Lecture #17

Small comment about last example:

Feedback linearization $\not\Rightarrow$ I/O linearization since the output map, which was originally linear, may become nonlinear.

In the prev. example: suppose control objective is to design tracking controller for $y = x_2 = h(x)$

Full eqn:

$$\dot{z}_1 = z_2$$
$$\dot{z}_2 = v$$

output eqn becomes nonlinear!!

$$\boxed{y = \sin^{-1}\left(\frac{z_2}{a}\right)}$$

Question:

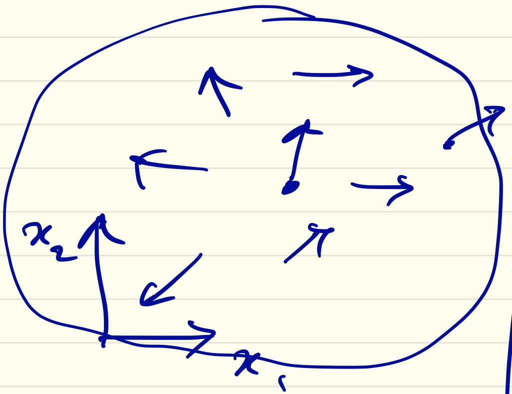
What control systems can be feedback linearized / Input-output linearized

Need: Structural conditions

[HOLD ON]

Geometric Control Theory

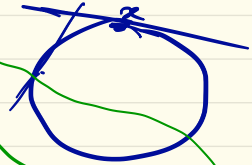
Differential Geometry Ideas in Control



Manifold ($\mathcal{M}$)

Tangent Space ($T_x \mathcal{M}$)

circle or sphere

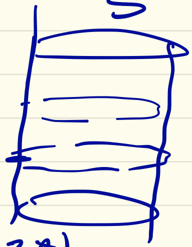


Pendulum :

$$x = \underbrace{[0, 2\pi)}_{S^1} \times \mathbb{R}$$

$$\times \mathbb{R}$$

cylinder



$$\dot{\theta}_1 = f_1(\theta_1, \theta_2)$$

$$\dot{\theta}_2 = f_2(\theta_1, \theta_2)$$

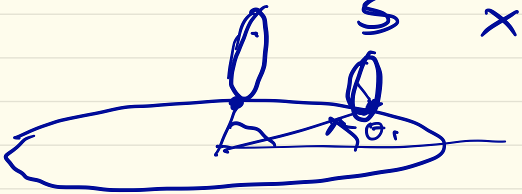
$$\dot{x} = f(\underline{x})$$

$$\underline{x} \in \mathcal{M}$$

$$\underline{\dot{x}} \in T_x \mathcal{M}$$

$$x = [0, 2\pi) \times [0, 2\pi)$$

$$S^1 \times S^1 = \mathbb{T}$$



Torus

Lie Derivative

vector field ($\underline{f}(\underline{x})$) : $\quad f : \mathcal{M} \longmapsto T\mathcal{M}$

Scalar field ($\lambda(\underline{x})$) : $\quad \lambda : \mathcal{M} \longmapsto \mathbb{R}$

Lie Derivative of a scalar valued function $\lambda(\underline{x})$ w.r.t. vector field

$$\underbrace{L_{\underline{f}} \lambda(x)}_{\text{Rate of change of } \lambda(\underline{x}) \text{ in the dir}^{n} \ \underline{f}} = \langle \nabla\lambda, \underline{f} \rangle = \sum_{i=1}^{n} \frac{\partial \lambda}{\partial x_i} f_i(\underline{x})$$

$$= \sum_{i=1}^{n} f_i \frac{\partial \lambda}{\partial x_i}$$

$$= \underbrace{\sum_{i=1}^{n} f_i \frac{\partial}{\partial x_i} \lambda}_{f^{n} \text{ of } \underline{x} \text{ (new scalar field)}}$$

we have already seen something like this in Lyapunov theorems:

$$\dot{V} = \langle \nabla V, \underline{f} \rangle = L_{\underline{f}} V(\underline{x})$$

- Higher order Lie derivative w.r.t. different vector fields. (possibly)

$$L_{\underline{g}} L_{\underline{f}} \lambda(\underline{x}) = \langle \nabla L_f \lambda, g \rangle$$
$$= \left\langle \frac{\partial}{\partial \underline{x}} (L_f \lambda), \underline{g} \right\rangle$$

$\underline{f}$ & $\underline{g}$ could be different vector fields

- Can also iterate Lie derivatives w.r.t. same vector field.

$$L_{\underline{f}}^{k} \lambda(\underline{x}) = \left\langle \frac{\partial}{\partial \underline{x}} \left(L_f^{k-1} \lambda \right), f \right\rangle$$
$$= \langle \nabla L_f^{k-1} \lambda, f \rangle$$

Convention

$$L_{\underline{f}}^{0} \lambda(\underline{x}) = \lambda(\underline{x})$$

(zeroth order derivative = original scalar field) vector field $\xrightarrow[\text{operating on}]{}$ Scalar field $\xrightarrow[\text{generates}]{}$ New scalar field.

vector field $\xrightarrow[\text{operating on}]{}$ another vector field $\xrightarrow[\text{generates}]{}$ New vector field.

Think

Two vector fields $\underline{f}$ & $\underline{g}$ both on same state space $\underline{x} \subseteq \mathbb{R}^n$

Want: New vector field:

Operation: $\boxed{\text{Lie Bracket}}$ operation

$$[\underline{f}, \underline{g}](\underline{x}) := \underbrace{\frac{\partial \underline{g}}{\partial \underline{x}}}_{\text{Jacobian}} \underline{f}(\underline{x}) - \underbrace{\frac{\partial \underline{f}}{\partial \underline{x}}}_{\text{Jacobian}} g(x)$$

Again, can do repeated operation (bracketing of $\underline{g}$) with same vector field $\underline{f}$:

$$[\underline{f}, [\underline{f}, [\underline{f}, [\underline{f}, \underline{g}]]] \ldots] \leftarrow \text{Confusing notation}$$

Instead, we write the notation :

$$\mathrm{ad}_{\underline{f}}^{k}\,\underline{g}\,(\underline{x}) := [\underline{f}, \mathrm{ad}_{\underline{f}}^{k-1}\underline{g}]\,(\underline{x})$$

$$\forall\, k \geq 1$$

Again

$$\mathrm{ad}_{\underline{f}}^{0}\,\underline{g}(\underline{x}) = \underline{g}(\underline{x}).$$

Example LTI system $\dot{x} = Ax$, $\dot{x} = Bx$

$$\underline{f}(\underline{x}) \equiv A\,\underline{x}, \quad \underline{g}(\underline{x}) = B\,\underline{x}$$

$$[\underline{f}, \underline{g}](\underline{x}) = \mathrm{ad}_{\underline{f}}\,\underline{g}(\underline{x})$$

$$= \frac{\partial \underline{g}}{\partial \underline{x}}\underline{f}(\underline{x}) - \frac{\partial \underline{f}}{\partial \underline{x}}\underline{g}(\underline{x})$$

$$= BA\underline{x} - AB\underline{x}$$

$$= \underbrace{(BA - AB)}\,\underline{x}$$

This matrix is called commutator matrix

Properties of Lie Bracket: Given vector fields $\underline{f}, \underline{g}, \underline{h}$ on some state space $\mathcal{X}$.

Lie Bracket is

① $\boxed{\text{Bilinear over } \mathbb{R}}$ (i.e.) for $\underbrace{\underline{f}_1, \underline{f}_2, \underline{g}_1, \underline{g}_2}_{\text{vector fields}}$, $r_1, r_2 \in \mathbb{R}$

we have

$$[r_1 \underline{f}_1 + r_2 \underline{f}_2, \underline{g}_1] = r_1 [\underline{f}_1, \underline{g}_1] + r_2 [\underline{f}_2, \underline{g}_1]$$

$$\& \; [\underline{f}_1, r_1 \underline{g}_1 + r_2 \underline{g}_2] = r_1 [\underline{f}_1, \underline{g}_1] + r_2 [\underline{f}_1, \underline{g}_2]$$

More generally $\forall$ scalar fields $\lambda(\underline{x})$ & $\mu(\underline{x})$

$$[\lambda(\underline{x}) \underline{f}(\underline{x}), \mu(\underline{x}) \underline{g}(\underline{x})]$$

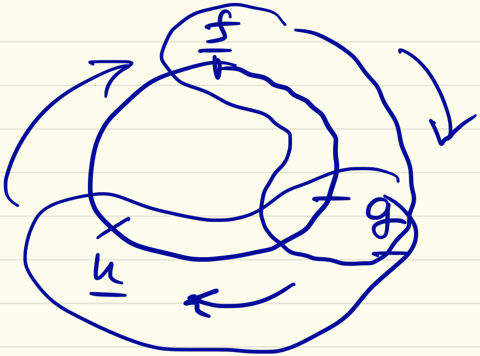
$$= \lambda(\underline{x}) \mu(\underline{x}) [\underline{f}, \underline{g}] + \lambda \underbrace{(L_f \mu)}_{\substack{\text{new}\\\text{scalar}\\\text{field}}} g - \mu \underbrace{(L_g \lambda)}_{\substack{\text{new}\\\text{scalar}\\\text{field}}} f$$

② Lie Bracket is Skew symmetric:

$$[\underline{f}, \underline{g}] = -[\underline{g}, \underline{f}]$$

③ Jacobi Identity:

$$[[\underline{f}, \underline{g}], \underline{h}] + [[\underline{g}, \underline{h}], \underline{f}] + [[\underline{h}, \underline{f}], \underline{g}] = 0$$



cyclic order

● Space of vector fields is a vector space over the reals

(why: sum of vector fields $\underline{f}_1(\underline{x}) + \underline{f}_2(\underline{x})$ is also vec. field

$\lambda(\underline{x})\,\underline{f}(\underline{x})$ is also vec. field

Given a set of vector fields:

$$\underline{f}_1(\underline{x}), \underline{f}_2(\underline{x}), \ldots, \underline{f}_m(\underline{x}) \quad \text{where } \underline{x} \in \mathcal{X} \subseteq \mathbb{R}^n$$

we define

$$\underbrace{\Delta}_{\substack{\text{we call this} \\ \text{"distribution"}}} := \text{span}\{\underline{f}_1, \underline{f}_2, \ldots, \underline{f}_m\}$$

We say $\Delta(\underline{x})$ is the value of Δ at $\underline{x}$

Elements of Δ at pt. $\underline{x}$, are of the form:

$$\lambda_1(\underline{x})\underline{f}_1(\underline{x}) + \lambda_2(\underline{x})\underline{f}_2(\underline{x}) + \ldots + \lambda_m(\underline{x})\underline{f}_m(\underline{x})$$

where $\lambda_i(\underline{x})$ are all smooth functions of $\underline{x}$.

→ A distribution Δ = smooth assignment of a subspace of $\mathbb{R}^n$ to each pt. $\underline{x}$

→ At each FIXED $\underline{x}$, $\Delta(\underline{x})$ is a subspace of $\mathbb{R}^n$.

→ We can define "sum":

$$(\Delta_1 + \Delta_2)(\underline{x}) = \Delta_1(\underline{x}) + \Delta_2(\underline{x})$$ (Pointwise sum, preserves smoothness)

Also, intersection

$$(\Delta_1 \cap \Delta_2)(\underline{x}) = \underbrace{\Delta_1(\underline{x}) \cap \Delta_2(\underline{x})}$$

(may not preserve smoothness)

→ $\Delta_1 \subset \Delta_2$ if $\Delta_1(\underline{x}) \subset \Delta_2(\underline{x}) \ \forall \ \underline{x} \in \mathcal{X}$

→ $\underline{f} \in \Delta$ if $\underline{f}(\underline{x}) \in \Delta(\underline{x}) \ \forall \ \underline{x} \in \mathcal{X}$

→ $\dim(\Delta)(\underline{x})$ = dimension of subspace $\Delta(\underline{x})$

→ Suppose I make the matrix

$$\underbrace{\underbrace{F}_{\in \mathbb{R}^{n\times m}}}_{\text{stacking of vector fields}} = \left[\underbrace{\underline{f}_1}_{n\times 1} \;\middle|\; \underbrace{\underline{f}_2}_{n\times 1} \;\middle|\; \cdots \;\middle|\; \underbrace{\underline{f}_m}_{n\times 1} \right]$$

$F(\underline{x})$ ← matrix whose columns are vector fields

$$\Delta(\underline{x}) = \text{Image}(F(\underline{x}))$$

If a distribution Δ is spanned by the columns of F, then

$$\dim(\Delta)(\underline{x}) = \underbrace{\text{rank}(F(\underline{x}))}_{\text{may be } f^{n} \text{ of } \underline{x}}$$

Example : $\mathcal{X} = \mathbb{R}^3$

$m = 3$ vector fields in $\mathbb{R}^3$

$$\underline{f}_1(x_1, x_2, x_3) = \begin{pmatrix} x_1 \\ 1 + x_3 \\ 1 \end{pmatrix}$$

$$\underline{f}_2(x_1, x_2, x_3) = \begin{pmatrix} x_1 x_2 \\ (1 + x_3) x_2 \\ x_2 \end{pmatrix}$$

$$\underline{f}_3(x_1, x_2, x_3) = \begin{pmatrix} x_1 \\ x_1 \\ 0 \end{pmatrix}$$

$$F(\underline{x}) = \left[\underline{f}_1 \;\middle|\; \underline{f}_2 \;\middle|\; \underline{f}_3 \right] = \begin{bmatrix} x_1 & x_1 x_2 & x_1 \\ 1 + x_3 & (1 + x_3) x_2 & x_1 \\ 1 & x_2 & 0 \end{bmatrix}$$

Second col. $=$ x_2 (first col.)

$\text{rank}(F(\underline{x})) \leq 2$

If $x_1 \neq 0$, then 1^{st} & 3^{rd} col. are indep.

$\Rightarrow \text{rank}(F(\underline{x})) = 2 \quad \forall\, \underline{x} \in \mathbb{R}^3 : x_1 \neq 0$

$\therefore$ Columns of F span the distribution Δ given by:

$$\Delta(x) = \begin{cases} \text{span}\left(\begin{bmatrix} 0 \\ 1+x_3 \\ 1 \end{bmatrix}\right) & \text{if } x_1 = 0 \\ \text{span}\left(\begin{bmatrix} x_1 \\ 1+x_3 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}\right) & \text{if } x_1 \neq 0 \end{cases}$$

$\therefore \dim(\Delta)(\underline{x}) = 2 \quad \forall\, \underline{x} \in \mathbb{R}^3 \setminus \underbrace{\{x_1 = 0\}}_{\text{plane}}$.

End of example.

Involutive Distribution :

$\Delta = \underset{\text{span}}{\{} \underline{f}_1, \ldots, \underline{f}_m \}$

Δ is called "involutive" if

$\forall \ \underline{f}_i, \underline{f}_j \in \Delta, \quad i, j = 1, \ldots, m, \quad \underline{x} \in \mathbb{R}^n$

we have $[\underline{f}_i, \underline{f}_j] \in \Delta$

Example: $\mathcal{X} = \mathbb{R}^3, \quad m = 2,$

$\Delta = \text{span}\{\underline{f}_1, \underline{f}_2\}$

$$\underline{f}_1(\underline{x}) = \begin{pmatrix} 2x_2 \\ 1 \\ 0 \end{pmatrix}, \quad \underline{f}_2(\underline{x}) = \begin{pmatrix} 1 \\ 0 \\ x_2 \end{pmatrix}.$$

$\dim(\Delta)(\underline{x}) = 2 \ \forall \ \underline{x} \in \mathbb{R}^3$

But $[\underline{f}_1, \underline{f}_2](\underline{x}) = \frac{\partial \underline{f}_2}{\partial \underline{x}} \underline{f}_1 - \frac{\partial f_1}{\partial x} f_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$$F(\underline{x}) = [\underline{f}_1 \mid \underline{f}_2 \mid [\underline{f}_1, \underline{f}_2]] = \begin{bmatrix} 2x_2 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & x_2 & 1 \end{bmatrix}$$

$\Rightarrow \text{rank}(F(\underline{x})) = 3 \Rightarrow$ NOT involutive.

Relative Degree: (SISO)

$\underline{x} \in \mathbb{R}^n$
$u, y \in \mathbb{R}$

$$\dot{\underline{x}} = f(\underline{x}) + \underline{g}(\underline{x})u$$
$$y = h(\underline{x})$$

This system is said to have relative degree r @ $\underline{x} = \underline{x}^\circ$ if

(i) $L_{\underline{g}} L_{\underline{f}}^k h(\underline{x}) = 0 \ \forall \ \underline{x}$ in a nhbd. of x° & all $k < r-1$

(ii) $L_g L_f^{r-1} h(\underline{x}^\circ) \neq 0$

Example: (LTI system)

$$\left.\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned}\right\} \text{SISO}$$

$$\left.\begin{aligned} L_f^k h(x) &= CA^k \underline{x} \\ L_g L_f^k h(x) &= CA^k B \end{aligned}\right\}$$

we need $r \in \mathbb{N}$ s.t.

$CA^k B = 0 \ \forall \ k < r-1$

$CA^{r-1} B \neq 0$

$$r = \left| \deg(\text{Numerator polynomial of } G(s)) - \deg(\text{Denominator polynomial of } G(s)) \right|$$

where

$$G(s) = C\,(sI - A)^{-1}B.$$