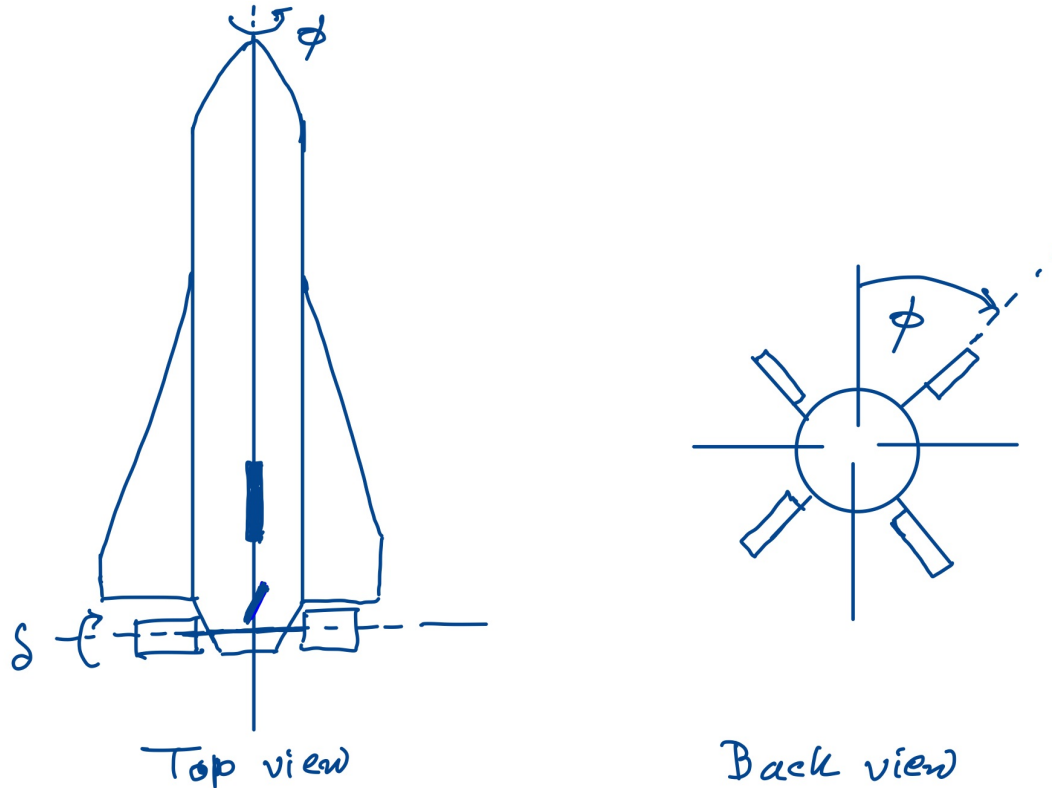


## Problem 1. [100 points] Regulating Roll

To regulate the rolling motion of a missile, a common strategy is to actuate its hydraulic-powered ailerons. The control  $u$  is the aileron deflection rate (rad/s). The state vector  $(\delta, \omega, \phi)^T$  comprises of the aileron deflection angle  $\delta$  (rad), the roll angular velocity  $\omega$  (rad/s), and the roll angle  $\phi$  (rad), respectively. See Fig. below.



The control design objective is to

$$\underset{u(\cdot)}{\text{minimize}} \quad \int_0^\infty \frac{1}{2} \left( \frac{\delta^2}{\delta_0^2} + \frac{\phi^2}{\phi_0^2} + \frac{u^2}{u_0^2} \right) dt$$

subject to the equations of rolling motion

$$\dot{\delta} = u, \quad \dot{\omega} = -\frac{1}{\tau}\omega + \frac{q}{\tau}\delta, \quad \dot{\phi} = \omega,$$

where  $\tau$  is the rolling time constant (s), and  $q$  is the aileron effectiveness ( $\text{s}^{-1}$ ). The weights  $(\delta_0, \phi_0, u_0)$  are known constants.

### (a) [20 + 20 = 40 points] Analysis

Let  $\sigma := u_0 P_{11}$ , where  $P_{11}$  is the  $(1, 1)$  entry of the symmetric matrix  $P_\infty$  that solves the associated CARE. **Prove that the optimal feedback control**

$$u^{\text{opt}} = -u_0^2 \begin{bmatrix} \frac{\sigma}{u_0} & \frac{\tau}{2q} \left( \sigma^2 - \frac{1}{\delta_0^2} \right) & \frac{1}{u_0 \phi_0} \end{bmatrix} \begin{bmatrix} \delta \\ \omega \\ \phi \end{bmatrix},$$

and that  $\sigma$  solves a quartic equation

$$\sigma^4 + \frac{4}{u_0 \tau} \sigma^3 + \frac{4}{u_0^2 \tau^2} \left( 1 - \frac{u_0^2 \tau^2}{2 \delta_0^2} \right) \sigma^2 - \frac{4}{u_0 \tau} \left( \frac{2q}{\phi_0 u_0} + \frac{1}{\delta_0^2} \right) \sigma + \left( \frac{1}{\delta_0^4} - \frac{4}{\delta_0^2 u_0^2 \tau^2} - \frac{8q}{\phi_0 u_0^3 \tau^2} \right) = 0.$$

### (b) [(10 + 10) + (10 + 10 + 10) + 10 = 60 points] Numerics

Consider the parameter values  $\tau = 1$  s,  $Q = 10$   $\text{s}^{-1}$ ,  $u_0 = \pi$  rad/s,  $\delta_0 = \pi/12$  rad,  $\phi_0 = \pi/180$  rad.

(i) **Write a MATLAB code to verify that the system with the above parameters is controllable (hence stabilizable) and observable (hence detectable).** For this purpose, you may use MATLAB commands such as `rank(ctrb(A,B))` and `rank(observ(A,C))` for appropriately defined matrices A,B,C. **What can you conclude about the nature of the matrix  $P_\infty$  from this controllability and observability verifications? Give reasons to support your answer.** (Hint: see Lec. 11)

(ii) Use the same MATLAB code to **compute the four roots** of the quartic equation derived in part (a). You may use the MATLAB command `roots()`. **Which of these four roots should be used to compute  $P_{11}$  and why? Use your reasoning to report the  $3 \times 3$  matrix  $P_\infty$ .**

(iii) To verify the  $P_\infty$  obtained in part (b)(ii), **compute the same using MATLAB command `icare` in the same code.** Submit your MATLAB code.