AMS 231 – Spring 2018 – Lecture 1 Notes

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Some Math Shorthand

$\forall$ (for all), $\exists$ (there exists), $\nexists$ (does not exist), $\exists!$ (there exists unique)

These shorthand symbols are good for taking quick notes and classroom teaching. Please avoid them in writing technical article (conference/journal paper), thesis or dissertation; instead use natural language such as "for all", "there exists" etc.

Standard Math Notations

$\in$ (belongs to), $:=$ (defined as), $\equiv$ (equivalent/identically equal to),

$\implies$ (implies), $\not\Longrightarrow$ (does not imply) $\iff$ (if and only if),

$(\cdot,\cdot]$ (left-side open and right-side closed interval), $[\cdot,\cdot]$ (both side closed interval),

$(\cdot,\cdot)$ (both side open interval), $\mathbb{R}^n, \mathbb{R}^n_+, \mathbb{R}^n_{++}$(real, non-negative real, and positive real,

vectors in n dimensions, respectively), $\mathbb{R}^{m\times n}$ (real matrices of size $m \times n$), $\mathbb{S}^1$ (angles in $[0, 2\pi)$)

Our Notations for Scalar, Vector, Matrix and Set

Scalar x, vector $\underline{x}$, matrix X, set $\mathscr{X}$

Abbreviations

i.e. (that is), LHS (left-hand-side), RHS (right-hand-side), IVP (initial value problem), a.k.a. (also known as)

Modeling Dynamical Systems

- Mostly continuous time dynamical systems in this course
- We only consider finite dimensional dynamics (ODEs, not PDEs)
- Specifically, first order vector ODE IVPs of the form

$$\dot{\underline{x}}(t) = \underline{f}\left(\underline{x}(t), t\right), \qquad \underline{x}(0) = \underline{x}_0 \text{ (given initial condition)}$$

Here independent variable is the scalar (time) $t \in (0, t_{\text{final}}]$, dependent variable is the vector $\underline{x} \in \mathscr{X} \subseteq \mathbb{R}^n$.

- Often we write casually as $\dot{\underline{x}} = \underline{f}\left(\underline{x}, t\right)$. As usual, $\dot{\underline{x}}$ means $\frac{\mathrm{d}}{\mathrm{d}t}\,\underline{x}$.

Geometrically, the solution $\underline{x}(t)$ is a curve parametrized by t. As an example, for $n = 2$, we can visualize the curve as

- More elaborate way to write the same ODE IVP is

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix} = \begin{pmatrix} f_1\left(x_1(t), x_2(t), \dots, x_n(t), t\right) \\ f_2\left(x_1(t), x_2(t), \dots, x_n(t), t\right) \\ \vdots \\ f_n\left(x_1(t), x_2(t), \dots, x_n(t), t\right) \end{pmatrix}, \qquad \begin{pmatrix} x_1(0) \\ x_2(0) \\ \vdots \\ x_n(0) \end{pmatrix} = \begin{pmatrix} x_{01} \\ x_{02} \\ \vdots \\ x_{0n} \end{pmatrix}.$$

- Why considering first order vector ODE is enough?

An n-th order ODE can be re-written as a system of n first order ODEs (see example below).

Example 1.1: Simple pendulum in air

Assumptions: (A1) rod/chord is massless and rigid, (A2) the bob is a point mass m, (A3) pivot is fixed and frictionless, (A4) motion is contained in 2D.

Dynamics: From Euler's second law (Newton's second law for rotational motion), we know that

$$\frac{\mathrm{d}}{\mathrm{d}t}(\text{angular momentum}) = \text{external torque}.$$

Suppose we attach a 2D coordinate system to the bob, with radial unit vector $\widehat{e}_r$, and tangential unit vector $\widehat{e}_\theta$ as shown. Then at time t, the position vector of the bob is $\underline{r}(t) = \ell\,\widehat{e}_r$, the translational velocity is $\underline{v}(t) = \dot{\ell}\,\widehat{e}_r + \ell\dot{\theta}\,\widehat{e}_\theta = \ell\dot{\theta}\,\widehat{e}_\theta$. The angular momentum, by definition, is $\underline{r}(t) \times m\underline{v}(t) = m\ell^2\dot{\theta}\widehat{k}$, where $\widehat{k}$ is the unit vector perpendicular and out of the plane. Thus the LHS in Euler's second law is $m\ell^2\ddot{\theta}\,\widehat{k}$. The external torque in the RHS equals

$$\underbrace{\underline{r}(t) \times \left(mg\cos\theta\,\widehat{e}_r - mg\sin\theta\,\widehat{e}_\theta\right)}_{\text{torque due to gravity}} - \underbrace{b\dot{\theta}\,\widehat{k}}_{\text{torque due to air friction}}$$
$$= \left(-mg\ell\sin\theta - b\dot{\theta}\right)\widehat{k}.$$

Equating the LHS and RHS results the second order ODE

$$\ddot{\theta} = -\frac{g}{\ell}\sin\theta - \frac{b}{m\ell^2}\dot{\theta}.$$

To rewrite the above second order ODE as a system of 2 first order ODEs, let $x_1 := \theta$, $x_2 := \dot{x}_1 = \dot{\theta}$. This results the vector IVP

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ -\alpha\sin x_1 - \beta x_2 \end{pmatrix}, \qquad \begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} = \begin{pmatrix} \theta(0) \\ \dot{\theta}(0) \end{pmatrix},$$

where the parameters $\alpha := \frac{g}{\ell} > 0, \beta := \frac{b}{m\ell^2} > 0$. Notice that here $\underline{x} \in \mathcal{X} \equiv \mathbb{S}^1 \times \mathbb{R}$, i.e., $\mathscr{X}$ is a cylinder.

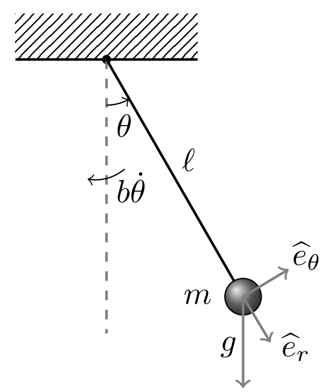

Figure 1: Pendulum in Example 1

Systems Theoretic Terminology

- In the first order vector ODE IVP, the vector $\underline{x} \in \mathcal{X} \subseteq \mathbb{R}^n$ is called the **state vector**; the set $\mathcal{X}$ is called the **state space**; the **dimension of the state space** is n.
- We call $\underline{f}(\underline{x}, t)$ a **vector field**.
- An **autonomous** dynamical system is one where the vector field $\underline{f}$ has no explicit time dependence, i.e., $\dot{\underline{x}} = \underline{f}(\underline{x})$. A **non-autonomous** dynamical system is one where the vector field $\underline{f}$ has explicit time dependence, i.e., $\dot{\underline{x}} = \underline{f}(\underline{x}, t)$ (as in the general form of the IVP).
- A **linear** (dynamical) system is one where the vector field $\underline{f}$ is linear in state vector $\underline{x}$.
- Combining the above two nomenclature, an **autonomous linear system**, a.k.a. **linear time invariant (LTI) system**, is the ODE IVP $\dot{\underline{x}} = A\underline{x}$, $\underline{x}(0) = \underline{x}_0$ (given), where $A \in \mathbb{R}^{n \times n}$ is a constant matrix.

  Similarly, a **non-autonomous linear system**, a.k.a. **linear time variant (LTV) system**, is the ODE IVP $\dot{\underline{x}} = A(t)\underline{x}$, $\underline{x}(0) = \underline{x}_0$ (given), where the entries of matrix $A(t) \in \mathbb{R}^{n \times n}$ are functions of time.

Groups Encountered Often in Control Applications

- **Orthogonal group**, denoted as $\mathrm{O}(n)$, is the group of $n \times n$ orthogonal matrices, where the group operation is given by matrix multiplication. Notice that an orthogonal matrix has determinant either $+1$ or -1. The subgroup of $n \times n$ orthogonal matrices with determinant $+1$, are called **special orthogonal group**, a.k.a. **rotation group**, denoted as $\mathrm{SO}(n)$. The term **rotation group** is motivated by the fact that each element of $\mathrm{SO}(n)$ denotes rotation in $\mathbb{R}^n$. For example, elements of $\mathrm{SO}(2)$ are rotations in 2D about a point; elements of $\mathrm{SO}(3)$ are rotations in 3D about a line, etc.

  A matrix R is orthogonal if $RR^\top = R^\top R = I$, the identity matrix. In other words, an orthogonal matrix is one whose inverse equals transpose.

  One way to parameterize elements of $\mathrm{SO}(n)$ is via the so-called **Euler angles**. For example, a matrix $R \in \mathrm{SO}(2)$ can be parameterized as
$$R = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}, \quad \theta \in \mathbb{S}^1,$$
  whose action on any vector in $\mathbb{R}^2$ results in circular displacement by an angle θ. In 2D, only one angle θ suffices since planar rotations are rotations about an axis perpendicular to the plane.

A matrix $R \in \mathrm{SO}(3)$ can be parameterized as the composition of three individual axis rotation matrices $R_\mathrm{x}, R_\mathrm{y}$ and R_z, where

$$R_\mathrm{x} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{bmatrix}, \quad R_\mathrm{y} = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix},$$

$$R_\mathrm{z} = \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad R = R_\mathrm{z} R_\mathrm{y} R_\mathrm{x}.$$

Other compositions are possible too, such as $R = R_\mathrm{z} R_\mathrm{y} R_\mathrm{z}$.

Thus in 3D, the Euler angles are the triple (ϕ, θ, ψ); and the action of R on any vector in $\mathbb{R}^3$ results in spherical displacement.

- **Special Euclidean group**, denoted as $\mathrm{SE}(n)$, is the group of rigid body (translational and rotational) transformations in $\mathbb{R}^n$. Its elements are the pairs $(R, \underline{r}) \in \mathrm{SO}(n) \times \mathbb{R}^n$, i.e., the matrix-vector pairs specifying translation and rotation.

  For example, $\mathrm{SE}(3)$ is the group of translational and rotational transformations in 3D. It turns out that $\mathrm{SE}(3)$ is also a differentiable manifold, i.e., $\mathrm{SE}(3)$ is a Lie group. Its tangent space is the Lie algebra $\mathfrak{se}(3)$, whose elements are $(\underline{v}, \underline{\omega}) \in \mathbb{R}^6$, i.e., translational and angular velocities.

Instead of representing as a tuple $(R, \underline{r})$, it is convenient to express an element of $\mathrm{SE}(n)$ in the so called **homogeneous representation**:

$$\begin{pmatrix} R & \underline{r} \\ 0_{1\times n} & 1 \end{pmatrix}.$$

We can then think of the action of an element in $\mathrm{SE}(n)$ on a vector $\underline{x} \in \mathbb{R}^n$, as a linear map given by the homogeneous representation that acts on the lifted vector $\begin{pmatrix} \underline{x} \\ 1 \end{pmatrix} \in \mathbb{R}^{n+1}$.

A group that is also differentiable manifold is called a **Lie group**.