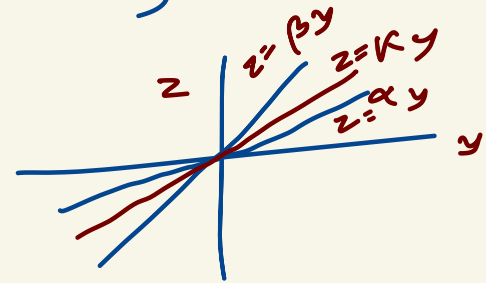
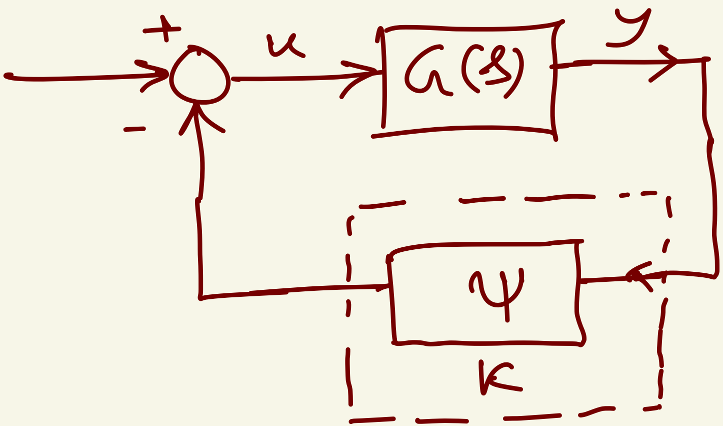
3 conjectures (all wrong in general):

• (1949) Aizerman's conjecture:

(True for $n = 1, 2$; False for $n \geq 3$)

$D = 0$, $m = 1$ (single I/O)





where $K \in [\alpha, \beta]$

If the closed-loop is G.A.S. for $\psi(y) = Ky \ \forall K \in [\alpha, \beta]$

then it is also GAS for all $\psi \in$ Sector$[\alpha, \beta]$.

• (1954) Kalman's Conjecture:

(True for $n = 1, 2, 3$. False for $n \geq 4$)

Again assume $D = 0$, $m = 1$ (Single I/O).

$\psi(t, y) \equiv \psi(y)$, $\psi(0) = 0$.

If the closed loop is G.A.S. for $\psi(y) = ky$
$\forall k \in [\alpha, \beta]$ then it is also GAS
$\forall \ \psi(y)$ s.t. $\alpha \leq \psi'(y) \leq \beta$.

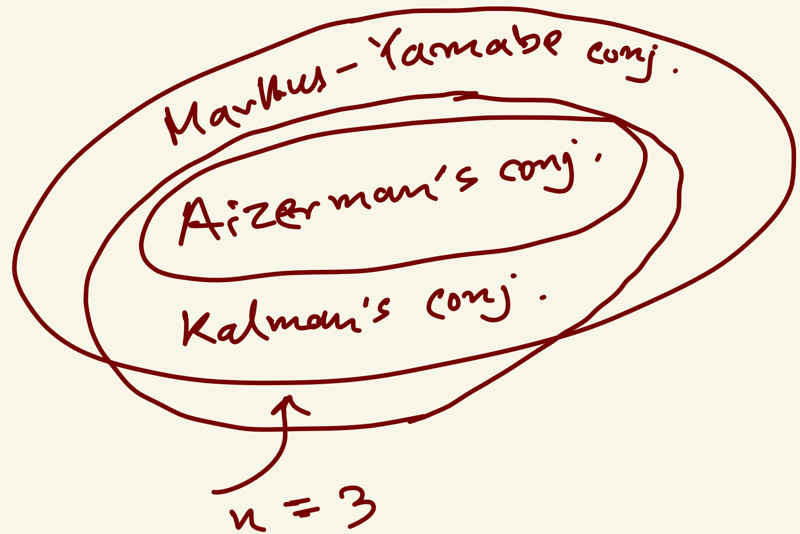
• (1960) Markus – Yamabe Conjecture:

(True for $n = 1, 2$. False for $n \geq 3$)

Consider $\dot{\underline{x}} = f(\underline{x})$, $\underline{f} : \mathbb{R}^n \mapsto \mathbb{R}^n$, f is C^1

Suppose $\underline{x}^* = \underline{0}$ is a fixed point ($\Leftrightarrow f(\underline{0}) = \underline{0}$)

If the Jacobian matrix $\left[\frac{\partial f}{\partial \underline{x}}\right]$ is Hurwitz $\forall\ \underline{x} \in \mathbb{R}^n$, then $\underline{x}^* = \underline{0}$ is GAS.



(Multivariate) Circle Criterion:

The system
$$\dot{\underline{x}} = A\underline{x} + B\underline{u}$$
$$\underline{y} = C\underline{x} + D\underline{u}$$
$$\underline{u} = -\underline{\psi}(t, \underline{y})$$

- (A, B, C, D) minimal
- $\psi : [0, \infty) \times \mathbb{R}^m \mapsto \mathbb{R}^m$
- $\underline{u} = -\psi(t, C\underline{x} + D\underline{u})$ has unique solution $\forall$ pairs $(t, \underline{x})$.

is absolutely stable if

- either $\left\{ \underline{\psi} \in \text{Sector}[\underline{\alpha}, \underline{\infty}]\right.$, and $G(s)\left(I + \text{diag}(\underline{\alpha})\, G(s)\right)^{-1}$ is SPR.
- or $\left\{ \underline{\psi} \in \text{Sector}[\underline{\alpha}, \underline{\beta}]\right.$, $\underline{\beta} \geq \underline{\alpha}$ (elementwise) and $\left(I + \text{diag}(\underline{\beta})\, G(s)\right)\left(I + \text{diag}(\underline{\alpha})\, G(s)\right)^{-1}$ is SPR

Proof: Strategy: prove for $\underline{\psi} \in \text{Sector}[\underline{0}, \underline{\infty}]$

Then argue that this is enough since $\underline{\psi} \in \text{Sector}[\underline{\alpha}, \underline{\infty}]$ and $\underline{\psi} \in \text{Sector}[\underline{\alpha}, \underline{\beta}]$ can be obtained by "loop transformations".

Proof for $\underline{\psi} \in \text{Sector}[\underline{0}, \underline{\infty}]$:

Setting $\underline{\alpha} = \underline{0} \Rightarrow \underline{\psi} \in [\underline{0}, \underline{\infty}]$ and $G(s)$ is SPR

$\Updownarrow$ KYP lemma

$\exists P \succ 0, L, W$ s.t.

$$\begin{cases} PA + A^T P = -LL^T - \varepsilon P \\ PB - C^T = -LW \\ D + D^T = W^T W \end{cases}$$

$\Updownarrow$

$G(s)$ is strictly passive with storage function $V(x) = \frac{1}{2} x^T P x$

$$\boxed{\underline{y}^T \underline{\psi}(t, \underline{y}) \geq 0}$$

To show absolute stability, let us use the quadratic Lyapunov function that is same as the storage function for the LTI subsystem in the loop:

$$V(\underline{x}) = \frac{1}{2}\,\underline{x}^T P \underline{x} \;\longleftarrow\; \text{Candidate Lyapunov function}$$

$$\geq 0$$

$$\dot{V} = \frac{1}{2}\,\underline{x}^T P \dot{\underline{x}} + \frac{1}{2}\,\dot{\underline{x}}^T P \underline{x}$$

$$= \frac{1}{2}\,\underline{x}^T P\big(Ax - B\psi(t,y)\big)^T + \frac{1}{2}\big(Ax - B\psi(t,y)\big)^T P \underline{x}$$

$$= -\frac{1}{2}\,\underbrace{\varepsilon}_{>0}\,\underbrace{\underline{x}^T P \underline{x}}_{>0} - \frac{1}{2}\big(L\underline{x} + W\underline{u}\big)^T\big(L\underline{x} + W\underline{u}\big) - \underbrace{\underline{y}^T \psi(t,\underline{y})}_{=(Cx+D\underline{u})^T\underline{u}}$$

(after some algebra)

(uses the KYP lemma equations)

$\therefore \dot{V} \leq -\frac{1}{2} \underbrace{\varepsilon}_{>0} \underbrace{\underline{x}^T P \underline{x}}_{>0} < 0$ (since $y^T \psi(t,y) \geq 0$)

$\therefore$ Origin is GUAES

If this inequality holds locally, then <u>locally</u> UAES.

To handle $\underline{\psi} \in [\underline{\alpha}, \underline{\infty}]$ and $\underline{\psi} \in [\underline{\alpha}, \underline{\beta}]$

can do loop transforms to $\underline{\tilde{\psi}} \in [\underline{0}, \underline{\infty}]$

(See Khalil's book, fig. 7.2, 7.3 in ch. 7).

Other graphical condition: <u>Popov criterion</u>.

One application:

$$\boxed{\underset{x}{\text{minimize}} \quad f(x)}, \quad x \in \mathbb{R}^d$$

Gradient Descent:

$$x_{k+1} = x_k - \alpha \nabla f(x_k) \quad O(1/k)$$

Nesterov's Accelerated Gradient Descent:

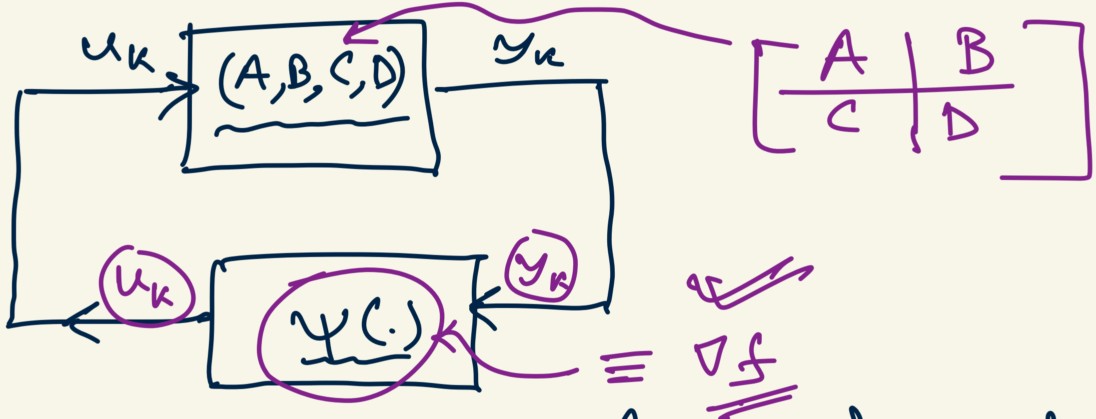
$$\left.\begin{aligned} x_{k+1} &= y_k - \alpha \nabla f(y_k) \\ y_k &= (1+\beta)x_k - \beta x_{k-1} \end{aligned}\right\} O(1/k^2)$$

Heavy Ball:

$$\left.\begin{aligned} x_{k+1} = x_k - \alpha \nabla f(x_k) + {} \\ \beta(x_k - x_{k-1}) \\ \alpha, \beta > 0 \end{aligned}\right\} O(1/k)$$

$$k = 0, 1, 2, \ldots$$

Rewrite algorithms as control systems:



$$\left.\begin{aligned} \underline{\xi}_{k+1} &= A\,\underline{\xi}_k + B\,\underline{u}_k \\ \underline{y}_k &= C\,\underline{\xi}_k + D\,\underline{u}_k \end{aligned}\right\} \quad \psi : \mathbb{R}^d \mapsto \mathbb{R}^d$$

$$\underline{u}_k = \underline{\psi}(\underline{y}_k)$$

$$\underline{\psi} \equiv \nabla f$$

Gradient Descent:

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] = \left[\begin{array}{c|c} I_d & -\alpha I_d \\ \hline I_d & 0_d \end{array}\right]$$

Nesterov's accelerated Gradient Descent:

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] = \left[\begin{array}{cc|c} (1+\beta) I_d & -\beta I_d & -\alpha I_d \\ I_d & 0_d & 0_d \\ \hline (1+\beta) I_d & -\beta I_d & 0_d \end{array}\right]$$

Similarly Heavy Ball . . . .

$f \in S(m, L) \iff f$ is m-strongly convex

$$f(y) \geq f(x) + \langle \nabla f, y - x \rangle + \frac{m}{2} \|y - x\|^2$$

$\forall (x, y) \in \mathrm{dom}(f)$

$K = \frac{L}{m}$

and ∇f is Lipschitz

$$\|\nabla f(x) - \nabla f(y)\|_2 \leq L \|x - y\|_2$$

<u>Problem in machine learning:</u>

We know $f \in S(m, L)$

but may not exactly know / compute ∇f.

But ∇f is L-Lipschitz

$\Updownarrow$

$$\boxed{\|\underline{u}_k - \underline{u}_{opt}\|_2 \leq L \|\underline{y}_k - \underline{y}_{opt}\|_2}$$

$\forall (y_{opt}, u_{opt})$ satisfying $\underline{u}_{opt} = \nabla f(y_{opt})$

$$\Updownarrow$$

$$\begin{pmatrix} y_k - y_{opt} \\ u_k - u_{opt} \end{pmatrix}^T \begin{pmatrix} L^2 I_d & O_d \\ O_d & -I_d \end{pmatrix} \begin{pmatrix} y_k - y_{opt} \\ u_k - u_{opt} \end{pmatrix} \geqslant 0$$

$\forall k = 0, 1, 2 \ldots$

pointwise quadratic inequality

So far, <u>nonlinear systems theory</u>.

From now on, how to <u>design controllers?</u>

Things that will NOT be covered:

→ feedback passivation

→ Sliding mode control / Variable structure control

Will do:

Lyapunov style

- Stabilization via feedback using Control Lyapunov function (CLF)
- Backstepping controller design (again uses Lyapunov for controller synthesis)

Geometric control

- Feedback linearization controller design
- Nonlinear controllability

Design idea #1: Feedback stabilization

Problem statement: Given $\dot{\underline{x}} = f(t, \underline{x}, \underline{u})$

Design $\boxed{\underline{u} = \underline{u}(t, \underline{x})}$

s.t. $\underline{x} = \underline{0}$ is UAS for the closed-loop.

controller

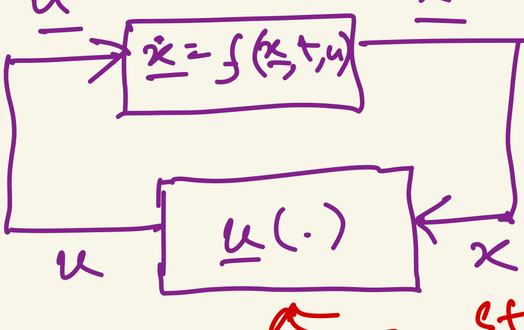
State feedback

Static state feedback

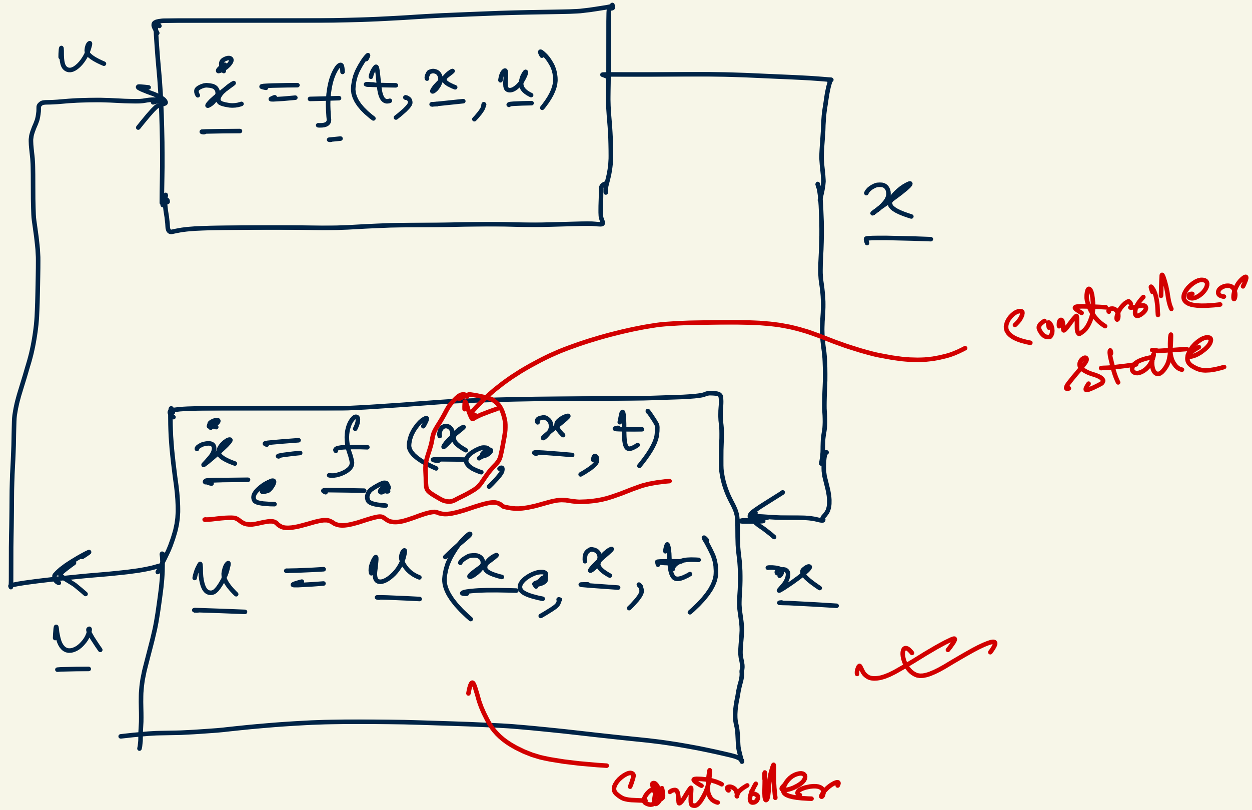
feedback/controller/policy

control/input/action

$\underline{u}$ is a static map.



static state feedback controller



Dynamic state feedback controller