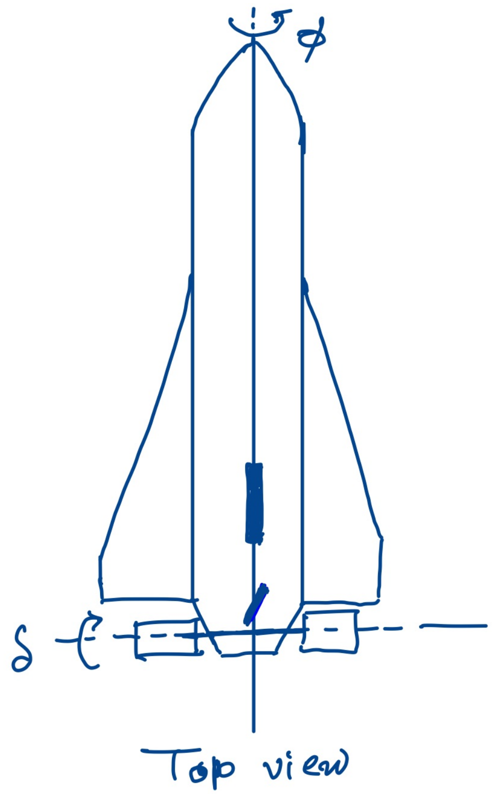
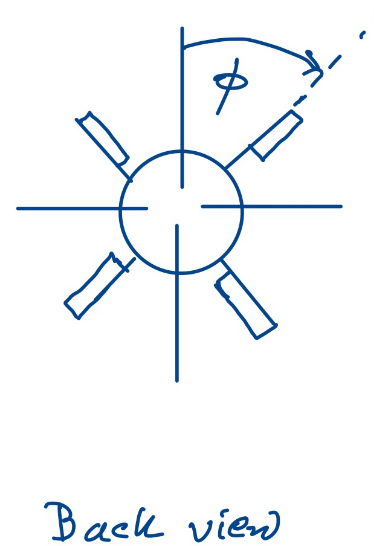
Problem 1. [100 points] Regulating Roll

To regulate the rolling motion of a missile, a common strategy is to actuate its hydraulic-powered ailerons. The control u is the aileron deflection rate (rad/s). The state vector $(\delta, \omega, \phi)^\top$ comprises of the aileron deflection angle δ (rad), the roll angular velocity ω (rad/s), and the roll angle ϕ (rad), respectively. See Fig. below.





The control design objective is to

$$\underset{u(\cdot)}{\text{minimize}} \quad \int_0^\infty \frac{1}{2}\left(\frac{\delta^2}{\delta_0^2} + \frac{\phi^2}{\phi_0^2} + \frac{u^2}{u_0^2}\right) \mathrm{d}t$$

subject to the equations of rolling motion

$$\dot{\delta} = u, \quad \dot{\omega} = -\frac{1}{\tau}\omega + \frac{q}{\tau}\delta, \quad \dot{\phi} = \omega,$$

where τ is the rolling time constant (s), and q is the aileron effectiveness constant (s^{-1}). The weights (δ_0, ϕ_0, u_0) are known constants.

(a) [20 + 20 = 40 points] Analysis

Let $\sigma := u_0 P_{11}$, where P_{11} is the $(1,1)$ entry of the symmetric matrix P_∞ that solves the associated CARE. **Prove that** the **optimal feedback control**

$$u^{\text{opt}} = -u_0^2 \begin{bmatrix} \dfrac{\sigma}{u_0} & \dfrac{\tau}{2q}\left(\sigma^2 - \dfrac{1}{\delta_0^2}\right) & \dfrac{1}{u_0\phi_0} \end{bmatrix} \begin{bmatrix} \delta \\ \omega \\ \phi \end{bmatrix},$$

and **that σ solves a quartic equation**

$$\sigma^4 + \frac{4}{u_0\tau}\sigma^3 + \frac{4}{u_0^2\tau^2}\left(1 - \frac{u_0^2\tau^2}{2\delta_0^2}\right)\sigma^2 - \frac{4}{u_0\tau}\left(\frac{2q}{\phi_0 u_0} + \frac{1}{\delta_0^2}\right)\sigma + \left(\frac{1}{\delta_0^4} - \frac{4}{\delta_0^2 u_0^2\tau^2} - \frac{8q}{\phi_0 u_0^3\tau^2}\right) = 0.$$

(b) [(10 + 10) + (10 + 10 + 10) + 10 = 60 points] Numerics

Consider the parameter values $\tau = 1$ s, $q = 10$ s^{-1}, $u_0 = \pi$ rad/s, $\delta_0 = \pi/12$ rad, $\phi_0 = \pi/180$ rad.

(i) **Write a MATLAB code to verify that the system with the above parameters is controllable (hence stabilizable) and observable (hence detectable)**. For this purpose, you may use MATLAB commands such as `rank(ctrb(A,B))` and `rank(obsv(A,C))` for appropriately defined matrices `A,B,C`. **What can you conclude about the nature of the matrix P_∞ from this controllability and observability verifications? Give reasons to support your answer.** (Hint: see Lec. 11)

(ii) Use the same MATLAB code to **compute the four roots** of the quartic equation derived in part (a). You may use the MATLAB command `roots()`. **Which of these four roots should be used to compute P_{11} and why? Use your reasoning to report the 3×3 matrix P_∞.**

(iii) To verify the P_∞ obtained in part (b)(ii), **compute the same using MATLAB command `icare` in the same code**. Submit your MATLAB code.