1.a. We consider the state space of the model of a car with n trailers defined as

$$\mathbf{x} := (x, y, \theta_0, \theta_1, \ldots, \theta_n)^T.$$

Physically we have $x, y \in \mathbb{R}$ and $\theta_0, \ldots, \theta_n \in S^1$. The $n+1$ dimensional torus, $\mathbb{T}^{n+1}$, is defined as the cross product of $n+1$ circles, i.e. $\mathbb{T}^{n+1} := S_1 \times \cdots \times S_1$. Therefore $\mathbf{x} \in \mathscr{X} := \mathbb{R}^2 \times \mathbb{T}^{n+1}$.

1.b. Define the functions

$$f_n := 1, \qquad f_i := \prod_{j=i+1}^{n} \cos(\theta_j - \theta_{j-1}), \quad 0 \le i \le n-1.$$

To find the differential equations governing the model, we start with the car. We directly control the angular velocity, hence $\dot{\theta}_n = \omega$.

Now for the $(n-1)$st cart, consider the hitch at the center of the nth (car) axle. The car has a velocity with magnitude v, at an angle relative to the rod connecting the car and the $(n-1)$st cart given by $\theta_n - \theta_{n-1}$. The angular velocity of the hitch, relative to the center of the $(n-1)$st axle, is the same as the angular velocity $\dot{\theta}_{n-1}$, since the two axles are connected by a rigid rod. Then from basic physics the angular velocity is

$$\dot{\theta}_{n-1} = \frac{v \sin(\theta_n - \theta_{n-1})}{l} = v f_n \sin(\theta_n - \theta_{n-1}).$$

Similarly for the $(n-2)$nd cart, the center of the $(n-1)$st axle is connected by a rigid rod. The $(n-1)$st cart has some velocity at an angle θ_{n-1}, so the angle relative to the rod is given by $\theta_{n-1} - \theta_{n-2}$. The magnitude of the velocity of the $(n-1)$st cart is the portion of velocity of the nth cart (car) parallel to the wheels of the $(n-1)$st cart. This is precisely $\cos(\theta_n - \theta_{n-1})$ times the magnitude of the velocity of the car, that is $v\cos(\theta_n - \theta_{n-1})$. Hence the angular velocity $\dot{\theta}_{n-2}$ is given by

$$\dot{\theta}_{n-2} = \frac{v \cos(\theta_n - \theta_{n-1}) \sin(\theta_{n-1} - \theta_{n-2})}{l} = v f_{n-1} \sin(\theta_{n-1} - \theta_{n-2}).$$

We go down the chain constructing the velocities just as for the $(n-2)$nd cart. The velocity of the ith cart has magnitude of $\cos(\theta_{i+1} - \theta_i)$ times the magnitude of the velocity of the $ik+1)$st cart. By induction, this is

$$\cos(\theta_{i+1} - \theta_i) v \prod_{j=i+2}^{n} \cos(\theta_j - \theta_{j-1} = v \prod_{j=i+1}^{n} \cos(\theta_j - \theta_{j-1}) = v f_i.$$

The direction of the velocity of the ith cart relative to the direction of the velocity of the $(i-1)$st cart is $\theta_i - \theta_{i-1}$. Consequently the $(i-1)$st angular velocity is given by

$$\dot{\theta}_{i-1} = \frac{v f_i \sin(\theta_i - \theta_{i-1})}{l} = v f_i \sin(\theta_i - \theta_{i-1}).$$

As for $\dot{x}$ and $\dot{y}$, these are the linear velocities of the 0th cart. As before, by induction the magnitude of the velocity is $v f_0$. Since the wheels are constrained to be parallel to the velocity of the cart, the direction is exactly θ_0. Then $\dot{x}$ is simply the x component of that velocity, and $\dot{y}$ is simply the y component:

$$\dot{x} = v f_0 \cos\theta_0, \qquad \dot{y} = v f_0 \sin\theta_0.$$

Therefore the control system corresponding to this model can be expressed in the drift-free form

$$\dot{\mathbf{x}} = \mathbf{g}_1(\mathbf{x})u_1 + \mathbf{g}_2(\mathbf{x})u_2,$$

where $u_1 = v$, $u_2 = \omega$, and

$$\mathbf{g}_1(\mathbf{x}) = [f_0\cos\theta_0, f_0\sin\theta_0, f_1\sin(\theta_1-\theta_0), \ldots, f_{i+1}\sin(\theta_{i+1}-\theta_i), \ldots, f_n\sin(\theta_n-\theta_{n-1}), 0]^T,$$

$$\mathbf{g}_2(\mathbf{x}) = \Bigg[\underbrace{0,\ldots,0}_{(n+2)\text{ times}}, 1\Bigg]^T.$$

1.c. Now we consider the special case $n = 1$. In this case,

$$\mathbf{x} = (x, y, \theta_0, \theta_1)^T \in \mathbb{R}^2 \times \mathbb{T} = \mathbb{R}^2 \times \mathbb{S}^1 \times \mathbb{S}^1.$$

Thus the tangent space of the domain is $\mathcal{T}_{\mathbf{x}}\mathscr{X} = \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R} = \mathbb{R}^4$. We explicitly write the dynamics as $\dot{\mathbf{x}} = \mathbf{g}_1(\mathbf{x})v + \mathbf{g}_2(\mathbf{x})\omega$ with

$$\mathbf{g}_1(\mathbf{x}) = \begin{bmatrix} \cos(\theta_1-\theta_0)\cos\theta_0 \\ \cos(\theta_1-\theta_0)\sin\theta_0 \\ \sin(\theta_1-\theta_0) \\ 0 \end{bmatrix}, \qquad \mathbf{g}_2(\mathbf{x}) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

Clearly $\mathscr{X}$ is connected and both $\mathbf{g}_1(\mathbf{x})$ and $\mathbf{g}_2(\mathbf{x})$ are analytic. Then to prove that the system is globally controllable, it remains to show that $\mathrm{Lie}\{\mathbf{g}_1, \mathbf{g}_2\}(\mathbf{x}) = \mathcal{T}_{\mathbf{x}}\mathscr{X} = \mathbb{R}^4$ for all $\mathbf{x} \in \mathscr{X}$. First we calculate

$$\begin{aligned}
[\mathbf{g}_1, \mathbf{g}_2] &= \frac{\partial \mathbf{g}_2}{\partial \mathbf{x}}\mathbf{g}_1 - \frac{\partial \mathbf{g}_1}{\partial \mathbf{x}}\mathbf{g}_2 \\
&= \mathbf{0}_{4\times4}\mathbf{g}_1 - \begin{bmatrix} 0 & 0 & \sin(\theta_1-\theta_0)\cos\theta_0 - \cos(\theta_1-\theta_0)\sin\theta_0 & -\sin(\theta_1-\theta_0)\cos\theta_0 \\ 0 & 0 & \sin(\theta_1-\theta_0)\sin\theta_0 + \cos(\theta_1-\theta_0)\cos\theta_0 & -\sin(\theta_1-\theta_0)\sin\theta_0 \\ 0 & 0 & -\cos(\theta_1-\theta_0) & \cos(\theta_1-\theta_0) \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} \sin(\theta_1-\theta_0)\cos\theta_0 \\ \sin(\theta_1-\theta_0)\sin\theta_0 \\ -\cos(\theta_1-\theta_0) \\ 0 \end{bmatrix}.
\end{aligned}$$

Next we construct the still more heinous Lie bracket,

$$
\begin{aligned}
[\mathbf{g}_1, [\mathbf{g}_1, \mathbf{g}_2]] &= \frac{\partial [\mathbf{g}_1, \mathbf{g}_2]}{\partial \mathbf{x}} \mathbf{g}_1 - \frac{\partial \mathbf{g}_1}{\partial \mathbf{x}} [\mathbf{g}_1, \mathbf{g}_2] \\
&= \begin{bmatrix} 0 & 0 & -\cos(\theta_1 - \theta_0)\cos\theta_0 - \sin(\theta_1 - \theta_0)\sin\theta_0 & \cos(\theta_1 - \theta_0)\cos\theta_0 \\ 0 & 0 & -\cos(\theta_1 - \theta_0)\sin\theta_0 + \sin(\theta_1 - \theta_0)\cos\theta_0 & \cos(\theta_1 - \theta_0)\sin\theta_0 \\ 0 & 0 & -\sin(\theta_1 - \theta_0) & \sin(\theta_1 - \theta_0) \\ 0 & 0 & 0 & 0 \end{bmatrix} \mathbf{g}_1 \\
&\quad - \begin{bmatrix} 0 & 0 & \sin(\theta_1 - \theta_0)\cos\theta_0 - \cos(\theta_1 - \theta_0)\sin\theta_0 & -\sin(\theta_1 - \theta_0)\cos\theta_0 \\ 0 & 0 & \sin(\theta_1 - \theta_0)\sin\theta_0 + \cos(\theta_1 - \theta_0)\cos\theta_0 & -\sin(\theta_1 - \theta_0)\sin\theta_0 \\ 0 & 0 & -\cos(\theta_1 - \theta_0) & \cos(\theta_1 - \theta_0) \\ 0 & 0 & 0 & 0 \end{bmatrix} [\mathbf{g}_1, \mathbf{g}_2] \\
&= \begin{bmatrix} -\sin(\theta_1 - \theta_0)\cos(\theta_1 - \theta_0)\cos\theta_0 - \sin^2(\theta_1 - \theta_0)\sin\theta_0 \\ -\sin(\theta_1 - \theta_0)\cos(\theta_1 - \theta_0)\sin\theta_0 + \sin^2(\theta_1 - \theta_0)\cos\theta_0 \\ -\sin^2(\theta_1 - \theta_0) \\ 0 \end{bmatrix} \\
&\quad - \begin{bmatrix} -\sin(\theta_1 - \theta_0)\cos(\theta_1 - \theta_0)\cos\theta_0 + \cos^2(\theta_1 - \theta_0)\sin\theta_0 \\ -\sin(\theta_1 - \theta_0)\cos(\theta_1 - \theta_0)\sin\theta_0 - \cos^2(\theta_1 - \theta_0)\cos\theta_0 \\ \cos^2(\theta_1 - \theta_0) \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} -\sin\theta_0 \\ \cos\theta_0 \\ -1 \\ 0 \end{bmatrix}.
\end{aligned}
$$

Note that by computing the Jacobian $\partial[\mathbf{g}_1, \mathbf{g}_2]/\partial\mathbf{x}$, it is easy to see that $(\partial[\mathbf{g}_1, \mathbf{g}_2]/\partial\mathbf{x})\,\mathbf{g}_2 = \mathbf{0}$, hence $[\mathbf{g}_2, [\mathbf{g}_1, \mathbf{g}_2]] = \mathbf{0}$. Consequently this bracket adds nothing to the span of

$$\{\mathbf{g}_1, \mathbf{g}_2, [\mathbf{g}_1, \mathbf{g}_2], [\mathbf{g}_1, [\mathbf{g}_1, \mathbf{g}_2]]\}.$$

It follows that $\Delta_3 = \mathrm{span}\{\mathbf{g}_1, \mathbf{g}_2, [\mathbf{g}_1, \mathbf{g}_2], [\mathbf{g}_1, [\mathbf{g}_1, \mathbf{g}_2]]\}$. Equivalently, it is equal to the span of the columns of the matrix

$$
\begin{aligned}
\mathbf{M} &:= \left[\ \mathbf{g}_1 \mid \mathbf{g}_2 \mid [\mathbf{g}_1, \mathbf{g}_2] \mid [\mathbf{g}_1, [\mathbf{g}_1, \mathbf{g}_2]]\ \right] \\
&= \begin{bmatrix} \cos(\theta_1 - \theta_0)\cos\theta_0 & 0 & \sin(\theta_1 - \theta_0)\cos\theta_0 & -\sin\theta_0 \\ \cos(\theta_1 - \theta_0)\sin\theta_0 & 0 & \sin(\theta_1 - \theta_0)\sin\theta_0 & \cos\theta_0 \\ \sin(\theta_1 - \theta_0) & 0 & -\cos(\theta_1 - \theta_0) & -1 \\ 0 & 1 & 0 & 0 \end{bmatrix}.
\end{aligned}
$$

Δ_3 is involutive if and only if $\mathbf{M}$ is non-singular. To check this condition, we calculate the determinant

$$
\begin{aligned}
\det \mathbf{M}(\mathbf{x}) &= \begin{vmatrix} \cos(\theta_1-\theta_0)\cos\theta_0 & 0 & \sin(\theta_1-\theta_0)\cos\theta_0 & -\sin\theta_0 \\ \cos(\theta_1-\theta_0)\sin\theta_0 & 0 & \sin(\theta_1-\theta_0)\sin\theta_0 & \cos\theta_0 \\ \sin(\theta_1-\theta_0) & 0 & -\cos(\theta_1-\theta_0) & -1 \\ 0 & 1 & 0 & 0 \end{vmatrix} \\
&= +1 \begin{vmatrix} \cos(\theta_1-\theta_0)\cos\theta_0 & \sin(\theta_1-\theta_0)\cos\theta_0 & -\sin\theta_0 \\ \cos(\theta_1-\theta_0)\sin\theta_0 & \sin(\theta_1-\theta_0)\sin\theta_0 & \cos\theta_0 \\ \sin(\theta_1-\theta_0) & -\cos(\theta_1-\theta_0) & -1 \end{vmatrix} \\
&= \sin(\theta_1-\theta_0)\left[\sin(\theta_1-\theta_0)\cos^2\theta_0 + \sin(\theta_1-\theta_0)\sin^2\theta_0\right] \\
&\quad - \left[-\cos(\theta_1-\theta_0)\right]\left[\cos(\theta_1-\theta_0)\cos^2\theta_0 + \cos(\theta_1-\theta_0)\sin^2\theta_0\right] \\
&\quad - \left[\sin(\theta_1-\theta_0)\cos(\theta_1-\theta_0)\sin\theta_0\cos\theta_0 - \sin(\theta_1-\theta_0)\cos(\theta_1-\theta_0)\sin\theta_0\cos\theta_0\right] \\
&= \sin^2(\theta_1-\theta_0)\left(\cos^2\theta_0+\sin^2\theta_0\right) + \cos^2(\theta_1-\theta_0)\left(\cos^2\theta_0+\sin^2\theta_0\right) - 0 \\
&= 1 \neq 0.
\end{aligned}
$$

This holds for all $\mathbf{x} \in \mathscr{X}$, so Δ_3 is involutive everywhere. Thus the Lie Algebra Rank condition is satisfied everywhere: $\mathrm{Lie}\{\mathbf{g}_1, \mathbf{g}_2\}(\mathbf{x}) = \mathcal{T}_{\mathbf{x}}\mathscr{X}$ for all $\mathbf{x} \in \mathscr{X}$, from which it follows that the system is globally controllable for $n = 1$.

1.d. For the case $n = 1$, it is clear that Δ_3 is involutive while Δ_1 and Δ_2, since bracketing of $\mathbf{g}_1, \mathbf{g}_2$, and $[\mathbf{g}_1, \mathbf{g}_2]$ generated vector fields outside of Δ_1 and Δ_2, as seen from the determinant calculation in part (c). In addition, any further brackets would have to be inside the Δ_3. This is because $\mathbf{M}(\mathbf{x})$ already has four linearly independent columns for all $\mathbf{x}$, and cannot gain any new ones since it only has four rows. Thus for all $k \geq 3$, $\dim(\Delta_k) = \dim(\Delta_3)$, so the degree of non-holonomy is $k = 3$.

2.a. We consider the following nonlinear system with state $\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2$ and single input $u \in \mathbb{R}$:

$$
\begin{cases} \dot{x}_1 = x_1 + \dfrac{x_2}{1+x_1^2} \\ \dot{x}_2 = -x_2 + u \end{cases} \tag{1}
$$

We can write this in the form $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x})u$, where

$$
\mathbf{f}(\mathbf{x}) = \begin{bmatrix} x_1 + \dfrac{x_2}{1+x_1^2} \\ -x_2 \end{bmatrix}, \qquad \mathbf{g}(\mathbf{x}) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.
$$

In this first part, using feedback linearization we design a controller to globally stabilize the origin. Of course, we need to check if this is possible. To do this, first we calculate

$$
\begin{aligned}
[\mathbf{f},\mathbf{g}] &= \frac{\partial \mathbf{g}}{\partial \mathbf{x}}\mathbf{f} - \frac{\partial \mathbf{f}}{\partial \mathbf{x}}\mathbf{g} \\
&= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 + \dfrac{x_2}{1+x_1^2} \\ -x_2 \end{bmatrix} - \begin{bmatrix} 1 - \dfrac{2x_1x_2}{(1+x_1^2)^2} & \dfrac{1}{1+x_1^2} \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} -\dfrac{1}{1+x_1^2} \\ 1 \end{bmatrix}.
\end{aligned}
$$

Then we define the matrix

$$
\mathbf{M} := \left[\ \mathbf{g} \mid [\mathbf{f},\mathbf{g}]\ \right] = \begin{bmatrix} 0 & -\dfrac{1}{1+x_1^2} \\ 1 & 1 \end{bmatrix},
$$

and let Δ be the distribution defined by the span of the columns of $\mathbf{M}$. The determinant of $\mathbf{M}$ is simply

$$
\det \mathbf{M}(\mathbf{x}) = 0 \cdot 1 - 1\left(-\frac{1}{1+x_1^2}\right) = \frac{1}{1+x_1^2} \neq 0
$$

for all $\mathbf{x} \in \mathbb{R}^2$. Thus $\mathbf{M}(\mathbf{0})$ has rank $2 = n$ and moreoever, Δ is involutive in all of $\mathbb{R}^2$. Then the system (1) is feedback linearizable.

To find this transformation, we start by solving for $\lambda(\mathbf{x})$ from the PDE system

$$
L_{\mathbf{g}}\lambda(\mathbf{x}) = 0, \qquad L_{[\mathbf{f},\mathbf{g}]}\lambda(\mathbf{x}) \neq 0.
$$

Written explicitly, the first (zero) condition requires

$$
0 = \langle \nabla\lambda, \mathbf{g} \rangle = \frac{\partial \lambda}{\partial x_1}(0) + \frac{\partial \lambda}{\partial x_2}(1) = \frac{\partial \lambda}{\partial x_2}.
$$

Thus $\lambda(\mathbf{x})$ is a function of x_1 only. Using this, the second (non-zero) condition requires

$$
0 \neq \langle \nabla\lambda, [\mathbf{f},\mathbf{g}] \rangle = \frac{\partial \lambda}{\partial x_1}\left(-\frac{1}{1+x_1^2}\right) + \frac{\partial \lambda}{\partial x_2}(1) = -\frac{\partial \lambda}{\partial x_1}\frac{1}{1+x_1^2}.
$$

We know that $-1/(1+x_1^2)$ is non-zero everywhere, so we can pick $\lambda(\mathbf{x}) = \lambda(x_1)$ to be any function such that $\partial\lambda/\partial x_1 \neq 0$. We choose the simplest one,

$$
\lambda(\mathbf{x}) = x_1.
$$

To double check that this choice of λ gives us the correct relative degree, we differentiate $\dot{y} = \dot{\lambda}(x_1) = \dot{x}_1 = x_1 + x_2/(1+x_1^2)$, which does not have u explicitly, but

$$
\ddot{y} = \dot{x}_1 + \frac{\dot{x}_2}{1+x_1^2} - \frac{2x_1x_2}{(1+x_1^2)^2} = x_1 + \frac{u}{1+x_1^2} - \frac{2x_1x_2}{(1+x_1^2)^2}.
$$

$\ddot{y}$ depends explicitly on u for all $\mathbf{x} \in \mathbb{R}^2$, hence the relative degree is indeed $r = 2 = n$ globally, as expected. Next we set the control $u = \alpha(\mathbf{x}) + \beta(\mathbf{x})v$ for a new input v, where

$$\begin{aligned}
\alpha(\mathbf{x}) &= -\frac{L_{\mathbf{f}}^2\lambda(\mathbf{x})}{L_{\mathbf{g}}L_{\mathbf{f}}\lambda(\mathbf{x})} \\
&= -\frac{\langle \nabla \langle \nabla\lambda, \mathbf{f}\rangle, \mathbf{f}\rangle}{\langle \nabla \langle \nabla\lambda, \mathbf{f}\rangle, \mathbf{g}\rangle} \\
&= -\frac{\left\langle \nabla\left(\begin{bmatrix}1\\0\end{bmatrix}^T \begin{bmatrix}x_1 + x_2/(1+x_1^2)\\ -x_2\end{bmatrix}\right), \mathbf{f}\right\rangle}{\langle \nabla \langle \nabla\lambda, \mathbf{f}\rangle, \mathbf{g}\rangle} \\
&= -\frac{\langle \nabla[x_1 + x_2/(1+x_1^2)], \mathbf{f}\rangle}{\langle \nabla \langle \nabla\lambda, \mathbf{f}\rangle, \mathbf{g}\rangle} \\
&= -\frac{\begin{bmatrix}1 - 2x_1x_2/(1+x_1^2)^2\\ 1/(1+x_1^2)\end{bmatrix}^T \begin{bmatrix}x_1 + x_2/(1+x_1^2)\\ -x_2\end{bmatrix}}{\begin{bmatrix}1 - 2x_1x_2/(1+x_1^2)^2\\ 1/(1+x_1^2)\end{bmatrix}^T \begin{bmatrix}0\\1\end{bmatrix}} \\
&= -\frac{x_1 + x_2/(1+x_1^2) - 2x_1^2x_2/(1+x_1^2)^2 - 2x_1x_2^2/(1+x_1^2)^3 - x_2/(1+x_1^2)}{1/(1+x_1^2)} \\
&= -x_1(1+x_1^2) + \frac{2x_1x_2}{1+x_1^2}\left(x_1 + \frac{x_2}{1+x_1^2}\right),
\end{aligned}$$

and

$$\beta(\mathbf{x}) = \frac{1}{L_{\mathbf{g}}L_{\mathbf{f}}\lambda(\mathbf{x})} = \frac{1}{1/(1+x_1^2)} = 1 + x_1^2.$$

The state transformation $\mathbf{z} = \boldsymbol{\tau}(\mathbf{x})$ is given by

$$\boldsymbol{\tau}(\mathbf{x}) = \begin{pmatrix}\lambda(\mathbf{x})\\ L_{\mathbf{f}}\lambda(\mathbf{x})\end{pmatrix} = \begin{pmatrix}x_1\\ x_1 + \dfrac{x_2}{1+x_1^2}\end{pmatrix}.$$

This gives us transformed dynamics

$$\begin{cases}\dot{z}_1 = z_2\\ \dot{z}_2 = v\end{cases} \tag{2}$$

To stabilize the linearized system (2), we let $v = -k_1z_1 - k_2z_2$ and use the characteristic equation of the closed loop system to design the feedback gains:

$$\begin{aligned}
0 &= \det[s\mathbf{I} - (\mathbf{A} - \mathbf{BK})] \\
&= \det\left[\begin{pmatrix}s & 0\\ 0 & s\end{pmatrix} - \left[\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix} - \begin{pmatrix}0\\1\end{pmatrix}\begin{pmatrix}k_1 & k_2\end{pmatrix}\right]\right] \\
&= \det\begin{pmatrix}s & -1\\ k_1 & s+k_2\end{pmatrix} \\
&= s^2 + k_2s + k_1.
\end{aligned}$$

This gives us the locations of the closed-loop poles:

$$s_{1,2} = \frac{-k_2 \pm \sqrt{k_2^2 - 4k_1}}{2}.$$

We only care about stability, so it suffices to choose $k_2 > 0$ and $k_1 \neq 0$. For example, we could pick $k_2 = 2$ and $k_1 = 1$ so that $s_1 = s_2 = -1$, both in the left-half plane. This gives us stability. The linear controller is then

$$v = -z_1 - 2z_2 = -x_1 - 2\left(x_1 + \frac{x_2}{1+x_1^2}\right) = -3x_1 - \frac{2x_2}{1+x_1^2}.$$

Finally we substitute this back into the equation for u, giving us the controller which stabilizes the origin of the full nonlinear system (1):

$$\begin{aligned} u_{fl} &= \alpha(\mathbf{x}) + \beta(\mathbf{x})v \\ &= -x_1(1+x_1^2) + \frac{2x_1x_2}{1+x_1^2}\left(x_1 + \frac{x_2}{1+x_1^2}\right) - (1+x_1^2)\left(3x_1 + \frac{2x_2}{1+x_1^2}\right) \\ &= -2x_1(1+x_1^2) - 2\left(x_1 + \frac{x_2}{1+x_1^2}\right)\left(1 + x_1^2 - \frac{x_1x_2}{1+x_1^2}\right). \end{aligned} \tag{3}$$

2.b. As an alternative method, we use integrator backstepping to design a stabilizing controller for (1). To write it in a form suitable for integrator backstepping design, we let $v := -x_2 + u$, so

$$\begin{cases} \dot{x}_1 = f(x_1) + g(x_1)x_2 = x_1 + \dfrac{1}{1+x_1^2}x_2 \\ \dot{x}_2 = v \end{cases} \tag{4}$$

Taking x_2 as an input for the scalar equation $\dot{x}_1 = x_1 + \frac{1}{1+x_1^2}x_2$, we observe that we can stabilize x_1 using

$$x_2 = \phi(x_1) = -2x_1(1+x_1^2),$$

since this input gives

$$\dot{x}_1 = x_1 + \frac{1}{1+x_1^2}[-2x_1(1+x_1^2)] = -x_1.$$

Define a candidate Lyapunov function $V(x) := x_1^2/2$, which is positive definite and radially unbounded with

$$\dot{V}(x) = x_1\dot{x}_1 = -x_1^2.$$

Hence the origin of $\dot{x}_1 = x_1 + \frac{1}{1+x_1^2}x_2$ with $x_2 = \phi(x_1) = -2x_1(1+x_1^2)$ is G.A.S. Now we use the change of variables

$$z_2 = x_2 - \phi(x_1) = x_2 + 2x_1(1+x_1^2),$$

which transforms (4) into the form

$$\begin{cases} \dot{x}_1 = x_1 + \dfrac{1}{1+x_1^2}[z_2 + \phi(x_1)] = -x_1 + \dfrac{1}{1+x_1^2}z_2 \\ \dot{z}_2 = \dot{x}_2 - \dot{\phi}(x_1) = v + 2(1+3x_1^2)\dot{x} = v + 2(1+3x_1^2)\left(-x_1 + \dfrac{1}{1+x_1^2}z_2\right) \end{cases} \tag{5}$$

Next we consider a candidate composite Lyapunov function,

$$V_c(x_1, z_2) := \frac{1}{2}x_1^2 + \frac{1}{2}z_2^2.$$

$V_c(\cdot)$ is clearly positive definite and radially unbounded. Further,

$$\begin{aligned}
\dot{V}_c(x_1, z_2) &= x_1\dot{x}_1 + z_2\dot{z}_2 \\
&= x_1\left(-x_1 + \frac{1}{1+x_1^2}z_2\right) + z_2\left[v + 2(1+3x_1^2)\left(-x_1 + \frac{1}{1+x_1^2}z_2\right)\right] \\
&= -x_1^2 + z_2\left[v + \frac{x_1 z_2}{1+x_1^2} + 2(1+3x_1^2)\left(-x_1 + \frac{z_2}{1+x_1^2}\right)\right].
\end{aligned}$$

Then we select

$$v = -z_2 - \left[\frac{x_1 z_2}{1+x_1^2} + 2(1+3x_1^2)\left(-x_1 + \frac{z_2}{1+x_1^2}\right)\right],$$

which gives us

$$\dot{V}_c(x_1, z_2) = -x_1^2 - z_2^2 \leq 0$$

for all $(x_1, z_2) \in \mathbb{R}^2$, with equality if and only if $(x_1, z_2) = (0, 0)$. Hence this choice of v makes the origin of the transformed system (5) G.A.S. Moreover, since $\phi(0) = 0$, it follows that $z_2 = 0$ if and only if $x_2 = 0$, so this same choice of v globally stabilizes the origin of (4). Finally, for the original system this gives us

$$\begin{aligned}
u_{bs} = v + x_2 &= x_2 - z_2 - \left[\frac{x_1 z_2}{1+x_1^2} + 2(1+3x_1^2)\left(-x_1 + \frac{z_2}{1+x_1^2}\right)\right] \\
&= x_2 - [x_2 + 2x_1(1+x_1^2)] \\
&\quad - \left[\frac{x_1[x_2 + 2x_1(1+x_1^2)]}{1+x_1^2} + 2(1+3x_1^2)\left(-x_1 + \frac{x_2 + 2x_1(1+x_1^2)}{1+x_1^2}\right)\right] \\
&= -2x_1(2 + x_1 + 4x_1^2) - \frac{x_2}{1+x_1^2}(2 + x_1 + 6x_1^2). \qquad (6)
\end{aligned}$$

2.c. Lastly we compare the performance of the two controllers derived, u_{fl} (3) and u_{bs} (6). Fig. 1 shows the trajectories $(x_1(t), x_2(t))$ of the system (1) with these two controllers. Both x_1 and x_2 are zeroed, as we designed. There is also some overshoot, particularly with the x_2 variable, that was present for all initial conditions except $(0, 0)$. We note that u_{bs}, however, has significantly less overshoot, but approximately the same settling time as u_{fl}. Thus our integrator backstepping design has achieves better performance.

To guage the efficiency, we plot the control u against $\|\mathbf{x}\|_2$. This is a rather strange way of plotting the control effort. To do this, we chose $\mathbf{x}$ along the line $x_2 = x_1$ for $x_1 \in \left(-3/\sqrt{2}, 0\right)$, then calculated u_{fl} and u_{bs} for these $\mathbf{x}$ values. We plotted the controls against the norms for those $\mathbf{x}$ values. Fig. 2 shows this plot.

Even though u is not a function of $\|\mathbf{x}\|_2$ directly, fig. 2 does give us a good idea of the control effort at any given point, and other graphs with different choices of $\mathbf{x}$ values yield qualitatively similar results. Fig. 2 that in general, for any given $\mathbf{x}$, the control effort of u_{bs} is higher than that of u_{fl}. This suggests that u_{fl} is more efficient.

On the other hand, it is possible that the speed of the integrator backstepping controller drives it into more efficient regions more quickly. Fig. 2 also gives a plot of total control $\int_0^t |u(\tau)| d\tau$ versus t for the representative trajectory shown in fig. 1. This shows the total control effort to be roughly the same for both u_{bs} and u_{fl}. Note, however, that peak effort for u_{bs} is still, in general, higher. We conclude that the controller u_{bs} has better performance than u_{fl}, without needing more control effort overall.

Figure 1: Trajectories of the system (1) with controllers u_{fl} derived by feedback linearization (3) and u_{bs} derived by integrator backstepping (6). The initial condition used is $x_0 = (1, 3)$; similar results hold for other initial conditions. Left: $x_1(t)$. Right: $x_2(t)$.

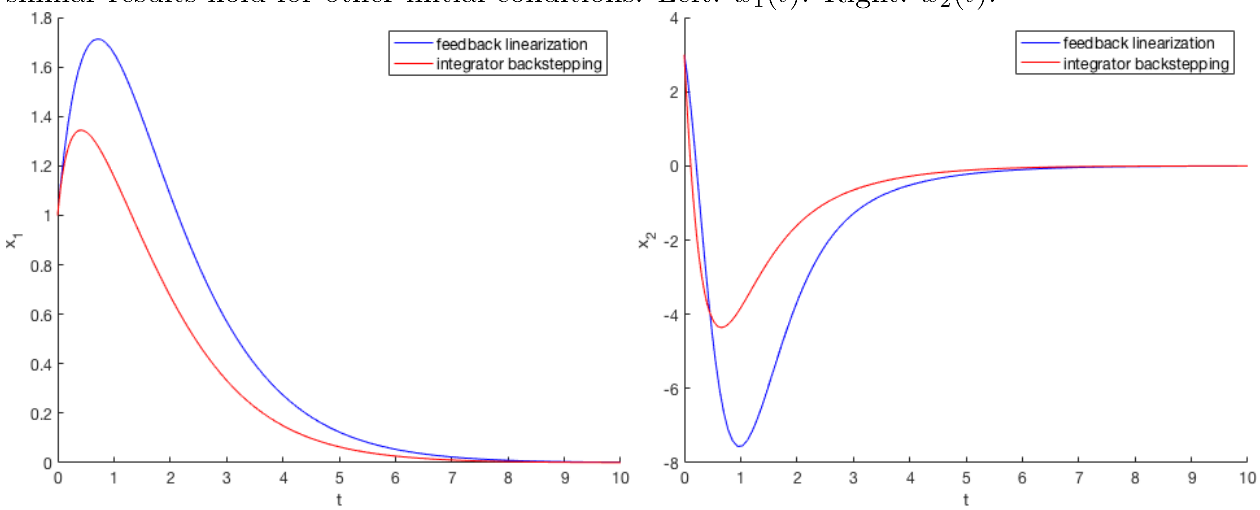

Figure 2: Control efforts u_{fl} derived by feedback linearization (3) and u_{bs} derived by integrator backstepping (6). Left: controls u_{fl} and u_{bs} plotted against $\|\mathbf{x}\|_2$ for $\mathbf{x}$ on the line $x_2 = x_1$ for $x_1 \in \left(-3/\sqrt{2}, 0\right)$. Similar results hold for other $\mathbf{x}$. We see that in general, $|u_{fl}| < |u_{bs}|$ for given $\mathbf{x}$. Right: total control effort $\int_0^t |u(\tau)| d\tau$ versus t for initial condition used is $x_0 = (1, 3)$; similar results hold for other initial conditions. This shows that u_{bs} ends up requiring less total effort than u_{fl} in general.