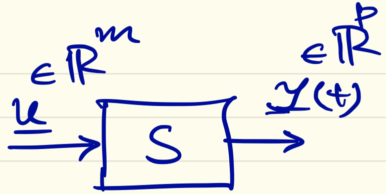
Lecture #13

$$\underline{\dot{x}} = \underline{f}(\underline{x}, \underline{u}, t)$$

$$\underline{y} = \underline{h}(\underline{x}, \underline{u}, t)$$

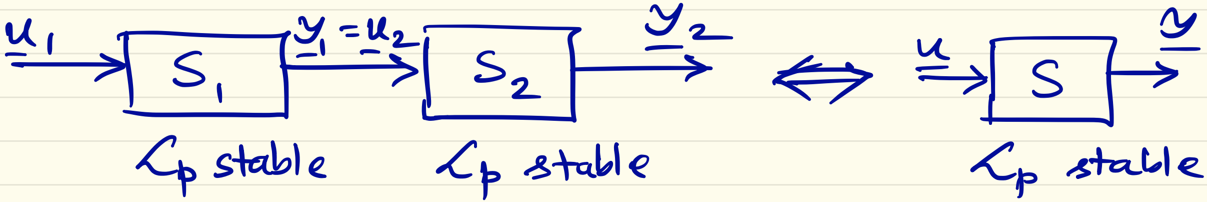


Compositional Results for $\mathcal{L}_p$ stability (Interconnection of Nonlin. Systems):

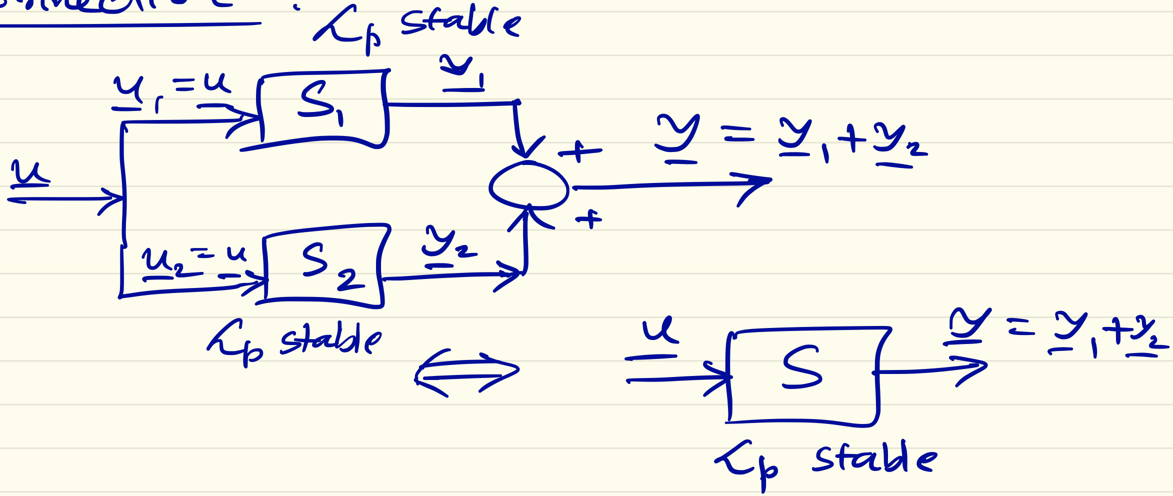
$$S_1 \begin{cases} \underline{\dot{x}}_1 = \underline{f}_1(\underline{x}_1, \underline{u}_1, t) \\ \underline{y}_1 = \underline{h}_1(\underline{x}_1, \underline{u}_1, t) \end{cases}$$

$$S_2 \begin{cases} \underline{\dot{x}}_2 = f_2(\underline{x}_2, \underline{u}_2, t) \\ \underline{y}_2 = \underline{h}_2(\underline{x}_2, \underline{u}_2, t) \end{cases}$$

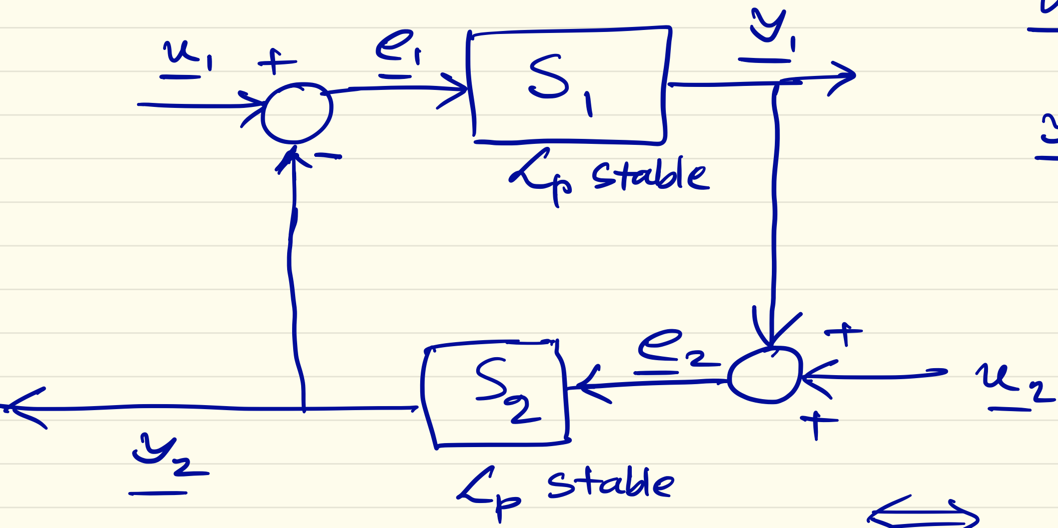
Series Connection:



Parallel Connection:



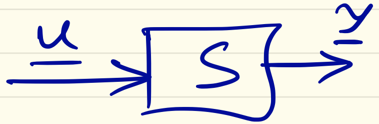
Feedback Connection:



$\underline{u}_1, \underline{y}_2, \underline{e}_1 \in \mathbb{R}^m$

$\underline{y}_1, \underline{u}_2, \underline{e}_2 \in \mathbb{R}^q$

$\Longleftrightarrow$



Is S $\mathcal{L}_p$ stable?

For S_1:

$$\|\underline{y}_1(t)\|_{\mathcal{L}} \leq \gamma_1 \|\underline{e}_1(t)\|_{\mathcal{L}} + \beta_1 \quad \forall\, \underline{e_1(t)} \in \mathcal{L}_p$$

$$\|\underline{y}_2(t)\|_{\mathcal{L}} \leq \gamma_2 \|\underline{e}_2(t)\|_{\mathcal{L}} + \beta_2 \quad \forall\, \underline{e}_2(t) \in \mathcal{L}$$

$\forall\, t \in [0, \infty)$

Assumption : System is well-defined

$$\Updownarrow$$

$\forall$ pair $\begin{pmatrix} \underline{u}_1 \\ \underline{u}_2 \end{pmatrix} \in \mathcal{L} \times \mathcal{L}$

$\exists$ unique outputs $\begin{pmatrix} e_1 \\ y_2 \end{pmatrix} \in \mathcal{L} \times \mathcal{L}$

& $\begin{pmatrix} e_2 \\ y_1 \end{pmatrix} \in \mathcal{L} \times \mathcal{L}$.

Let $\underline{u} := \begin{pmatrix} \underline{u}_1 \\ \underline{u}_2 \end{pmatrix} \in \mathbb{R}^{m+q}$ & $\underline{y} := \begin{pmatrix} \underline{y}_1 \\ \underline{y}_2 \end{pmatrix} \in \mathbb{R}^{m+q}$

$\underline{e} := \begin{pmatrix} \underline{e}_1 \\ \underline{e}_2 \end{pmatrix} \in \mathbb{R}^{m+q}$

Exercise :

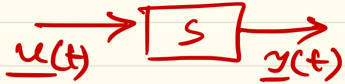
$\underline{u} \longmapsto \underline{y}$ is finite gain $\mathcal{L}$ stable

$\Updownarrow$ (iff)

$\underline{u} \longmapsto \underline{e}$ is finite gain $\mathcal{L}$ stable

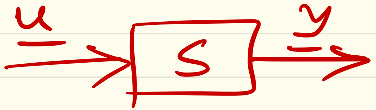
Thm: (Small Gain Theorem)

Feedback connection is finite gain $\mathcal{L}$ stable



if $\boxed{\gamma_1 \gamma_2 < 1}$

Passivity



$\underline{u}(t), \underline{y}(t) \in \mathbb{R}^m$

$$\langle \underline{u}(t), \underline{y}(t) \rangle = \underline{u}^T(t)\, \underline{y}(t) = \underline{y}^T(t)\, \underline{u}(t)$$

$$\int_0^T \underbrace{\langle \underline{u}(t), \underline{y}(t) \rangle}_{\substack{P(t) \\ \text{instantaneous} \\ \text{power}}} dt = \text{Energy of any control system}$$

$$\int_0^T \underbrace{F\,dx}_{W(t)}\, dt$$

Example:

- $\underline{u}(t)$ could be current
  $y(t)$ " " voltage
- $u(t)$ could be force
  $y(t)$ " " displacement

Passive System:

Proposition: The following sentences are equivalent

① The system S is "passive".

$\Updownarrow$

② (Rate inequality) $\exists\, C^1$ f^{n} $V(\underline{x}, t) \geq 0$ called "storage function" s.t. $\forall\, t > 0$, we have

2.1 $\dot{V} < 0$

2.2 $\underbrace{u^T(t)\, y(t)}_{\text{supply rate of energy}} \geq \underbrace{\dot{V} =: \frac{d}{dt} V = \langle \nabla V, \underline{f} \rangle}_{\text{Storage rate}}$

(if equality) we say S is "lossless"

③ **Dissipativity inequality** : The system satisfies :

$$\int_0^t (\underline{u}(\tau))^T \underline{y}(\tau)\, d\tau \geq V(\underline{x}(t), t) - V(\underline{x}(0), 0)$$

$\Updownarrow$

④ **Dissipation equality** :

$$\underbrace{\int_0^t (\underline{u}(\tau))^T \underline{y}(\tau)\, d\tau}_{\text{Energy supplied up until time } t} = \underbrace{V(\underline{x}(t), t) - V(\underline{x}(0), 0)}_{\text{Energy stored}} + \underbrace{d(\underline{x}(t))}_{\text{Energy dissipated}}$$

In short,

$$\boxed{\text{supply} = \text{storage} + \text{dissipation}}$$

(If dissipation $= 0$, $\Longleftrightarrow$ System is Lossless)

Related notions :

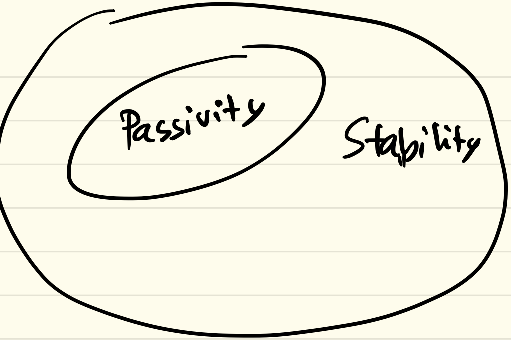
- Strictly passive (SP) if $\underline{u}^T \underline{y} \geq \dot{V} + \psi(\underline{x})$ for some pos.-def. f$^{\underline{n}}$ $\psi(\cdot)$

- input-feedforward passive (IFP) if $\underline{u}^T \underline{y} \geq \dot{V} + u^T \xi(\underline{u})$ for some $\xi(\cdot)$

- input strictly passive (ISP) if IFP + Extra cond$^{\underline{n}}$ $\left(u^T \xi(\underline{u}) > 0 \quad \forall \ \underline{u} \neq 0 \right)$

- output-feedback passive (OFP) if $\underline{u}^T \underline{y} \geq \dot{V} + \underline{y}^T \eta(\underline{y})$ for some $\eta(\cdot)$

- output strictly passive (OSP) if OFP + Extra cond$^{\underline{n}}$ $(y^T \eta(y) > 0 \ \forall \ \underline{y} \neq 0)$



Theorem #1:

If S is passive with storage function $V(\cdot)$, then origin of the unforced system

$$\left(\dot{\underline{x}} = \underline{f}(\underline{x}, \underline{0})\right)$$

is stable

$$\dot{\underline{x}} = f(\underline{x}, \underline{u})$$
$$\underline{y} = \underline{h}(\underline{x}, \underline{u})$$
$$\underline{x} \in \mathbb{R}^n$$
$$\underline{u} \in \mathbb{R}^m$$

Thm #2:

If S is OSP with

$$(u(t))^T y(t) \geq \dot{V} + \delta \, y(t)^T y(t)$$

$$y(t)^T y(t) = \|y(t)\|_2^2$$

for some $\delta > 0$,

then S is finite gain $\mathcal{L}_2$ stable.

& the gain $\leq (1/\delta)$

Def^{n}.: (**Zero State Observability**) We say S is **Zero state Observable** (ZSO) if no solution of $\dot{\underline{x}} = \underline{f}(\underline{x}, \underline{0})$ can stay identically in the set $\{ \underline{x} \in \mathbb{R}^n \mid \underline{h}(\underline{x}, \underline{0}) = \underline{0} \}$ other than the trivial solution $\underline{x}(t) \equiv \underline{0} \;\; \forall t$.

Thm #3: If S is <u>either</u> SP (Strictly passive)

<u>or</u> OSP + ZSO

then origin of $\dot{\underline{x}} = \underline{f}(\underline{x}, 0)$ is A.S.

If in addition, the storage $f^{\underline{n}}$ is rad. unbdd. then GAS.

Thm # 4: Feedback preserves passivity.
(same picture of connecting S_1 & S_2 in feedback structure)

(If S_1 & S_2 are passive $\Rightarrow$ S is passive)

Thm # 5: Feedback connection of two OSP subsystems with

$$\underline{e}_i^T(t)\,\underline{y}_i(t) \geq \dot{V}_i + \delta_i \|\underline{y}_i(t)\|_2^2, \quad \delta_i > 0$$

for $i = 1, 2$

for some storage functions $V_i(\underline{x}_i)$

Then the system S is finite gain $\mathcal{L}_2$ stable with $\mathcal{L}_2$ gain $\leq \dfrac{1}{\min(\delta_1, \delta_2)}$

Thm #6 : (Neither subsystem is "nice" but overall system is "nice")

Consider the same feedback diagram & suppose

$$(\underline{e}_i(t))^T \underline{y}_i \geq \dot{V}_i + \epsilon_i \underbrace{\| \underline{e}_i(t) \|_2^2}_{\substack{\parallel \\ e_i^T e_i}} + \delta_i \underbrace{\| \underline{y}_i(t) \|_2^2}_{\substack{\parallel \\ \underline{y}_i^T \underline{y}_i}}$$

for some storage functions $V_i(\underline{x}_i)$ for $i = 1, 2$

Then the overall system $\underline{u} \longmapsto \underline{y}$ is finite gain $\mathcal{L}_2$ stable if

$$\epsilon_1 + \delta_2 > 0 \quad \& \quad \epsilon_2 + \delta_1 > 0 .$$