Lec. 6 (04/16/2020)

Application #1 : Model invalidation using Barrier Certificates :

Model: $\dot{\underline{x}} = \underline{f}(t, \underline{x}, \underline{p})$, $\quad \underline{x} \in \mathcal{X} \subseteq \mathbb{R}^n$, $\quad \underline{p} \in \mathcal{P} \subseteq \mathbb{R}^p$ ---- (*)

$\underline{x}$: state vector, $\underline{p}$: parameter vector

Two sets of measurements:

at $t = 0$, $\quad \underline{x}_0 \in \mathcal{X}_0 \subseteq \mathcal{X}$ ✓

at $t = T$, $\quad \underline{x}_T \in \mathcal{X}_T \subseteq \mathcal{X}$ ✓

Invalidation problem:

Given model $\dot{\underline{x}} = \underline{f}(t, \underline{x}, \underline{p})$, parameter set $\mathcal{P}$, trajectory information $\{\mathcal{X}_0, \mathcal{X}_T, \mathcal{X}\}$, prove that

$\forall \underline{p} \in \mathcal{P}, \quad \nexists\ \underline{x}(t)$ such that $\underline{x}_0 \in \mathcal{X}_0$

$\underline{x}_T \in \mathcal{X}_T$

$\underline{x}(t) \in \mathcal{X}$

$\forall\, t \in [0, T]$

If such a proof is found, we say the model & parameter set $\mathcal{P}$ are invalidated / falsified by data $\{\mathcal{X}_0, \mathcal{X}_T, \mathcal{X}\}$.

Theorem: Let the model $(*)$, and the sets $\mathcal{P}, \mathcal{X}_0, \mathcal{X}_T, \mathcal{X}$ be given, with $\underline{f}(t, \underline{x}, \underline{p})$ being continuous in t and $\underline{x}$.

Suppose $\exists\ B(t, \underline{x}, \underline{p}) : [0, T] \times \mathcal{X} \times \mathcal{P} \mapsto \mathbb{R}$ that is differentiable w.r.t. $t, \underline{x}$, such that

① $B(T, \underline{x}_T, \underline{p}) - B(0, \underline{x}_0, \underline{p}) > 0$

$$\forall\, (\underline{x}_T, \underline{x}_0, \underline{p}) \in \mathcal{X}_T \times \mathcal{X}_0 \times \mathcal{P}.$$

② $\left\langle \frac{\partial B}{\partial \underline{x}}, \underline{f} \right\rangle + \frac{\partial B}{\partial t} \leq 0 \quad \forall\, (t, \underline{x}, \underline{p}) \in [0, T] \times \mathcal{X} \times \mathcal{P}.$

Then the model (*) and its associated parameter set $\mathcal{P}$ are invalidated by $\{\mathcal{X}_0, \mathcal{X}_T, T\}$.

We refer the function $B(t, \underline{x}, \underline{p})$ as <u>Barrier certificate.</u>

Proof: (By contradiction)

Suppose, if possible, $\exists B(t, \underline{x}, \underline{p})$ satisfying ①, ②, while at the same time the model is valid.

(i.e.) $\exists \underline{p} \in \mathcal{P}$, $\underline{x}_0 \in \mathcal{X}_0$, $\underline{x}_T \in \mathcal{X}_T$ satisfying $\dot{\underline{x}} = f(t, \underline{x}, \underline{p})$

But ② says: $\underbrace{\frac{dB}{dt}}_{\text{Total derivative}} \leq 0 \quad \forall\, t \in [0, T]$

$$\Rightarrow B(T, \underline{x}_T, \underline{p}) \leq B(0, \underline{x}_0, \underline{p})$$

$\Rightarrow$ contradicts ①.

(Proved.)

Example #1:

Model:
$$\begin{cases} \dot{x}_1 = \underbrace{x_1 + 2x_2}_{= f_1(x_1, x_2)} \\ \dot{x}_2 = \underbrace{x_1 x_2 - 0.5 x_2^2}_{f_2(x_1, x_2)} \end{cases} \quad \underline{x} \in \mathbb{R}^2$$

Measurement data: $T = 1$, $\mathcal{X}_0 = [-1, 1]^2$

$\mathcal{X}_T = [-1, 1] \times [3, 5]$

Prove: Model is false/invalid.

Proof: Consider $B(\underline{x}) = -0.25 x_1^2 + x_2 - 2$

The zero level set of $B(\underline{x})$ provides a "barrier"

i.e., any trajectory starting in $\mathcal{X}_0$ can never cross this level set to reach $\mathcal{X}_T$

Why?
$$\frac{\partial B}{\partial x_1} \underbrace{\dot{x}_1}_{f_1} + \frac{\partial B}{\partial x_2} \underbrace{\dot{x}_2}_{f_2} + \cancel{\frac{\partial B}{\partial t}}^{\,0}$$

$$= -\frac{1}{2}(x_1^2 + x_2^2) \leq 0$$

$B(x_0) < 0 \quad \forall x_0 \in \mathcal{X}_0 = [-1, 1]^2$

$B(x_T) > 0 \quad \forall x_T \in \mathcal{X}_T = [-1, 1] \times [3, 5]$

Example: Model $\left\{ \dot{x} = -p x^3 \right\}$ $x \in \mathbb{R}$, $P = [0.5, 2]$.

Data: $x_0 = [0.85, 0.95]$, $x_T = [0.55, 0.65]$, $T = 4$.

Prove: Model is false.

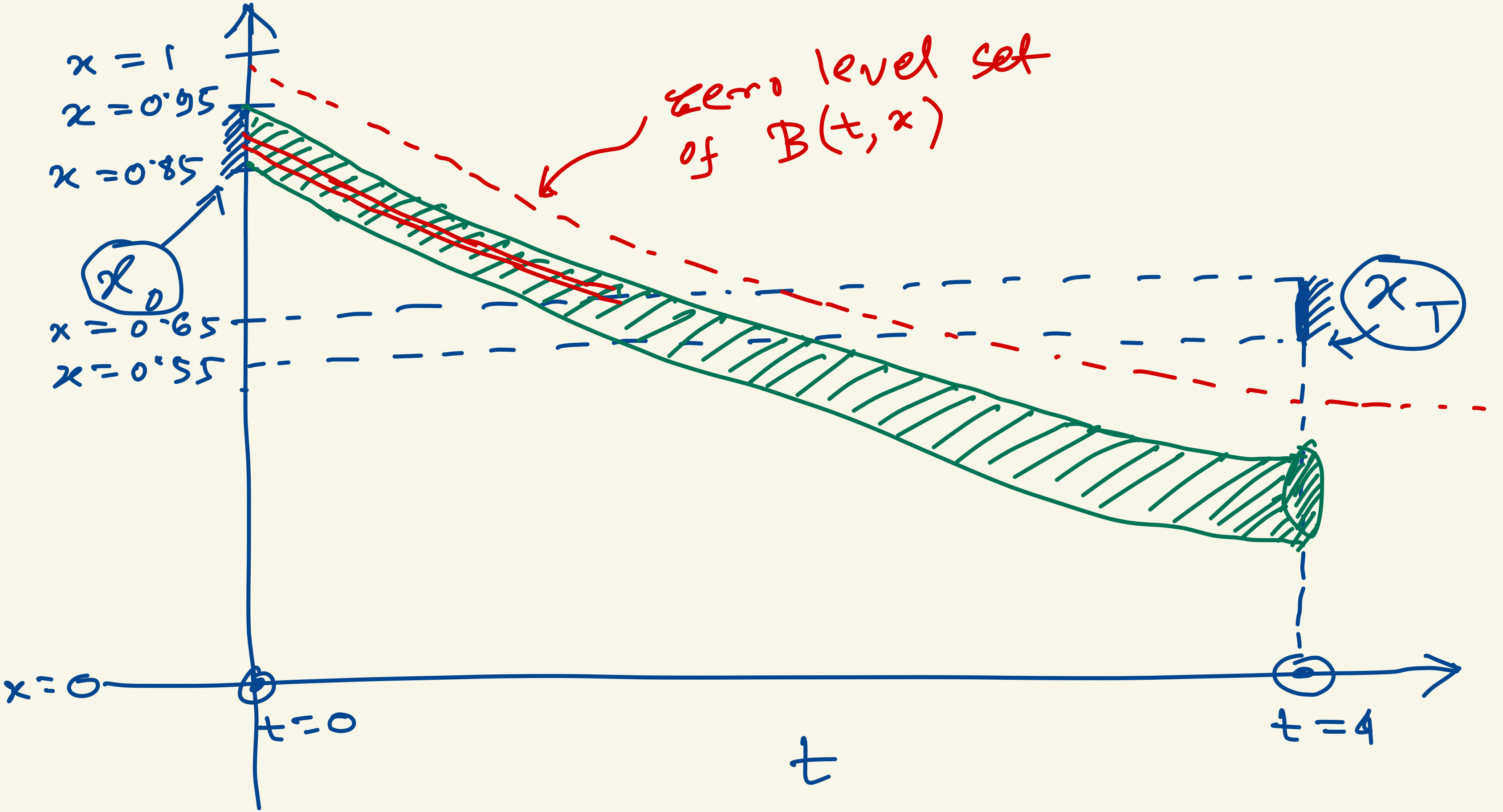
Proof: $B(t, x) = \underbrace{B_1(x)} + t \underbrace{B_2(x)}$

polynomials in x

Using SOS programming:

$$B_1(x) = (9.86)x^4 - (21.5)x^3 + (10.4)x^2 + (8.35)x$$

$$B_2(x) = (1.54)x^4 - (1.19)x^3 - (4.12)x^2 + (6.58)x - 1.78$$



x = 1
x = 0·95
x = 0·85
x₀
x = 0·65
x = 0·55
zero level set of B(t, x)
x_T
x = 0
t = 0
t = 4
t

Application # 2 : Using Lyapunov theory to estimate ROA (of $\underline{x}^* = \underline{0}$) :

ROA is the largest positively invariant set containing $\underline{x}^* = \underline{0}$.

Step 1: Get $V(\underline{x})$ such that $V(\cdot)$ is positive definite function

$V(0) = 0$,
$V(\cdot) > 0 \; \forall \underline{x} \neq 0$

Step 2: Compute $\dot{V} = \frac{d}{dt} V$

and check where $\dot{V} < 0$

Let us say, this happens $\forall \; \underline{x} \in \mathcal{D}$

(NO guarantee that $\mathcal{D}$ is positively invariant)

Step 3: Construct:

$$\Omega_c := \{ \underline{x} \in \mathbb{R}^n \mid V(\underline{x}) \leq c \}$$

such that ① Ω_c is compact
(closed and bounded)

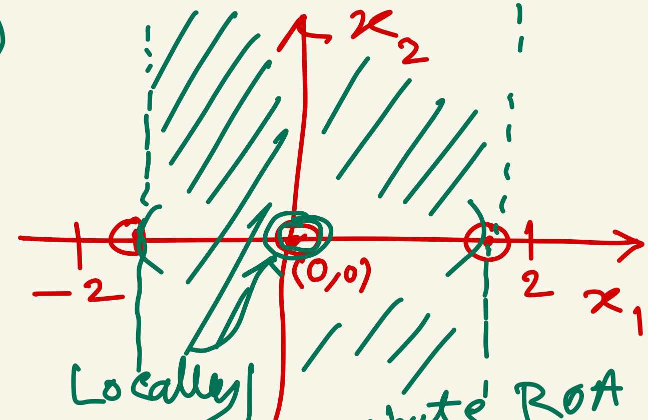
② $\Omega_c \subset \mathcal{D}$.

Then,
$$\boxed{\Omega_c \subseteq ROA}$$

Example 1 (for ROA):

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -x_1 + \frac{1}{3}x_1^3 - x_2$$

Notice that: $(0,0)$, $\underbrace{(\pm\sqrt{3},\, 0)}_{\text{Saddle points}}$



Locally A.S. } compute ROA for origin

$$V(\underline{x}) = \frac{1}{2}\underline{x}^T \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1 \end{bmatrix} \underline{x} + \int_0^{x_1} \left(y - \frac{1}{3}y^3\right) dy$$

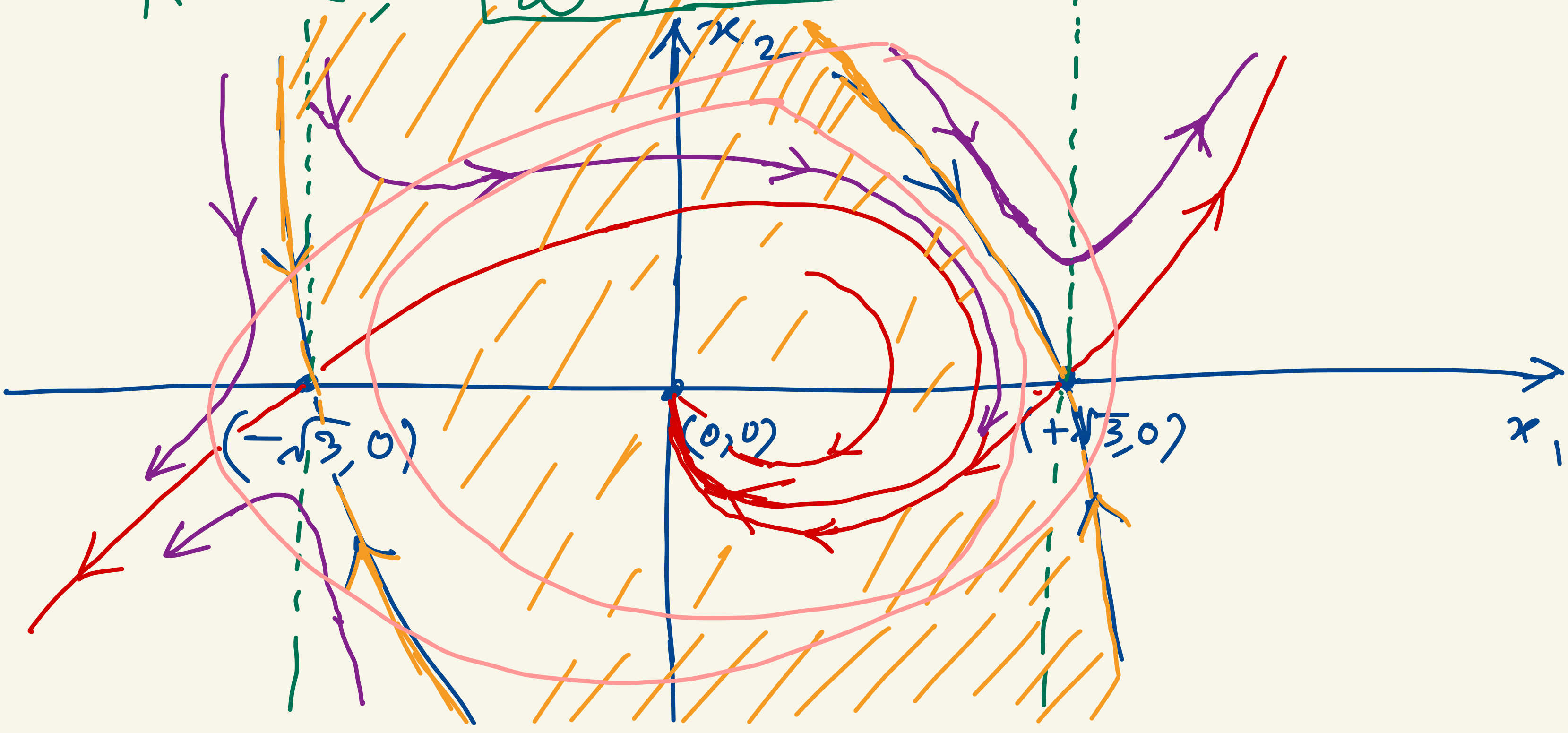
$$= \frac{3}{4}x_1^2 - \frac{1}{12}x_1^4 + \frac{1}{2}x_1 x_2 + \frac{1}{2}x_2^2 \quad \Big\} \; V(\underline{0}) = 0.$$

$$\dot{V} = \underline{-\frac{1}{2}x_1^2\left(1 - x_1^2/3\right) - \frac{1}{2}x_2^2}$$

If $\mathcal{D} := \{ \underline{x} \in \mathbb{R}^2 \mid -\sqrt{3} < x_1 < +\sqrt{3} \}$

then $V > 0$, and $\dot{V} < 0 \quad \forall \underline{x} \in \mathcal{D} \setminus \{\underline{0}\}$

However, $\boxed{\mathcal{D} \not\subset \text{ROA}}$



In our theorem, the set $\{\underline{x} \in \mathbb{R}^n \mid V(\underline{x}) \leq c\}$ for a particular choice of c, may have more than one connected components, there can only be one bounded component, and in <u>ROA approximation</u>, that is the component we work with.

<u>Example 2</u>: <u>(for approximating ROA)</u>

Consider
$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -x_1 - x_2 - (2x_2 + x_1)(1 - x_2^2)$$

(a) Use
$$V(x_1, x_2) = 5x_1^2 + 2x_1 x_2 + 2x_2^2$$
$$= (x_1 \quad x_2)\begin{bmatrix} 5 & 1 \\ 1 & 2 \end{bmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

to show $\underline{0}$ is A.S.

(b) Let $\mathcal{S} := \{\underline{x} \in \mathbb{R}^2 \mid V(\underline{x}) \leq 5\} \cap \{\underline{x} \in \mathbb{R}^2 \mid |x_2| \leq 1\}$.

Prove $\mathcal{S}$ is an estimate / inner approximation of ROA for $\underline{x}^* = \underline{0}$.

<u>Soln</u>: (a) $\dot{V} = \frac{\partial V}{\partial x_1} f_1 + \frac{\partial V}{\partial x_2} f_2$

$$= -2x_1^2 + 4x_1 x_2 - 2x_2^2 - 2(x_1 + 2x_2)^2 \underbrace{(1 - x_2^2)}$$

<u>For $|x_2| \leq 1$</u>,

we get $\dot{V}(\underline{x}) \leq \underbrace{-2(x_1^2 - 2x_1 x_2 + x_2^2)}$

$$= -2(x_1 - x_2)^2 \leq 0$$

Consider $\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}$ such that $\dot{V} = 0 \Rightarrow x_1(t) - x_2(t) \equiv 0 \quad \forall t$

$\Rightarrow \quad \dot{x}_1(t) - \dot{x}_2(t) \equiv 0$

$\Rightarrow \quad x_2 + x_1 + x_2 + (x_1 + 2x_2)(1 - x_2^2) \equiv 0$

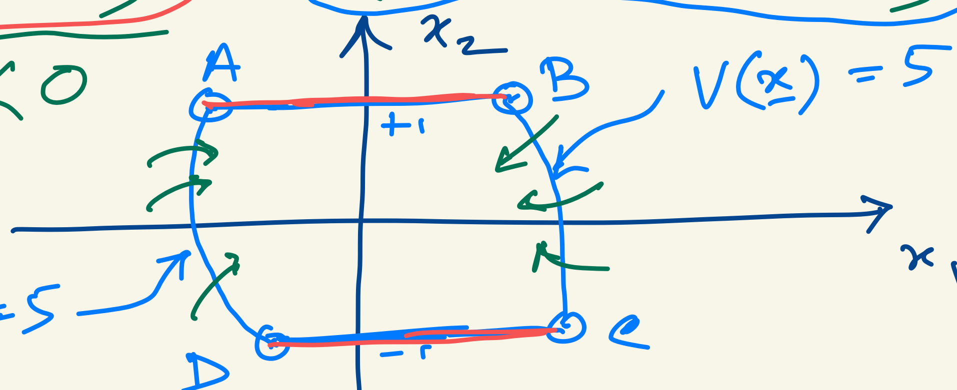
$\Rightarrow \quad 3x_2 \underline{(2 - x_2^2)} \equiv 0 \qquad (\because |x_2| \le 1)$

$\Rightarrow \quad \boxed{x_2(t) \equiv 0}$ by LaSalle Invariance, origin in A.S.

(b) We are given:

$$S := \underbrace{\{|x_2| \le 1\}}_{\dot{V} < 0} \cap \{V(x) \le \cancel{a}\ 5\}$$

This set is compact



Since BC & DA are part of the Lyapunov surface $V(\underline{x}) = 5$, and $\dot{V} \leq 0$, trajectories cannot leave $\mathcal{S}$ through BC & DA

To show: no trajectory can leave $\mathcal{S}$ through lines AB & CD:

$$\dot{V}\Big|_{|x_2|=1} = \langle \nabla V, f \rangle \Big|_{|x_2|=1} \leq 0$$

$\therefore$ No leakage through AB, BC, CD or DA.

$\therefore$ $\mathcal{S}$ is positively invariant

$\therefore$ $\mathcal{S} := \{ |x_2| \leq 1 \} \cap \{ V(\underline{x}) \leq 5 \}$ satisfies $\mathcal{S} \subset$ ROA.