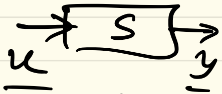
Lecture #14

Designing Controllers

Design Idea #1: Passivity Based Control:

Thm (*) Suppose you have

$$\left.\begin{aligned} \dot{\underline{x}} &= \underline{f}(\underline{x}, \underline{u}) \\ \underline{y} &= \underline{h}(\underline{x}) \end{aligned}\right\} S$$



Set up:

$$\left\{\begin{aligned} &\underline{f} \text{ is Loc. Lip. in } (\underline{x}, \underline{u}) \\ &\underline{h} \text{ is continuous in } \underline{x} \end{aligned}\right\} \quad \forall \underline{x} \in \mathbb{R}^n, \ \forall \underline{u} \in \mathbb{R}^m$$

$$\left\{\begin{aligned} f(\underline{0}, \underline{0}) &= \underline{0} \\ \underline{h}(\underline{0}) &= \underline{0} \end{aligned}\right.$$

If the system S is

1. passive with rad. unbdd. storage $f^{\underline{n}}$ $V(\cdot)$
2. ZSO

Then origin $\underline{x} = \underline{0}$ of the forced system can be GLOBALLY Stabilized by output feedback

$$\underline{u}(\underline{y}) = -\underline{\phi}(\underline{y})$$

where $\underline{\phi}(\cdot)$ is ANY locally Lip. $f^{\underline{n}}$

s.t. $\boxed{\underline{\phi}(\underline{0}) = \underline{0}}$ & $\boxed{\underline{y}^T \underline{\phi}(\underline{y}) > 0 \ \forall \ \underline{y} \neq \underline{0}}$

Proof: Idea: Use the storage function $V(\cdot)$ as the Lyapunov $f^{\underline{n}}$ for the closed-loop system:

closed-loop system: $\underline{\dot{x}} = \underline{f}(\underline{x}, -\underline{\phi}(\underline{y}))$

$$\dot{V} = \langle \nabla V, \underline{f}(\underline{x}, -\underline{\phi}(\underline{y})) \rangle$$

$$\leq -\underline{y}^T \underline{\phi}(\underline{y}) \leq 0$$

$\Rightarrow \dot{V} \leq 0$ & $\dot{V} = 0$ iff $\underline{y} = \underline{0}$

By ZSO, $\underline{y}(t) \equiv 0 \Rightarrow \underline{u}(t) \equiv \underline{0} \Rightarrow \underline{x}(t) \equiv 0$ $\therefore$ by LaSalle GAS.

"Not given, nice System", How to make it "nice".

"Nice" ≡ "Passive"

How to passivate ?

→ either by choice of output

→ or by feedback (Feedback Passivation)

→ both

Example: (By Choice of output)

$$\underbrace{\underline{\dot{x}}}_{\in \mathbb{R}^n} = \underbrace{\underline{f}(\underline{x})}_{\in \mathbb{R}^n} + \underbrace{\underline{g}(\underline{x})}_{\in \mathbb{R}^{n\times m}}\underbrace{\underline{u}}_{\in \mathbb{R}^m}$$

(control affine system)

No output is defined

Suppose

$$\langle \nabla V, \underline{f} \rangle \leq 0 \quad \forall\ \underline{x} \in \mathbb{R}^n$$

Choose output as $\underset{m\times 1}{\underline{y}} = \underline{h}(\underline{x}) := \langle \nabla V, g \rangle^T$

$$= \underbrace{g^T}_{m\times n} \underbrace{\nabla V}_{n\times 1}$$

Then

$$\dot{V} := \frac{d}{dt} V = \langle \nabla V, \text{RHS} \rangle$$

$$= \langle \nabla V, \underline{f}(\underline{x}) + \underline{g}(\underline{x})\,\underline{u} \rangle$$

$$\underline{\underline{\langle \nabla V, f \rangle}} + \langle \nabla V, g\underline{u} \rangle \leq \underbrace{\underline{y}^T \underline{u}}_{\parallel\!\parallel\!\parallel}$$

$$(\nabla V)^T g\, \underline{u}$$

If in addition, zSO, then closed loop is GAS.

$\therefore$ The choice $\underline{y} = g^T \nabla V$ makes the closed-loop system $(\underline{u} \longmapsto \underline{y})$ passive.

<u>Example</u>:

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1^3 + u\end{aligned}\Bigg\}$$

Constraint:

$|u| \le \underbrace{u_{max}}_{>0 \text{ (given)}}$

$$V(x_1, x_2) = \frac{1}{4} x_1^4 + \frac{1}{2} x_2^2$$

with $u = 0$, $\dot{V} = x_1^3 x_2 - x_2 x_1^3 = 0$

Take $y = \underline{h}(\underline{x}) := \underset{(0 \;\; 1)\begin{pmatrix} \cdots \\ x_2 \end{pmatrix}}{g^T \nabla V} = \frac{\partial V}{\partial x_2} \cdot 1$

$$= x_2$$

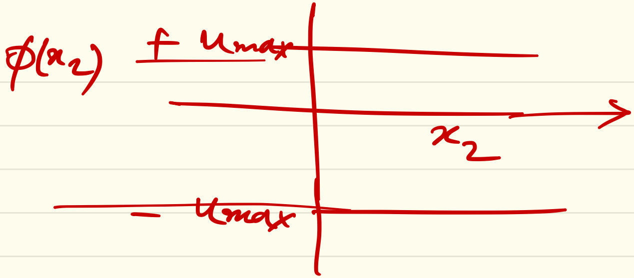
with $u = 0$, $y(t) \equiv 0 \Rightarrow \underline{x}(t) \equiv 0$

$\therefore$ Conditions of Thm (*) are satisfied.

$\therefore$ We can choose an output feedback

$$u = -\phi(y) = -\phi(x_2)$$

$$\text{s.t. } |u| \le u_{max}$$



$$u = -u_{max} \underbrace{sat(x_2)}_{\text{Saturation } f^n}$$

$$sat(z) = \begin{cases} +1 & \text{if } z \geq +1 \\ z & \text{if } z \in (-1,1) \\ -1 & \end{cases}$$

$$u = -\left(\frac{2u_{max}}{\pi}\right) \arctan(x_2)$$

$$|u| \leq u_{max}$$

Problem with "choosing output approach":
still requires open-loop stability $(\langle \nabla V, \underline{f} \rangle \leq 0)$

Feedback Passivation:

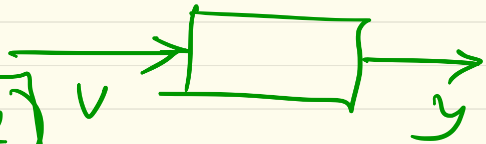
Given, $\begin{cases} \dot{\underline{x}} = \underline{f}(\underline{x}) + \underline{g}(\underline{x})\,\underline{u} \\ \underline{y} = \underline{h}(\underline{x}) \end{cases}$

If a feedback control of the form

$$\underline{u} = \alpha(\underline{x}) + \beta(\underline{x})\,\underline{v}$$ exists

s.t. the new system

$$\boxed{\begin{aligned} \dot{\underline{x}} &= \underline{f}(\underline{x}) + \underline{g}(x)\left(\alpha(\underline{x}) + \beta(\underline{x})\,\underline{v}\right) \\ \underline{y} &= \underline{h}(\underline{x}) \end{aligned}}$$



satisfies conditions of Thm (*), then we can globally stabilize origin by output feedback

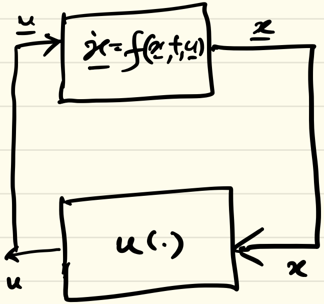
$$\underline{v} = -\phi(\underline{y})$$

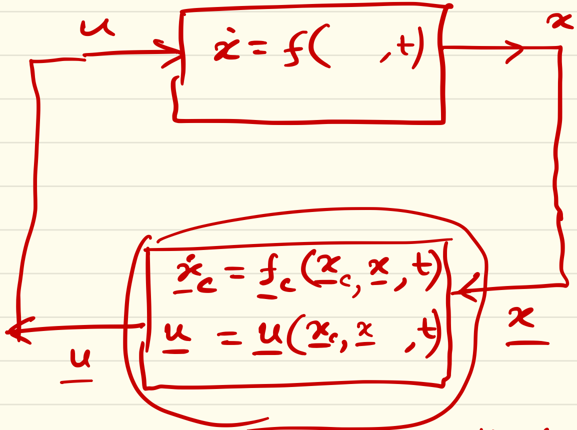
Design Idea # 2 : Feedback Stabilization

Problem Statement: Given $\dot{\underline{x}} = f(\underline{x}, t, \underline{u})$

Design $\underline{u} = \underbrace{\underline{u}(\underline{x}, t)}_{\text{State feedback}}$ s.t. $\underline{x} = \underline{0}$ is UAS for the closed loop.

Static feedback





Dynamic feedback

Feedback Stabiliz^n

	Static Feedback	Dynamic Feedback
State Feedback	$\underline{u} = \underline{u}(\underline{x}, t)$	$\underline{\dot{x}}_c = \underline{f}_c(\underline{x}_c, \underline{x}, t)$ $\underline{u} = \underline{u}(\underline{x}_c, \underline{x}, t)$
Output Feedback	$\underline{u} = \underline{u}(\underline{y}, t)$	$\underline{\dot{x}}_c = \underline{f}_c(\underline{x}_c, \underline{y}, t)$ $\underline{u} = \underline{u}(\underline{x}_c, \underline{y}, t)$

we want to stabilize $\begin{pmatrix} \underline{x} \\ \underline{x}_c \end{pmatrix} = \begin{pmatrix} \underline{0} \\ \underline{0} \end{pmatrix}$

Special Case : Single input $u \in \mathbb{R}$

Static state feedback

$$u = u(\underline{x}, \cancel{t}), \qquad u : \mathbb{R}^n \mapsto \mathbb{R}$$

$$= \psi(\underline{x})$$

continuous.

Feedback Stabilization for single input system using Continuous static state feedback.

$$\underbrace{\dot{\underline{x}}}_{\in \mathbb{R}^n} = \underbrace{\underline{f}(\underline{x})}_{\in \mathbb{R}^n} + \underbrace{\underline{g}(x)}_{\in \mathbb{R}^n} u$$

scalar

Idea: (Converse Lyap. Thm)

If $u = \psi(\underline{x})$ exists, s.t.

① $\psi(\cdot)$ is continuous

② origin of $\underline{\dot{x}} = \underline{f}(\underline{x}) + \underline{g}(\underline{x})\,\psi(\underline{x})$ is A.S.

then $\exists\, V(\underline{x})$ s.t.

$$\dot{V} = \langle \nabla V, \underline{f}(\underline{x}) + \underline{g}(\underline{x})\,\psi(\underline{x}) \rangle < 0$$

$$\forall x \in \mathcal{D} \setminus \{0\}$$

Single input case: ($u \in \mathbb{R}$)

Def^{n}: Control Lyapunov Function (C.L.F.)

$\exists$ C^1 pos. def. f^{n} $V(\underline{x})$ is called C.L.F. for dynamical system

$$\underline{\dot{x}} = \underline{f}(\underline{x}) + \underline{g}(\underline{x})\, u \quad \text{if}$$

$$\langle \nabla V, g \rangle = 0 \;\; \forall\, x \in \mathcal{D} \setminus \{0\}$$

$$\Rightarrow \langle \nabla V, \underline{f} \rangle < 0$$

It is called GLOBAL C.L.F if

this condition

+

rad. unbdd. holds

for $\mathcal{D} \equiv \mathbb{R}^n$.

Thm. (Sontag's Formula)

Let $V(\underline{x})$ be C.L.F. for $\dot{\underline{x}} = \underline{f}(\underline{x}) + \underline{g}(\underline{x})u$

Then origin is stabilizable by $u = \psi(\underline{x})$

where $\psi(\cdot)$ is continuous and

$$\psi(\underline{x}) = \begin{cases} 0 & \text{if } \langle \nabla V, g \rangle = 0 \\ -\dfrac{\langle \nabla V, \underline{f} \rangle + \sqrt{(\langle \nabla V, f \rangle)^2 + (\langle \nabla V, g \rangle)^4}}{\langle \nabla V, g \rangle} & \text{otherwise } (\langle \nabla V, g \rangle \neq 0) \end{cases}$$

Proof: $\dot{\underline{x}} = \underline{f}(\underline{x}) + \underline{g}(\underline{x})\,\psi(\underline{x})$

$$\dot{V} = \langle \nabla V, \underline{f}(\underline{x}) + \underline{g}(\underline{x})\,\psi(\underline{x}) \rangle$$

If $\underline{x} \neq 0$ & $\frac{\partial V}{\partial x} g(x) \neq 0$

then $\dot{V} = \frac{\partial V}{\partial \underline{x}} \underline{f} - \left[\frac{\partial V}{\partial \underline{x}} f + \sqrt{(\langle \nabla V, f \rangle)^2 + \langle \nabla V, g \rangle^4} \right]$

$$= -\sqrt{\langle \nabla V, f \rangle^2 + \langle \nabla V, g \rangle^4}$$

$$< 0$$

Example: $\dot{x} = x - x^3 + u$

$V(x) = \frac{1}{2} x^2$ is CLF

$\frac{\partial V}{\partial x} g = x$, $\frac{\partial V}{\partial x} f = x(x - x^3)$

Question:

$u = \psi(x)$

What is $\psi(\cdot)$?

$$- \frac{\frac{\partial V}{\partial x} f + \sqrt{\left(\frac{\partial V}{\partial x} f\right)^2 + \left(\frac{\partial V}{\partial x} g\right)^4}}{\frac{\partial V}{\partial x} g}$$

$$= - \frac{x(x - x^3) + \sqrt{x^2(x - x^3)^2 + x^4}}{x}$$

$$= -x + x^3 - x\sqrt{(1 - x^2)^2 + 1}$$

$$= \psi(x)$$