Kinematic Bicycle Model and Controllers

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1 Plant Model

The kinematic bicycle model has four states (x, y, v, ψ) and two control inputs (a_c, δ), and is given by

$$\dot{x} = v\cos(\psi + \beta) \tag{1a}$$
$$\dot{y} = v\sin(\psi + \beta) \tag{1b}$$
$$\dot{v} = a_c \tag{1c}$$
$$\dot{\psi} = \frac{v}{\ell_{\text{rear}}}\sin\beta \tag{1d}$$

where the sideslip angle

$$\beta = \arctan\left(\frac{\ell_{\text{rear}}}{\ell_{\text{front}} + \ell_{\text{rear}}}\tan\delta\right).$$

The 4×1 state vector comprises of the inertial position (x, y) for the vehicle's center-of-mass, its speed v, and the vehicle's inertial heading angle ψ. The 2×1 control vector comprises of the acceleration a_c, and the front steering wheel angle δ.

The parameters $\ell_{\text{front}}, \ell_{\text{rear}}$ are the distances of the vehicle's center-of-mass to the front and rear axles, respectively.

2 Control Objective and Constraints

We consider the problem of tracking a desired path given as a sequence of N waypoint tuples $\left\{\left(x_d^{(i)}, y_d^{(i)}, v_d^{(i)}\right)\right\}_{i=1}^{N}$, i.e., a sequence of desired positions and speeds.

A constraint on the acceleration input is $|a_c| \leq 1$.

3 Controllers

3.1 PID and Stanley Controller

We consider a full state feedback controller that first computes the nearest waypoint to the measured position (x, y) as

$$i^{\text{opt}} = \underset{i\in[N]}{\arg\min}\left(x_d^{(i)} - x\right)^2 + \left(y_d^{(i)} - y\right)^2 \tag{2}$$

and let

$$\left(x_d^{\text{opt}}, y_d^{\text{opt}}, v_d^{\text{opt}}\right) := \left(x_d^{(i^{\text{opt}})}, y_d^{(i^{\text{opt}})}, v_d^{(i^{\text{opt}})}\right). \tag{3}$$

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We next implement a PID controller composed with a saturation function for the acceleration command:

$$a_c(t) = \mathrm{sat}\left(k_p\left(v_d^{\mathrm{opt}}(t) - v(t)\right) + k_i\int_0^t \left(v_d^{\mathrm{opt}}(s) - v(s)\right)\,\mathrm{d}s + k_d\frac{\mathrm{d}}{\mathrm{d}t}\left(v_d^{\mathrm{opt}}(t) - v(t)\right)\right) \tag{4}$$

with suitable gains (k_p, k_i, k_d), where the stauration function

$$\mathrm{sat}(q) := \begin{cases} -1 & \text{for} \quad q < -1, \\ q & \text{for} \quad -1 < q < 1, \\ +1 & \text{for} \quad q > 1. \end{cases}$$

For the front steering wheel angle command, we implement a Stanley controller:

$$\mathrm{dist}(t) := \left\| \begin{pmatrix} x_d^{\mathrm{opt}}(t) - x(t) \\ y_d^{\mathrm{opt}}(t) - y(t) \end{pmatrix} \right\|_2, \qquad \delta(t) = \psi(t) + \arctan\left(\frac{k\,\mathrm{dist}(t)}{v(t)}\right), \tag{5}$$

where k is a suitable control gain.

3.2 MPC

The MPC or model predictive controller works by solving the following optimization problem to determine the control output at a given timestep:

$$\min_u \quad \sum_{i=0}^{H_p}(z_i - z_{\mathrm{ref},i})^\top Q(z_i - z_{\mathrm{ref},i}) + \sum_{i=0}^{H_p-1}\left[(u_i - u_{i-1})^\top \bar{R}(u_i - u_{i-1}) + u_i^\top R u_i\right] \tag{6a}$$

$$\text{s.t.} \quad z_0 = z(t), u_{-1} = u(t - t_s) \tag{6b}$$

$$z_{i+1} = f(z_i, u_i) \tag{6c}$$

$$u_{\min,i} \le u_i \le u_{\max,i} \tag{6d}$$

$$\dot{u}_{\min,i} \le \frac{u_i - u_{i-1}}{t_s} \le \dot{u}_{\max,i} \tag{6e}$$

where $z_i \in \mathbb{R}^3$ contains the crosstrack, ψ, and v error terms, $u_i \in \mathbb{R}^2$ contains the control inputs a, δ, and $Q \in \mathbb{R}^{3\times3}, \bar{R}, R \in \mathbb{R}^{2\times2}$ are diagonal weighting matrices. Thus, the problem is to minimize the sum of the crosstrack, ψ, and v errors, along with the magnitude and rapidity in change of the control inputs, over a period of time from 0 to the time horizon H_p. Once solved, the minimizer u is applied as the control input, and the problem is solved again at the next timestep.
Constraint (6b) dictates that z_0 is the error vector at time t, and similarly for u_{-1}. Constraint (6c) states that the system state evolves according to the function f, which in our case is the KBM system (1). The last two constraints impose limits on the control outputs.

4 Figures

The following figures overlay the KBM states and control inputs for both the PID + Stanley and MPC controllers vs. time. Figure 3 shows the desired path overlayed with the trajectory followed by both controllers. Despite variety in control inputs and speed maintained, both controllers appear to track the path very well.

Figure 1: Vehicle states vs. time

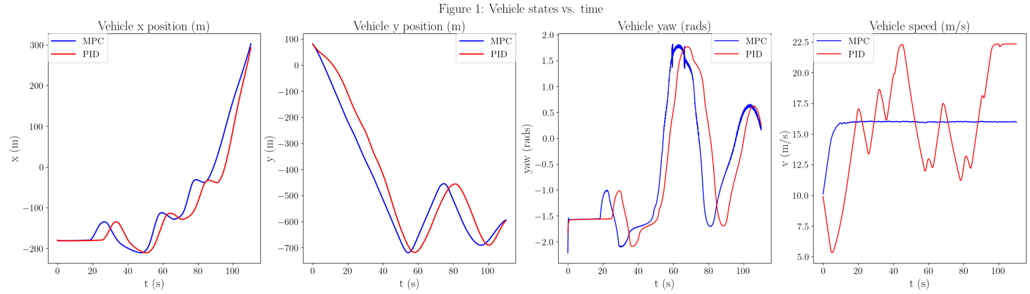

Figure 2: Control outputs vs. time

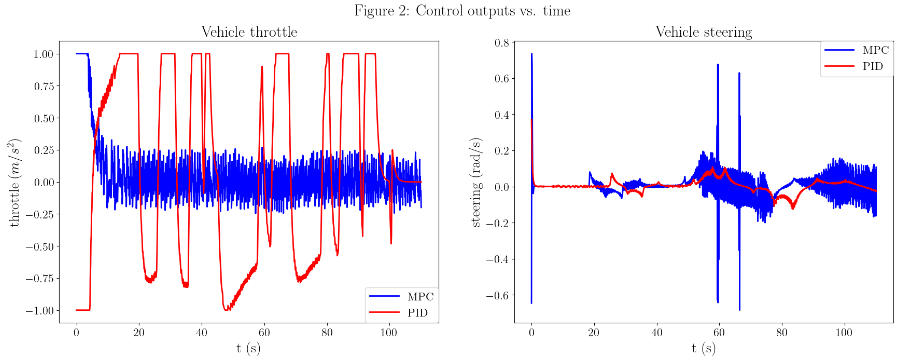

Figure 3: Desired vs. actual trajectory