Lec. 5 (04/14/2020)

Last example contd.

By direct differentiation, we verify that solution of our LTV system with $A(t) = e^{\Omega t} B e^{-\Omega t}$ is :

$$\underline{x}(t) = \underbrace{e^{\Omega t} \, e^{(-\Omega + B)t}}_{\Phi(t,0)} \, \underline{x}_0 , \quad t \geq 0 .$$

Let us take : $B = \begin{pmatrix} -1 & -4 \\ 0 & -1 \end{pmatrix}$

$\Rightarrow \text{eig}(A(t)) = -1, -1 \ \forall \ t \geq 0.$

$\therefore A(t)$ is Hurwitz for all $t \geq 0$.

However, $-\Omega + B = \begin{pmatrix} -1 & -3 \\ -1 & -1 \end{pmatrix}$

$\Rightarrow \text{eig}(-\Omega + B) = -1 \pm \sqrt{3}$.

$\Rightarrow$ $\therefore$ we can find $\underline{x}_0$ s.t. $\|\underline{x}(t)\|_2 \to \infty$ as $t \to \infty$, meaning origin of LTV is NOT A.S.

Next, set $B = \begin{pmatrix} -1 & 4 \\ 1 & -1 \end{pmatrix}$

$$\Rightarrow \quad \text{eig}\left(\underline{\underline{A(t)}}\right) = \text{eig}\left(\underline{\underline{B}}\right) = +1, -3, \quad \forall t \geq 0.$$

$A(t)$ NOT Hurwitz for any $t \geq 0$.

However, the matrix $-\Omega + B = \begin{pmatrix} -1 & 5 \\ 0 & -1 \end{pmatrix}$ -2

has both eig. values $-1, -1$.

Since $e^{\Omega t}$ is bounded, we conclude origin is GAUES

$\exp(A)\exp(B) \overset{?}{=} \exp(A+B)$ if and only if A and B commutes

$\Updownarrow$

$AB = BA$

LTI System Stability :

HW #1, p3 : part (c) : sufficiency of $V(x) = x^T P x$

(quadratic Lyapunov function)

can be solved numerically in software like cvx

$\Longleftrightarrow$

$$\boxed{\begin{aligned} &\text{find } P \\ &\text{s.t. } P \succ 0, \ \mathcal{L}(P) = A^T P + PA \prec 0 \end{aligned}}$$

LMI (feasibility problem)

(Linear Matrix inequality)

Converse Lyapunov Theorem

part (d) : Necessity of quadratic Lyap. function for LTI

Similarly, discrete time LTI.

LTV case: ($\dot{\underline{x}} = A(t)\,\underline{x}$, $\underline{x}(t_0) = \underline{x}_0$)

→ S iff $\underbrace{\sup_{t \geq t_0} \|\Phi(t,t_0)\|_2}_{=:\, c(t_0)} < \infty$

→ AS iff $\lim_{t\to\infty} \|\Phi(t,t_0)\|_2 = 0$

→ UAS iff $\sup_{t_0 \geq 0} c(t_0) = \underbrace{\sup_{t_0 \geq 0} \sup_{t \geq t_0} \|\Phi(t,t_0)\|}_{=:\, c} < \infty$

→ GUAS ⟺ ES iff $\exists\, \alpha, \beta > 0$ s.t.

$$\|\Phi(t,t_0)\| \leq \alpha \exp(-\beta(t - t_0)) \quad \forall\, t \geq t_0 \geq 0$$

LTV case: Lyapunov Thm (Sufficiency):

Consider $\dot{\underline{x}} = A(t)\underline{x}$, $\underline{x}(t_0) = \underline{x}_0$,
where $A(t)$ is continuous and bounded function of t as $t \geq t_0 \geq 0$.

In this case, GUAS $\Leftrightarrow$ ES.

If there exists continuously differentiable, bounded, $P(t) > 0$, ($\Leftrightarrow$ $0 < c_1 I \leq P(t) \leq c_2 I$ $\forall\ t \geq t_0 \geq 0$)

that solves the Lyapunov matrix differential equation

$$-\dot{P}(t) = (A(t))^T P(t) + P(t)A(t) + Q(t)$$

for any $Q(t) > 0$ continuous in t (next page)

$\left(\Leftrightarrow \quad 0 < c_3 I \leq Q(t), \quad \forall\, t \geq t_0 \geq 0\right).$

then $\underline{x}^* = \underline{0}$ is <u>GES</u> (and hence <u>GUAES</u>).

<u>Proof</u>: Choose $V(t, \underline{x}) = \boxed{\underline{x}^T P(t)\, \underline{x}}$

Clearly,
- V is pos. def. f^n.
- $V(0) = 0$.
- V is radially unbounded.

Also, under the stated conditions on matrix $P(t) > 0$, we have:

$$c_1 \|\underline{x}\|^2 \leq V(t, \underline{x}) \leq c_2 \|\underline{x}\|^2 \quad \forall\, t \geq t_0 \geq 0.$$

Also,

$$\dot{V} = \dot{\underline{x}}^T P(t)\, \underline{x} + \underline{x}^T \dot{P}(t)\, \underline{x} + \underline{x}^T P(t)\, \dot{\underline{x}}$$

$$= \underline{x}^T \left(\dot{P}(t) + A(t)^T P(t) + P(t) A(t) \right) \underline{x}$$

$$= - \underline{x}^T Q(t) \underline{x}$$

$$\leq - c_3 \|\underline{x}\|_2^2 \quad \forall\ t \geq t_0 \geq 0$$

Since the above holds for all $\underline{x} \in \mathbb{R}^n$,

$\therefore$ by ES theorem for non-autonomous theorem, the origin $\underline{x}^* = \underline{0}$ is GES ($\Leftrightarrow$ GUAES).

$\therefore$ $V(t, \underline{x}) = \underline{x}^T P(t) \underline{x}$ serves as Lyapunov certificate

in fact necessary.

Necessity / Converse Lyapunov Theorem for LTV:

Theorem: Let $\underline{x}^* = \underline{0}$ be ES fixed point for $\dot{\underline{x}} = A(t)\underline{x}$. Suppose $A(t)$ is continuous, bounded. Let $Q(t) > 0$, continuous and bounded. Then there exists C^1, bounded matrix function $P(t)$ that satisfies:

$$-\dot{P}(t) = (A(t))^T P(t) + P(t) A(t) + Q(t),$$

and $V(t, \underline{x}) = \underline{x}^T P(t) \underline{x}$ serves as Lyapunov certificate

($\Leftrightarrow$ $V(t, \underline{x})$ satisfies all the required conditions of Lyapunov function)

So far, all the (sufficient) Lyapunov Theorems we mentioned, are "Lyapunov's Direct Method".

Lyapunov's Indirect Method:

Idea: Suppose $\underline{x}^* = \underline{0}$ is fixed point of $\dot{\underline{x}} = f(t, \underline{x})$.

Get Jacobian: $A(t) = \frac{\partial f}{\partial \underline{x}}(t, \underline{x}) \Big|_{\underline{x}^* = \underline{0}}$

If $\underline{x}^* = \underline{0}$ is ES for $\boxed{\dot{\underline{x}} = A(t)\,\underline{x}}$, then $\underline{x}^* = \underline{0}$ is ES for $\dot{\underline{x}} = f(t, \underline{x})$.

We're not going to cover indirect Lyapunov method.

Application of Lyapunov-like concepts outside stability problems:

Overview: 2 Applications:

- → Barrier certificates → collision avoidance
  → Model invalidation
- → Estimating ROA (region-of-attraction)

Next thing: Computing Lyapunov functions