AMS 231: Nonlinear Control Theory: Winter 2018

Details on Lecture #9 ©Abhishek Halder

Counterexample showing $A(t)$ being Hurwitz (or not) for all $t \geq t_0 \geq 0$ is irrelevant for ascertaining stability of LTV system

Consider the LTV system

$$\underline{\dot{x}} = A(t)\underline{x}(t), \qquad \underline{x}(t_0 = 0) = \underline{x}_0, \qquad \underline{x} \in \mathbb{R}^2.$$

Now consider a skew-symmetric matrix $\Omega := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, and notice that $e^{\Omega t} = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$ is an orthogonal matrix. Then let the LTV dynamics be governed by

$$A(t) := e^{\Omega t} B e^{-\Omega t},$$

and consequently, eigenvalues of $A(t)$ are same as those of B for all $t \geq 0$. By direct differentiation, one can verify that the solution of the LTV system is given by

$$\underline{x}(t) = \underbrace{e^{\Omega t} e^{(-\Omega + B)t}}_{\Phi(t,0)} \underline{x}_0, \qquad t \geq 0.$$

First, let us take $B = \begin{pmatrix} -1 & -4 \\ 0 & -1 \end{pmatrix}$; thus $A(t)$ has both eigenvalues equal to -1 for all $t \geq 0$, that is, Hurwitz for all t. However, the matrix $-\Omega + B = \begin{pmatrix} -1 & -3 \\ -1 & -1 \end{pmatrix}$ has eigenvalues $-1 \pm \sqrt{3}$, and hence we can find $\underline{x}_0$ such that $\parallel \underline{x}(t) \parallel_2 \to \infty$ as $t \to \infty$, meaning the origin of the LTV system is NOT A.S.

Next, let us take $B = \begin{pmatrix} -1 & 4 \\ 1 & -1 \end{pmatrix}$; thus $A(t)$ has eigenvalues 1 and -3 for all $t \geq 0$, that is, NOT Hurwitz for any t. However, the matrix $-\Omega + B = \begin{pmatrix} -1 & 5 \\ 0 & -1 \end{pmatrix}$ has both eigenvalues equal to -1. Since $e^{\Omega t}$ is bounded, we conclude the origin is G.A.U.E.S.