Lec. 4 (04/09/2020)

Stability of non-autonomous System:

Autonomous	Non-autonomous
$\forall \epsilon > 0,\ t_0 \geq 0,\ \exists\, \delta = \delta(\epsilon) > 0$	$\forall \epsilon > 0,\ t_0 \geq 0,\ \exists\, \delta = \delta(\epsilon, t_0) > 0$
s.t. $\lVert \underline{x}(t_0) \rVert_2 < \delta \Rightarrow \lVert \underline{x}(t) \rVert_2 < \epsilon$	s.t. $\lVert \underline{x}(t_0) \rVert_2 < \delta \Rightarrow \lVert \underline{x}(t) \rVert_2 < \epsilon$
$\forall\, t \geq t_0$	$\forall\, t \geq t_0 \geq 0$

Now,

- δ may depend on t_0
- S/AS/GAS all will depend on t_0, in general

Special case:

If for non-autonomous, δ turns out to be independent of t_0, then we say system is <u>Uniformly stable (US)</u> in time.

Example: (S but not US) : 1D

$\dot{x} = (6t\sin t - 2t)\,x$ } $x^* = 0$ is a fixed pt.

$\Rightarrow x(t) = x_0 \exp\left(\int_{t_0}^{t} (6\tau \sin\tau - 2\tau)\, d\tau \right)$

$= x_0 \exp\left(6\sin t - 6t\cos t - t^2 \underline{- 6\sin t_0 + 6 t_0 \cos t_0 + t_0^2}\right)$

Since t_0 is given/fixed, the term $(-t^2)$ inside $\exp(\cdot)$ will dominate as t becomes large

$\Rightarrow \exp(\underline{\quad}\,)$ is bounded $\forall\, t \geq t_0$ by some constant $c(t_0)$

$\Rightarrow |x(t)| < |x_0|\, c(t_0) \quad \forall\, t \geq t_0$

$\forall\, \epsilon > 0$, choose $\delta := \delta(\epsilon, t_0) = \dfrac{\epsilon}{c(t_0)}$ $\Rightarrow$ proves $x^* = 0$ is stable.

Suppose $t_0 = 2n\pi$ for $n = 0, 1, 2, \ldots$

Then $x(t_0+\pi) = \dfrac{x(t_0)}{x_0} \exp\left((4n+1)(6-\pi)\pi\right)$

$\Rightarrow$ whenever $x_0 \neq 0$, $\displaystyle\lim_{n\to\infty} \frac{x(t_0+\pi)}{x_0} \longrightarrow +\infty$

$\therefore$ For any $\epsilon > 0$, $\nexists\, \delta$ that is indep. of t_0, which can prove stability.

$\therefore$ Origin is S but not US.

Conceptually, we understand:

$$\underbrace{S,\ US,\ UAS,\ GUAS}_{\text{Theorems in the spirit of Lyapunov}}$$

Theorem (Lyapunov-like Theorem for non-autonomous)

(US/UAS/GUAS)

Suppose $\mathcal{D} \subseteq \mathbb{R}^n$ contains $\underline{x}^* = \underline{0}$.

Let $V : [0, \infty) \times \mathcal{D} \mapsto \mathbb{R}_{\geq 0}$ be a $C^1(\mathcal{D})$ function of $\underline{x}$.

such that $\forall\, t \geq t_0 \geq 0$, and $\forall\, \underline{x} \in \mathcal{D}$

① $$\boxed{W_1(\underline{x}) \leq V(t, \underline{x}) \leq W_2(\underline{x})}$$

where $W_1(\cdot), W_2(\cdot)$ are continuous pos. def. f^{n}s of $\underline{x}$ in $\mathcal{D}$.

② $$\dot{V} = \frac{d}{dt} V(t, \underline{x}) = \frac{\partial V}{\partial t} + \langle \nabla V, \underline{f}(t, \underline{x}) \rangle \leq 0$$

Then $\underline{x}^* = \underline{0}$ is US.

- If ① + ②' $\Rightarrow$ $\boxed{\dot{V} = \frac{\partial V}{\partial t} + \langle \nabla V, \underline{f}(t, \underline{x}) \rangle \le -W_3(\underline{x}), \quad W_3(\cdot) \text{ pos. def.}}$

Then $\underline{x}^*$ is UAS

- If $\mathcal{D} \equiv \mathbb{R}^n$ AND $W_1(\underline{x})$ is radially unbounded in $\underline{x}$, then $\underline{x}^* = \underline{0}$ is GUAS

ES (Exponential Stability)

①'' $\longrightarrow$ $k_1 \|\underline{x}\|_2^a \le V(t, \underline{x}) \le k_2 \|\underline{x}\|_2^a$

+

②'' $\longrightarrow$ $\dot{V} = \frac{\partial V}{\partial t} + \langle \nabla V, \underline{f}(t, \underline{x}) \rangle$

$$\le -k_3 \|\underline{x}\|_2^a$$

$\forall\, t \ge t_0 \ge 0, \quad \forall\, \underline{x} \in \mathcal{D}$ where $k_1, k_2, k_3, a > 0$

Then $\underline{x}^* = \underline{0}$ is ES! If $\mathcal{D} = \mathbb{R}^n$, then GES

Example #1 : (1D, GUAS)

$$\dot{x} = -(1+g(t))\,x^3, \quad x(0) = x_0 \text{ given}$$

where $g(t)$ is continuous in t. and $g(t) \geq 0 \;\forall\, t \geq 0$

Prove that

① $x^* = 0$ is a fixed point $(f(t,0) = 0 \;\forall\, t \geq 0)$

② $x^* = 0$ is GUAS.

Proof of part (2): Take $V(t,x) = \underbrace{\frac{1}{2}x^2}_{\text{radially unbounded}}$

Then

$$\dot{V} = \cancel{\frac{\partial V}{\partial t}}^{\,0} + \frac{\partial V}{\partial x} f(t,x)$$

$$= -(1+g(t))\,x^4 \leq -\cancel{W_3(x)} \; x^4$$

$\forall\, x \in \mathbb{R},\ \forall\, t \geq 0$

$\therefore$ Origin is GUAS

Example (GES) $\dot{x}_1 = -x_1 - g(t)x_2$

$\dot{x}_2 = x_1 - x_2$

where $g(t)$ is C^1 in t, and $0 \leq g(t) \leq \underset{>0}{k}$,

and $\dot{g}(t) \leq g(t) \quad \forall \; t \geq 0$.

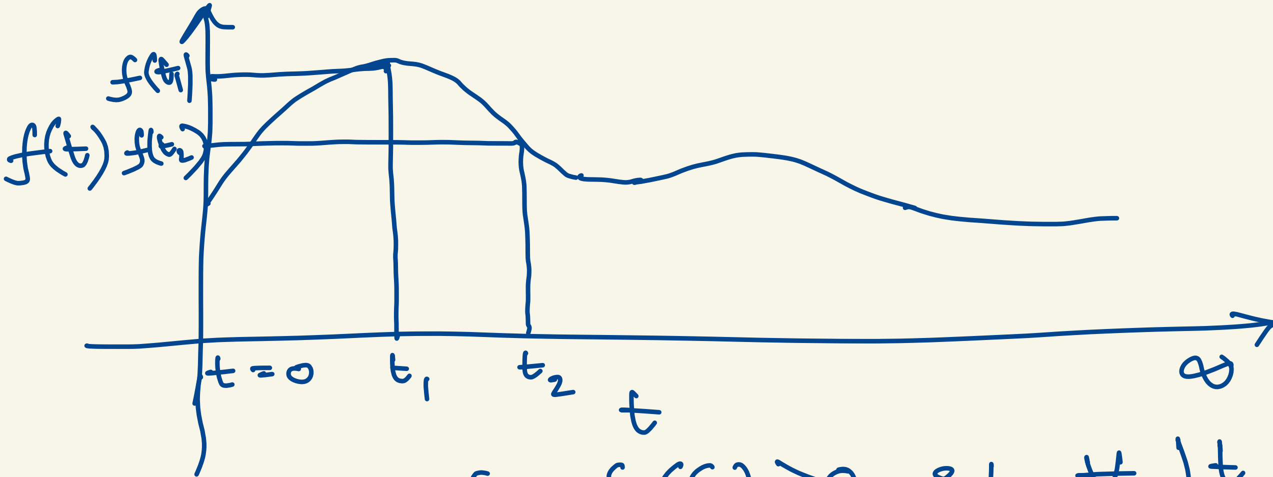
By taking $V(t, \underline{x}) = x_1^2 + (1 + g(t))\, x_2^2$

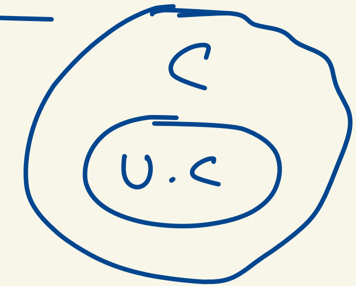
prove that origin is GES.

What if in UAS theorem, we can only show $\dot{V} \leq 0$?

Barbalat's Lemma (1959) comes in here.

Uniformly Continuous function of time : (U.C.)





$\forall \epsilon > 0, \exists \delta = \delta(\epsilon) > 0$, s.t. $\forall |t_2 - t_1| < \delta \Rightarrow |f(t_2) - f(t_1)| < \epsilon$

Barbalat's Lemma (1959) :

$|f(t_2) - f(t_1)| < K|t_2 - t_1|$

Let $f : [0, \infty) \mapsto [0, \infty)$ be a uniformly continuous f^{n}. with $\int_0^\infty f(t)\,dt < \infty$. Then $\boxed{\lim_{t \to \infty} f(t) = 0}$.

In the control context, apply Barbalat's Lemma for $f \mapsto \dot{f}$

Explicitly: If $f(t)$ is C^1 in time,

and $\lim_{t\to\infty} f(t) < \infty$,

and $\dot{f}(t)$ is uniformly continuous,

Then $\lim_{t\to\infty} \dot{f}(t) = 0$.

Proof: (contradiction)

Background: f converges $\not\Rightarrow \lim_{t\to\infty} \dot{f} \to 0$

e.g. $f(t) = \exp(-t)\sin(\exp(2t))$.

$\lim_{t\to\infty} f(t) = 0 < \infty$. BUT $\lim_{t\to\infty} \dot{f}$ does not exist.

So, what additional assumption is needed to guarantee that f converges $\Longrightarrow \dot{f}$ converges

Ans: Uniform continuity of $\dot{f}$.

Corollary of Barbalat's Lemma:

If ① $f(t)$ is C^1

② $\lim_{t\to\infty} f(t) < \infty$ exists

replaces uniform continuity ③ $\ddot{f}(t)$ exists and is bounded

$\Leftrightarrow |\ddot{f}(t)| < \infty \ \forall\ t$

Then, $\lim_{t\to\infty} \dot{f}(t) = 0$

Application: Take $V(t, \underline{x})$

Check if ① $\dot{V}(t, \underline{x}) \leq 0$

② $\dot{V}(t, \underline{x})$ is uniformly continuous

($\Leftarrow$ $\ddot{V}$ exists and bounded)

Then $\lim_{t \to \infty} \dot{V}(t, \underline{x}) = 0$.

Example: (closed-loop dynamics of adaptive control systems)

$$\dot{e} = -e + \theta\, w(t) \qquad \underline{x} = \begin{pmatrix} e \\ \theta \end{pmatrix}$$

$$\dot{\theta} = -e\, w(t)$$

$w(t)$ is bounded and continuous in t ($|w| < \infty \ \forall t$)

Prove: $\boxed{\lim_{t \to \infty} e(t) = 0}$

<u>Proof:</u> Consider $V(t, x) = \frac{1}{2}(e^2 + \theta^2)$

$$\Rightarrow \dot{V} = \underbrace{\cancel{\frac{\partial V}{\partial t}}}_{0} + \langle \nabla V, \underline{f}(t, \underline{x}) \rangle$$

$$= -2e^2 \leq 0$$

$$\Rightarrow V(t) \leq V(0)$$

$\Rightarrow$ $e(t)$ and $\theta(t)$ are bounded

(NOT clear what if anything will converge)

BUT cannot apply La Salle

Let's check uniform continuity of $\dot{V}$

$$\Rightarrow \ddot{V} = -4e(-e + \theta w) < \infty$$

<u>since $w < \infty$ given</u> & <u>e, θ bounded proved</u>

$\Rightarrow$ $\dot{V}$ is uniformly continuous

$\Rightarrow$ Barbalat's Lemma says $\lim_{t\to\infty} \dot{V}(t, \underline{x}) = 0$

$$\Leftrightarrow \lim_{t\to\infty} (-2e^2) = 0$$

$$\Leftrightarrow \underline{e(t)} \xrightarrow{t \uparrow \infty} 0$$

But $\not\Rightarrow$ A.S.

because $\theta(t)$ is only bounded.

Stability Conditions for Linear Systems

Linear Time Invariant (LTI)

$$\dot{\underline{x}} = \boxed{\underline{f}(\underline{x}) \equiv A\underline{x}}$$

$A \in \mathbb{R}^{n \times n}$ constant matrix

$\underline{x}(t_0) = x_0$

$$S \Leftarrow \boxed{AS \Leftrightarrow GAS \Leftrightarrow ES}$$

$$\Updownarrow$$

A is Hurwitz

$$\Updownarrow$$

$$\mathrm{Re}(\lambda_i(A)) < 0 \quad \forall i = 1, \ldots, n$$

$$\underline{x}(t) = \exp(A(t - t_0))\, \underline{x}_0$$

Linear Time Varying (LTV)

$$\dot{\underline{x}} \boxed{= \underline{f}(t, \underline{x}) = A(t)\, \underline{x}}$$

$\underline{x}(t_0) = \underline{x}_0$

$$\boxed{\begin{aligned} \frac{d}{dt} \Phi(t, t_0) &= A(t)\, \Phi(t, t_0) \\ \Phi(t, t_0) &= I \end{aligned}}$$

State transition matrix

$$\underline{x}(t) = \Phi(t, t_0)\, \underline{x}_0$$

In general, we can write Φ as Peano–Baker series:

$$\Phi(t,t_0) = I + \int_{t_0}^{t} A(\tau_1)\,d\tau_1 + \int_{t_0}^{t} A(\tau_1) \int_{t_0}^{\tau_1} A(\tau_2)\,d\tau_2\,d\tau_1$$

$$+ \int_{t_0}^{t} A(\tau_1) \int_{t_0}^{\tau_1} A(\tau_2) \int_{t_0}^{\tau_2} A(\tau_3)\,d\tau_3\,d\tau_2\,d\tau_1 + \cdots$$

Two special cases:

- A matrix is constant $\Leftrightarrow$ $\Phi(t,t_0) = \exp\left(A(t-t_0)\right)$
- If $A(t)$ and $\int_{t_0}^{t} A(\tau)\,d\tau$ commute, then

$$\Phi(t,t_0) = \exp\left(\int_{t_0}^{t} A(\tau)\,d\tau\right)$$

Counterexamples for LTV stability:

$$A(t) = P(t)\, \underset{\text{Hurwitz (constant matrix)}}{B}\, P(t)^{-1} \quad , \quad P \text{ invertible}$$

Hurwitz
(constant matrix)

$\exp(\text{Skew symm.}) = \text{orthogonal}$

$$\Omega := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \underbrace{\exp(t\Omega)}_{P(t)} = \begin{pmatrix} \cos(t) & -\sin t \\ \sin t & \cos t \end{pmatrix}$$

$$\therefore \quad \underline{\underline{A(t)}} = e^{t\Omega}\, \underline{\underline{B}}\, e^{-t\Omega}$$