Lecture # 19

Recall: SISO normal form (Pg. 9, Lecture #18)

Relative deg. $r \leq n$

$$\dot{\underline{x}} = \underline{f}(\underline{x}) + \underline{g}(\underline{x})u$$

$$y = h(\underline{x})$$

$\underline{x} \in \mathbb{R}^n$

$u, y \in \mathbb{R}$

If Both state & O/p. eq^{n}s are given.
then can do
Input-output linearization
(may be local)

$$\dot{z}_1 = z_2$$
$$\dot{z}_2 = z_3$$
$$\vdots$$
$$\dot{z}_{r-1} = z_r$$
$$\dot{z}_r = \boxed{b(\underline{z}) + a(\underline{z})\, u} =: v$$
$$\dot{z}_{r+1} = q_{r+1}(\underline{z})$$
$$\vdots$$
$$\dot{z}_n = q_n(\underline{z})$$

& $y = z_1$

mapping

$(\underset{\mathbb{R}^n}{\underline{x}}, \underset{\mathbb{R}}{u}) \mapsto (\underset{\mathbb{R}^n}{\underline{z}}, \underset{\mathbb{R}}{v})$

$$\underset{n\times 1}{\underline{z}} = \begin{pmatrix} \underline{\xi} \\ \\ \underline{\eta} \end{pmatrix}$$

If O/p. eq^{n} NOT given
then we say (partial) feedback linearization for $r < n$

& full state feedback lineariz^{n} for $r = n$

$$\underset{r\times 1}{\underline{\xi}} := \begin{pmatrix} z_1 \\ \vdots \\ z_r \end{pmatrix}, \quad \underset{(n-r)\times 1}{\underline{\eta}} := \begin{pmatrix} z_{r+1} \\ \vdots \\ z_n \end{pmatrix}, \quad \underline{z}_{n\times 1} = \begin{pmatrix} \underline{\xi} \\ \hline \underline{\eta} \end{pmatrix}_{n\times 1}$$

$$\left.\begin{aligned} \dot{\underline{\xi}} &= \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & & 0 \\ \vdots & & & & & \vdots \end{bmatrix} \underline{\xi} + \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} v \\ \dot{\underline{\eta}} &= \underline{q}(\underline{z}) = \underline{q}(\underline{\xi}, \underline{\eta}) \end{aligned}\right\} \text{after I/O lineariz}^{\underline{n}}$$

old state $(\underline{x}) \in \mathbb{R}^n$ $\longmapsto$ new state $(\underline{z}) \in \mathbb{R}^n$

$\begin{pmatrix}\text{old o/p eq}^{\underline{n}} \\ y = h(\underline{x})\end{pmatrix}$ $\longmapsto$ new o/p. eq$^{\underline{n}}$: $\boxed{y = z_1}$

② Zeroing the output Problem:

Question: Find, if possible, all pairs $\underbrace{(\underline{x}_0,\ \underline{u})}_{\text{original coordinates}}$

Initial Condition ($\underline{x}_0$), Control ($\underline{u}$)

s.t. $y(t) \equiv 0 \quad \forall\, t$

identically ($\equiv$)

(NOT just trivial pair
($\underline{x}_0 = 0$, $u(t) \equiv 0$)
(assuming $f(0) = 0$ for unforced)

Analysis:

First, do I/O linearization.

$$y(t) = z_1(t) \equiv 0 \quad \forall\, t$$

$$\Rightarrow \dot{z}_1 = 0 \Rightarrow \dot{z}_1(t) = \dot{z}_2(t) = \dots = \dot{z}_r(t) = 0 \ \forall\, t$$

$$\Rightarrow \underline{\xi}(t) = 0 \quad \forall\, t$$

On the other hand,
$u(t)$ must be <u>unique</u> sol^n of the nonlinear algebraic eq^n:

$$\dot{\cancel{z_r}}_{\to 0} = b(\underbrace{\cancel{\underline{\xi}(t)}}_{\underline{0}}, \underline{\eta}(t)) + a(\underbrace{\cancel{\underline{\xi}(t)}}_{0}, \underline{\eta}(t))\, u(t)$$

$$\Leftrightarrow \boxed{0 = b(\underline{0}, \underline{\eta}) + a(0, \underline{\eta})\, u} \quad \forall\, t$$

Also, $\dot{\underline{\eta}}(t) = \underline{q}(\underline{0}, \underline{\eta}(t)) \quad \forall\, t$

$\therefore$ If we want $y(t) \equiv 0 \;\forall\, t$, then the (original co-ordinate) initial condition $\underline{x}_0$ must be such that $\underline{\xi}(0) = \underline{0}$

& $\underline{\eta}(0) = \underline{\eta}_0$ (arbitrary)

According to the value of $\underline{\eta}_0$ chosen, the input must be

$$u(t) = -\frac{b(\underline{0}, \underline{\eta}(t))}{a(\underline{0}, \underline{\eta}(t))}, \quad a \text{ is invertible } a(\cdot) \neq 0 \ \forall t$$

where $\eta(t)$ solves the IVP:

$$\begin{cases} \dot{\underline{\eta}} = \underline{q}(\underline{0}, \underline{\eta}(t)) \\ \underline{\eta}(0) = \underline{\eta}_0 \end{cases}$$

← called "zero dynamics"

$\therefore$ For each set of initial data $(\underline{\xi}(0), \underline{\eta}(0)) = (\underline{0}, \underline{\eta}_0)$, the input $u(t)$ is unique to hold

$$y(t) \equiv 0 \ \forall t$$

Analogy with LTI system

- Rel. degree $(r) = \big|(\text{deg. of num. poly. of } G(s)) - (\text{deg. of deno. poly. of } G(s))\big|$

For $r \leq n$

$$\frac{(s+1)(s+3)}{s^2 - 3s + 7}$$

$$\underbrace{G(s)}_{\text{SISO transfer fn}} = C(sI - A)^{-1}B = C\,\frac{\operatorname{adj}(sI - A)}{\det(sI - A)}\,B$$

$\Leftrightarrow \quad r = |\#\text{ of zeros} - \#\text{ of poles}|$

- If $r = n$, then # of zeros $= 0$

e.g. $G(s) = \dfrac{5}{(s+1)(s-2)^2}$

In general, $(r<n)$

$$G(s) = K \frac{b_0 + b_1 s + \dots + b_{n-r-1} s^{n-r-1} + s^{n-r}}{a_0 + a_1 s + \dots + a_{n-1} s^{n-1} + s^n}$$

Minimal realization problem $(tf \longmapsto ss)$

NOT unique

Given $G(s)$, find (A, B, C) s.t. the triple (A,B,C) is both controllable & observable

→ This has unique answer of the form:

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ K \end{bmatrix},$$

$$C = [\, b_0 \;\; b_1 \;\; \cdots \;\; b_{n-r-1} \;\; 1 \;\; 0 \;\; \cdots \;\; 0 \,]$$

Now take it in normal form:

Define $\tau(.)$:

$$
\begin{aligned}
z_1 &= Cx \\
z_2 &= CAx \\
&\vdots \\
z_r &= CA^{r-1}x \\
\hline
z_{r+1} &= x_1 \\
z_{r+2} &= x_2 \\
&\vdots \\
z_n &= x_{n-r}
\end{aligned}
$$

$\text{Jac}(\tau)$ nonsingular

$\Updownarrow$

"admissible"

$\Updownarrow$

$\underline{z} = \underline{\tau}(\underline{x})$

$\text{Jac}(\underline{\tau})$

$$
\begin{bmatrix} [\ * \ ] & \begin{bmatrix} 1 & & 0 \\ & 1 & \\ * & & 1 \end{bmatrix} \\ I & 0 \end{bmatrix}
$$

$\Leftrightarrow \det(\text{Jac}(\underline{\tau})) \neq 0$

Normal form

$$
\begin{aligned}
\dot{z}_1 &= z_2 \\
\dot{z}_2 &= z_3 \\
&\vdots \\
\dot{z}_{r-1} &= z_r \\
\dot{z}_r &= R\xi + S\eta + ku \\
\dot{\eta} &= P\xi + Q\eta
\end{aligned}
$$

$R, S \rightarrow$ row vectors

$P, Q \rightarrow$ matrices.

$\therefore$ Zero dynamics of LTI : $\quad \underbrace{\dot{\underline{\eta}}}_{(n-r)\times 1} = Q \underbrace{\underline{\eta}}_{(n-r)\times 1}$

with

$$Q = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -b_0 & -b_1 & -b_2 & \cdots & -b_{n-r-1} \end{bmatrix}$$

~~alg.~~ Lin. exercise:

Show that $\mathrm{eig}(Q) =$ zeros of $G(s)$

$\therefore$ for LTI, zero dyn. $\Leftrightarrow \dot{\underline{\eta}} = Q \underline{\eta}$

where Q has spectrum same as the zeros of transfer f^{n}

Remark: If instead of $y(t) \equiv 0 \ \forall t$, we want $y(t) \equiv y_{ref}(t) \ \forall t$, we call it "Reproduce Ref. O/p. problem".

Then again, $\underline{\xi}(0) = \underline{\xi}_{ref}(0)$

$$\underline{\eta}(0) = \underline{\eta}_0 \ \text{(arbitrary)}$$

where $z_i(t) = y_{ref}^{(i-1)}(t) \quad \forall t, \quad 1 \le i \le r$

Set

$$\underline{\xi}_{ref}(t) = \begin{pmatrix} y_{ref}(t) \\ y_{ref}^{(1)}(t) \\ \vdots \\ y_{ref}^{(r-1)}(t) \end{pmatrix}$$

Then

$$y_{ref}^{(r)}(t) = b(\underline{\xi}_{ref}(t), \eta(t)) + a(\underline{\xi}_{ref}(t), \eta(t))\, u(t)$$

After setting $\xi(0) = \xi_{ref}(0)$

$$\underline{\eta}(0) = \underline{\eta}_0 \quad (\text{arbitrary})$$

then

$$\underset{\text{unique}}{\underline{u(t)}} = \frac{y_{ref}^{(r)}(t) - b(\underline{\xi}_{ref}(t), \underline{\eta}(t))}{a(\underline{\xi}_{ref}(t), \underline{\eta}(t))}$$

where $\underline{\eta}(t)$ solves the new IVP:

$$\dot{\underline{\eta}}(t) = q(\underline{\xi}_{ref}(t), \underline{\eta}(t))$$

$$\underline{\eta}(0) = \underline{\eta}_0$$

Again $u(t)$ is unique for given $(\xi_{ref}(t), \eta_0)$

$(\underline{\xi}_0, \underline{\eta}_0)$

Importance of zero dynamics:

Asymptotic Stabilization Problem:

Given $\dot{\underline{x}} = \underline{f}(\underline{x}) + \underline{g}(\underline{x})\,u$, with $\underline{f}(\underline{0}) = \underline{0}$

$\underline{x} \in \mathbb{R}^n$

$u \in \mathbb{R}$

Find feedback control with $u(0)=0$

$u = u(x)$ feedback s.t. closed-loop is A.S. at $\underline{x} = \underline{0}$

To solve it: First do (partial) feedback linearization

Thm: If zero dynamics is A.S. then (partial) feedback linearized system is also (i.e.) A.S. at $\underline{x} = \underline{0}$.

ξ is thought of as input

For GAS stabilization you need $\dot{\eta} = q(\xi, \eta)$ dynamics to be ISS.
(GAS of zero dyn. is NOT enough)

Controllability

Consider affine nonlinear control system:

$$\underline{\dot{x}} = \underline{f}(\underline{x}) + \sum_{i=1}^{m} g_i(\underline{x})\, u_i \,, \qquad \underline{x} \in \mathcal{X} \subseteq \mathbb{R}^n, \quad \underline{u} \in \mathcal{U} \subseteq \mathbb{R}^m$$

$\underline{f}(\underline{x}), g_1(\underline{x}), \ldots, g_m(\underline{x})$ are all assumed to be "analytic" vectorfields

We say:

$\underline{f}(\underline{x})$ as "drift vector field"

& $g_1(\underline{x}), \ldots, g_m(\underline{x})$ as "input vector fields"

In general, $\mathcal{X}$ & $\mathcal{U}$ can be manifolds

We say the system is <u>controllable</u> if for given pair $(\underline{x}_0, \underline{x}_1) \in \mathcal{X} \times \mathcal{X}$, $\exists\, T > 0$, & $\exists$ an admissible $u(t) \in \mathcal{U}$ $\forall\, t \in [0,T]$

s.t. $\quad x(t=0) = \underline{x}_0$ & $x(t=T) = \underline{x}_1$