AMS 231: Nonlinear Control Theory: Winter 2018

Homework #3

Name: ....

Due: February 13, 2018

NOTE: Please show all the steps in your solution. Turn in a hard copy of your HW stapled with this as cover sheet with your name written in the above field. Please submit your HW in class on the due date.

Problem 1

Lyapunov Theory for Continuous-time LTI System

($(2+7)+(2+2+2)+3+2+(10+10)+5 = 45$ **points)**

Consider the n-dimensional LTI system

$$\dot{\underline{x}} = A\,\underline{x}, \qquad A \in \mathbb{R}^{n\times n}, \qquad \underline{x}(0) = \underline{x}_0.$$

Since origin is the unique fixed point, the notions of A.S. and G.A.S. coincide.

(a) Write down the explicit solution for $\underline{x}(t)$ in terms of A, t and $\underline{x}_0$. Substituting this solution in the condition $\lim\limits_{t\to\infty} \underline{x}(t) = \underline{0}$, prove that origin is G.A.S. iff all eigenvalues of A lie in the open left half (complex) plane. Such a matrix is called Hurwitz. (Hint: Be careful about non-diagonalizable A.)

(b) A symmetric matrix P is called "positive (resp. negative) definite matrix" if $\underline{x}^\top P \underline{x} >$ (resp. $<$) 0 for all $\underline{x} \in \mathbb{R}^n$. Symbolically, we write $P \succ$ (resp. $\prec$) 0, which needs to be understood as matrix inequality. So for example, $P_1 \succ P_2$ means that $P_1 - P_2 \succ 0$.

For any given $P \succ 0$, prove that $V(\underline{x}) = \underline{x}^\top P \underline{x}$ is a positive definite function. Geometrically, what do the level sets of such a function V represent? Argue whether such V is radially bounded or unbounded.

(c) Motivated by your arguments in part (b), use $V(\underline{x}) = \underline{x}^\top P \underline{x}$ as the Lyapunov function to prove that the LTI system is G.A.S. if the matrix function $\mathcal{L}(P) := A^\top P + PA \prec 0$. This condition is called "Lyapunov matrix inequality".

(d) Argue that the condition $A^\top P + PA \prec 0$ in part (c) is equivalent to the statement: for any $Q \succ 0$, there exists $P \succ 0$ that solves the linear matrix equation $\mathcal{L}(P) = -Q$. This equation is called "Lyapunov (algebraic) matrix equation".

(e) We have shown in parts (b), (c), (d) that existence of solution for the Lyapunov matrix equation (equivalently, Lyapunov matrix inequality) implies G.A.S. i.e., A is Hurwitz. Now prove the converse, i.e., if A is Hurwitz then for any $Q \succ 0$, there exists unique $P \succ 0$ that solves $\mathcal{L}(P) = -Q$. (Hint: Prove existence by construction. Prove uniqueness by contradiction.)

(f) For an LTI system with A Hurwitz, prove that if $Q_1 \succ Q_2$, then $P_1 \succ P_2$.

Solution

(a) The solution is $\underline{x}(t) = e^{At}\underline{x}_0$.

Let us use the symbol $\mathbb{C}_-^\circ$ to denote the open left-half complex plane. Substituting the above solution into the GAS condition $\lim\limits_{t\to\infty} \underline{x}(t) = \underline{0}$ results in the requirement $\lim\limits_{t\to\infty} e^{At} = 0$.

If A is diagonalizable, then there exists non-singular matrix T such that $A = T\Lambda T^{-1}$, where the diagonal matrix $\Lambda := \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$, and λ_i denotes the i-th eigenvalue of A. Then $e^{At} = T\,\mathrm{diag}(e^{\lambda_1 t}, \ldots, e^{\lambda_n t})\,T^{-1}$. Therefore,

$$\lim_{t\to\infty} e^{At} = \lim_{t\to\infty} \mathrm{diag}(e^{\lambda_1 t}, \ldots, e^{\lambda_n t}) = 0 \;\Leftrightarrow\; \lim_{t\to\infty} e^{\lambda_i t} = 0 \;\Leftrightarrow\; \lambda_i \in \mathbb{C}_-^\circ \quad \text{for all} \quad i = 1, \ldots, n.$$

If A is non-diagonalizable, then there exists non-singular matrix T such that $A = TJT^{-1}$, where the block-diagonal Jordan matrix $J := \mathrm{diag}(J_1, \ldots, J_n)$, where each Jordan block J_i is upper-triangular with λ_i along its diagonal. Consequently, we have $e^{At} = T\mathrm{diag}(e^{J_1 t}, \ldots, e^{J_n t})T^{-1}$. Therefore,

$$\lim_{t\to\infty} e^{At} = \lim_{t\to\infty} \mathrm{diag}(e^{J_1 t}, \ldots, e^{J_n t}) = 0 \Leftrightarrow \lim_{t\to\infty} e^{J_i t} = \lim_{t\to\infty} e^{\lambda_i t} = 0 \Leftrightarrow \lambda_i \in \mathbb{C}_-^\circ \quad \text{for all} \quad i = 1, \ldots, n.$$

Hence the statement.

(b) Clearly, $V(\underline{0}) = 0$ iff $\underline{x} = \underline{0}$; and thanks to P being a positive definite matrix, we also have $V(\underline{x}) = \underline{x}^\top P\underline{x} > 0$ for all $\underline{x} \neq \underline{0}$. Therefore, V is a positive definite function of vector $\underline{x}$.

Geometrically, the level sets of such a function $V(\underline{x}) = \underline{x}^\top P\underline{x}$ are ellipsoids.

Radially unbounded since $\lim\limits_{r\to\infty} \underline{x}^\top P\underline{x} = \lim\limits_{r\to\infty} \mathrm{trace}\left(\left(\underline{x}\underline{x}^\top\right) P\right) = \mathrm{trace}\left(\left(\lim\limits_{r\to\infty} \left(\underline{x}\underline{x}^\top\right)\right) P\right) = \infty$.

(c) From part (b), we know that $V(\underline{x}) = \underline{x}^\top P\underline{x}$ satisfies $V(\underline{x}) = 0$ iff $\underline{x} = 0$, and > 0 for all $\underline{x} \neq 0$. Now

$$\dot{V} = \langle \nabla V, A\underline{x}\rangle = (2P\underline{x})^\top A\underline{x} = 2\,\underline{x}^\top PA\,\underline{x} = \underline{x}^\top\left(A^\top P + PA\right)\underline{x},$$

where the last step follows from the fact that a scalar (in this case $\underline{x}^\top PA\,\underline{x}$) must be equal to its own transpose. (Alternatively, we can derive the same expression from the product rule of differentiation: $\dot{V} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\underline{x}^\top P\underline{x}\right) = \dot{\underline{x}}^\top P\underline{x} + \underline{x}^\top P\dot{\underline{x}} = (A\underline{x})^\top P\underline{x} + \underline{x}^\top PA\underline{x}$, etc.)

From Lyapunov's theorem, A.S. is guaranteed if $\dot{V} < 0 \, \forall \, \underline{x} \neq \underline{0} \Leftrightarrow \mathcal{L}(P) := A^\top P + PA \prec 0$. Since origin is the unique fixed point for LTI system, and our choice of V is radially unbounded (as shown in part (b)), hence by Barbashin-Krasovskii theorem, G.A.S. is guaranteed if $\mathcal{L}(P) \prec 0$.

(d) This follows from the fact that negative of a positive definite matrix is negative definite, i.e., $Q \succ 0 \Leftrightarrow -Q \prec 0$. Therefore, $\mathcal{L}(P) \prec 0 \Leftrightarrow \mathcal{L}(P) = -Q$ for $Q \succ 0$.

(e) **(Existence)** Let us consider the ansatz

$$P = \int_0^\infty e^{A^\top t} \, Q \, e^{At} \, \mathrm{d}t.$$

Because A is Hurwitz, the above integral converges. Since $Q \succ 0$, and e^{At} is non-singular, hence $P \succ 0$. Now

$$\begin{aligned}\mathcal{L}(P) = A^\top P \, + \, PA = \int_0^\infty \left(A^\top e^{A^\top t} \, Q \, e^{At} + e^{A^\top t} \, Q \, e^{At} A\right) \, \mathrm{d}t &= \int_0^\infty \frac{\mathrm{d}}{\mathrm{d}t}\left(e^{A^\top t} \, Q \, e^{At}\right) \, \mathrm{d}t \\ &= \left(\lim_{t \to \infty} e^{A^\top t} \, Q \, e^{At}\right) - Q = -Q,\end{aligned}$$

where we used the fact that for A Hurwitz, we have $\lim\limits_{t \to \infty} e^{A^\top t} \, Q \, e^{At} = 0$. Since our ansatz P is $\succ 0$ and satisfies $\mathcal{L}(P) = -Q$ for any $Q \succ 0$, hence existence is guaranteed.

(Uniqueness) If possible, let us assume that $\mathcal{L}(P) = -Q$ admits two distinct solutions $P_1, P_2 \succ 0$, $P_1 \neq P_2$, for any given $Q \succ 0$. This implies

$$\mathcal{L}(P_1) - \mathcal{L}(P_2) = A^\top (P_1 - P_2) + (P_1 - P_2) \, A = 0.$$

Pre-multiplying the above by $e^{A^\top t}$ and post-multiplying by e^{At}, we get

$$e^{A^\top t} \left(A^\top (P_1 - P_2) + (P_1 - P_2) \, A\right) e^{At} = \frac{\mathrm{d}}{\mathrm{d}t}\left(e^{A^\top t} \, (P_1 - P_2) \, e^{At}\right) = 0 \Leftrightarrow e^{A^\top t} \, (P_1 - P_2) \, e^{At} = M,$$

where the matrix M does not depend on time t. Since the above must hold for all $t \geq 0$, evaluating the same at $t \to \infty$ gives $M = 0$ (again, here we used the fact that A is Hurwitz). On the other hand, at $t = 0$, we have $P_1 - P_2 = M = 0$, which contradicts our hypothesis that $P_1 \neq P_2$. Therefore, if A is Hurwitz, and a solution $P \succ 0$ exists for $\mathcal{L}(P) = -Q$, it must be unique.

(f) We have $A^\top P_1 + P_1 A = -Q_1$, and $A^\top P_2 + P_2 A = -Q_2$. Now $Q_1 \succ Q_2 \succ 0$ implies

$$e^{A^\top t} \, Q_1 \, e^{At} \succ e^{A^\top t} \, Q_2 \, e^{At} \; \Rightarrow \; \int_0^\infty e^{A^\top t} \, Q_1 \, e^{At} \, \mathrm{d}t \; \succ \; \int_0^\infty e^{A^\top t} \, Q_2 \, e^{At} \, \mathrm{d}t \; \Leftrightarrow \; P_1 \succ P_2.$$

Notice that the converse is not true.

Problem 2

Global Uniform Asymptotic Exponential Stability for Continuous-time LTV System **(25 points)**

Consider the LTV system

$$\dot{\underline{x}} = A(t)\underline{x}, \qquad \underline{x}(t_0) = \underline{x}_0,$$

where $A(t)$ is a continuous bounded function of t for all $t \geq t_0 \geq 0$. In this case, the notions of GUAS and ES coincide.

Prove that **if** there exists continuously differentiable, bounded, positive definite $P(t)$ (in other words, $0 \prec c_1 I \preceq P(t) \preceq c_2 I$, $\forall t \geq t_0 \geq 0$) that solves the linear matrix differential equation

$$-\dot{P}(t) = (A(t))^\top P(t) + P(t)A(t) + Q(t),$$

for any $Q(t)$ that is continuous and positive definite (in other words, $0 \prec c_3 I \preceq Q(t)$, $\forall t \geq t_0 \geq 0$), **then** the origin is G.E.S. (and thus G.U.A.E.S.)

Solution

Consider the Lyapunov function $V(\underline{x}, t) = \underline{x}^\top P(t)\underline{x}$, which is a positive definite and radially unbounded function, and vanishes only at origin (unique fixed point). Under the stated conditions on matrix $P(t)$, we also have

$$c_1 \parallel \underline{x} \parallel_2^2 \leq V(\underline{x}, t) \leq c_2 \parallel \underline{x} \parallel_2^2, \qquad \forall t \geq t_0 \geq 0.$$

Furthermore,

$$\begin{aligned} \dot{V} = \dot{\underline{x}}^\top P(t)\underline{x} + \underline{x}^\top \dot{P}(t)\underline{x} + \underline{x}^\top P(t)\dot{\underline{x}} &= \underline{x}^\top \left(\dot{P}(t) + (A(t))^\top P(t) + P(t)A(t)\right)\underline{x} \\ &= -\underline{x}^\top Q(t)\underline{x} \\ &\leq -c_3 \parallel \underline{x} \parallel_2^2, \qquad \forall t \geq t_0 \geq 0. \end{aligned}$$

Since the above conditions hold for all $\underline{x} \in \mathbb{R}^n$, by the exponential stability theorem for non-autonomous systems (see Lecture 8 notes, pg. 6), the origin is G.E.S. (and thus G.U.A.E.S.)

Problem 3

Region of Attraction **(5+10+15 = 30 points)**

Consider the nonlinear system

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -x_1 - x_2 + x_1^3.$$

(a) Find all isolated fixed points.

(b) By taking $V(x_1, x_2) = \frac{1}{2}x_2^2 + \int_0^{x_1} (y - y^3)\, dy$ as the Lyapunov function, prove that origin is asymptotically stable. (Hint: You may need to use LaSalle invariance theorem.)

(c) Use your answer in part (b) to estimate the region of attraction for origin.

Solution

(a) Setting $\dot{x}_1^* = 0, \dot{x}_2^* = 0$ yield three isolated fixed points: $(x_1^*, x_2^*) = (0, 0), \quad (-1, 0), \quad (+1, 0)$.

(b) The function $V(x_1, x_2) = \frac{1}{2}x_2^2 + \frac{1}{2}x_1^2 - \frac{1}{4}x_1^4$ is positive definite in the region $|x_1| < \sqrt{2}$ (this follows by setting $\frac{1}{2}x_1^2 - \frac{1}{4}x_1^4 > 0$). On the other hand, we have $\dot{V} = -x_2^2 \leq 0$, which guarantees stability but not necessarily A.S. for the origin. We notice that $\dot{V} = 0 \Rightarrow x_2(t) \equiv 0 \Rightarrow x_1(t) - x_1^3(t) \equiv 0$. Thus for $|x_1| < 1$, we can invoke LaSalle invariance theorem to conclude that origin is (locally) A.S. in $\mathscr{D} := \{(x_1, x_2) \in \mathbb{R}^2 : |x_1| < 1\}$.

(c) We seek an estimate for the region of attraction of the form $\Omega_c := \{(x_1, x_2) \in \mathbb{R}^2 : V(x_1, x_2) \leq c\}$, where an upper bound for the level $c > 0$ needs to be determined such that Ω_c is compact, positively invariant, and $\Omega_c \subset \mathscr{D}$. In this case, positive invariance condition is subsumed by compactness and $\Omega_c \subset \mathscr{D}$ conditions.

For our choice of $V(x_1, x_2)$, the condition $\Omega_c \subset \mathscr{D}$ is satisfied for $0 < c < \frac{1}{4}$. To show this explicitly, consider the level set

$$V(x_1, x_2) = c \quad \Leftrightarrow \quad \frac{1}{2}x_2^2 + \frac{1}{2}x_1^2 - \frac{1}{4}x_1^4 = c.$$

Since $x_2^2 \geq 0$, hence the above tells us

$$\frac{1}{2}\left(x_1^2\right)^2 - x_1^2 + 2c \geq 0 \quad \Leftrightarrow \quad (x_1^2 - 1)^2 + (4c - 1) \geq 0.$$

For $\Omega_c \in \mathscr{D}$, we need $|x_1| < 1 \Rightarrow x_1^2 - 1 < 0$, which combined with the above inequality results

$$0 > x_1^2 - 1 \geq -\sqrt{1 - 4c}, \quad \text{and} \quad 1 - 4c > 0;$$

the latter solved for $c > 0$ yields the bound $0 < c < \frac{1}{4}$. For such choices of c, the level set with boundary $V(x_1, x_2) = c$ is guaranteed to be within $\mathscr{D}$ (hence bounded) and closed (since $V(x_1, x_2)$ is positive definite there), therefore compact. See Fig. 1 and 2 below.

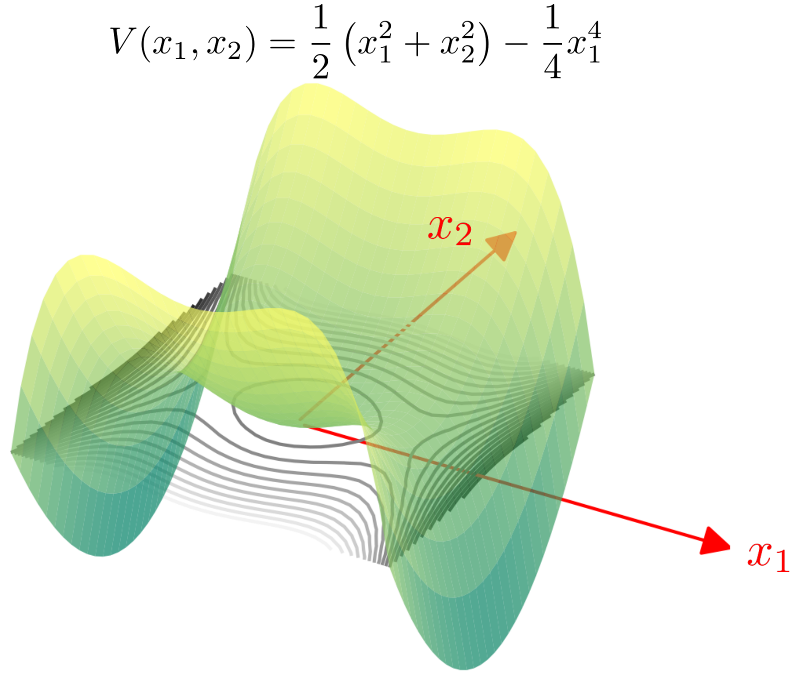

Figure 1: 3D surface plot of $V(x_1, x_2)$ as well as its level sets. The function V becomes negative for $|x_1| > \sqrt{2}$.

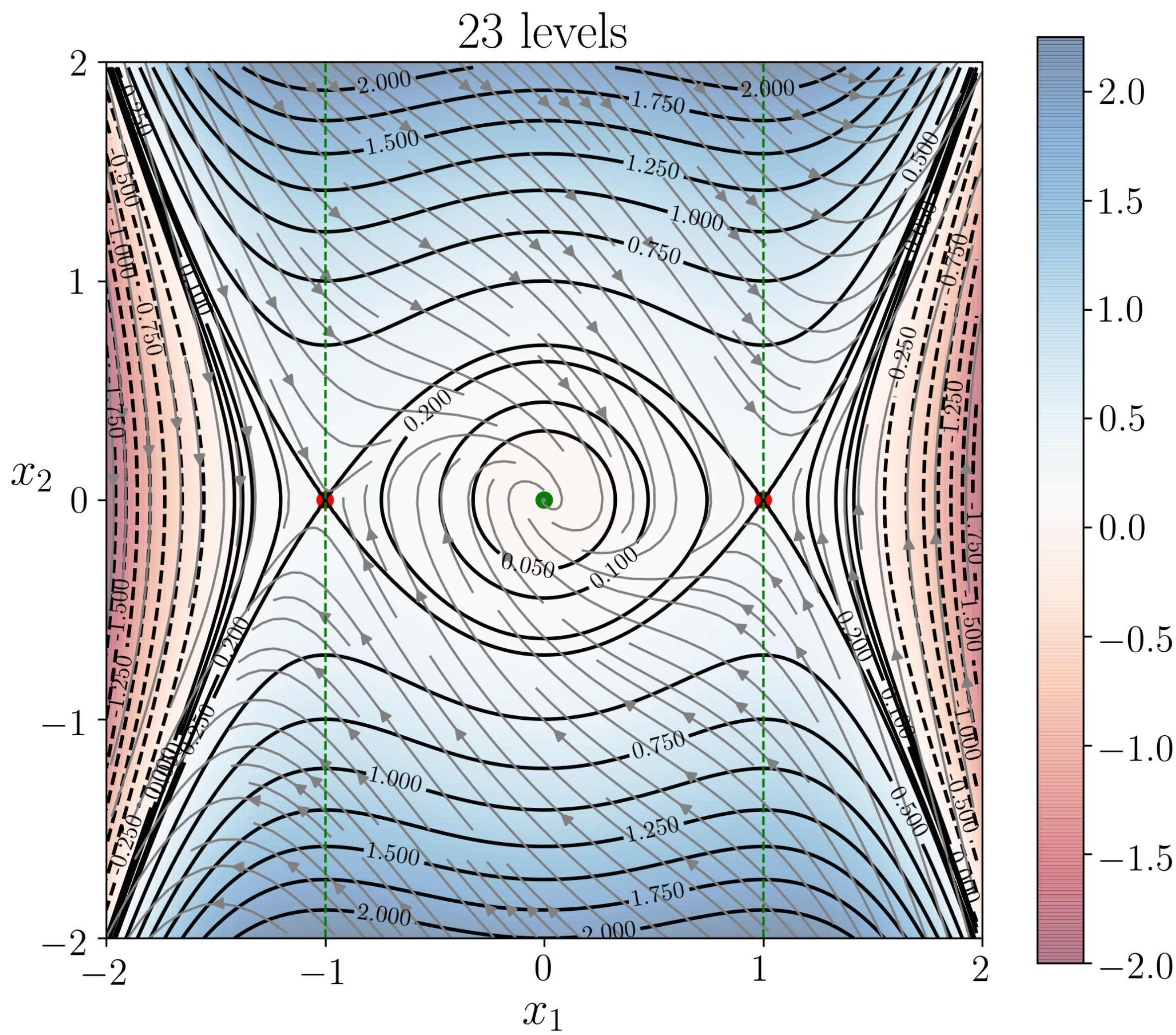

Figure 2: Shown here are 23 contours (solid black lines for positive c, dashed black lines for negative c) of the function $V(x_1, x_2)$, superimposed with the given vector field. Blue (red) color denotes region where V is > 0 (< 0). The open set $\mathscr{D}$ is the infinite vertical strip strictly inside the dashed green lines $x_1 = \pm 1$. Our inner estimate for the region of attraction is $\Omega_c := \{(x_1, x_2) \in \mathbb{R}^2 : V(x_1, x_2) \leq c\}$ where $0 < c < \frac{1}{4}$ (the "eye-shaped" set inside $\mathscr{D}$).