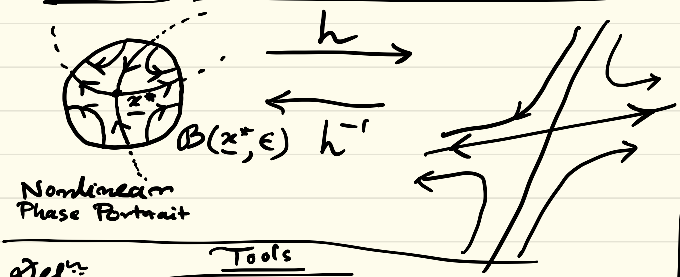
Lecture #4

Planar Nonlinear (Autonomous) Systems

Recap: Hartman – Grobman Thm.



Limit Cycle

Sustained Oscillation with fixed time period T

$$\tilde{x}(t+T) = \tilde{x}(t)$$

$\forall t > 0$

for some fixed $T > 0$

Tools

Defn: Invariant Set:

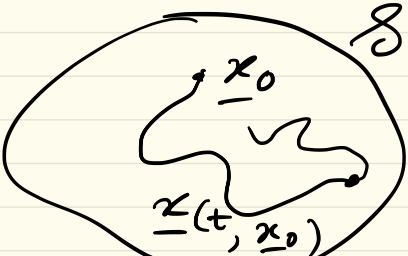
Let $\underline{x}(t, x_0)$ denote a trajectory for $\dot{\underline{x}} = \underline{f}(\underline{x})$, $\underline{x}(0) = \underline{x}_0$. A set $S \subset \mathbb{R}^n$ is positively (resp. negatively) invariant in time if

never leave

$$\forall \underline{x}_0 \in S, \ \underline{x}(t, \underline{x}_0) \in S \ \forall t \geq 0$$

$\underbrace{(\text{resp. } \forall t \leq 0)}$

eventually leave



$\mathbb{R}^n$

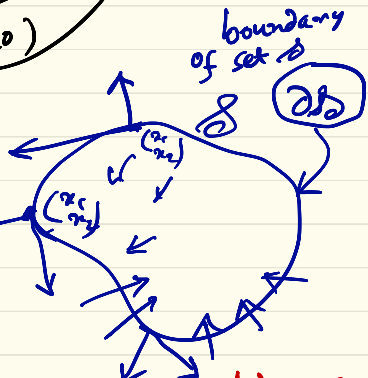
$$\dot{\underline{x}} = \underline{f}(\underline{x})$$

If

$$\boxed{\underline{f}(\underline{x}) \cdot \underline{n}(\underline{x}) \leq 0}$$

$\forall x \in \partial S$

then (+ve) invariance in time



Example of "(+ve) Invariant in Time" set: ① Trivial fixed pt. $\{\underline{x}^*\}$ positively invariant

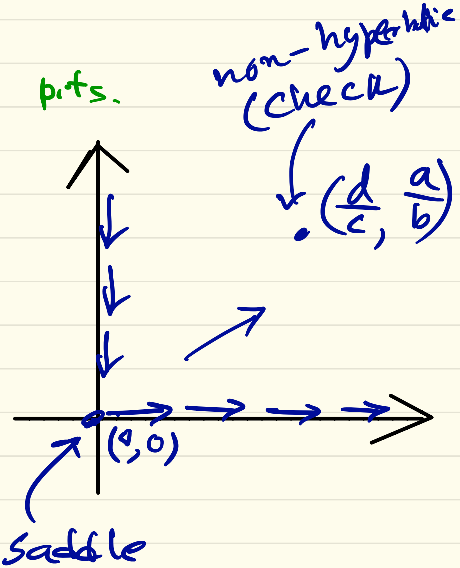
② $$\dot{x}_1 = (a - b x_2) x_1, \quad \dot{x}_2 = (c x_1 - d) x_2 \quad \Big\} \; a, b, c, d > 0$$

check: 2 isolated fixed pts.

$$\underline{x}_1^* = \begin{pmatrix} x_{11}^* \\ x_{12}^* \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\underline{x}_2^* = \begin{pmatrix} x_{21}^* \\ x_{22}^* \end{pmatrix} = \begin{pmatrix} d/c \\ a/b \end{pmatrix}$$

$\mathcal{S} := \mathbb{R}^2_+$ is positively invariant } check exercise



Existence of Limit Cycle

Thm: (Poincaré Bendixson Criterion)

$\dot{\underline{x}} = \underline{f}(\underline{x})$, $\underline{x} \in \mathbb{R}^2$, Construct $\mathcal{S}$

Suppose ① $\mathcal{S}$ is compact (closed & bdd.)

② +ve-ly invariant in time for $\dot{\underline{x}} = \underline{f}(\underline{x})$

(contd.)

③ $\mathcal{S}$ contains either No fixed pt. or ONE fixed pt. that is UNSTABLE

$\left(\max \mathrm{Re}\left(\mathrm{eig}\left(J\big|_{\underline{x}^*}\right)\right)\right) > 0$

(focus OR node)

If ①, ②, ③ satisfied, then $\mathcal{S}$ contains a limit cycle

Existence Thm., DOES NOT say uniqueness of limit cycle (could be multiple limit cycles)

How to APPLY this Thm. in practice:

① Strategy #1: Define a closed & simple curve

$$\phi(\underline{x}) = c$$

some constant

consider the set

$$\mathcal{S} := \{\underline{x} \in \mathbb{R}^2 : \phi(\underline{x}) \leq c\}$$

$$\partial \mathcal{S} = \{\underline{x} \in \mathbb{R}^2 : \phi(\underline{x}) = c\}$$

Vector field pointing inward @ $\underline{x}$ if $\langle \underline{f}(\underline{x}), \nabla\phi(\underline{x})\rangle < 0$

outward @ $\underline{x}$ if > 0

tangent @ $\underline{x}$ if $= 0$

Idea (strategy #1)

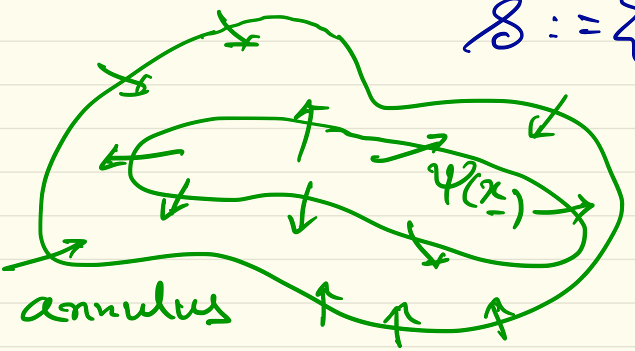
Take set

$$\mathcal{S} := \{ x \in \mathbb{R}^2 : \phi(\underline{x}) \leq c \}$$

Show: $\langle \underline{f}(\underline{x}), \nabla\phi(\underline{x})\rangle \leq 0$ on $\underline{\phi(\underline{x}) = c}$

Idea (Strategy #2)

$$\mathcal{S} := \{ \phi(\underline{x}) \leq c \ \& \ \psi(\underline{x}) \geq d \}$$



$\underline{\phi}(\underline{x})$

Trajectories are trapped inside

if $\langle \underline{f}(\underline{x}), \nabla\phi(\underline{x})\rangle \leq 0$ on $\phi(\underline{x}) = c$

$\langle \underline{f}(\underline{x}), \nabla\psi(\underline{x})\rangle \geq 0$ on $\psi(\underline{x}) = d$

Example : $\dot{x}_1 = x_1 + x_2 - x_1(x_1^2 + x_2^2)$

$\dot{x}_2 = -2x_1 + x_2 - x_2(x_1^2 + x_2^2)$

→ Origin is UNIQUE fixed pt.

→ $$J\Big|_{(0,0)} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix}\Bigg|_{(0,0)}$$

$$= \begin{bmatrix} 1 - 3x_1^2 - x_2^2 & 1 - 2x_1x_2 \\ -2 - 2x_1x_2 & 1 - x_1^2 - 3x_2^2 \end{bmatrix}\Bigg|_{(0,0)}$$

$$= \begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix}$$

$$\text{eig}\left(J\Big|_{(0,0)}\right) = \underbrace{1 \pm j\sqrt{2}}_{}$$

∴ (0,0) is hyperbolic fixed pt.

∴ Linearized analysis ⇒ UNSTABLE FOCUS

Take $\mathcal{S} := \{\phi(\underline{x}) \le c\}$ where

$\phi(x_1, x_2) := x_1^2 + x_2^2$ & $c \in \mathbb{R}_+$

On the bndy. ∂S ($\Leftrightarrow \phi(\underline{x}) = c$),
we have

$$\frac{\partial \phi}{\partial x_1} f_1 + \frac{\partial \phi}{\partial x_2} f_2 \Longleftrightarrow \langle \underline{f}, \nabla \phi \rangle$$

$$= 2(x_1^2 + x_2^2) - 2(x_1^2 + x_2^2)^2 - \underline{2x_1 x_2}$$

$$\le \underbrace{2(x_1^2 + x_2^2) - 2(x_1^2 + x_2^2)^2 + \underline{(x_1^2 + x_2^2)}}$$

$$= 2c - 2c^2 + c$$

$$= \underline{3c - 2c^2 < 0}$$

where we used $|2x_1 x_2| \le x_1^2 + x_2^2$

If $c \ge \frac{3}{2}$, then

all traj. are trapped

inside $\mathcal{S} := \{\underline{x} \in \mathbb{R}^2 : x_1^2 + x_2^2 \le c\}$

By Poincaré-Bendixson Criterion,
$\exists$ a limit cycle in S.

Converse Thm. (Ruling out / Proving non-existence of limit cycle)

Recall: A set is simply connected if $\nexists$ any "hole" inside it

$\Updownarrow$

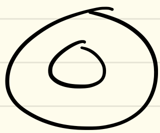
If the set can be smoothly contracted to a point

e.g. ① Interior of circle is simply connected

② The annulus

$$0 < c_1 \le x_1^2 + x_2^2 \le c_2$$

is NOT simply connected



Statement of Bendixson – Dulac
(1901) (1933)

Standing assumption: $\underline{f}$ is C^1 function of $\underline{x}$ (check).

Thm. statement:

Let $\mathcal{D} \subseteq \mathbb{R}^2$ be simply connected, such that

$$\nabla \cdot \underline{f} := \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} = \mathrm{tr}(\text{Jacobian})$$

satisfies

① is $\neq 0$ in any subset of $\mathcal{D}$

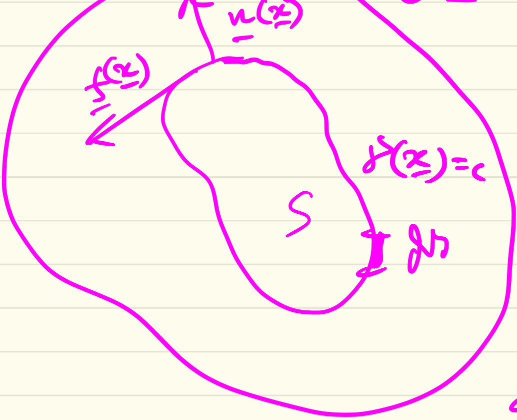
② does NOT change sign in $\mathcal{D}$

Then. $\nexists$ any limit cycle in $\mathcal{D}$.

Proof: (By contradiction)

Suppose, if possible, $\gamma(x_1, x_2) = c$ is a limit cycle in $\mathcal{D}$.

$\mathcal{D} \subseteq \mathbb{R}^2$



$\underline{f}(\underline{x})$ is tangent to curve $g(\underline{x}) = c$

If $\underline{n}(\underline{x})$ is the normal vector at $\underline{x}$, then

$$\langle \underline{f}(\underline{x}), \underline{n}(\underline{x}) \rangle = 0$$

But Gauss Divergence Thm.

$$\oint_\gamma \underbrace{\langle \underline{f}(\underline{x}), \underline{n}(\underline{x}) \rangle}_{=0} ds$$

$$= \iint_S \underbrace{\nabla \cdot f}_{\neq 0} \, dx_1 \, dx_2$$

$\Rightarrow$ Contradiction!

Example: (Bendixson - Dulac criterion)

$$\dot{x}_1 = f_1(x_1, x_2) = x_2$$
$$\dot{x}_2 = f_2(x_1, x_2) = a x_1 + b x_2 - x_1^2 x_2 - x_1^3$$

Let $\mathcal{D} \equiv \mathbb{R}^2$ (whole plane)

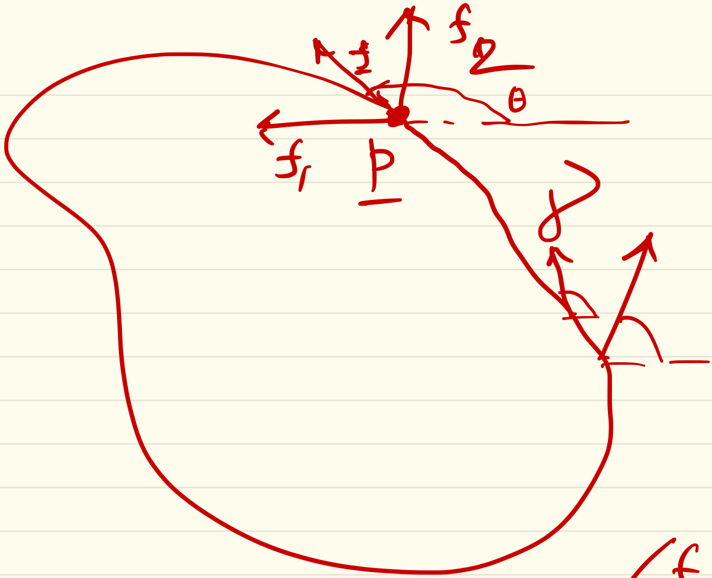
$$\nabla \cdot \underline{f} = \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} = b - x_1^2$$

If parameter $b < 0$, then NO limit cycle.

Index Theory

(Milnor, "Topology from a Differential Viewpoint") → Reference

Defn (Index) Let γ be a simple closed curve in $\mathbb{R}^2$ NOT passing through ANY fixed pt. $\underline{x}^*$ of $\underline{\dot{x}} = \underline{f}(\underline{x})$



$$\theta_{\underline{f}}(x_1, x_2) := \tan^{-1}\left(\frac{f_2(x_1,x_2)}{f_1(x_1,x_2)}\right)$$

Traverse γ in C.C. dirⁿ.

Upon returning to original posⁿ, $\underline{\text{it}}$ must have rotated by
vector field
amount $2\pi k$,

$k \in \mathbb{Z}$

(angle is measured +ve C.C.)

Then k is the index of γ w.r.t. dynamics $\underline{\dot{x}} = \underline{f}(\underline{x})$

(i.e.) $$\kappa_{\underline{f}}(\gamma) = \frac{1}{2\pi} \oint_{\gamma} d\theta_f(x_1, x_2)$$

where $\theta_f(x_1, x_2) := \tan^{-1}\left(\frac{f_2(x_1,x_2)}{f_1(x_1,x_2)}\right)$

More explicitly,

$$\kappa_{\underline{f}}(\gamma) = \frac{1}{2\pi} \oint_{\gamma} \frac{f_1\, df_2 - f_2\, df_1}{f_1^2 + f_2^2}$$

If γ is chosen to encircle a SINGLE ISOLATED fixed pt $\underline{x}^*$, then we say κ is the INDEX of $\underline{x}^*$.

$$\kappa_{\underline{f}}(\underline{x}^*)$$

Lemma (Exercise)

1) Index of {node, focus, center} $= +1$

" " hyperbolic saddle $= -1$

" " limit cycle $= +1$

" " closed curve NOT encircling $\underline{x}^*$ $= 0$

If $\dot{\underline{x}} = \underline{f}(\underline{x})$ has m ISOLATED fixed pt.s. then

$$K_f = \sum_{j=1}^{m} K_f(\underline{x}_j^*)$$

enclosed by γ