Lec. 7 (04/21/2020)

Construction of Lyapunov Functions :
(How to engineer)

Three methods
well-known in literature

Method # 1	Method # 2	Method # 3
(Variable Gradient Method)	(Krasovskii Method)	SOS (Sum-of-suqeares method)

Method # 1 : Variable Gradient Method :

To construct $V(\underline{x})$ for $\dot{\underline{x}} = \underline{f}(\underline{x})$

Let $\underbrace{\underline{g}(\underline{x})} := \nabla_{\underline{x}} V \equiv \dfrac{\partial V}{\partial \underline{x}}$

we know: $$\dot{V} = \left\langle \frac{\partial V}{\partial \underline{x}}, \underline{f} \right\rangle = \langle \underline{g}, \underline{f} \rangle$$

Idea: Instead of engineering V (scalar valued $f^{\underline{n}}$), let us instead engineer $\underline{g}$ (vector valued $f^{\underline{n}}$)

Claim: If $\underline{g}(\underline{x})$ is gradient of a scalar $f^{\underline{n}}$ then its Jacobian $\left[\dfrac{\partial \underline{g}}{\partial \underline{x}}\right]$ is a symmetric matrix.

$$\Leftrightarrow \quad \frac{\partial g_i}{\partial x_j} = \frac{\partial g_j}{\partial x_i} \quad \forall\ i,j = 1, \ldots, n$$

Why? If $\underline{g} = \nabla_{\underline{x}} V$, then $\left[\frac{\partial \underline{g}}{\partial \underline{x}}\right] = \text{Hess}(V)$
$\Rightarrow$ Symmetric.

Construct $\underline{g}$ such that ① $\left[\frac{\partial \underline{g}}{\partial \underline{x}}\right]$ is symmetric

② $\langle \underline{g}, \underline{f} \rangle < 0$.

Then,
$$V(\underline{x}) = \oint_{\underline{0}}^{\underline{x}} \langle \underline{g}(\underline{y}), d\underline{y} \rangle$$

Integral taken over any path joining $\underline{0}$ to $\underline{x}$

(line integral of gradient vector field is independent of the path)

For example, if we take integral along the co-ordinate axes, then

$$V(\underline{x}) = \int_0^{x_1} g_1(y_1, 0, \ldots, 0)\, dy_1 + \int_0^{x_2} g_2(x_1, y_2, 0, \ldots, 0)\, dy_2$$

$$+ \ldots + \int_0^{x_n} g_n(x_1, x_2, \ldots, x_{n-1}, y_n)\, dy_n$$

By leaving some parameters in $\underline{g}(\underline{x})$ undetermined, try to choose the parameters such that $V(\underline{x})$ becomes positive def. function.

<u>Example:</u>

$$\boxed{\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= -h(x_1) - a x_2\end{aligned}}$$

spl. case of this
→ pendulum
→ HW 2, p 2

$a = 1$

$\boxed{h(x_1) = x_1 - x_1^3}$

$a > 0$, $h(\cdot)$ is locally Lipschitz

$h(0) = 0$.

$\boxed{y\,h(y) > 0}$ $\forall \{y \neq 0 \mid -b < y < c\}$

$b, c > 0$.

we want ① $\boxed{\dfrac{\partial g_1}{\partial x_2} = \dfrac{\partial g_2}{\partial x_1}}$

② $\dot{V} = \langle \underline{g}, \underline{f} \rangle$

$= g_1(\underline{x})x_2 - g_2(\underline{x})\left(h(x_1) + a(x_2)\right)$

$< 0 \quad \forall \; \underline{x} \neq \underline{0}$.

③ $V(\underline{x}) = \int_0^{\underline{x}} \langle g(\underline{y}), d\underline{y} \rangle > 0 \quad \forall \; \underline{x} \neq 0$.

Try: $g(\underline{x}) = \begin{pmatrix} \alpha(\underline{x})\, x_1 + \beta(\underline{x})\, x_2 \\ \gamma(\underline{x})\, x_1 + \delta(\underline{x})\, x_2 \end{pmatrix} = \begin{pmatrix} g_1(x_1, x_2) \\ g_2(x_1, x_2) \end{pmatrix}$

$$\frac{\partial g_1}{\partial x_2} = \frac{\partial g_2}{\partial x_1}$$

where $\alpha(\cdot)$, $\beta(\cdot)$, $\gamma(\cdot)$, $\delta(\cdot)$ are to be determined.

From symmetry requirement:

$$\beta(\underline{x}) + \frac{\partial \alpha}{\partial x_2} x_1 + \frac{\partial \beta}{\partial x_2} x_2 = \gamma(\underline{x}) + \frac{\partial \gamma}{\partial x_1} x_1 + \frac{\partial \delta}{\partial x_1} x_2$$

From $\dot{V}$:

$$\dot{V}(\underline{x}) = \alpha(\underline{x}) x_1 x_2 + \beta(\underline{x}) x_2^2 - a\gamma(\underline{x}) x_1 x_2 - a\delta(\underline{x}) x_2^2 - \delta(\underline{x}) x_2 h(x_1) - \gamma(\underline{x}) x_1 h(x_1)$$

Let us cancel the cross terms:

To cancel the cross terms:

$$\boxed{\alpha(\underline{x})x_1 - a\gamma(\underline{x})x_1 - \delta(\underline{x})h(x_1) = 0}$$

$$\Rightarrow \dot{V} = -[a\delta(\underline{x}) - \beta(\underline{x})]x_2^2 - \gamma(\underline{x})x_1 h(x_1)$$

To simplify further, fix $\underbrace{\beta(\underline{x}) = \beta,\ \gamma(\underline{x}) = \gamma,\ \delta(\underline{x}) = \delta}_{\text{constants}}$:

$$\alpha(\underline{x}) = a\gamma + \frac{\delta h(x_1)}{x_1} \Rightarrow \underbrace{\alpha(\underline{x}) \equiv \alpha(x_1)}_{f^n \text{ of } x_1 \text{ alone}}$$

$$\Rightarrow \boxed{\frac{\partial\alpha}{\partial x_2} = 0}$$

Then, simplify the symmetry requirement condition:

$$\beta + \cancel{\frac{\partial\alpha}{\partial x_2}}^{\,0} x_1 + \cancel{\frac{\partial\beta}{\partial x_2}}^{\,0} x_2 = \cancel{\gamma(\underline{x})}^{\,\gamma} + \cancel{\frac{\partial\gamma}{\partial x_1}}^{\,0} x_1 + \cancel{\frac{\partial\delta}{\partial x_1}}^{\,0} x_2$$

$$\Rightarrow \beta + 0 + 0 = \gamma + 0 + 0 \Rightarrow \boxed{\beta = \gamma}$$

Therefore, $g(\underline{x}) = \begin{bmatrix} a\gamma x_1 + \delta h(x_1) + \gamma x_2 \\ \gamma x_1 + \delta x_2 \end{bmatrix}$

Now, $V(\underline{x}) = \int_0^{x_1} g_1(y_1, 0)\, dy_1 + \int_0^{x_2} g_2(x_1, y_2)\, dy_2$

$$= \int_0^{x_1} \left(a\gamma y_1 + \delta h(y_1)\right) dy_1 + \int_0^{x_2} \left(\gamma x_1 + \delta y_2\right) dy_2$$

$$= \frac{1}{2} \underline{x}^T P \underline{x} + \delta \int_0^{x_1} h(y)\, dy,$$ where $P := \begin{bmatrix} a\gamma & \gamma \\ \gamma & \delta \end{bmatrix}$

For $V(\cdot)$ to be positive definite, we need:

$$a\gamma > 0, \quad a\gamma\delta - \gamma^2 > 0 \Leftrightarrow \gamma(a\delta - \gamma) > 0$$

Choose $\gamma > 0 \Rightarrow \boxed{a\delta > \gamma > 0}$.

$\therefore$ Choosing $\delta > 0$, $0 < \gamma < a\delta$, ensures $V(\underline{x})$ is a pos. def. function.

and $\dot{V}$ is a neg. def. function.

For example, taking $\gamma = \kappa a \delta$, for some $0 < \kappa < 1$,

gives
$$V(\underline{x}) = \frac{\delta}{2}\underline{x}^T \begin{bmatrix} \kappa a^2 & \kappa a \\ \kappa a & 1 \end{bmatrix} \underline{x} + \delta \int_0^{x_1} h(y)\,dy$$

as a Lyapunov function over domain

$$\mathcal{D} := \{\underline{x} \in \mathbb{R}^2 \mid -b < x_1 < c\}$$

Method # 2 : Krasovskii's Method :

Theorem : Let $\underline{x}^* = \underline{0}$ be a fixed point for $\dot{\underline{x}} = \underline{f}(\underline{x})$. If $\exists$ $P > 0$ $(\Leftrightarrow \underline{x}^T P \underline{x} > 0 \ \forall \ \underline{x} \neq \underline{0})$

such that $$\boxed{\left[\frac{\partial \underline{f}}{\partial \underline{x}}\right]^T \underline{\underline{P}} + \underline{\underline{P}}\left[\frac{\partial \underline{f}}{\partial \underline{x}}\right]} < 0 \quad (= -Q \text{ where } Q > 0)$$

constant matrix

$\forall \ \underline{x} \in \mathcal{D} \subseteq \mathbb{R}^n$

then $\underline{x}^* = \underline{0}$ is A.S, with $V(\underline{x}) = (\underline{f}(\underline{x}))^T P \underline{f}(\underline{x})$

- If in addition, $\mathcal{D} \equiv \mathbb{R}^n$, and $V(\underline{x})$ is radially unbounded, then $\underline{x}^* = \underline{0}$ is GAS.

Proof: (Sketch) $\quad \underline{V(\underline{x}) = (\underline{f}(\underline{x}))^T P \, \underline{f}(\underline{x})}$

$V(\underline{x}) = 0$ iff $\underline{f}(\underline{x}) = \underline{0} \Leftrightarrow \underline{x}^* = \underline{0}$

$V(\cdot) > 0$ for $\underline{x} \neq \underline{0}$

$$\dot{V} = \langle \nabla V, \underline{f} \rangle$$

$$= 2 (\underline{f}(\underline{x}))^T \underbrace{\left\{ \left(\frac{\partial \underline{f}}{\partial \underline{x}}\right)^T P + P \left(\frac{\partial f}{\partial \underline{x}}\right) \right\}} \underline{f}(\underline{x})$$

If < 0

then $\dot{V} < 0 \ \forall \ \underline{x} : \underline{f}(\underline{x}) = \underline{0} \Leftrightarrow \underline{x} \neq \underline{x}^*$

<u>Example:</u>

$(P = ??)$

(Try $P = I$)

$$\left.\begin{aligned} \dot{x}_1 &= -7x_1 + 4x_2 \\ \dot{x}_2 &= x_1 - x_2 - x_2^5 \end{aligned}\right\}$$

$$V(\underline{x}) = f^T \cancel{P} f \quad (P \to I)$$

$$= f^T f$$

$$= (-7x_1 + 4x_2)^2 + \underbrace{(x_1 - x_2 - x_2^5)^2}_{\text{Radially unbounded.}} \geq 0$$

<u>Jacobian:</u>

$$\frac{\partial \underline{f}}{\partial \underline{x}} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix}$$

$$= \begin{bmatrix} -7 & 4 \\ 1 & (-1 - 5x_2^4) \end{bmatrix}$$

$$\left[\frac{\partial f}{\partial x}\right]^T \cancel{P}\, I + \cancel{P}\, I \left[\frac{\partial f}{\partial x}\right]$$

$$= \begin{bmatrix} -7 & 4 \\ 1 & (-1-5x_2^4) \end{bmatrix}^T + \begin{bmatrix} -7 & 4 \\ 1 & (-1-5x_2^4) \end{bmatrix}$$

$$\underbrace{= - \underbrace{\begin{bmatrix} 14 & -5 \\ -5 & 2(1+5x_2^4) \end{bmatrix}}_{>0}}_{<0}$$

$\boxed{14>0}$

$\det(\cdot) = 3 + 140 x_2^4 > 0$

$\therefore\ \underline{x}^* = \underline{0}$ is GAS.

Method # 3: Sum-of-squares Polynomials (SOS) / SOS programming / optimization

Lyapunov style Theorems ask to establish

① $V(\underline{x}) > 0 \quad \forall \; \underline{x} \neq \underline{0}$

② $-\dot{V}(\underline{x}) > 0 \quad \forall \; \underline{x} \neq \underline{0}$.

If LTI then LMI: $P > 0$, $A^T P + PA < 0$

$\underline{f}(\underline{x}) = A\underline{x}$

$V(\underline{x}) = \underline{x}^T P \, \underline{x}$

<u>Idea:</u> Search $V(\underline{x})$ over non-negative polynomials

<u>Polynomial</u>

is a linear combination of monomials:

<u>Monomial</u>

Product of power of variables: $(x^2 y z^3)$

<u>Example:</u>

$$p(x_1, x_2) = x_1^2 - 2x_1x_2^2 + 2x_2^4 + 2x_1^3x_2 - 7x_2 + 8$$

polynomial of degree 4

<u>SOS polynomial</u>: A polynomial $p(\underline{x})$ is SOS if $\exists$ other polynomials $g_1(\underline{x}), g_2(\underline{x}), \ldots, g_r(\underline{x})$ such that

$$p(\underline{x}) = \sum_{i=1}^{r} (g_i(\underline{x}))^2$$

Fact: All SOS polynomials are $\geq 0 \;\; \forall \underline{x} \in \mathbb{R}^n$

Converse: Are all ≥ 0 polynomials SOS? No

Converse is NOT true:

Counter-example: (Motzkin Polynomial)

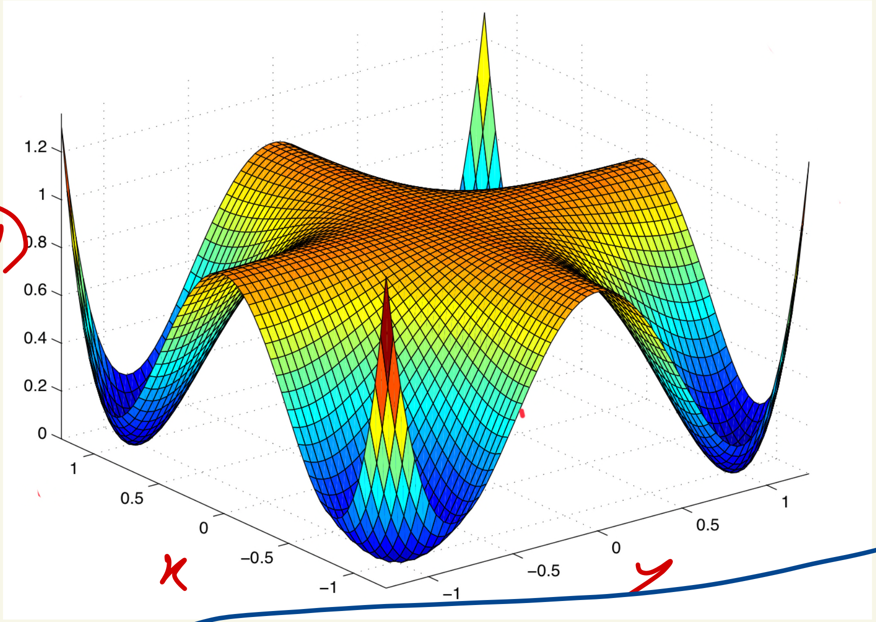
$$p(x,y) = \underline{x^2y^4 + x^4y^2 + 1} - 3x^2y^2$$

Claim #1: $p \geq 0 \;\; \forall (x,y) \in \mathbb{R}^2$.

Proof:

$$\underbrace{A.M.}_{\text{arithmetic mean}} \geq \underbrace{G.M}_{\text{Geometric mean}} \qquad \left(\frac{x_1+x_2+x_3}{3} \geq (x_1x_2x_3)^{1/3} \text{ for all } x_1, x_2, x_3 \geq 0 \right)$$

$$\frac{x^2y^4 + x^4y^2 + 1}{3} \geq \left(x^2y^4 \cdot x^4y^2 \cdot 1\right)^{1/3}$$



$$p(x,y) = x^2y^4 + x^4y^2 + 1 - 3x^2y^2$$

Claim #2: $p(x,y)$ is NOT SOS.

$$\underbrace{(x^2+y^2)^2}_{h^2}\, p(x,y) = x^2y^2(x^2+y^2+1)(x^2+y^2-2)^2 + (x^2-y^2)^2$$

Hilbert's 17th Problem : (1900, Intl. Congress of Mathematics)

$$p(\underline{x}) \geq 0 \overset{??}{\iff} \text{SOS}$$

Artin (1927)

For all polynomials $p(\underline{x}) \geq 0$, there exists polynomial $h(\underline{x})$ such that $h^2 p$ is SOS.

$$p = \frac{g_1^2 + g_2^2 + \ldots + g_r^2}{h^2}$$

$$= \frac{\text{SOS}}{(\text{poly})^2} = \left(\frac{g_1}{h}\right)^2 + \ldots + \left(\frac{g_r}{h}\right)^2$$

$= \text{SOS}$ of rationals

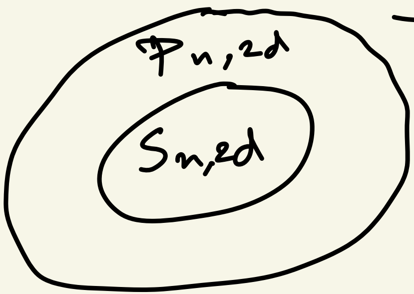
Example:
$$p(x,y) = \underbrace{2x^4 + 2x^3y - x^2y^2 + 5y^4}$$ is SOS

Why: $$= \underbrace{\frac{1}{2}\left[(2x^2 - 3y^2 + xy)^2 + (y^2 + 3xy)^2\right]}$$

Question: How to get SOS decomposition (even if it exists)

Definitions: $P_{n,2d}$: All ≥ 0 polynomials in $\underline{x} \in \mathbb{R}^n$ of degree $2d$

$S_{n,2d}$: All SOS polynomials in $\underline{x} \in \mathbb{R}^n$ of degree $2d$



Theorem (Hilbert, 1888)

$P_{n,2d} = S_{n,2d}$ if and only if
- either $n=1$ (univariate polynomials)
- or $2d=2$ (quadratic polynomials)
- or $(n,2d)=(2,4)$ (bivariate quartic)

Let $[\underline{x}]_d$ be the column of monomials of degree $\leq d$

i.e., $[\underline{x}]_d = [1, x_1, x_2, \ldots, x_n, x_1^2, x_1x_2, x_1x_3, \ldots, \ldots, x_n^d]^T$

Fact: Any polynomial of degree $\leq 2d$ can be written as

$$p(\underline{x}) = [\underline{x}]_d^T \, M \, [\underline{x}]_d,$$

$M = M^T$ symmetric

To be continued (next lecture)