Lec. 8 (04/23/2020)

Example: $p(x,y) = \underline{2x^2y^2}$ $(d = 4, n = 2)$

$$[x]_{d=2} = \underbrace{\begin{pmatrix} 1 \\ x \\ y \\ x^2 \\ xy \\ y^2 \end{pmatrix}}_{\text{vector of monomials}},$$

Claim: $p(x,y) = 2x^2y^2$

$$= (\cdot)^T \underbrace{M}_{\text{Symmetric matrix}} (\cdot)$$

But M need not be unique

e.g., $M_1 = \begin{bmatrix} \boxed{\text{zeros}(5,5)} & \begin{matrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 & 0 & 1 & 0 \end{matrix} & 0 \end{bmatrix}$ will do $\Big\}$ $M_2 = \begin{bmatrix} & & \text{symmetrize} & & & \\ & & & & & \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ will do

Theorem: A polynomial $p(\underline{x})$ with degree $2d$ is SOS iff $\exists M \succcurlyeq 0$ such that

$$p(\underline{x}) = [\underline{x}]_d^T \, M \, [\underline{x}]_d$$

Example: $p(x,y) = 2x^4 + 2x^3y - x^2y^2 + 5y^4$

$2d = 4 \Leftrightarrow d = 2, \quad n = 2$

$$[\underline{x}]_d = \begin{pmatrix} x^2 \\ xy \\ y^2 \end{pmatrix}$$

$$p(x,y) = \begin{pmatrix} x^2 \\ xy \\ y^2 \end{pmatrix}^T \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{12} & m_{22} & m_{23} \\ m_{13} & m_{23} & m_{33} \end{bmatrix} \begin{pmatrix} x^2 \\ xy \\ y^2 \end{pmatrix}$$

$$= m_{11} x^4 + 2m_{12} x^3 y + (m_{22} + 2m_{13}) x^2 y^2 + \boxed{2m_{23}} x y^3 + m_{33} y^4$$

$\therefore p(x,y)$ is SOS iff ① $M \succcurlyeq 0$,

② $m_{11} = 2$, $2m_{12} = 2$, $2m_{13} + m_{22} = -1$, $m_{33} = 5$, $2m_{23} = 0$.

SDP problem: (Semi-definite Programming problem)

$\underbrace{}_{\text{SDP (LMI)}}$

$$\min_{X} \ \mathrm{trace}(C^T X)$$

such that

$$\boxed{X \succeq 0}$$

$$\mathrm{trace}(A_i X) = b_i, \quad i = 1, \ldots, m$$

So, finding SOS polynomial $\Longleftrightarrow$ Solving SDP/LMI feasibility problem

For searching SOS Lyapunov functions, (for polynomial vector fields)

we can set (1) V is SOS

(2) $-\dot{V}$ is also SOS

$$= -\langle \nabla V, \underline{f} \rangle$$

needs to be polynomial

SOSTOOLS

MATLAB toolbox (free)

requires SeDuMi (an SDP solver)

Example

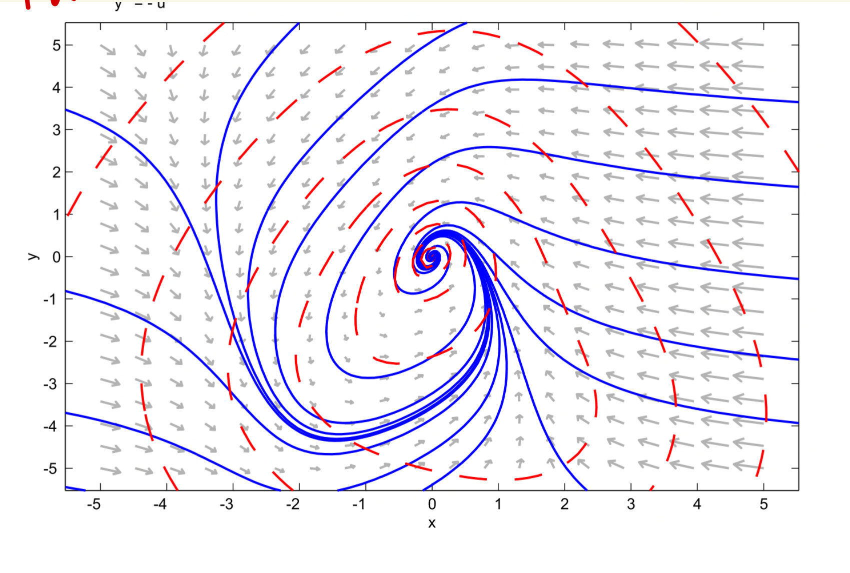
$$\dot{x} = -y + \frac{3}{2}x^2 - \frac{1}{2}x^3$$
$$\dot{y} = 3x - y$$

Find $V(\cdot)$ SOS, $-\dot{V}(\cdot)$ SOS

supply $\underline{f}$, $2d$ to software

$$V(x,y) = \sum_{0 \le j+k \le 4} c_{jk} x^j y^k \quad \text{for } 2d = 4$$

Red dash: 4th order SOS poly V



→ Polynomial vector field

→ origin unique fixed point GAS

→ There does not exist a polynomial Lyapunov function

possible

Counter-example:

$$\boxed{\begin{aligned} \dot{x} &= -x + xy, \\ \dot{y} &= -y \end{aligned}}$$

Origin in GAS But no poly. $V(\cdot)$ exists.

IEEE CDC 2011 paper, 2 pg. paper.

"A Globally Asymptotically Stable Polynomial Vector Field with No Polynomial Lyapunov Function". by (Ahmadi, Krstic, Parrilo)

(Princeton) (UC San Diego) (MIT)

Proof of GAS: $V(x,y) = \ln(1+x^2) + y^2$

$V(\underline{0}) = 0, \quad V(\cdot) > 0 \;\forall (x,y) \neq (0,0)$

$$\dot{V} = \langle \nabla V, \underline{f} \rangle = -\frac{x^2 + 2y^2 + x^2y^2 + (x-2y)^2}{1+x^2}$$

$$< 0 \quad \forall (x,y) \neq (0,0)$$

$V(\cdot)$ is rad. unbdd. $\longrightarrow \therefore$ GAS.

Impossibility of poly. $V(\cdot)$:

$$\dot{V} < 0 \Leftrightarrow \frac{d}{dt} V < 0 \Leftrightarrow \underline{V(x(t), y(t)) < V(x(0), y(0))}$$

$$x(t) = x_0 \exp\left(y_0 - y_0 \exp(-t) - t\right)$$

$$y(t) = y_0 \exp(-t)$$

Accounting Input $\underline{u}$

$$\underline{f} : \underbrace{\mathbb{R}^n}_{\substack{\text{state space}\\ (x)}} \times \underbrace{\mathbb{R}^m}_{\substack{\text{control space}\\ (u)}} \times [0, \infty) \longmapsto \mathbb{R}^n$$

$$\dot{\underline{x}} = f(\underline{x}, \underline{u}, t)$$

piecewise continuous in t, Locally Lipschitz in both $\underline{x}$, $\underline{u}$

Suppose, $\underline{u}(t) \in \mathbb{R}^m$ is a bdd. f^n of t $\forall\, t \geq 0$

Suppose also, $\underline{x}^* = \underline{0}$ is GAS for $\underbrace{\dot{\underline{x}} = \underline{f}(\underline{x}, \underline{0}, t)}_{}$

unforced system.

Question: Does this mean:

bdd. $\underline{u}(t) \Longrightarrow$ bdd. $\underline{x}(t)$

$$\underbrace{\text{Input-to-state}}_{\text{(ISS)}} \text{ Stability}$$

Answer: Yes for LTI system.

$$\dot{\underline{x}} = A\underline{x} + B\underline{u}$$

$$\Rightarrow \underline{x}(t) = e^{A(t-t_0)}\underline{x}_0 + \int_{t_0}^{t} e^{A(t-\tau)} B\, u(\tau)\, d\tau$$

Given origin is GAS for unforced $\Longleftrightarrow$ A is Hurwitz

$$\Leftrightarrow \quad \| e^{A(t-t_0)} \underline{x}_0 \|_2 \le \alpha e^{-\beta(t-t_0)} \| \underline{x}_0 \|_2$$

If $u(t) \neq 0$

Then

$$\boxed{\| \underline{x}(t) \|_2} \le \alpha e^{-\beta(t-t_0)} \| \underline{x}_0 \|_2 + \int_0^t \alpha e^{-\beta(t-\tau)} \| B \|_2 \| \underline{u}(\tau) \|_2 \, d\tau$$

$$\le \boxed{\alpha e^{-\beta(t-t_0)} \| \underline{x}_0 \|_2 + \frac{\alpha \| B \|_2}{\beta} \sup_{t_0 \le \tau \le t} \| \underline{u}(\tau) \|_2}$$

where $\| u(\tau) \|_2 \le \sup\limits_{t_0 \le \tau \le t} \| u(\tau) \|_2$, $0 < \beta < -\max(\mathrm{Re}(\lambda_i))$

$\therefore$ For LTI, if the unforced system is GAS, then for $u(t) \neq 0$, bdd. $\underline{u} \Rightarrow$ bdd. $\underline{x}(t)$.

NOT true for nonlinear system :

Counterexample : (1D)

$$\dot{x} = -3x + (1+2x^2)u, \quad x(0) = 2.$$

Unforced system : $\dot{x} = -3x \Rightarrow$ origin is GAES.

forced system : $u(t) \equiv 1 \quad \forall t \geq 0.$

$$\Rightarrow x(t) = \frac{3-e^t}{3-2e^t}$$

is unbounded even though $u(t) \equiv 1 < \infty.$

finite escape time @ $t = \ln(3/2)$

$x(t) \to +\infty$

Class K function	Class K_∞ function	Class KL function
• $\alpha : [0, a] \mapsto \mathbb{R}^+$ • $\alpha(\cdot)$ is continuous • $\alpha(0) = 0$ • $\alpha(\cdot)$ is strictly increasing	• $\alpha : \mathbb{R}^+ \mapsto \mathbb{R}^+$ • $\alpha(\cdot)$ is continuous • $\alpha(0) = 0$ • $\alpha(\cdot)$ is strictly increasing • $\lim_{r \to \infty} \alpha(r) = \infty$.	• $\beta : [0, a] \times \mathbb{R}^+ \mapsto \mathbb{R}^+$ • For fixed s, $\beta(r, s) \in$ class K w.r.t. r • For fixed r, $\beta(r, s)$ is decreasing w.r.t. s • $\lim_{s \to \infty} \beta(r, s) = 0$.

ISS definition (Global/Local):

Consider $\underline{\dot{x}} = \underline{f}(\underline{x}, \underline{u}, t)$. We say, this system is ISS if $\exists$ $\beta \in KL$ and $\gamma \in K$ such that $\forall\ \underline{x}_0 \in \mathcal{D}(\mathbb{R}^n)$, we have:

$$\|\underline{x}(t)\|_2 \leq \beta\left(\|\underline{x}_0\|, t - t_0\right) + \underline{\gamma}\underline{\left(\sup_{t_0 \leq \tau \leq t} \|u(\tau)\|_2\right)} \quad \text{------- } (*)$$

Implications of ISS (Input to state stability) def^n:

① If $u(t) \equiv 0\ \forall\ t \geq 0$, then $(*)$ implies

$$\|\underline{x}(t)\|_2 \leq \beta\left(\|\underline{x}_0\|, t - t_0\right) \quad \forall\ t \geq t_0 \geq 0$$
$$\forall\ \underline{x}_0 \in \mathcal{D}(\mathbb{R}^n)$$

$$\Rightarrow \lim_{t \to \infty} \|\underline{x}(t)\|_2 \leq \underbrace{\lim_{t \to \infty} \beta\left(\|\underline{x}_0\|, t - t_0\right)}_{= 0} \quad (\because \beta \in KL)$$

$\Rightarrow \quad \lim_{t\to\infty} \|\underline{x}(t)\|_2 = 0 \quad (\Leftrightarrow \text{(G) UAS})$

$\Rightarrow$ Origin of the unforced system is (G)UAS.

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② In general, bdd. input $\not\Rightarrow$ bdd. state

ISS stability Theorem in the sense of Lyapunov:

Theorem: A C^1 function $V : \mathcal{D} \mapsto \mathbb{R}$ is called an "ISS Lyapunov function" on $\mathcal{D}$ if $\exists$ $\alpha_1(\cdot), \alpha_2(\cdot) \in \mathcal{K}_\infty$ and $\rho(\cdot) \in \mathcal{K}$ such that

(next pg.)

① $\alpha_1(\|\underline{x}\|) \leq V(\underline{x}) \leq \alpha_2(\|\underline{x}\|) \;\forall\, \underline{x} \in \mathcal{D}$

② $\dot{V} = \dfrac{\partial V}{\partial t} + \langle \nabla V, \underline{f} \rangle \leq -W_3(\|\underline{x}\|)$

$\forall\; \|\underline{x}\| \geq \rho(\|\underline{u}\|) > 0$

$\forall\; \underline{x} \in \mathcal{D},\; \forall\, \underline{u} \in \mathcal{U}$

and $W_3(\cdot)$ is a pos. definite function.

If $\exists$ such an ISS Lyapunov function $V(\cdot)$ for $\dot{\underline{x}} = f(\underline{x}, \underline{u}, t)$, then the system is ISS with

$$\gamma = \alpha_1^{-1} \circ \alpha_2 \circ \rho.$$

Example: $\dot{x} = -x^3 + u(t)$

Origin of unforced system $\dot{x} = -x^3$ is GAS

To show ISS: Let ISS Lyapunov function be:

$$V(x) = \frac{1}{2}x^2$$

$$\alpha_1(|x|) = \alpha_2(|x|) = V(x)$$

$$\dot{V} = -x^4 + xu$$

$$= -x^4 + \theta x^4 - \theta x^4 + xu$$

$$\leq -\underbrace{(1-\theta)x^4}_{W_3(|x|)} \quad \forall \ |x| \geq \underbrace{\left(\frac{|u|}{\theta}\right)^{1/3}}_{\rho(|u|)}$$

$$\forall \quad 0 < \theta < 1 .$$

$\therefore$ This system is ISS with

$$\gamma(r) = \underbrace{\alpha_1^{-1} \circ \alpha_2}_{= Id} \circ \rho = Id \circ \rho = \rho(r) = \left(\frac{r}{\theta}\right)^{1/3}.$$

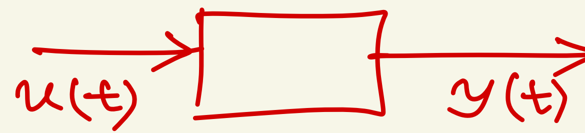
$$\underline{\dot{x}} = \underline{f}(\underline{x}, \underline{u}, t)$$

Ideas for stability

ISS

$$\left(\text{If } \|u\| < \infty \ \forall t \overset{??}{\Longrightarrow} \|x\| < \infty \ \forall t \right)$$

Input - Output (IO) Stability



$\mathcal{L}_p$ stability.