Lecture #1 (03/31/2020)

$$\boxed{\dot{\underline{x}} = \underline{f}(t, \underline{x})},$$

$\underline{x} \in \mathcal{X} \subseteq \mathbb{R}^n$ — $\mathcal{X}$: State space, $\underline{x}$: State vector

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{pmatrix} = \begin{pmatrix} f_1(t, x_1, x_2, \ldots, x_n) \\ f_2(t, x_1, x_2, \ldots, x_n) \\ \vdots \\ f_n(t, x_1, x_2, \ldots, x_n) \end{pmatrix},$$

vector ODE of size $n \times 1$ } without control, just dynamics

$$\underbrace{\underbrace{\underline{x}(0)}_{n\times 1} = \underbrace{\underline{x}_0}_{\text{given}}}_{\text{Initial condition}}$$

Autonomous ODE: $\underline{f}$ has no explicit t dependence

i.e., $\underline{f} \equiv \underline{f}(\underline{x}) \Leftrightarrow \boxed{\dot{\underline{x}} = \underline{f}(\underline{x})}$

Non-autonomous ODE: $\underline{f} \equiv \underline{f}(t, \underline{x})$

Controlled ODE:

$$\underset{\sim}{\dot{x}} = \underset{\sim}{f}(\underbrace{t}_{\text{time}}, \underbrace{\underset{\sim}{x}}_{\text{state vector}}, \underbrace{\underset{\sim}{u}}_{\text{control vector}}), \quad \underbrace{\underset{\sim}{x}(0) = \underset{\sim}{x}_0 \text{ (given)}}_{\text{Initial condition}}$$

$$\underbrace{\underset{\sim}{x}}_{n\times 1} \in \underbrace{\mathcal{X}}_{\text{state space}} \subseteq \mathbb{R}^n, \qquad \underbrace{\underset{\sim}{u}}_{m\times 1} \in \underbrace{\mathcal{U}}_{\text{control space}} \subseteq \mathbb{R}^m, \quad \text{Typically } m < n$$

$$\underset{\sim}{y} = \underset{\sim}{h}(t, \underset{\sim}{x}, \underset{\sim}{u})$$

$$\underbrace{\underset{\sim}{y}}_{\text{output vector}} \in \underbrace{\mathcal{Y}}_{\text{Output Space}} \subseteq \mathbb{R}^p$$

$$\left.\begin{aligned} \underset{\sim}{\dot{x}} &= \underset{\sim}{f}(t, \underset{\sim}{x}, \underset{\sim}{u}) \\ \underset{\sim}{y} &= \underset{\sim}{h}(t, \underset{\sim}{x}, \underset{\sim}{u}) \end{aligned}\right\}$$

State eq^n.

output/measurement equation

Linear Control System:

$$\underline{\dot{x}} = \underline{f}(t, \underline{x}, \underline{u}) \equiv A(t)\,\underline{x} + B(t)\,\underline{u}$$

$$\underline{y} = \underline{h}(t, \underline{x}, \underline{u}) \equiv C(t)\,\underline{x} + D(t)\,\underline{u}$$

continuous time

Discrete-time control system:

Linear

$$\underline{x}(k+1) = A(k)\,\underline{x}(k) + B(k)\,\underline{u}(k)$$

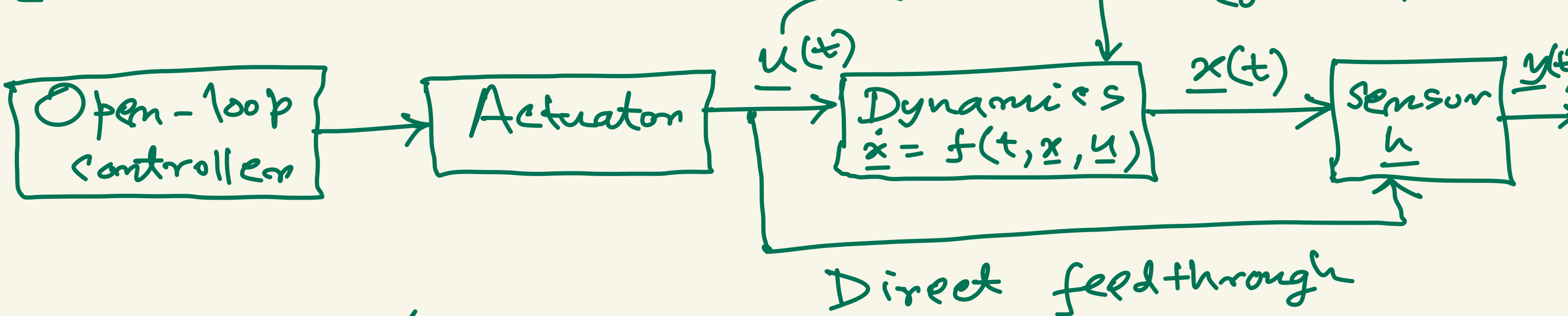
$$\underline{y}(k) = C(k)\,\underline{x}(k) + D(k)\,\underline{u}(k)$$

Nonlinear

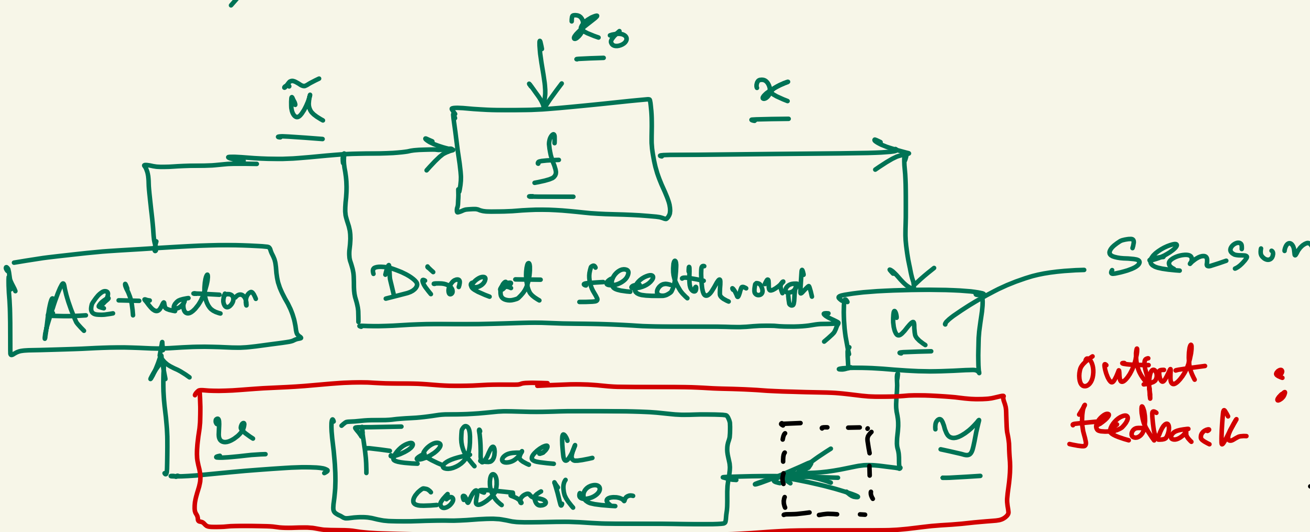
$$\underline{x}(k+1) = \underline{f}(k, \underline{x}(k), \underline{u}(k))$$

$$\underline{y}(k) = \underline{h}(k, \underline{x}(k), \underline{u}(k))$$

Open-loop control system:



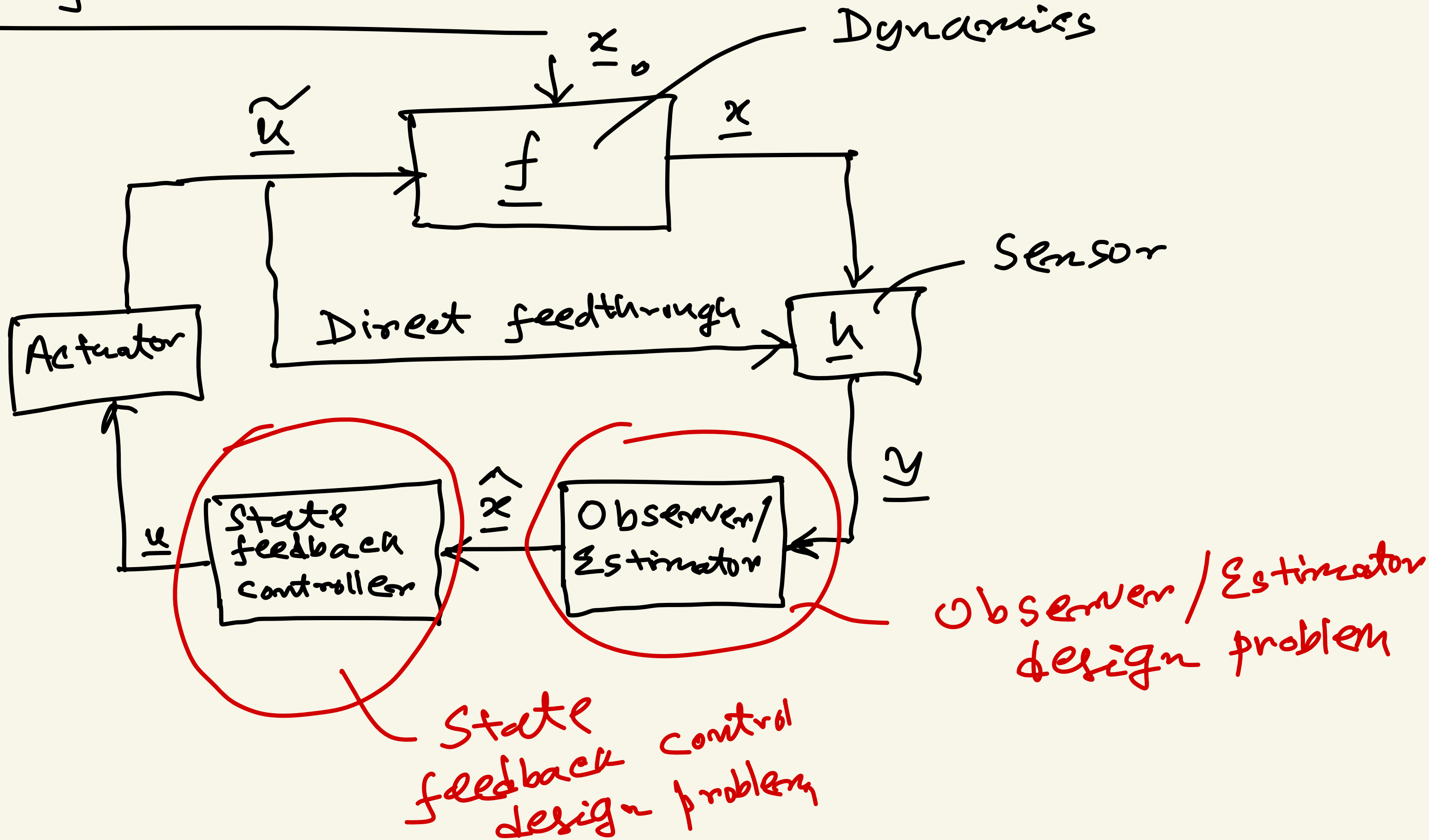
Closed-loop / Feedback control system:



Output feedback : $\underline{y} \mapsto \underline{u}$

$\hat{\underline{y}} \mapsto \hat{\underline{u}}$

State-feedback control:

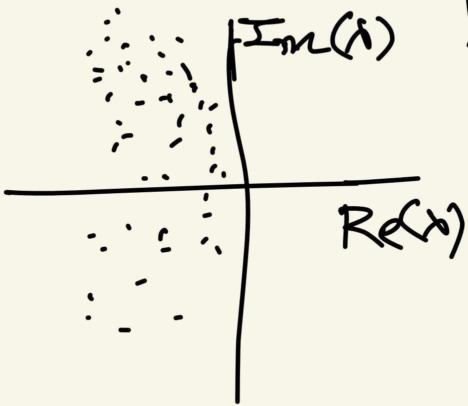


Linear System Stability:

$$\underbrace{\dot{\underline{x}}}_{n\times 1} = \underbrace{A}_{n\times n}\,\underbrace{\underline{x}}_{n\times 1}$$

Stable $\Updownarrow$ $\operatorname{Re}(\lambda_i(A)) < 0$ for all $i = 1, \dots, n$

A is Hurwitz stable

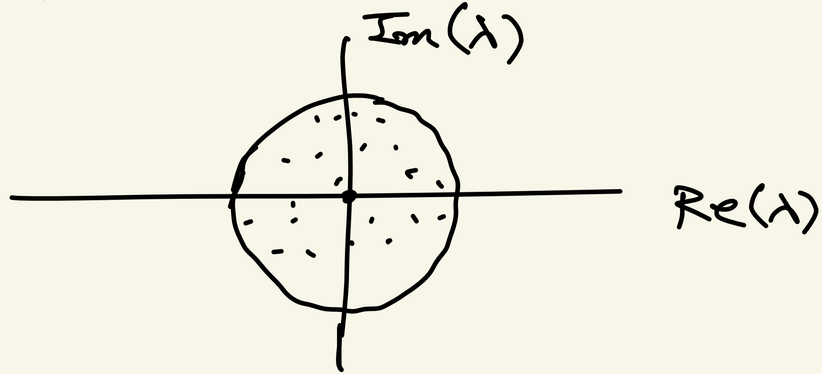


$$\underbrace{\underline{x}(k+1)}_{n\times 1} = \underbrace{A}_{n\times n}\,\underbrace{\underline{x}(k)}_{n\times 1}$$

Stable $\Updownarrow$ $\rho(A) := \max_i |\lambda_i(A)| < 1$

A is Schur-Cohn stable

$\Leftrightarrow$ $|\lambda_i(A)| < 1$ for all $i = 1, \dots, n$



Lyapunov Stability Theory:

Set up: Autonomous ODEs: $\dot{\underline{x}} = \underline{f}(\underline{x})$, $\underline{x} \in \mathcal{X} \subseteq \mathbb{R}^n$

$\underline{x}(0) = \underline{x}_0$ (given),

W.L.O.G. let $\underline{x}^* = \underbrace{\underline{0}}_{n\times 1}$ be a fixed point

Definition:

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Stable (S)	Asymptotically stable (A. S.)	Globally Asymp. Stable (G. A. S.)
$\underline{x}^* = \underline{0}$ is STABLE if for all $\epsilon > 0$, there exists $\delta = \delta(\epsilon) > 0$ s.t. $\|\underline{x}(0)\|_2 < \delta$ $\Rightarrow \|\underline{x}(t)\|_2 < \epsilon$ for all $t > 0$. $\Updownarrow$ If starting from $\underline{x}(0) \in B(\underline{x}^*, \delta)$, $\underline{x}^* = \underline{0}$, then $\underline{x}(t)$ does NOT leave $B(\underline{x}^*, \epsilon)$	$\|\underline{x}(0)\|_2 < \delta$ (Given δ) $\Downarrow$ $\lim_{t\to\infty} \underline{x}(t) = \underline{0} (\equiv \underline{x}^*)$ (convergence if we wait long enough) when A.S., we say $B(\underline{x}^*, \delta)$ is a Region of Attraction (RoA) "The" RoA = Largest such δ ball.	If δ is arbitrary (i.e.) $B(\underline{x}^*, \delta) \equiv \mathbb{R}^n$ $\Updownarrow$ $\forall\, \underline{x}(0) \in \mathbb{R}^n$, $\lim_{t\to\infty} \underline{x}(t) = \underline{0}$.

Stay arbitrarily close to $\underline{x}^* = \underline{0}$

(Staying close is NOT good enough for stability)

Example: (A.S. but GAS)

Multiple isolated stable (locally) fixed points } Each will have their local ROA

Example: (Pendulum): $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in [0, 2\pi) \times \mathbb{R}$

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = \underbrace{-\alpha \sin x_1}_{\text{gravity}} - \underbrace{\beta x_2}_{\text{damping}}, \quad \alpha > 0$$

$$\underbrace{\underline{x}_1^* = \begin{pmatrix} x_{11}^* \\ x_{12}^* \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}}, \quad \underline{x}_2^* = \begin{pmatrix} x_{21}^* \\ x_{22}^* \end{pmatrix} = \begin{pmatrix} \pi \\ 0 \end{pmatrix}.$$

- Stable but NOT A.S. if $\beta = 0$.
- Stable and A.S. if $\beta > 0$