AMS 231: Nonlinear Control Theory: Winter 2018

Homework #4

Name: ....

Due: February 22, 2018

NOTE: Please show all the steps in your solution. Turn in a hard copy of your HW stapled with this as cover sheet with your name written in the above field. Please submit your HW in class on the due date.

Problem 1

State Space Computation of $\mathcal{H}_\infty$ Norm **(25+20+15+20 = 80 points)**

In class (Lecture 12), we derived that the worst-case $\mathcal{L}_2$ gain of a stable LTI system is

$$\gamma_{\text{LTI}} = \parallel G(j\omega) \parallel_\infty := \sup_{\omega \in \mathbb{R}} \sigma_{\max}\left(G(j\omega)\right), \qquad j := \sqrt{-1}, \qquad G(s) = C(sI - A)^{-1}B + D,$$

where $G(j\omega)$ is the associated transfer matrix. However, this frequency domain formula is inconvenient for computing γ_{LTI}, since it requires solving a nonlinear optimization problem in ω. The purpose of this exercise is to demonstrate an alternate method for computing γ_{LTI} using state space formulation, for the case $D = 0$ (no direct feedthrough).

(a) By specializing the $\mathcal{L}_2$ gain theorem for nonlinear systems (Lecture 12 notes, page 8 and 9) for $f(\underline{x}) = A\underline{x}$, $g(\underline{x}) = B$, $h(\underline{x}) = C\underline{x}$, and $V(\underline{x}) = \underline{x}^\top P \underline{x}$ where $P \succ 0$, prove that **if** the following optimization problem:

$$\begin{aligned} &\underset{\gamma, P}{\text{minimize}} \quad \gamma \\ &\text{subject to} \quad \gamma > 0, \quad P \succ 0, \quad PA + A^\top P + \frac{1}{\gamma^2} PBB^\top P + C^\top C \preceq 0, \end{aligned}$$

has unique solution, **then** the answer of this optimization problem gives the tightest upper bound of $\mathcal{L}_2$ gain γ_{LTI}. (In fact, when the triple (A, B, C) is minimal, meaning both controllable and observable, then the answer of this optimization problem equals γ_{LTI}, and hence equals $\parallel G(j\omega) \parallel_\infty$. But you can ignore this detail).

(b) At first glance, it may seem that the optimization problem in part (a) is nonlinear in both variables: scalar γ and matrix P, due to the last inequality constraint. However, this difficulty can be overcome via the following lemma.

Lemma: Consider real square matrices Q, R, S with Q and R symmetric. The linear matrix inequality $\begin{bmatrix} Q & S \\ S^\top & R \end{bmatrix} \succeq 0$ is equivalent to (if and only if) $R \succ 0$ and $Q - SR^{-1}S^\top \succeq 0$.

Prove this lemma.

(c) Using the lemma in part (b), and introducing $\sigma := \gamma^2$, show that the optimization problem derived in part (a) is equivalent to the following optimization problem:

$$\begin{aligned} &\underset{\sigma, P}{\text{minimize}} \quad \sigma \\ &\text{subject to} \quad \sigma > 0, \quad P \succ 0, \quad \begin{bmatrix} A^\top P + PA + C^\top C & PB \\ B^\top P & -\sigma I \end{bmatrix} \preceq 0, \end{aligned}$$

which is linear in both variables σ and P. Here I denotes the identity matrix of appropriate dimension.

(d) The type of optimization problem derived in part (c) is called semi-definite programming (SDP) problem that minimizes linear objective subject to linear matrix inequalities. SDPs are convex optimization problems, and can be solved efficiently via software like `cvx` in `MATLAB`.

Download `cvx` from `http://cvxr.com/cvx/download/` and follow installation instructions in `http://cvxr.com/cvx/doc/install.html`. To understand how to specify an optimization problem in `cvx`, you may want to take a look at: `http://cvxr.com/cvx/examples/`

Then write a `MATLAB` code to compute the $\mathcal{H}_\infty$ norm of the following stable, controllable and observable linear system (see partial code) in two ways: by using `cvx` to solve the optimization in part (c), and by using `MATLAB` command `norm(sys,inf)` to solve the frequency domain optimization problem. Report the $\mathcal{H}_\infty$ norms computed from the two methods, and submit your code.

Partial MATLAB Code

```
clear; clc;

A = [-2 1 0 0;
     -6 -6 0 3;
     0 0 -1 1;
     0 0 0 -2];
B = [0;1;1;2];
C = [0 6 2 -8; 2 -3 4 5];
```

```
D = 0;

sys = ss(A,B,C,D);

eig(A) % is stable?
rank(ctrb(sys))==length(A) % is controllable?
rank(obsv(sys))==length(A) % is observable?

dim=size(B); n_x = dim(1); n_u = dim(2);
```

Problem 2

Input-to-State Stability (ISS) **(**$4\times 5 = 20$ **points)**

Consider the scalar nonlinear systems

$$\text{(a)}\,\dot{x} = -(1+u)x^3, \quad \text{(b)}\,\dot{x} = -(1+u)x^3 - x^5, \quad \text{(c)}\,\dot{x} = -x + x^2u, \quad \text{(d)}\,\dot{x} = x - x^3 + u.$$

Which systems are input-to-state stable (ISS) and which are not? Give reasons.