AMS 231: Nonlinear Control Theory: Winter 2018

Homework #5

Name: ....

Due: March 06, 2018

NOTE: Please show all the steps in your solution. Turn in a hard copy of your HW stapled with this as cover sheet with your name written in the above field. Please submit your HW in class on the due date.

Problem 1

Stabilizing Controllers $(10 + 10 + (5 + 5) + 10 + (5 + 5) + 10 + 10 = 70$ **points)**

Consider simple scalar control system

$$\dot{x} = -x^3 + u, \qquad x, u \in \mathbb{R}.$$

We want to design (static) state feedback control $u = u(x)$ such that origin of the closed-loop system is G.A.S. We will design multiple stabilizing controllers for this system, and compare their performance.

(a) Design a **feedback linearizing controller** $u_{\mathrm{FL}}(x)$ by applying "cancel the nonlinearity and get a stable linear closed-loop system" idea.

(b) Prove that a **linear feedback controller** $u_{\mathrm{L}}(x) = -x$ makes the origin of the closed-loop system G.A.S. (Hint: use $V(x) = \frac{1}{2}\,x^2$ and the Barbashin-Krasovskii theorem.)

(c) Give two reasons why the controller $u_{\mathrm{L}}(x)$ in part (b) is a better controller than $u_{\mathrm{FL}}(x)$ in part (a). (Hint: think rate-of-convergence of the closed-loop system, and magnitude of control signal for large x.)

(d) The answer in part (c) tells us that it is better not to kill "friendly nonlinearity". Consider another design idea: **doing nothing controller**, i.e., $u_0(x) \equiv 0$ for all $x \in \mathbb{R}$. Prove that $u_0(x)$ also makes the origin G.A.S.

(e) Give <u>one advantage</u> and <u>one disadvantage</u> of $u_0(x)$ compared to $u_{\mathrm{L}}(x)$. (Hint: again think in terms of the hint in part (c)).

(f) Design another stabilizing controller $u_{\mathrm{S}}(x)$ using **Sontag's formula**. (Hint: use the Lyapunov function in part (b) as the CLF.)

(g) From your answer in part (f), argue that near $x = 0$, we have $u_{\mathrm{S}}(x) \approx u_{\mathrm{L}}(x)$; and for $|x| \to \infty$, we have $u_{\mathrm{S}}(x) \approx u_0(x)$, and therefore, $u_{\mathrm{S}}(x)$ outperforms all the previous controllers.

Solution

(a) Motivated by the "cancel the nonlinearity and get a stable linear closed-loop system" idea, we take $u_{\mathrm{FL}}(x) = x^3 - x$ (more generally, can take $u_{\mathrm{FL}}(x) = x^3 - kx, k > 0$). This results a closed-loop system $\dot{x} = -x$, which makes the origin G.A.S.

(b) For $u_{\mathrm{L}}(x) = -x$, we get the closed-loop system $\dot{x} = -x^3 - x$. Taking $V(x) = \frac{1}{2}\, x^2$ (positive definite function, radially unbounded), we get $\dot{V} = -x^4 - x^2$ for all $x \neq 0$. By Barbashin-Krasovskii theorem, this guarantees that the origin is G.A.S.

(c) The controller in part (b) entails *faster rate of convergence* than the controller in part (a). **One way to see this** is to compare the $\dot{V}$ for the two closed-loop systems for same $V(x) = \frac{1}{2}\, x^2$. For the controller in part (a), we get $\dot{V} = -x^2$, whereas the controller in part (b) yields $\dot{V} = -x^4 - x^2 < -x^2$ for all $x \neq 0$. **Another way to see this** is to actually solve the scalar closed-loop systems: $x_{\mathrm{FL}}(t) = x_0 e^{-t}$, $x_{\mathrm{L}}(t) = \pm x_0 \left((1 + x_0^2) \left(e^{2t} - \frac{x_0^2}{1+x_0^2} \right) \right)^{-1/2}$, and notice that the latter decays faster than e^{-t}.

Second reason to prefer the controller in part (b) over the controller in part (a) is that for large $|x|$ (far from the origin), larger control effort is needed for $u_{\mathrm{FL}}(x)$ than $u_{\mathrm{L}}(x)$. This is illustrated in the following plot.

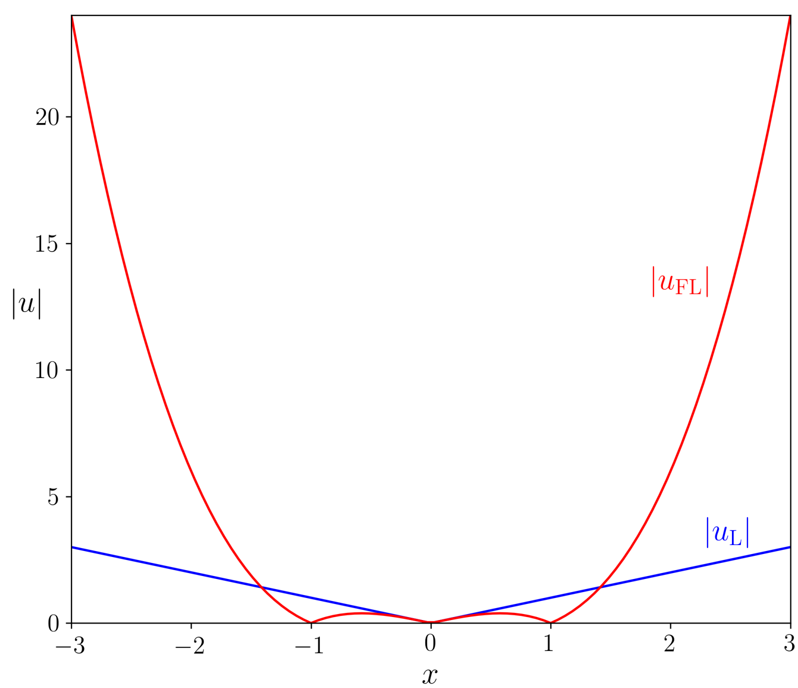

Figure 1: The plot of x versus the magnitude of control effort $|u|$. Note that for large $|x|$, we have $|u_{\mathrm{FL}}| >> |u_{\mathrm{L}}|$.

(d) The controller $u_0(x) \equiv 0$ for all x, results the closed-loop dynamics $\dot{x} = -x^3$, which again by taking $V(x) = \frac{1}{2}\,x^2$, yields $\dot{V} = -x^4 < 0$ for all $x \neq 0$, thereby establishing G.A.S. for origin via Barbashin-Krasovskii theorem.

(e) The advantage of $u_0(x)$ is that the control effort is always zero (smaller than any other control strategy). Disadvantage of $u_0(x)$ is that for small $|x|$ (near the origin), the rate of convergence is slower than that resulting from $u_{\rm L}(x)$. This can be seen by solving the closed-loop system for $u_0(x)$ as $x(t) = \pm\dfrac{x_0}{\sqrt{1+2tx_0^2}}$, and comparing with $x_{\rm L}(t)$ above. (Can also compare $|\dot{x}|$ for small $|x|$ to get the same conclusion.)

(f) By taking $V(x) = \frac{1}{2}\,x^2$ to be the CLF as per hint, we use Sontag's formula to get the stabilizing controller $u_{\rm S}(x) = \dfrac{x^4 - \sqrt{x^8 + x^4}}{x} = x^3 - x\sqrt{x^4+1}$, for $x \neq 0$. Notice that the formula automatically captures $u_{\rm S}(x) = 0$ for $x = 0$.

(g) Expanding $u_{\rm S}(x)$ in Taylor series about $x = 0$ yields

$$-x + x^3 - \frac{x^5}{2} + \frac{x^9}{8} + O\left(x^{11}\right).$$

Therefore, up to first order, $u_{\rm S}(x) \approx u_{\rm L}(x)$ near $x = 0$.

Next, expanding $u_{\rm S}(\frac{1}{x})$ in Taylor series about $x = 0$ (equivalent to expanding $u_{\rm S}(x)$ around $|x| = \infty$), to get

$$-\frac{1}{2x} + \frac{1}{8x^5} - \frac{1}{16x^9} + O\left(\frac{1}{x^{11}}\right),$$

which tells us that $u_{\rm S}(x) \approx u_0(x)$ as $|x| \to \infty$, hence the claim. The following plot compares all the feedback controllers.

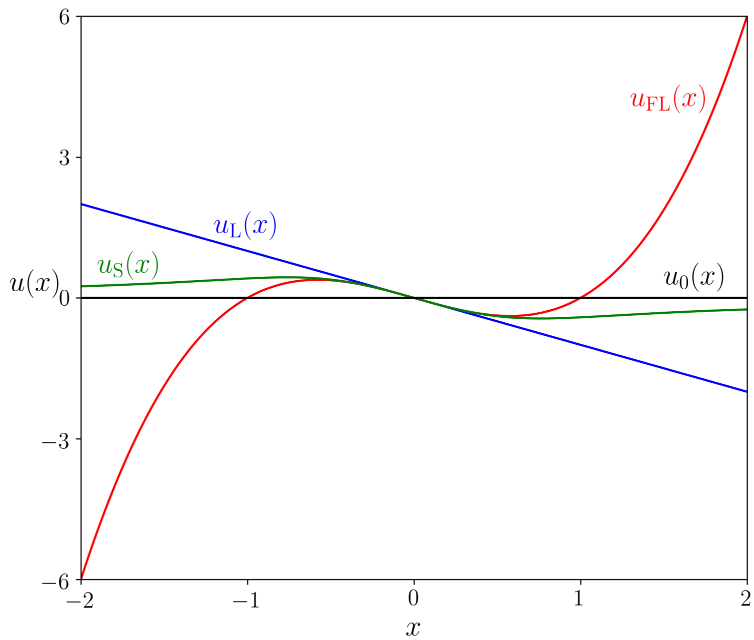

Figure 2: Comparison of the feedback controllers for Problem 1.

Problem 2

Integrator Backstepping **(30 points)**

Consider the following 3 state control system which is a modification of the worked out example in Lecture 16 with an additional integrator at the input side:

$$
\begin{aligned}
\dot{x}_1 &= x_1^2 - x_1^3 + x_2, \\
\dot{x}_2 &= x_3, \\
\dot{x}_3 &= u.
\end{aligned}
$$

Design an integrator backstepping controller to make the origin G.A.S.

Solution

From the worked out backstepping example in Lecture 16, we know that the second order system

$$
\begin{aligned}
\dot{x}_1 &= x_1^2 - x_1^3 + x_2, \\
\dot{x}_2 &= x_3,
\end{aligned}
$$

with x_3 as input, has global stabilizing controller

$$x_3 = -x_1 - (1 + 2x_1)\left(x_1^2 - x_1^3 + x_2\right) - \left(x_2 + x_1 + x_1^2\right) =: \phi\left(x_1, x_2\right),$$

and

$$V\left(x_1, x_2\right) = \frac{1}{2}x_1^2 + \frac{1}{2}\left(x_2 + x_1 + x_1^2\right)^2$$

is the corresponding Lyapunov function.

To backstep further, introduce the change-of-variable

$$z_3 := x_3 - \phi\left(x_1, x_2\right)$$

to get

$$
\begin{aligned}
\dot{x}_1 &= x_1^2 - x_1^3 + x_2, \\
\dot{x}_2 &= \phi\left(x_1, x_2\right) + z_3, \\
\dot{z}_3 &= u - \frac{\partial \phi}{\partial x_1}\left(x_1^2 - x_1^3 + x_2\right) - \frac{\partial \phi}{\partial x_2}\left(\phi + z_3\right).
\end{aligned}
$$

Letting $V_c := V\left(x_1, x_2\right) + \dfrac{1}{2}\, z_3^2$, we obtain

$$
\begin{aligned}
\dot{V}_c &= \frac{\partial V}{\partial x_1}\left(x_1^2 - x_1^3 + x_2\right) + \frac{\partial V}{\partial x_2}\left(\phi + z_3\right) + z_3\left(u - \frac{\partial \phi}{\partial x_1}\left(x_1^2 - x_1^3 + x_2\right) - \frac{\partial \phi}{\partial x_2}\left(\phi + z_3\right)\right) \\
&= -x_1^2 - x_1^4 - \left(x_2 + x_1 + x_1^2\right)^2 + z_3\left(\frac{\partial V}{\partial x_2} + u - \frac{\partial \phi}{\partial x_1}\left(x_1^2 - x_1^3 + x_2\right) - \frac{\partial \phi}{\partial x_2}\left(\phi + z_3\right)\right),
\end{aligned}
$$

and therefore, we can set

$$u\left(x_1, x_2, x_3\right) = -\frac{\partial V}{\partial x_2} + \frac{\partial \phi}{\partial x_1}\left(x_1^2 - x_1^3 + x_2\right) + \frac{\partial \phi}{\partial x_2}\left(\phi + z_3\right) - z_3$$

resulting

$$\dot{V}_c = -x_1^2 - x_1^4 - \left(x_2 + x_1 + x_1^2\right)^2 - z_3^2 \leq -\{\underbrace{x_1^2 + \left(x_2 + x_1 + x_1^2\right)^2 + z_3^2}_{\text{positive definite function}}\}.$$

Thus, the controller $u\left(x_1, x_2, x_3\right)$ above makes the origin G.A.S., and the associated Lyapunov function is

$$\begin{aligned} V_c\left(x_1, x_2, x_3\right) &= V\left(x_1, x_2\right) + \frac{1}{2}\left(x_3 - \phi\left(x_1, x_2\right)\right)^2 \\ &= \frac{1}{2}x_1^2 + \frac{1}{2}\left(x_2 + x_1 + x_1^2\right)^2 + \frac{1}{2}\left[x_3 + x_1 + \left(1 + 2x_1\right)\left(x_1^2 - x_1^3 + x_2\right) + \left(x_2 + x_1 + x_1^2\right)\right]^2. \end{aligned}$$