AMS 231: Nonlinear Control Theory: Winter 2018

Homework #2

Name: ....

Due: February 06, 2018

NOTE: Please show all the steps in your solution. Turn in a hard copy of your HW stapled with this as cover sheet with your name written in the above field. Please submit your HW in class on the due date.

Problem 1

Limit Cycle in Planar Nonlinear Systems $((3+7)+10+15=35$ **points)**

Consider the system

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ -(2b-g(x_1))ax_2 - a^2x_1 \end{pmatrix},$$

where the parameters $a, b > 0$, and the function $g(x_1) := \begin{cases} 0 & \text{for } |x_1| > 1, \\ k & \text{for } |x_1| \leq 1, \end{cases} \quad k \in \mathbb{R}.$

(a) Find all fixed points. Determine which are hyperbolic and which are non-hyperbolic.

(b) Show, using Bendixson's criterion, that there are no limit cycles if $k < 2b$.

(c) Show, using Poincaré-Bendixson criterion, that there is a limit cycle if $k > 2b$.

Problem 2

Lyapunov Stability in Continuous Time $(1+(2+2)+(15+2+3)+20=45$ **points)**

Dynamics of a rotating rigid spacecraft is given by the Euler equation

$$J_1\,\dot{\omega}_1 = (J_2 - J_3)\,\omega_2\omega_3 \;+\; \tau_1,$$
$$J_2\,\dot{\omega}_2 = (J_3 - J_1)\,\omega_3\omega_1 \;+\; \tau_2,$$
$$J_3\,\dot{\omega}_3 = (J_1 - J_2)\,\omega_1\omega_2 \;+\; \tau_3,$$

where the parameters $J_1, J_2, J_3 > 0$ denote the principal moments of inertia; the state vector $(\omega_1, \omega_2, \omega_3)$ denotes the spacecraft's angular velocity along its principal axes; and the control vector (τ_1, τ_2, τ_3) denotes the torque input applied about the principal axes.

(a) For $\tau_1 = \tau_2 = \tau_3 = 0$, prove that origin is a fixed point.

(b) For $\tau_1 = \tau_2 = \tau_3 = 0$, how many fixed points other than the origin are there? What physical motions do they correspond to?

(c) For $\tau_1 = \tau_2 = \tau_3 = 0$, show that origin is stable. Is it asymptotically stable? Why/why not?

(d) For $i = 1, 2, 3$, consider the feedback control law $\tau_i = -k_i \omega_i$, where $k_i > 0$ are constants. Prove that origin of the closed-loop system is globally asymptotically stable (G.A.S).

Problem 3

Lyapunov Stability in Discrete Time **($5 + 10 + 5 = 20$ points)**

For discrete-time autonomous nonlinear system $\underline{x}(k+1) = \underline{f}(\underline{x}(k))$, one can derive a Lyapunov stability theorem analogous to the continuous-time case, by simply replacing the condition $\dot{V} <$ (or $\leq$) 0 to its discrete-time counterpart: $V(k+1) <$ (or $\leq$) $V(k)$, where $V(k) := V(\underline{x}(k))$, while keeping the other conditions (positive definiteness/semi-definiteness) same.

Consider the nonlinear system

$$x_1(k+1) = \frac{\alpha\, x_2(k)}{1 + (x_1(k))^2}, \qquad x_2(k+1) = \frac{\beta\, x_1(k)}{1 + (x_2(k))^2},$$

where the parameters α, β satisfy $0 < \alpha^2 < 1$, $0 < \beta^2 < 1$.

(a) Prove that origin is a fixed point.

(b) Prove that origin is asymptotically stable (A.S).

(c) Prove that origin is globally asymptotically stable (G.A.S).