Planar Dynamical Systems

⊙ Planar LTI systems

$\dot{\underline{x}} = A\underline{x}, \quad \underline{x} \in \mathbb{R}^2$

modal co-ordinates   state vector

$$\boxed{\underset{\in \mathbb{R}^2}{\underline{z}} := T^{-1}\underline{x},} \qquad T := [\underline{v}_1 | \underline{v}_2]$$

$\mathrm{eig}(A) = \{\lambda_i, \underline{v}_i\}_{i=1}^{2}$

$$\dot{\underline{z}} = T^{-1}\dot{\underline{x}} = T^{-1}A\underline{x}$$
$$= T^{-1}A(T\underline{z})$$
$$= \underbrace{(T^{-1}AT)}_{M}\underline{z}$$

$$\boxed{\dot{x} = Ax} \xrightarrow[z = T^{-1}x]{} \boxed{\dot{\underline{z}} = M\underline{z}}$$

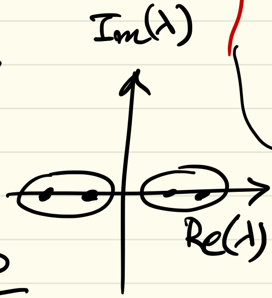
M has Jordan Canonical Form

Case I: $\lambda_1 \neq \lambda_2$ (real)

$\dot{z}_1 = \lambda_1 z_1, \quad \dot{z}_2 = \lambda_2 z_2$

$$\left.\begin{aligned} z_1(t) &= z_{10}e^{\lambda_1 t} \\ z_2(t) &= z_{20}e^{\lambda_2 t} \end{aligned}\right\}$$

$$\left(\frac{z_1}{z_{10}}\right)^{\lambda_2} = \left(\frac{z_2}{z_{20}}\right)^{\lambda_1}$$



Case I.1. $\underbrace{\lambda_1, \lambda_2 < 0}_{\text{stable node}}$

Case I.2: $\lambda_1, \lambda_2 > 0$ (unstable node)

Case I.3: $\lambda_1 < 0 < \lambda_2$ (saddle node)

$M := T^{-1}AT \qquad T := [\underline{v}_1 | \underline{v}_2]$, eig(A)

Case I:

$$M = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

$\lambda_1 \neq \lambda_2$ (two real distinct roots)

"Node"

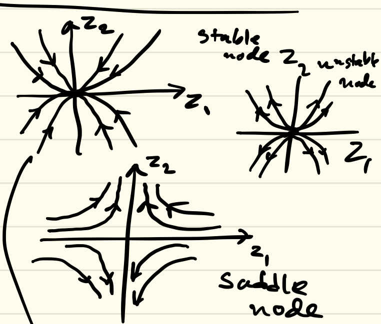
Case II: $\lambda_1 = \lambda_2 = \lambda$ (real but equal)

$$M = \begin{bmatrix} \lambda & \kappa \\ 0 & \lambda \end{bmatrix}, \quad \kappa \in \{0, 1\}$$

Case III: $\lambda_{1,2} = \alpha \pm j\beta$

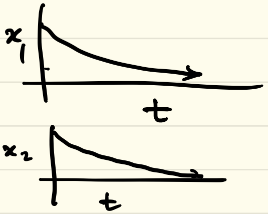
$$M = \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix}$$

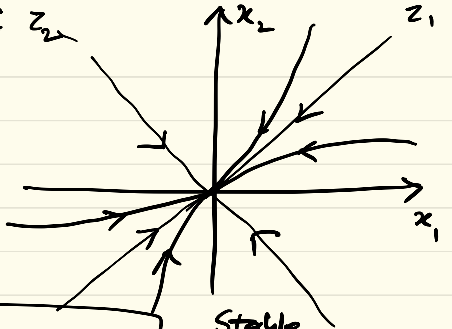
Case IV: Both or one of the $\lambda = 0$



$x(t) = c_1 e^{-2t}\underline{v}_1 + c_2 e^{-3t}\underline{v}_2$

Stable node :





Stable node in $x_1 - x_2$

Case II : $\lambda_1 = \lambda_2 = \lambda$ (real & equal)

$$M = T^{-1}AT = \begin{bmatrix} \lambda & k \\ 0 & \lambda \end{bmatrix}, \quad k \in \{0, 1\}$$

$$\dot{z} = Mz$$

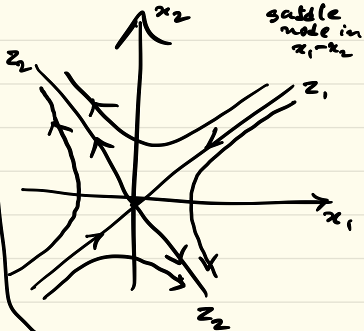
$$z_1(t) = z_{10} e^{\lambda t} + z_{20} t e^{\lambda t},$$

$$z_2(t) = z_{20} e^{\lambda t}$$

eliminate t:

$$\frac{z_1}{z_2} = \left(\frac{z_{10}}{z_{20}}\right) + \frac{k}{\lambda} \ln\left(\frac{z_2}{z_{20}}\right)$$

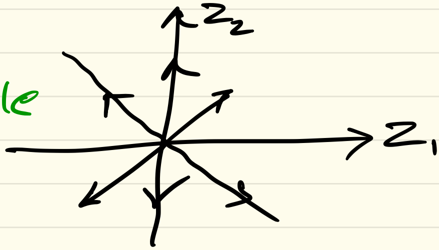
$\boxed{\text{Improper node}}$ → stable ($\lambda<0$)
→ unstable ($\lambda>0$)

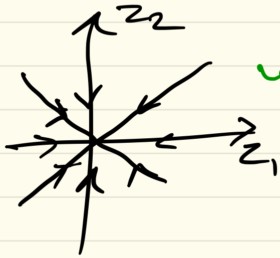


saddle node in $x_1 - x_2$

$K = 0$ → $\lambda > 0$ $\lambda < 0$

Stable

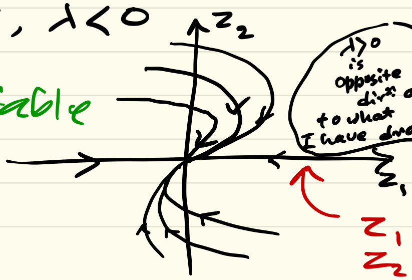




unstable

$K = 1, \lambda < 0$

stable



$\lambda > 0$ is opposite dir.ⁿ arrows to what I have drawn

z_1 axis is Invariant
z_2 axis is NOT

Case III Complex conjugate

$$r := \sqrt{z_1^2 + z_2^2}, \quad \theta := \tan^{-1}\left(\frac{z_2}{z_1}\right)$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \longmapsto \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \longmapsto \begin{pmatrix} r \\ \theta \end{pmatrix}$$

$$M = T^{-1}AT = \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix}$$

$$\left.\begin{aligned} \dot r &= \alpha r \\ \dot\theta &= \beta \end{aligned}\right\} \lambda_1, \lambda_2 = \alpha \pm j\beta$$

$$\underbrace{r(t) = r_0 e^{\alpha t}, \ \theta(t) = \theta_0 + \beta t}_{\text{logarithmic spiral}}$$

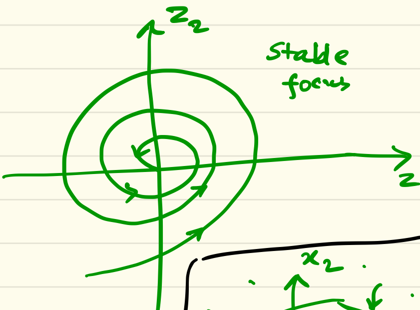
in $z_1 z_2$ plane

$\boxed{\text{Origin is called “Focus”}}$

$\alpha < 0$ (Stable focus)

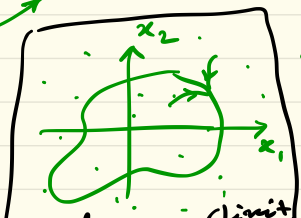
$\alpha > 0$ (unstable focus)

$\alpha = 0$ (Center)

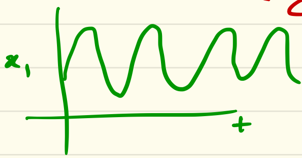


∞ number of circular orbits

(An LTI system cannot have ISOLATED limit cycle)



nonlin. case (limit cycle)

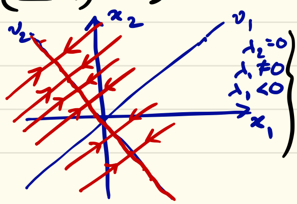


$$e^{-\alpha t} \cos \beta t$$

Case IV : One OR both $\lambda = 0$ ($\underline{A \text{ is singular}}$)

(Subspace of equib$^{\underline{m}}$)

(IV. 1) only one $\lambda = 0$



$\left.\begin{aligned} \lambda_2 &= 0 \\ \lambda_1 &\neq 0 \\ \lambda_1 &< 0 \end{aligned}\right\} x(t) = c_1 e^{\lambda_1 t} + c_2 v_2$

If $\lambda_2 = 0$, $\lambda_1 > 0$ then dir$^{\underline{n}}$ of arrows reverse

(IV.2) Both $\lambda = 0$

$\boxed{\text{IV.2.1}}$

trivial case $\leftarrow$ $A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

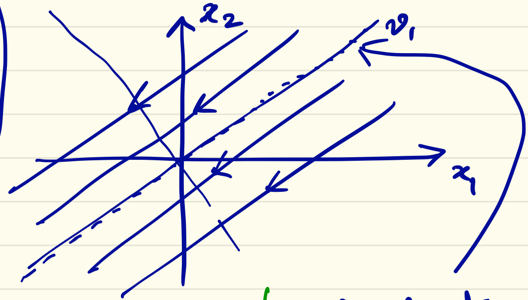
plane $\mathbb{R}^2$ is equib^{m}

$x(t) = c_1 \underline{v}_1 + c_2 \underline{v}_2$

$\dim(\ker(A)) = 2$

$\boxed{\text{IV.2.2}}$ $\dim(\ker(A)) = 1$

$\underline{x}(t) = (c_1 + c_2 t)\, \underline{v}_1 + c_2 \underline{w}_1$



$\underline{v}_1$ vector contains a continuum of unstable fixed pt.s & flow moves parallel to $\underline{v}_1$

Summary

LTI system in 2D

→ (proper node) $\lambda_1 \neq \lambda_2$ (real) < stable, unstable, saddle } 3

→ Improper node $\lambda_1 = \lambda_2$ (real) < Stable < $k=1$, $k=0$; unstable < $k=1$, $k=0$ } 4

→ Focus → stable < ↻, ↺; → unstable < ↻, ↺ } 4

→ Center < ↻, ↺ } 2

→ Singular matrix ($\det(A) = 0$) [subspace / line of equib^{m}]

- only one $\lambda = 0$ < attracting, repelling } 2
- Both $\lambda = 0$ < whole plane → 1; unstable subspace (flow moves $\parallel^{R}$ to the unstable subspace) → which way to move } 2

Total 18 phase portraits (one of them is trivial, corresponds to A = zero matrix)

→ 17 Non-trivial cases

$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$; $\det(A - \lambda I) = 0$

$$\Rightarrow \boxed{\lambda^2 - (a_{11} + a_{22})\lambda + (a_{11}a_{22} - a_{21}a_{12}) = 0}$$

characteristic polynomial of the state matrix

$$\Rightarrow \lambda^2 - tr(A)\lambda + \det(A) = 0$$

Discriminant of this quadratic eqn:

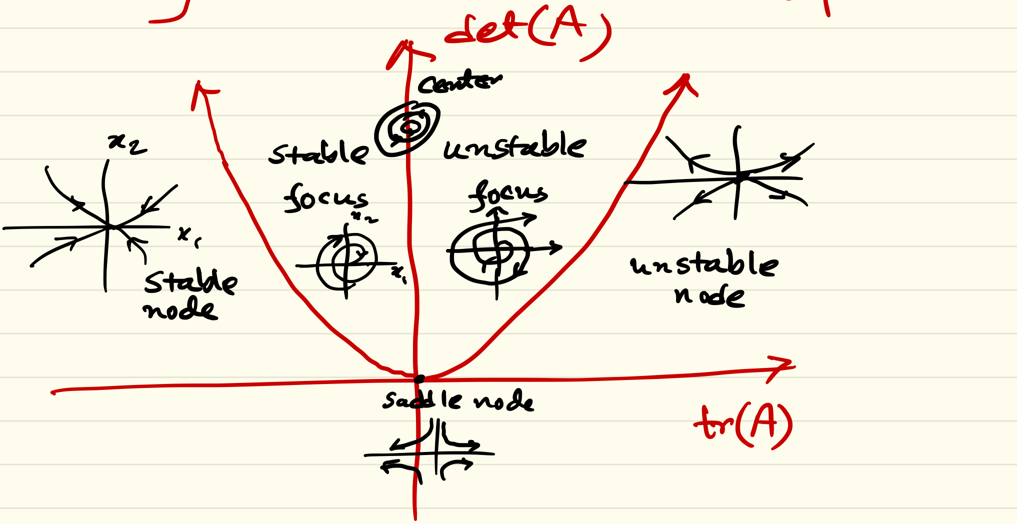
$$(tr(A))^2 - 4.1.\det(A)$$

Bifurcation curve:

discriminant $= 0$

$$\underbrace{\det(A)}_{y} = \frac{1}{4}\big(\underbrace{tr(A)}_{x}\big)^2$$

} equation of a parabola that is open upwards



Lineariz^{n} of Nonlin. System About Equib^{m} $\underline{x}^*$

$$\dot{\underline{x}} = \underline{f}(\underline{x}), \quad \underline{x}(0) = \underline{x}_0$$

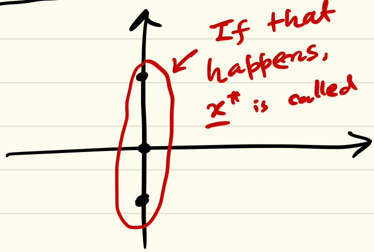
$\underline{x}^*$ s.t. $\underline{f}(\underline{x}^*) = \underline{0}$

$$\left.\frac{\partial \underline{f}}{\partial \underline{x}}\right\} \text{Jacobian matrix}$$

$$\underbrace{A(\underline{x}^*)}_{\text{Jacobian evaluated at } \underline{x}^*} = \left.\begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \frac{\partial f_1}{\partial x_3} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \frac{\partial f_2}{\partial x_3} & \cdots & \frac{\partial f_2}{\partial x_n} \\ & & & \ddots & \\ \frac{\partial f_n}{\partial x_1} & \cdots & \cdots & & \frac{\partial f_n}{\partial x_n} \end{bmatrix}\right|_{\underline{x} = \underline{x}^*}$$

$$\boxed{\dot{y} = A(\underline{x}^*)\, y}$$

Linearized system



If that happens, $\underline{x}^*$ is called non-hyperbolic equib^{m}.

Thm. Hartman-Grobman (1960)
Thm.

If Jacobian @ $\underline{x}^*$ has NO zero OR pure imaginary eig., then $\exists$ a homeomorphism (continuous map with continuous inverse) from a nhbd. U of $\underline{x}^*$ to $\mathbb{R}^n$:

$$h : U \mapsto \mathbb{R}^n.$$

The homeomorphism map "h" takes trajectory of the nonlinear system: $\dot{\underline{x}} = \underline{f}(\underline{x})$ to trajectory of the linearized system:

$$\dot{y} = A(\underline{x}^*)y$$

In particular, $h(\underline{x}^*) = \underline{0}$

Examples

$\dot{x} = -x^3$	$\dot{x} = x^2$
$\forall x_0 \neq 0$ converges to $x^* = 0$ as $t \to \infty$	only $x_0 < 0$ converge to $x^* = 0$ as $t \to \infty$

Both have $x^* = 0$ as UNIQUE EQUILIB^m (fixed pt.)

Both have lineariz^n $= 0$

$$\left(\because \frac{d}{dx}(-x^3)\Big|_{x^*=0} = [-3x^2]\Big|_{x^*=0} = 0 \quad \& \right.$$

$$\left. \frac{d}{dx}(x^2)\Big|_{x^*=0} = [2x]\Big|_{x^*=0} = 0 \right)$$

Moral: Hartman-Grobman Thm. says that if the fixed pt. $\underline{x}^*$ is non-hyperbolic, then linear analysis is inconclusive EVEN LOCALLY. (around $\underline{x}^*$)