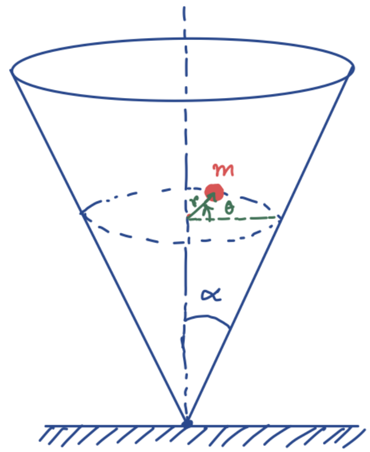
Problem 1. [50 points] Particle inside a Cone



Consider a particle of mass m (in red) sliding down (due to Earth's gravity) along the frinctionless inner surface of a right circular cone whose vertex is fixed to the ground, and its axis is vertical, see Figure. The half angle at the apex is equal to α, as shown.

Suppose that at a generic time t, the distance of the particle from the vertical axis is $r(t)$, and its angular position in the conic cross-section is $\theta(t)$, as shown in the Figure.

(a) [30 points] Equation of motion

Use Euler-Lagrange equation to **prove that** there exists a constant k such that $r(t)$ satisfies the ODE

$$\ddot{r} - \frac{k\sin^2\alpha}{m^2 r^3} + g\sin\alpha\cos\alpha = 0,$$

where g is the (constant) acceleration due to gravity.

Solution for 1(a):

In cylindrical coordinates, the position of the particle is $\left(r, \theta, \frac{r}{\tan\alpha}\right)$. By Lec. 5, p. 3, the kinetic energy is

$$T = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2 + \frac{\dot{r}^2}{\tan^2\alpha}\right) = \frac{1}{2}m\left(\frac{\dot{r}^2}{\sin^2\alpha} + r^2\dot{\theta}^2\right).$$

The potential energy is

$$V = mg\frac{r}{\tan\alpha}.$$

Therefore, the Lagrangian

$$\mathcal{L} = T - V = \frac{1}{2}m\left(\frac{\dot{r}^2}{\sin^2\alpha} + r^2\dot{\theta}^2\right) - \frac{mgr}{\tan\alpha}.$$

We have EL equations:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\mathcal{L}}{\partial\dot{r}}\right) - \frac{\partial\mathcal{L}}{\partial r} = 0 \qquad (*), \qquad \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\mathcal{L}}{\partial\dot{\theta}}\right) - \frac{\partial\mathcal{L}}{\partial\theta} = 0 \qquad (**).$$

Substituting our Lagrangian in $(*)$, we get

$$\frac{m\ddot{r}}{\sin^2\alpha} - mr\dot{\theta}^2 + \frac{mg}{\tan\alpha} = 0 \quad\Rightarrow\quad m\ddot{r} - mr\dot{\theta}^2\sin^2\alpha + mg\sin\alpha\cos\alpha = 0 \quad (***).$$

Substituting our Lagrangian in $(**)$, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(mr^2\dot{\theta}\right) = 0 \quad\Rightarrow\quad mr^2\dot{\theta} = c \text{ (some constant)} \quad\Rightarrow\quad \dot{\theta}^2 = \frac{k}{m^2r^4} \text{ where } k := c^2 \text{ (constant)} \quad (****).$$

In $(***)$, we substitute for $\dot{\theta}^2$ using $(****)$, to obtain

$$\ddot{r} - \frac{k\sin^2\alpha}{m^2r^3} + g\sin\alpha\cos\alpha = 0,$$

as desired.

(b) [(4 + 4 + 2) + 10 = 20 points] Conserved quantities

(b.1) **Compute** the total energy E, and the Hamiltonian H at a generic time t. Is $H = E$ in this case?

(b.2) Use your answer in part (b.1) to conclude if any of the two quantities among H and E is a constant of motion.

Solution for 1(b.1)

The total energy $E = T + V = \frac{1}{2}m\left(\frac{\dot{r}^2}{\sin^2\alpha} + r^2\dot{\theta}^2\right) + \frac{mgr}{\tan\alpha}$.

The Hamiltonian

$$H = \frac{\partial\mathcal{L}}{\partial\dot{r}}\dot{r} + \frac{\partial\mathcal{L}}{\partial\dot{\theta}}\dot{\theta} - \mathcal{L} = \frac{m\dot{r}^2}{\sin^2\alpha} + mr^2\dot{\theta}^2 - \frac{1}{2}\frac{m\dot{r}^2}{\sin^2\alpha} - \frac{1}{2}mr^2\dot{\theta}^2 + \frac{mgr}{\tan\alpha} = \frac{1}{2}\frac{m\dot{r}^2}{\sin^2\alpha} + \frac{1}{2}mr^2\dot{\theta}^2 + \frac{mgr}{\tan\alpha}.$$

Therefore, for this problem, $H = E$ at all times t.

Solution for 1(b.2)

From part (b.1), directly taking time derivative, we get

$$\frac{\mathrm{d}H}{\mathrm{d}t} = \frac{\mathrm{d}E}{\mathrm{d}t} = m\dot{r}\left(\frac{\ddot{r}}{\sin^2\alpha} + r\dot{\theta}^2 + \frac{g}{\tan\alpha}\right) + mr^2\dot{\theta}\ddot{\theta} = m\dot{r}\left(2r\dot{\theta}^2\right) - 2mr\dot{r}\dot{\theta}^2 = 0,$$

wherein the last but one equality used $(***)$ in part 1(a), and that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(mr^2\dot{\theta}\right) = 0 \quad \Rightarrow \quad 2mr\dot{r}\dot{\theta} + mr^2\ddot{\theta} = 0 \quad \Rightarrow \quad mr^2\dot{\theta}\ddot{\theta} = -2mr\dot{r}\dot{\theta}^2.$$

Therefore, in this problem, $H = E =$ constant at all times t.

Problem 2. [50 points] Brachistochrone as OCP

In this exercise, we will solve Brachistochrone as an OCP.

(a) [10 points] Reformulation

To reformulate the Brachistochrone problem of going from point $A \equiv (0,0)$ to point $B \equiv (x_1, y_1)$ as a **minimum time optimal control problem** with terminal time T free, and terminal state $(x(T), y(T))$ fixed, consider the control to be the angle θ that the velocity vector V (tangent to the curve) makes at a generic location (x, y), with respect to the horizontal direction. **Clealry write down the OCP in this case** following the general template.

Solution for 2(a):

The optimal control problem is to

$$\underset{\theta(\cdot)}{\text{minimize}} \quad \int_0^T 1 \, \mathrm{d}t$$

subject to $\dot{x} = V\cos\theta$, $\dot{y} = V\sin\theta$, where $V = \sqrt{2gy}$. The initial condition $(x(0), y(0)) \equiv (0,0)$, and the terminal condition $(x(T), y(T)) \equiv (x_1, y_1)$ are given. There is no terminal cost ($\phi \equiv 0$); the terminal constraint is

$$\psi(x(T), T) := \begin{pmatrix} x(T) - x_1 \\ y(T) - y_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

(b) [30 points] Two identities

Prove that the optimal position tuple $(x(t), y(t))$ at any time $t \in [0, T]$, is given by

$$x(t) = x_1 + \frac{y_1}{2\cos^2\theta(T)}\left\{2\left(\theta(T) - \theta(t)\right) + \sin(2\theta(T)) - \sin(2\theta(t))\right\}, \qquad y(t) = y_1 \frac{\cos^2\theta(t)}{\cos^2\theta(T)}.$$

The above two identities allow (numerically) solving for $(\theta(t), \theta(T))$ as function of $(x(t), y(t))$. This helps define the optimal state feedback: $\theta(t)$ as function of $(x(t), y(t))$.

Solution for 2(b):

The Hamailtonian $H = 1 + \sqrt{2gy}\,(\lambda_1\cos\theta + \lambda_2\sin\theta)$ gives the necessary conditions:

$$\dot{\lambda}_1 = -\frac{\partial H}{\partial x} = 0, \qquad \dot{\lambda}_2 = -\frac{\partial H}{\partial y} = -\frac{g}{V}(\lambda_1\cos\theta + \lambda_2\sin\theta),$$
$$0 = \frac{\partial H}{\partial \theta} = -\lambda_1 V\sin\theta + \lambda_2 V\cos\theta.$$

From the costate ODEs, $\lambda_1 =$ constant. Since $\mathrm{d}x(T) = 0$, and $dT \neq 0$, the transversality condition yields $H(T) = 0$. But since $H(t)$ has no explcit time dependence, it must be constant along the optimal trajectory, which combined with $H(T) = 0$ gives $H(t) = 0$ for all $t \in [0, T]$. In other words,

$$H = 1 + \lambda_1 V\cos\theta + \lambda_2 V\sin\theta = 0.$$

Eliminating λ_2 from (2) and (3), we obtain

$$\lambda_1 = -\frac{\cos\theta}{V}.$$

Likewise, eliminating λ_1 from (2) and (3), we obtain

$$\lambda_2 = -\frac{\sin\theta}{V}.$$

Furthermore, combining (4) with $\dot{\lambda}_1 = 0$, we get

$$0 = \dot{\lambda}_1 = \frac{\partial\lambda_1}{\partial\theta}\dot{\theta} + \frac{\partial\lambda_1}{\partial y}\dot{y} = \frac{\sin\theta}{V}\dot{\theta} + \frac{g\sin\theta\cos\theta}{V^2}$$
$$\Rightarrow \quad 0 = \dot{\theta} + \frac{g}{V}\cos\theta \quad \Rightarrow \quad \dot{\theta} = -\frac{g}{V}\cos\theta.$$

Now, evaluating (4) at $t = t$ and $t = T$, and using the fact that λ_1 is constant, we obtain

$$\frac{\cos\theta(t)}{\sqrt{y(t)}} = \frac{\cos\theta(T)}{\sqrt{y_1}} \quad \Rightarrow \quad y(t) = y_1\frac{\cos^2\theta(t)}{\cos^2\theta(T)}.$$

Next, we use $\dot{x} = V\cos\theta$, together with (7), to get

$$V\cos\theta = \dot{x} = \frac{\mathrm{d}x}{\mathrm{d}\theta}\dot{\theta} = \frac{\mathrm{d}x}{\mathrm{d}\theta}\times\left(-\frac{g}{V}\cos\theta\right) \quad \Rightarrow \quad \frac{\mathrm{d}x}{\mathrm{d}\theta} = -2y = -\frac{2y_1}{\cos^2\theta(T)}\cos^2\theta,$$

wherein the last equality follows from (8). Integrating (9), we have

$$\int_{x=x}^{x=x_1} \mathrm{d}x = -\frac{2y_1}{\cos^2\theta(T)}\int_{\theta=\theta}^{\theta=\theta(T)} \cos^2\theta\,\mathrm{d}\theta$$
$$\Rightarrow \quad x(t) = x_1 + \frac{y_1}{2\cos^2\theta(T)}\left\{2\left(\theta(T)-\theta(t)\right) + \sin(2\theta(T)) - \sin(2\theta(t))\right\}.$$

We conclude by noting that (8) and (10) are the desired identities.

(c) [2 + 3 + 5 = 10 points] Properties of optimal solution

(c.1) **Prove that** the optimal control $\theta(t)$ satisfies $\dot{\theta} =$ constant.

(c.2) **Prove that** $\theta(0) = \frac{\pi}{2}$. Give a physical interpretation of this result.

(c.3) Letting $\phi := \pi - 2\theta$, use your answer in part (b) to **deduce that** the optimal curve is a cycloid.

Solution for 2(c.1):

From equations (4) and (7) above, we get $\dot{\theta} = g\lambda_1 =$ constant, assuming g is constant as in the original Brachistochrone formulation.

Solution for 2(c.2):

Evaluating equation (8) above at $t = 0$ gives $\theta(0) = \frac{\pi}{2}$. Vertical initial heading is natural since the terminal point B is below the initial point A. In other words, to minimize the time to go, it is optimal to increase the vertical coordinate as fast as possible.

Solution for 2(c.3):

In equations (8) and (10) above, we substitute θ as function of ϕ, and define a constant

$$a := \frac{y_1}{2\cos^2\theta(T)} = \frac{y_1}{1-\cos\phi(T)}.$$

Then (8) gives

$$y = a\left[1+\cos(\pi-\phi)\right] = a\left(1-\cos\phi\right).$$

Similarly, (10) gives

$$x - x_1 + a\left[\phi(T) - \sin\phi(T)\right] = a(\phi - \sin\phi).$$

The above two equations give the parametric form of a cycloid passing through (x_1, y_1). Here, $\theta(T)$ (and thus $\phi(T)$) is such that the cycloid passes though point A with coordinate $(0, 0)$.