Lecture #11

$\dot{\underline{x}} = f(\underline{x}(t))$, $\quad \dot{\underline{x}} = f(\underline{x}, t)$, $\quad \boxed{\dot{\underline{x}} = f(\underline{x}, t, \underline{u})}$

$$\underline{f} : \underbrace{\mathbb{R}^n}_{\substack{\text{State space} \\ (\mathcal{X})}} \times \underbrace{\mathbb{R}^m}_{\substack{\text{control Space} \\ (\mathcal{U})}} \times [0, \infty) \longmapsto \mathbb{R}^n$$

p. w. continuous in t, & Locally Lip. in both $\underline{x}$, $\underline{u}$

Suppose $\underline{u}(t) \in \mathbb{R}^m$ is a bdd. f^n of t $\forall t \geq 0$

Suppose also, $\underline{x}^* = \underline{0}$ is GAS for $\underbrace{\dot{\underline{x}} = \underline{f}(\underline{x}, t, \underline{0})}_{\text{unforced system}}$

Question : Does this mean

bdd. $\underline{u}(t) \Longrightarrow$ bdd. $\underline{x}(t)$?

$\underbrace{\text{Input-to-state}}_{\text{(ISS)}}$ Stability

Answer: Yes for LTI system

$$\underline{\dot{x}} = A\underline{x}(t) + B\underline{u}(t)$$

$$\Rightarrow \underline{x}(t) = e^{A(t-t_0)}\underline{x}_0 + \int_{t_0}^{t} e^{A(t-\tau)} B u(\tau) d\tau$$

Given, origin is GAS

$\Updownarrow$

A is Hurwitz $\left(\| e^{A(t-t_0)} x_0 \|_2 \leq \alpha e^{-\beta(t-t_0)} \|x_0\| \right)$

This gives

$$\|x(t)\| \leq \alpha e^{-\beta(t-t_0)} \|x_0\| + \int_{t_0}^{t} \alpha e^{-\beta(t-\tau)} \|B\| \, \|u(\tau)\| d\tau$$

$$\leq \alpha e^{-\beta(t-t_0)} \|x_0\| + \frac{\alpha\|B\|}{\beta} \sup_{t_0 \leq \tau \leq t} \|u(\tau)\|$$

where:

$$\|u(\tau)\| \leq \sup_{t_0 \leq \tau \leq t} \|u(\tau)\|.$$

$$0 < \beta < -\max(\mathrm{Re}(\lambda_i(A)))$$

NOT true for nonlin. system

Counter-example: (1D)

$$\dot{x} = -3x + (1+2x^2)u, \quad x(0) = 2$$

unforced system: $\dot{x} = -3x \Rightarrow$ origin is GAES

forced system: $u(t) \equiv 1 \ \forall\, t \geq 0,$

$$\Rightarrow x(t) = \frac{3 - e^t}{3 - 2e^t}$$

is unbounded even though $u(t) \equiv 1 < \infty$.

finite escape @ $t = \ln(3/2)$

$x(t) \to +\infty$

NOT ISS.

Class $\mathcal{K}$ fn	Class $\mathcal{K}_\infty$ fn	Class $\mathcal{KL}$ fn
$\alpha : [0, a] \mapsto \mathbb{R}^+$	$\alpha : \mathbb{R}^+ \mapsto \mathbb{R}^+$	$\beta : [0, a] \times \mathbb{R}^+ \mapsto \mathbb{R}^+$
• α is continuous	• α is continuous	• For fixed s, $\beta(r, s) \in$ class $\mathcal{K}$ w.r.t. r
• $\alpha(0) = 0$	• $\alpha(0) = 0$	• For fixed r, $\beta(r, s) \downarrow$ w.r.t. s (decreasing)
• $\alpha(\cdot)$ is strictly increasing $\uparrow$	• $\alpha(\cdot)$ is strictly increasing	• $\lim_{s \to \infty} \beta(r, s) = 0$
	• $\lim_{r \to \infty} \alpha(r) = \infty$	

ISS defn (Global/Local)

Consider $\underline{\dot{x}} = \underline{f}(\underline{x}, t, \underline{u})$. This system is ISS if $\exists\ \boxed{\beta \in \mathcal{KL}}$ & $\boxed{\gamma \in \mathcal{K}}$ s.t. $\forall x_0 \in \mathcal{D}(\mathbb{R}^n)$ we have:

$$\|x(t)\| \le \beta(\|x_0\|, t - t_0) + \gamma\left(\sup_{t_0 \le \tau \le t} \|u(\tau)\|\right) \quad (*)$$

Implications of ISS def^n:

① If $u(t) \equiv 0 \; \forall t$, then $(*)$ implies

$$\|x(t)\| \leq \beta(\|x_0\|, t-t_0) \quad \forall t \geq t_0 \geq 0$$

$$\forall x_0 \in \mathcal{D}(\subseteq \mathbb{R}^n)$$

$$\Rightarrow \lim_{t \to \infty} \|x(t)\| \leq \lim_{t \to \infty} \underbrace{\beta(\|x_0\|, t-t_0)}_{=0}$$

$$(\because \beta \in \text{class } \mathcal{KL})$$

$$\Rightarrow \lim_{t \to \infty} \|x(t)\| = 0 \quad (\Leftrightarrow (G)\,UAS)$$

$\Rightarrow$ Origin of the unforced system is (G)UAS

② In general, bdd. input $\not\Rightarrow$ bdd. State.

ISS Stability Thm. in the sense of Lyapunov.

Thm: A C^1 f^n $V : \mathcal{D} \mapsto \mathbb{R}$ is called an ISS Lyapunov f^n on $\mathcal{D}$ if $\exists$ $\boxed{\alpha_1(\cdot), \alpha_2(\cdot) \in \mathcal{K}_\infty}$ and $\boxed{\rho(\cdot) \in \mathcal{K}}$ s.t.

① $$\alpha_1(\|\underline{x}\|) \leq V(\underline{x}) \leq \alpha_2(\|\underline{x}\|) \quad \forall \underline{x} \in \mathcal{D}$$

② $$\dot{V} = \frac{\partial V}{\partial t} + \langle \nabla V, \underline{f} \rangle \leq -W_3(\|\underline{x}\|)$$

$$\forall \|\underline{x}\| \geq \rho(\|\underline{u}\|) > 0$$

$$\forall x \in \mathcal{D}, \ \forall u \in \mathcal{U}$$

and $W_3(\cdot)$ is a pos. def. function.

If $\exists$ an ISS Lyap. f^{n} $V(\cdot)$ for $\dot{\underline{x}} = \underline{f}(\underline{x}, t, \underline{u})$, then the system is ISS with

$$\gamma = \alpha_1^{-1} \circ \alpha_2 \circ \rho .$$

Example :
$$\dot{x} = -x^3 + u(t)$$

origin of $\dot{x} = -x^3$ is GAS

To show ISS : Let ISS Lyap. f^{n} be

$$V(x) = \frac{1}{2} x^2$$

$$\alpha_1(\|x\|) = \alpha_2(\|x\|) = V(x)$$

$$\dot{V} = -x^4 + xu$$

$$= \underbrace{-x^4 + \underline{\underline{\theta x^4}}}\ \underline{\underline{-\theta x^4}} + xu$$

$$\leq -(1-\theta)x^4 \quad \forall \ |x| \geq \left(\frac{|u|}{\theta}\right)^{1/3} \ \forall\, 0<\theta<1$$

$\therefore$ This system is ISS with

$$\gamma(r) = \alpha_1^{-1} \circ \alpha_2 \circ \rho$$
$$= \left(\frac{r}{\theta}\right)^{1/3}$$

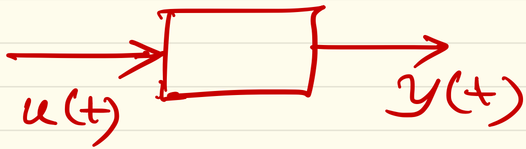
$$\underline{\dot{x}} = \underline{f}(\underline{x}, \underline{u}, t)$$

Ideas for Stability

ISS

(If $\|u\| < \infty \ \forall t \overset{?}{\Longrightarrow} \|x(t)\| < \infty \ \forall t$)

Input - Output Stability



$\mathcal{L}_p$ stability

Signal $\mathcal{L}_p$ norm :

$$\|\underline{u}(t)\|_{\mathcal{L}_p[0,\infty)} = \left(\int_0^\infty \|\underline{u}(t)\|_{\ell_p}^p \, dt\right)^{1/p} < \infty$$

Infinite horizon $\mathcal{L}_p$ norm

$\forall \; p \in [1, \infty)$

$[0, T] \rightarrow$ finite horizon norm

e.g. $p = 2$:

$$\|\underline{u}(t)\|_{\mathcal{L}_2[0,\infty)} = \left(\int_0^\infty \underline{u}^T(t)\, \underline{u}(t)\, dt\right)^{1/2}$$

$\underline{u}(t) \in \mathcal{L}_2[0,\infty)$ if $\|\underline{u}(t)\|_{\mathcal{L}_2[0,\infty)} < \infty$

e.g. $p = \infty$

$$\|\underline{u}(t)\|_{\mathcal{L}_\infty[0,\infty)} = \sup_{\infty > t \geq 0} \|u(t)\|_\infty < \infty$$