Lecture #6

From Lecture #5, we need
Locally Lip + "Something"

$\Downarrow$

Thm: Let $\underline{f}(\underline{x}, t)$ be p.w. continuous in time t & locally Lip. in $\underline{x}$ $\forall\, t \geq t_0$, and $\forall\, \underline{x} \in \mathcal{D} \subset \mathbb{R}^n$.

Let $\mathcal{W} \subset \mathcal{D}$ be a compact (closed & bdd.) set such that $\underline{x}_0 \in \mathcal{W}$.

Suppose $\mathcal{W}$ is (+ve)-ly invariant in time.

(This is the "something")

[Recall defn. of (+ve)-ly invariant from earlier lectures]

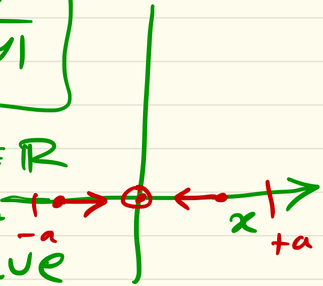
Then $\underbrace{\exists!\ \text{sol}^{\underline{n}}}_{\underline{x}(t,\, \underline{x}_0)}$ $\forall\, t \geq t_0$.

whenever $\underline{x}_0 \in \mathcal{W}$

Example: $\dot{x} = -x^3 = f(x)$ (again)

Notice that $\text{sgn}(x(t))$ is opposite to $\text{sgn}(\dot{x}(t))$

$$\boxed{\text{sgn}(y) := \frac{y}{|y|}}$$



$\therefore$ If $x_0 = a \in \mathbb{R}$ then the $\text{sol}^{\underline{n}}$ cannot leave the set

$$\mathcal{W} := \{x \in \mathbb{R} : |x| \leq |a|\}$$

$\mathcal{W}$ is compact

$\therefore$ Without computing the $\text{sol}^{\underline{n}}$ of ODE, we conclude $\exists!\ x(t)$ $\forall\, t \geq 0$

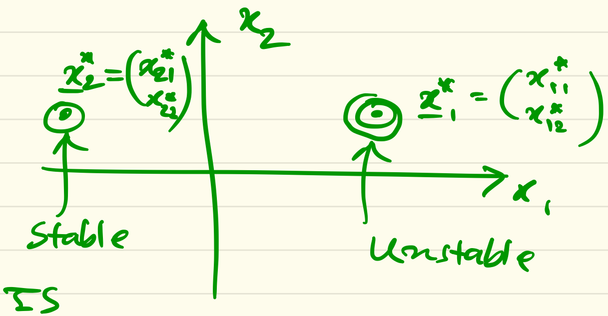
Lyapunov Stability Theory

Setup: Autonomous ODEs $\dot{\underline{x}} = f(\underline{x})$, $\underline{x} \in \mathbb{R}^n$, $\underline{x}(0) = \underline{x}_0$ (given) } WLOG Let $\underline{x}^* = \underline{0}$ be a fixed pt.

Defn: ⟶ Stronger

Stable (S)	Asymptotically Stable (A.S)	Globally Asymp. Stable G.A.S
$\underline{x}^* = \underline{0}$ is STABLE if $\forall\, \epsilon > 0,\ \exists\, \delta = \delta(\epsilon) > 0$ s.t. $\Vert\underline{x}(0)\Vert_2 < \delta \Rightarrow \Vert\underline{x}(t)\Vert_2 < \epsilon$ $\forall\, t \geq 0$ $\Updownarrow$ Start from $\underline{x}(0) \in \mathcal{B}(\underset{\underset{\underline{0}}{\parallel}}{\underline{x}^*}, \delta)$ $\Downarrow$ don't leave $\underline{x}(t) \in \mathcal{B}(\underset{\underset{\underline{0}}{\parallel}}{\underline{x}^*}, \epsilon)$ $\Updownarrow$ Stay ARBITRARILY close (staying close is NOT enough for stability) $\Updownarrow$ NOT stable = Unstable	$\Vert x(0)\Vert_2 < \delta$ $\Downarrow$ $\lim_{t\to\infty} \underline{x}(t) = \underline{0}$ ($\Leftrightarrow \underline{x}^*$) (Convergence in time) When A.S., we say $\mathcal{B}(\underset{\underset{(\underline{x}^*)}{\parallel}}{\underline{0}}, \delta)$ is a region of attraction (ROA) "The" ROA = Largest such $\mathcal{B}(\underline{0}, \delta)$	Replace $\mathcal{B}(\underline{0}, \delta)$ by $\mathbb{R}^n$, in A.S. $\forall\, x(0) \in \mathbb{R}^n$, $\lim_{t\to\infty} \underline{x}(t) = \underline{0}$

Example (A.S. but NOT G.A.S.)



Is $\underline{x}_2^*$ A.S. ?

Is $\underline{x}_2^*$ G.A.S. ?

Example (Pendulum)

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = \underbrace{-\alpha \sin x_1}_{\text{gravity}} \underbrace{- \beta x_2}_{\text{damping}}$$

Suppose, $\alpha > 0$,

$\beta = 0 \rightarrow \underline{x}_1^* = \begin{pmatrix} x_{11}^* \\ x_{12}^* \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\underline{x}_2^* = \begin{pmatrix} x_{21}^* \\ x_{22}^* \end{pmatrix} = \begin{pmatrix} \pi \\ 0 \end{pmatrix}$

Stable but NOT A.S.

$\beta > 0 \rightarrow$ Stable A.S.

A.S. does not say rate-of-convergence

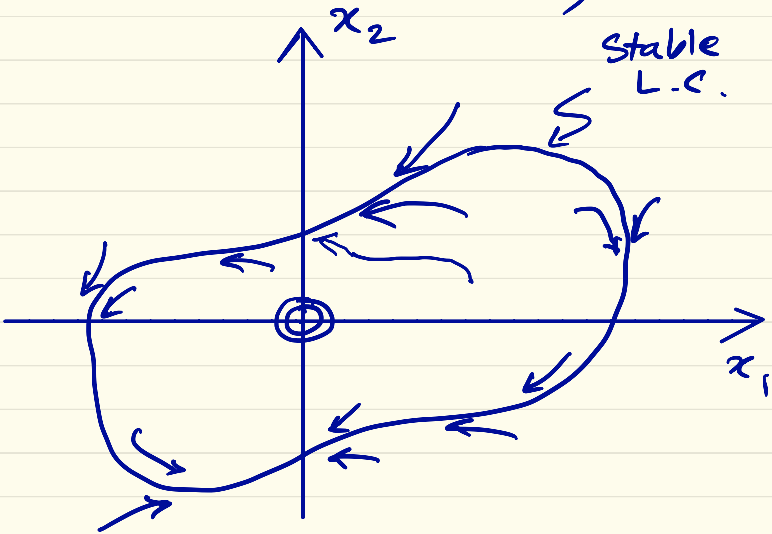
except LTI $\Rightarrow$ A.S. implies E.S.

(exponential convergence)

Example: (Van der pol Oscillator)

$$\left.\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 + (1 - x_1^2)\, x_2\end{aligned}\right\}$$

$\underline{x}^* = (0,0)$ is unique fixed pt.



Even though $\underline{x}(t) \not\to \infty$

still we say Origin is NOT stable

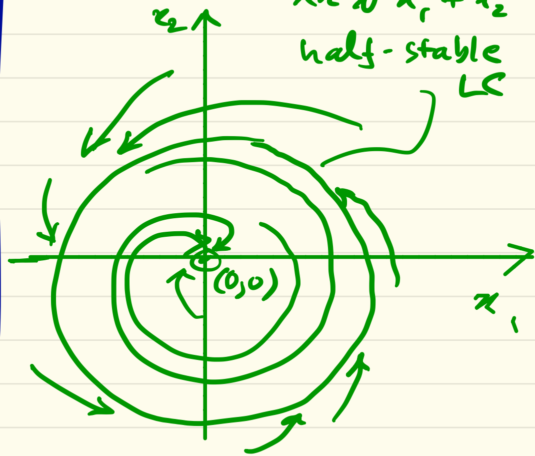
Staying close is NOT enough,

("arbitrarily" close)

Example: (HW1 P2)

$$\dot{x}_1 = -x_1 - \frac{x_2}{\ln\sqrt{x_1^2 + x_2^2}}$$

$$\dot{x}_2 = -x_2 + \frac{x_1}{\ln\sqrt{x_1^2 + x_2^2}}$$



Origin Unique F.P.

→ Stable ✓

→ A.S. ✓

→ But not GAS

→ The ROA: open unit disc

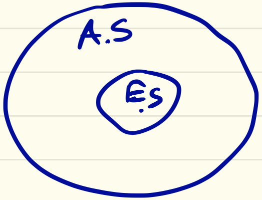
Exponential Stability (Def^{n})

$\underline{x}^* = 0$ is Exp. Stable (E.S)

if $\forall\, \underline{x}(0) \in \mathcal{B}(\underline{0}, \delta)$,

$\exists\, \epsilon_1, \epsilon_2$ s.t. $\forall\, t > 0$,

$$\|\underline{x}(t)\|_2 \leq \underbrace{\epsilon_1 e^{-\epsilon_2 t}}_{\epsilon}$$



Example: (NOT E.S.)

$\dot{x} = -x^2, \quad x(0) = 1$

$x(t) = \dfrac{1}{1+t}$

As $t \to \infty$, $x(t) \to 0$
(A.S.) ✓

BUT slower than e^{-t}

(Rate-of-convergence

How to prove Stability / A.S / G.A.S. etc.

Thm. (Lyapunov, 1892)

Let $\underline{x}^* = \underline{0}$ be a fixed pt. for $\dot{\underline{x}} = \underline{f}(\underline{x})$, $\underline{x} \in \mathbb{R}^n$

and $\mathcal{D} \subset \mathbb{R}^n$ be a domain containing $\underline{x}^*$

Let $V : \mathcal{D} \mapsto \mathbb{R}_0^+$ be a C^1 function of $\underline{x}$ such that $(V(\underline{x}))$

Positive semi definite f^{n}:

① $V(\underline{0}) = 0$ (here $\underline{0} \equiv \underline{x}^*$)

② $V(\underline{x}) > 0 \;\; \forall\, \underline{x} \in \mathcal{D} \setminus \{\underline{x}^*\}$ ($\{\underline{x}^*\} \equiv \{0\}$)

③ $\dot{V}(\underline{x}) \leq 0 \;\; \forall\, x \in \mathcal{D}$

① ✓

② ✓

③' $\dot{V}(\underline{x}) < 0 \;\; \forall\, x \in \mathcal{D} \setminus \{0\}$ ($\{0\} \equiv \{x^*\}$)

If ① + ② + ③' then we say $V(\underline{x})$ is positive definite f^{n}

If $V(\underline{x})$ is P.S.D. then $\underline{x}^*=\underline{0}$ is STABLE

If $V(\underline{x})$ is P.D. then $\underline{x}^*=\underline{0}$ is A.S.

The function $V(\underline{x})$ is called Lyapunov $f^{\underline{n}}$.

If $\dot{V} \leq 0$, then we are NOT sure if $\lim_{t\to\infty} x(t)=0$ or NOT. (A.S.)

but still can guarantee "stable".

Interpretation of the cond$^{\underline{n}}$ $\dot{V} \leq 0$

$\dot{V} \leq 0 \iff$ Whenever a traj. crosses a level set of $V(\underline{x})$, ($V(\underline{x}) = c$ for some $c>0$) if then moves inside the set

(+ve)ly invariant in time

$$\Omega_c := \{\underline{x} \in \mathbb{R}^n : V(\underline{x}) \leq c\}$$

& can never come out of Ω_c again.

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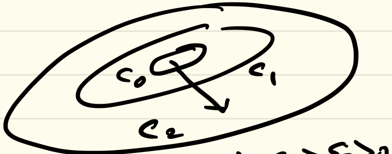
① As c decreases, the set $V(\underline{x}) = c$ shrinks to origin

$\Updownarrow$

$x(t) \to 0$ as $t \to \infty$



$c_2 > c_1 > c_0 > 0$

Level sets of Lyapunov function shown

<u>Example</u> : (Pendulum in air)

$\dot{x}_1 = x_2$

$\dot{x}_2 = -\alpha \sin x_1$ $\left(\beta = 0 \right)$ without damping

$V(\underline{x}) = V(x_1, x_2)$

$= \underbrace{\alpha(1-\cos x_1)}_{\text{Potential energy}} + \underbrace{\tfrac{1}{2} x_2^2}_{\text{Kinetic energy}}$

$V(0,0) = 0$

$V(x_1, x_2) \geq 0 \quad \forall \ x_1, x_2$

$> 0 \quad \forall \ x_1, x_2 \neq x_1^*, x_2^*$

$\dot{V}(\underline{x}) := \frac{d}{dt} V(\underline{x})$

$= \frac{\partial}{\partial \underline{x}} V(\underline{x}) \cdot \frac{d\underline{x}}{dt}$

$= (\nabla V)^T \underline{f}(\underline{x})$

$= \left(+ \alpha \sin x_1 \, \hat{e}_{x_1} + x_2 \, \hat{e}_{x_2} \right)^T$

$\left(x_2 \, \hat{e}_{x_1} \quad -\alpha \sin x_1 \, \hat{e}_{x_2} \right)$

$= \alpha x_2 \sin x_1 - \alpha x_2 \sin x_1 = 0$ but NOT < 0

$\therefore$ $(0,0)$ is STABLE but NOT A.S.