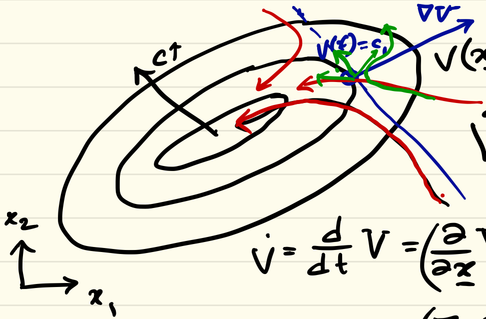
Lecture #7



$V(\underline{x}) = c, \quad c > 0$

$\dot{V} \le 0$

$\Updownarrow$

$$\dot{V} = \frac{d}{dt} V = \left(\frac{\partial V}{\partial \underline{x}}\right) \cdot \underline{f}(\underline{x})$$

$$= (\nabla V)^T \underline{f}$$

$$= \langle \nabla V, \underline{f} \rangle \le 0$$

S. (Stability)

A.S. (Asymp. Stability)

G.A.S. (Globally Asymp. Stable)

$V(\underline{x})$ is radially unbounded

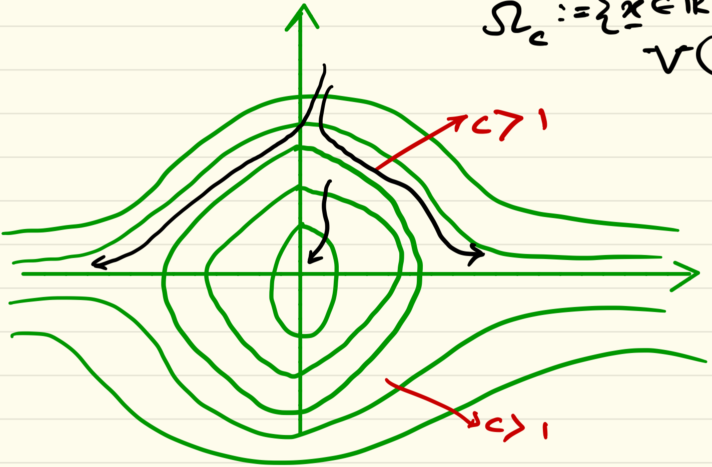
$\left(V(\underline{x}) \to \infty \text{ as } \underline{\underline{\|x\|_2 \to \infty}} \right)$

GAS $\Longrightarrow$ unique fixed pt.

$\not\Longleftarrow$ " (+A.S.)

<u>Example</u> : (Extra condⁿ; radial unbdd.ness)

$$V(\underline{x}) = \frac{x_1^2}{1+x_1^2} + x_2^2 \quad , \underline{x} \in \mathbb{R}^2$$



$\Omega_c := \{ \underline{x} \in \mathbb{R}^2 : V(\underline{x}) \le c \}$ ← For large c

→ NOT compact (unbounded set)

The curves $V(\underline{x}) = c$ are NOT closed.

Ω_c should be in the interior of some ball $B(0, r)$

$\Updownarrow$

c must satisfy $c < \inf_{\|x\|_2 \ge r} V(\underline{x})$

If $\ell := \lim_{r\to\infty} \inf_{\|x\|_2 \geq r} V(\underline{x}) < \infty$

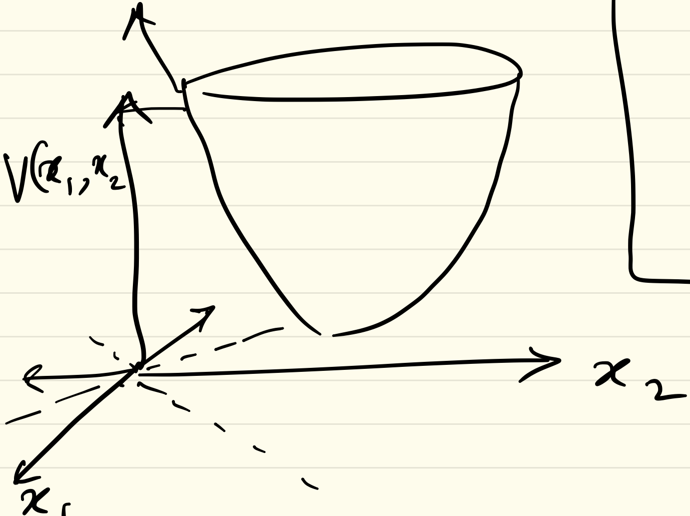
then Ω_c will be bdd. for $c < \ell$.

In our example,

$$\ell := \lim_{r\to\infty} \inf_{\|x\|_2 = r} \left(\frac{x_1^2}{1+x_1^2} + x_2^2 \right)$$

$$= \lim_{|x_1|\to\infty} \frac{x_1^2}{1+x_1^2} = 1$$

Ω_c is bdd. ONLY for $c < 1$



Barbashkin - Krasovskii Thm.

A.S. + $\underbrace{\lim_{\|x\|_2 \to \infty} V(\underline{x}) = \infty}_{}$

(radial unbdd.ness)

$\Updownarrow$

G.A.S.

Example: (Pendulum with damping)

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -\alpha \sin x_1 - \beta x_2$$

$$V(x_1, x_2) = \alpha (1 - \cos x_1) + \frac{1}{2} x_2^2$$

$$V(0,0) = 0$$

$$V(x_1, x_2) > 0 \quad \forall (x_1, x_2) \in (\mathbb{R} \times S^1) \setminus \{0,0\}$$

$$\begin{aligned}
\mathring{V} &= \langle \nabla V, f \rangle \\
&= \frac{\partial V}{\partial x_1} f_1 + \frac{\partial V}{\partial x_2} f_2 \\
&= (\alpha \sin x_1) x_2 + (x_2)(-\alpha \sin x_1 - \beta x_2) \\
&= \cancel{\alpha x_2 \sin x_1 - \alpha x_2 \sin x_1} - \beta x_2^2 \\
&= -\beta x_2^2 \leq 0
\end{aligned}$$

LaSalle Invariance Principle/Thm.

(Handling the case $\dot{V} \leq 0$)

Thm: Let $\Omega \subset \mathcal{D}$ be a compact set s.t. Ω is (+ve)-ly invariant in time set w.r.t. $\dot{\underline{x}} = \underline{f}(\underline{x})$.

Let $V : \mathcal{D} \mapsto \mathbb{R}$ be C^1 f^{n}. such that

① $\dot{V} \leq 0 \ \forall \ \underline{x} \in \Omega$

② Let $E \subset \Omega$ s.t. $\dot{V} = 0 \ \forall \ \underline{x} \in E$

③ Let M be the largest invariant set in E.

Then, every sol^{n} starting in Ω, approach M, as $t \to \infty$ ($\Leftrightarrow$ M is A.S.)

Corollary (Lasalle for fixed pts)

Let $\underline{x}^* = \underline{0}$ be a fixed pt.

Let $V : \mathcal{D} \mapsto \mathbb{R}_0^+$

① V is pos. def. ($V(\underline{0}) = 0$, $V(\underline{x} \neq \underline{0}) > 0$)

② $\dot{V} \leq 0$ in $\mathcal{D}$

③ $S := \{\underline{x} \in \mathcal{D} : \dot{V}(\underline{x}) = 0\}$ and suppose NO sol^{n} can stay identically in S other than the trivial sol^{n} $\underline{x}(t) = \underline{0}$; Then Origin is A.S.

Pendulum with damping

$$\dot{V} = -\beta x_2^2 \leq 0$$

$$\therefore S := \{(x_1, x_2) : x_2 = 0\}$$

Example (LaSalle Invariance for Limit Cycle (A.S.))

$$\dot{x}_1 = x_2 - x_1 \underline{(x_1^4 + 2x_2^2 - 10)}$$

$$\dot{x}_2 = -x_1^3 - 3x_2^5 \underline{(x_1^4 + 2x_2^2 - 10)}$$

Prove that the set $x_1^4 + 2x_2^2 = 10$ is invariant (Stable L.C.)

To show invariance:

$$\frac{d}{dt}(x_1^4 + 2x_2^2 - 10)$$

$$= -(4x_1^4 + 12x_2^6)\underbrace{(x_1^4 + 2x_2^2 - 10)}_{=0}$$

$= 0$ (on the set)

$\therefore$ Motion on the invariant set:

$$\left.\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1^3 \end{aligned}\right\}$$

Why A.S.

$$\underbrace{V := (x_1^4 + 2x_2^2 - 10)^2}_{\text{measures distance to L.C.}}$$

$$\dot V = \frac{\partial V}{\partial x_1} f_1 + \frac{\partial V}{\partial x_2} f_2$$

$$= -8\left(x_1^4 + 2x_2^2 - 10\right)^2 \left(x_1^4 + 3x_2^6\right)$$

$$\le 0$$

$$E := \{ \underline{x} \in \mathbb{R}^2 : \dot V = 0 \}$$

$$= \{ \underline{x} \in \mathbb{R}^2 : \text{either } x_1^4 + 2x_2^2 = 10 \text{ or } x_1^4 + 3x_2^6 = 0 \}$$

$$= E_1 \cup E_2$$

$$= \underbrace{\{(x_1, x_2) : x_1^4 + 2x_2^2 - 10 = 0\}}_{\text{L C.}} \cup \underbrace{\{(x_1, x_2) : x_1^4 + 3x_2^6 = 0\}}_{\text{iff } x_1 = 0,\ x_2 = 0 \text{ origin}}$$

What is M, $M \equiv E_1$

$\therefore$ M is $\underline{\text{A.S.}}$

Very Briefly: Chetaev's Thm. (used to prove $\underline{x}^* = \underline{0}$ is unstable)

Let $\mathcal{D} \subset \mathbb{R}^n$ be a domain that contains $\underline{x}^* = \underline{0}$.

Let $V : \mathcal{D} \mapsto \mathbb{R}$ be a C^1 fn such that

① $V(\underline{0}) = 0$

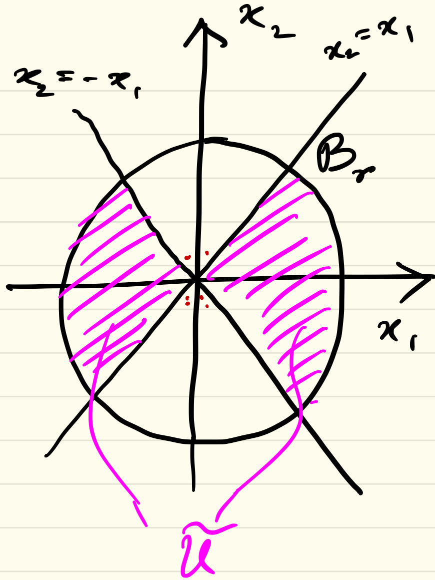
② $V(\underline{x}_0) > 0$ for some $\underline{x}_0$ with arbitrarily small $\|\underline{x}\|_2$

③ Choose $r > 0$ such that the ball $B_r := \{\underline{x} \in \mathbb{R}^n : \|\underline{x}\| \leq r\}$ is contained in $\mathcal{D}$, and let

$$\tilde{\mathcal{U}} := \{\underline{x} \in B_r : V(\underline{x}) > 0\}$$

Suppose $\dot{V} > 0 \;\forall\, \underline{x} \in \tilde{\mathcal{U}}$

Then $\underline{x}^* = \underline{0}$ is unstable!



$$V(x_1, x_2) = \frac{1}{2}(x_1^2 - x_2^2)$$

$$\dot{V} > 0 \;\forall\, \underline{x} \in \tilde{\mathcal{U}}$$

$\tilde{\mathcal{U}} \subset B_r$

- non-empty
- its bndry is the surface $V(\underline{x}) = 0$ & the sphere $\|\underline{x}\|_2 = r$

$V(0) = 0 \Rightarrow$ origin $\in \partial\tilde{\mathcal{U}}$

Exercise : (Chetaev's Thm)

$$\dot{x}_1 = x_1 + g_1(x_1, x_2)$$
$$\dot{x}_2 = -x_2 + g_2(x_1, x_2)$$

where $g_1(\cdot)$ & $g_2(\cdot)$ are locally Lip. in $\underline{x}$ such that $\begin{cases} |g_1(\underline{x})| \le k\|\underline{x}\|^2 \text{ \&} \\ |g_2(\underline{x})| \le k\|\underline{x}\|^2 \end{cases}$ in a nhbd. of the origin

$$\Rightarrow g_1(0,0) = g_2(0,0) = 0$$

$\Updownarrow$

$(0,0)$ is fixed pt.

Use the f^{n}:

$$V(x_1, x_2) = \frac{1}{2}(x_1^2 - x_2^2)$$

to show that origin is <u>Unstable</u>.