Lecture #5

NOT to be covered:

→ Degree Theory (generalization of index theory in $\mathbb{R}^n$)

Bifurcation

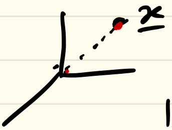
$$\dot{\underline{x}} = \underline{f}(\underline{x}, \underline{p})$$

Math Preliminaries

Vector norms: ℓ_p norm

$$\|\underline{x}\|_{\ell_p} := \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p}$$

$$\|\underline{x}\|_\infty \le 1 \cdot \|\underline{x}\|_2 \le 1 \cdot \|\underline{x}\|_1 \le \sqrt{n}\, \|\underline{x}\|_2 \le n\, \|\underline{x}\|_\infty$$



$$\|\underline{x}\|_2 := \sqrt{x_1^2 + \dots + x_n^2}$$

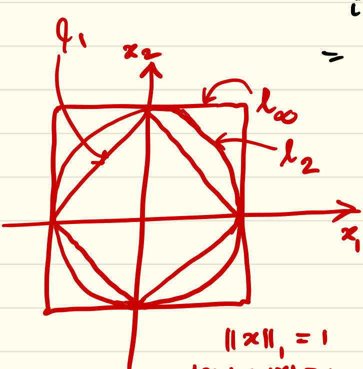
$$\|\underline{x}\|_\infty := \max_i |x_i|$$

$$\underline{x} = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}, \quad \|\underline{x}\|_2 = \sqrt{1 + (-2)^2 + 3^2} = \sqrt{14}$$

$$\|\underline{x}\|_\infty = \max\{|1|, |-2|, |3|\} = 3$$

$$\|\underline{x}\|_1 = \sum_i |x_i| = 1 + |-2| + 3 = 6$$

On $\mathbb{R}^n$, all (vector) norms are equivalent



$\|x\|_1 = 1$

$|x_1| + |x_2| = 1$

$\alpha, \beta \to$ integer

$$\boxed{\|\underline{x}\|_\alpha \asymp \|\underline{x}\|_\beta}$$

equiv.

$\exists\, c_1, c_2$ s.t.

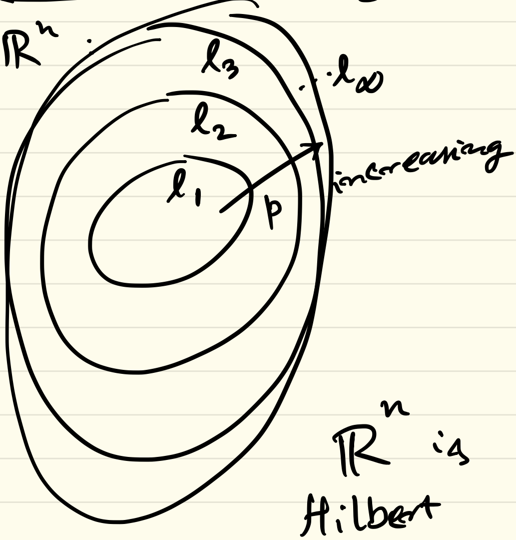
$$\|\underline{x}\|_\alpha \le c_1 \|\underline{x}\|_\beta$$

$$\& \ \|\underline{x}\|_\beta \le c_2 \|\underline{x}\|_\alpha$$

This says two norms $\|\cdot\|_\alpha$ & $\|\cdot\|_\beta$ are equivalent

ℓ_p spaces $\{\underline{x} \in \mathbb{R}^n : \|\underline{x}\|_p < \infty\}$

In $\mathbb{R}^n$:



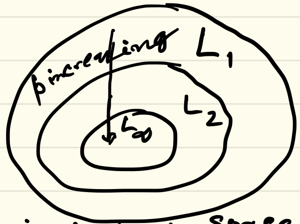
$\mathbb{R}^n$ is Hilbert space

Function norms

L_p spaces : $f(\underline{x})$

$$\|f\|_{L_p} := \left(\int_{\mathbb{R}^n} |f(\underline{x})|^p \, d\underline{x} \right)^{\frac{1}{p}}$$

$f \in L_p$ if $\|f\|_{L_p} < \infty$



(opposite to the picture on left)

L_2 is Hilbert Space
L_1 is NOT

Matrix norms:

$$\|A\|_p := \sup_{\underline{x} \neq 0} \frac{\|A\underline{x}\|_p}{\|\underline{x}\|_p}$$

$$\|A\|_2 = \text{spectral norm} = \sqrt{\lambda_{max}(AA^T)}$$

$$\|A\|_1 = \max_{j=1,\dots,n} \sum_{i=1}^{m} |a_{ij}|$$

$$\|A\|_\infty := \max_{i=1,\dots,m} \sum_{j=1}^{n} |a_{ij}|$$

$\langle \underline{a}, \underline{b} \rangle = \underline{a}^T \underline{b}$

$\|A\|_F$ Frobenius norm

$$= \sqrt{tr(AA^T)} = \sqrt{\sum_i \sum_j a_{ij}^2}$$

$$\langle A, B \rangle := tr(A^T B)$$

Existence & Uniqueness of Nonlin. Systems

$$\underline{\dot{x}} = \underline{f}(\underline{x}, t), \quad \underline{x}(0) = \underline{x}_0 \ \text{(given)}$$

Example:

$$\dot{x} = x^{1/3}, \quad x(0) = 0, \quad 1D$$

$$\Rightarrow x(t) = \left(\frac{2t}{3}\right)^{3/2} \quad \forall t$$

$$x(t) = 0 \quad \forall t$$

Non-unique !

Thm. "Local" Existence & Uniqueness Thm.

$$\underline{\dot{x}} = \underline{f}(\underline{x}, t), \quad \underline{x}(0) = \underline{x}_0 \ \text{(known)}$$

Local Lipschitz:

Let $\underline{f}(\underline{x}, t)$ be p.w. Cont. in t and "Locally" Lipschitz in $\underline{x}$

$\Updownarrow$

If $\mathcal{B}(\underline{x}_0, r) := \{ \underline{x} \in \mathbb{R}^n \mid \|\underline{x} - \underline{x}_0\| \le r \}$

&

$$\|f(\underline{x}, t) - f(\underline{y}, t)\|_{l_p} \le \lambda \|\underline{x} - \underline{y}\|_{l_p}$$

$$\forall \underline{x}, \underline{y} \in \mathcal{B}(\underline{x}_0, \underline{r})$$

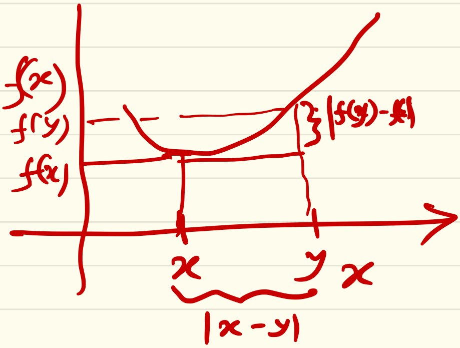
$$\forall t \in [t_0, t_1]$$

Then

$$\underline{f} \in \mathrm{Lip}(\|\cdot\|_p, \lambda)$$

Lipschitz constant

$$\frac{\|f(\underline{x}, t) - f(\underline{y}, t)\|_{l_p}}{\|\underline{x} - \underline{y}\|_{l_p}} \le \lambda$$



$$\frac{|f(y) - f(x)|}{|x - y|} \le \lambda$$

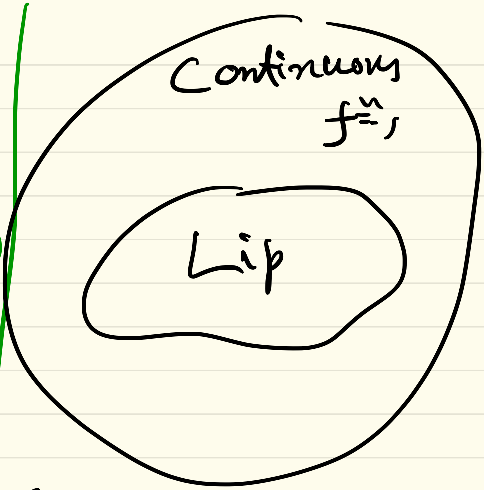
If $\overbrace{\text{Locally Lipschitz}}^{\text{(in } \underline{x}\text{)}}$, then $\exists\, \delta > 0$ s.t. the initial value problem (IVP) has UNIQUE solⁿ $\forall t \in (0, \delta]$

If $\underline{\text{Globally Lipschitz (in } \underline{x}\text{)}}$

(Replace "$\forall\, \underline{x}, \underline{y} \in B(\underline{x}_0, r)$"
with
"$\forall\, \underline{x}, \underline{y} \in \mathbb{R}^n$")

Then UNIQUE solⁿ

$\forall\, t \in (0, t_f]$



Example:
$f(x) = x^{1/3}$
continuous BUT
NOT Lip @ $x = 0$
(has unbounded slope)

Lemma:

If $\underline{f}(\underline{x}, t)$ and Jacobian $\frac{\partial \underline{f}}{\partial \underline{x}}(\underline{x}, t)$ are BOTH continuous on $[0, t_1] \times \mathbb{R}^n$

Then $\underline{f}$ is Globally Lipschitz on $[0, t_1] \times \mathbb{R}^n$ iff

$\left\| \left[\frac{\partial \underline{f}}{\partial \underline{x}} \right] \right\|_p$ is uniformly bounded on $[0, t_1] \times \mathbb{R}^n$

Example: $\underline{f}(\underline{x}) = \begin{pmatrix} -x_1 + x_1 x_2 \\ x_2 - x_1 x_2 \end{pmatrix}$

is C^1 on $\mathbb{R}^2$

NOT Globally Lipschitz

since $\left[\frac{\partial f}{\partial x}\right]$ is NOT uniformly bdd. on $\mathbb{R}^2$.

But on any compact (closed & bdd.) subset of $\mathbb{R}^2$, $\underline{f}$ is Lipschitz.

e.g. Prove that $\underline{f}$ is Lip. over $\mathcal{S} := \{\underline{x} \in \mathbb{R}^2 : |x_1| \leq a_1, |x_2| \leq a_2\}$

Compute the Lip. Constant λ.

Sol$^{\underline{n}}$: Jacobian $\begin{bmatrix} -1+x_2 & x_1 \\ -x_2 & 1-x_1 \end{bmatrix}$

$$\left\|\frac{\partial f}{\partial x}\right\|_\infty = \max\left\{|-1+x_2| + |x_1|,\ |x_2| + |1-x_1|\right\}$$

All pt.s on $\mathcal{S}$ satisfy

$$\left\|\frac{\partial f}{\partial x}\right\|_\infty \leq 1 + a_1 + a_2$$

$\therefore \lambda \left(\text{Lip. const}\right) = 1 + a_1 + a_2$

Locally Lip. $\leftarrow$ Necessary condⁿ for UNIQUENESS

Globally Lip $\leftarrow$ Suff. condⁿ for UNIQUENESS

<u>example</u>: $\dot{x} = -x^3 = f(x)$

$x(t_0) = x_0$

NOT Globally Lip. ($-3x^2$ NOT globally bdd.)

But unique solⁿ

$$x(t) = \mathrm{sgn}(x_0)\sqrt{\frac{x_0^2}{1+2x_0^2(t-t_0)}}$$

$\forall\, t \geq t_0$