Lecture # 16

Integrator Backstepping Thm.

Consider

$$\dot{\underline{\eta}} = \underline{f}(\underline{\eta}) + \underline{g}(\underline{\eta})\xi \quad \text{---} (*)$$

$$\dot{\xi} = u \quad \text{---} (**)$$

$\left.\right\} \underline{\eta} \in \mathbb{R}^n, \ \xi, u \in \mathbb{R}$

Let $\xi = \phi(\underline{\eta})$ be a stabilizing state feedback control for $(*)$ with $\phi(0) = 0$, & let $V(\underline{\eta})$ be a Lyapunov f^{n} for this feedback control, i.e.,

$$\frac{\partial V}{\partial \eta}\left(\underline{f}(\underline{\eta}) + \underline{g}(\eta)\,\phi(\underline{\eta})\right) \leq -\underbrace{W(\underline{\eta})}_{p.d.} \quad \forall\ \eta \in \mathcal{D}$$

(If $\mathcal{D} \equiv \mathbb{R}^n$ then G.A.S.)

Then the state feedback controller

$$u = \frac{\partial \phi}{\partial \eta}\left(\underline{f}(\underline{\eta}) + \underline{g}(\underline{\eta})\xi\right) - \frac{\partial V}{\partial \eta} g(\eta) - k\left(\xi - \phi(\underline{\eta})\right)$$

makes the origin in $\mathbb{R}^{n+1}$ A.S. with Lyap. f^{n}

$$V_c(\underline{\eta}, \xi) = V(\eta) + \frac{1}{2}\left(\xi - \phi(\underline{\eta})\right)^2.$$

Example : (Integrator Backstepping)

$$\left\{\begin{array}{l} \boxed{\dot{x}_1 = \overbrace{x_1^2 - x_1^3}^{f} + \overbrace{x_2}^{g \equiv 1}} \\ \dot{x}_2 = u \end{array}\right\} \quad \underline{x} \in \mathbb{R}^2, \; u \in \mathbb{R}$$

Let $\boxed{\begin{array}{l} \eta = x_1 \\ \xi = x_2 \end{array}}$

Start with this subsystem

Design $x_2 = \phi(x_1)$ such that $x_1 = 0$ stable

Can do this by choosing

e.g. $x_2 = \phi(x_1) = -x_1^2 - x_1$

killing unfriendly nonlinearity

This makes the first subsystem:

$$\dot{x}_1 = -x_1 - x_1^3$$

$$\& \; V(x_1) = \frac{1}{2} x_1^2 \Rightarrow \dot{V} = -x_1^2 - x_1^4 \leq -x_1^2 \quad \forall x_1 \in \mathbb{R}$$

$\therefore$ origin of the subsystem $\dot{x}_1 = -x_1 - x_1^3$ is GES (AS needs LaSalle)

To backstep, let

$$z_2 := x_2 - \phi(x_1)$$
$$= x_2 + x_1 + x_1^2$$

to get new system:

$$\dot{x}_1 = -x_1 - x_1^3 + z_2$$
$$\dot{z}_2 = u + (1+2x_1)(-x_1 - x_1^3 + z_2)$$

Take $V_c(x) = \frac{1}{2}x_1^2 + \frac{1}{2}z_2^2$

$$= \frac{1}{2}x_1^2 + \frac{1}{2}(x_2 + x_1 + x_1^2)^2$$

$$\Rightarrow \dot{V}_c = -x_1^2 - x_1^4 + z_2\left\{x_1 + (1+2x_1)(-x_1 - x_1^3 + z_2) + u\right\}$$

Take:

$$u = -x_1 - (1+2x_1)(-x_1 - x_1^3 + z_2) - z_2$$

$\Rightarrow \quad \dot{V}_c = -x_1^2 - x_1^4 - z_2^2 \Rightarrow$ origin G(A)ES

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$$\left.\begin{aligned}\dot{x}_1 &= x_1^2 - x_1^3 + x_2\\ \dot{x}_2 &= x_3\\ \dot{x}_3 &= u\end{aligned}\right\}$$

$$x_3 = -x_1 - (1+2x_1)(x_1^2 - x_1^3 + x_2) - (x_2 + x_1 + x_1^2)$$

$$=: \phi(x_1, x_2)$$

$$\& \; V(x_1, x_2) = \frac{1}{2}x_1^2 + \frac{1}{2}(x_2 + x_1 + x_1^2)^2$$

We need to further backstep:

$$z_3 = x_3 - \phi(x_1, x_2)$$

Find $u = u(x_1, x_2, x_3)$

and $V_c(x_1, x_2, x_3)$

Can work out same idea for slightly more general form:

$$\left.\begin{cases} \dot{\underline{\eta}} = \underline{f}(\underline{\eta}) + \underline{g}(\underline{\eta})\xi_s \\ \dot{\xi}_s = \underbrace{f_a(\underline{\eta},\xi_s) + g_a(\underline{\eta},\xi_s)u} \end{cases}\right\} \quad \begin{array}{l} \xi_s \in \mathbb{R}, \quad u \in \mathbb{R} \\ \underline{\eta} \in \mathbb{R}^n, \\ \text{assumption: } g_a(\underline{\eta},\xi_s) \neq 0 \end{array}$$

Can simply call "this whole thing" as new control u_a, i.e., the input transformation $u \longmapsto u_a$ given by

$$u = \frac{1}{g_a(\underline{\eta},\xi_s)}\left(u_a - f_a(\underline{\eta},\xi_s)\right)$$

brings ~~this~~ $\boxed{\text{system}}$ back to (*)-(**) form in the Thm.

Can do backstepping on that to get:

$$u = \phi_c(\underline{\eta},\xi_s) \sim \text{stabilizing controller}$$

overall feedback

$$= \frac{1}{g_a(\underline{\eta},\xi_s)}\left\{\frac{\partial\phi}{\partial\eta}\left(\underline{f}(\underline{\eta}) + \underline{g}(\underline{\eta})\xi_s\right) - \frac{\partial V}{\partial\eta}\underline{g}(\underline{\eta}) - K(\xi_s - \phi(\underline{\eta})) - f_a(\underline{\eta},\xi_s)\right\}$$

$$V_c(\underline{\eta},\xi_s) = V(\underline{\eta}) + \frac{1}{2}\left(\xi_s - \phi(\underline{\eta})\right)^2$$

overall Lyap. fn

We can generalize further: Can apply backstepping recursively for any nonlinear system of the form:

overall state vector $\begin{pmatrix} \underline{x} \\ \underline{z} \end{pmatrix} \in \mathbb{R}^{n+k}$

$$\dot{\underline{x}} = \underline{f}_0(\underline{x}) + \underline{g}_0(\underline{x}) z_1$$
$$\dot{z}_1 = f_1(\underline{x}, z_1) + g_1(\underline{x}, z_1) z_2$$
$$\dot{z}_2 = f_2(\underline{x}, z_1, z_2) + g_2(\underline{x}, z_1, z_2) z_3$$
$$\vdots$$
$$\dot{z}_{k-1} = f_{k-1}(\underline{x}, z_1, \ldots, z_{k-1}) + g_{k-1}(\underline{x}, z_1, \ldots, z_{k-1}) z_k$$
$$\dot{z}_k = f_k(\underline{x}, z_1, \ldots, z_k) + g_k(\underline{x}, z_1, \ldots, z_k) u$$

Here

$\underline{x} \in \mathbb{R}^n$, $z_i \in \mathbb{R}\ \forall\, i = 1, \ldots, k$, $f_0 : \mathbb{R}^n \mapsto \mathbb{R}^n$

$g_0 : \mathbb{R}^n \mapsto \mathbb{R}^n$

$f_i(\underline{0}) = 0\ \forall\, i = 0, 1, \ldots, k$

Assumption is that $g_i(\underline{x}, z_1, \ldots, z_j) \neq 0\ \forall\, 1 \leq j \leq k$

Same strategy: start from top-most eq$^{\underline{n}}$ & then back-step.

Block BackStepping : (Even further generalization)

$$\dot{\underline{\eta}} = \underbrace{\underline{f}(\underline{\eta})}_{n\times 1} + \underbrace{G(\underline{\eta})}_{n\times m} \underbrace{\underline{\xi}}_{m\times 1} \quad \text{--- (1)}$$

$$\dot{\underline{\xi}} = \underline{f_a}(\underline{\eta}, \underline{\xi}) + G_a(\underline{\eta}, \underline{\xi})\, \underline{u} \quad \text{--- (2)}$$

$$\underline{\eta} \in \mathbb{R}^n, \quad \underline{\xi} \in \mathbb{R}^m, \quad u \in \mathbb{R}^m, \quad m > 1$$

Here overall state vector $\begin{pmatrix} \underline{\eta} \\ \underline{\xi} \end{pmatrix} \in \mathbb{R}^{m+n}$

Suppose (1) can be stabilized by

$$\underline{\xi} = \phi(\underline{\eta}) \text{ with } \phi(\underline{0}) = 0$$

& Let $V(\underline{\eta})$ be the Lyap. f^n for that subsystem.

$f_a \in \mathbb{R}^m$

$G_a \in \mathbb{R}^{m\times m}$, $\quad G \in \mathbb{R}^{n\times m}$, $\quad$ Assumption is that G_a is non-singular

The overall stabilizing state feedback is

$$\underbrace{\underline{u}}_{m\times 1} = \underbrace{G_a^{-1}}_{m\times m}\Bigg[\underbrace{\frac{\partial \overset{m\times 1}{\phi}}{\partial \underset{n\times 1}{\eta}}}_{\substack{\text{Jacobian} \\ m\times n}}\underbrace{\Big(\underbrace{\underline{f}}_{n\times 1} + \underbrace{G}_{n\times m}\,\underbrace{\xi_s}_{m\times 1}\Big)}_{n\times 1} - \underbrace{\underbrace{G^T}_{m\times n}\,\underbrace{\frac{\partial V}{\partial \eta}}_{n\times 1}}_{m\times 1} - \underbrace{\underline{f}_a}_{m\times 1} - \underbrace{K\Big(\underbrace{\xi_s - \phi}_{m\times 1}\Big)}_{m\times 1}\Bigg]$$

for some $K > 0$

$$V_c : \mathbb{R}^{m+n} \mapsto \mathbb{R}^+_0$$

$$\& \quad \underbrace{V_c(\underline{\xi}, \underline{\eta})}_{\text{overall Lyap. f}^{\underline{n}}} = V(\underline{\eta}) + \frac{1}{2}\left(\xi_s - \phi(\underline{\eta})\right)^T\left(\xi_s - \phi(\underline{\eta})\right)$$

Geometric Control Theory

Design Idea #5
Feedback linearization

Motivation: SISO system

(will later generalize to MIMO)

$$\left\{\begin{aligned} \dot{\underline{x}} &= \underline{f}(\underline{x}) + \underline{g}(\underline{x})\,u \\ y &= h(\underline{x}) \end{aligned}\right. \qquad \begin{aligned} &\underline{x} \in \mathbb{R}^n \\ &u, y \in \mathbb{R} \end{aligned}$$

(SISO_original)

Question: Does there exist change-of-variable

$$\underline{z} := \begin{bmatrix} \eta \\ \underline{\xi} \end{bmatrix} = \tau(\underline{x}),$$

where $\tau(\cdot)$ is a diffeomorphism

$\Updownarrow$

(τ is C^1

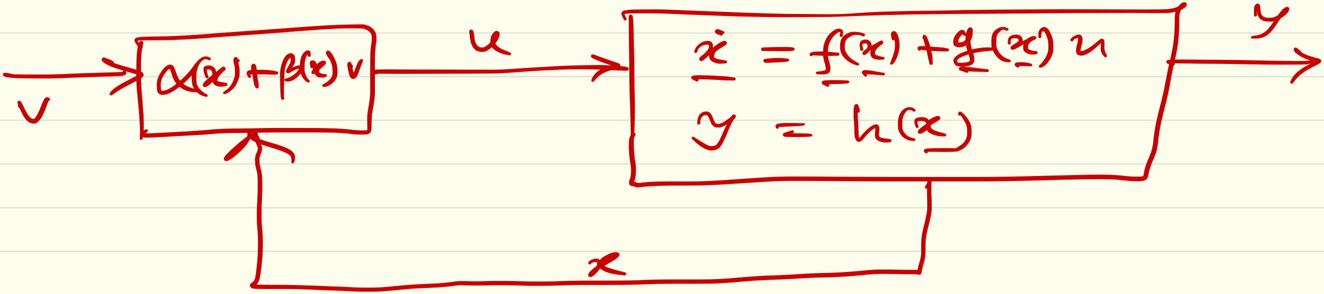
τ^{-1} is also C^1)

and a control

$$u = \alpha(\underline{x}) + \beta(\underline{x})\, v$$

such that $(T(\cdot), \alpha(\cdot), \beta(\cdot))$ transforms (SISO_original) into

<u>either</u> (Form 1) $\quad \dot{\underline{z}} = A \underline{z} + B v$



<u>or</u> (Form 2)

$$\dot{\eta} = f_0(\eta, \xi)$$
$$\dot{\xi} = A \xi + B v$$
$$y = C \xi$$

If the ANSWER is Yes for $\boxed{\text{Form 1}}$, then we say the original nonlinear system (SISO_original) is

"<u>feedback Linearizable</u>"

If Yes for $\boxed{\text{Form 2}}$, we say (SISO_original) is "<u>Input – Output Linearizable</u>"

(State) Feedback linearization :

<u>Static</u> $(\tau, (\alpha, \beta))$

$$\underline{z} := \begin{bmatrix} \eta \\ \underline{\xi} \end{bmatrix} = \tau(\underline{x})$$

$$u = \alpha(x) + \beta(x)\, v$$

<u>Dynamic</u> $(\tau, (\alpha, \beta, \gamma, \delta))$

$$\underline{z} := \begin{bmatrix} \eta \\ \underline{\xi} \end{bmatrix} = \tau(\underline{x})$$

$$\dot{x}_c = \gamma(\underline{x}, \underline{x}_c) + \delta(\underline{x}, \underline{x}_c)\, v$$

$$u = \alpha(\underline{x}, \underline{x}_c) + \beta(\underline{x}, \underline{x}_c)\, v$$

Example : Pendulum in air

$$x_1 \equiv \theta \qquad x_2 = \dot{x}_1 = \dot{\theta}$$

$$\left.\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\alpha \sin x_1 - \beta x_2 + u \end{aligned}\right\} \alpha, \beta > 0$$

" Cancel the nonlinearity & get stable linear closed-loop system" approach:

$$u = \underbrace{+\alpha \sin x_1}_{\alpha(x)} + \underbrace{1}_{\beta(x)} \cdot v = \alpha(x) + \beta(x)\, v$$

Then we get:

$$\left.\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\beta x_2 + v \end{aligned}\right\} \text{LTI control system.}$$

Can design

$$v = -k_1 x_1 - k_2 x_2 \Rightarrow \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{bmatrix} 0 & 1 \\ -k_1 & -k_2-\beta \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Choose k_1, k_2 s.t. Hurwitz even if $\beta = 0$

Cannot always kill nonlinearity

Example

$$\dot{x}_1 = a \sin x_2$$
$$\dot{x}_2 = -x_1^2 + u$$

Cannot choose "u" to cancel "$a \sin x_2$".

But consider

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \tau(\underline{x}) = \left\{ \begin{matrix} x_1 \\ a \sin x_2 \end{matrix} \right\} = \left\{ \begin{matrix} x_1 \\ \dot{x}_1 \end{matrix} \right\}$$

Then dynamics in new variables:

$$\dot{z}_1 = z_2$$
$$\dot{z}_2 = a(\cos x_2)\dot{x}_2$$
$$= a(\cos x_2)(-x_1^2 + u)$$

$$\dot{z}_1 = z_2$$
$$\dot{z}_2 = v$$

This is feedback linearized

Now can kill nonlinearity by control

$$u = \underbrace{x_1^2}_{\alpha(x)} + \underbrace{\frac{1}{a \cos x_2}}_{\beta(x)} v$$

(well defined for $-\pi/2 < x_2 < +\pi/2$)