Lecture # 11 (05/05/2020)

SPR characterization theorem:

Let $G(s)$ be $p \times p$ proper rational transfer matrix.

Let $\det\left(G(s) + G^T(-s) \neq 0\right)$.

Then $G(s)$ is SPR iff

1′ $G(s)$ is Hurwitz $\Leftrightarrow$ poles of all elements of $G(s)$ have negative real part

2′ $(G + G^*)(j\omega) > 0 \quad \forall\, \omega \in \mathbb{R}$

3′ either $G(\infty) + G^T(\infty) > 0$

or $\begin{cases} G(\infty) + G^T(\infty) \geq 0 \\ \text{and } \lim\limits_{\omega \to \infty} \omega^2 M^T\left((G + G^*)(j\omega)\right) M > 0 \end{cases}$

$\forall$ full rank M of size $p \times (p - q)$ (contd) such that $M^T\left(G(\infty) + G^T(\infty)\right) M = 0$

where $q := \text{rank}\left(G(\infty) + G^T(\infty)\right)$.

SPR for $p = 1$ (scalar/transfer function case):

(2') becomes $\text{Re}\left(G(j\omega)\right) > 0 \quad \forall\, \omega \in \mathbb{R}$

(3') " $\begin{cases} \text{either } G(\infty) > 0 \\ \text{or} \quad G(\infty) = 0 \text{ and } \lim\limits_{\omega\to\infty} \omega^2 \text{Re}(G(j\omega)) > 0 \end{cases}$.

Exercise:

- $G(s) = \dfrac{1}{s}$ is PR but not SPR (because $\dfrac{1}{s-\epsilon}$ has a pole in $\text{Re}(s) > 0$ for any $\epsilon > 0$)

- Show that $G(s) = \dfrac{1}{s+a}$, $a > 0$ is PR, also SPR.

- $G(s) = \dfrac{1}{s^2+s+1}$ is not PR

- $G(s) = \dfrac{1}{s+1}\begin{bmatrix} s & 1 \\ -1 & 2s+1 \end{bmatrix}$ is SPR.

- $G(s) = \begin{bmatrix} \dfrac{s}{s+1} & \dfrac{1}{s+2} \\ -\dfrac{1}{s+2} & \dfrac{2}{s+1} \end{bmatrix}$ is SPR.

It is possible to check PRness / SPRness in state space.

Positive Real Lemma: Consider minimal LTI system (A, B, C, D) with $m \times m$ transfer matrix $G(s)$

$$\left(\Leftrightarrow A \in \mathbb{R}^{n\times n},\ B \in \mathbb{R}^{n\times m},\ C \in \mathbb{R}^{m\times n},\ D \in \mathbb{R}^{m\times m} \right)$$

$G(s)$ is PR $\Leftrightarrow$ $\exists \begin{cases} P > 0, & P \in \mathbb{R}^{n\times n} \\ L \in \mathbb{R}^{n\times m}, & W \in \mathbb{R}^{m\times m} \end{cases}$

s.t.

$$PA + A^T P = -LL^T \quad \ldots\ldots (*)$$

$$PB - C^T = -LW \quad \ldots\ldots (**)$$

$$D + D^T = W^T W \quad \ldots\ldots (***)$$

Gives a factorization of $D+D^T$. If $D = 0$, then $W = 0$.

Remark: We can rewrite (*), (**), (***) as :

$$\begin{bmatrix} -PA - A^T P & C^T - PB \\ C - B^T P & D + D^T \end{bmatrix} = \begin{bmatrix} L \\ W^T \end{bmatrix} \begin{bmatrix} L^T & W \end{bmatrix} \succeq 0$$

KYP Lemma : Same setting as PR Lemma.

$G(s)$ is SPR $\Leftrightarrow$ $\exists \left\{ \begin{array}{l} P \succ 0, \ P \in \mathbb{R}^{n \times n} \\ L \in \mathbb{R}^{n \times m}, \ W \in \mathbb{R}^{m \times m}, \ \varepsilon > 0 \end{array} \right.$

s.t. $PA + A^T P = -LL^T - \varepsilon P \quad \ldots (*)'$

$PB - C^T = -LW \quad \text{---} (**)'$

$D + D^T = W^T W \quad \text{---} (***)'$

Proof of KYP lemma (uses PR lemma):

($\Rightarrow$) Suppose $\exists\, P > 0$, L, W, $\varepsilon > 0$ satisfying $(*)'$, $(**)'$, $(***)'$.

Set $\mu := \varepsilon/2$, and recall:

$$G(s-\mu) = C(sI - \mu I - A)^{-1}B + D.$$

From $(*)'$, we have:

$$P(A+\mu I) + (A+\mu I)^T P = -LL^T$$

Then, from PR lemma, $G(s-\mu)$ is PR

$\Rightarrow$ $G(s)$ is SPR. $\blacksquare$

($\Leftarrow$) Suppose $G(s)$ is SPR.

Then, $\exists\, \mu > 0$ s.t. $G(s-\mu)$ is PR.

By PR lemma, $\exists\, P > 0$, L, W s.t.

$$\begin{cases} P(A+\mu I) + (A+\mu I)^T P = -LL^T \\ (**)' \\ (***)' \end{cases}$$

Setting $\varepsilon := 2\mu$, we recover $(*)'$, $(**)'$, $(***)'$. $\blacksquare$

Summary of Passivity Results for LTI system:

PR/KYP Lemma

$\Updownarrow$

$G(s)$ is PR/SPR

$\Updownarrow$

LTI system is Passive/Strictly passive with quadratic storage function $V(x) = \frac{1}{2}x^T P x$

$\Updownarrow$ (next pg.)

LTI dissipativity with quadratic storage function.

Energy stored upto time t

(i.e.) $$\underbrace{\int_0^t (u(\tau))^T y(\tau)\, d\tau}_{\text{Externally supplied energy until time } t} = \boxed{V(x(t)) - V(x_0)} - \frac{1}{2}\int_0^t x^T(\tau)\,(A^TP+PA)\,x(\tau)\, d\tau$$

$$= \frac{1}{2}\int_0^t x^T(\tau)\, Q\, x(\tau)\, d\tau$$

(where $Q \succ 0$ and $-Q := A^TP + PA$)

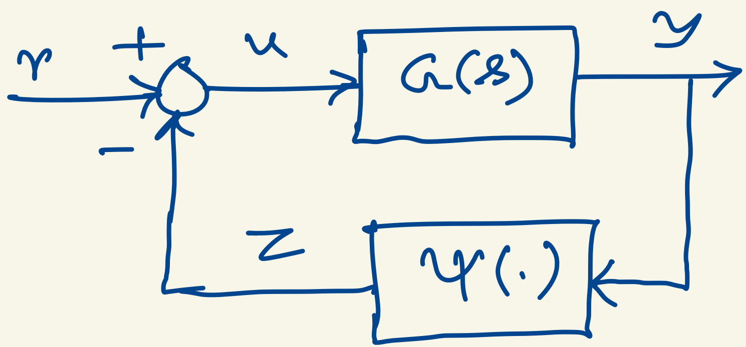
Dissipated energy

Here,

$$V(x) = \frac{1}{2}x^T P x$$

Absolute Stability

LTI system with nonlinearity in the feedback loop :



We will analyze the case $r = 0$.

$$\dot{\underline{x}} = A\underline{x} + B\underline{u}, \quad \underline{x}(0) = \underline{x}_0$$
$$\underline{y} = C\underline{x} + D\underline{u},$$
$$\underline{u} = -\underline{\psi}(t, \underline{y})$$

$\underline{x} \in \mathbb{R}^n$; $\underline{u}, \underline{y} \in \mathbb{R}^m$

Example :

If $D = 0$, then

$$\dot{\underline{x}} = Ax - B\psi(t, Cx)$$

(separates linear part & nonlinear part of closed-loop dynamics)

Setting of the problem:

- The static memoryless nonlinearity $\psi : [0, \infty) \times \mathbb{R}^m \longmapsto \mathbb{R}^m$ is piecewise continuous w.r.t. time t, and locally Lipschitz in $\underline{y}$.

- Assume (A, B, C, D) minimal ($\Leftrightarrow$ (A, B) is controllable pair, (A, C) is observable pair)

- Assume that $\underline{u} = -\psi(t, C\underline{x} + D\underline{u})$ has unique solution $\underline{u}$ for all pairs $(t, \underline{x})$.
(Can show that this always holds for $D = 0$).

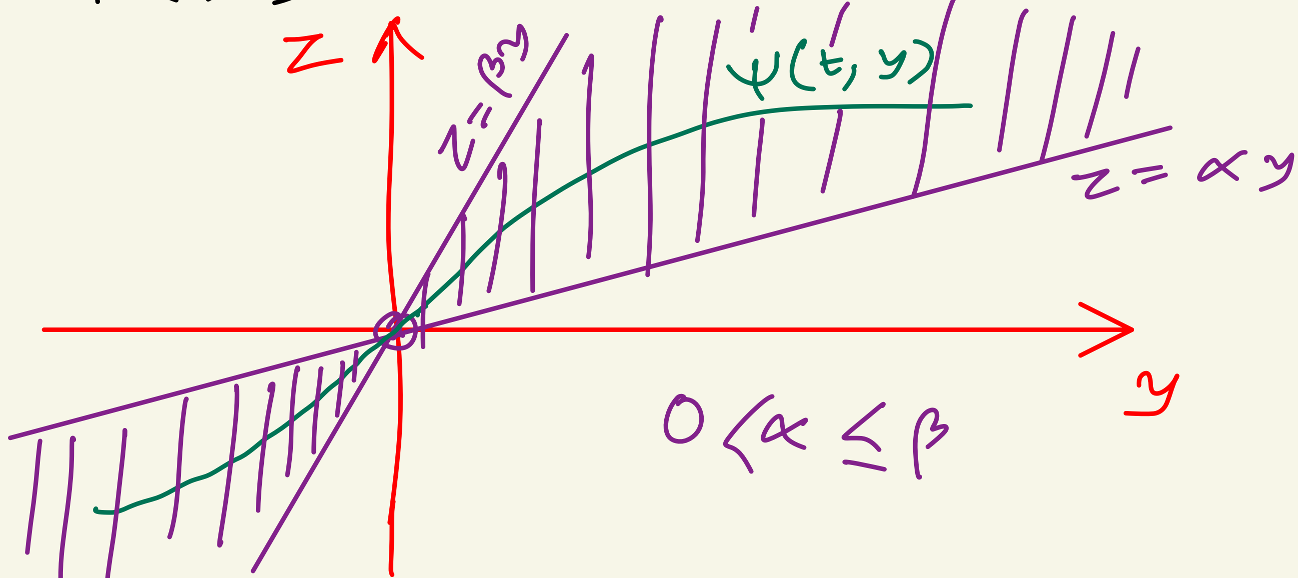
Motivating Sector bounded nonlinearity $\psi(\cdot)$:

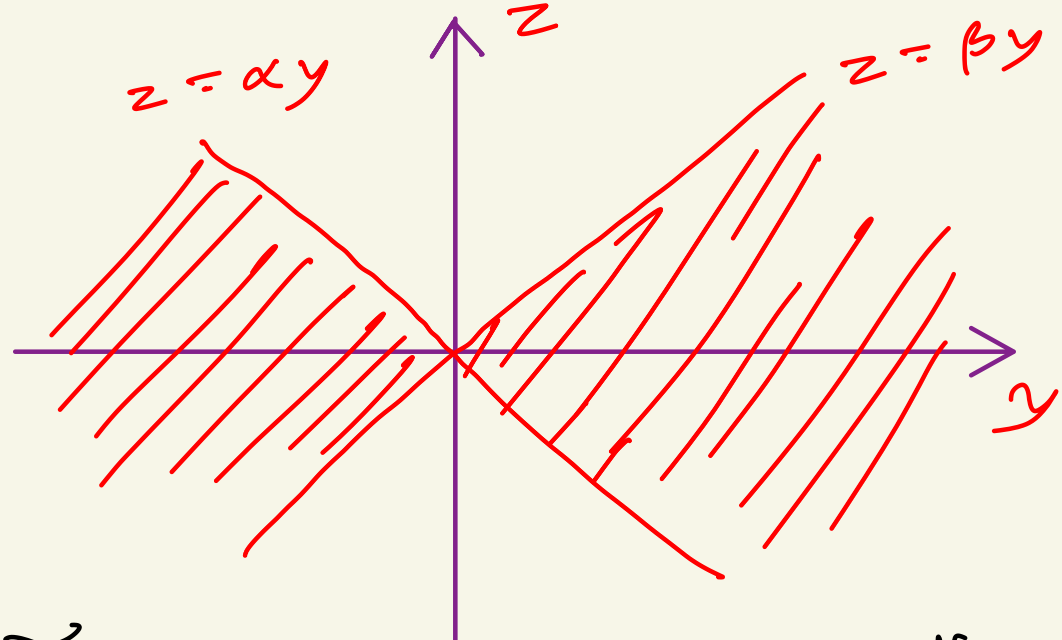
Scalar case: Consider $z = \psi(t, y)$ s.t.

$$\boxed{\alpha y \le \psi(t, y) \le \beta y}$$

$\forall (t, y)$ where $\alpha \le \beta$.

Graph of $z = \psi(t, y)$ looks like





$\alpha < 0 < \beta$

To say $\psi(t, y)$ satisfies the sector bounded nonlinearity with interval $[\alpha, \beta]$, is same as writing a quadratic inequality:

$$(z - \alpha y)(z - \beta y) \leq 0 \quad \forall (y, z) \text{ s.t. } z = \psi(t, y) \ \forall t.$$

for sector $[\alpha, \beta]$

<u>Example:</u> (1D)

$\psi \in$ Sector $[-1,+1] \Leftrightarrow |\psi(y)| \leq |y|$.

<u>Definition:</u>

Let $\underline{\alpha} := (\alpha_1, \ldots, \alpha_m)$

$\underline{\beta} := (\beta_1, \ldots, \beta_m)$

$\psi(t, y) : [0, \infty) \times \mathbb{R}^m \mapsto \mathbb{R}^m$

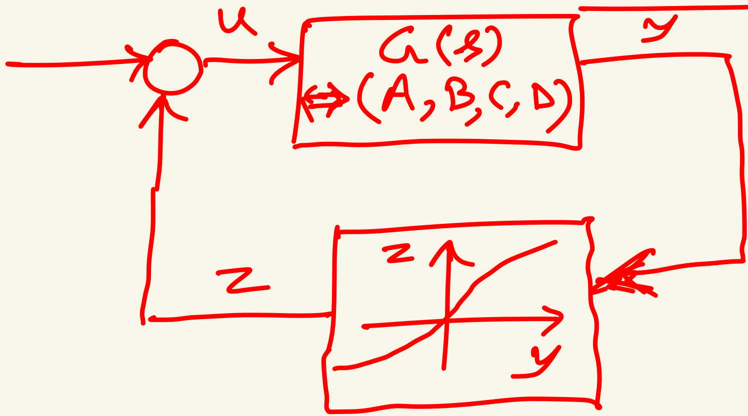
A static memoryless function ψ is said to belong to sector

- $[\underline{0}, \underline{\infty}]$ if $\underline{y}^T \underline{\psi}(t, \underline{y}) \geq 0$
- $[\underline{\alpha}, \underline{\infty}]$ if $\underline{y}^T (\underline{\psi}(t, \underline{y}) - \underline{\alpha} \odot \underline{y}) \geq 0$
- $[\underline{0}, \underline{\beta}]$ if $\underline{\psi}^T(t, \underline{y}) (\psi(t, y) - \beta \odot \underline{y}) \leq 0$

• $[\underline{\alpha}, \underline{\beta}]$ if $\left(\underline{\Psi}(t,\underline{y}) - \underline{\alpha} \odot \underline{y}\right)^T \left(\underline{\Psi}(t,\underline{y}) - \underline{\beta} \odot \underline{y}\right) \leq 0$.

If in any case, the inequality is strict, then we write the sector notation as $(\underline{0}, \underline{\infty})$, $(\underline{\alpha}, \underline{\infty})$, $(\underline{0}, \underline{\beta})$, or $(\underline{\alpha}, \underline{\beta})$.

Lure's Problem



$\Psi \in$ Sector $[\underline{\alpha}, \underline{\beta}]$

time-varying and/or uncertain nonlinearities in the loop

Goal: Prove stability using only sector info. (i.e., not for a specific system, but for a family of nonlinearities Ψ)

Defn. : We say the system is Absolutely stable if origin of this system is GUAS for all ψ in the given sector.

It is (locally) absolutely stable if origin is UAS in some domain $\mathcal{D}$.

Lyapunov function $V(x) = \frac{1}{2} x^T P x$. (next class).