Lecture #10

Construction of Lyap. f^n

① Variable Gradient Method (uploaded)

② Krasovskii's Method :

Thm: Let $\underline{x}^* = \underline{0}$ be a fixed pt. for $\dot{\underline{x}} = \underline{f}(\underline{x})$.

If $\exists\ P > 0$ ($\Leftrightarrow \underline{x}^T P \underline{x} > 0\ \forall\, \underline{x} \neq \underline{0}$), s.t.

$$\left[\frac{\partial \underline{f}}{\partial \underline{x}}\right]^T P + P \left[\frac{\partial \underline{f}}{\partial \underline{x}}\right] < 0$$

($= -Q$ where $Q > 0$)

$\forall\ \underline{x} \in \mathcal{D} \subseteq \mathbb{R}^n$

then $\underline{x}^* = \underline{0}$ is A.S.

- If in addition, $\mathcal{D} = \mathbb{R}^n$, & $V(\underline{x})$ is rad. unbdd. then $\underline{x}^* = \underline{0}$ is GAS

Proof (sketch)

$$V(\underline{x}) = (\underline{f}(\underline{x}))^T P (\underline{f}(\underline{x}))$$

$V(\underline{x}) = 0$ iff $\underline{f}(\underline{x}) = 0 \Leftrightarrow \underline{x}^* = 0$

$$\dot{V} = \langle \nabla V, \underline{f}(\underline{x}) \rangle$$

$$= 2\left(\underline{f}(\underline{x})\right)^T \underbrace{\left\{ \left(\frac{\partial \underline{f}}{\partial \underline{x}}\right)^T P + P\left(\frac{\partial \underline{f}}{\partial \underline{x}}\right) \right\}}_{< 0} \underline{f}(\underline{x})$$

$$< 0 \ \forall x : \underline{f}(\underline{x}) \neq 0 \iff \underline{x} \neq \underline{x}^*$$

<u>**Example**</u>: (Try $P = I$)

$$\left.\begin{aligned} \dot{x}_1 &= -7x_1 + 4x_2 \\ \dot{x}_2 &= x_1 - x_2 - x_2^5 \end{aligned}\right\} \quad V(\underline{x}) = f^T f$$

$$= (-7x_1 + 4x_2)^2 + (x_1 - x_2 - x_2^5)^2$$

≥ 0 & Rad. unbdd.

Jacobian:

$$\frac{\partial \underline{f}}{\partial \underline{x}} = \begin{bmatrix} -7 & 4 \\ 1 & -1 - 5x_2^4 \end{bmatrix}$$

$$\underbrace{\left[\frac{\partial \underline{f}}{\partial \underline{x}}\right]^T + \left[\frac{\partial \underline{f}}{\partial \underline{x}}\right]}_{<0} = \underbrace{- \begin{bmatrix} 14 & -5 \\ -5 & 2(1+5x_2^4) \end{bmatrix}}_{>0}$$

$\therefore \ \underline{x}^* = \underline{0}$ is GAS

$tr(\cdot) > 0$

$det(\cdot) = 3 + 140 x_2^4 > 0$

SOS (Sum-of-squares) Polynomials

(SOS Programming / Optimization)

Lyap. style Thm.s ask to establish non-negativity of f^{ns}:

$$\left\{ \begin{array}{ll} V(\underline{x}) > 0 & \forall \underline{x} \neq \underline{0} \\ -\dot{V} > 0 & \forall \underline{x} \neq \underline{0} \end{array} \right.$$

If $\underline{f}(\underline{x}) = A\underline{x}$

$V(\underline{x}) = \underline{x}^T P \underline{x}$

$$\Updownarrow$$

$$\boxed{\begin{array}{l} P > 0 \\ A^T P + PA < 0 \end{array}} \quad \text{(LMI)}$$

Idea: Search $V(\underline{x})$ over non-neg. polynomials

Polynomial: is a linear combination of monomials

Monomial: Product of power of variables ($x^2 y z^3$)

$$p(x_1, x_2) = x_1^2 - 2x_1x_2^2 + 2x_2^4 + 2x_1^3x_2 - 7x_2 + 8$$

polynomial of deg. 4

SOS polynomial: A polynomial $p(\underline{x})$ is SOS if $\exists$ other polynomials $g_1(\underline{x}), g_2(\underline{x}), \ldots, g_r(\underline{x})$ s.t.

$$p(\underline{x}) = \sum_{i=1}^{r} (g_i(\underline{x}))^2$$

Fact: All SOS polynomials are $\geq 0 \ \forall \underline{x} \in \mathbb{R}^n$

Converse is NOT true :

Counter-example : (Motzkin Polynomial)

$$p(x,y) = x^2y^4 + x^4y^2 + 1 - 3x^2y^2$$

Claim #1 : $p(x,y) \geq 0 \quad \forall \; x, y \in \mathbb{R}^2$

Proof : AM $\geq$ GM for $(x^2$

$$\frac{x^2y^4 + x^4y^2 + 1}{3} \geqslant \left(x^2y^4 \cdot x^4y^2 \cdot 1\right)^{1/3}$$

Claim #2 : $p(x,y)$ is NOT SOS

Introduce a non. neg. factor (x^2+y^2+1)

Now observe :

$$(x^2+y^2+1)\, p(x,y) = \text{SOS}$$

$$= (x^2y - y)^2 + (xy^2 - x)^2 + (x^2y^2 - 1)^2 + \tfrac{1}{4}(xy^3 - x^3y)^2 + \tfrac{3}{4}(xy^3 + x^3y - 2xy)^2$$

$\therefore$ $p(x,y)$ is NOT SOS.

Example :

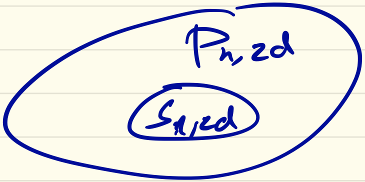
$p(x,y) = 2x^4 + 2x^3y - x^2y^2 + 5y^4$ is SOS

why : $p(x,y) = \frac{1}{2}\left[(2x^2 - 3y^2 + xy)^2 + (y^2+3xy)^2\right]$

Q. How to get SOS decomposition (even if it exists)

Defⁿ : $P_{n,2d}$: All pos. def. polynomials in $\underline{x} \in \mathbb{R}^n$ of deg. $2d$

$S_{n,2d}$: All SOS polynomials in $\underline{x} \in \mathbb{R}^n$ of deg. $2d$



Thm (Hilbert, 1888)

$P_{n,2d} = S_{n,2d}$ iff $\begin{cases} \text{either } n=1 \text{ (univariate polynomial)} \\ \text{or } d=2 \text{ (quadratic poly)} \\ \text{or } (n,2d)=(2,4) \text{ (bivariate quartic)} \end{cases}$

Let $[x]_d$ be the column of monomials of deg. $\leq d$

(i.e.) $[\underline{x}]_d := [1, x_1, x_2, \ldots, x_n, x_1^2, x_1x_2, x_1x_3, \ldots, \ldots x_n^d]^T$

Fact: Any polynomial of deg. $\leq 2d$ can be written as

$$p(\underline{x}) = [\underline{x}]_d^T M [\underline{x}]_d$$

for some $M = M^T$.

e.g. $p(x,y) = 2x^2y^2 \quad (d=4,\ n=2)$

$$[\underline{x}]_{d=2} = \begin{pmatrix} 1 \\ x \\ y \\ x^2 \\ xy \\ y^2 \end{pmatrix} ; \quad \text{claim } p(x,y) = 2x^2y^2 = (\cdot)^T M (\cdot)$$

But $M = M^T$ is NOT unique, e.g.

$$M_1 = \underbrace{\begin{bmatrix} \text{zeros}(4,6) \\ 0 \ 0 \ 0 \ 0 \ 2 \ 0 \\ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \end{bmatrix}}_{6\times 6} \quad \text{will do}$$

Also

$$M_2 = \begin{bmatrix} \boxed{\text{zeros}(5,5)} & \begin{matrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{matrix} \\ 0 \ 0 \ 0 \ 1 \ 0 & 0 \end{bmatrix} \quad \text{will also do}$$

<u>Thm</u>: A polynomial $p(\underline{x})$ with deg. $2d$ is SOS iff $\exists\, M \succeq 0$ s.t.

$$p(\underline{x}) = [\underline{x}]_d^T \, M \, [\underline{x}]_d$$

<u>Example</u> $p(x,y) = 2x^4 + 2x^3y - x^2y^2 + 5y^4$

$$2d = 4 \Leftrightarrow d = 2$$

$$[\underline{x}]_d = \begin{pmatrix} x^2 \\ xy \\ y^2 \end{pmatrix},$$

$$p(x,y) = \begin{pmatrix} x^2 \\ xy \\ y^2 \end{pmatrix}^T \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{12} & m_{22} & m_{23} \\ m_{13} & m_{23} & m_{33} \end{bmatrix} \begin{pmatrix} x^2 \\ xy \\ y^2 \end{pmatrix}$$

$$= m_{11}x^4 + 2m_{12}x^3y + (m_{22} + 2m_{13})x^2y^2 + 2m_{23}xy^3 + m_{33}y^4$$

$\therefore p(x,y)$ is SOS iff

① $M \succeq 0$

② $m_{11} = 2$

$2m_{12} = 2$

$2m_{13} + m_{22} = -1$

$2m_{23} = 0$

$m_{33} = 5$

SDP problem (Semi-definite Programming)

$$\min_{X} \ tr(C^T X)$$

$$\left.\begin{aligned} \text{s.t.} \quad & X \succeq 0 \\ & tr(A_i X) = b_i, \quad i = 1, \ldots, m \end{aligned}\right\}$$

For Lyap. f^n (for poly. vector field)

we can set ① V is SOS

② $-\dot{V}$ is also SOS

$$= -\langle \nabla V, \underbrace{\underline{f}(\underline{x})}_{\text{needs to be polynomial}} \rangle$$

SOSTOOLS

MATLAB toolbox (free)

requires SeDuMi (an SDP solver)