**AMS 229: Convex Optimization**

Fall 2018

# Midterm Exam

Name:____________________________________ Student ID:________________________

For this exam you only need a pen/pencil. No electronic device is allowed. Write your answers in the spaces provided. If you need more space, work on the other side of the page.

**Problem 1** (10 + 5 + 5 = 20 points)

A real $n \times n$ matrix $X$ is called doubly stochastic if its elements are non-negative, and all row sums and column sums equal unity, that is,

$$X_{ij} \geq 0, \qquad \sum_{i=1}^{n} X_{ij} = \sum_{j=1}^{n} X_{ij} = 1, \quad \text{for all } i, j = 1, \ldots, n.$$

In this problem, we investigate the set $\mathcal{X} = \{X \in \mathbb{R}^{n \times n} \mid X \text{ is doubly stochastic}\}$.

(a) Prove that $\mathcal{X}$ is a convex set.

(b) Is $\mathcal{X}$ a cone? Why/why not?

(c) Is $\mathcal{X}$ a polytope (also known as polyhedron)? Why/why not?

**Problem 2** (5 + 5 + 5 + 5 = 20 points)

Consider the following optimization problem, which we call problem (P). Here, $\boldsymbol{A} \in \mathbb{R}^{m \times n}$, $\boldsymbol{b} \in \mathbb{R}^{m}$, $\boldsymbol{c} \in \mathbb{R}^{n}$ are given.

$$\begin{aligned} \underset{\boldsymbol{x} \in \mathbb{R}^n}{\text{minimize}} \quad & \boldsymbol{c}^\top \boldsymbol{x} \\ \text{subject to} \quad & \boldsymbol{A}\boldsymbol{x} \preceq \boldsymbol{b}, \\ & x_i \in \{0, 1\}, \quad i = 1, \ldots, n. \end{aligned}$$

(a) Is problem (P) a convex optimization problem? Why/Why not?

(b) Consider the following modification of problem (P), which we call problem (Q).

$$\begin{aligned} \underset{\boldsymbol{x} \in \mathbb{R}^n}{\text{minimize}} \quad & \boldsymbol{c}^\top \boldsymbol{x} \\ \text{subject to} \quad & \boldsymbol{A}\boldsymbol{x} \preceq \boldsymbol{b}, \\ & 0 \leq x_i \leq 1, \quad i = 1, \ldots, n. \end{aligned}$$

Is problem (Q) convex? If "yes", then what kind of convex optimization problem is it? If “no”, then explain why.

(c) Suppose the optimal value for problem (P) is $p_{\text{opt}}$, and the same for problem (Q) is $q_{\text{opt}}$. Write an inequality between $p_{\text{opt}}$ and $q_{\text{opt}}$. Justify your answer.

(d) Suppose we do not know how to solve problem (P). Instead, we solve problem (Q), and suppose we find that its minimizer $\boldsymbol{x}^q_{\text{opt}} \in \{0, 1\}$. Can we then conclude that $p_{\text{opt}} = q_{\text{opt}} = \boldsymbol{c}^\top \boldsymbol{x}^q_{\text{opt}}$, that is, problem (Q) solves problem (P)? Explain.

**Problem 3** (5 × 2 = 10 points)

For each the following statements, ONLY ONE among the three options are correct. Choose the correct option for each. You DO NOT need to provide any explanation.

(a) For a non-convex function $f$, its tri-conjugate $f^{***}$ satisfies
(i) $f^{***} = f$, the original non-convex function
(ii) $f^{***} = f^{*}$, which is a convex function
(iii) $f^{***} = f^{**}$, which is a convex function

(b) At each point on the boundary of a compact convex set, a supporting hyperplane
(i) may not exist
(ii) exists
(iii) exists and is unique

(c) The negative entropy function $f(\boldsymbol{x})$ given by

$$f(\boldsymbol{x}) = \sum_{i=1}^{n} x_i \log x_i, \qquad \mathrm{dom}(f) = \{\boldsymbol{x} \in \mathbb{R}^n_{>0} \mid \mathbf{1}^\top \boldsymbol{x} = 1\},$$

(i) is quasiconcave
(ii) is quasiconvex but not convex
(iii) both quasiconvex and convex

(d) A function is concave if and only if its hypograph is
(i) convex
(ii) open
(iii) closed

(e) The dual cone of $\mathbb{S}^n_+$ equals
(i) $\mathbb{S}^n_+$
(ii) $\mathbb{S}^n_-$
(iii) $\mathbb{S}^n$

# Some useful information

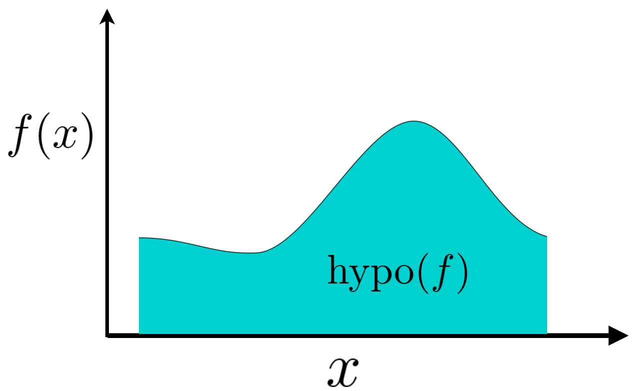
- Convex conjugate or Legendre-Fenchel transform of function $f(\boldsymbol{x})$ is

$$f^*(\boldsymbol{y}) = \sup_{\boldsymbol{x} \in \mathrm{dom}(f)} \left( \boldsymbol{y}^\top \boldsymbol{x} - f(\boldsymbol{x}) \right).$$

  $f^*$ is convex even if $f$ is not.

- Suppose $f$ is a twice differentiable function. Then $f$ is convex (resp. concave) if and only if $\mathrm{dom}(f)$ is convex, and Hessian of $f$ is positive (resp. negative) semidefinite everywhere in $\mathrm{dom}(f)$.

- Hypograph of a function $f$ is the set of points lying on or below its graph.




- Given a cone $\mathcal{K}$, its dual cone $\mathcal{K}^*$ is given by

$$\mathcal{K}^* = \{ y \mid \langle y, x \rangle \geq 0, \text{ for all } x \in \mathcal{K} \}.$$

  $\mathcal{K}^*$ is convex even if $\mathcal{K}$ is not.