Lecture #8

Lyapunov Theory for Non-autonomous Systems

Set up: $\underline{\dot{x}} = \underline{f}(\underline{x}, t)$, $\underline{x}(t_0) = \underline{x}_0$ (given)

→ $\underline{f}$ is p.w. continuous in t, and locally Lip. in $\underline{x}$

→ $\underline{x}^* = \underline{0}$ is a fixed pt. if $\underline{f}(\underline{x}^*, t) = \underline{0}$ $\forall\ t \geq t_0 \geq 0$

(in $\underline{f}(\underline{x}^*, t)$, $\underline{x}^*$ is replaced by $\underline{0}$)

→ $\underline{f} : [0,\infty) \times \mathcal{D} \longmapsto \mathbb{R}^n$, where $\underline{x}^* \in \mathcal{D}$.

Autonomous System $\underline{x}(t-t_0)$	Non-autonomous System $\underline{x}(t, t_0)$
Sol^{n} $\underline{x}$ is a f^{n} of $(t-t_0)$	Sol^{n} is a f^{n} of t & t_0
e.g. $\dot{x} = -x$, $x(t_0) = x_0$	e.g. $\dot{x} = -\dfrac{x}{1+t}$ $\quad \boxed{x = x_0\,\dfrac{1+t_0}{1+t}}$
$\Rightarrow x(t) = x_0 \exp(-(t-t_0))$	$\Rightarrow [\ln x]_{x_0}^{x} = \int_{t_0}^{t} -\dfrac{dt}{1+t}$, $= -\int_{1+t_0}^{1+t} \dfrac{d\tau}{\tau} = \ln\left(\dfrac{1+t_0}{1+t}\right)$
	$1+t =: \tau$ $\Rightarrow dt = d\tau$

Stability of non-autonomous system.

Autonomous: $\forall \epsilon > 0,\ t_0 \geq 0,\ \exists\ \delta = \delta(\epsilon) > 0$

s.t. $\|\underline{x}(t_0)\|_2 < \delta \Rightarrow \|\underline{x}(t)\|_2 < \epsilon$

$\forall\ t \geq t_0$

Non-autonomous:

$\forall\ \epsilon > 0,\ t_0 \geq 0\ ,\ \exists\ \delta = \delta(\epsilon, t_0) > 0$

s.t. $\|\underline{x}(t_0)\|_2 < \delta \Rightarrow \|\underline{x}(t)\|_2 < \epsilon\ \ \forall\ t \geq t_0$

- δ depends on t_0
- S/ AS/ GAS all will depend on t_0

Spl. case: If for non-autonomous, δ is independent of t_0, then we call the system Uniformly Stable (US) in time

Example: (stable but NOT US) 1D

$$\left.\begin{aligned}\dot{x} &= \underbrace{\alpha(t)}\, x \\ &= (6t\sin t - 2t)\, x\end{aligned}\right\}\quad x^* = 0 \text{ is fixed pt.}$$

$$\Rightarrow x(t) = x_0 \exp\left(\int_{t_0}^{t} (6\tau\sin\tau - 2\tau)\, d\tau \right)$$

$$= x_0 \exp\big(6\sin t - 6t\cos t - t^2 - 6\sin t_0 + 6t_0\cos t_0 + t_0^2\big)$$

$\because t_0$ is given/fixed, the term $(-t^2)$ inside $\exp(\cdot)$ will dominate as $t \uparrow$

$\Rightarrow \exp(\cdot)$ is bdd. $\forall\, t \geq t_0$ by some constant $c(t_0) \Longrightarrow |x(t)| < |x_0|\, c(t_0)$

$\forall\, t \geq t_0$

$\forall\, \epsilon > 0$, choose $\delta := \delta(\epsilon, t_0)$

$$= \frac{\epsilon}{c(t_0)} \Rightarrow \text{proves } x^* = 0 \text{ is stable.}$$

Suppose, $t_0 = 2n\pi$, for $n = 0, 1, 2 \ldots$

Then $x(t_0+\pi) = x(t_0) \exp\left((4n+1)(6-\pi)\pi\right)$

$\Rightarrow$ whenever $x_0 \neq 0$, $\displaystyle\lim_{n\to\infty} \frac{x(t_0+\pi)}{x_0} \to +\infty$

$\therefore$ For any $\epsilon > 0$, $\nexists\ \delta$ that is indep. of t_0, that can prove stability

$\therefore$ Origin is S but NOT US.

Conceptually, we understand

$$\text{S},\ \underbrace{\text{US},\ \text{UAS},\ \text{GUAS}}_{\text{Theorems in the spirit of Lyapunov}}$$

① Thm (Lyapunov-like Thm. for non-autonomous) (US/UAS/GUAS)

Suppose $\mathcal{D} \subseteq \mathbb{R}^n$ contains $\underline{x}^* = \underline{0}$.

Let $V : [0,\infty) \times \mathcal{D} \mapsto \mathbb{R}_0^+$ be C^1 f^n of $\underline{x}$

s.t.

$\forall\ t \geq t_0 \geq 0$

& $\forall\ \underline{x} \in \mathcal{D}$

① $W_1(\underline{x}) \leq V(\underline{x}, t) \leq W_2(\underline{x})$

where W_1 & W_2 are continuous pos. def. f^n of $\underline{x}$ in $\mathcal{D}$

② $\dot{V} = \frac{d}{dt} V = \frac{\partial V}{\partial t} + \langle \nabla V, \underline{f}(t, \underline{x}) \rangle \leq 0$

Then $\underline{x}^*$ is US.

If ① & ②'

②' ≡

$\dot{V} = \frac{\partial V}{\partial t} + \langle \nabla V, \underline{f}(\underline{x}, t) \rangle$
$\leq -W_3(\underline{x})$

Then $\underline{x}^*$ is UAS

If $\mathcal{D} \equiv \mathbb{R}^n$ AND $W_1(\underline{x})$ is radially unbdd. (in $\underline{x}$)

then $\underline{x}^* = \underline{0}$ is GUAS

E S (Exponential Stability):

1' $\rightarrow$ $$k_1 \|\underline{x}\|_2^a \leq V(t, \underline{x}) \leq k_2 \|\underline{x}\|_2^a$$

+

2'' $\rightarrow$ $$\dot{V} = \frac{\partial V}{\partial t} + \langle \nabla V, \underline{f}(\underline{x}, t) \rangle$$

$$\leq -k_3 \|\underline{x}\|_2^a$$

$$\left.\begin{array}{l} \forall\, t \geq t_0 \geq 0 \\ \forall\, \underline{x} \in \mathcal{D} \end{array}\right\}$$ where $k_1, k_2, k_3, a > 0$

Then $\underline{x}^* = \underline{0}$ is E.S.

If 1' + 2'' hold for $\mathcal{D} \equiv \mathbb{R}^n$

then G E S

Example#1 (1D, GUAS)

$$\dot{x} = -(1+g(t))\,x^3, \quad x(0) = x_0$$

where $g(t)$ is continuous (in t) and $g(t) \geq 0 \ \forall\ t \geq 0$

Prove that

① $x^* = 0$ is a fixed pt. $(f(t,0) = 0 \ \forall\ t \geq 0)$

② $x^* = 0$ is GUAS

Proof of part (2): Take $V(x,t) = \underbrace{\tfrac{1}{2}x^2}_{\text{radially unbdd.}}$

Then $\dot{V} = \underbrace{\cancel{\frac{\partial V}{\partial t}}}_{0} + \frac{\partial V}{\partial x} f(x,t)$

$$= -(1+g(t))\,x^4 \leq -x^4 \quad \forall\ x \in \mathbb{R},\ \forall\ t \geq 0$$

Taking

$W_1(x) = W_2(x) = V(x)$

$W_3(x) = x^4$

$\therefore$ Origin is GUAS

Exercise : (GES) $\dot{x}_1 = -x_1 - g(t)\,x_2$
$$\dot{x}_2 = x_1 - x_2$$

where $g(t)$ is C^1 in t, and $0 \leq g(t) \leq \underbrace{k}_{>0}$.

and $\dot{g}(t) \leq g(t) \;\; \forall \; t \geq 0$

By taking, $V(\underline{x}, t) = x_1^2 + (1 + g(t))\, x_2^2$

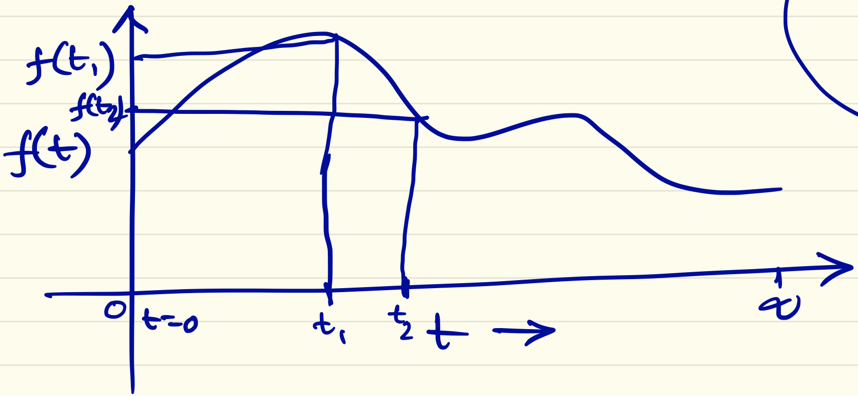
prove that origin is GES.

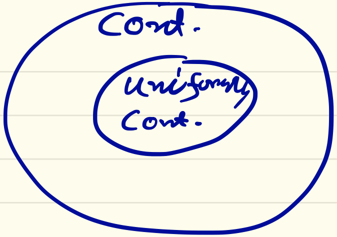
What if in UAS Thm, I can only show $\dot{V} \leq 0$?

Barbalat's Lemma (1959)

comes in here

Uniformly Continuous f^n (of time):





$\forall \epsilon > 0, \exists \delta = \delta(\epsilon) > 0$ s.t.
$\forall |t_2 - t_1| < \delta \Rightarrow |f(t_2) - f(t_1)| < \epsilon$

Barbalat's Lemma (1959)

Let $f : [0, \infty) \mapsto [0, \infty)$ be a uniformly continuous f^n with $\int_0^\infty f(t)\,dt < \infty$. Then

$$\boxed{\lim_{t \to \infty} f(t) = 0}$$

In the control context, apply Barbalat's Lemma for $f \longmapsto \dot{f}$

Explicitly, If $f(t)$ is C^1 (in time)

$$\& \lim_{t\to\infty} f(t) = \alpha < \infty$$

$\&$ $\dot{f}(t)$ is uniformly continuous

Then $\lim_{t\to\infty} \dot{f}(t) = 0$

Proof: (Contradiction)

Background: NOTE: f converges $\not\Rightarrow \lim_{t\to\infty} \dot{f} \to 0$

e.g. $f(t) = e^{-t} \sin(e^{2t})$

$$\lim_{t\to\infty} f(t) = 0 < \infty$$

BUT $\lim_{t\to\infty} \dot{f} = +\infty$

What additional assumption is needed to say-

f converges $\Longrightarrow$ $\dot{f}$ converges

Ans. Uniform continuity of $\dot{f}$

Corollary of Barbalat's Lemma :

If (1) $f(t)$ is C^1

(2) $\lim_{t\to\infty} f(t) = \text{exists} < \infty$

(replace (3) by uniform continuity) $\ddot{f}(t)$ exists & is bdd. $\left(|\ddot{f}(t)| < \infty \quad \forall t \right)$

Then $\lim_{t\to\infty} \dot{f}(t) = 0$

Appl^n: Take $V(\underline{x},t)$

Check if
- $\dot{V}(\underline{x},t) \le 0$ (neg. semi-definite)
- $\dot{V}(\underline{x},t)$ is uniformly continuous

($\Rightarrow \ddot{V}$ exists & bdd.)

Then $\lim_{t\to\infty} \dot{V}(\underline{x},t) = 0$.

Example: (Closed-loop dynamics of adaptive control system)

$$\dot{e} = -e + \theta w(t)$$
$$\dot{\theta} = -e w(t)$$

$\underline{x} = \begin{pmatrix} e \\ \theta \end{pmatrix}$

$w(t)$ is bdd. & continuous in t $(|w(t)| < \infty)$

Prove : $\lim_{t\to\infty} e(t) = 0$

Proof: Consider $V(\underline{x},t) = \frac{1}{2}(e^2 + \theta^2)$

$\Rightarrow \quad \dot{V} = \underbrace{\cancel{\frac{\partial V}{\partial t}}}_{0} + \langle \nabla V, \underline{f}(\underline{x},t) \rangle$

$= -2e^2 \leq 0$

$\Rightarrow \quad V(t) \leq V(0)$

$\Rightarrow$ $e(t)$ & $\theta(t)$ are bdd.

(NOT clear what/if anything converge)

BUT cannot apply LaSalle

Let's check uniform continuity of $\dot{V}$

$\Rightarrow \quad \ddot{V} = -4e(-e+\theta w) < \infty$

(why w bdd (given), e & θ bdd. proved)

$\Rightarrow \dot{V}$ is uniformly Continuous

$\Rightarrow$ Barbalat's Lemma say $\lim\limits_{t \to \infty} \dot{V}(\underline{x}, t) = 0$

$$\Rightarrow \lim_{t \to \infty} (-2e^2) = 0$$

$$\therefore e(t) \to 0$$

BUT $\not\Rightarrow$ A.S.

because $\theta(t)$ is only bdd.