AMS 231: Nonlinear Control Theory: Winter 2018

Homework #3

Name: ....

Due: February 13, 2018

NOTE: Please show all the steps in your solution. Turn in a hard copy of your HW stapled with this as cover sheet with your name written in the above field. Please submit your HW in class on the due date.

Problem 1

Lyapunov Theory for Continuous-time LTI System

$((2+7)+(2+2+2)+3+2+(10+10)+5=45$ **points)**

Consider the n-dimensional LTI system

$$\dot{\underline{x}} = A\,\underline{x}, \qquad A \in \mathbb{R}^{n\times n}, \qquad \underline{x}(0) = \underline{x}_0.$$

Since origin is the unique fixed point, the notions of A.S., G.A.S. and E.S. coincide.

(a) Write down the explicit solution for $\underline{x}(t)$ in terms of A, t and $\underline{x}_0$. Substituting this solution in the condition $\lim\limits_{t\to\infty} \underline{x}(t) = \underline{0}$, prove that origin is G.A.S. iff all eigenvalues of A lie in the open left half (complex) plane. Such a matrix is called Hurwitz. (Hint: Be careful about non-diagonalizable A.)

(b) A symmetric matrix P is called "positive (resp. negative) definite matrix" if $\underline{x}^\top P \underline{x} >$ (resp. $<$) 0 for all $\underline{x} \in \mathbb{R}^n$. Symbolically, we write $P \succ$ (resp. $\prec$) 0, which needs to be understood as matrix inequality. So for example, $P_1 \succ P_2$ means that $P_1 - P_2 \succ 0$.

For any given $P \succ 0$, prove that $V(\underline{x}) = \underline{x}^\top P \underline{x}$ is a positive definite function. Geometrically, what do the level sets of such a function V represent? Argue whether such V is radially bounded or unbounded.

(c) Motivated by your arguments in part (b), use $V(\underline{x}) = \underline{x}^\top P \underline{x}$ as the Lyapunov function to prove that the LTI system is G.A.S. if the matrix function $\mathcal{L}(P) := A^\top P + PA \prec 0$. This condition is called "Lyapunov matrix inequality".

(d) Argue that the condition $A^\top P + PA \prec 0$ in part (c) is equivalent to the statement: for any $Q \succ 0$, there exists $P \succ 0$ that solves the linear matrix equation $\mathcal{L}(P) = -Q$. This equation is called "Lyapunov (algebraic) matrix equation".

(e) We have shown in parts (b), (c), (d) that existence of solution for the Lyapunov matrix equation (equivalently, Lyapunov matrix inequality) implies G.A.S. i.e., A is Hurwitz. Now prove the converse, i.e., if A is Hurwitz then for any $Q \succ 0$, there exists unique $P \succ 0$ that solves $\mathcal{L}(P) = -Q$. (Hint: Prove existence by construction. Prove uniqueness by contradiction.)

(f) For an LTI system with A Hurwitz, prove that if $Q_1 \succ Q_2$, then $P_1 \succ P_2$.

Problem 2

Global Uniform Asymptotic Exponential Stability for Continuous-time LTV System (25 points)

Consider the LTV system

$$\dot{\underline{x}} = A(t)\underline{x}, \qquad \underline{x}(t_0) = \underline{x}_0,$$

where $A(t)$ is a continuous bounded function of t for all $t \geq t_0 \geq 0$. In this case, the notions of GUAS and ES coincide.

Prove that **if** there exists continuously differentiable, bounded, positive definite $P(t)$ (in other words, $0 \prec c_1 I \preceq P(t) \preceq c_2 I$, $\forall t \geq t_0 \geq 0$) that solves the linear matrix differential equation

$$-\dot{P}(t) = (A(t))^\top P(t) + P(t)A(t) + Q(t),$$

for any $Q(t)$ that is continuous and positive definite (in other words, $0 \prec c_3 I \preceq Q(t)$, $\forall t \geq t_0 \geq 0$), **then** the origin is G.E.S. (and thus G.U.A.E.S.)

Problem 3

Region of Attraction (5+10+15 = 30 points)

Consider the nonlinear system

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -x_1 - x_2 + x_1^3.$$

(a) Find all isolated fixed points.

(b) By taking $V(x_1, x_2) = \frac{1}{2}x_2^2 + \int_0^{x_1} (y - y^3)\,\mathrm{d}y$ as the Lyapunov function, prove that origin is asymptotically stable. (Hint: You may need to use LaSalle invariance theorem.)

(c) Use your answer in part (b) to estimate the region of attraction for origin.