Lecture # 9

Stability Cond^{n}. (Linear Systems)

LTI

S $\Leftrightarrow$ AS $\Leftrightarrow$ GAS $\Leftrightarrow$ ES

A is Hurwitz

$$\Updownarrow$$

$$\max_i \left(Re(\lambda_i(A))\right) < 0$$

$$\Updownarrow$$

Lyapunov Theory applied to LTI (HW3 p1)

$$\dot{\underline{x}} = A\underline{x}, \quad \underline{x}(t_0) = \underline{x}_0$$

$$\underline{x}(t) = \exp(At)\, \underline{x}_0$$

LTV

$$\boxed{\text{what cond}^{n}\text{. on } \Phi(t,t_0)\ ?}$$

$$\dot{\underline{x}} = A(t)\, \underline{x}(t),$$

$$\underline{x}(t_0) = \underline{x}_0$$

$$\underline{x}(t) = \Phi(t,t_0)\, \underline{x}_0$$

where

$$\frac{d}{dt}\Phi(t,t_0) = A(t)\, \Phi(t,t_0),$$

$$\Phi(t_0,t_0) = I$$

Peano - Baker Series

$$\Phi(t,t_0) = I + \int_{t_0}^{t} A(\tau_1)\, d\tau_1 + \int_{t_0}^{t} A(\tau_1) \int_{t_0}^{\tau_1} A(\tau_2)\, d\tau_2\, d\tau_1$$

$$+ \int_{t_0}^{t} A(\tau_1) \int_{t_0}^{\tau_1} A(\tau_2) \int_{t_0}^{\tau_2} A(\tau_3)\, d\tau_3\, d\tau_2\, d\tau_1 + \cdots$$

Spl. Cases of P.B. Series

① A is const. $\Leftrightarrow \Phi(t,t_0) = e^{A(t-t_0)}$

② $\Phi(t,t_0) = \exp\left(\int_{t_0}^{t} A(\tau)\, d\tau\right)$

iff the matrices $A(t)$ & $\int_{t_0}^{t} A(\tau) d\tau$ Commute

$$\Leftrightarrow \left[A(t), \int_{t_0}^{t} A(\tau)\, d\tau\right] = 0$$

$$\Leftrightarrow A(t)\left(\int_{t_0}^{t} A(\tau)\, d\tau\right) = \left(\int_{t_0}^{t} A(\tau)\, d\tau\right) A(t)$$

LTV is

→ S iff $\underbrace{\sup_{t \geq t_0} \|\Phi(t,t_0)\|}_{=: c(t_0)} < \infty$

→ AS iff $\lim_{t \to \infty} \|\Phi(t,t_0)\| = 0$

→ UAS iff $\sup_{t_0 \geq 0} c(t_0) = \sup_{t_0 \geq 0} \sup_{t \geq t_0} \|\Phi(t,t_0)\|$

$=: c < \infty$

→ GUAS ⟺ ES iff $\exists\, \alpha, \beta > 0$ s.t.

$$\|\Phi(t,t_0)\| \leq \alpha \exp(-\beta(t-t_0)) \quad \forall\, t \geq t_0 \geq 0$$

Omit: ① Converse Lyapunov Thm.

② Lyapunov's Indirect Method

Using Lyap. Theory to Estimate ROA (of $\underline{x}^* = \underline{0}$)

ROA is the largest (+ve)-ly invariant set containing $\underline{x}^* = \underline{0}$.

Step 1: Get $V(\underline{x})$ such that $V(\cdot)$ is positive definite

Step 2: Compute $\dot{V} := \frac{d}{dt} V$

& check where $\dot{V} < 0$

say $\forall\ \underline{x} \in \Omega$ this happens

(NO guarantee that Ω is (+ve)-ly invariant)

• Step 3: Construct:

$$\Omega_c := \{ \underline{x} \in \mathbb{R}^n \mid V(\underline{x}) \leq c \}$$

such that ① Ω_c is compact
(closed & bounded)

② $\Omega_c \subset \mathcal{D}$

Then $\boxed{\Omega_c \subseteq ROA}$

Example:

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -x_1 + \frac{1}{3}x_1^3 - x_2$$

$$V(\underline{x}) = \frac{1}{2}x^T \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1 \end{bmatrix} \underline{x} + \int_0^{x_1} (y - \frac{1}{3}y^3)\,dy$$

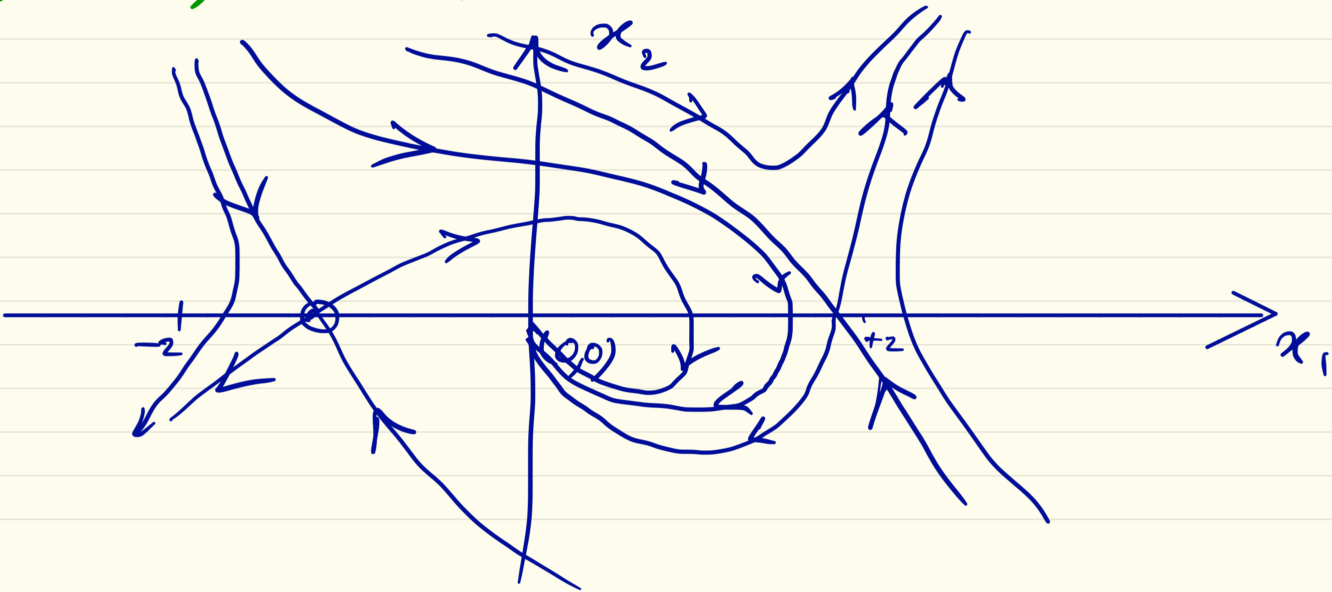
$$= \frac{3}{4}x_1^2 - \frac{1}{12}x_1^4 + \frac{1}{2}x_1 x_2 + \frac{1}{2}x_2^2$$

& $\dot{V} = -\frac{1}{2}x_1^2\left(1 - \frac{x_1^2}{3}\right) - \frac{1}{2}x_2^2$

If $\mathcal{D} := \{\underline{x} \in \mathbb{R}^2 : -\sqrt{3} < x_1 < +\sqrt{3}\}$

Then $V > 0$ & $\dot{V} < 0$ $\forall\, \underline{x} \in \mathcal{D} \setminus \{\underline{0}\}$

However, $\mathcal{D} \not\subset$ ROA



Example #2 : Consider $\dot{x}_1 = x_2$

$$\dot{x}_2 = -x_1 - x_2 - (2x_2+x_1)(1-x_2^2)$$

(a) Use $V(x_1, x_2) = 5x_1^2 + 2x_1 x_2 + 2x_2^2$

$$= (x_1 \quad x_2)\begin{bmatrix} 5 & 1 \\ 1 & 2 \end{bmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

to show $\underline{0}$ is A.S.

(b) Let

$$\mathcal{S} := \{ \underline{x} \in \mathbb{R}^2 \mid V(\underline{x}) \leq 5 \} \cap \{ \underline{x} \in \mathbb{R}^2 \mid |x_2| \leq 1 \}$$

Prove $\mathcal{S}$ is an estimate of ROA.

Sol^n. (a) $\dot{V} = \frac{\partial V}{\partial x_1} f_1 + \frac{\partial V}{\partial x_2} f_2$

$$= -2x_1^2 + 4x_1 x_2 - 2x_2^2 - 2(x_1 + 2x_2)^2 (1 - x_2^2)$$

For $|x_2| \le 1$,

we get

$$\dot{V}(\underline{x}) \le \underbrace{-2\left(x_1^2 - 2x_1 x_2 + x_2^2\right)}$$

$$= -2(x_1 - x_2)^2 \le 0$$

Consider $\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}$ where $\dot{V} = 0$

$$\Rightarrow x_1(t) - x_2(t) \equiv 0$$

$$\Rightarrow \dot{x}_1(t) - \dot{x}_2(t) \equiv 0$$

$$\Rightarrow x_2(t) + x_1 + x_2 + (x_1 + 2x_2)(1 - x_2^2)$$

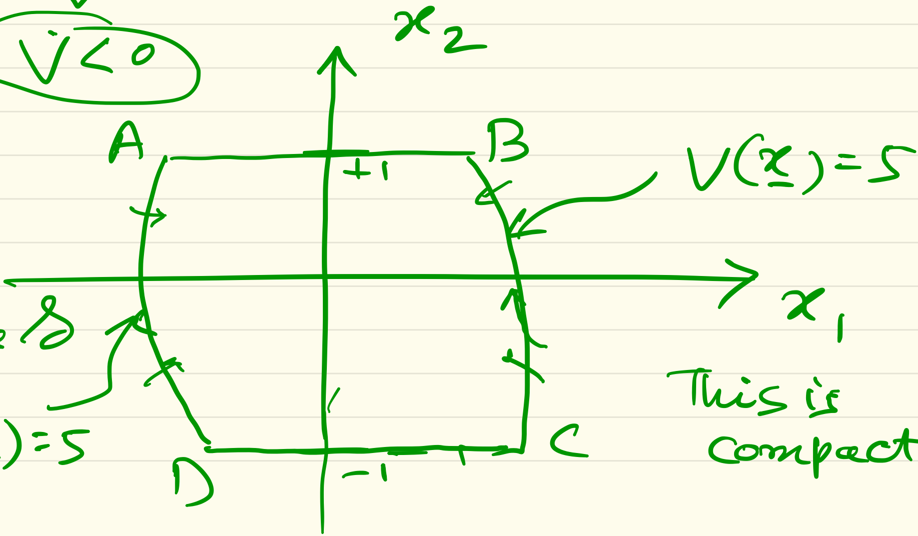
$$\Rightarrow \quad 3x_2(t)\left(2 - x_2^2(t)\right) \equiv 0$$

$$\Rightarrow \quad x_2(t) \equiv 0$$ by Lasalle Origin is A.S.

(b) We are given.

$$\mathcal{S} := \underbrace{\{|x_2| \le 1\}}_{\dot V < 0} \cap \{V(\underline{x}) \le 5\}$$

$\because$ BC & DA are part of the Lyap. surface $V(\underline{x}) = 5$, & $\dot V \le 0$, traj. cannot leave $\mathcal{S}$ through BC & DA



This is compact

To show: no trajectory can leave $\mathcal{S}$ through lines AB & CD :

$$\dot{V}\Big|_{|x_2|=1} = \langle \nabla V, f \rangle \Big|_{|x_2|=1} \leq 0$$

$\therefore$ No leakage through AB, BC, CD, or DA

$\therefore$ $\mathcal{S}$ is (+ve)-ly invariant

$$\therefore \mathcal{S} := \{ |x_2| \leq 1 \} \cap \{ V(\underline{x}) \leq 5 \}$$

satisfies

$$\mathcal{S} \subset ROA .$$