Lecture #15

$M^T = M \in \mathbb{R}^{n \times n}$

Positive semi definite if $\forall\ \underline{x} \in \mathbb{R}^n$, we get

$$\underline{x}^T M \underline{x} \geq 0 \iff M \succeq 0$$

$$M_1 \succeq M_2 \iff M_1 - M_2 \succeq 0 \iff \underline{x}^T (M_1 - M_2) \underline{x} \geq 0$$

$$\text{eig}(M) = \lambda_1, \dots, \lambda_n \geq 0$$

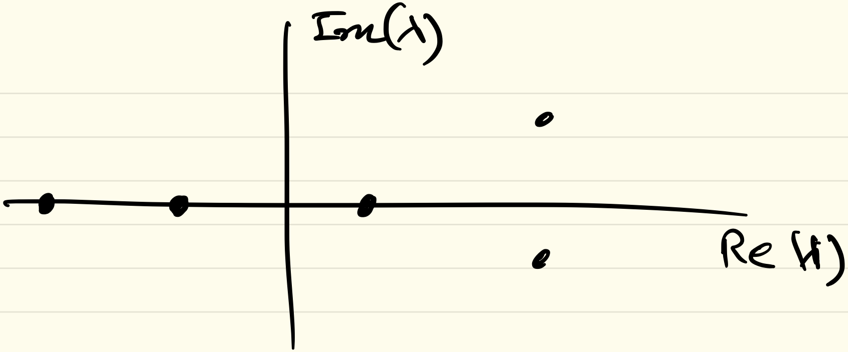
$$\begin{pmatrix} 1 & & \\ & 2 & \\ & & -3 \end{pmatrix}$$

Transformation of matrices:

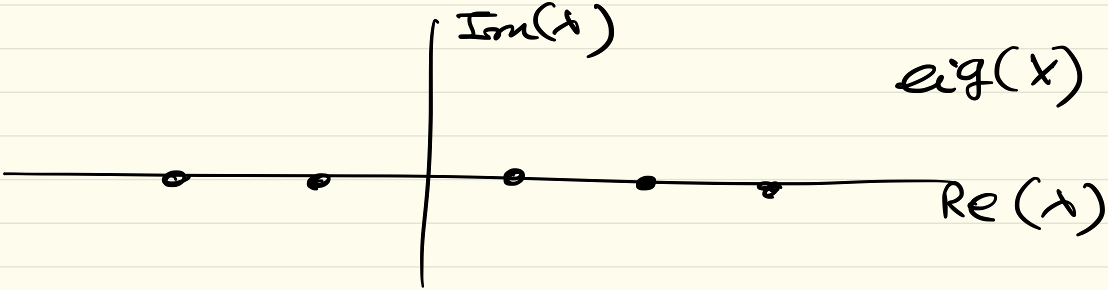
- $X \mapsto T X T^{-1}$ T is non-singular

  $\text{eig}(X) = \text{eig}(T X T^{-1})$: similarity transformation

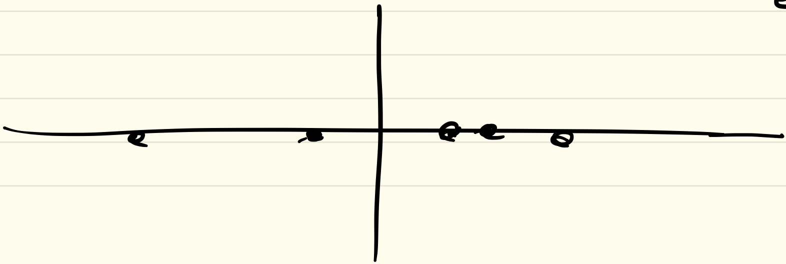
- $X \mapsto T X T^T$: Congruence transformation,



Symmetric X, eig(X) are reals



eig(TXT^T)



Sliding Mode Control / Variable Structure Control

Example:

$\underline{x} \in \mathbb{R}^2$

$$\left\{\begin{array}{l} \dot{x}_1 = x_2 \\ \dot{x}_2 = f(\underline{x}) + g(\underline{x})\, u \end{array}\right\}$$

state, $u \in \mathbb{R}$

where f & g are unknown nonlinear function

& $g(x) \geq g_0 > 0 \quad \forall\, \underline{x} \in \mathbb{R}^n$

want to design feedback $u(\cdot)$ to stabilize origin of closed-loop.

$$x^{(n)} = f(\underline{x}) + g(\underline{x})\, u$$

$$\Leftrightarrow \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{pmatrix} = \begin{pmatrix} x_2 \\ x_3 \\ \\ x_n \\ f(x) + g(x)u \end{pmatrix}$$

Let

$$s := a_1 x_1 + x_2 = 0$$

$x_2 = -a_1 x_1$

manifold

On this manifold:

$$\dot{x}_1 = (x_2) = -a_1 x_1$$

$$\text{Manifold } s := \{ \underline{x} \in \mathbb{R}^2 : a_1 x_1 + x_2 = 0 \}$$

Choosing $\longrightarrow a_1 > 0$ will ensure "sliding"

And choosing $|a_1|$, will allow us to vary the rate-of-convergence.

<u>NOTE</u>: The motion/dynamics of s is independent of f & g.

<u>Also</u>: the variable "s" satisfies

$$\dot{s} = a_1 \dot{x}_1 + \dot{x}_2 = a_1 x_2 + \underline{f}(\underline{x}) + \underline{g}(\underline{x})\, u$$

Suppose f & g satisfy the inequality:

$$\left| \frac{a_1 x_2 + f(x)}{g(x)} \right| \leq \rho(x) \quad \forall\ \underline{x} \in \mathbb{R}^2$$

for some known function $\rho(\cdot)$

Now choose $V = \frac{1}{2} s^2$ for dynamics

$$\dot{s} = a_1 x_2 + f(x) + g(x)\, u$$

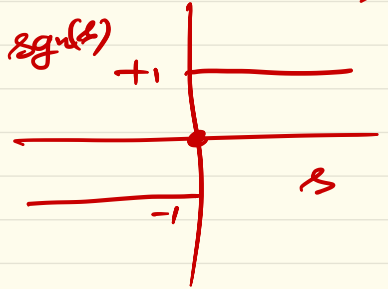
Then $\dot{V} = s\dot{s}$

$$= s\left(a_1 x_2 + f(x)\right) + g(x)\, s\, u$$

$$\le g(\underline{x})\, |s|\, \rho(x) + g(x)\, s\, u$$

(assumption inequality)

$$\mathrm{sgn}(s) := \begin{cases} +1, & s>0 \\ 0, & s=0 \\ -1, & s<0 \end{cases}$$



Take $u = -\beta(x)\,\mathrm{sgn}(s)$

where

$$\beta(x) \ge \rho(x) + \beta_0, \quad \beta_0 > 0$$

This gives

$$\dot{V} \le \underbrace{g(x)\, |s|\, \rho(x) - g(x)\left(\rho(x)+\beta_0\right) s \cdot \mathrm{sgn}(s)}$$

$$= -g(x)\, \beta_0\, |s|$$

$$\le -g_0\, \beta_0\, |s| \quad \left(\because g(x) \ge g_0 > 0\right)$$

$\Rightarrow s\dot{s} \le -g_0\beta_0 |s|$ (Differential inequality)

$\Rightarrow D^+|s| \le -g_0\beta_0$

upper right hand derivative

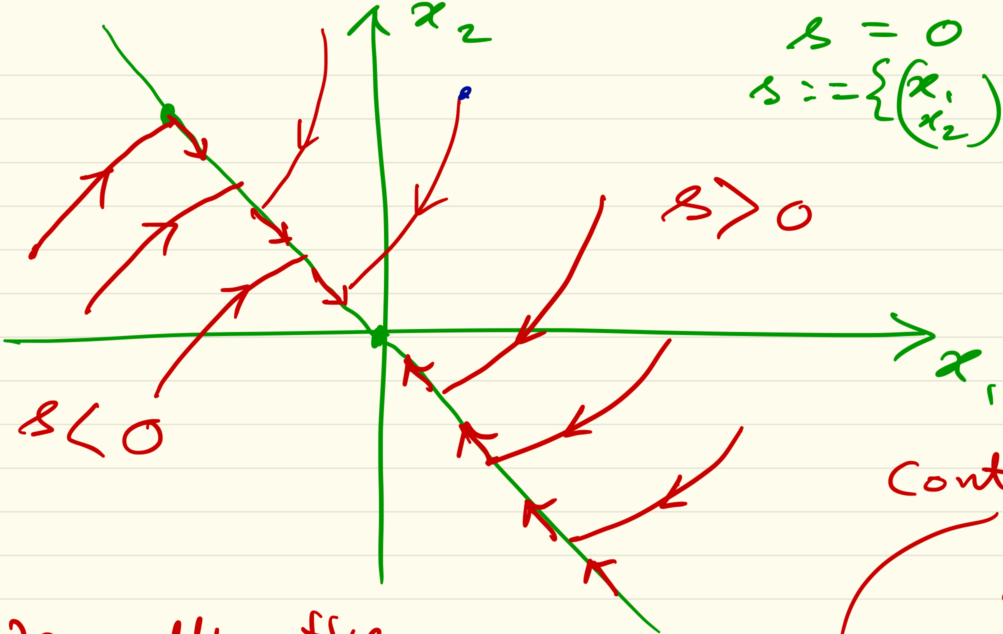
$$D^+ v(t) := \limsup_{h \to 0+} \frac{v(t+h) - v(t)}{h}$$

If $v(t)$ is differentiable, $D^+ \equiv \frac{d}{dt}$

$\Rightarrow$ (Comparison Lemma)

$\Rightarrow$ Trajectory $s(t)$ reaches $s = 0$ (our manifold) in finite time

Already shown, once on $s=0$, then cannot leave if $(\dot{V} \le -g_0\beta_0|s|)$



$s=0$

$S := \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2 : a_1 x_1 + x_2 = 0 \right\}$

$x_2 = -a_1 x_1$

$a_1 > 0$

2 phases:

→ Reach phase

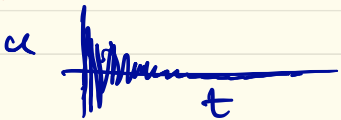
→ Sliding phase

Control is discontinuous at $s=0$

We call the control $u = -\beta(x)\,\mathrm{sgn}(s)$ as "sliding mode control"

Pro: Sliding phase is indep. of f & g.

Difficulty/Con: Chattering (Discretization, Delay)

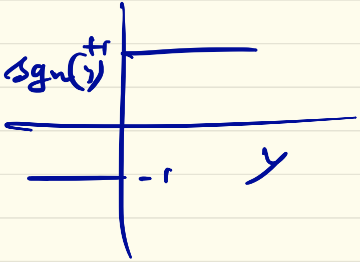


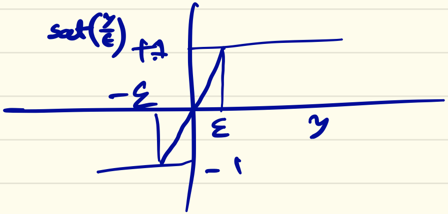
approx. control law
(sliding mode)

$$u = -\beta(x)\,\mathrm{sgn}(s)$$

$$\approx -\beta(x)\,\mathrm{sat}\left(\frac{s}{\epsilon}\right)$$

where $\mathrm{sat}(y) := \begin{cases} y & \text{if } y \le 1 \\ \mathrm{sgn}(y) & \text{if } |y| > 1 \end{cases}$





Backstepping

Integrator Backstepping (Single Input)

$$\left.\begin{aligned} \dot{\underline{\eta}} &= \underline{f}(\underline{\eta}) + g(\eta)\,\xi \\ \dot{\xi} &= u \end{aligned}\right\} \quad \left.\begin{aligned} &\eta \in \mathbb{R}^n \\ &\xi \in \mathbb{R} \\ &u \in \mathbb{R} \end{aligned}\right\} \quad \text{state } \begin{Bmatrix} \eta \\ \hline \dot{\xi} \end{Bmatrix} \in \mathbb{R}^{n+1}$$

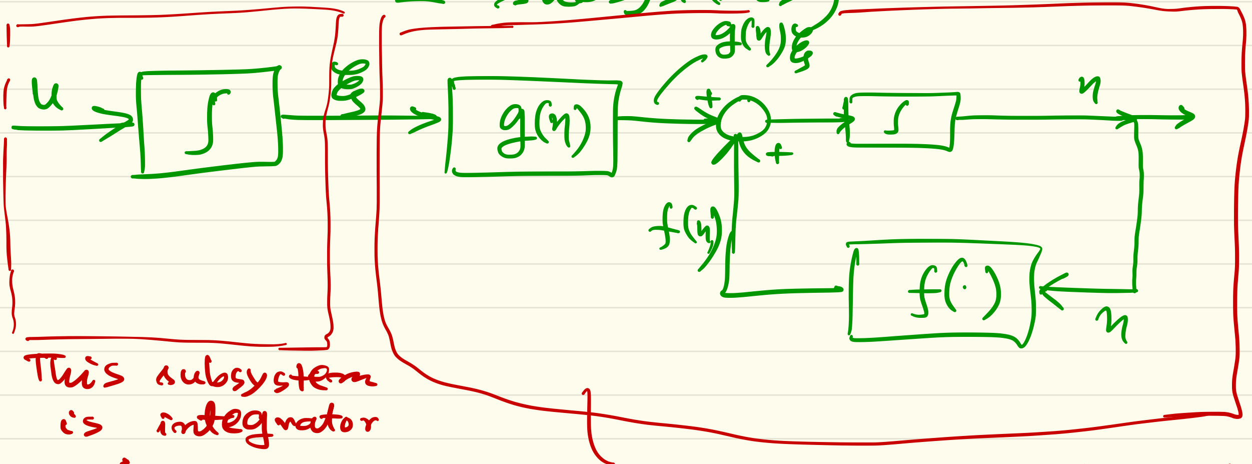
$f : \mathcal{D} \mapsto \mathbb{R}^n, \quad \underline{f}(\underline{0}) = \underline{0}$

$g : \mathcal{D} \mapsto \mathbb{R}$

Objective: Design state feedback law to stabilize origin in $\mathbb{R}^{n+1}$, $\begin{pmatrix} \eta \\ \xi \end{pmatrix} = \begin{pmatrix} \underline{0} \\ 0 \end{pmatrix}$.

Assumption: Both f & g are known.

Diagram: (Can be thought of as cascade of 2 subsystems)



This subsystem is integrator

$\dot{\xi} = u$

This subsystem "thinks" ξ is its input

$$\left(\underbrace{\dot{\underline{\eta}} = \underline{f}(\underline{\eta}) + \underline{g}(\underline{\eta})\xi}_{} \right)$$

Suppose, I can stabilize this subsystem by some feedback law $\xi = \phi(\underline{\eta})$ with $\phi(0) = 0$

$\Updownarrow$

$\dot{\underline{\eta}} = \underline{f}(\underline{\eta}) + \underline{g}(\underline{\eta})\, \phi(\underline{\eta})$ is A.S.

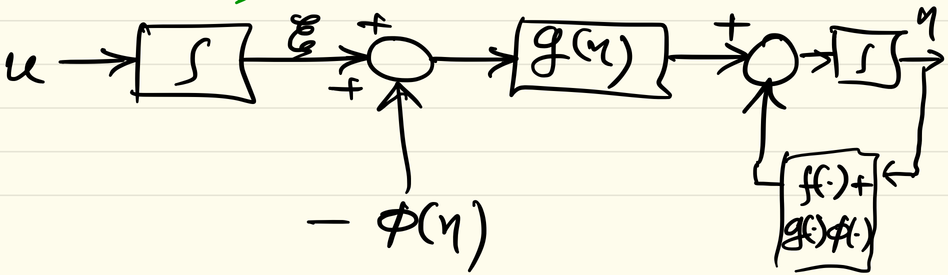
Suppose further we know $V(\eta)$ which is C^1, p.d. s.t.

$$\left\langle \underbrace{\nabla_{\eta} V}_{\frac{\partial V}{\partial \underline{\eta}}} , \underline{f}(\underline{\eta}) + \underline{g}(\underline{\eta})\, \phi(\underline{\eta}) \right\rangle \le -W(\underline{\eta}) \quad \forall \eta \in \mathcal{D}$$

p.d.

Now let's add & subtract $g(\eta)\phi(\eta)$ to the RHS of $\dot{\underline{\eta}} = \underline{f}(\underline{\eta}) + \underline{g}(\underline{\eta})\,\underline{\xi}$ to get

$$\begin{cases} \dot{\eta} = \big(f(\eta) + g(\eta)\phi(\eta)\big) + g(\eta)\big(\xi - \phi(\eta)\big) \\ \dot{\xi} = u \end{cases}$$



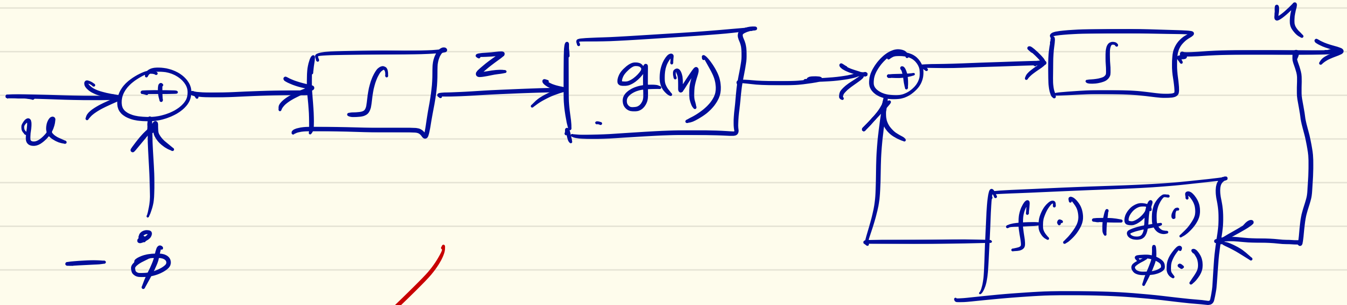
Introduce a change-of-variable:

$$z := \xi - \phi(\eta)$$

This gives

$$\dot{\eta} = \big(f(\eta) + g(\eta)\,\phi(\eta) \big) + g(\eta)\, z$$

$$\dot{z} = u - \dot{\phi}$$



prev picture $\longmapsto$ This picture is "backstepping" $-\phi(\eta)$ through the integrator

We know f, g, ϕ :

$\therefore$ We can compute $\dot{\phi} = \frac{\partial \phi}{\partial \eta}\left(f(\eta) + g(\eta)\xi\right)$

Let $v := u - \dot{\phi}$, then the system reduces to

$$\begin{cases} \dot{\eta} = \left(f(\eta) + g(\eta)\phi(\eta)\right) + g(\eta) z \\ \dot{z} = v \end{cases}$$

Similar to what we started with, except the first component now has A. S. origin when input $= 0$

Idea : Exploit this to design v to stabilize overall system.

use : $V_c(\underline{\eta}, \xi) = V(\eta) + \frac{1}{2} z^2$

Then

$$\dot{V}_e = \frac{\partial V}{\partial \eta}\left(f(\eta) + g(\eta)\phi(\eta)\right) + \frac{\partial V}{\partial \eta} g(\eta) z + z v$$

$$\leq -W(\eta) + \frac{\partial V}{\partial \eta} g(\eta) z + z v$$

Choose: $v = -\frac{\partial V}{\partial \eta} g(\eta) - kz, \quad k > 0$

This makes: $\dot{V}_e \leq -W(\eta) - kz^2$

$\Rightarrow \begin{pmatrix} \eta \\ \xi \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is A.S. ($\because$ $\phi(0) = 0$ La Salle etc.)

Sub. back v, z, $\dot{\phi}$, to get original feedback control:

$$u = \frac{\partial \phi}{\partial \eta}\left(f(\eta) + g(\eta)\xi \right) - \frac{\partial V}{\partial \eta} g(\eta) - k(\xi - \phi(\eta))$$