Lecture # 12

$$\underset{\mathbb{R}^m}{\underline{u}(t)} \longrightarrow \boxed{S} \longrightarrow \underline{y}(t) \in \mathbb{R}^q \qquad q \neq m$$

Control system viewed as operator S:

$$S : \underset{\mathbb{R}^m}{\underline{u}(t)} \mapsto \underset{\mathbb{R}^q}{\underline{y}(t)}$$

$\mathcal{L}_p$ gain of a nonlinear system:

$$\underbrace{\frac{\|\underline{y}(t)\|_{\mathcal{L}_p[0,\infty)}}{\|\underline{u}(t)\|_{\mathcal{L}_p[0,\infty)}}} \leq \gamma$$

Infinite horizon $(t \in [0,\infty))$

Finite horizon $\mathcal{L}_p[0,T]$

$\mathcal{L}_p$ gain

If $\mathcal{L}_p$ gain $< \infty$ $\forall\, u(t) \in \mathcal{L}_p$, we say the system is finite gain $\mathcal{L}_p$ stable

Worst-case $\mathcal{L}_p$ gain:

$$\gamma := \sup_{\substack{u(t) \in \mathcal{L}_p \\ u(t) \neq 0}} \frac{\|\underline{y}(t)\|_{\mathcal{L}_p[0,\infty)}}{\|\underline{u}(t)\|_{\mathcal{L}_p[0,\infty)}} = \sup_{\substack{\underline{u}(t) \in \mathcal{L}_p \\ \|u\|_{\mathcal{L}_p} = 1}} \|y\|$$

γ is a system property, not a function of particular experiment / simulation / input ($p = 2 \Leftrightarrow$ finite energy)

Example: (Worst-case $\mathcal{L}_2$ gain of LTI system):

$$\left\{\begin{aligned} \dot{\underline{x}} &= A\underline{x} + B\underline{u} \\ \underline{y} &= C\underline{x} + D\underline{u} \end{aligned}\right\}$$

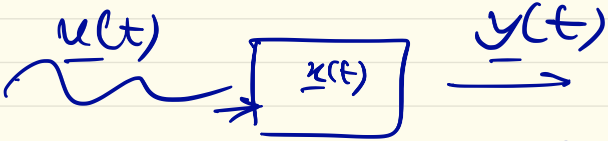
A is Hurwitz

$\underline{x} \in \mathbb{R}^n$

$\underline{u} \in \mathbb{R}^m$

$\underline{y} \in \mathbb{R}^q$

$\underline{x}(0) = \underline{0}$



$$sX(s) = AX(s) + BU(s) \Rightarrow (sI - A)X(s) = BU(s)$$

$$\Rightarrow X(s) = (sI-A)^{-1}BU(s)$$

$$Y(s) = CX(s) + DU(s)$$

$$= \left[C(sI-A)^{-1}B + D\right]U(s)$$

$$\Rightarrow \frac{Y(s)}{U(s)} = C(sI-A)^{-1}B + D$$

Transfer function $= G(s)$

$$G(s) = C(sI-A)^{-1}B + D$$
$$= \frac{C\,\mathrm{adj}(sI-A)}{\det(sI-A)} B + D$$
} Transfer matrix

$G(j\omega)$, $\quad G^*(j\omega) = \left(G(-j\omega)\right)^T$, $\quad j = \sqrt{-1}$

<u>Claim</u>:

$$\underbrace{\gamma_{LTI}}_{} := \sup_{\substack{\|\underline{u}(t)\|_{\mathcal{L}_2} \\ \underline{u}(t) \neq 0}} \frac{\|\underline{y}(t)\|_{\mathcal{L}_2}}{\|\underline{u}(t)\|_{\mathcal{L}_2}}$$

Worst-case $\mathcal{L}_2$ gain of LTI system

$$= \sup_{\omega \in \mathbb{R}} \|G(j\omega)\|_2$$

$$= \sup_{\omega \in \mathbb{R}} \underbrace{\sigma_{max}(G(j\omega))}_{\text{maximum singular value}}$$

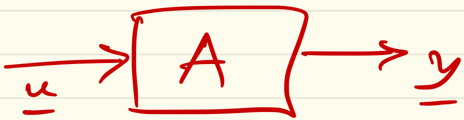
$$\boxed{\text{Remember that } \sigma_{max}(M) := \sqrt{\lambda_{max}(M^* M)}}$$

Recap. matrix norm: induced

$$\underline{y} = A\underline{u}$$

$$\|A\|_2 := \sup_{\underline{u} \neq 0} \frac{\|A\underline{u}\|_2}{\|\underline{u}\|_2} = \sup_{\underline{u}^T\underline{u}=1} \frac{\|A\underline{u}\|_2}{1}$$

$$= \sqrt{\lambda_{max}(A^TA)} = \sigma_{max}(A)$$



Therefore, $\mathcal{L}_p$ gain of a static linear system $\underline{y} = A\underline{u}$ is simply the induced p-norm of matrix A.

$$\gamma_{LTI} := \sup_{\substack{\underline{u}(t) \in \mathcal{L}_2 \\ \underline{u}(t) \neq 0}} \frac{\|\underline{y}(t)\|_{\mathcal{L}_2}}{\|\underline{u}(t)\|_{\mathcal{L}_2}}$$

$$= \boxed{\sup_{\omega \in \mathbb{R}} \underbrace{\sigma_{max}(G(j\omega))}_{= \sqrt{\lambda_{max}(G^*(j\omega)\,G(j\omega))}}} = \|G\|_\infty$$

$\mathcal{H}_\infty$ norm of $G(j\omega)$

Proof: (part of proof: $\dfrac{\|\underline{y}(t)\|_{\mathcal{L}_2}}{\|\underline{u}(t)\|_{\mathcal{L}_2}} \leq \sup_{\omega \in \mathbb{R}} \sigma_{max}(G(j\omega))$)

Numerator of LHS.

$$= \|\underline{y}(t)\|^2_{\mathcal{L}_2} = \int_0^{\infty} (\underline{y}(t))^T \underline{y}(t)\, dt$$

$\underline{y}(t) \rightarrow Y(j\omega)$

(Parseval's Thm) $$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} Y^*(j\omega)\, Y(j\omega)\, d\omega$$

Parseval's Thm.

$$\int_0^{\infty} (\underline{y}(t))^T y(t)\, dt = \frac{1}{2\pi}\int_{-\infty}^{+\infty} Y^*(j\omega)\, Y(j\omega)\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} U^*(j\omega)\, G^T(-j\omega)\, G(j\omega)\, U(j\omega)\, d\omega$$

$$\boxed{Y(j\omega) = G(j\omega)U(j\omega)}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} tr\left[U^*(j\omega)\, G^T(-j\omega)\, G(j\omega)\, U(j\omega)\right] d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} tr\left[\{U(j\omega)\, U^*(j\omega)\}\{G^T(-j\omega)\, G(j\omega)\}\right] d\omega$$

$$\operatorname{tr}(AB) \leq \sqrt{\operatorname{tr}(A^2)\,\operatorname{tr}(B^2)}$$

$$\leq \sqrt{(\operatorname{tr}(A))^2\,(\operatorname{tr}(B))^2} = \operatorname{tr}(A)\,\operatorname{tr}(B)$$

$$\boxed{A \equiv U U^*} \qquad \boxed{B \equiv G^* G} \; \boxed{= G^T(-j\omega)\, G(j\omega)}$$

$$\therefore \| y(t) \|_{\mathcal{L}_2}^2 = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \operatorname{tr}\Big((U U^*)\,(G^* G) \Big)\, d\omega$$

$$\leq \frac{1}{2\pi} \int_{-\infty}^{+\infty} U^*(j\omega)\, U(j\omega)\; \sigma_{max}^2\big(G(j\omega)\big)\, d\omega$$

$$\leq \Big(\sup_{\omega \in \mathbb{R}} \| G(j\omega) \|_2^2 \Big) \underbrace{\frac{1}{2\pi} \int_{-\infty}^{+\infty} U^*(j\omega)\, U(j\omega)\, d\omega}_{\| u(t) \|_{\mathcal{L}_2}^2}$$

$$\Rightarrow \frac{\| y(t) \|_{\mathcal{L}_2}^2}{\| u(t) \|_{\mathcal{L}_2}^2} \leq \| G \|_\infty^2$$

$$\mathrm{tr}\,(U^* U\, a^* a) \le \underbrace{\mathrm{tr}\,(U U^*)}_{\substack{\| \\ \underbrace{\mathrm{tr}(U^* U)}_{\substack{\| \\ U^* U}}}}\ \underbrace{\mathrm{tr}\,(a^* a)}_{\le \lambda_{max}(a^* a)}$$

We have shown :

$$\frac{\| \underline{y}(t) \|_{\mathcal{L}_2}^2}{\| \underline{u}(t) \|_{\mathcal{L}_2}^2} \le \| a \|_\infty^2$$

$$\Rightarrow \frac{\| \underline{y}(t) \|_{\mathcal{L}_2}}{\| \underline{u}(t) \|_{\mathcal{L}_2}} \le \| a \|_\infty = \gamma_{LTI} \quad \forall\, \underline{u}(t) \in \mathcal{L}_2$$

$$\sup_{u(t) \in \mathcal{L}_2} \frac{\| \underline{y}(t) \|_{\mathcal{L}_2}}{\| \underline{u}(t) \|_{\mathcal{L}_2}} = \| a \|_\infty = \gamma_{LTI}$$

(we skip the proof that the worst-case gain is equal to $\| a \|_\infty$)

<u>Thm</u>: (upper bound γ of $\mathcal{L}_2$ gain for nonlinear system)

<u>Setup</u>:

$$\left.\begin{aligned} \dot{\underline{x}} &= \underline{f}(\underline{x}) + \underline{g}(\underline{x})\,\underline{u} \\ \underline{y} &= \underline{h}(\underline{x}) \end{aligned}\right\} \qquad \begin{aligned} \underline{x} &\in \mathbb{R}^n \\ \underline{u} &\in \mathbb{R}^m \\ \underline{y} &\in \mathbb{R}^q \end{aligned}$$

$h : \mathbb{R}^n \mapsto \mathbb{R}^q$

- $g(\cdot)$ & $h(\cdot)$ are continuous
- $\underline{f}(\cdot)$ is Lip. (locally) in $\underline{x}$
- $\underline{f}(\underline{0}) = \underline{h}(\underline{0}) = \underline{0}$

<u>Statement</u>: If $\exists$ C^1 f^n $V(\underline{x})$ s.t.

① $V(\cdot)$ is positive semi-definite $\left(\begin{array}{l} V(0) = 0 \text{ iff } \underline{x} = 0 \\ V(x \neq 0) > 0 \end{array} \right)$

&

② V satisfies the <u>partial differential inequality</u> (PDI)

$$\left(\frac{\partial V}{\partial \underline{x}}\right)^T \underline{f}(\underline{x}) + \frac{1}{2\gamma^2}\left(\frac{\partial V}{\partial \underline{x}}\right)^T \underline{g}(\underline{x})\left(\underline{g}(\underline{x})\right)^T\left(\frac{\partial V}{\partial \underline{x}}\right) + \frac{1}{2}\left(h(\underline{x})\right)^T h(\underline{x}) \leq 0$$

$$\forall\ \underline{x} \in \mathbb{R}^n$$

Then the system is finite gain $\mathcal{L}_2$ stable $\forall\ \underline{x}_0 \in \mathbb{R}^n$ & its $\mathcal{L}_2$ gain $\dfrac{\|\underline{y}(t)\|_{\mathcal{L}_2}}{\|\underline{u}(t)\|_{\mathcal{L}_2}} \leq \gamma$.

Remark: 1 This Thm. does NOT assume any stability about unforced system.

2 The PDI is called Hamilton - Jacobi inequality.

Proof of Thm.

$$\dot{V} = \left\langle \nabla V, \underline{f}(\underline{x}) + \underline{g}(\underline{x})\,\underline{u} \right\rangle$$

$$= \left(\frac{\partial V}{\partial \underline{x}}\right)^T \underline{f}(\underline{x}) + \left(\frac{\partial V}{\partial \underline{x}}\right)^T \underline{g}(\underline{x})\,\underline{u}$$

$$\left.\begin{aligned} &= \left(\frac{\partial V}{\partial \underline{x}}\right)^T \underline{f}(\underline{x}) - \frac{\gamma^2}{2}\left\| \underline{u} - \frac{1}{\gamma^2}\left(\underline{g}(\underline{x})\right)^T\left(\frac{\partial V}{\partial \underline{x}}\right)\right\|_2^2 \\ &\quad + \frac{1}{2\gamma^2}\left(\frac{\partial V}{\partial \underline{x}}\right)^T \underline{g}(\underline{x})\left(\underline{g}(\underline{x})\right)^T\left(\frac{\partial V}{\partial \underline{x}}\right) \\ &\quad + \frac{\gamma^2}{2}\|\underline{u}\|_2^2 \end{aligned}\right\} \text{Completion of squares} \quad \cdots (*)$$

Substitute the Hamilton-Jacobi PDI in $(*)$:

$$\Rightarrow \dot{V} \leq \frac{\gamma^2}{2}\|\underline{u}\|_2^2 - \frac{1}{2}\|\underline{y}\|_2^2 - \frac{\gamma^2}{2}\left\| \underline{u} - \frac{1}{\gamma^2}\left(\underline{g}(\underline{x})\right)^T\left(\frac{\partial V}{\partial \underline{x}}\right)\right\|_2^2$$

$$\Rightarrow \dot{V} \leq \frac{\gamma^2}{2}\|\underline{u}\|_2^2 - \frac{1}{2}\|\underline{y}\|_2^2$$

Integrating both sides from $t=0$ to $t=T$ yields:

$$V(\underline{x}(T)) - V(\underline{x}(0)) \leq \frac{\gamma^2}{2}\int_0^T \|\underline{u}(t)\|_2^2\,dt - \frac{1}{2}\int_0^T \|\underline{y}(t)\|_2^2\,dt$$

(contd.) Since $V(\underline{x}(t)) \geq 0 \;\forall\, t \geq 0$, the last inequality in the prev. page gives:

$$\underbrace{\int_0^T \|\underline{y}(t)\|_2^2\, dt} \leq V(\underline{x}(T)) + \int_0^T \|\underline{y}(t)\|_2^2\, dt \leq \underbrace{\gamma^2 \int_0^T \|\underline{u}(t)\|_2^2\, dt + 2V(\underline{x}(0))}$$

collect together

& use the inequality $\sqrt{a^2+b^2} \leq a+b \;\forall\, (a,b) \in \mathbb{R}^2_{\geq 0}$

$$\Downarrow$$

$$\|\underline{y}(t)\|_{\mathcal{L}_2[0,T]} \leq \gamma \|\underline{u}(t)\|_{\mathcal{L}_2[0,T]} + \underbrace{\sqrt{2V(\underline{x}(0))}}_{\substack{\text{non-zero} \\ \text{bias} \\ \text{unless} \\ V(\underline{x}(0))=0}}$$

(Proved)