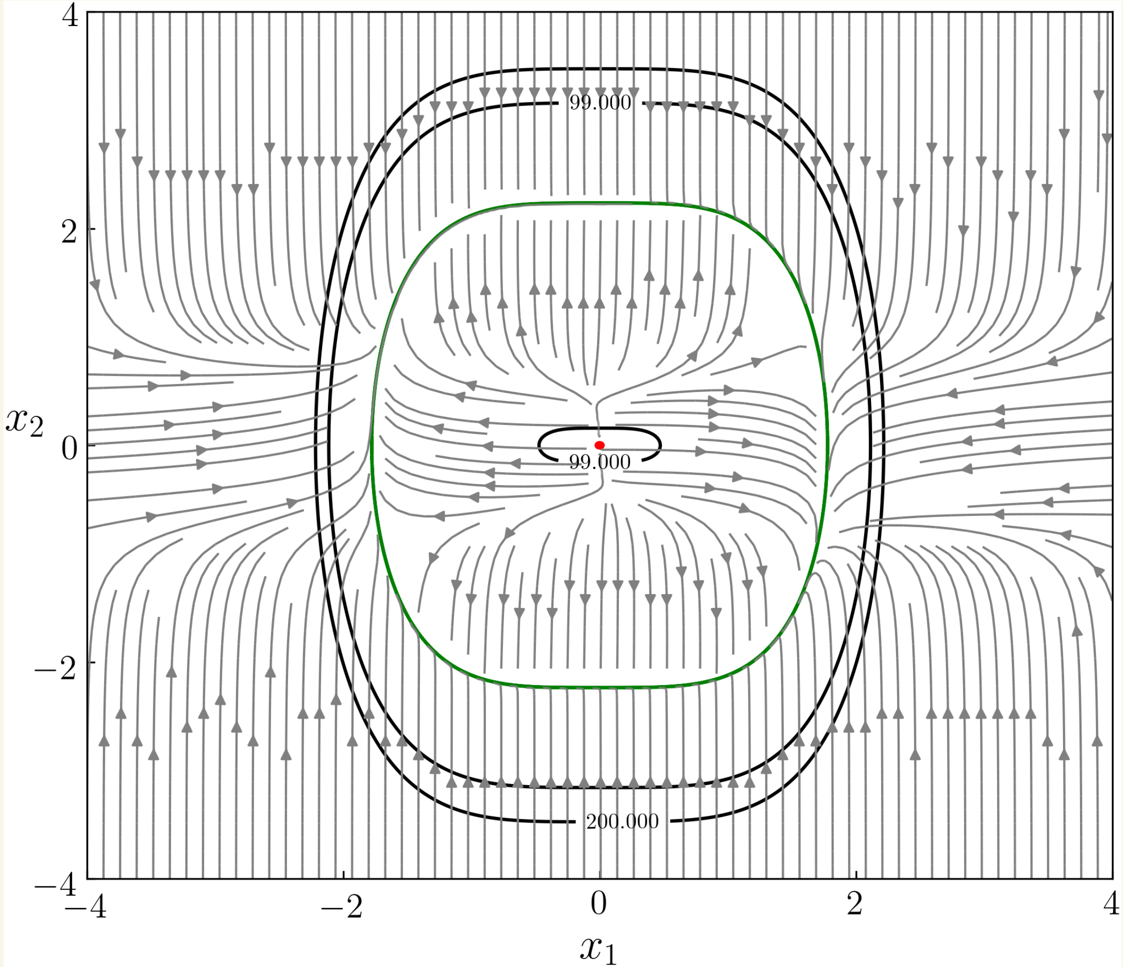
But the choice of $\varepsilon > 0$ is arbitrary,

so origin is an <u>unstable fixed point.</u>

In this case, $M = \Sigma_1 = \Sigma \subset \Omega_c := \{\underline{x} \in \mathbb{R}^2 \mid V(\underline{x}) \le 100 - \varepsilon, \varepsilon > 0\}$

fig. <u>next page</u>

$\Omega := \{\underline{x} \in \mathbb{R}^2 \mid V(x) \leq 200\}$ is the disc enclosing the limit cycle & the origin



A.S. limit cycle

unstable origin

$\Omega := \{\underline{x} \in \mathbb{R}^2 \mid V(x) \leq 99\}$ is the annular region that excludes origin

- Chetaev's Theorem (used to prove that $\underline{x}^* = \underline{0}$ is unstable)

Let $\mathcal{D} \subset \mathbb{R}^n$ be a domain that contains $\underline{x}^* = \underline{0}$.

Let $V : \mathcal{D} \mapsto \mathbb{R}$ be a $C^1(\mathcal{D})$ function such that

① $V(\underline{0}) = 0$.

② $V(\underline{x}_0) > 0$ for some $\underline{x}_0$ with arbitrarily small $\|\underline{x}_0\|_2$.

③ Choose $r > 0$ such that the ball

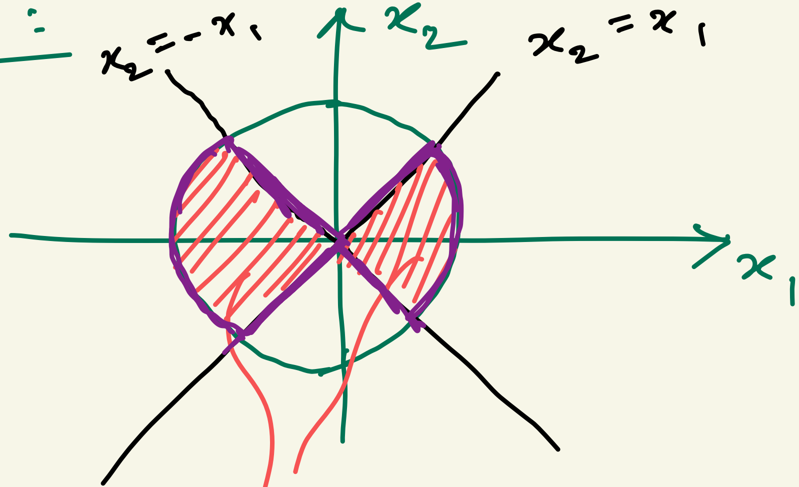
$$\mathcal{B}_r := \{\underline{x} \in \mathbb{R}^n \mid \|\underline{x}\|_2 \leq r\}$$ is contained in $\mathcal{D}$,

and let $\tilde{\mathcal{U}} := \{\underline{x} \in \mathcal{B}_r \mid V(\underline{x}) > 0\}$

Suppose $\dot{V} > 0 \ \forall \ \underline{x} \in \tilde{\mathcal{U}}$.

Then, $\underline{x}^* = \underline{0}$ is unstable

Example:



$$V(x_1, x_2) = \frac{1}{2}(x_1^2 - x_2^2)$$

$$V(\underline{0}) = 0$$

$$\boxed{\dot{V} > 0 \ \forall\, \underline{x} \in \tilde{\mathcal{U}}}$$

$\tilde{\mathcal{U}} \subset B_r$

non-empty

its boundary is the surface $V(\underline{x}) = 0$ and the sphere $\|\underline{x}\|_2 = r$

$V(\underline{0}) = 0 \Rightarrow$ origin $\in \partial \tilde{U}$

$$\dot{x}_1 = x_1 + g_1(x_1, x_2)$$
$$\dot{x}_2 = -x_2 + g_2(x_1, x_2)$$

where $g_1(\cdot)$ & $g_2(\cdot)$ are locally Lipschitz in $\underline{x}$ such that in a neighborhood $\mathcal{D}$ of the origin, we get

$$|g_1(\underline{x})| \leq K \|\underline{x}\|_2^2$$
$$|g_2(\underline{x})| \leq K \|\underline{x}\|_2^2$$

$$g_1(0,0) = g_2(0,0) = 0$$

$\Updownarrow$

((0,0) is fixed point)

Use the function:

$$V(x_1, x_2) = \frac{1}{2}(x_1^2 - x_2^2)$$

to show that origin is unstable.

Lyapunov Theory for non-autonomous systems:

Set up: $\dot{\underline{x}} = \underline{f}(t, \underline{x})$, $\underline{x}(t_0) = \underline{x}_0$ (given)

→ $\underline{f}$ is piecewise continuous in t, and locally Lipschitz in $\underline{x}$

→ $\underline{x}^* = \underline{0}$ is a fixed point if $\underline{f}(t, \underset{\underline{0}}{\cancel{\underline{x}^*}}) = \underline{0}$

for all $t \geq t_0 \geq 0$

→ $\underline{f} : [0, \infty) \times \mathcal{D} \mapsto \mathbb{R}^n$, where $\underline{x}^* \in \mathcal{D}$.

Autonomous system $\underline{x}(t - t_0)$	Non-autonomous system $\underline{x}(t, t_0)$
Next pg.	Next pg.

Autonomous System $\underline{x}(t-t_0)$	Non-autonomous System $\underline{x}(t, t_0)$
Solⁿ: $\underline{x}$ is a fⁿ of $(t-t_0)$	Solⁿ $\underline{x}$ is a fⁿ of t & t_0
e.g. $\dot{x} = -x$, $x(t_0) = x_0$ $\Rightarrow x(t) = x_0 \exp(-(t-t_0))$	e.g. $\dot{x} = -\dfrac{x}{1+t}$ $\Rightarrow x(t) = x_0\left(\dfrac{1+t_0}{1+t}\right)$

Stability of non-autonomous System:

Autonomous	Non-autonomous
$\forall \epsilon > 0,\ t_0 \geq 0,\ \exists \delta = \delta(\epsilon) > 0$ s.t. $\lVert \underline{x}(t_0) \rVert_2 < \delta \Rightarrow \lVert \underline{x}(t) \rVert_2 < \epsilon$ $\forall t \geq t_0$	$\forall \epsilon > 0$