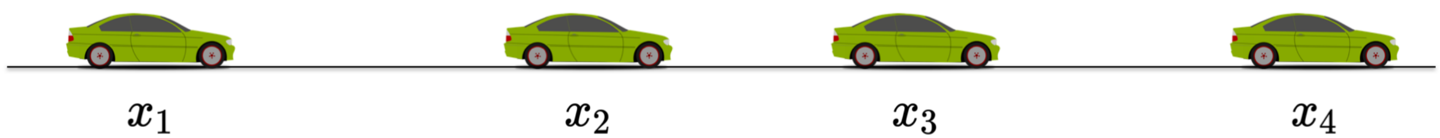
Problem 1. [100 Points] Platooning

In a single lane straight road, N vehicles are moving to the right with respective 1D position coordinates $x_1, x_2, \ldots, x_N$. See Fig. showing an example scenario for $N = 4$.



Suppose that all vehicles have identical discrete time controlled dynamics $x_i(k+1) = x_i(k) + h(u_i(k) - v)$, $i = 1, 2, \ldots, N$, for time index $k = 0, \ldots, T-1$. The parameter $h > 0$ is given constant sampling time. The control u_i can be thought of as the speed of the ith vehicle, and v is a known speed limit.

Here is the high level question of interest: what should be the optimal controls such that all consecutive vehicles maintain a separation close to some (known) desired distance d at all times? A separation smaller than d may be unsafe, and thus undesirable. A separation more than d reduces traffic throughput, and therefore also undesirable. We also want all vehicles to move at a speed close to the known speed limit v.

(a) [35 points] OCP formulation

Motivated by the aforesaid objective, consider minimizing

$$\frac{1}{2}\sum_{i=1}^{N-1}(x_{i+1}(T) - x_i(T) - d)^2 + \frac{1}{2}\sum_{k=0}^{T-1}\left\{\sum_{i=1}^{N-1}(x_{i+1}(k) - x_i(k) - d)^2 + \sum_{i=1}^{N}(u_i(k) - v)^2\right\}$$

subject to $x_i(k+1) = x_i(k) + h(u_i(k) - v)$, $i = 1, 2, \ldots, N$. Consider the final time T fixed.

Recast this problem as discrete time finite horizon LQ tracking by clearly defining the **state vector x and its dimension**, the **control vector u and its dimension**, the **output vector y and its dimension**, the **desired output trajectory y_d to track**, the **system matrices A, B, C**, and the **weight matrices M, Q, R in the cost function.**

Hint: Take a look at Lec. 10, p. 3 to see how LQ tracking problem was formulated in the continuous time case. See also part (b) to get a hint on the problem structure.

(b) [35 points] Discrete time LQ tracking solution

Extend the derivation in Lec. 10, p. 15-23 for the tracking case:

$$\underset{\{u_k\}_{k=0}^{T-1}}{\text{minimize}} \quad \frac{1}{2}\left\{(y(T)-y_d(T))'M(y(T)-y_d(T)) + \sum_{k=0}^{T-1}(y(k)-y_d(k))'Q(y(k)-y_d(k)) + (u(k))'Ru(k)\right\}$$

subject to $x(k+1) = Ax(k) + Bu(k),\ y(k) = Cx(k),\ k = 0, 1, \ldots, T-1.$

Hint: Just like the continuous time LQ tracking solution given in Lec. 10, here too you should get optimal control as a sum of a linear state feedback term, and a feedforward term. You need to derive a backward **vector** recursion for the feedforward control. Also derive how the Riccati backward recursion needs to be modified in this case, compared to the same for LQR.

(c) [30 points] Optimal control for platooning

Apply your answer in part (b) to the formulation in part (a), to **compute and plot** $y^{\text{opt}}(k)$ superimposed with $y_d(k)$. Also plot $u^{\text{opt}}(k)$. To make these plots, fix $T = 200$, $h = 0.01$, $N = 4$, and the initial conditions $x_1(0) = 0$ ft, $x_2(0) = 250$ ft, $x_3(0) = 480$ ft, $x_4(0) = 780$ ft.

Please submit your single MATLAB code generating these plots.