Problem 1 [55 points]

Minimum Energy State Transfer for a Nonlinear System

(a) [15 points] Warm up exercise on linear time varying ODE

This basic ODE exercise does not require any knowledge of control theory, but will soon be useful in a nonlinear optimal control problem that follows.

Consider a linear time varying ODE initial value problem (IVP) in unknown $z(t) \in \mathbb{R}^m$, given by

$$\dot{z} = \Omega(t)z, \quad z(0) = z_0 \text{ (known initial vector)},$$

where $\Omega(t)$ is given time-varying skew-symmetric matrix of size $m \times m$. **Prove that** the IVP solution is of the form

$$z(t) = Q(t)z_0, \quad \text{where matrix } Q(t) \text{ orthogonal for all time } t.$$

Hint: Think about the state transition matrix.

(b) [10 + 2 + 3 = 15 points] Nonlinear OCP

(i) Consider a nonlinear OCP with **final time T fixed**, given by

$$\begin{aligned}
&\min_{u(\cdot)} \int_0^T \|u\|_2^2 \, \mathrm{d}t \\
&\dot{x} = F(x)u, \quad F(x) := \left[\, f_1(x), f_2(x), \dots, f_m(x) \right] \in \mathbb{R}^{n \times m}, \quad x \in \mathbb{R}^n, \quad u \in \mathbb{R}^m, \\
&x(t=0) = x_0 \text{ (given)}, \quad x(t=T) = x_T \text{ (given)},
\end{aligned}$$

where the nonlinear vector fields $f_j(x) \in \mathbb{R}^n$ for all $j = 1, \dots, m$. We assume that the system is controllable (this can be checked by some known conditions involving the f_j's, but these details are unnecessary for this problem).

Clearly write down the first order necessary conditions of optimality for this problem, that is, the state and costate equations, PMP, transversality and boundary conditions **satisfied by the optimal solution of this OCP**.

(ii) **Without doing any calculation, argue** if the Hamiltonian along the optimal solution, for this problem, is constant or not.

(iii) Write the **optimal Hamiltonian only as a function of the optimal control** (meaning your expression should not be a function of any other variable). **Show all calculations**.

(c) [(10 + 5) + 5 + 5 = 25 points] Optimal control is unitary

(i) Define the Lie bracket of two vector fields $f(x), g(x)$ as $[f, g] := \frac{\partial g}{\partial x} f - \frac{\partial f}{\partial x} g$. Use your answer in part (b)(i) to **prove that** the optimal control $u^{\text{opt}}(t)$ for the OCP in part (b) solves an IVP

$$\dot{u}^{\text{opt}} = M(x^{\text{opt}}, \lambda^{\text{opt}})\, u^{\text{opt}}, \quad u^{\text{opt}}(t=0) = u_0^{\text{opt}}.$$

Derive the (i, j)th entry of the matrix $M \in \mathbb{R}^{m \times m}$ in terms of x^{opt}, λ^{opt} and Lie brackets involving $f_1, \ldots, f_m$.

(ii) Use your answer in part (c)(i) together with part (a), to **prove that** the optimal control is unitary (i.e., norm preserving):

$$\|u^{\text{opt}}(t)\|_2 = \|u_0^{\text{opt}}\|_2.$$

(iii) Suppose we somehow (say, numerically) compute the optimal control $u^{\text{opt}}(t)$ for the OCP in part (b)(i). **How will this optimal control change** if we change the Lagrangian in the cost-to-go from $\|u\|_2^2$ to $\phi(\|u\|_2^2)$, where $\phi : \mathbb{R} \mapsto \mathbb{R}$ is an arbitrary monotone function? **Give one sentence explanation to support your answer**.

Problem 2 [45 points]

Optimal Economic Reform

Suppose the scalar state $x(t)$ of national economy is governed by the second order ODE

$$\ddot{x} = -\alpha^2 x + u, \quad \alpha \in \mathbb{R} \setminus \{0\}, \quad t \geq 0, \quad x(0) = \dot{x}(0) = 0,$$

where $u(t)$ is the effort a Government puts at time t for economic reform.

Suppose the Goverment would like to maximize its chance of getting re-elected at the **fixed** terminal time T, by bringing the national economy at a healthy state at the time of re-election, while not spending too much effort in economic reform during its tenure, i.e.,

$$\underset{u(\cdot)}{\text{maximize}} \quad x(T) - \int_0^T u^2 \, \mathrm{d}t.$$

In practice, the Government may want to maximize an increasing function of the above cost, but we will ignore such details.

(a) [5 + 5 + 8 = 18 points] Standard form

(i) **Define the state vector** and **write the second order ODE in state space form**, i.e., as a controlled vector first order ODE.

(ii) Use your answer in part (a)(i) to clearly **rewrite the OCP in standard form**. **Identify terminal cost/terminal constraint**, if any.

(iii) Write the **Hamiltonian, the costate ODEs, the PMP**, and **the transversality condition** for the OCP in part (a)(ii).

(b) [15 + (10 + 2) = 27 points] Solution of the OCP

(i) Find the costates in terms of t, α, T.

Hint: use the transversality conditions to solve the costate ODE initial value problem.

(ii) Compute the **optimal economic reform** $u^{\text{opt}}(t)$ for the Government. **Also compute the optimal terminal reform** $u^{\text{opt}}(T)$.