# AMS 231: Nonlinear Control Theory: Winter 2018

## Homework #4

Name: ..................................................................................................

Due: February 22, 2018

NOTE: Please show all the steps in your solution. Turn in a hard copy of your HW stapled with this as cover sheet with your name written in the above field. Please submit your HW in class on the due date.

## Problem 1

**State Space Computation of $\mathcal{H}_\infty$ Norm** (25+20+15+20 = 80 **points)**

In class (Lecture 12), we derived that the worst-case $\mathcal{L}_2$ gain of a stable LTI system is

$$\gamma_{\text{LTI}} = \| G(j\omega) \|_\infty := \sup_{\omega\in\mathbb{R}} \sigma_{\max}\left(G(j\omega)\right), \qquad j := \sqrt{-1}, \qquad G(s) = C(sI-A)^{-1}B + D,$$

where $G(j\omega)$ is the associated transfer matrix. However, this frequency domain formula is inconvenient for computing $\gamma_{\text{LTI}}$, since it requires solving a nonlinear optimization problem in $\omega$. The purpose of this exercise is to demonstrate an alternate method for computing $\gamma_{\text{LTI}}$ using state space formulation, for the case $D = 0$ (no direct feedthrough).

(a) By specializing the $\mathcal{L}_2$ gain theorem for nonlinear systems (Lecture 12 notes, page 8 and 9) for $f(\underline{x}) = A\underline{x}$, $g(\underline{x}) = B$, $h(\underline{x}) = C\underline{x}$, and $V(\underline{x}) = \frac{1}{2}\,\underline{x}^\top P \underline{x}$ where $P \succ 0$, prove that **if** the following optimization problem:

$$\begin{aligned} &\underset{\gamma,P}{\text{minimize}} \quad \gamma \\ &\text{subject to} \quad \gamma > 0, \quad P \succ 0, \quad PA + A^\top P + \frac{1}{\gamma^2} PBB^\top P + C^\top C \preceq 0, \end{aligned}$$

has unique solution, **then** the answer of this optimization problem gives the tightest upper bound of $\mathcal{L}_2$ gain $\gamma_{\text{LTI}}$. (In fact, when the triple $(A, B, C)$ is minimal, meaning both controllable and observable, then the answer of this optimization problem equals $\gamma_{\text{LTI}}$, and hence equals $\| G(j\omega) \|_\infty$. But you can ignore this detail).

(b) At first glance, it may seem that the optimization problem in part (a) is nonlinear in both variables: scalar $\gamma$ and matrix $P$, due to the last inequality constraint. However, this difficulty can be overcome via the following lemma.

**Lemma:** Consider real square matrices $Q, R, S$ with $Q$ and $R$ symmetric. The linear matrix inequality $\begin{bmatrix} Q & S \\ S^\top & R \end{bmatrix} \succeq 0$ is equivalent to (if and only if) $R \succ 0$ and $Q - SR^{-1}S^\top \succeq 0$.

Prove this lemma.

(c) Using the lemma in part (b), and introducing $\sigma := \gamma^2$, show that the optimization problem derived in part (a) is equivalent to the following optimization problem:

$$\underset{\sigma, P}{\text{minimize}} \quad \sigma$$

$$\text{subject to} \quad \sigma > 0, \quad P \succ 0, \quad \begin{bmatrix} A^\top P + PA + C^\top C & PB \\ B^\top P & -\sigma I \end{bmatrix} \preceq 0,$$

which is linear in both variables $\sigma$ and $P$. Here $I$ denotes the identity matrix of appropriate dimension.

(d) The type of optimization problem derived in part (c) is called semi-definite programming (SDP) problem that minimizes linear objective subject to linear matrix inequalities. SDPs are convex optimization problems, and can be solved efficiently via software like `cvx` in `MATLAB`.

Download `cvx` from `http://cvxr.com/cvx/download/` and follow installation instructions in `http://cvxr.com/cvx/doc/install.html`. To understand how to specify an optimization problem in `cvx`, you may want to take a look at: `http://cvxr.com/cvx/examples/`

Then write a `MATLAB` code to compute the $\mathcal{H}_\infty$ norm of the following stable, controllable and observable linear system (see partial code) in two ways: by using `cvx` to solve the optimization in part (c), and by using `MATLAB` command `norm(sys,inf)` to solve the frequency domain optimization problem. Report the $\mathcal{H}_\infty$ norms computed from the two methods, and submit your code.

## Solution

(a) We specialize the $\mathcal{L}_2$ gain theorem for nonlinear systems (Lecture 12 notes, page 8 and 9) for $f(\underline{x}) = A\underline{x}$, $g(\underline{x}) = B$, $h(\underline{x}) = C\underline{x}$, and $V(\underline{x}) = \frac{1}{2}\,\underline{x}^\top P \underline{x}$ where $P \succ 0$. Clearly, $V(\underline{0}) = 0$, $V(\underline{x} \neq \underline{0}) > 0$, and the Hamilton-Jacobi PDI reduces to

$$\frac{1}{2}\,\underline{x}^\top \left[ PA + A^\top P + \frac{1}{\gamma^2} PBB^\top P + C^\top C \right] \underline{x} \leq 0,$$

which will hold for all $\underline{x} \in \mathbb{R}^n$ iff

$$PA + A^\top P + \frac{1}{\gamma^2} PBB^\top P + C^\top C \preceq 0.$$

Thus for any $\gamma > 0$ to be an upper bound of the $\mathcal{L}_2$ gain of the LTI system, it must satisfy the feasibility conditions

$$\gamma > 0, \qquad P \succ 0, \qquad PA + A^\top P + \frac{1}{\gamma^2} PBB^\top P + C^\top C \preceq 0.$$

The tightest upper bound is obtained by minimizing $\gamma$ subject to the above constraints. Hence the statement. (**Additional info regarding the qualifier "tightest":** The fact that this bound cannot be improved by a different choice of Lyapunov function follows from a converse Lyapunov theorem for LTI system – something we will not cover in this course.)

(b) We are going to use three basic facts from linear algebra:

**Fact 1:** For a *symmetric matrix* $X$, the congruence transformation $X \mapsto MXM^\top$ via any invertible matrix $M$, preserves the matrix inertia, i.e., the numbers of positive, negative and zero eigenvalues of $X$, equal to the same for $MXM^\top$. (Sometimes this fact is referred to as "Sylvester's law of inertia".)

**Fact 2:** A block triangular matrix is non-singular iff its diagonal blocks are non-singular.

**Fact 3:** A block diagonal matrix is positive semi-definite iff its diagonal blocks are positive semi-definite.

Now let us give the proof.

($\Rightarrow$)

Suppose that $X := \begin{bmatrix} Q & S \\ S^\top & R \end{bmatrix} \succeq 0$, where $Q, R$ are symmetric, and $R$ is invertible (which is implicit in the inequality involving $R^{-1}$). Notice that **Fact 1** specialized to symmetric sign-definite matrices says that "sign-definiteness is preserved under congruence transformation via any non-singular matrix". Now let $M := \begin{bmatrix} I & -SR^{-1} \\ 0 & I \end{bmatrix}$, which by **Fact 2** is non-singular. Then by **Fact 1**, we have

$$X := \begin{bmatrix} Q & S \\ S^\top & R \end{bmatrix} \succeq 0 \Leftrightarrow MXM^\top = \begin{bmatrix} I & -SR^{-1} \\ 0 & I \end{bmatrix} \begin{bmatrix} Q & S \\ S^\top & R \end{bmatrix} \begin{bmatrix} I & 0 \\ -R^{-1}S^\top & I \end{bmatrix} = \begin{bmatrix} Q - SR^{-1}S^\top & 0 \\ 0 & R \end{bmatrix} \succeq 0,$$

which by **Fact 3** further implies $Q - SR^{-1}S^\top \succeq 0$ and $R \succeq 0$. Now, $R \succeq 0$ together with the requirement that it is also invertible yields $R \succ 0$. Therefore, $\begin{bmatrix} Q & S \\ S^\top & R \end{bmatrix} \succeq 0 \Rightarrow R \succ 0$ and $Q - SR^{-1}S^\top \succeq 0$. ■

($\Leftarrow$)

Suppose that $R \succ 0$ and $Q - SR^{-1}S^\top \succeq 0$. Then by **Fact 3**, $Y := \begin{bmatrix} Q - SR^{-1}S^\top & 0 \\ 0 & R \end{bmatrix} \succeq 0$. On the other hand, by **Fact 2**, the matrix $N := \begin{bmatrix} I & SR^{-1} \\ 0 & I \end{bmatrix}$ is non-singular. Therefore, by **Fact 1**, we have

$$NYN^\top = \begin{bmatrix} I & SR^{-1} \\ 0 & I \end{bmatrix} \begin{bmatrix} Q - SR^{-1}S^\top & 0 \\ 0 & R \end{bmatrix} \begin{bmatrix} I & 0 \\ R^{-1}S^\top & I \end{bmatrix} = \begin{bmatrix} Q & S \\ S^\top & R \end{bmatrix} \succeq 0.$$

■

(c) Taking $\sigma := \gamma^2$, this follows immediately from the lemma in part (b) with $R := \sigma I \succ 0$, $Q := -(A^\top P + PA + C^\top C)$, and $S := -PB$. Notice that such choices make $Q$ and $R$ symmetric matrices, as needed to apply the lemma. Furthermore, by the same lemma, the pair of matrix inequalities

$$R := \sigma I \succ 0, \qquad \text{and} \qquad (-Q) - (-S)R^{-1}(-S^\top) \succeq 0,$$

is equivalent to the single block matrix inequality

$$\begin{bmatrix} -\left(A^\top P + PA + C^\top C\right) & -PB \\ -B^\top P & \sigma I \end{bmatrix} \succeq 0 \qquad \Leftrightarrow \qquad \begin{bmatrix} A^\top P + PA + C^\top C & PB \\ B^\top P & -\sigma I \end{bmatrix} \preceq 0,$$

as claimed.

(d) Both `MATLAB` `norm(sys,inf)` and `cvx` SDP solutions produce the same answer: $||G(j\omega)||_\infty = 12.1843$.

**MATLAB Code:**

```
clear; clc;

A = [-2 1 0 0;
     -6 -6 0 3;
     0 0 -1 1;
     0 0 0 -2];
B = [0;1;1;2];
C = [0 6 2 -8; 2 -3 4 5];
D = 0;

sys = ss(A,B,C,D);
```

```

eig(A) % is stable?
rank(ctrb(sys))==length(A) % is controllable?
rank(obsv(sys))==length(A) % is observable?

dim=size(B); n_x = dim(1); n_u = dim(2);

cvx_begin sdp
    % declare variables
    variable P(n_x,n_x) symmetric;
    variable sig;
    % objective
    minimize sig;
    subject to
        % constraints
        sig >= 0;
        P >= 0;
        [A'*P+P*A+C'*C     P*B;
          B'*P      -sig*eye(n_u)] <= 0;
cvx_end

Hinf_cvx = sqrt(sig);

Hinf_matlab = norm(sys,inf);
```

## Problem 2

**Input-to-State Stability (ISS)** **($4 \times 5 = 20$ points)**

Consider the scalar nonlinear systems

$$\text{(a)}\ \dot{x} = -(1+u)x^3, \quad \text{(b)}\ \dot{x} = -(1+u)x^3 - x^5, \quad \text{(c)}\ \dot{x} = -x + x^2 u, \quad \text{(d)}\ \dot{x} = x - x^3 + u.$$

Which systems are input-to-state stable (ISS) and which are not? Give reasons.

## Solution

(a) Not ISS since $u(t) \equiv$ constant $> 1$ with $x_0 > 0$ leads to $\lim_{t\to\infty} x(t) = \infty$.

(b) Let $V(x) = \frac{1}{2}x^2$. Then

$$\dot{V} = -x^4 + ux^4 - x^6 \leq -x^4, \quad \forall\, |x| > \sqrt{u}.$$

Invoking the ISS Lyapunov theorem (see Lecture 11, pg. 6) with $\alpha_1(|x|) = \alpha_2(|x|) = V(x)$, $W_3(|x|) = x^4$, and $\rho(|u|) = \sqrt{u}$, we conclude that the system is ISS with $\gamma(r) = \rho(r) = \sqrt{r}$.

(c) Not ISS since $u(t) \equiv 1$ with $x_0 > 0$ leads to $\lim_{t\to\infty} x(t) = \infty$.

(d) Not ISS since the origin of the unforced system ($u(t) \equiv 0$) is unstable.