Problem 1 [55 points]

Minimum Energy State Transfer for a Nonlinear System

(a) [15 points] Warm up exercise on linear time varying ODE

This basic ODE exercise does not require any knowledge of control theory, but will soon be useful in a nonlinear optimal control problem that follows.

Consider a linear time varying ODE initial value problem (IVP) in unknown $z(t) \in \mathbb{R}^m$, given by

$$\dot{z} = \Omega(t)z, \quad z(0) = z_0 \text{ (known initial vector)},$$

where $\Omega(t)$ is given time-varying skew-symmetric matrix of size $m \times m$. **Prove that** the IVP solution is of the form

$$z(t) = Q(t)z_0, \quad \text{where matrix } Q(t) \text{ orthogonal for all time } t.$$

Hint: Think about the state transition matrix.

Solution for Part 1(a):

From the definition of the state transition matrix (STM), we have $z(t) = \Phi(t,0)z_0$. So we need to demonstrate that the STM $\Phi(t,0)$ is orthogonal for all $t \geq 0$.

We know that $\dot{z} = \Omega(t)z \;\Rightarrow\; \dot{\Phi} = \Omega\Phi$ (Lec. 8, p. 4, properties of STM). Thus, $\dot{\Phi}' = -\Phi'\Omega$ where $'$ denotes matrix transpose.

Now

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\Phi'(t,0)\Phi(t,0)\right) = \dot{\Phi}'(t,0)\Phi(t,0) + \Phi'(t,0)\dot{\Phi}(t,0) = -\Phi'(t,0)\Omega(t)\Phi(t,0) + \Phi'(t,0)\Omega(t)\Phi(t,0) = 0,$$

which implies

$$\Phi'(t,0)\Phi(t,0) = \text{constant} = \underbrace{\Phi'(0,0)}_{=I}\underbrace{\Phi(0,0)}_{=I} = I,$$

wherer we used Lec. 8, p. 4, properties of STM. This completes the proof.

(b) [10 + 2 + 3 = 15 points] Nonlinear OCP

(i) Consider a nonlinear OCP with **final time T fixed**, given by

$$\min_{u(\cdot)} \int_0^T \frac{1}{2}\|u\|_2^2 \, \mathrm{d}t$$
$$\dot{x} = F(x)u, \quad F(x) := \left[\, f_1(x), f_2(x), \ldots, f_m(x) \right] \in \mathbb{R}^{n\times m}, \quad x \in \mathbb{R}^n, \quad u \in \mathbb{R}^m,$$
$$x(t=0) = x_0 \text{ (given)}, \quad x(t=T) = x_T \text{ (given)},$$

where the nonlinear vector fields $f_j(x) \in \mathbb{R}^n$ for all $j = 1, \ldots, m$. We assume that the system is controllable (this can be checked by some known conditions involving the f_j's, but these details are unnecessary for this problem).

Clearly write down the first order necessary conditions of optimality for this problem, that is, the state and costate equations, PMP, transversality and boundary conditions **satisfied by the optimal solution of this OCP**.

(ii) **Without doing any calculation, argue** if the Hamiltonian along the optimal solution, for this problem, is constant or not.

(iii) Write the **optimal Hamiltonian only as a function of the optimal control** (meaning your expression should not be a function of any other variable). **Show all calculations**.

Solution for Part 1(b):

(i) The Hamiltonian

$$H = \frac{1}{2}u'u + \lambda' F(x)u = \frac{1}{2}u'u + \lambda' \sum_{j=1}^{m} f_j(x)u_j.$$

The state ODE is $\dot{x} = F(x)u$. The costate ODE is $\dot{\lambda} = -\dfrac{\partial H}{\partial x} = -\left(\displaystyle\sum_{j=1}^{m} \underbrace{\frac{\partial f_j}{\partial x}}_{\text{Jacobian of } f_j} u_j\right)' \lambda.$

The PMP gives $0 = \dfrac{\partial H}{\partial u} = u + \dfrac{\partial}{\partial u}\left(\lambda' F(x)u\right) = u + F(x)'\lambda \;\Rightarrow\; u^{\text{opt}} = -F(x^{\text{opt}})'\lambda^{\text{opt}}.$

The transversality condition yields $0 + 0 = 0$, giving no new information.

(ii) The Hamiltonian along the optimal solution is constant since this is a **time invariant OCP**, meaning neither the Lagrangian, nor the controlled vector field has any explicit dependence on t.

(iii) From PMP, $\left(u^{\text{opt}}\right)' = -\left(\lambda^{\text{opt}}\right)' F(x^{\text{opt}})$. Therefore, the Hamiltonian along optimal solution is

$$H^{\text{opt}} = \frac{1}{2}\left(u^{\text{opt}}\right)'u^{\text{opt}} + \left(\lambda^{\text{opt}}\right)' F(x^{\text{opt}})u^{\text{opt}} = \frac{1}{2}\left(u^{\text{opt}}\right)'u^{\text{opt}} - \left(u^{\text{opt}}\right)'u^{\text{opt}} = -\frac{1}{2}\|u^{\text{opt}}\|_2^2.$$

This is already telling us that the norm of optimal control must be constant w.r.t. time since H^{opt} is constant w.r.t. time (from part 1(b)(ii)).

(c) [(10 + 5) + 5 + 5 = 25 points] Optimal control is unitary

(i) Define the Lie bracket of two vector fields $f(x), g(x)$ as $[f, g] := \frac{\partial g}{\partial x} f - \frac{\partial f}{\partial x} g$. Use your answer in part (b)(i) to **prove that** the optimal control $u^{\text{opt}}(t)$ for the OCP in part (b) solves an IVP

$$\dot{u}^{\text{opt}} = M(x^{\text{opt}}, \lambda^{\text{opt}})\, u^{\text{opt}}, \quad u^{\text{opt}}(t=0) = u_0^{\text{opt}}.$$

Derive the (i, j)th entry of the matrix $M \in \mathbb{R}^{m \times m}$ in terms of x^{opt}, λ^{opt} and Lie brackets involving $f_1, \ldots, f_m$.

(ii) Use your answer in part (c)(i) together with part (a), to **prove that** the optimal control is unitary (i.e., norm preserving):

$$\|u^{\text{opt}}(t)\|_2 = \|u_0^{\text{opt}}\|_2.$$

(iii) Suppose we somehow (say, numerically) compute the optimal control $u^{\text{opt}}(t)$ for the OCP in part (b)(i). **How will this optimal control change** if we change the Lagrangian in the cost-to-go from $\|u\|_2^2$ to $\phi(\|u\|_2^2)$, where $\phi : \mathbb{R} \mapsto \mathbb{R}$ is an arbitrary monotone function? **Give one sentence explanation to support your answer.**

Solution for Part 1(c):

(i) From part 1(b)(i), notice that the components of the optimal control are $u_i^{\text{opt}} = -\langle \lambda^{\text{opt}}, f_i \rangle,\ i = 1, 2, \ldots, m.$

Taking its time derivative, we get

$$\begin{aligned}
\dot{u}_i^{\text{opt}} &= -\frac{\mathrm{d}}{\mathrm{d}t}\langle \lambda^{\text{opt}}, f_i \rangle \\
&= -\langle \dot{\lambda}^{\text{opt}}, f_i \rangle - \left\langle \lambda^{\text{opt}}, \frac{\mathrm{d}f_i}{\mathrm{d}t} \right\rangle \\
&= \underbrace{\left\langle \left(\sum_{j=1}^{m} \frac{\partial f_j}{\partial x^{\text{opt}}} u_j^{\text{opt}} \right)' \lambda^{\text{opt}}, f_i \right\rangle}_{\text{from costate ODE}} - \underbrace{\left\langle \lambda^{\text{opt}}, \frac{\partial f_i}{\partial x^{\text{opt}}} \dot{x}^{\text{opt}} \right\rangle}_{\text{by chain rule}} \\
&= \left(\lambda^{\text{opt}}\right)' \left(\sum_{j=1}^{m} \frac{\partial f_j}{\partial x^{\text{opt}}} u_j^{\text{opt}} \right) f_i - \left(\lambda^{\text{opt}}\right)' \frac{\partial f_i}{\partial x^{\text{opt}}} \underbrace{\sum_{j=1}^{m} f_j u_j^{\text{opt}}}_{F(x^{\text{opt}})u^{\text{opt}}} \\
&= \left(\lambda^{\text{opt}}\right)' \sum_{j=1}^{m} \left[f_i, f_j\right] u_j^{\text{opt}}.
\end{aligned}$$

Therefore,

$$\dot{u}^{\text{opt}} = \begin{pmatrix} 0 & \left(\lambda^{\text{opt}}\right)' [f_1, f_2] & \dots & \left(\lambda^{\text{opt}}\right)' [f_1, f_m] \\ \left(\lambda^{\text{opt}}\right)' [f_2, f_1] & 0 & \dots & \left(\lambda^{\text{opt}}\right)' [f_2, f_m] \\ \vdots & \vdots & \ddots & \vdots \\ \left(\lambda^{\text{opt}}\right)' [f_m, f_1] & \dots & \dots & 0 \end{pmatrix} u^{\text{opt}}.$$

Thus, the (i,j)th entry of the matrix M equals $\left(\lambda^{\text{opt}}\right)' \left[f_i, f_j\right]$ where $i, j = 1, 2, \dots, m$.

(ii) Since M is skew-symmetric, hence from part 1(a), we conclude that $u^{\text{opt}}(t) = Q(t) u_0^{\text{opt}}$ where matrix $Q(t)$ is orthogonal for all $t \geq 0$. Therefore,

$$\|u^{\text{opt}}(t)\|_2 = \|u_0^{\text{opt}}\|_2.$$

(iii) Since the optimal control for the OCP in part 1(b)(i) is unitary, the optimal control will not change if the Lagrangian is changed to $\phi(\|u\|_2^2)$ for monotone ϕ.

Problem 2 [45 points]

Optimal Economic Reform

Suppose the scalar state $x(t)$ of national economy is governed by the second order ODE

$$\ddot{x} = -\alpha^2 x + u, \quad \alpha \in \mathbb{R} \setminus \{0\}, \quad t \geq 0, \quad x(0) = \dot{x}(0) = 0,$$

where $u(t)$ is the effort a Government puts at time t for economic reform.

Suppose the Goverment would like to maximize its chance of getting re-elected at the **fixed** terminal time T, by bringing the national economy at a healthy state at the time of re-election, while not spending too much effort in economic reform during its tenure, i.e.,

$$\underset{u(\cdot)}{\text{maximize}} \quad x(T) - \int_0^T u^2 \, \mathrm{d}t.$$

In practice, the Government may want to maximize an increasing function of the above cost, but we will ignore such details.

(a) [5 + 5 + 8 = 18 points] Standard form

(i) **Define the state vector** and **write the second order ODE in state space form**, i.e., as a controlled vector first order ODE.

(ii) Use your answer in part (a)(i) to clearly **rewrite the OCP in standard form**. **Identify terminal cost/terminal constraint**, if any.

(iii) Write the **Hamiltonian, the costate ODEs, the PMP**, and **the transversality condition** for the OCP in part (a)(ii).

Solution for Part 2(a):

(i) The state vector is $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} := \begin{pmatrix} x \\ \dot{x} \end{pmatrix} \in \mathbb{R}^2$. The controlled ODE in state space form is

$$\dot{x} = \begin{pmatrix} 0 & 1 \\ -\alpha^2 & 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u, \qquad u \in \mathbb{R}.$$

(ii) The OCP in standard form is

$$\underset{u(\cdot)}{\text{minimize}} \quad -x_1(T) + \int_0^T u^2 \, \mathrm{d}t$$

subject to

$$\dot{x} = \begin{pmatrix} 0 & 1 \\ -\alpha^2 & 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u,$$

and final time T fixed. The terminal cost $\phi(x_1(T), T) \equiv -x_1(T)$. There is no terminal constraint, i.e., $\psi \equiv 0$.

(iii) The Hamiltonian $H = u^2 + \lambda_1 x_2 + \lambda_2 \left(-\alpha^2 x_1 + u\right)$. The costate ODEs are

$$\dot{\lambda}_1 = -\frac{\partial H}{\partial x_1} = \alpha^2 \lambda_2, \quad \dot{\lambda}_2 = -\frac{\partial H}{\partial x_2} = -\lambda_1.$$

The PMP gives $0 = \dfrac{\partial H}{\partial u} = 2u + \lambda_2$, which implies $u^{\text{opt}} = -\lambda_2^{\text{opt}}/2$.

Since $\mathrm{d}T = 0$ and $\mathrm{d}x(T) \neq 0$, the transversality condition gives $\dfrac{\partial \phi}{\partial x(T)} = \begin{pmatrix} -1 \\ 0 \end{pmatrix} = \begin{pmatrix} \lambda_1(T) \\ \lambda_2(T) \end{pmatrix}$.

(b) [15 + (10 + 2) = 27 points] Solution of the OCP

(i) Find the costates in terms of t, α, T.

Hint: use the transversality conditions to solve the costate ODE initial value problem.

(ii) Compute the **optimal economic reform** $u^{\text{opt}}(t)$ for the Government. **Also compute the optimal terminal reform** $u^{\text{opt}}(T)$.

Solution for Part 2(b):

(i) From the costate ODEs, we have $\ddot{\lambda}_1 = \alpha^2 \dot{\lambda}_2 = -\alpha^2 \lambda_1$, that is, $\ddot{\lambda}_1 + \alpha^2 \lambda_1 = 0$, which gives

$$\lambda_1(t) = a \cos(\alpha t) + b \sin(\alpha t),$$

where the constants a, b are to be determined from the terminal values of the costates. Consequently,

$$\lambda_2(t) = \frac{1}{\alpha^2}\dot{\lambda}_1 = \frac{1}{\alpha}(-a\sin(\alpha t) + b\cos(\alpha t)) .$$

To determine the constants a, b, we now use the terminal values of the costates (coming from transversality): $\lambda_1(T) = -1$, $\lambda_2(T) = 0$. This gives

$$\begin{pmatrix} \cos(\alpha T) & \sin(\alpha T) \\ -\frac{1}{\alpha}\sin(\alpha T) & \frac{1}{\alpha}\cos(\alpha T) \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \cos(\alpha T) & \sin(\alpha T) \\ -\frac{1}{\alpha}\sin(\alpha T) & \frac{1}{\alpha}\cos(\alpha T) \end{pmatrix}^{-1} \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} a \\ b \end{pmatrix} = \frac{1}{\frac{1}{\alpha}\left(\cos^2(\alpha T) + \sin^2(\alpha T)\right)} \begin{pmatrix} \frac{1}{\alpha}\cos(\alpha T) & -\sin(\alpha T) \\ \frac{1}{\alpha}\sin(\alpha T) & \cos(\alpha T) \end{pmatrix} \begin{pmatrix} -1 \\ 0 \end{pmatrix} = \begin{pmatrix} -\cos(\alpha T) \\ -\sin(\alpha T) \end{pmatrix}.$$

(ii) Combining the PMP and part 2(b)(i), we obtain

$$u^{\mathrm{opt}}(t) = -\lambda_2^{\mathrm{opt}}(t)/2 = -\frac{1}{2\alpha}(-a\sin(\alpha t) + b\cos(\alpha t)) = \frac{1}{2\alpha}(\sin(\alpha T)\cos(\alpha t) - \cos(\alpha T)\sin(\alpha t)) = \frac{1}{2\alpha}\sin(\alpha(T-t)).$$

Thus, the optimal terminal reform $u^{\mathrm{opt}}(T) = 0$.