Problem 1. [50 points] Discrete Time Finite Horizon DP

Before start solving this problem, please review Lec. 16, p. 4-8.

Given a controlled dynamical system

$$x_{k+1} = x_k + u_k, \quad u_k \in \{-1, 0\}, \quad k = 0, 1,$$

with the initial condition $x_0 \equiv 1$. Consider the problem

$$\underset{\gamma \in \Gamma}{\text{minimize}} \quad \alpha\left(x_1^2 + x_2^2\right) + (1-\alpha)\left(u_0^2 + u_1^2\right)$$

subject to the above dynamics, where Γ is the collection of all history dependent randomized policies, and α is a given constant satisfying $0 \leq \alpha \leq 1$.

(a) [(2+2+2) + (3+3+3) = 15 points] State space and costs

(i) Clearly write down the state space $\mathcal{X}_k$ for each $k = 0, 1, 2$.

(ii) Looking at the objective, clearly write down the costs c_k for each $k = 0, 1, 2$.

(b) [35 points] Optimal cost-to-go

Apply DP recursions to compute the optimal cost-to-go (a.k.a. value function) V_0. Your final answer should be in the form:

$$V_0 = \begin{cases} f(\alpha) & \text{if} \quad \alpha \leq \alpha_0, \\ g(\alpha) & \text{if} \quad \alpha > \alpha_0, \end{cases}$$

where you need to explicitly compute $f(\cdot), g(\cdot), \alpha_0$.

Problem 2. [50 points] Continuous Time DP

(a) [25 points]

Suppose we have a continuous time DP problem given by the HJB PDE initial value problem

$$\frac{\partial V}{\partial \tau} + H_{\text{opt}}\left(P\frac{\partial V}{\partial x} + q\right) = 0, \qquad V(\tau = 0, x) = \phi(x), \quad x \in \mathbb{R}^n,$$

where $H_{\text{opt}}(z)$ is a convex and 1-coercive function of $z \in \mathbb{R}^n$. The matrix $P \succ 0$, and the vector $q \in \mathbb{R}^n$ are given.

Derive a Hopf-Lax formula for the solution $V(\tau, x)$.

(b) [25 points]

Consider the continuous time OCP

$$\underset{\gamma \in \Gamma}{\text{minimize}} \quad \|x(T)\|_2 + \int_0^T \frac{1}{2}\|u\|_2^2 \, \mathrm{d}t$$

subject to $\dot{x} = u$, where $x \in \mathbb{R}^n$. Prove that the value function is of the form

$$V(t, x) = \begin{cases} a(t, x) & \text{if} \quad \|x\|_2 \leq T - t, \\ b(t, x) & \text{if} \quad \|x\|_2 > T - t. \end{cases}$$

Explicitly determine the functions $a(t, x)$ and $b(t, x)$.