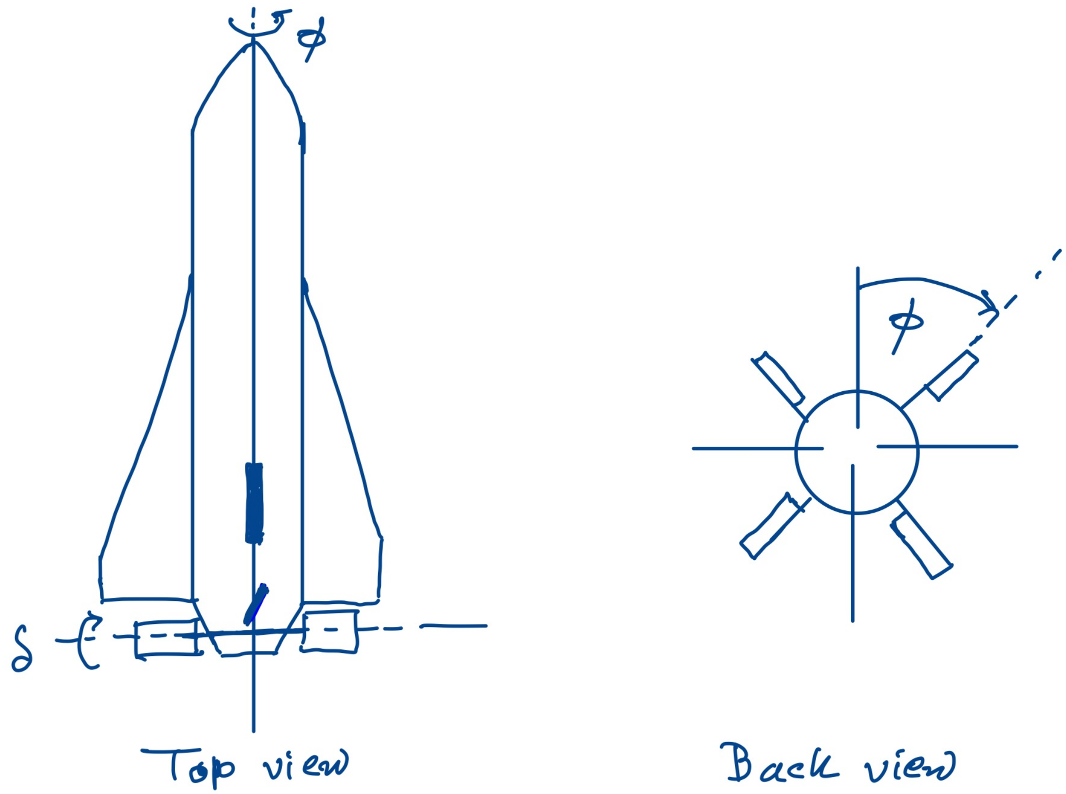
Problem 1. [100 points] Regulating Roll

To regulate the rolling motion of a missile, a common strategy is to actuate its hydraulic-powered ailerons. The control u is the aileron deflection rate (rad/s). The state vector $(\delta, \omega, \phi)^{\top}$ comprises of the aileron deflection angle δ (rad), the roll angular velocity ω (rad/s), and the roll angle ϕ (rad), respectively. See Fig. below.



The control design objective is to

$$\underset{u(\cdot)}{\text{minimize}} \quad \int_0^\infty \frac{1}{2}\left(\frac{\delta^2}{\delta_0^2} + \frac{\phi^2}{\phi_0^2} + \frac{u^2}{u_0^2}\right) \mathrm{d}t$$

subject to the equations of rolling motion

$$\dot{\delta} = u, \quad \dot{\omega} = -\frac{1}{\tau}\omega + \frac{q}{\tau}\delta, \quad \dot{\phi} = \omega,$$

where τ is the rolling time constant (s), and q is the aileron effectiveness constant (s^{-1}). The weights (δ_0, ϕ_0, u_0) are known constants.

(a) [20 + 20 = 40 points] Analysis

Let $\sigma := u_0 P_{11}$, where P_{11} is the $(1,1)$ entry of the symmetric matrix P_∞ that solves the associated CARE. **Prove that** the **optimal feedback control**

$$u^{\text{opt}} = -u_0^2 \begin{bmatrix} \dfrac{\sigma}{u_0} & \dfrac{\tau}{2q}\left(\sigma^2 - \dfrac{1}{\delta_0^2}\right) & \dfrac{1}{u_0\phi_0} \end{bmatrix} \begin{bmatrix} \delta \\ \omega \\ \phi \end{bmatrix},$$

and **that σ solves a quartic equation**

$$\sigma^4 + \frac{4}{u_0\tau}\sigma^3 + \frac{4}{u_0^2\tau^2}\left(1 - \frac{u_0^2\tau^2}{2\delta_0^2}\right)\sigma^2 - \frac{4}{u_0\tau}\left(\frac{2q}{\phi_0 u_0} + \frac{1}{\delta_0^2}\right)\sigma + \left(\frac{1}{\delta_0^4} - \frac{4}{\delta_0^2 u_0^2\tau^2} - \frac{8q}{\phi_0 u_0^3\tau^2}\right) = 0.$$

Solution for part (a):

This is an infinite horizon LQ regulation problem with state equation

$$\begin{pmatrix} \dot{\delta} \\ \dot{\omega} \\ \dot{\phi} \end{pmatrix} = \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ q/\tau & -1/\tau & 0 \\ 0 & 1 & 0 \end{bmatrix}}_{A} \underbrace{\begin{pmatrix} \delta \\ \omega \\ \phi \end{pmatrix}}_{x} + \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}_{B} u,$$

and cost weight matrices

$$Q = \begin{pmatrix} 1/\delta_0^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1/\phi_0^2 \end{pmatrix} \geq 0, \qquad R = 1/u_0^2 > 0.$$

The associated CARE is $0 = A'P_\infty + P_\infty A - P_\infty B R^{-1} B' P_\infty + Q$. Since P_∞ is symmetric, we have 6 independent entries of the 3×3 matrix $P_\infty \equiv [P_{ij}]$ to determine from the CARE:

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & q/\tau & 0 \\ 0 & -1/\tau & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} P_{11} & P_{12} & P_{13} \\ P_{12} & P_{22} & P_{23} \\ P_{13} & P_{23} & P_{33} \end{bmatrix} + \begin{bmatrix} P_{11} & P_{12} & P_{13} \\ P_{12} & P_{22} & P_{23} \\ P_{13} & P_{23} & P_{33} \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ q/\tau & -1/\tau & 0 \\ 0 & 1 & 0 \end{bmatrix}$$
$$+ \begin{bmatrix} P_{11} \\ P_{12} \\ P_{13} \end{bmatrix} u_0^2 \begin{bmatrix} P_{11} & P_{12} & P_{13} \end{bmatrix} + \begin{bmatrix} 1/\delta_0^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1/\phi_0^2 \end{bmatrix}.$$

Equating the entries in the LHS with the entries in the RHS gives

$$\frac{2q}{\tau}P_{12} - u_0^2 P_{11} + \frac{1}{\delta_0^2} = 0, \tag{1}$$
$$\frac{q}{\tau}P_{22} - \frac{1}{\tau}P_{12} + P_{13} - u_0^2 P_{11}P_{12} = 0, \tag{2}$$
$$\frac{q}{\tau}P_{23} - u_0^2 P_{11}P_{13} = 0, \tag{3}$$
$$-\frac{1}{\tau}P_{12} + P_{13} + \frac{q}{\tau}P_{22} - u_0^2 P_{11}P_{12} = 0, \tag{4}$$
$$-\frac{1}{\tau}P_{22} + P_{23} - \frac{1}{\tau}P_{22} + P_{23} - u_0^2 P_{12}^2 = 0, \tag{5}$$
$$-\frac{1}{\tau}P_{23} + P_{33} - u_0^2 P_{12}P_{13} = 0, \tag{6}$$
$$\frac{q}{\tau}P_{23} - u_0^2 P_{11}P_{13} = 0, \tag{7}$$
$$-\frac{1}{\tau}P_{23} + P_{33} - u_0^2 P_{12}P_{13} = 0, \tag{8}$$
$$-u_0^2 P_{13}^2 + \frac{1}{\phi_0^2} = 0. \tag{9}$$

Notice that (3) is same as (7). Also, (4) is same as (2). Furthermore, (6) is same as (8). So we really have 9-3=6 equations to solve for 6 unknowns.

Recall that $P_{11} = \sigma/u_0$ (given). We get

$$P_{13} = \frac{1}{u_0\phi_0}, \qquad \text{(form (9))}$$
$$P_{23} = \frac{\tau}{q\phi_0}\sigma, \qquad \text{(form (7))}$$
$$P_{12} = \frac{\tau}{2q}\left(\sigma^2 - \frac{1}{\delta_0^2}\right), \qquad \text{(from (1))}$$
$$P_{33} = \frac{1}{q\phi_0}\sigma + \frac{u_0\tau}{2q\phi_0}\left(\sigma^2 - \frac{1}{\delta_0^2}\right), \qquad \text{(from (8))}$$
$$P_{22} = \frac{\tau}{2q^2}(u_0\tau\sigma + 1)\left(\sigma^2 - \frac{1}{\delta_0^2}\right) - \frac{\tau}{qu_0\phi_0}. \qquad \text{(from (2))}$$

Substituting the expressions for P_{23}, P_{12}, P_{22} above into (5), we obtain

$$-\frac{1}{q^2}(u_0\tau\sigma + 1)\left(\sigma^2 - \frac{1}{\delta_0^2}\right) + \frac{2}{qu_0\phi_0} + \frac{2\tau}{q\phi_0}\sigma - \frac{u_0^2\tau^2}{4q^2}\left(\sigma^2 - \frac{1}{\delta_0^2}\right)^2 = 0.$$

In the LHS above, rearranging the quartic polynomial in σ as a monic polynomial, we obtain the desired equation given in the problem statement.

The optimal feedback control

$$\begin{aligned} u^{\text{opt}} &= -R^{-1} B' P_\infty x \\ &= -u_0^2 \begin{bmatrix} P_{11} & P_{12} & P_{13} \end{bmatrix} \begin{pmatrix} \delta \\ \omega \\ \phi \end{pmatrix} \\ &= -u_0^2 \begin{bmatrix} \dfrac{\sigma}{u_0} & \dfrac{\tau}{2q}\left(\sigma^2 - \dfrac{1}{\delta_0^2}\right) & \dfrac{1}{u_0\phi_0} \end{bmatrix} \begin{pmatrix} \delta \\ \omega \\ \phi \end{pmatrix}, \end{aligned}$$

as desired.

(b) [(10 + 10) + (10 + 10 + 10) + 10 = 60 points] Numerics

Consider the parameter values $\tau = 1$ s, $q = 10$ s^{-1}, $u_0 = \pi$ rad/s, $\delta_0 = \pi/12$ rad, $\phi_0 = \pi/180$ rad.

(i) **Write a MATLAB code to verify that the system with the above parameters is controllable (hence stabilizable) and observable (hence detectable)**. For this purpose, you may use MATLAB commands such as `rank(ctrb(A,B))` and `rank(obsv(A,C))` for appropriately defined matrices `A,B,C`. **What can you conclude about the nature of the matrix P_∞ from this controllability and observability verifications? Give reasons to support your answer.** (Hint: see Lec. 11)

(ii) Use the same MATLAB code to **compute the four roots** of the quartic equation derived in part (a). You may use the MATLAB command `roots()`. **Which of these four roots should be used to compute P_{11} and why? Use your reasoning to report the 3×3 matrix P_∞.**

(iii) To verify the P_∞ obtained in part (b)(ii), **compute the same using MATLAB command `icare` in the same code**. Submit your MATLAB code.

Solution for part (b):

(i) Please see the posted MATLAB code `UCSC-AM232-S21-HW5.m` in CANVAS Files section, folder: HW. Since both `rank(ctrb(A,B))` and `rank(obsv(A,C))` equals three, we conclude that the system is both controllable (hence stabilizable) and observable (hence detectable). Therefore, the symmetric P_∞ solving the associated CARE, is not only unique positive semidefinite, but is in fact strictly positive definite (see Lec. 11, p. 13). In other words, unique strict positive definiteness is a consequence of stabilizability + observability (instead of mere detectability).

(ii) The same MATLAB code, using the command `roots()`, gives the following four roots of the quartic polynomial equation in σ derived in part (a).

$$\sigma = 8.5689, 0.1216, -4.9818 \pm 5.6527i.$$

For our purpose, we can immdeiately discard the complex conjugate pair. Now the question becomes out of the two remaining real roots: $8.5689, 0.1216$, which one is admissible.

We recall that a symmetric matrix P_∞ is positive definite iff all its leading principal minors are > 0. We numerically verify that the root 0.1216 fails this criterion. However, the root 8.5689 satisfies this criterion, and the corresponding matrix is

$$P_\infty = \begin{bmatrix} 2.7276 & 2.9418 & 18.2378 \\ 2.9418 & 6.3897 & 49.0962 \\ 18.2378 & 49.0962 & 578.6189 \end{bmatrix}.$$

(iii) Using MATLAB command `icare` in the same code, we get exactly the P_∞ reported above, thus verifying our solution.