AMS 231: Nonlinear Control Theory: Winter 2018

Homework #6

Name: ....

Due: March 15, 2018

NOTE: Please show all the steps in your solution. Turn in a hard copy of your HW stapled with this as cover sheet with your name written in the above field. Please submit your HW in class on the due date.

Problem 1

Feedback Linearization **(**$20+20+20+20+(10+2+3)+5 = 100$ **points)**

In this exercise, you will apply the Theorem (pg. 14) and step-by-step recipe (pg. 15–16) for feedback linearization given in Lecture 18 notes. Consider the nonlinear system

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{pmatrix} x_3\,(1+x_2) \\ x_1 \\ x_2\,(1+x_1) \end{pmatrix} + \begin{pmatrix} 0 \\ 1+x_2 \\ -x_3 \end{pmatrix} u, \qquad \underline{x} \in \mathbb{R}^3, \quad u \in \mathbb{R}. \tag{1}$$

(a) Prove that the system is locally feedback linearizable around $\underline{x} = \underline{0}$. (Hint: see Step 1 in pg. 15, Lecture 18 notes.)

(b) Show that a solution for $\lambda(\underline{x})$ in the Theorem (pg. 14, Lecture 18 notes) is given by $\lambda(\underline{x}) = x_1$. (Hint: Notice that Step 2 in pg. 15, Lecture 18 notes gives a system of $n-1$ first order PDEs, and one PDE not-equal-to-zero condition, where $\underline{x} \in \mathbb{R}^n$. You will need to consider all of them simultaneously. Also, the solution for $\lambda(\underline{x})$ for this PDE system is non-unique – which is a good thing since our Theorem only requires existence but there can be multiple feedback linearizing controllers corresponding to different admissible $\lambda(\underline{x})$.)

(c) By directly computing relative degree, prove that the state equation (1) above, augmented with the output equation $y = \lambda(\underline{x}) = x_1$, indeed has relative degree 3 (that is, satisfies $r = n$ condition) at the point $\underline{x} = \underline{0}$.

(d) Use your answer in part (b), to compute the feedback linearizing transformation tuple $(\underline{\tau}(\cdot), \alpha(\cdot), \beta(\cdot))$. (Hint: use steps 3 and 4 in pg. 16, Lecture 18 notes.)

(e) Show that the Jacobian of $\underline{\tau}$ is non-singular at $\underline{x} = \underline{0}$. What does it mean in terms of feedback linearization? Submit a plot of the surface of the form $x_3 = \phi(x_1, x_2)$ where the Jacobian of $\underline{\tau}(\cdot)$ is singular.

(f) Clearly write down the feedback linearized control system in new co-ordinates with state $\underline{z}$ and control v, where $\underline{z} = \underline{\tau}(\underline{x})$ and $u = \alpha(\underline{x}) + \beta(\underline{x})v$.

Solution

(a) To prove that (1) is feedback linearizable, we need to verify the two constructive conditions in the Theorem given in pg. 14, Lecture 18 notes. To verify condition (i), we notice that the state dimension $n = 3$, and construct the matrix

$$M(\underline{x}) = \left[\underline{g}(\underline{x}) \qquad \text{ad}_{\underline{f}}\, \underline{g}(\underline{x}) \qquad \text{ad}^2_{\underline{f}}\, \underline{g}(\underline{x})\right],$$

where

$$\underline{g}(\underline{x}) = \begin{pmatrix} 0 \\ 1+x_2 \\ -x_3 \end{pmatrix},$$

$$\text{ad}_{\underline{f}}\, \underline{g}(\underline{x}) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{pmatrix} x_3\,(1+x_2) \\ x_1 \\ x_2\,(1+x_1) \end{pmatrix} - \begin{bmatrix} 0 & x_3 & 1+x_2 \\ 1 & 0 & 0 \\ x_2 & 1+x_1 & 0 \end{bmatrix} \begin{pmatrix} 0 \\ 1+x_2 \\ -x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ x_1 \\ -(1+x_1)(1+2x_2) \end{pmatrix},$$

$$\text{ad}^2_{\underline{f}}\, \underline{g}(\underline{x}) = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ -(1+2x_2) & -2(1+x_1) & 0 \end{bmatrix} \begin{pmatrix} x_3\,(1+x_2) \\ x_1 \\ x_2\,(1+x_1) \end{pmatrix} - \begin{bmatrix} 0 & x_3 & 1+x_2 \\ 1 & 0 & 0 \\ x_2 & 1+x_1 & 0 \end{bmatrix} \begin{pmatrix} 0 \\ x_1 \\ -(1+x_1)(1+2x_2) \end{pmatrix}$$

$$= \begin{pmatrix} -x_3x_1 + (1+x_1)(1+x_2)(1+2x_2) \\ x_3(1+x_2) \\ -x_3(1+x_2)(1+2x_2) - 3x_1(1+x_1) \end{pmatrix}.$$

Therefore, we have

$$\text{rank}\,(M(\underline{x} = \underline{0})) = \text{rank}\left(\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}\right) = 3,$$

and hence the condition (i) is satisfied.

To verify condition (ii), notice that

$$\mathrm{ad}_{\underline{g}}\,\mathrm{ad}_{\underline{f}}\,\underline{g}(\underline{x}) = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ -(1+2x_2) & -2(1+x_1) & 0 \end{bmatrix} \begin{pmatrix} 0 \\ 1+x_2 \\ -x_3 \end{pmatrix} - \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{pmatrix} 0 \\ x_1 \\ -(1+x_1)(1+2x_2) \end{pmatrix}$$
$$= \begin{pmatrix} 0 \\ -x_1 \\ -(1+x_1)(3+2x_2) \end{pmatrix},$$

and thus

$$\begin{aligned}
&\mathrm{rank}\left(\begin{bmatrix}\underline{g}(\underline{x}) & \mathrm{ad}_{\underline{f}}\,\underline{g}(\underline{x}) & \mathrm{ad}_{\underline{g}}\,\mathrm{ad}_{\underline{f}}\,\underline{g}(\underline{x})\end{bmatrix}_{\underline{x}=\underline{0}}\right) \\
&= \mathrm{rank}\left(\begin{bmatrix} 0 & 0 & 0 \\ 1+x_2 & x_1 & -x_1 \\ -x_3 & -(1+x_1)(1+2x_2) & -(1+x_1)(3+2x_2) \end{bmatrix}_{\underline{x}=\underline{0}}\right) \\
&= \mathrm{rank}\left(\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & -1 & -3 \end{bmatrix}\right) \\
&= 2.
\end{aligned}$$

Hence, $\mathrm{ad}_{\underline{g}}\,\mathrm{ad}_{\underline{f}}(\underline{x}=\underline{0}) \in \Delta := \mathrm{span}\{\underline{g}, \mathrm{ad}_{\underline{f}}\,\underline{g}\}(\underline{x}=\underline{0})$, i.e., the distribution Δ is involutive near $\underline{x}=\underline{0}$, thereby verifying condition (ii).

By the Theorem, the system (1) is feedback linearizable around $\underline{x}=\underline{0}$.

(b) We need to find a scalar field $\lambda(\underline{x})$ that solves the following system of first order PDEs:

$$\begin{aligned}
L_{\underline{g}}\lambda(\underline{x}) = \left\langle \frac{\partial\lambda}{\partial\underline{x}}, \underline{g}(\underline{x})\right\rangle = 0, \quad &\Leftrightarrow \quad (1+x_2)\,\frac{\partial\lambda}{\partial x_2} \;-\; x_3\frac{\partial\lambda}{\partial x_3} = 0, \\
L_{\mathrm{ad}_{\underline{f}}\,\underline{g}}\lambda(\underline{x}) = \left\langle \frac{\partial\lambda}{\partial\underline{x}}, \mathrm{ad}_{\underline{f}}\,\underline{g}(\underline{x})\right\rangle = 0, \quad &\Leftrightarrow \quad x_1\frac{\partial\lambda}{\partial x_2} \;-\; (1+x_1)\,(1+2x_2)\,\frac{\partial\lambda}{\partial x_3} = 0,
\end{aligned}$$

and the not-equal-to-zero condition:

$$L_{\mathrm{ad}^2_{\underline{f}}\,\underline{g}}\lambda(\underline{x}=\underline{0}) = \left\langle \frac{\partial\lambda}{\partial\underline{x}}, \mathrm{ad}^2_{\underline{f}}\,\underline{g}(\underline{x})\right\rangle\Big|_{\underline{x}=\underline{0}} \neq 0 \quad\Leftrightarrow\quad \frac{\partial\lambda}{\partial x_1}\Big|_{\underline{x}=\underline{0}} \neq 0.$$

Eliminating $\dfrac{\partial\lambda}{\partial x_2}$ from the system of PDEs, we get

$$\left[(1+x_1)\,(1+x_2)\,(1+2x_2) - x_3x_1\right]\frac{\partial\lambda}{\partial x_3} = 0,$$

for which to hold $\forall\,(x_1,x_2,x_3)$ around $\underline{x}=\underline{0}$, we must have $\dfrac{\partial\lambda}{\partial x_3}=0$. On the other hand, substituting $\dfrac{\partial\lambda}{\partial x_3}=0$ back in the system of PDEs, arguing likewise, we obtain $\dfrac{\partial\lambda}{\partial x_2}=0\ \forall\,(x_1,x_2,x_3)$ around $\underline{x}=\underline{0}$. Therefore, any solution to the PDE system must be of the form $\lambda=\lambda(x_1)$ such that $\frac{\partial\lambda}{\partial x_1}|_{x_1=0}\neq 0$ (due to the not-equal-to-zero condition). The candidate solution $\lambda(\underline{x})=x_1$ satisfies this, and hence is an admissible solution.

(c) For $\lambda(\underline{x})=x_1$, by direct calculation, we get

$$L_{\underline{g}}\,\lambda(\underline{x})=0,\quad L_{\underline{g}}\,L_{\underline{f}}\,\lambda(\underline{x})=0,\quad L_{\underline{g}}\,L_{\underline{f}}^2\,\lambda(\underline{x})=(1+x_1)\,(1+x_2)\,(1+2x_2)-x_3x_1,$$

and $L_{\underline{g}}\,L_{\underline{f}}^2\,\lambda(\underline{x}=\underline{0})=1\neq 0$. Therefore, the relative degree r satisfies $r-1=2\Rightarrow r=3$.

(d) From steps 3 and 4 in pg. 16, Lecture 18 notes, for $\lambda(\underline{x})=x_1$, we get

$$\alpha(\underline{x})=-\frac{L_{\underline{f}}^3\,\lambda(\underline{x})}{L_{\underline{g}}\,L_{\underline{f}}^2\,\lambda(\underline{x})}=\frac{-x_3^2(1+x_2)-x_2x_3(1+x_2)^2-x_1(1+x_1)(1+2x_2)-x_1x_2(1+x_1)}{(1+x_1)\,(1+x_2)\,(1+2x_2)-x_3x_1},$$

$$\beta(\underline{x})=\frac{1}{L_{\underline{g}}\,L_{\underline{f}}^2\,\lambda(\underline{x})}=\frac{1}{(1+x_1)\,(1+x_2)\,(1+2x_2)-x_3x_1},$$

and

$$\underline{\tau}\,(\underline{x})=\begin{pmatrix}\lambda(\underline{x})\\ L_{\underline{f}}\lambda(\underline{x})\\ L_{\underline{f}}^2\lambda(\underline{x})\end{pmatrix}=\begin{pmatrix}x_1\\ x_3(1+x_2)\\ x_3x_1+(1+x_1)(1+x_2)x_2\end{pmatrix}.$$

(e) The Jacobian

$$\left.\frac{\partial\underline{\tau}}{\partial\underline{x}}\right|_{\underline{x}=\underline{0}}=\begin{bmatrix}1 & 0 & 0\\ 0 & x_3 & 1+x_2\\ x_3+(1+x_2)x_2 & (1+x_1)(1+2x_2) & x_1\end{bmatrix}_{\underline{x}=\underline{0}}=\begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix},$$

and hence is non-singular (since all columns are linearly independent).

The implication of non-singular Jacobian is that the change-of-state-co-ordinate transformation $\underline{x}\mapsto\underline{z}:=\underline{\tau}(\underline{x})$ is locally invertible around $\underline{x}=\underline{0}$, as promised by the theory of feedback linearization.

We observe that

$$\det\left(\frac{\partial\underline{\tau}}{\partial\underline{x}}\right)=x_3x_1-(1+x_1)\,(1+x_2)\,(1+2x_2)\,.$$

Notice that the singularity condition, $x_3x_1-(1+x_1)\,(1+x_2)\,(1+2x_2)=0$ is precisely what makes $\alpha(\cdot),\beta(\cdot)$ undefined in part (d), and $\lambda(\cdot)$ undefined in part (b). The singularity surface is

$$x_3=\left(1+\frac{1}{x_1}\right)(1+x_2)\,(1+2x_2)\,,$$

which is plotted below.

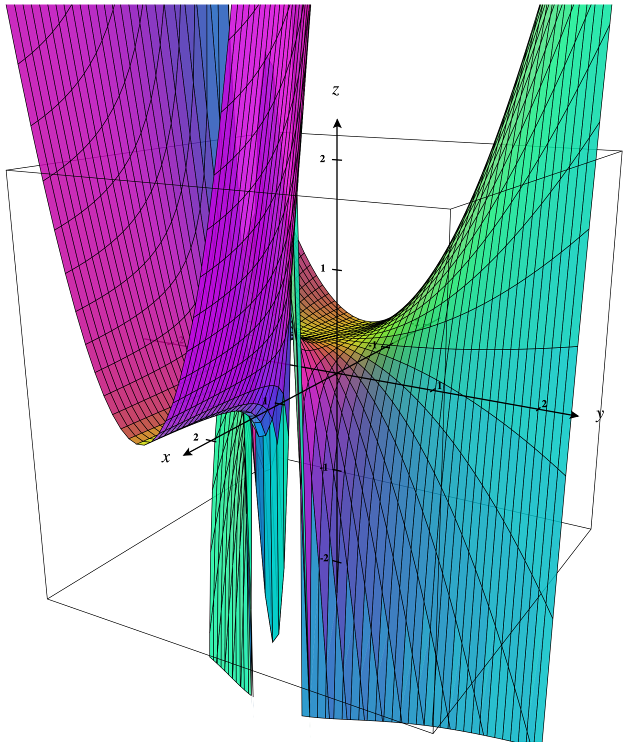

Figure 1: Plot of the singularity surface for $\underline{\tau}$.

(f) The feedback linearized control system in new co-ordinates is

$$\begin{pmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \end{pmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} v,$$

where $\underline{z} = \underline{\tau}(\underline{x})$ and $u = \alpha(\underline{x}) + \beta(\underline{x})v$, as given in part (d).