AMS 231: Nonlinear Control Theory: Winter 2018

Homework #2

Name: ....

Due: January 23, 2018

NOTE: Please show all the steps in your solution. Turn in a hard copy of your HW stapled with this as cover sheet with your name written in the above field. Please submit your HW in class on the due date.

Problem 1

Limit Cycle in Planar Nonlinear Systems **($(3+7)+10+15=35$ points)**

Consider the system

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ -(2b - g(x_1))ax_2 - a^2 x_1 \end{pmatrix},$$

where the parameters $a, b > 0$, and the function $g(x_1) := \begin{cases} 0 & \text{for } |x_1| > 1, \\ k & \text{for } |x_1| \leq 1, \end{cases}$ $k \in \mathbb{R}$.

(a) Find all fixed points. Determine which are hyperbolic and which are non-hyperbolic.

(b) Show, using Bendixson's criterion, that there are no limit cycles if $k < 2b$.

(c) Show, using Poincaré-Bendixson criterion, that there is a limit cycle if $k > 2b$.

Solution

(a) The dynamics is

$$\dot{x}_1 = x_2,$$

$$\dot{x}_2 = \begin{cases} -2abx_2 + akx_2 - a^2 x_1, & \text{for } x_1 \in [-1, 1], \\ -2abx_2 - a^2 x_1, & \text{for } x_1 \in (-\infty, -1) \cup (1, \infty). \end{cases}$$

Solving for fixed point (x_1^*, x_2^*) by setting $\dot{x}_1^* = 0$, $\dot{x}_2^* = 0$ yields $(0, 0)$ as the unique solution, i.e., origin is the unique fixed point.

Notice that this is a switched linear system. Inside the vertical strip $x_1 \in [-1, 1]$, we have one linear system, and outside the strip we have another. The flow switches dynamics right outside the lines $x_1 = \pm 1$.

Let $\mathbb{1}_{\mathscr{S}}$ denote the indicator function of a set $\mathscr{S}$, i.e., $\mathbb{1}_{\mathscr{S}} = 1$ if $x \in \mathscr{S}$, and zero otherwise. The Jacobian evaluated at the fixed point is

$$\begin{pmatrix} 0 & 1 \\ -a^2 & a(k\mathbb{1}_{x_1 \in [-1,1]} - 2b) \end{pmatrix}\Bigg|_{(0,0)} = \begin{pmatrix} 0 & 1 \\ -a^2 & a(k - 2b) \end{pmatrix},$$

which has eigenvalues

$$\lambda_{1,2} = a\left(\frac{k-2b}{2}\right) \pm a\sqrt{\left(\frac{k-2b}{2}\right)^2 - 1}.$$

Clearly, the fixed point (origin) is hyperbolic iff $k \neq 2b$.

(b) $\nabla \cdot \underline{f} = a(k\mathbb{1}_{x_1 \in [-1,1]} - 2b)$, which is < 0 for $k < 2b$. Therefore, by Bendixson's criterion, for $k < 2b$, the dynamics admits no limit cycle in $\mathbb{R}^2$ (which is simply connected).

(c) For $k > 2b$, since the eigenvalues have positive real part, the origin is locally unstable. Thus, to apply Poincaré-Bendixon theorem, all that remains is to construct a compact set that includes the origin and is positively invariant in time w.r.t. the switched LTI dynamics.

To construct the set, we track the trajectory starting from a point A$= (0, p)$, as shown in *blue* in the phase portrait in next page. At the starting point A, we have $f_1 = x_2 > 0$ and $f_2 = (k - 2b)ax_2 > 0$, implying the trajectory starts off from point A with positive slope. Within the segment $0 \leq x_1 \leq 1$, the trajectory will continue to have positive slope, provided p is large enough, until it arrives at B $= (1, \beta(p))$. As the trajectory leaves point B, we have $f_2 = -2abx_2 - a^2x_1 < 0$, meaning the trajectory will turn around forming the curve BCD. Let D $= (1, -\gamma(p))$ and consider the motion on the curve BCD. Let $V(\underline{x}) := a^2x_1^2 + x_2^2$ be defined on the domain $x_1 \geq 1$. Then

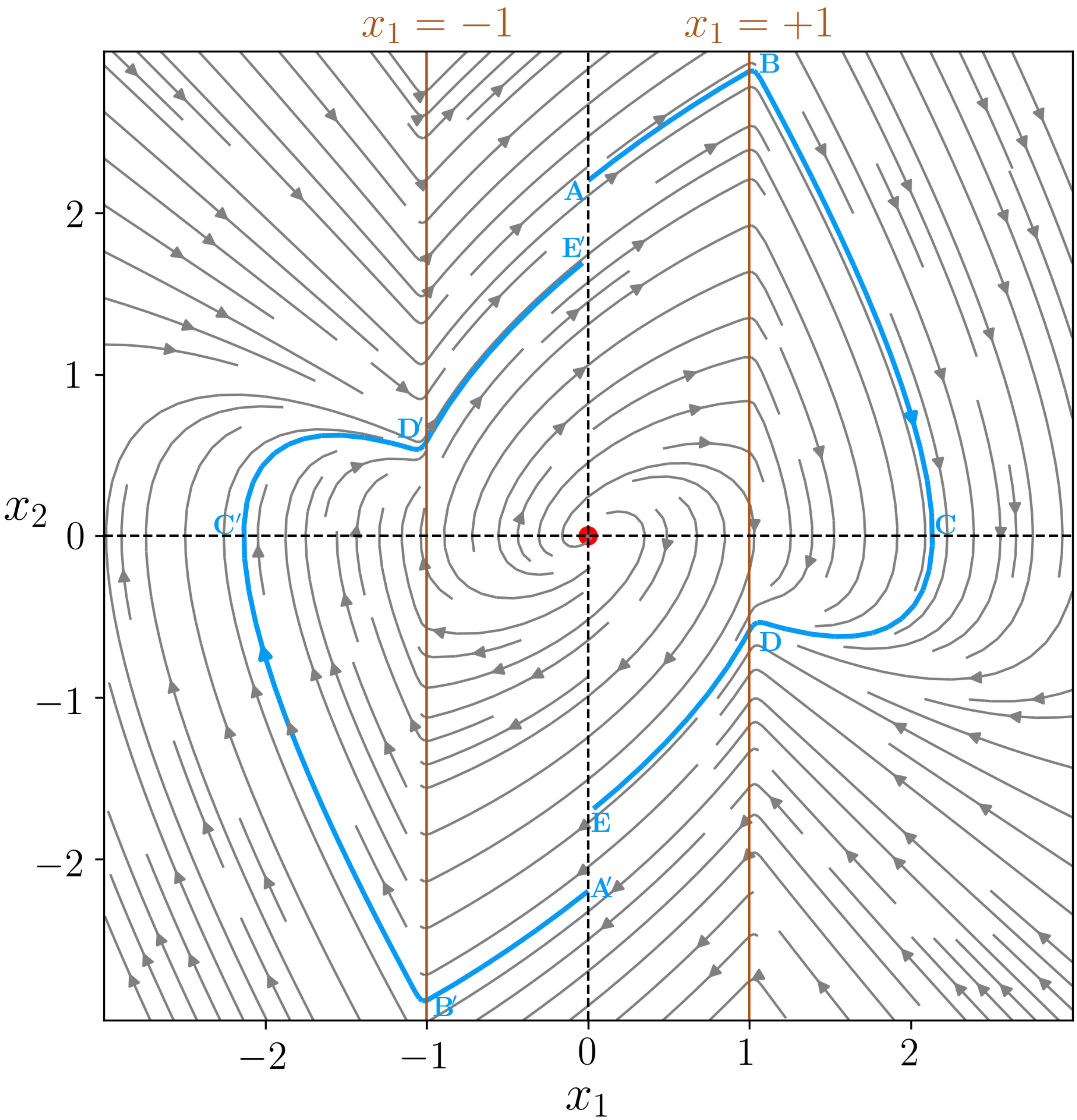
$$\dot{V} = 2a^2x_1x_2 - 4abx_2^2 - 2a^2x_1x_2 = -4abx_2^2 \leq 0, \qquad \text{and} \qquad V(\mathrm{D}) - V(\mathrm{B}) = \int_{\mathrm{BD}} \dot{V}\,\mathrm{d}t.$$

Since $V(\mathrm{D}) - V(\mathrm{B}) = a^2 + \gamma^2(p) - a^2 - \beta^2(p)$, we get

$$\gamma^2(p) - \beta^2(p) = -4ab\int_{\mathrm{BD}} x_2^2\,\frac{\mathrm{d}t}{\mathrm{d}x_2}\,\mathrm{d}x_2 = 4ab\int_{\mathrm{BD}} \frac{x_2^2}{a^2x_1 + 2abx_2}\,\mathrm{d}x_2.$$

As p increases, the arc BCD moves to the right and the domain of integration increases. It follows that $\gamma^2(p) - \beta^2(p)$ decreases as p increases, and

$$\lim_{p\to\infty}\left(\gamma^2(p) - \beta^2(p)\right) \to -\infty \quad \text{as} \quad p \to \infty.$$


$x_1 = -1$
$x_1 = +1$
A
B
C
D
E
A′
B′
C′
D′
E′
x_2
x_1
2
1
0
−1
−2
−2
−1
0
1
2

Therefore, for sufficiently large p, by the time the trajectory reaches the point $\mathrm{E} = (0, -\delta(p))$ on the x_2-axis, we have $\delta(p) < p$.

Now notice that the dynamics has reflective symmetry about the origin, i.e., if $(x_1(t), x_2(t))$ is a solution, then so is $(-x_1(t), -x_2(t))$. This allows us to construct a compact set whose boundary is composed of segment ABCDE, its reflection about origin (the segment A′B′C′D′E′), and the segments EA′ and AE′ along the x_2-axis. In other words, the compact set enclosed by the closed curve ABCDEA′B′C′D′E′A is positively invariant in time. Since the origin is within this set, and is locally unstable, by Poincaré-Bendixon theorem, there exists a limit cycle in this set.

Problem 2

Lyapunov Stability in Continuous Time **($1 + (2 + 2) + (15 + 2 + 3) + 20 = 45$ points)**

Dynamics of a rotating rigid spacecraft is given by the Euler equation

$$
\begin{aligned}
J_1\,\dot{\omega}_1 &= (J_2 - J_3)\,\omega_2\omega_3 \;+\; \tau_1,\\
J_2\,\dot{\omega}_2 &= (J_3 - J_1)\,\omega_3\omega_1 \;+\; \tau_2,\\
J_3\,\dot{\omega}_3 &= (J_1 - J_2)\,\omega_1\omega_2 \;+\; \tau_3,
\end{aligned}
$$

where the parameters $J_1, J_2, J_3 > 0$ denote the principal moments of inertia; the state vector $(\omega_1, \omega_2, \omega_3)$ denotes the spacecraft's angular velocity along its principal axes; and the control vector (τ_1, τ_2, τ_3) denotes the torque input applied about the principal axes.

(a) For $\tau_1 = \tau_2 = \tau_3 = 0$, prove that origin is a fixed point.

(b) For $\tau_1 = \tau_2 = \tau_3 = 0$, how many fixed points other than the origin are there? What physical motions do they correspond to?

(c) For $\tau_1 = \tau_2 = \tau_3 = 0$, show that origin is stable. Is it asymptotically stable? Why/why not?

(d) For $i = 1, 2, 3$, consider the feedback control law $\tau_i = -k_i\omega_i$, where $k_i > 0$ are constants. Prove that origin of the closed-loop system is globally asymptotically stable (G.A.S).

Solution

(a) The substitution $\omega_1 = \omega_2 = \omega_3 = 0$ satisfies the fixed point equations $\dot{\omega}_1 = \dot{\omega}_2 = \dot{\omega}_3 = 0$. Hence the claim.

(b) Solving the the fixed point equations $\dot{\omega}_1 = \dot{\omega}_2 = \dot{\omega}_3 = 0$ yields

$$\begin{aligned}
\omega_1^* = \omega_2^* = 0, \omega_3^* = \Omega_3 \text{ (arbitrary real constant);}\\
\omega_2^* = \omega_3^* = 0, \omega_1^* = \Omega_1 \text{ (arbitrary real constant);}\\
\omega_3^* = \omega_1^* = 0, \omega_2^* = \Omega_2 \text{ (arbitrary real constant).}
\end{aligned}$$

Thus there are infinite fixed points other than the origin.

The physical motion corresponding to each of these (non-origin) fixed point is spinning about one principal axis at a constant rate. Since the constants $\Omega_1, \Omega_2, \Omega_3$ can be positive or negative, the spinning could be clockwise or anti-clockwise about that axis.

(c) Let $V(\omega_1, \omega_2, \omega_3) = \frac{1}{2}\left(J_1\omega_1^2 + J_2\omega_2^2 + J_3\omega_3^2\right)$, and notice that $V(\cdot)$ is a positive definite function of the state vector $(\omega_1, \omega_2, \omega_3) \in \mathbb{R}^3$. Now

$$\dot{V} = \left[J_1(J_2 - J_3) + J_2(J_3 - J_1) + J_3(J_1 - J_2)\right]\omega_1\omega_2\omega_3 = 0.$$

Therefore, by Lyapunov's theorem, the origin is stable (S).

The origin is not asymptotically stable (A.S.). This is beacause $\dot{V}$ is not strictly less than zero.

(d) The closed-loop dynamics becomes

$$\begin{aligned}
J_1\,\dot{\omega}_1 &= (J_2 - J_3)\,\omega_2\omega_3 \;-\; k_1\omega_1,\\
J_2\,\dot{\omega}_2 &= (J_3 - J_1)\,\omega_3\omega_1 \;-\; k_2\omega_2,\\
J_3\,\dot{\omega}_3 &= (J_1 - J_2)\,\omega_1\omega_2 \;-\; k_3\omega_3,
\end{aligned}$$

and by taking the same Lyapunov function as in part (c), we now get

$$\dot{V} = -\sum_{i=1}^{3} k_i\omega_i^2 < 0.$$

Therefore, by Lyapunov's theorem, the origin is A.S.

To prove G.A.S., we need to show two things: uniqueness of origin as the fixed point, and radial unboundedness of $V(\cdot)$. For $i = 1, 2, 3$, multiplying the i-th fixed point equation by ω_i^*, then summing the resulting equations, we get

$$\sum_{i=1}^{3} k_i\left(\omega_i^*\right)^2 = 0, \qquad k_i > 0,$$

which is possible iff $\omega_1^* = \omega_2^* = \omega_3^* = 0$. Thus, origin is the unique fixed point of the closed-loop system. Furthermore, our choice of $V(\cdot)$ as positively weighted sum-of-squares, is radially unbounded. Hence by Barbashin-Krasovskii theorem, the origin of the closed-loop system is G.A.S.

Problem 3

Lyapunov Stability in Discrete Time $(5 + 10 + 5 = 20$ **points)**

For discrete-time autonomous nonlinear system $\underline{x}(k+1) = \underline{f}(\underline{x}(k))$, one can derive a Lyapunov stability theorem analogous to the continuous-time case, by simply replacing the condition $\dot{V} <$ (or $\leq$) 0 to its discrete-time counterpart: $V(k+1) <$ (or $\leq$) $V(k)$, where $V(k) := V(\underline{x}(k))$, while keeping the other conditions (positive definiteness/semi-definiteness) same.

Consider the nonlinear system

$$x_1(k+1) = \frac{\alpha\, x_2(k)}{1 + (x_1(k))^2}, \qquad x_2(k+1) = \frac{\beta\, x_1(k)}{1 + (x_2(k))^2},$$

where the parameters α, β satisfy $0 < \alpha^2 < 1$, $0 < \beta^2 < 1$.

(a) Prove that origin is a fixed point.

(b) Prove that origin is asymptotically stable (A.S).

(c) Prove that origin is globally asymptotically stable (G.A.S).

Solution

(a) The substitution $(x_1^*, x_2^*) = (0, 0)$ satisfies the fixed point equations

$$x_1^* = \frac{\alpha\, x_2^*}{1 + (x_1^*)^2}, \qquad x_2^* = \frac{\beta\, x_1^*}{1 + (x_2^*)^2}.$$

Hence origin is a fixed point.

(b) Let us choose the Lyapunov function $V(x_1, x_2) = x_1^2 + x_2^2$, which is positive definite in $\mathbb{R}^2$. Now

$$\begin{aligned}
V(k+1) - V(k) &= \frac{\alpha^2 (x_2(k))^2}{\left(1 + (x_1(k))^2\right)^2} + \frac{\beta^2 (x_1(k))^2}{\left(1 + (x_2(k))^2\right)^2} - (x_1(k))^2 - (x_2(k))^2 \\
&= \left(\frac{\alpha^2}{\left(1 + (x_1(k))^2\right)^2} - 1\right)(x_2(k))^2 + \left(\frac{\beta^2}{\left(1 + (x_2(k))^2\right)^2} - 1\right)(x_1(k))^2 \\
&\leq \left(\alpha^2 - 1\right)(x_2(k))^2 + \left(\beta^2 - 1\right)(x_1(k))^2,
\end{aligned}$$

which is < 0 for all $(x_1, x_2) \in \mathbb{R}^2 \setminus \{(0,0)\}$, since $0 < \alpha^2 < 1$, $0 < \beta^2 < 1$; and is $= 0$ for $(x_1, x_2) = (0,0)$. Let $\mathscr{S}$ be the set of (x_1, x_2) such that $V(k+1) - V(k) = 0$ (see Lecture 7 notes about Corollary of LaSalle invariance theorem for fixed point). In this case, the set $\mathscr{S} = \{(0,0)\}$ is singleton, and no solution (x_1, x_2) can stay identically in $\mathscr{S}$ other than the trivial solution $(0,0)$; hence by LaSalle invariance theorem, the origin is A.S.

(c) Let us first demonstrate that origin is the unique fixed point. Any fixed point (x, y) needs to satisfy

$$x = \frac{\alpha\, y}{1 + x^2} \Rightarrow x + x^3 = \alpha y, \quad \text{and} \quad y = \frac{\beta\, x}{1 + y^2} \Rightarrow \beta x = y + y^3.$$

From the second equation, we get $x = \frac{1}{\beta}(y + y^3)$, which upon substituting into the first results a polynomial equation in y, given by

$$y \left\{ \frac{1}{\beta^3} y^8 + \frac{3}{\beta^3} y^6 + \frac{3}{\beta^3} y^4 + \left(\frac{1}{\beta^3} + \frac{1}{\beta} \right) y^2 + \left(\frac{1}{\beta} - \alpha \right) \right\} = 0.$$

The only possible real root of the above is $y = 0$, since the polynomial factor in curly braces is a sum of even powered monomials with positive coefficients and hence (by DesCartes' rule of sign) does not admit any real root. Now $y = 0$ implies $x = \frac{1}{\beta}(0 + 0^3) = 0$. Hence origin is the unique fixed point.

On the other hand, our choice of Lyapunov function $V(x_1, x_2) = x_1^2 + x_2^2 = r^2$ is radially unbounded. These, together with the fact that origin is A.S. that we have established in part (b), allow us to invoke Barbashin-Krasovskii theorem, to conclude that the origin is G.A.S.