Lec. 9 (04/28/2020)

Signal $\mathcal{L}_p$ norm:

$$\underbrace{\| \underline{u}(t) \|_{\mathcal{L}_p[0,\infty)}}_{\text{Infinite horizon } \mathcal{L}_p \text{ norm}} = \left(\int_0^\infty \| \underline{u}(t) \|_{\ell_p}^p \, dt \right)^{1/p} < \infty$$

$$\forall \, p \in [1, \infty)$$

$$\underbrace{\| \underline{u}(t) \|_{\mathcal{L}_p[0,T]}}_{\text{Finite horizon } \mathcal{L}_p \text{ norm}} = \left(\int_0^T \| \underline{u}(t) \|_{\ell_p}^p \, dt \right)^{1/p}$$

For example, if $p = 2$, infinite horizon $\mathcal{L}_2$ norm:

$$\| \underline{u}(t) \|_{\mathcal{L}_2[0,\infty)} = \left(\int_0^\infty u^T(t) \, u(t) \, dt \right)^{1/2}, \quad p = 2 \text{ here}$$

We say, $\underline{u}(t) \in \mathcal{L}_2[0,\infty)$ if $\|\underline{u}(t)\|_{\mathcal{L}_2[0,\infty)} < \infty$.

<u>Another example:</u>

$$p = \infty, \quad \|\underline{u}(t)\|_{\mathcal{L}_\infty[0,\infty)} = \sup_{0 \le t < \infty} \|u(t)\|_\infty < \infty$$

Think of control systems as an input-output map:

$$\underline{u}(t) \in \mathbb{R}^m \longrightarrow \boxed{S} \longrightarrow \underline{y}(t) \in \mathbb{R}^q, \quad q \neq m$$

Viewing control system as an operator S

$$S : \underbrace{\underline{u}(t)}_{\in \mathbb{R}^m} \longmapsto \underbrace{\underline{y}(t)}_{\in \mathbb{R}^q}$$

$\mathcal{L}_p$ gain of a nonlinear system: (Infinite horizon)

$$\underbrace{\frac{\|\underline{y}(t)\|_{\mathcal{L}_p[0,\infty)}}{\|\underline{u}(t)\|_{\mathcal{L}_p[0,\infty)}}}_{\mathcal{L}_p \text{ gain of the system}} \leq \gamma \quad \text{for some constant } \gamma > 0.$$

If $\mathcal{L}_p$ gain $< \infty$ $\forall\ u(t) \in \mathcal{L}_p$, we say the system is finite gain $\mathcal{L}_p$ stable.

Worst-case $\mathcal{L}_p$ gain:

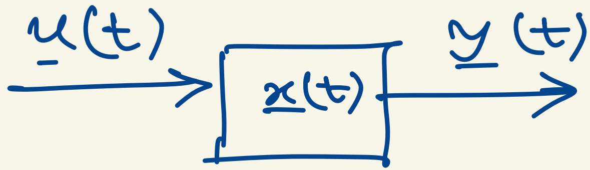
$$\gamma := \sup_{\substack{u(t) \in \mathcal{L}_p \\ u(t) \not\equiv 0}} \frac{\|\underline{y}(t)\|_{\mathcal{L}_p[0,\infty)}}{\|\underline{u}(t)\|_{\mathcal{L}_p[0,\infty)}}$$

Worst-case $\mathcal{L}_p$ gain γ is a system property, not a function of particular experiment/simulation/input ($p = 2 \Leftrightarrow$ finite energy)

Example:

Worst-case $\mathcal{L}_2$ gain of LTI system:

$$\left.\begin{aligned}\dot{\underline{x}} &= A\underline{x} + B\underline{u}\\ \underline{y} &= C\underline{x} + D\underline{u}\end{aligned}\right\} \quad \left.\begin{aligned}&\underline{x} \in \mathbb{R}^n\\ &\underline{u} \in \mathbb{R}^m\\ &\underline{y} \in \mathbb{R}^q\end{aligned}\right\} \quad \begin{aligned}&A \text{ is Hurwitz}\\ &\underline{x}(0) = \underline{0}.\end{aligned}$$



$$\left.\begin{aligned} s\,X(s) - \cancel{\underline{x}(0)}^{\,0} &= A\,X(s) + B\,U(s)\\ Y(s) &= C\,X(s) + D\,U(s)\end{aligned}\right\} \text{Eliminate } X(s)$$

$$(sI - A)X(s) = BU(s)$$

$$\Rightarrow \boxed{X(s) = (sI - A)^{-1} BU(s)}$$

$$Y(s) = C\boxed{X(s)} + DU(s)$$

$$= \boxed{C(sI - A)^{-1} BU(s)} + DU(s)$$

$$\Rightarrow \underbrace{\frac{Y(s)}{U(s)} = \underline{C(sI - A)^{-1} B + D}}_{\text{Transfer Matrix } G(s)}$$

$$= C\frac{\operatorname{adj}(sI - A)}{\det(sI - A)} B + D$$

$$G(s) \xrightarrow{s \equiv j\omega} G(j\omega)$$

$$\left.\begin{aligned} G^*(j\omega) = \left(G(-j\omega)\right)^T, \end{aligned}\right\} j = \sqrt{-1}.$$

Claim:

$$\gamma_{LTI} := \sup_{\substack{\|u(t)\|_2 \in \mathcal{L}_2 \\ \underline{u}(t) \not\equiv \underline{0}}} \frac{\|\underline{y}(t)\|_{\mathcal{L}_2}}{\|\underline{u}(t)\|_{\mathcal{L}_2}}$$

Worst-case $\mathcal{L}_2$ gain for an LTI system

result

$$= \sup_{\omega \in \mathbb{R}} \|G(j\omega)\|_2$$

$$= \sup_{\omega \in \mathbb{R}} \sigma_{max}\left(G(j\omega)\right)$$

$\|G\|_\infty$ ($\mathcal{H}_\infty$ norm of G)

$$\boxed{\sigma_{max}(M) := \sqrt{\lambda_{max}(M^* M)}}$$

For static linear input-output map:

$$\boxed{\underline{y} = A\,\underline{u}} \longleftrightarrow \underline{u} \longrightarrow \boxed{S} \longrightarrow \underline{y}$$

$$\sup_{\underline{u} \neq \underline{0}} \frac{\|\underline{y}\|_2}{\|\underline{u}\|_2} = \sup_{\underline{u} \neq 0} \frac{\|A\,\underline{u}\|_2}{\|\underline{u}\|_2} = \sup_{\underline{u}^T\underline{u} = 1} \frac{\|A\,\underline{u}\|_2}{1}$$

$$=: \|A\|_2$$

$$= \sqrt{\lambda_{max}(A^TA)}$$

$$=: \sigma_{max}(A)$$

Therefore,
the $\mathcal{L}_p$ gain of
a static linear system
$\underline{y} = A\,\underline{u}$ is simply the
induced p-norm of matrix A.

Theorem:

Set up:

$$\left.\begin{aligned} \dot{\underline{x}} &= \boxed{\underline{f}(\underline{x}) + \underline{g}(\underline{x})\,\underline{u}} \\ \underline{y} &= \underline{h}(\underline{x}) \end{aligned}\right\} \quad \begin{aligned} &\underline{x} \in \mathbb{R}^n \\ &\underline{u} \in \mathbb{R}^m \\ &\underline{y} \in \mathbb{R}^q \\ &h : \mathbb{R}^n \mapsto \mathbb{R}^q \end{aligned}$$

- $g(\cdot)$ and $h(\cdot)$ are continuous
- $f(\cdot)$ is locally Lipschitz in $\underline{x}$
- $\underline{f}(\underline{0}) = \underline{0}$, $\underline{h}(\underline{0}) = \underline{0}$

Statement: If $\exists\ \gamma > 0$, and C^1 function $V(\underline{x})$ such that

① V is a pos. definite function $\left(\begin{aligned} &V(\underline{0}) = 0, \\ &V(x \neq 0) > 0 \end{aligned} \right)$

and

(next pg.)

② V satisfies the partial differential inequality:

$$\langle \nabla V, \underline{f} \rangle + \frac{1}{2\gamma^2} \left(\frac{\partial V}{\partial \underline{x}}\right)^T \underline{g}\, \underline{g}^T \left(\frac{\partial V}{\partial \underline{x}}\right) + \frac{1}{2} h^T h \leq 0$$

$$\forall\, \underline{x} \in \mathbb{R}^n.$$

Then, the nonlinear system is finite gain $\mathcal{L}_2$ stable $\forall\, \underline{x}_0 \in \mathbb{R}^n$, and its $\mathcal{L}_2$ gain $\frac{\| \underline{y}(t) \|_{\mathcal{L}_2}}{\| \underline{u}(t) \|_{\mathcal{L}_2}} \leq \gamma$.

Remarks:

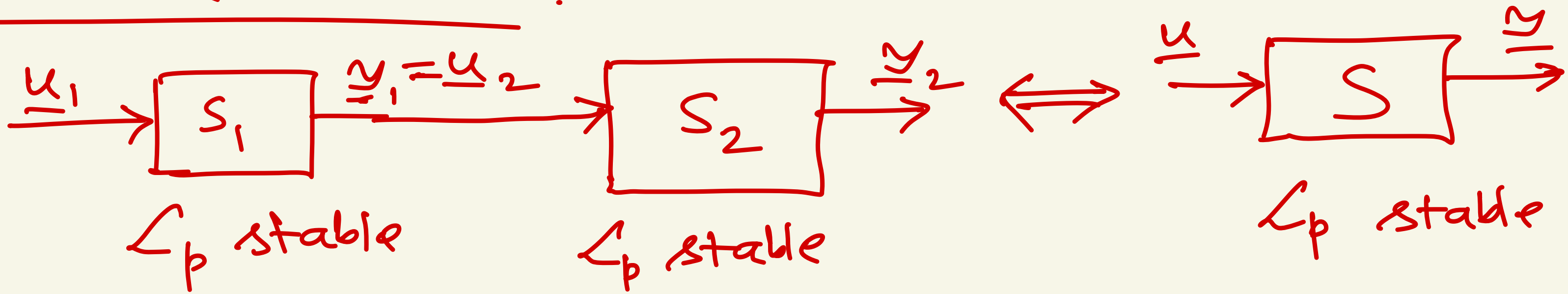
① Theorem does NOT assume any stability about unforced system

② The inequality is called Hamilton-Jacobi inequality

Compositional results for $\mathcal{L}_p$ stability (Interconnections of nonlinear systems)

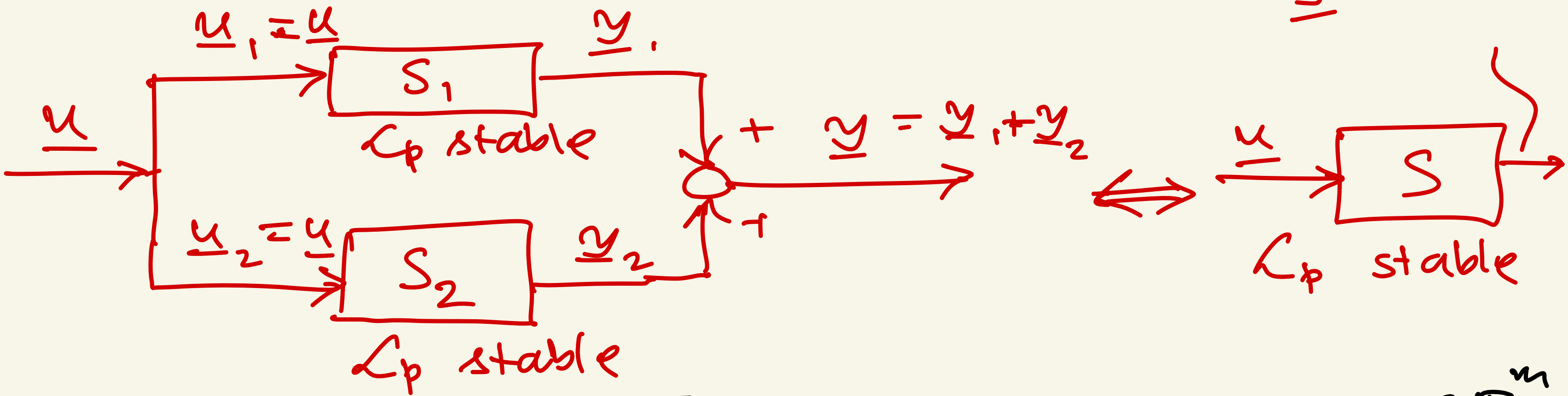
$$S_1 \begin{cases} \dot{\underline{x}}_1 = \underline{f}_1(\underline{x}_1, \underline{u}_1, t) \\ \underline{y}_1 = \underline{h}_1(\underline{x}_1, \underline{u}_1, t) \end{cases} \qquad S_2 \begin{cases} \dot{\underline{x}}_2 = \underline{f}_2(\underline{x}_2, \underline{u}_2, t) \\ \underline{y}_2 = \underline{h}_2(\underline{x}_2, \underline{u}_2, t) \end{cases}$$

Series connection :



Parallel Connection:

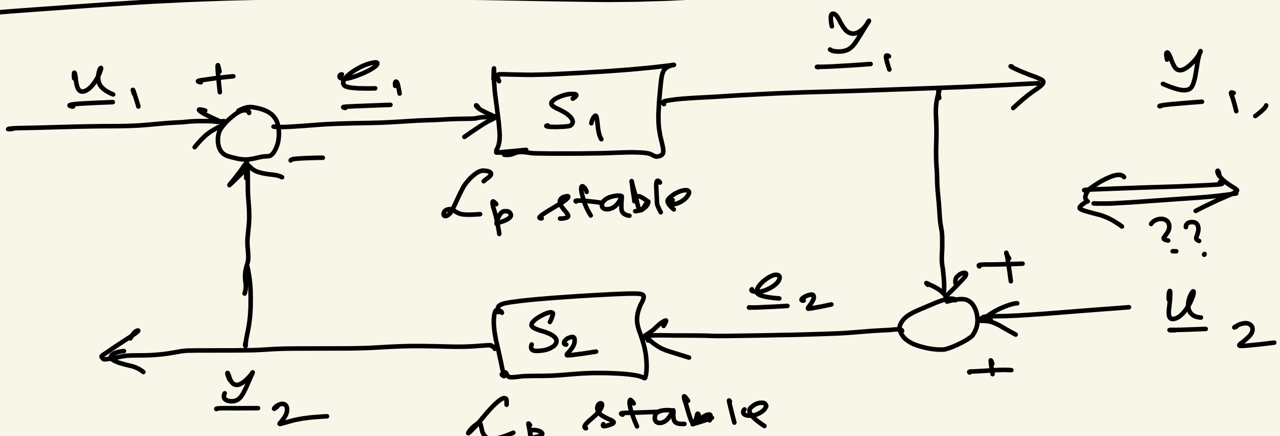
$\underline{y} = \underline{y}_1 + \underline{y}_2$

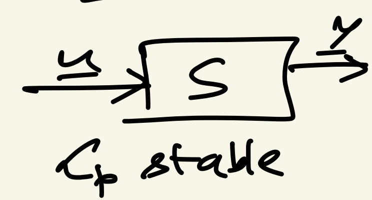


Feedback Connection:

$\underline{u}_1, \underline{y}_2, \underline{e}_1 \in \mathbb{R}^m$

$\underline{y}_1, \underline{u}_2, \underline{e}_2 \in \mathbb{R}^q$





For S_1: $\mathcal{L}_p$ stable:

$$\|\underline{y}_1(t)\|_{\mathcal{L}_p} \leq \gamma_1 \|\underline{e}_1(t)\|_{\mathcal{L}_p} + \beta_1 \quad \forall\ \underline{e}_1(t) \in \mathcal{L}_p$$

For S_2: $\mathcal{L}_p$ stable:

$$\|\underline{y}_2(t)\|_{\mathcal{L}_p} \leq \gamma_2 \|\underline{e}_2(t)\|_{\mathcal{L}_p} + \beta_2 \quad \forall\ \underline{e}_2(t) \in \mathcal{L}_p$$

For all $t \in [0, \infty)$

Assumption: System is well-defined:

$\forall$ pair $\begin{pmatrix} \underline{u}_1 \\ \underline{u}_2 \end{pmatrix} \in \mathcal{L}_p \times \mathcal{L}_p$

$\exists$ unique outputs $\begin{pmatrix} \underline{e}_1 \\ \underline{y}_2 \end{pmatrix} \in \mathcal{L}_p \times \mathcal{L}_p$

and $\begin{pmatrix} \underline{e}_2 \\ \underline{y}_1 \end{pmatrix} \in \mathcal{L}_p \times \mathcal{L}_p$.

Let $\underline{u} := \begin{pmatrix} \underline{u}_1 \\ \underline{u}_2 \end{pmatrix} \in \mathbb{R}^{m+q}$, $\underline{y} := \begin{pmatrix} \underline{y}_1 \\ \underline{y}_2 \end{pmatrix} \in \mathbb{R}^{m+q}$

$\underline{e} := \begin{pmatrix} \underline{e}_1 \\ \underline{e}_2 \end{pmatrix} \in \mathbb{R}^{m+q}$

Lemma: $\underline{u} \mapsto \underline{y}$ is finite gain $\mathcal{L}_p$ stable

$$\Updownarrow$$

$\underline{u} \mapsto \underline{e}$ is " " $\mathcal{L}_p$ "

Theorem: (Small gain theorem)

Feedback connection is finite gain $\mathcal{L}_p$ stable if $\boxed{\gamma_1 \gamma_2 < 1}$

$\underline{u} \to \boxed{S} \to \underline{y}$