AMS 231: Nonlinear Control Theory: Winter 2018

Take Home Exam

Name: ....

Deadline: March 22, 2018

NOTE: Please show all the steps in your solution. Turn in a hard copy of your Exam stapled with this as cover sheet with your name written in the above field. Please submit (either in person to instructor's office, or by scanned email attachment to the instructor) your Exam no later than the deadline.

Problem 1

Modeling and Controllability **($3 + 30 + 25 + 2 = 60$ points)**

Figure 1: A car with multiple direct-hooked passive trailers.

Consider a car with n direct-hooked passive trailers, which you may have seen in the airports carrying passenger luggage (Fig. 1). To mathematically model this system, we represent both the car and the trailers as having two driving wheels connected by an axle, as shown in Fig. 2.

Each trailer is hooked up in the middle point of the axle of the previous body by a rigid bar of length $\ell = 1$. To describe the state, we fix an inertial coordinate system, shown in the left bottom corner of Fig. 2.

Suppose (x, y) is the coordinate of the mid-point of the axle of the last trailer; θ_n is the angle that the car's pair of wheels make with the inertial horizontal axis; and θ_i, $0 \leq i \leq n-1$, is the angle that the $(n-i)^{\text{th}}$ trailer's pair of wheels make with the inertial horizontal axis. The state

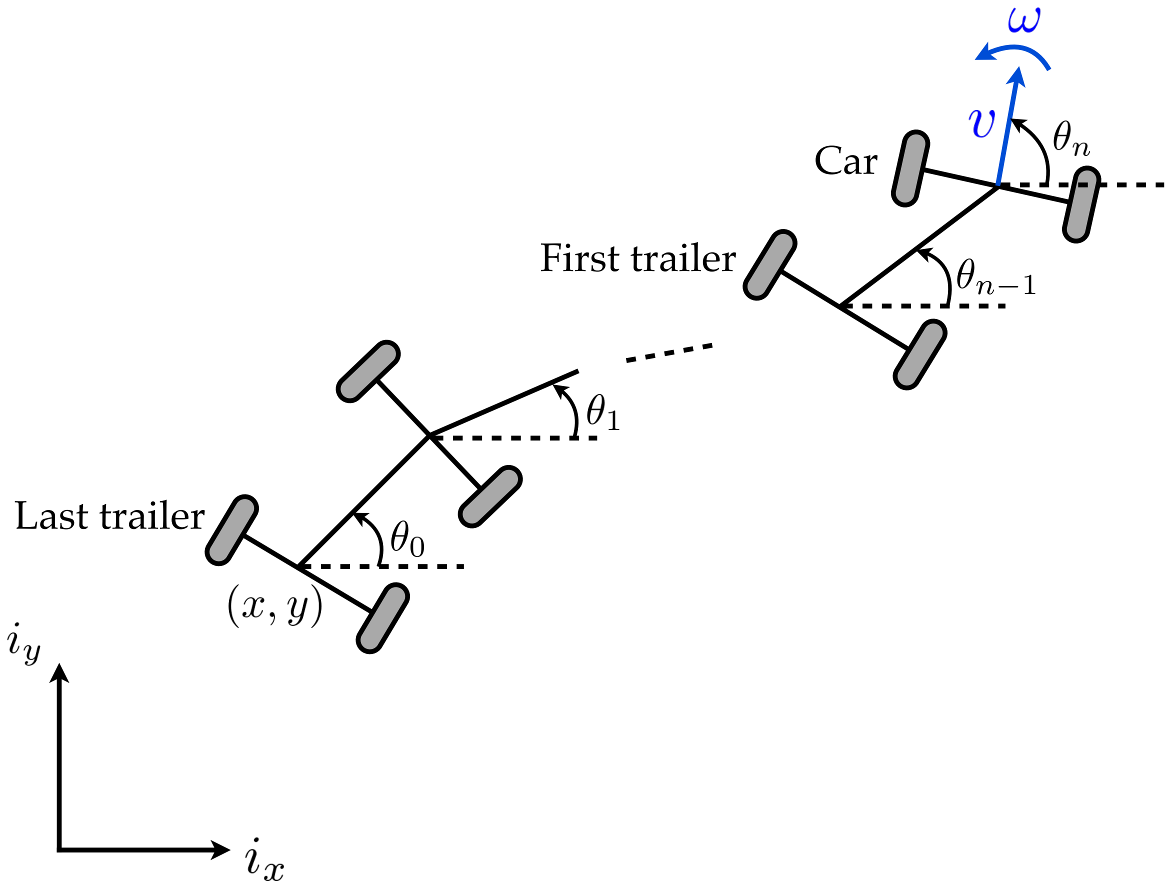

Figure 2: Model of a car with n trailers system. The inertial coordinate system is denoted as i_x-i_y. The control variables are shown in blue.

vector of the system is

$$\underline{x} := \{x, y, \theta_0, \theta_1, \dots, \theta_n\}^\top.$$

The control vector consists of the car's translational velocity v and angular velocity ω, i.e., $(u_1, u_2) := (v, \omega)$.

The wheels of each body (1 car and n trailers) are constrained to roll without slipping, i.e., the velocity of each body is parallel to the direction of its wheels. Let $f_n := 1$, and

$$f_i := \cos(\theta_{i+1} - \theta_i)\cos(\theta_{i+2} - \theta_{i+1}) \dots \cos(\theta_n - \theta_{n-1}) = \prod_{j=i+1}^{n} \cos(\theta_j - \theta_{j-1}), \quad 0 \le i \le n-1.$$

(a) The state space $\mathscr{X}$ of the system is: (choose the only one correct option, and give reasons)

(i) $\mathbb{R}^2 \times \mathrm{SO}(n+1)$

(ii) $\mathbb{R}^2 \times \mathrm{SE}(n+1)$

(iii) $\mathbb{R}^2 \times \mathbb{T}^{n+1}$

(iv) $\mathbb{R}^{n+3}$

(b) Prove that the control system corresponding to the above model can be expressed in the drift-free form:

$$\underline{\dot{x}} = g_1(\underline{x})\, u_1 \;+\; g_2(\underline{x})\, u_2,$$

where the input vector fields are

$$g_1(\underline{x}) = \left[f_0 \cos\theta_0,\ f_0 \sin\theta_0,\ f_1 \sin\left(\theta_1 - \theta_0\right),\ \ldots,\ f_{i+1} \sin\left(\theta_{i+1} - \theta_i\right),\ \ldots,\ f_n \sin\left(\theta_n - \theta_{n-1}\right),\ 0\right]^\top,$$

$$g_2(\underline{x}) = \Bigg[\underbrace{0,\ \ldots,\ 0}_{(n+2)\text{ times}},\ 1\Bigg]^\top.$$

(c) For the special case of $n = 1$, i.e., the car with one trailer, show that the system is globally controllable.

(d) What is the degree of nonholonomy for the case in part (c)? Give reasons.

Problem 2

Controller Synthesis $(15 + 15 + 10 = 40$ **points)**

Consider the nonlinear system

$$\dot{x}_1 = x_1 + \frac{x_2}{1 + x_1^2},$$
$$\dot{x}_2 = -x_2 + u.$$

Design globally stabilizing feedback controller for the origin using

(a) feedback linearization,

(b) backstepping,

(c) compare the two controllers in parts (a) and (b) by plotting x_1 versus t, and x_2 versus t for the respective closed-loop systems. Also plot u versus $||\underline{x}||_2$ for the two controllers and comment on their performance.