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Lamperti Transform For Power Systems

Kenneth Caluya

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We consider the system of 2nd-order SDEs given by

$$m_i \ddot{\theta} p + \gamma_i \dot{\theta}_i = P_i - \sum_{j=1}^{n} k_{ij} \sin(\theta_i - \theta_j) + \sigma_i \times \text{ stochastic forcing } i = 1, \ldots, n.$$

We can transform the system into

$$\begin{aligned} \mathrm{d}\boldsymbol{\theta} &= \boldsymbol{\omega}\, \mathrm{d}t \\ \mathrm{d}\boldsymbol{\omega} &= -\mathrm{diag}(\boldsymbol{\gamma} \oslash \boldsymbol{m})\, \boldsymbol{\omega} - \nabla_{\boldsymbol{\theta}} V(\boldsymbol{\theta}) + \mathrm{diag}(\boldsymbol{\sigma} \oslash \boldsymbol{m})\, \mathrm{d}\boldsymbol{w} \end{aligned} \tag{1}$$

where $\boldsymbol{\theta} = (\theta_1, \theta_2, \ldots, \theta_n)$ and $\boldsymbol{\omega} = (\dot{\theta}_1, \dot{\theta}_2, \ldots, \dot{\theta}_n)$ and the potential function is given by

$$V(\theta) := \sum_{i=1}^{n} \frac{1}{m_i} P_i \theta_i + \sum_{(i,j) \in \mathcal{E}} \frac{1}{m_i} k_{ij} (1 - \cos(\theta_i - \theta_j)) \tag{2}$$

Theorem 1 (Lamperti Transform)**.** *Let X_t be the solution to the (Ito) SDE*

$$\mathrm{d}\boldsymbol{X}_t = \boldsymbol{f}(\boldsymbol{X}_t, t)\, \mathrm{d}t + \boldsymbol{\beta}(\boldsymbol{X}_t, t)\, \boldsymbol{R}(t) \mathrm{d}\boldsymbol{W}_t \tag{3}$$

where $\boldsymbol{X}_t, w_t \in \mathbb{R}^d$, $\boldsymbol{R}(t) \in \mathbb{R}^{d \times d}$ is any matrix function of t and $\boldsymbol{\beta} \in \mathbb{R}^{d \times d}$ is a diagonal matrix whose diagonal elements which we denote

$$\beta_{i,i}(\boldsymbol{X}_t.t) = \beta_i(X_{i,t}, t),$$

i.e., each diagonal element depend only on the ith component of $\boldsymbol{X}_t$. Then, the transformation defined by

$$Z_{i,t} = \psi_i(X_{i,t}, t) = \int \frac{1}{\beta_i(x,t)}\, \mathrm{d}x \bigg|_{x = X_{i,t}} \tag{4}$$

will result in the diffusion process

$$\mathrm{d}Z_{i,t} = \left(\frac{\partial}{\partial t}\psi_i(x,t)\bigg|_{x=X_{i,y}} + \frac{f_i(\boldsymbol{\psi}^{-1}(\boldsymbol{Z},t).t)}{\beta_i(\psi_i^{-1}(Z_{i,t},t),t)} + \frac{1}{2}\frac{\partial}{\partial x}\beta_i(\psi_i^{-1}(Z_{i,t},t),t) \right) \mathrm{d}t + \sum_{j=1}^{d} r_{i,j}(t)\, \mathrm{d}w_{j,t} \tag{5}$$

where $r_{i,j}(t)$ are the elements of $\boldsymbol{R}(t)$ and $\boldsymbol{X}_t = \psi^{-1}(\boldsymbol{Z},t)$.

Proposition 2. *Let $d = 2n$ and set*

$$\boldsymbol{X}_t = \begin{bmatrix} \boldsymbol{\theta} \\ \boldsymbol{\omega} \end{bmatrix}, \quad \boldsymbol{Z}_t := \begin{bmatrix} \boldsymbol{\xi} \\ \boldsymbol{\eta} \end{bmatrix}. \tag{6}$$

Let

$$\boldsymbol{\beta}(\boldsymbol{X},t) \equiv \begin{bmatrix} \boldsymbol{I} & 0 \\ 0 & \mathrm{diag}(\boldsymbol{\sigma} \oslash \boldsymbol{m}) \end{bmatrix}, \quad \boldsymbol{R}(t) \equiv \begin{bmatrix} 0 & 0 \\ 0 & \boldsymbol{I} \end{bmatrix} \tag{7}$$

then applying Theorem 1 to (1) results in the diffusion process

$$\mathrm{d}\boldsymbol{z}_t \tag{8}$$

Proof. Notice that performing the change of variable (4) results in

$$\boldsymbol{Z}_t = \boldsymbol{\psi}(\boldsymbol{X},t) = \begin{bmatrix} \boldsymbol{I} & 0 \\ 0 & \mathrm{diag}(\boldsymbol{m} \oslash \boldsymbol{\sigma}) \end{bmatrix} \boldsymbol{X}_t \tag{9}$$

so that

$$\boldsymbol{X}_t = \boldsymbol{\psi}^{-1}(\boldsymbol{Z},t) = \begin{bmatrix} \boldsymbol{I} & 0 \\ 0 & \mathrm{diag}(\boldsymbol{\sigma} \oslash \boldsymbol{m}) \end{bmatrix} \boldsymbol{Z}_t. \tag{10}$$

Using (6), the inverse transformation can be written component wise as

$$\theta_i = \xi_i, \quad \omega_i = \frac{\sigma_i}{m_i}\eta_i \tag{11}$$

Notice that since $\boldsymbol{\psi}$ and $\boldsymbol{\beta}$ are independent of t and x, then the first and third term in the drift of (5) vanish and we are left with the second term. Notice that we have

$$\frac{f_i(\boldsymbol{X}_t,t)}{\beta_i(\boldsymbol{X}_{i,t},t)} = \begin{cases} \omega_i & 1 \le i \le n \\ -\frac{\gamma_i}{m_i}\frac{m_i}{\sigma_i}\omega_i - \frac{m_i}{\sigma_i}\frac{P_i}{m_i} - \displaystyle\sum_{j=1}^{n} \frac{m_i}{\sigma_i}\frac{1}{m_i} k_{ij}\sin(\theta_i - \theta_j) & n+1 \le i \le 2n \end{cases} \tag{12}$$

which means that

$$\frac{f_i(\boldsymbol{\psi}(\boldsymbol{Z},t),t)}{\beta_i(\psi(Zi,t,t),t)} = \begin{cases} \frac{\sigma_i}{m_i}\eta_i & 1 \le i \le n \\ -\frac{\gamma_i}{m_i}\eta_i - \frac{P_i}{\sigma_i} - \sum_{j=1}^{n} \frac{1}{\sigma_i} k_{i,j}\sin(\xi_i - \xi_j) & n+1 \le i \le 2n \end{cases}. \tag{13}$$

This results in the diffusion process

$$\mathrm{d}\begin{bmatrix}\boldsymbol{\xi}\\ \boldsymbol{\eta}\end{bmatrix} = \begin{bmatrix}\mathrm{diag}(\boldsymbol{\sigma}\oslash\boldsymbol{m})\boldsymbol{\eta}\\ -\mathrm{diag}(\boldsymbol{\gamma}\oslash\boldsymbol{m})\boldsymbol{\eta} - \nabla_{\boldsymbol{\xi}}U(\boldsymbol{\xi})\end{bmatrix}\mathrm{d}t + \begin{bmatrix}0 & 0\\ 0 & \boldsymbol{I}\end{bmatrix}\mathrm{d}\boldsymbol{W}_t \tag{14}$$

$$:= \begin{bmatrix}\boldsymbol{G}_1\boldsymbol{\eta}\\ -\nabla_{\boldsymbol{\eta}}F(\boldsymbol{\eta}) - \nabla_{\boldsymbol{\xi}}U(\boldsymbol{\xi})\end{bmatrix}\mathrm{d}t + \begin{bmatrix}0 & 0\\ 0 & \boldsymbol{I}\end{bmatrix}\mathrm{d}\boldsymbol{W}_t \tag{15}$$

where $F(\boldsymbol{\eta}) = \frac{1}{2}\boldsymbol{\eta}^\top\boldsymbol{G}_2\boldsymbol{\eta}$ where the potential function U is given by

$$U(\boldsymbol{\xi}) := \sum_{i=1}^{n}\frac{P_i}{\sigma_i}\xi_i - \sum_{i,j\in\mathcal{E}}\frac{1}{\sigma_i}k_{i,j}(1-\cos(\xi_i-\xi_j)) \tag{16}$$

□

The Fokker-Planck equation of this SDE is

$$\frac{\partial\rho}{\partial t} = -\langle\boldsymbol{G}_1\boldsymbol{\eta},\nabla_{\boldsymbol{\xi}}\rho\rangle + \nabla_{\boldsymbol{\eta}}\cdot\left(\left(\nabla_{\boldsymbol{\eta}}F(\boldsymbol{\eta}) + \nabla_{\boldsymbol{\xi}}U(\boldsymbol{\xi})\right)\rho\right) + \Delta_{\boldsymbol{\eta}}\rho \tag{17}$$

0.1 Stationary Solution

To compute the stationary solution, we set

$$0 = \langle\boldsymbol{G}_1\boldsymbol{\eta},\nabla_{\boldsymbol{\xi}}\rho\rangle + \nabla_{\boldsymbol{\eta}}\cdot\left(\left(\nabla_{\boldsymbol{\eta}}F(\boldsymbol{\eta}) + \nabla_{\boldsymbol{\xi}}U(\boldsymbol{\xi})\right)\rho\right) + \Delta_{\boldsymbol{\eta}}\rho \tag{18}$$

and we have the ansatz

$$\rho_\infty \propto \exp\left(-(F(\boldsymbol{\eta}) + U(\boldsymbol{\xi})\right). \tag{19}$$

From this, we get the derivatives

$$\nabla_{\boldsymbol{\xi}}\rho_\infty = -\rho_\infty\nabla_{\boldsymbol{\xi}}U(\boldsymbol{\xi}), \quad \nabla_{\boldsymbol{\eta}}\rho_\infty = -\rho_\infty\nabla_{\boldsymbol{\eta}}F(\boldsymbol{\eta}). \tag{20}$$

Using (20), we obtain

$$\Delta_\eta\rho_\infty = \rho_\infty\|G_2\eta\|^2 - \rho_\infty\mathrm{tr}(G_2) \tag{21}$$

and

$$\nabla_{\boldsymbol{\eta}}\cdot\left(\left(\nabla_{\boldsymbol{\eta}}F(\boldsymbol{\eta}) + \nabla_{\boldsymbol{\xi}}U(\boldsymbol{\xi})\right)\right) = -\rho_\infty\|G_2\eta\|^2 - \rho_\infty\langle G_2\boldsymbol{\eta},\nabla_{\boldsymbol{\xi}}U(\boldsymbol{\xi}) + \rho_\infty\mathrm{tr}(G_2) \tag{22}$$

t which implies that (18) reduces to

$$0 = \rho_\infty\langle(G_2 - G_1)\boldsymbol{\eta},\nabla_{\boldsymbol{\xi}}U(\boldsymbol{\xi})\rangle. \tag{23}$$

In order for this to hold for all $\boldsymbol{\eta},\boldsymbol{\xi}$ we must have $G_2 = G_1$ which means that $\boldsymbol{\sigma} = \boldsymbol{\gamma}$, i.e., $\sigma_i = \gamma_i$ for all i.

0.2 Free Energy

Consider the functional

$$
\begin{aligned}
\Phi(\rho) &= \int \rho(\boldsymbol{\eta},\boldsymbol{\xi})\log\rho(\boldsymbol{\eta},\boldsymbol{\xi})\,\mathrm{d}\boldsymbol{\eta}\mathrm{d}\boldsymbol{\xi} + \beta\int \rho(\boldsymbol{\eta},\boldsymbol{\xi})U(\boldsymbol{\xi})\,\mathrm{d}\boldsymbol{\eta}\mathrm{d}\boldsymbol{\xi} + \beta\int \rho(\boldsymbol{\eta},\boldsymbol{\xi})F(\boldsymbol{\eta})\,\mathrm{d}\boldsymbol{\eta}\mathrm{d}\boldsymbol{\xi} \\
&= \beta\widetilde{\Phi}(\rho) \qquad (24)
\end{aligned}
$$

We compute

$$
\begin{aligned}
\frac{\mathrm{d}\Phi}{\mathrm{d}t} &= \int \frac{\partial\rho}{\partial t}\left(\log\rho + \beta F + \beta U\right)\,\mathrm{d}\boldsymbol{\eta}\mathrm{d}\boldsymbol{\xi}\,\mathrm{d}\boldsymbol{\eta}\mathrm{d}\boldsymbol{\xi} \\
&= \int \nabla_{\boldsymbol{\eta}}\cdot\left(\rho\left(\nabla_{\boldsymbol{\eta}}\log\rho + \nabla_{\boldsymbol{\eta}}F + \nabla_{\boldsymbol{\xi}}U\right)\right)\left(\log\rho + \beta F + \beta U\right)\,\mathrm{d}\boldsymbol{\eta}\mathrm{d}\boldsymbol{\xi} \\
&\quad - \int \langle G_1\boldsymbol{\eta}, \nabla_{\boldsymbol{\xi}}\rho\rangle\left(\log\rho + \beta F + \beta U\right)\,\mathrm{d}\boldsymbol{\eta}\mathrm{d}\boldsymbol{\xi} \\
&= \int -|\nabla_{\boldsymbol{\eta}}\log\rho|^2\rho - (\beta+1)\langle\nabla_{\boldsymbol{\eta}}\log\rho, \nabla_{\eta}F\rangle\rho - \beta|\nabla_{\boldsymbol{\eta}}F|^2\rho\,\mathrm{d}\boldsymbol{\eta}\mathrm{d}\boldsymbol{\xi} \\
&\int -\langle\nabla_{\boldsymbol{\xi}}U, \nabla_{\boldsymbol{\eta}}\log\rho\rangle\rho - \beta\langle\nabla_{\boldsymbol{\xi}}U, \nabla_{\boldsymbol{\eta}}F\rangle\rho + \langle G_1\boldsymbol{\eta}, \nabla_{\boldsymbol{\xi}}\log\rho\rangle\rho + \beta\langle G_1\boldsymbol{\eta}, \nabla_{\boldsymbol{\xi}}U\rangle\rho\,\mathrm{d}\boldsymbol{\eta}\mathrm{d}\boldsymbol{\xi} \\
&= \int -|\nabla_{\boldsymbol{\eta}}\log\rho|^2\rho - (\beta+1)\langle\nabla_{\boldsymbol{\eta}}\log\rho, \nabla_{\eta}F\rangle\rho - \beta|\nabla_{\boldsymbol{\eta}}F|^2\rho\,\mathrm{d}\boldsymbol{\eta}\mathrm{d}\boldsymbol{\xi} \\
&\quad + t\int -\beta\langle\nabla_{\boldsymbol{\xi}}U, \nabla_{\boldsymbol{\eta}}F\rangle\rho + \beta\langle G_1\boldsymbol{\eta}, \nabla_{\boldsymbol{\xi}}U\rangle\rho.
\end{aligned}
$$

A sufficient condition for $\frac{\mathrm{d}\Phi}{\mathrm{d}t} \leq 0$ is to set $G_!\boldsymbol{\eta} = \nabla_{\boldsymbol{\eta}}F = G_2\boldsymbol{\eta} \Rightarrow \boldsymbol{\sigma} = \boldsymbol{\gamma} \Rightarrow$ $\sigma_i = \gamma_i$ for all i.