Lecture # 20 (Last!!)

Recall:

① Any vector field : $\mathcal{X} \longmapsto T_{\underline{x}}\mathcal{X}$

② Affine control system : $\underline{\dot{x}} = \underbrace{\underline{f}(\underline{x})}_{\text{Drift vec. field}} + \sum_{i=1}^{m} \underbrace{g_i(\underline{x})}_{\text{Input vec. field}} u_i \,, \quad \underline{x} \in \mathbb{R}^n$

spl. case: Drift-free :

$$\boxed{\underline{\dot{x}} = \sum_{i=1}^{m} g_i(\underline{x}) u_i} \quad \text{- - - - -} (*)$$

$m < n$

$x \in \mathcal{X} \subseteq \mathbb{R}^n$

$\underline{u} \in \mathcal{U} \subseteq \mathbb{R}^m$

③ $\Delta(\underline{x}) := \operatorname{span}\{g_1, \ldots, g_m\}(\underline{x})$

Distribution

④ Δ thus constructed, may not be $\underbrace{\text{involutive}}$

(Lec. # 17, p. 14, Δ is closed under Lie bracketing)

→ If Δ not involutive, start adding new vec. fields by "bracketing", "bracket-of-bracketing" etc. till we reach <u>involutive closure</u>

(i.e.) $\Delta_1 = \Delta := \text{span}\{g_1, \ldots, g_m\}$

$\Delta_2 = \Delta_1 \cup \text{span}\{\text{Brackets}\}$

$\Delta_3 = \Delta_2 \cup \text{span}\{\text{Bracket-of-Brackets}\}$

etc.

$\vdots$

Nested sequence of distributions = "filtration"

→ Keep doing until: $\dim(\Delta_{k+1}) = \dim(\Delta_k)$

"No new dir^{n} being generated by Lie bracketing"

(reached involutive closure).

→ Smallest such $K \in \mathbb{N}$, we stop there
$(\dim(\Delta_{K+1}) = \dim(\Delta_K))$

is called "degree-of-nonholonomy".

⇒ Control System is called

→ Completely non-holonomic when $\dim(\Delta_K) = n$

→ partially " when $m < \dim(\Delta_K) < n$

→ holonomic when $\dim(\Delta_K) = m$

⇒ Lie Algebra:

$$\mathrm{Lie}\{g_1, \dots, g_m\} := \mathrm{span}\{\overbrace{\underbrace{g_1, g_2, \dots, g_m}_{\Delta_1}, [g_i, g_j]}^{\Delta_2 \text{ etc.}}, [g_i, [g_j, g_k]], \dots\}$$

$$= \mathrm{span}(\Delta_K)$$

→ $\text{Lie}\{g_1(\underline{x}), \ldots, g_m(\underline{x})\}(\underline{x})$ { Symbol for Lie algebra at $\underline{x} \in \mathcal{X}$

Rashevsky–Chow Thm. (Controllability Theorem)

Consider $\dot{\underline{x}} = \sum_{i=1}^{m} g_i(\underline{x})\, u_i$,

$\underline{x} \in \mathcal{X} \subseteq \mathbb{R}^n$

$\underline{u} \in \mathcal{U} \subseteq \mathbb{R}^m$

$m < n$

(i) $\mathcal{X}$ is connected

$g_i(\underline{x})$ are analytic $\forall\, i = 1, \ldots, m$

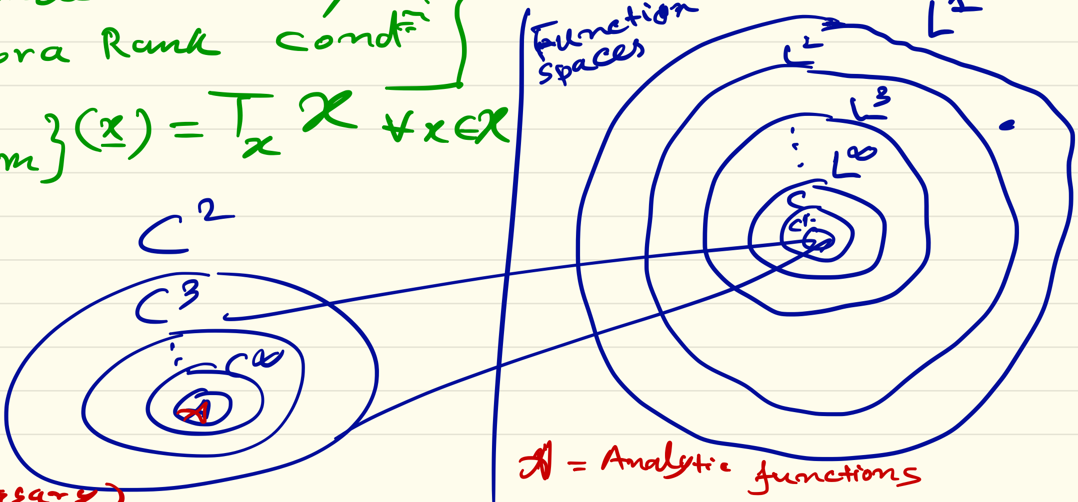
(ii) [Bracket-generating condⁿ / Hörmander condⁿ / Lie Algebra Rank condⁿ]

$$\text{Lie}\{g_1, \ldots, g_m\}(\underline{x}) = T_x \mathcal{X} \quad \forall\, x \in \mathcal{X}$$

$$\Updownarrow$$

controllable

If $g_i \in C^\infty$, then $\Downarrow$ (Suff. condⁿ, not necessary)



Example : HW #1 (p1(e)) Wheeled Mobile Robot / Dubin's Car

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{pmatrix} = \begin{pmatrix} V\cos\theta \\ V\sin\theta \\ \omega \end{pmatrix},$$

$$V := \tilde{v} = r\omega^{+}$$

$$\omega := \frac{r}{\ell}\tan\tilde{\phi} = \frac{r}{\ell}\omega^{-}$$

$$\boxed{\begin{aligned} u_1 &= V \\ u_2 &= \omega \end{aligned}}$$

$$\underset{\text{(state space)}}{\mathcal{X}} = \underbrace{\mathbb{R}^2 \times S^1}_{\text{connected}}$$

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{pmatrix} = \underbrace{\begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \end{pmatrix}}_{g_1} V + \underbrace{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}_{g_2} \omega, \quad \underbrace{\underline{x} := \begin{pmatrix} x \\ y \\ \theta \end{pmatrix}}_{\text{states}}$$

Controls

3 states
2 controls

$$T_{\underline{x}}(\mathbb{R}^2 \times S^1) = \underbrace{T_x\mathbb{R}^2}\times\underbrace{T_x S^1}$$

$$= \mathbb{R}^2 \times \mathbb{R} = \mathbb{R}^3$$

$$\underbrace{[g_1, g_2]}_{\text{Lie Bracket}} = ad_{g_1} g_2 = Jac(g_2)\, g_1 - Jac(g_1)\, g_2$$

$$= \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} g_1}_{} - \begin{bmatrix} 0 & 0 & -\sin\theta \\ 0 & 0 & \cos\theta \\ 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} -\sin\theta \\ \cos\theta \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} +\sin\theta \\ -\cos\theta \\ 0 \end{pmatrix}$$

$$\underbrace{\left[\, g_1 \;\middle|\; g_2 \;\middle|\; ad_{g_1} g_2 \,\right]}_{\text{rank (this matrix) = 3}} = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ \sin\theta & 0 & -\cos\theta \\ 0 & 1 & 0 \end{bmatrix}$$

$$\det\left[g_1 \,|\, g_2 \,|\, ad_{g_1} g_2\right] = \cos^2\theta + \sin^2\theta = 1 \neq 0 \quad \forall\, \underline{x} \in \mathcal{X}$$

$\therefore \operatorname{span}\{g_1, g_2, ad_{g_1} g_2\} = T_{\underline{x}}\mathcal{X} = \mathbb{R}^3 \ \forall \underline{x} \in \mathcal{X}$

$\Leftrightarrow \dim(\{g_1, g_2, ad_{g_1} g_2\}) = 3 \Leftrightarrow$ <u>Controllable</u>

deg. of non-holonomy $(k) = 2$

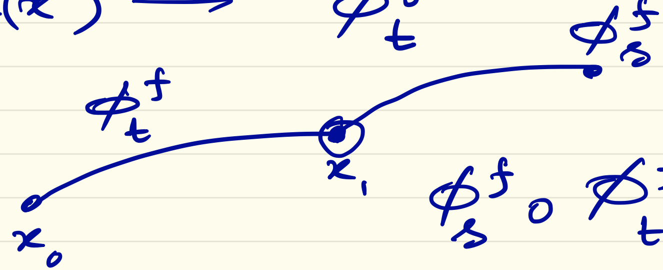
$$\dim(\Delta_2) = 3$$

(non-holonomic control system)

<u>Interpretating Lie Brackets as a measure of non-commutativity of flows:</u>

$\dot{x} = f(x) \xrightarrow{\text{Flow}} \phi^f_t$

Flows under same dynamics always commute



$$\underline{\phi^f_s \circ \phi^f_t} = \phi^f_{s+t} = \phi^f_t \circ \phi^f_s$$

For different dynamics, flows don't commute in general

$$\phi_t^f \circ \phi_t^g \underset{\text{(in general)}}{\neq} \phi_t^g \circ \phi_t^f \quad \forall t$$

In fact: (result)

$$\phi_t^f \circ \phi_t^g = \phi_t^g \circ \phi_t^f \quad \forall t$$

if and only if $\underbrace{[f, g]}_{ad_f g} = 0.$

Also, in general,

Flow of the Lie bracketed vector field

$$\overbrace{\phi_t^{[g_1, g_2]}} = \lim_{n\to\infty} \Big(\overbrace{\phi_{\sqrt{t/n}}^{-g_2} \circ \phi_{\sqrt{t/n}}^{-g_1} \circ \phi_{\sqrt{t/n}}^{g_2} \circ \phi_{\sqrt{t/n}}^{g_1}}^{\text{Infinitesimal primitives}} \Big)^n$$

$$= \frac{d}{dt}\Big|_{t=0} \left(\phi_{\sqrt{t}}^{-g_2} \circ \phi_{\sqrt{t}}^{-g_1} \circ \phi_{\sqrt{t}}^{g_2} \circ \phi_{\sqrt{t}}^{g_1} \right)$$

For $m = 2$, $\dot{\underline{x}} = g_1(\underline{x}) u_1 + g_2(\underline{x}) u_2$

the control

$$(u_1, u_2) = \begin{cases} (1, 0) & \forall\ t \in [0, h) \\ (0, 1) & \forall\ t \in [h, 2h) \\ (-1, 0) & \forall\ t \in [2h, 3h) \\ (0, -1) & \forall\ t \in [3h, 4h) \end{cases}$$

generates the motion:

$$\underline{x}(4h) = \underline{x}(0) + h^2 [g_1, g_2](\underline{x}) + O(h^3)$$

Proof: $\underline{x}(h) = \underline{x}(0) + h\, \dot{\underline{x}}(0) + \frac{1}{2} h^2\, \ddot{\underline{x}}(0) + \ldots$

Taylor series

$$= \underline{x}(0) + h\, g_1(\underline{x}_0) + \frac{1}{2} h^2 \left.\frac{\partial g_1}{\partial x}\right|_{x = x_0} g_1(\underline{x}_0) + \ldots$$

Similarly,

$$\underline{x}(2h) = \underline{x}(h) + h\, g_2(\underline{x}(h)) + \frac{1}{2} h^2 \left.\frac{\partial g_2}{\partial \underline{x}}\right|_{x = x(h)} g_2(\underline{x}(h))$$

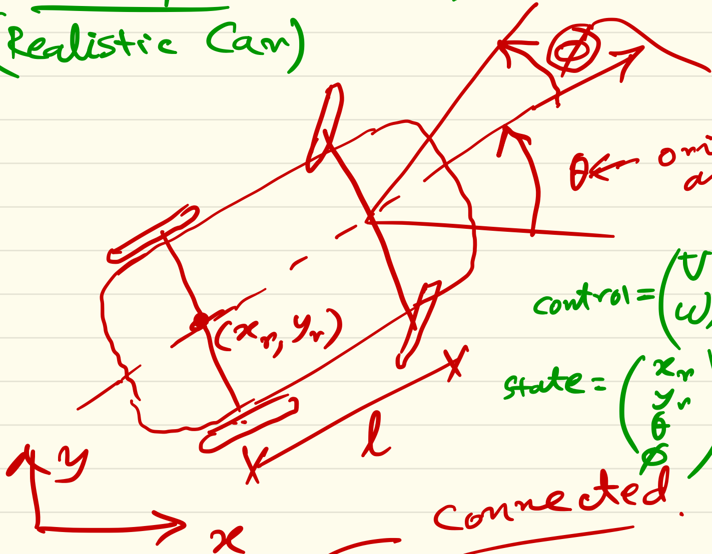
$\underline{x}(3h) = \ldots$

$$\underline{x}(4h) = x_0 + h^2\Big(\underbrace{\frac{\partial g_2}{\partial \underline{x}} g_1 - \frac{\partial g_1}{\partial \underline{x}} g_2}_{[g_1,\, g_2]}\Big) + O(h^3)$$

(Proved)

Example: HW 1, p1(a,b,c) → $\dot{\phi} = \omega$.

(Realistic Car)



$$\text{control} = \begin{pmatrix} v \\ \omega \end{pmatrix} \qquad \text{state} = \begin{pmatrix} x_r \\ y_r \\ \theta \\ \phi \end{pmatrix}$$

$$\begin{pmatrix} \dot{x}_r \\ \dot{y}_r \\ \dot{\theta} \\ \dot{\phi} \end{pmatrix} = \underbrace{\begin{pmatrix} \cos\theta \\ \sin\theta \\ \frac{1}{l}\tan\phi \\ 0 \end{pmatrix}}_{g_1} \underbrace{v}_{\text{Translational velocity}} + \underbrace{\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}}_{g_2} \underbrace{\omega}_{\text{steering angle's ang. vel.}}$$

State space: $\underbrace{\mathcal{X} = \mathbb{R}^2 \times S^1}_{\text{connected}} \times S^1 \setminus \{\pi/2\}$

$T_x \mathcal{X} = \mathbb{R}^4$

$g_1 \equiv$ Drive

$g_2 \equiv$ Steer

$$[g_1, g_2] = \mathrm{ad}_{g_1} g_2$$

$$= \mathrm{Jac}(g_1)\, g_2 - \mathrm{Jac}(g_2)\, g_1$$

$$= \begin{bmatrix} 0 & 0 & -\sin\theta & 0 \\ 0 & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & \frac{1}{l}\sec^2\phi \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} - \underset{4\times4}{0}\, g_1$$

$$= \underbrace{\begin{pmatrix} 0 \\ 0 \\ \frac{1}{l}\sec^2\phi \\ 0 \end{pmatrix}}_{\text{Rotate}}$$

only 3rd component non-zero

$\therefore$ Rotate = [Steer, Drive]

Drive / Tight parking Restriction: No forward space

Steer ” : Front wheel against curb.

Rotate ” : Other cars don't give enough radius to rotate

$$\underbrace{[g_1,[g_2,g_1]]}_{\mathrm{ad}_{g_1}\mathrm{ad}_{g_2} g_1} = [\text{Drive}, [\text{Steer}, \text{Drive}]] = [\text{Drive}, \text{Rotate}]$$

$$= \underbrace{\begin{pmatrix} \frac{1}{l}\sec^2\phi \sin\theta \\ -\frac{1}{l}\sec^2\phi\cos\theta \\ 0 \\ 0 \end{pmatrix}}_{\text{sliding}}$$

Only first two components non-zero

= Displacement $\perp^r$ to car

Discrete seq: (corresponding to sliding)

Steer → drive → Steer back → drive → Steer → drive back → steer back → Drive back

NOTICE. reverse steer in middle !!

not intuitive at all

Parking Thm.: One can get out of any parking lot larger than the size of car.

Proof Show that the matrix

$$\text{rank}\left[g_1 \,\middle|\, g_2 \,\middle|\, \underbrace{ad_{g_2} g_1}_{\substack{\shortparallel \\ [g_2, g_1]}} \,\middle|\, \underbrace{ad_{g_1} ad_{g_2} g_1}_{\substack{\shortparallel \\ [g_1, [g_2, g_1]]}} \right] = 4$$

(compute the 4x4 determinant)

Linear System Controllability

$$\dot{x} = Ax + Bu = Ax + \sum_{i=1}^{m} b_i u_i, \quad B = [b_1 | b_2 | \dots | b_m]$$

↳ controllability rank cond:

$$\underbrace{\text{rank}\left(B \,|\, AB \,|\, A^2B \,|\, \dots \,|\, A^{n-1}B\right) = n}_{\text{called Kalman Rank Condition}}$$

Linearized System Controllability

Linearize

$$\dot{x} = f(x) + \sum_{i=1}^{m} g_i(x)\, u$$

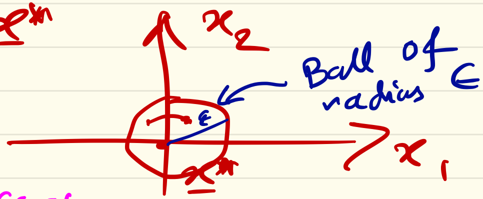
linearize around $x = x^*$, $f(x^*) = 0$

↳ linearize

$$\dot{z} = \tilde{A} z + \tilde{B} v$$

If linearized system is controllable then nonlin. system is LOCALLY controllable around x^*

(suff. cond: NOT necessary)



In other words, when the linearized system is NOT controllable, then it is still possible that the original non-linear system is controllable!

Exercise Dubin's Car linearize around $\underline{x} = 0$

Linearized rank $= 2 < 3$

But we showed that Dubin's car is nonlinearly controllable EVERYWHERE! (Globally)

So nonlinear controllability is much stronger concept than linear or linearized controllability