Lecture # 2 (04/02/2020)

A.S. does NOT, in general, say anything about the rate-of-convergence.

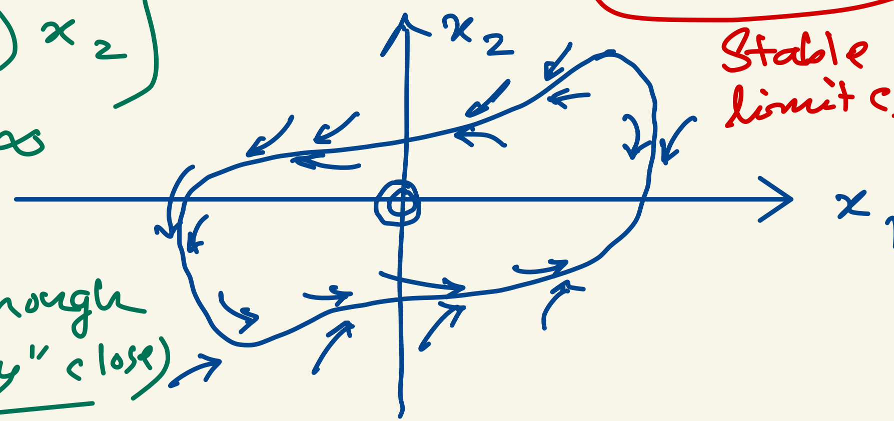
Except: Linear time invariant (LTI) $\underline{\dot{x}} = A\underline{x}$

- in that case, AS $\Rightarrow$ ES (exponential stability)

Example: (Vander pol Oscillator)

$$\left.\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 + (1 - x_1^2)\, x_2 \end{aligned}\right\} \quad \underline{x}^* = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ is the unique fixed point}$$

NOT stable



Stable limit cycle

Even though $\underline{x}(t) \not\to \infty$ still we say, origin is NOT stable

Staying close is NOT enough (we need to stay "arbitrarily" close)

Example:

$$\dot{x}_1 = -x_1 - \frac{x_2}{\ln\sqrt{x_1^2 + x_2^2}}$$

$$\dot{x}_2 = -x_2 + \frac{x_1}{\ln\sqrt{x_1^2 + x_2^2}}$$

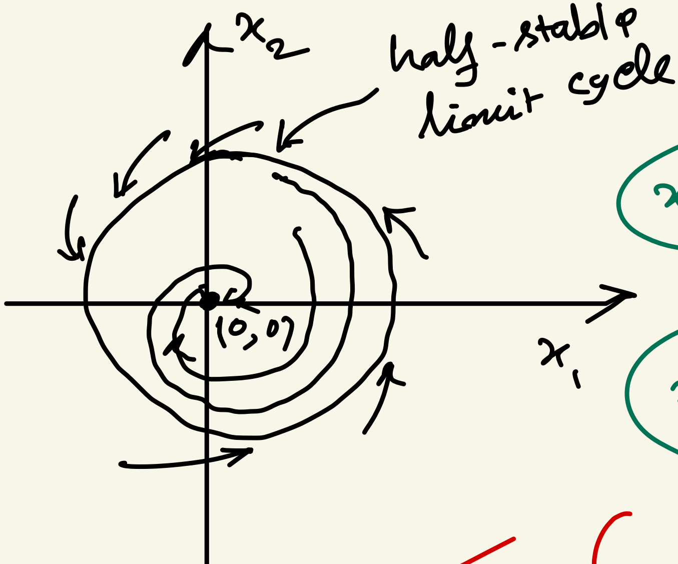
$$\left.\begin{aligned} x_1^* &= \frac{-x_2^*}{\ln\sqrt{x_1^{*2} + x_2^{*2}}} \\ x_2^* &= \frac{+x_1^*}{\ln\sqrt{x_1^{*2} + x_2^{*2}}} \end{aligned}\right\}$$

$$\Downarrow$$

$$\left.\begin{aligned} x_1^* x_2^* &= \frac{-(x_2^*)^2}{\ln\sqrt{(x_1^*)^2 + (x_2^*)^2}} \\ x_2^* x_1^* &= \frac{+(x_1^*)^2}{\ln\sqrt{(x_1^*)^2 + (x_2^*)^2}} \end{aligned}\right\}$$

$$(x_1^*)^2 + (x_2^*)^2 = 0 \Rightarrow \begin{pmatrix} x_1^* \\ x_2^* \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\therefore$ Unique fixed pt: $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$



→ Stable ✓

→ AS ✓

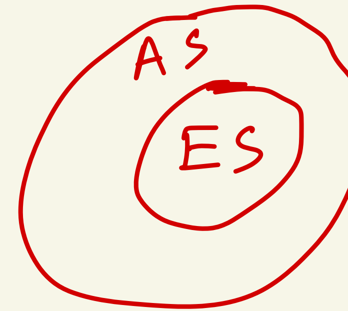
→ GAS ✗

→ The ROA: unit disc (open)

Def^n: Exponential stability

$\underline{x}^* = \underline{0}$ is exponentially stable (ES) if for all $\underline{x}(0) \in \mathcal{B}(\underline{0}, \delta)$, there exists $\epsilon_1, \epsilon_2 > 0$ such that for all $t > 0$,

$$\|\underline{x}(t)\|_2 \le \underline{\epsilon_1 \exp(-\epsilon_2 t)}.$$



Example: (Not ES)

$$\dot{x} = -x^2, \quad x(0) = 1$$

$$\Rightarrow x(t) = \frac{1}{1+t}, \quad \therefore \lim_{t \to \infty} x(t) = 0, \quad \therefore \text{AS}.$$

But convergence is slower than e^{-t}.

How to certify/guarantee S/AS/GAS etc.

Def$^{\underline{n}}$ of Positive (semi) definite Function:

Let $F : \mathcal{D} \subset \mathbb{R}^n \mapsto \mathbb{R}_{\geq 0}$ be a $C^1(\mathcal{D})$ function such that

① $F(\underline{0}) = 0$

② $F(\underline{x}) > 0 \quad \forall\ \underline{x} \in \mathcal{D} \setminus \{\underline{0}\}$

Then we say that F is positive definite function.

If we have ① + {② with $\geq$}, then we say F is a positive semi-definite function.

Similarly, if we have: ① + {② with $<$}, then we say F is negative definite function.

If we have ① + {② with $\leq$}, then we say F is negative semi-definite

Definition of Lyapunov function :

Let $V : \mathcal{D} \subset \mathbb{R}^n \mapsto \mathbb{R}_{\geq 0}$ be a $C^1(\mathcal{D})$ function such that

① + ② in prev. page } $V(\underline{x})$ is positive definite function

AND

either ③ } $\dot{V} \equiv \frac{dV}{dt}$ is negative semi-definite function

(i.e.), $\dot{V} \leq 0 \;\; \forall \; \underline{x} \in \mathcal{D}$

or

③′ $\dot{V}$ is negative definite funcition in $\mathcal{D} \setminus \{\underline{0}\}$

fixed point

(i.e.) $\dot{V} < 0 \;\; \forall \; \underline{x} \in \mathcal{D} \setminus \{\underline{0}\}$

we say $V(\underline{x})$ is a Lyapunov function if <u>either</u> ① + ② + ③ holds OR ① + ② + ③′ holds.

Theorem (Lyapunov, 1892):

Let $\underline{x}^* = \underline{0}$ be a fixed point for $\dot{\underline{x}} = \underline{f}(\underline{x}), x \in \mathcal{X} \subseteq \mathbb{R}^n$

Let $V: \mathcal{X} \mapsto \mathbb{R}_{\geq 0}$ be a C^1 function of $\underline{x}$, be a function such that

① + ② + ③ holds in the previous page.

Then $\underline{x}^* = \underline{0}$ is $\underline{\text{STABLE}}$ (S)

If V is such that ① + ② + ③' holds in the previous page.

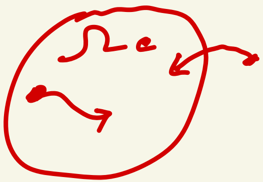
Then $\underline{x}^* = \underline{0}$ is A.S.

The function $V(\underline{x})$ is called $\underline{\text{Lyapunov function}}$.

If $\dot{V} \leq 0$ then we are NOT sure if $\lim_{t \to \infty} \underline{x}(t) = \underline{0}$ or NOT but at least can say $\underline{\text{Stable}}$

Interpretation of the condition $\dot{V} \le 0$

$\dot{V} \le 0 \iff$ Whenever a trajectory crosses a level set of $V(\underline{x})$ ($V(\underline{x}) = c$ for some $c > 0$) it then moves inside the set



$$\Omega_c := \{ \underline{x} \in \mathbb{R}^n \mid V(\underline{x}) \le c \}$$

and can never come out of Ω_c again in future.

$\iff$ The set Ω_c is (+ve)-ly invariant in time

As c decreases, the set $V(\underline{x}) = c$ shrinks to origin $\iff$ $\underline{x}(t) \to 0$ as $t \to \infty$

$c_2 > c_1 > c_0 > 0$

Example: (Simple pendulum)

$$\left.\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\alpha \sin(x_1), \\ &\alpha > 0. \end{aligned}\right\} \begin{aligned} &\beta = 0 \\ &\text{(no damping)} \end{aligned}$$

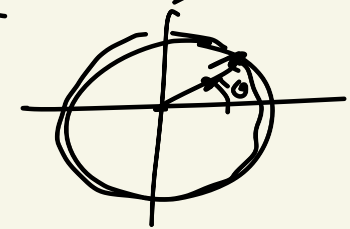
$$V(\underline{x}) = V(x_1, x_2) = \underbrace{\alpha(1-\cos x_1)}_{\text{Potential energy}} + \underbrace{\frac{1}{2}x_2^2}_{\text{Kinetic energy}}$$

$$V(0,0) \overset{?}{=} 0 \quad (\text{YES})$$

$$V(x_1,x_2) \overset{?}{>} 0 \quad \forall \underbrace{(x_1, x_2)}_{\in} \neq (0,0) \quad (\text{YES})$$

$$= [0, 2\pi) \times \mathbb{R}$$

$$= S^1 \times \mathbb{R}$$



$$\dot{V}(x_1, x_2) = \frac{d}{dt}V = \left(\frac{\partial V}{\partial \underline{x}}\right)^T \frac{d\underline{x}}{dt} = \boxed{\qquad\qquad} = (\nabla_{\underline{x}} V)^T \underline{f}(\underline{x})$$

$$\dot{V}(x_1, x_2) = (\nabla V)^T \underline{f}(\underline{x})$$

$$= \langle \nabla V, \underline{f} \rangle$$

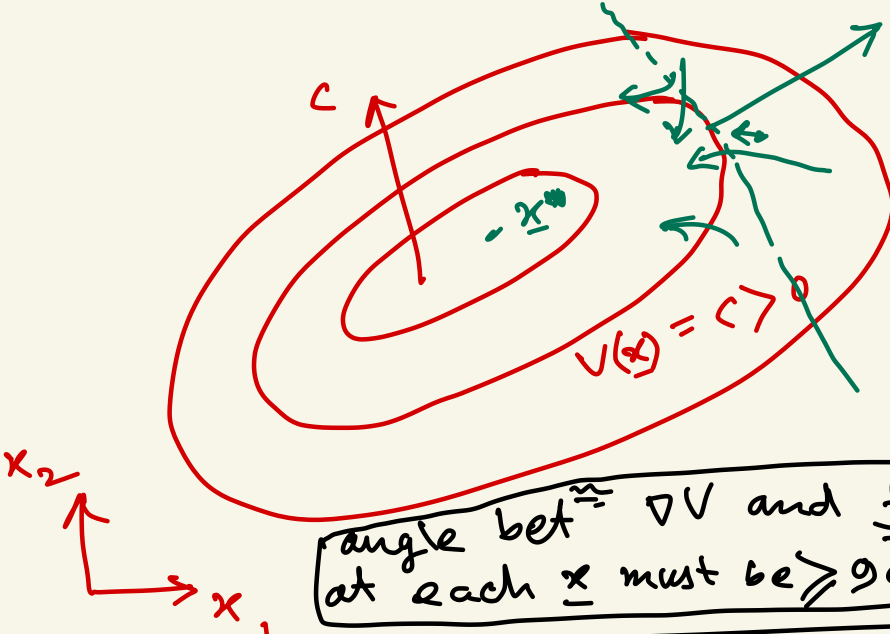
$$= \begin{pmatrix} \frac{\partial V}{\partial x_1} & \frac{\partial V}{\partial x_2} \end{pmatrix} \begin{pmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \end{pmatrix}$$

$$= \begin{pmatrix} +\alpha \sin x_1 & x_2 \end{pmatrix} \begin{pmatrix} x_2 \\ -\alpha \sin x_1 \end{pmatrix}$$

$$= \alpha x_2 \sin x_1 - \alpha x_2 \sin x_1$$

$$= 0 \quad \forall (x_1, x_2) \qquad \text{NOT} < 0$$

So, $(0,0)$ is <u>STABLE</u> (S).



$$\dot{V} \le 0$$

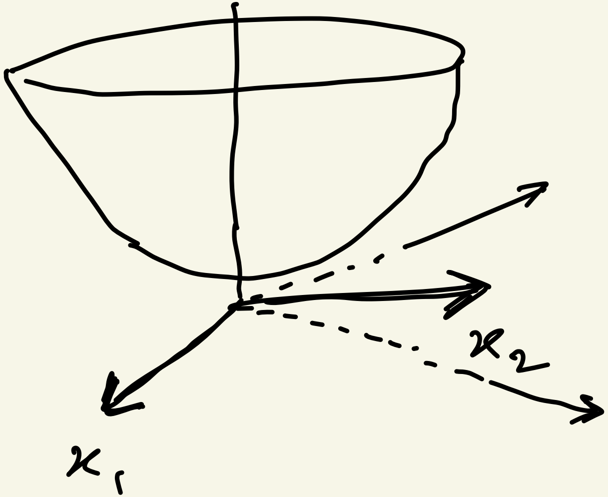
$$\Updownarrow$$

$$\frac{dV}{dt} = \left\langle \frac{\partial V}{\partial \underline{x}}, \underline{f} \right\rangle = \left(\frac{\partial V}{\partial \underline{x}}\right)^T \underline{f} = \left\langle \nabla V(\underline{x}), \underline{f}(\underline{x}) \right\rangle \le 0$$

$\boxed{\text{angle bet}^{\underline{n}}\ \nabla V \text{ and } \underline{f} \text{ at each } \underline{x} \text{ must be} \geq 90^\circ}$

Radially unbounded function:

We say $V(\underline{x})$ is radially unbounded if $V(\underline{x}) \to \infty$ as $\|\underline{x}\|_2 \to \infty$ for <u>all possible</u> radial directions.



GAS $\overset{?}{\Longrightarrow}$ unique fixed pt.

(yes)

GAS $\overset{?}{\Longleftarrow}$ unique fixed point + AS

(No)

NOT radially unbounded:

Example:

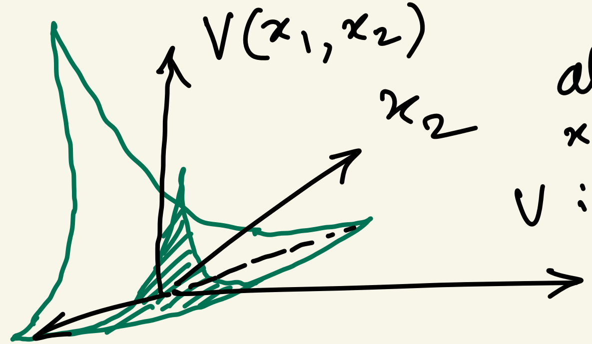
$$V(x_1, x_2) = \frac{x_1^2 + x_2^2}{1 + x_1^2 + x_2^2} + (x_1 - x_2)^2$$

$V(0,0) = 0$ (yes)

$V(x_1, x_2) \overset{?}{>} 0 \quad \forall (x_1, x_2) \neq (0,0)$

(yes)

This is NOT radially unbounded for all radial directions



along dir^n $x_1 = x_2$

V is bounded

Example: (Extra condition : radial unboundedness)

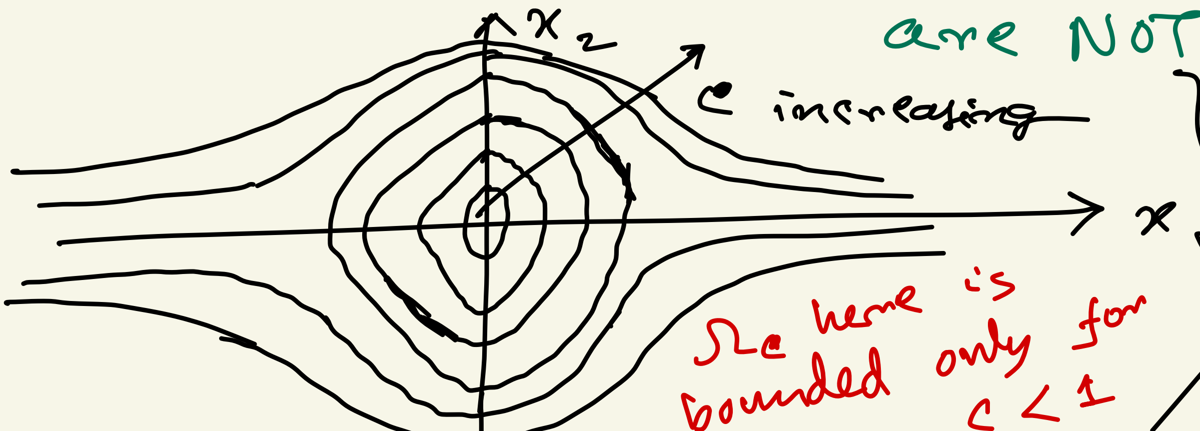
$$V(x_1, x_2) = \frac{x_1^2}{1 + x_1^2} + x_2^2 , \quad \underline{x} \in \mathbb{R}^2$$

$$\Omega_c := \{ \underline{x} \in \mathbb{R}^2 \mid V(\underline{x}) \leq c \} , \qquad c > 0$$

For large c, the set Ω_c is not compact (becomes unbounded)

$\Updownarrow$

For those large c, the curves $V(\underline{x}) = c$ are NOT closed



Ω_c should be in the interior of some ball $B(0, r)$

$\Updownarrow$

$$c < \inf_{\|\underline{x}\| \geq r} V(\underline{x})$$

If $\ell := \lim_{r \to \infty} \inf_{\|\underline{x}\|_2 \geq r} V(\underline{x}) < \infty$, then

Ω_c will be bounded for $c < \ell$.

In our example:

$$\ell := \lim_{r \to \infty} \inf_{\|\underline{x}\|_2 = r} \left(\frac{x_1^2}{1+x_1^2} + x_2^2 \right)$$

$$= \lim_{|x_1| \to \infty} \frac{x_1^2}{1+x_1^2} = 1.$$

Barbashin - Krasovskii Theorem :

$$\underbrace{\{A.S.\}}_{(①,\ ②,\ ③)} + \underbrace{\{\lim_{\|\underline{x}\|_2 \to \infty} V(\underline{x}) = \infty\}}_{\text{radial unboundedness}}$$

$$\Updownarrow$$

$$GAS$$

Two sided arrow
(in latex, \Leftrightarrow)
means
"if and only if"
or, "equivalent to".