AMS 231 – Spring 2018 – Lecture 2 Notes

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January 11, 2018

Modeling of Control Systems

- Recall that nonlinear dynamical ("unforced" $\equiv$ no control) system: $\underline{\dot{x}} = \underline{f}(\underline{x}, t)$, $\underline{x}(0) = \underline{x}_0$ (given), $\underline{x} \in \mathscr{X} \subseteq \mathbb{R}^n$

- Controlled dynamical system: $\underline{\dot{x}} = \underline{f}(\underline{x}, \underline{u}, t)$, $\underline{x}(0) = \underline{x}_0$ (given), $\underline{x} \in \mathscr{X} \subseteq \mathbb{R}^n$, $\underline{u} \in \mathscr{U} \subseteq \mathbb{R}^m$. The vector $\underline{u}$ is called the **control a.k.a. input vector**. In general, $\underline{u}$ is (either explicit or implicit function) of time, i.e., the control is a vector trajectory. Usually $m \neq n$.

Think physical examples: what are states and controls for a car, for a passenger aircraft? Be careful to distinguish between actuator and controller.

- To specify a control system, we often augment the ODE $\underline{\dot{x}} = \underline{f}(\underline{x}, \underline{u}, t)$ with an algebraic equation $\underline{y} = \underline{h}(\underline{x}, \underline{u})$, where the vector $\underline{y} \in \mathscr{Y} \subseteq \mathbb{R}^p$ is called the **measurement or output vector**. Clearly, $\underline{y}$ is also a function of time, i.e., a vector trajectory.

Intuitively, the map $\underline{h}$ is sensor model. It reflects the situation that we may not be able to directly measure the state $\underline{x}$, but only some nonlinear function of $\underline{x}$ and $\underline{u}$. A special case is $\underline{h}(\underline{x}, \underline{u}) = H\underline{x}$, where rows of H are p basis vectors in $\mathbb{R}^n$, $p < n$, which models the situation that some but not all states can be measured.

- **LTI control system** is one where the vector field $\underline{f}$ and the map $\underline{h}$ are linear in **both** $\underline{x}$ and $\underline{u}$. In continuous time, has the form: $\underline{\dot{x}} = A\underline{x} + B\underline{u}$, $\underline{y} = C\underline{x} + D\underline{u}$. In discrete time, has the form: $\underline{x}(k+1) = A\underline{x}(k) + B\underline{u}(k)$, $\underline{y}(k) = C\underline{x}(k) + D\underline{u}(k)$.

Clearly, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$, $D \in \mathbb{R}^{p\times m}$.

- Likewise, we can define **LTV control system**. In continuous time, has the form: $\underline{\dot{x}} = A(t)\underline{x} + B(t)\underline{u}$, $\underline{y} = C(t)\underline{x} + D(t)\underline{u}$. In discrete time, has the form: $\underline{x}(k+1) = A(k)\underline{x}(k) + B(k)\underline{u}(k)$, $\underline{y}(k) = C(k)\underline{x}(k) + D(k)\underline{u}(k)$.

Example 2.1: Controlled pendulum in air

Consider the simple pendulum example in Lecture 1, now with a motor attached at the pivot point that provides control torque $\underline{\tau}$, which can be manipulated by adjusting the amount of current being passed to the motor. The control torque opposes the restoring angular motion (for example, $\underline{\tau}$ can be used to "freeze" the pendulum at certain angle θ_{fixed} against its natural tendency to come back to $\theta = 0$), i.e., $\underline{\tau} = |\underline{\tau}|\,\widehat{k}$.

Comparing with Example 1.1 in Lecture 1 notes, the total external torque now equals

$$\left(-mg\ell \sin\theta - b\dot{\theta} + |\underline{\tau}|\right)\,\widehat{k},$$

and letting $u := |\underline{\tau}|$, the controlled dynamics becomes

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} f_1(x_1, x_2, u) \\ f_2(x_1, x_2, u) \end{pmatrix} = \begin{pmatrix} x_2 \\ -\alpha \sin x_1 - \beta x_2 + u \end{pmatrix},$$

which is in the standard form $\underline{\dot{x}} = \underline{f}(\underline{x}, \underline{u}, t)$. The initial conditions and parameters are as in Example 1.1. Since practical motors can only provide finite amount of torque, it is reasonable to hypothesize a bound $|\underline{\tau}| \leq \tau_{\max}$. Consequenltly $\mathscr{U} \equiv [-\tau_{\max}, \tau_{\max}] \subset \mathbb{R}$, and $u \in \mathscr{U}$. This is an example of autonomous nonlinear control system.

Open-loop and Closed-loop Control

- If $\underline{u}$ is specified as only an explicit function of time t, i.e., $\underline{u} = \underline{u}(t)$, then we call $\underline{u}$ an **open-loop control**.

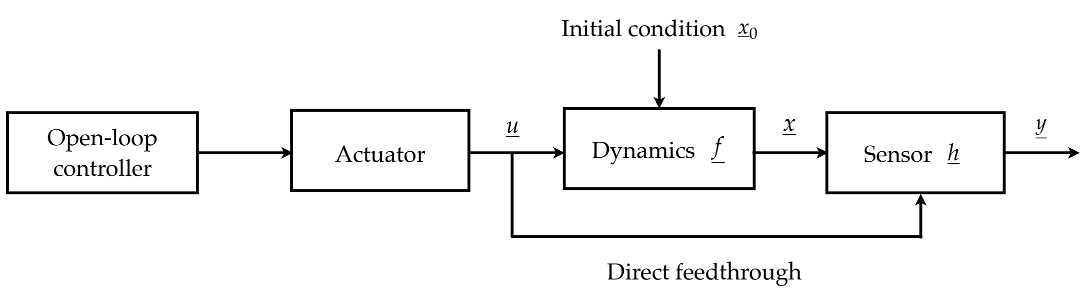
Intuitively, open-loop control is a time-table.

- If $\underline{u}$ is specified as only an explicit function of output $\underline{y}$, i.e., $\underline{u} = \underline{u}(\underline{y})$, then we call $\underline{u}$ **closed-loop control, a.k.a. feedback control**. In the special case $\underline{y} = \underline{x}$, the feedback control is referred as **state feedback**. If $\underline{y} \neq \underline{x}$, the feedback control is called **output feedback**. In general, we can have mixed open-loop and closed-loop control, i.e., $\underline{u} = \underline{u}(\underline{y}, t)$.

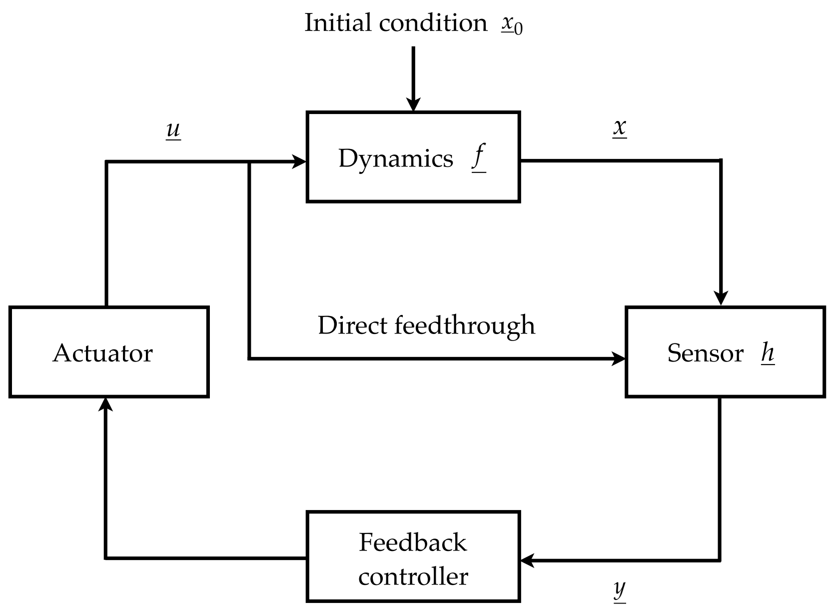
Intuitively, feedback is a policy for decision making.

Block Diagrams

- Open-loop control system:

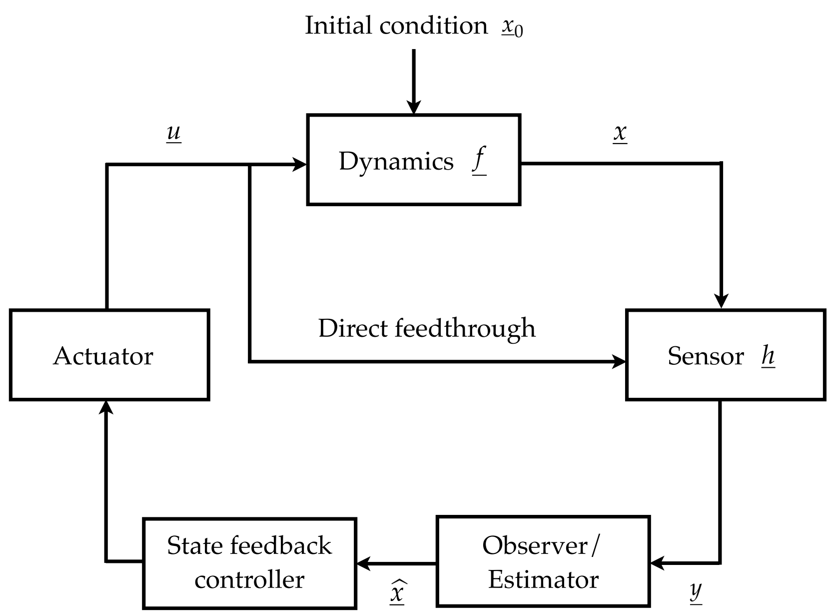


- Closed-loop a.k.a. feedback control system:



- The above block diagram assumes that the controller is of **output feedback** type (it is state feedback iff $\underline{y} = \underline{x}$). By introducing an observer/estimator in the loop, it is possible to perform feedback control with a **state feedback** controller even when $\underline{y} \neq \underline{x}$, as shown in the following block diagram. This is useful in practice since designing state feedback controller is often easier than designing output feedback controller.

Observer/estimator is an algorithm. It can be thought of as "virtual state sensor". Notice that if the state estimate $\underline{\widehat{x}}$ is away from true state $\underline{x}$, the control performance will degrade.



Two Common Types of Equilibria in Autonomous Systems

- **Fixed point** $\underline{x}^*$: in continuous time is a point satisfying $\dot{\underline{x}} = 0 \Leftrightarrow 0 = \underline{f}(\underline{x}^*)$; in discrete time is a point satisfying $\underline{x}(k+1) = \underline{x}(k) \Leftrightarrow \underline{x}^* = \underline{f}(\underline{x}^*)$. These algebraic equations may have multiple solutions, i.e., multiple isolated fixed points.
- **Limit cycle** $\underline{\widetilde{x}}$: in continuous time is a curve satisfying $\underline{\widetilde{x}}(t+T) = \underline{\widetilde{x}}(t)\ \forall t > 0$, for some fixed $T > 0$ (T is called time period); in discrete time is a pair of points satisfying $\underline{\widetilde{x}} = f \circ f(\underline{\widetilde{x}})$ (period-2 orbit). Again, these algebraic equations may admit multiple solutions.

Linear vs. Nonlinear (Autonomous) Dynamical Systems

Property	LTI system $\dot{\underline{x}} = A\underline{x}$	Nonlinear system $\dot{\underline{x}} = \underline{f}(\underline{x})$
# of fixed points	1 (if A is non-singular) or ∞ (otherwise) $\Leftrightarrow$ origin is the unique **isolated** fixed point	may have multiple **isolated** fixed points
State trajectory	$\underline{x}(t) = \underline{x}_0 e^{At} = \sum_{i=1}^{n} c_i e^{\lambda_i t}\underline{v}_i$, where $\{\lambda_i, \underline{v}_i\}_{i=1}^n$ are eigen pairs of A, and initial condition determines $\{c_i\}_{i=1}^n$	Usually no analytical solution
Finite escape time	Impossible, $\underline{x}(t) \to \infty$ may only happen in infinite time	$\underline{x}(t) \to \infty$ may happen in finite time
# of limit cycles	0 (origin is node or focus) or ∞ (origin is center)	may have multiple **isolated** limit cycles
Other types of equilibrium	Impossible	Possible (higher period orbits etc.)

Example 2.2: Finite escape time in nonlinear system

Consider scalar nonlinear system

$$\dot{x} = x^2,\ x(0) := x_0, \quad \Rightarrow \quad x(t) = \frac{x_0}{1 - x_0 t}.$$

At $t = \frac{1}{x_0}$, we have $x(t) \to \infty$.

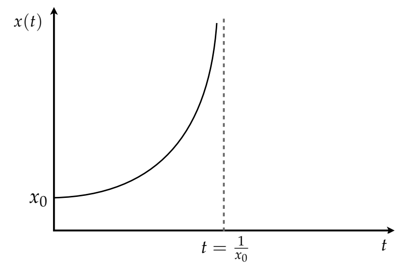


Figure 1: Finite escape time in Example 2.2