Linear Algebra.

Linear algebra is the branch of mathematics concerning linear equations such as:

$$a_1 x_1 + \dots + a_n x_n = b,$$

Vector Vs Scalar

A quantity that has magnitude but no particular direction is described as **scalar**.

It is a single Number

- In contrast to other objects in linear algebra, which are usually arrays of numbers.
- Represented in lower case italic x
  E.g. Let $x \in \mathbb{R}$ be the slope of the line
  - Defining a real-valued scalar
- E.g. let $n \in \mathbb{N}$ be the number

of units
- Defining a natural number scaler.

A quantity that has magnitude & acts in a particular direction is described as vector.

- vector is an array of numbers in order
- Each no. identified by an index
- Written in lower-case bold such as x

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

- If each element is in R then x is in R^n

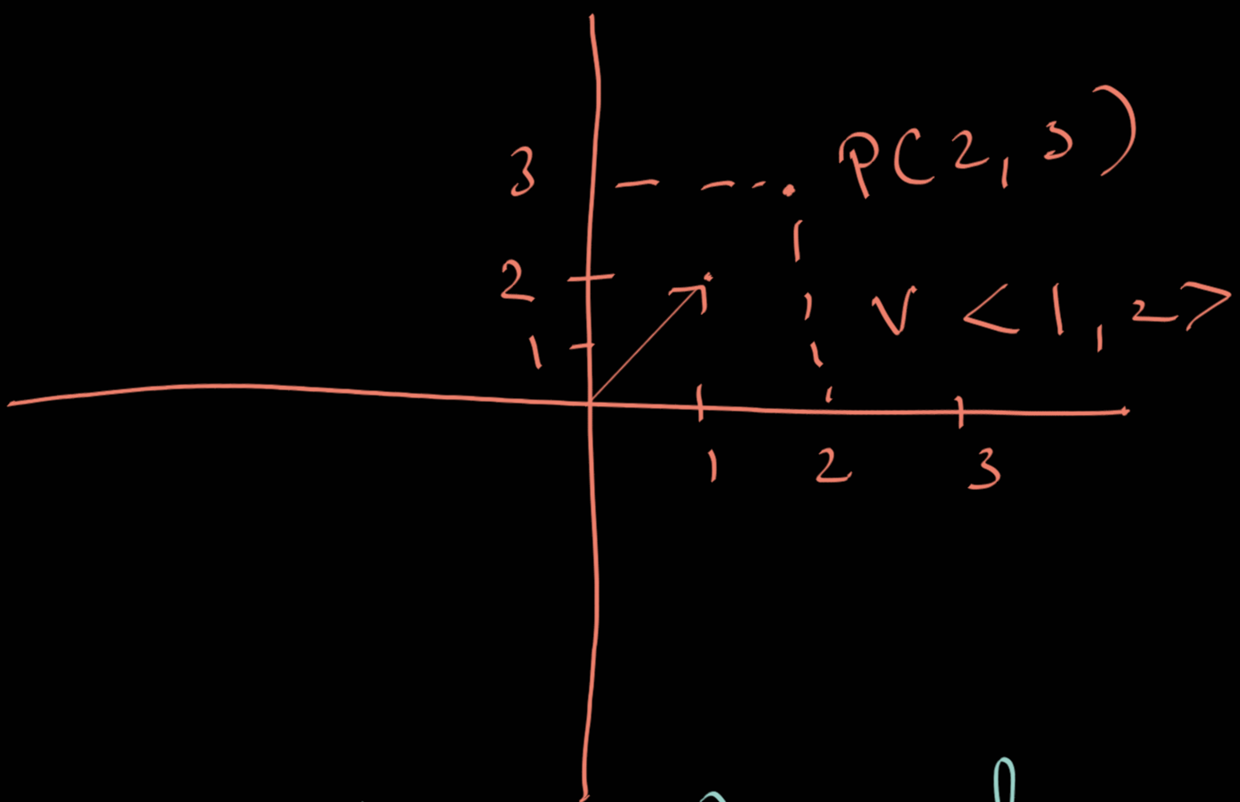
Vector vs Point

A ... a position in space

A point has position in space.

The only characteristic that distinguishes one point from another is its position.

A Vector has both magnitude & direction, but no fixed position in space. Geometrically, we draw points as dots & vectors as line segment with arrows.



Vector always starts from Origin.

Vector Operations

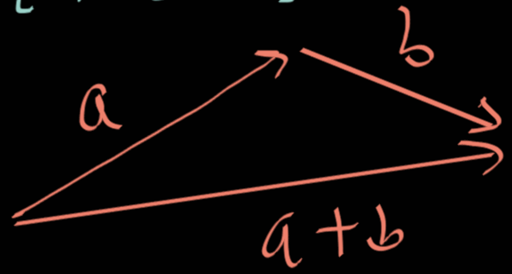
Addition

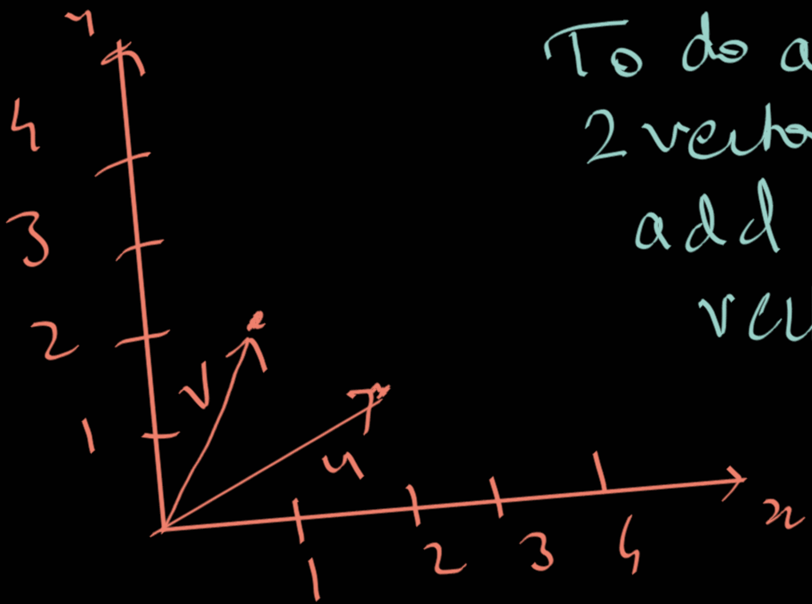
$\bar{u} = [x_1, y_1]$ $\bar{v} = [x_2, y_2]$

$\bar{u} + \bar{v} = [x_1 + x_2, y_1 + y_2]$

$\bar{u} = [2, 1]$

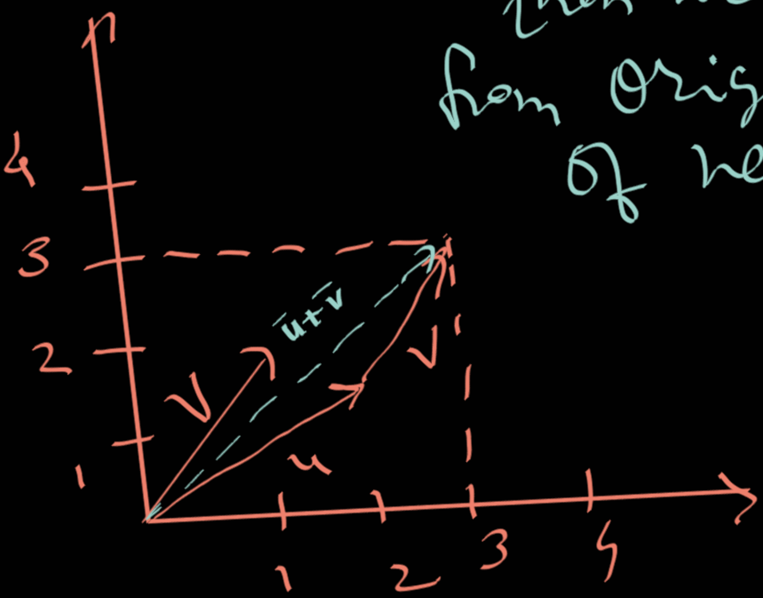
$\bar{v} = [1, 2]$





To do addition of 2 vectors we need to add tail of 2^{nd} vector in front of head of 1^{st} vector

then we see distance from origin to the end of new vector.



$\bar{u} + \bar{v} = [3, 3]$

$\bar{u} + \bar{v} = \bar{v} + \bar{u}$

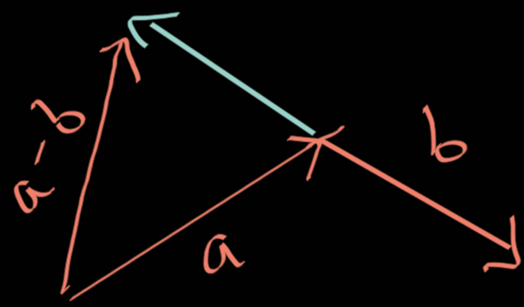
↓ Commutative property

↳ commutative property
of vector.

Subtraction

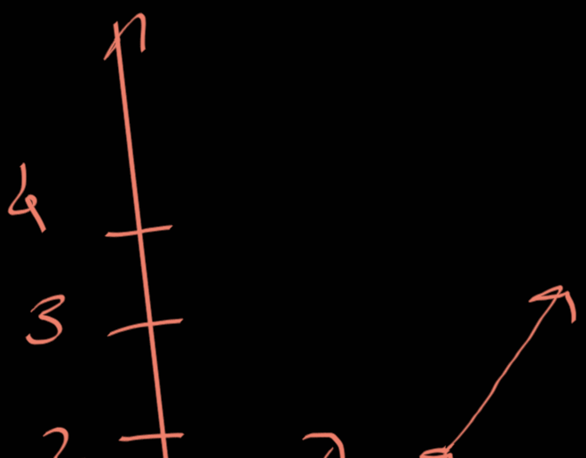
$\bar{u} = [2, 1]$

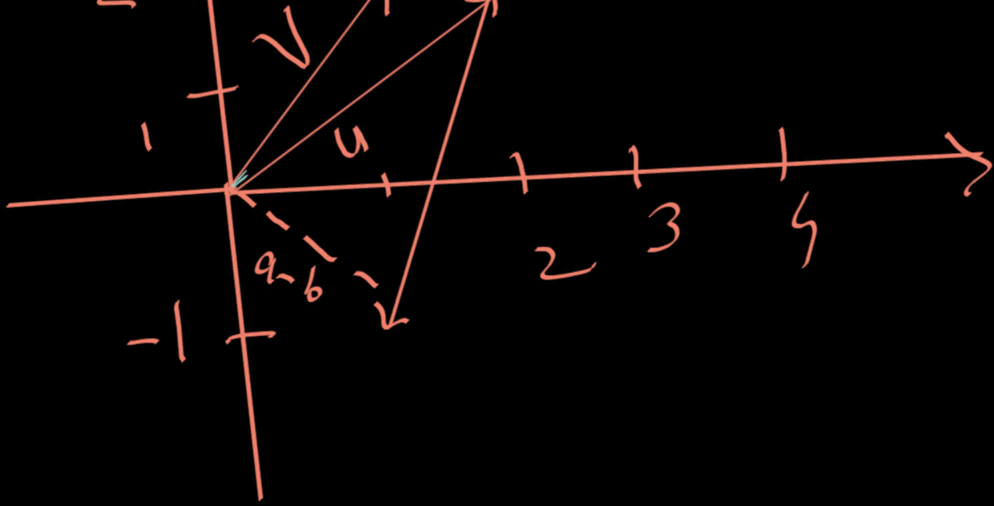
$\bar{v} = [1, 2]$



In vector subtraction we add tail of 2^{nd} vector in front of 1^{st} vector & we reverse the direction of 2^{nd} vector as we are doing subtraction. & then we see the distance from origin to end of this vector.

$\bar{u} - \bar{v} = [1, -1]$





Scalar Multiplication

if we have a vector & we multiply it with a scalar then it will boost its length in the form of positive manner or negative manner.

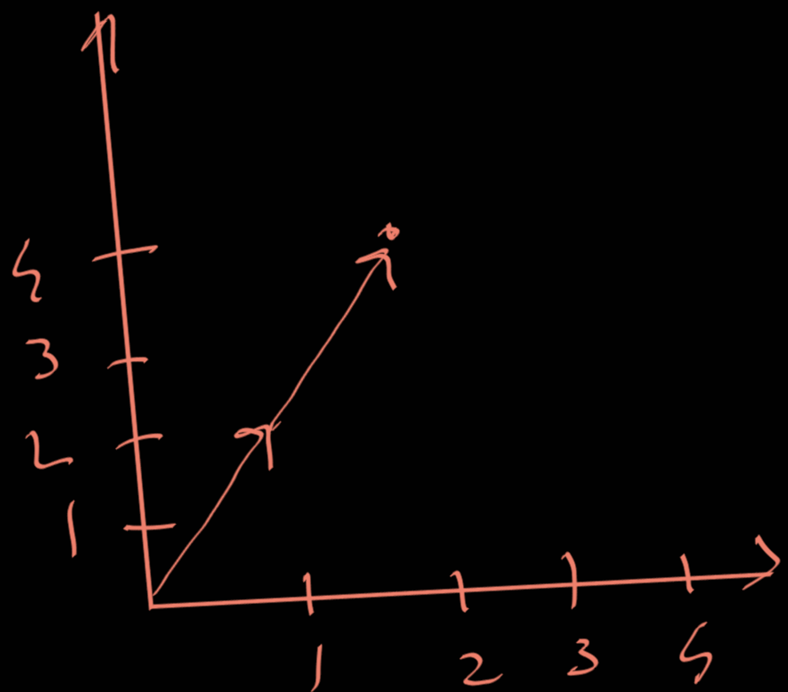
i.e. if we multiply by +ve no then it will move to positive region & so on.

$$\bar{u} = [1, 2]$$

$$= 2[1, 2]$$

scalar quantity

$$= \begin{bmatrix} 2, 4 \end{bmatrix}$$
$$x, y$$



Vector Operations Rules

Let u, v & w be vectors in the plane, and let c and d be scalars.

1) u + v is a vector in the plane.
— Closure under addition

2) u + v = v + u
Commutative Property

— commutative property
of addition.

but $u - v \neq v - u$

3) $(u+v)+w = u+(v+w)$

— Associative property of addition.

4) $u + 0 = u$

— Additive identity property

5) $u + (-u) = 0$

— additive inverse property.

Vector Norms

- How we can normalize our vectors

We normalize a vector & tells its max value.

- $P = 1$: The l_1-norm

in l_1 norm we just do sum of all variables including absolute. (-ve convert into +ve)

$$\|x\|_1 = |x_1| + |x_2| + \cdots + |x_n|$$

$P = 2$: The l_2-norm or Euclidean norm

$$\|x\|_2 = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$$

$$= \sqrt{x^T x}$$

$$[1, 2, 5] \times \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}$$

$p = \infty$; The l_∞ - norm

infinity norm

$$\|x\|_\infty = \max_{1 \le i \le n} |x_i|$$

In this norm we see max value in all elements

- first we need to absolute it then take max value.

l_1 norm, $x = \begin{bmatrix} 2 \\ 5 \\ -3 \end{bmatrix}$

$$\|x\|_1 = 10$$

l_2 norm $\|x\|_2 = \sqrt{4+25+9} \approx 6.16$

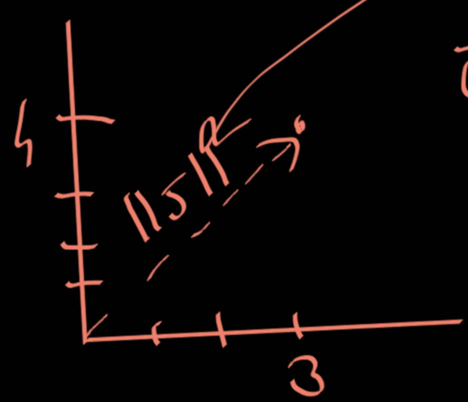
l_∞ norm $\|x\|_\infty = 5$

Unit Vector

A unit vector in a given direction is a vector with magnitude ONE in that direct.

Used to represent that direction of a vector

magnitude



$\bar{a} = [3, 4]$

$$\|a\| = \sqrt{3^2 + 4^2} = 5$$

$$\hat{a} = \frac{a_x}{\|a\|}, \quad \hat{a} = \frac{a_y}{\|a\|}$$

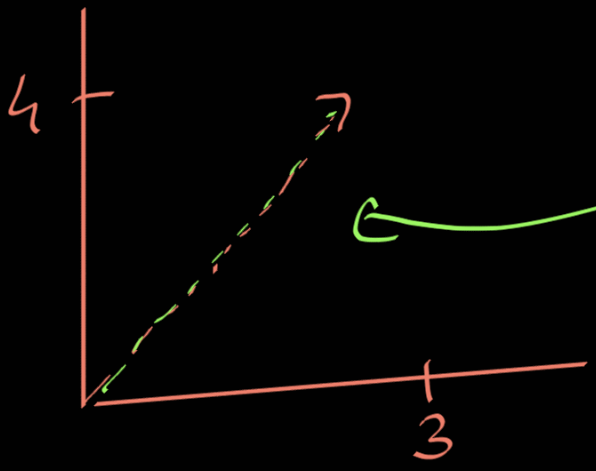
$$= 3/5, \quad 4/5$$

$$\|\hat{a}\| = \sqrt{(3/5)^2 + (4/5)^2}$$

$$= \sqrt{9/25 + 16/25}$$

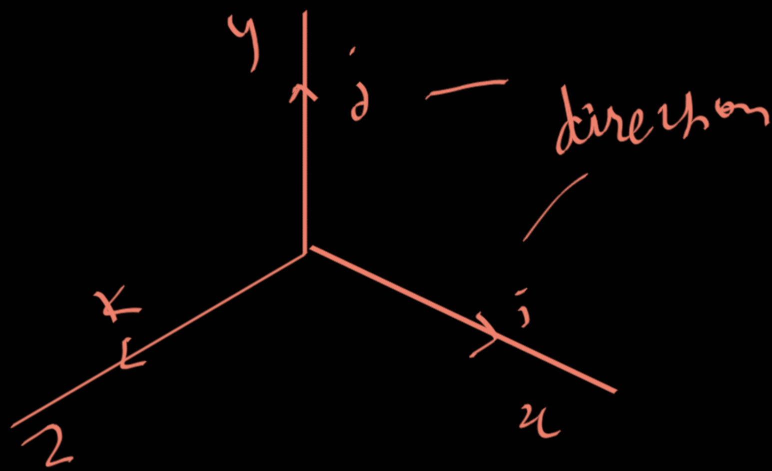
$$= \sqrt{\frac{25}{25}} = 1$$

direction



So unit vector magnitude is 1 & it represents direction.

$$\vec{a} = \overset{x}{2\hat{i}} + \overset{y}{3\hat{j}} + \overset{z}{\hat{k}}$$



$\hat{i}$ is direction of x, $\hat{j}$ is direction of y

$\hat{k}$ is direction of z

Magnitude of $|\vec{a}| = \sqrt{2^2+3^2+1^2}$

$$|\vec{a}| = \sqrt{4+9+1} = \sqrt{14}$$

unit vector in direction of $\vec{a} = \dfrac{1}{\text{magnitude of } \vec{a}} \times \vec{a}$

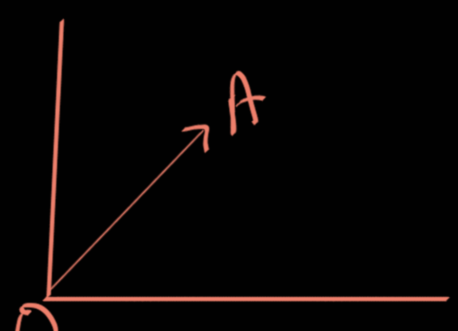
$$\hat{a} = \frac{1}{\sqrt{14}} [2\hat{i} + 3\hat{j} + 1\hat{k}]$$

$$\hat{a} = \frac{2}{\sqrt{14}}\hat{i} + \frac{3}{\sqrt{14}}\hat{j} + \frac{1}{\sqrt{14}}\hat{k}$$

$$\hat{a} = 1$$

Position Vector

A vector that starts from the Origin (O) is called a position vector.



Vector Operations

Vector to Vector Multiplication

Dot Product :-

$\bar{v} \cdot \bar{y} \Rightarrow$ result will be a Scalar quality.

$$\bar{a} \cdot \bar{b} = a_1 b_1 + a_2 b_2 + a_3 b_3 \text{ --- } a_n b_n$$

$$= \underbrace{[a_1, a_2, a_3 \text{ --- } a_n]}_{1 \times n} \underbrace{\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}}_{n \times 1}$$

When we multiply vector or matrix with other vector or matrix first we see if the colum of first vector &

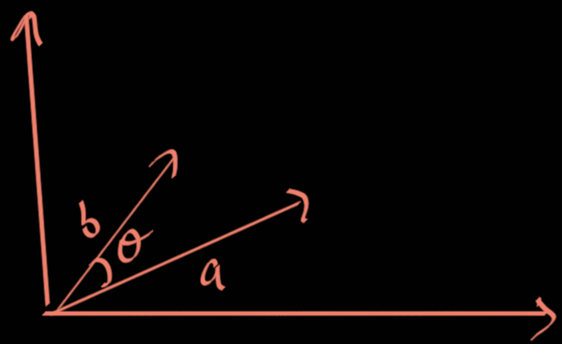
row of 2nd vector is same with respect to dimension. if they are same then only multiplication is possible.

$$a \cdot b = a^T \cdot b$$

$$\boxed{a \cdot b = \sum a_i b_i}$$ ← mathematical formula.

geometrically dot product

↳ $a \cdot b = ||a|| \times ||b|| \times \cos(\theta)$



When we do dot product geometrically of a & b then it will create a angle between them which is calculated using Cos θ.

$$\cos\theta = a \cdot b$$

$$\|a\| \, \|b\|$$

$$\boxed{\theta = \cos^{-1} \frac{a \cdot b}{\|a\| \, \|b\|}}$$

$a = (2, 2, -1)$

$b = (5, -3, 2)$

$\|a\| = \sqrt{2^2 + 2^2 + -1} = \sqrt{9} = 3$

$\|b\| = \sqrt{5^2 + -3^2 + 2^2} = \sqrt{38}$

$a \cdot b = (a_1 \times a_2) + (b_1 \times b_2) +$
$(c_1 \times c_2) = 2$

$\theta = \cos^{-1} a \cdot b / \|a\| \|b\|$

$= \cos^{-1} \frac{2}{3 \times \sqrt{38}}$

$\theta = 1.46$

π

Convert this into radian

$= 84^\circ$

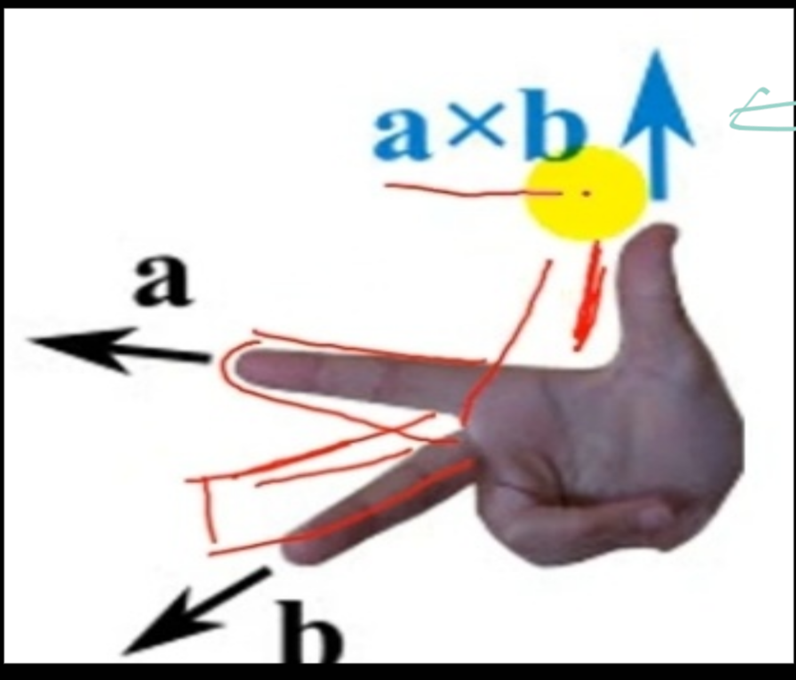
$a \cdot b = 3 \times \sqrt{38} \times 84$

Cross Product

$a \times b = |a||b| \sin(\theta)\, \hat{n}$

result of Cross Product of vector is vector itself (or we can say matrix) that's why we have unit vector $\hat{n}$ to show direction.

The Cross Product could point in the completely opposite direction & still be right angles to the two vectors, so we have the Right Hand Rule



direction $\hat{n}$

With your right-hand, point your index finger along vector a, & point your middle finger along vector b : the cross product goes in the direction of your thumb.

$a = (3, -3, 1)$ & $b = (4, 9, 2)$

first way to calculate cross product

Solution: Cross product is

↳

$$a \times b = \begin{vmatrix} i & j & k \\ 3 & -3 & 1 \\ 4 & 9 & 2 \end{vmatrix}$$

$$\begin{vmatrix} i & j & k \\ 3 & -3 & 1 \\ 4 & 9 & 2 \end{vmatrix}$$

→ ignore first row & columns

$= i(-3 \cdot 2 - 1 \cdot 9)$

$$\begin{vmatrix} i & j & k \end{vmatrix}$$

$$\begin{vmatrix} 3 & \cancel{-3} & 1 \\ 4 & \cancel{9} & 2 \end{vmatrix}$$

$$= j(3\cdot 2 - 1\cdot 4)$$

$$\begin{vmatrix} i & j & k \\ 3 & -3 & 1 \\ 4 & 9 & 2 \end{vmatrix}$$

$$= k(3\cdot 9 + 3\,4$$

$$= i(-3\cdot 2 - 1\cdot 9) - j(3\cdot 2 - 1\cdot 4)$$
$$+ k(3\cdot 9 + 3\cdot 4)$$

$$= -15i - 2j + 39k$$

2^{nd} way to Calculate Cross Product.

$a = \langle -4, 3, 0\rangle$ & $b = \langle 2, 0, 0\rangle$

$|a| = 5$ ← magnitude (norm 2)

$|b| = 2$

$a \cdot b = \langle 0, 0, -6\rangle$

$|a \cdot b| = 6$

$\sin^{-1}(6/(5*2)) = \sin^{-1}(3/5)$

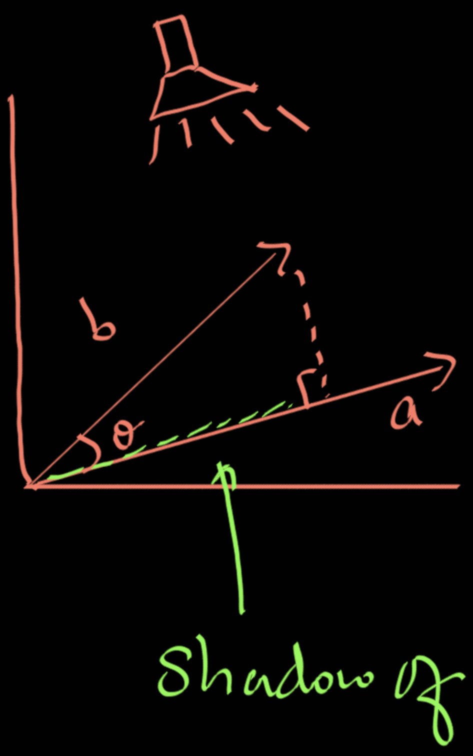
$= 36.87^\circ$

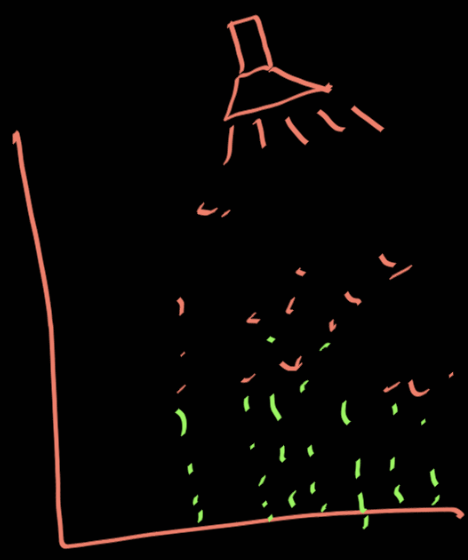
$a \times b = \|a\| \|b\| \sin\theta (n)$

$a \times b = 5 * 2 \sin(36.87)$

Vector Projection

↳ reduce multidimension into single dimension.





Projection of vector b on a

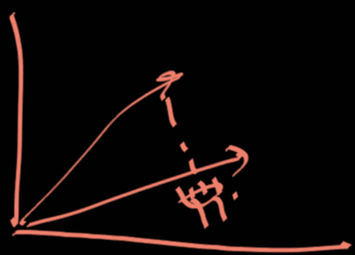
$$\text{Proj}_{\bar{a}} \bar{b} = |b| \cos\theta$$

$$a \cdot b = |a|\,|b| \cos\theta$$

$$\frac{a \cdot b}{|a|} = |b| \cos\theta$$

$$\vec{b} = \frac{a \cdot b}{|a|}$$

Projection of vector a on b



$$\vec{a} = \frac{a \cdot b}{|b|}$$

Find the Projection of vector $\bar{a} = \{1; 2\}$ on vector $\bar{b} = \{3; 4\}$.

Solution.

Calculate dot product of these vectors

$$\bar{a} \cdot \bar{b} = 1 \cdot 3 + 2 \cdot 4 = 3 + 8 = 11$$

Calculate magnitude of vector $\bar{b}$

$$|\bar{b}| = \sqrt{3^2 + 4^2} = \sqrt{9 + 16}$$
$$= \sqrt{25} = 5$$

Calculate vector Projection:-

$$\text{Projection}_{\bar{b}}\, \bar{a} = \frac{\bar{a} \cdot \bar{b}}{|\bar{b}|^2} \bar{b}$$

$$= \frac{11}{5^2} \{3; 4\}$$

$$= \frac{11}{25} \{3; 4\}$$

$$= \{1.32; 1.76\}$$

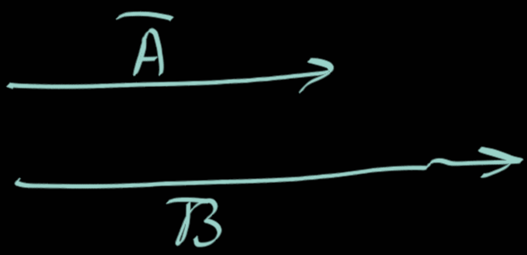
Calculate Scalar Projection

$$|\text{Proj}_{\bar{b}}\,\bar{a}| = \frac{\bar{a}\cdot\bar{b}}{|\bar{b}|} = \frac{11}{5} = 2.2$$

Types of Vectors

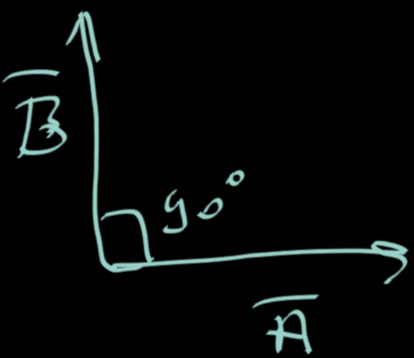
Parallel vector

$\bar{A} \parallel \bar{B}$



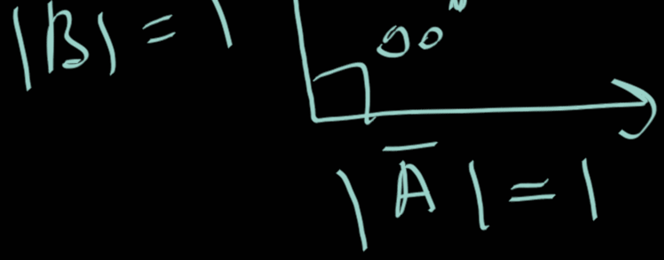
Perpendicular vector (Orthogonal)

$\bar{A} \perp \bar{B}$



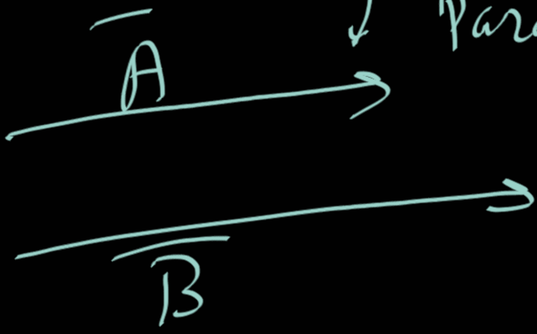
Orthonormal Vector

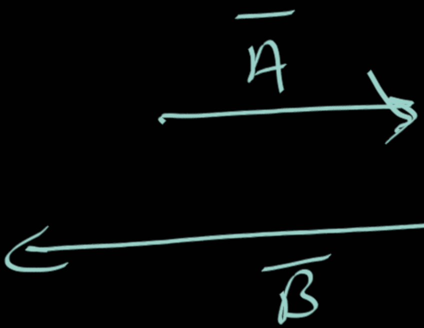
vector which are perpendicular to each other & their magnitude is 1



Like & Unlike vector

same as
↓ Parallel

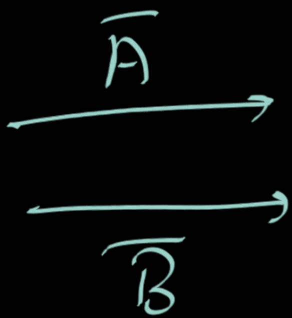




Unlike vector

Equal vector

$\bar{A} = \bar{B}$



direction & magnitude is same

Colinear Vector

On same axis but different direction

Unit Vector

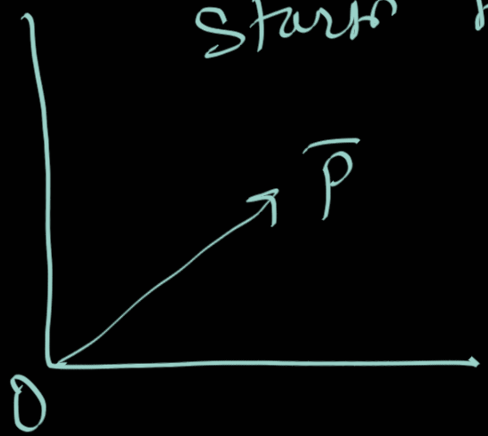
magnitude is 1

$|\bar{A}| = 1$

Position Vector

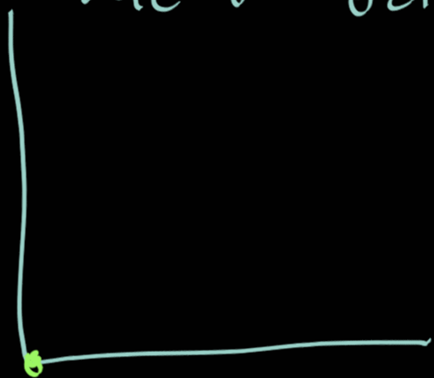
Starts from origin

$\overline{OP}$



Zero Vector

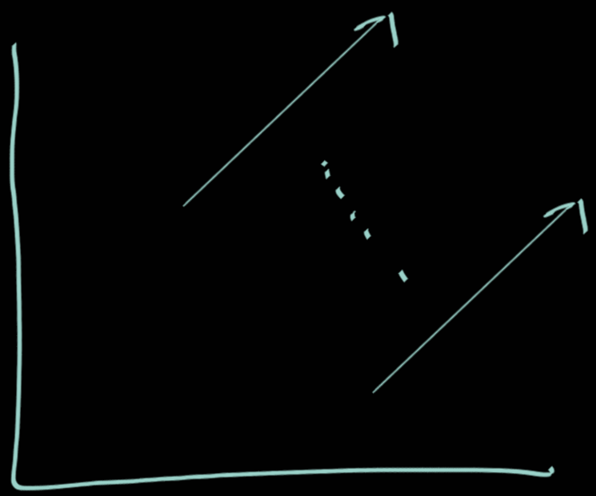
Lie on origin



Or Starts from Origin & ends at Origin

$\overline{AA}$

Free vector



When we move 1 vector to other position & it does not change its direction or magnitude then it is called Free vector

Negative vector

$\bar{A}$ 3 m

$\bar{B}$ 3 m

opposite direction but same magnitude

Vector Space

A vector space is a non empty set V of objects, called vectors, on which are defined two operations, called addition & multiplication by scalars (real numbers), subject to the ten rules listed below. The rules must hold for all vectors u, v & w in V & for all scalars c & d.

1) Sum of u & v, denoted by $u+v$, is in V

2) $u+v = v+u$

3) $(u+v)+w = u+(v+w)$

4) There is a zero vector 0 in V such that $u+0 = u$

5) For each u in V, there is a vector $-u$ in V, such that $u+(-u) = 0$

6) The scalar multiple of u by c, denoted by cu, is in V

7) $c(u+v) = cu + cv$.

8) $(c+d)u = cu + du$

9) $c(du) = (cd)u$

10) $1u = u$.

V is a collection of element that can be:

→ added together in any combination

→ multiply by scaler in any combination

Closures

① $\bar{a}, c$ then $c\bar{a} \in \bar{V}$

When we multiply scaler with

1 vector then result will be the value in Vector Space V

(2) $\bar{a} \in V$, $\bar{b} \in V$, then

$$\bar{a} + \bar{b} \in V$$

Linear Combination of Vectors

If one vector is equal to the sum of scalar multiples of other vectors, it is said to be a linear combination of the other vectors

$$\underset{a}{\begin{bmatrix} 11 \\ 16 \end{bmatrix}} = 2 \underset{b}{\begin{bmatrix} 1 \\ 2 \end{bmatrix}} + 3 \underset{c}{\begin{bmatrix} 3 \\ 4 \end{bmatrix}}$$

$$\begin{bmatrix} 2*1 + 3*3 \\ 2*2 + 3*4 \end{bmatrix} = \begin{bmatrix} 11 \\ 16 \end{bmatrix}$$

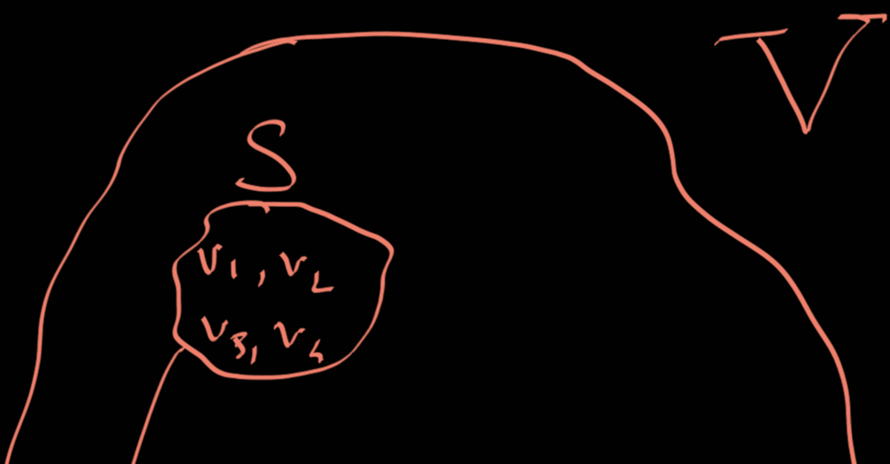
Note that 2b is scaler multiple &

3c is a scaler mutiple. Thus,

a is a linear combination of b & c.

Spanning set

Let S be a non-empty subset of vector space V. The set of all linear combination of elements of S is called linear span of S and is denoted by $\langle S \rangle$



$\langle S \rangle$

$a_1 v_1 + a_2 v_2$

$a_1 v_1$

$a_1 v_1 + a_2 v_2 + a_3 v_3$

$a \in \mathbb{R}$

$S = \{(1,1,0)\}$

$\langle S \rangle = x_1 v_1 + x_2 v_2 + x_3 v_3$

$= (v_1) + (v_2) + (0\, v_3)$

$5 \quad + \quad 6 \quad + \quad 7$

$\downarrow$

we can add any value & replace $v_1, v_2 v_3$

Tutorial 12, 13, 14 is pending.

Matrix Types

$$\begin{bmatrix} 1 & 4 & 0 \\ 8 & 15 & 3 \\ 1 & 9 & 2 \end{bmatrix} \begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{bmatrix}$$

3×3 $[a_{ij}]$ $m \times n$

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 8 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 8 \end{bmatrix}^T$$

1st row to 1st col, 2nd row to 2nd coln so on

Symmetric matrix :-

When we take Transpose of matrix & it is same as original matrix then it is Symmetric matrix

Square Matrix = $3*3$, $4*4$

Orthogonal matrix

When we take transpose of matrix & multiply it with Original matrix & if we get result as identity matrix then it is Orthogonal

$$A^TA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \leftarrow I_3$$

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Row Matrix $\rightarrow \begin{bmatrix} a & b & c \end{bmatrix}$

Column Matrix $\rightarrow \begin{bmatrix} a \\ b \\ c \end{bmatrix}$

Zero Matrix or Null Matrix $\rightarrow \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$

Diagonal Matrix $\rightarrow \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$

Scalar Matrix

$\rightarrow \begin{bmatrix} a^5 & 0 & 0 \\ 0 & a^5 & 0 \\ 0 & 0 & a^5 \end{bmatrix}$

Values are same is Scalar Matrix but different in Diagonal Matrix

Unit Matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

all diagonal values will be 1

Upper triangular Matrix

$$\rightarrow \begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{bmatrix}$$

Lowe Triangular Matrix

$$\begin{bmatrix} a & 0 & 0 \\ b & c & 0 \\ d & e & f \end{bmatrix}$$

Matrix Operations

Transpose Converts row vector to Column vector, vice versa.

$$\begin{bmatrix} 0 & 4 \\ 7 & 0 \\ 3 & 1 \end{bmatrix}^T = \begin{bmatrix} 0 & 7 & 3 \\ 4 & 0 & 1 \end{bmatrix}$$

$$(A^T)^T = A$$

Matrix addition & Subtraction.

if A & B are both m×n,
we form A + B :

$$\underbrace{\begin{bmatrix} 0 & 4 \\ 7 & 0 \\ 3 & 1 \end{bmatrix}}_{3\times 2} + \underbrace{\begin{bmatrix} 1 & 2 \\ 2 & 3 \\ 0 & 4 \end{bmatrix}}_{3\times 2} = \underbrace{\begin{bmatrix} 1 & 6 \\ 9 & 3 \\ 3 & 5 \end{bmatrix}}_{A+B}$$

Can add row or Column Vectors same way (but never to each other)

Matrix Subtraction is similar

$$\begin{bmatrix} 1 & 6 \\ 9 & 3 \end{bmatrix} - I$$

$$\begin{bmatrix} 1 & 6 \\ 9 & 3 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 6 \\ 9 & 2 \end{bmatrix}$$

Properties →

- Commutative: $A + B = B + A$
- associative: $(A+B) + C = A + (B+C)$,

  So we can write as $A + B + C$
- $A + 0 = 0 + A = A$; $A - A = 0$
- $(A+B)^T = A^T + B^T$

Scalar multiplication with matrix

We can multiply a number (scalar) by a matrix by multiplying every entry of the matrix by the scalar.

$$(-2)\begin{bmatrix} 1 & 6 \\ \end{bmatrix} = \begin{bmatrix} -2 & -12 \\ \end{bmatrix}$$

$$(-2)\begin{bmatrix} 9 & 3 \\ 6 & 0 \end{bmatrix} = \begin{bmatrix} -18 & -6 \\ -12 & 0 \end{bmatrix}$$

→ $(a+b)A = aA + bA$

→ $(ab)A = (a)(bA)$

→ $a(A+B) = aA + aB$

→ $0.A = 0$; $1.A = A$.

Matrix Vector Product

Very important special case of matrix multiplication: $y = Ax$

→ A is an $m \times n$ matrix

→ x is an n-vector

→ y is an m-vector

$$y_i = A_{i1}x_1 + \dots A_{in}x_n, \quad i = 1 \dots m$$

Can ~~think~~ of $y = Ax$ as

- a function that transforms n-vectors into m-vectors
- a set of m linear equations relating x to y.

Inner Product

→ if v is a row n-vector & w is a column n-vector, then vw makes sense, & has size 1×1, i.e. is a scalar

$$vw = v_1 w_1 + \cdots v_n w_n$$

→ if x & y are n-vectors, $x^T y$ is a scalar called inner product or dot product of x, y, and denoted $\langle x, y \rangle$ or $x \cdot y$:

$$\langle x, y \rangle = x^T y = x_1 y_1 + \cdots x_n y_n$$

Matrix Multiplication

Ex:-

$$\begin{bmatrix} 1 & 6 \\ 9 & 3 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1\times 0 + 6\times(-1) & 1\times(-1) + 6\times 2 \\ 9\times 0 + 3\times(-1) & 9\times(-1) + 3\times 2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 + (-6) & -1 + 12 \\ 0 + (-3) & -9 + 6 \end{bmatrix}$$

$$= \begin{bmatrix} -6 & 11 \\ -3 & -3 \end{bmatrix}$$

Ex2:-

$$\begin{bmatrix} 0 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 6 \\ 9 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 0\times 1 + (-1)\times 9 & 0\times 6 + (-1)\times 3 \\ -1\times 1 + 2\times 9 & -1\times 6 + 2\times 3 \end{bmatrix}$$

$$\begin{bmatrix} 0+(-9) & 0+(-3) \\ -1+(18) & -6+6 \end{bmatrix}$$

$$= \begin{bmatrix} -9 & -3 \\ 17 & 0 \end{bmatrix}$$

Matrix multiplication is not always Commutative: $AB \neq BA$

Hadamard Product Matrix

Hadmard product of two vectors is similar to matrix addition, elements corresponding to same row & column of given vectors/matrices are multiplied together to form a new vector/matrix.

$$\begin{bmatrix} 1 & 6 \\ 9 & 3 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ -1 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1\times 0 & 6\times -1 \\ 9\times -1 & 3\times 2 \end{bmatrix} = \begin{bmatrix} 0 & -6 \\ -9 & 6 \end{bmatrix}$$

Cross product Matrix

$$A = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} \qquad B = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$$

$A \times B$

$$\begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} \times \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$$

$$\begin{bmatrix} 3\times 6 - 1\times 5 \\ 1\times 4 - 2\times 6 \\ 3\times 4 - 2\times 5 \end{bmatrix}$$

$$\begin{bmatrix} 18-5 \\ 4-12 \\ 12-10 \end{bmatrix} = \begin{bmatrix} 12 \\ -8 \\ 2 \end{bmatrix}$$

Matrix Operations

Matrix Power

if matrix A is square, then product $A\,A$ makes sence. & denoted by A^2

k copies of A multiplied together gives A^k:

$$A^k = A\,A \ldots A$$

$$A^2 = A \cdot A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$$

$$A^3 = A^2 \cdot A = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 4 \\ 4 & 4 \end{bmatrix}$$

$$A^4 = A^3 \cdot A = \begin{bmatrix} 4 & 4 \\ 4 & 4 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 8 \\ 8 & 8 \end{bmatrix}$$

Determinant of Matrix

The determinant of matrix is a special value that is calculated from a square matrix. It can help you determine whether a matrix has an inverse, find the area of a triangle, & let you know if the system of equations has a unique solution.

- It is useful for solving linear equations, capturing how linear transformation change area or volume & changing variables in integrals.

- The determinant can be viewed as a function whose input is a square matrix & whose output is a number.

$$|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = (ad - bc) = x$$

$$|A| = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$$

3×3

$$= a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

$$= a(ei - fh) - b(di - fg) + c(dh - eg)$$

$$= aei + bfg + cdh - ceg - bdi - afh$$

$$\det\begin{bmatrix} 2 & -3 & 1 \\ 2 & 0 & -1 \\ 1 & 4 & 5 \end{bmatrix}$$

$$= 2 \cdot \det\begin{bmatrix} 0 & -1 \\ 4 & 5 \end{bmatrix} - (-3) \cdot \det\begin{bmatrix} 2 & -1 \\ 1 & 5 \end{bmatrix}$$

$$+1 \cdot \begin{bmatrix} 2 & 0 \\ 1 & 4 \end{bmatrix}$$

$$= 2[0-(-4)] + 3[10-(-1)] + 1[8-0]$$

$$= 2[0+4] + 3[10+1] + 1[8]$$

$$= 8 + 3(11) + 8$$

$$= 8 + 33 + 8$$

$$= 49$$

if matrix is 3

then 1^{st} value +ve , 2^{nd} -ve,

3^{rd} +ve

if matrix is 4

then +ve -ve ...

then +ve, −ve, +ve, −ve

& so on.

Matrix Inverse

if A is square, & (square) matrix F satisfies $FA = I$, then

- F is called inverse of A & denoted by A^{-1}

$$A^{-1}A = I$$

$$A^{-1} = \frac{1}{\det(A)}\left(\operatorname{adj}(A)\right)$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -c \\ -b & a \end{bmatrix}$$

$$\left[\ \right]^{-1} = \frac{1}{\ } \begin{bmatrix} 6 & -7 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 7 \\ 2 & 6 \end{bmatrix} = \frac{1}{4\times 6 - 7\times 2} \begin{bmatrix} & \\ -2 & 4 \end{bmatrix}$$

$$= \frac{1}{10} \begin{bmatrix} 6 & -7 \\ -2 & 4 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 0.6 & -0.7 \\ -0.2 & 0.4 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 7 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} 0.6 & 0.7 \\ -0.2 & 0.4 \end{bmatrix}$$

$$= \begin{bmatrix} 4\times 0.6 + 7\times(-0.2) & 4\times -0.7 + 0.4 \\ 2\times 0.6 + 6\times -0.2 & 2\times -0.7 + 6\times 0.4 \end{bmatrix}$$

$$= \begin{bmatrix} 2.4 - 1.4 & -2.8 + 2.8 \\ 1.2 - 1.2 & -1.4 + 2.4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Why do we need Inverse?

we don't have Concept of Divide matrix, But we can multiply by an inverse, which achieves the same thing.

Properties of Inverse

$\rightarrow (A^{-1})^{-1} = A$

$\rightarrow (AB)^{-1} = B^{-1} A^{-1}$

$\rightarrow (A^{T})^{-1} = (A^{-1})^{T}$

$\rightarrow I^{-1} = I$

$\rightarrow (aA)^{-1} = (1/a) A^{-1}$

→ if $y = Ax$, where $x \in R^n$ & A is invertible then $x = A^{-1}y$

$$A^{-1}y = A^{-1}Ax = Ix = x$$

Matrix Rank

Check this one later in Tutorial 18

Linear Equations & System Of Linear Equations

An equation in which variables highest power is 1

$x^1 + 4 = 0$ —— ①

$x^1 + y = 3$ — ②

$x = -4 \rightarrow$ put it in eq ①

$-4+4=0$

$\boxed{0=0}$

$x=3-9$

$x=-6 \rightarrow$ put in eq ②

$-6+y=3$

$\boxed{3=3}$

In general form :-

$$\boxed{a_1x_1+a_2x_2+\cdots a_nx_n=c}$$

System of Linear Equation.

A system (group) of two or more linear equations involving some variables is called system of Linear equations

$2x+7y=34$ — i

$8x - 4y = -1$ — ii

① ② ③

$x + y - z = -2$ — iii

$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$

$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$

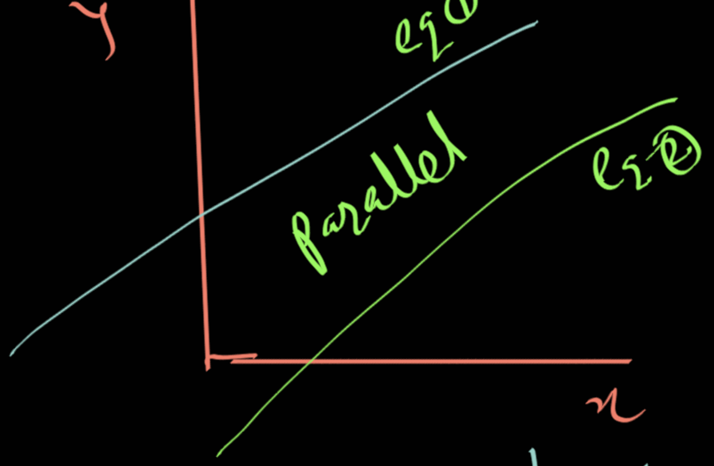
$Ax = b \rightarrow$ RHS

Matrix Unknowns

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

While solving system of linear equation one of three possible solutions possible

Inconsistent



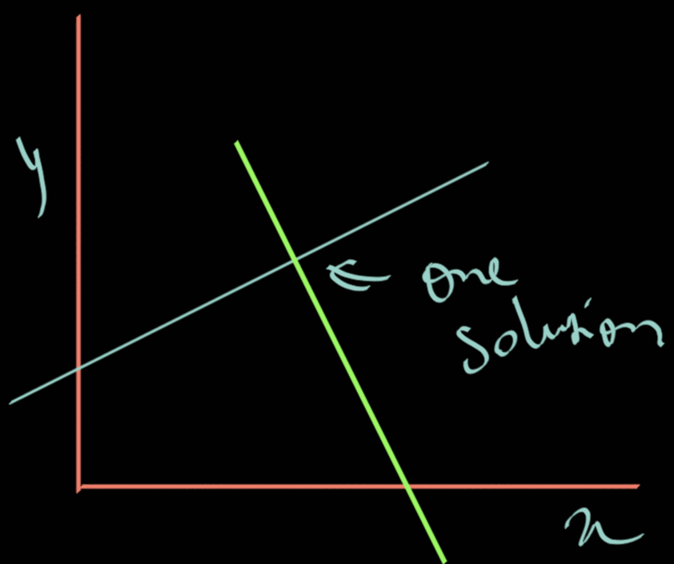
No Solution

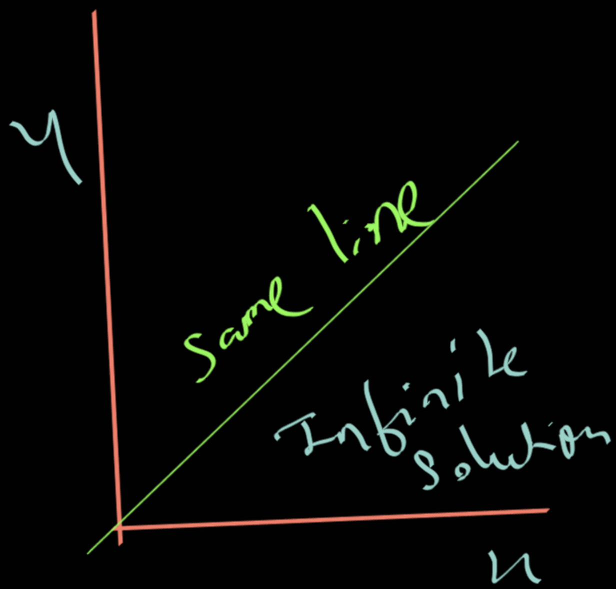
When we plot liner equations both are going parallel to infinite time, there is no intersection where we can plot x & y value

Consistent

Independent

Dependent





2 eq intersect at a point where we have 1 solution

eq¹ & eq² value are same.

System of Linear Equations

$$3x + 2y = 19$$

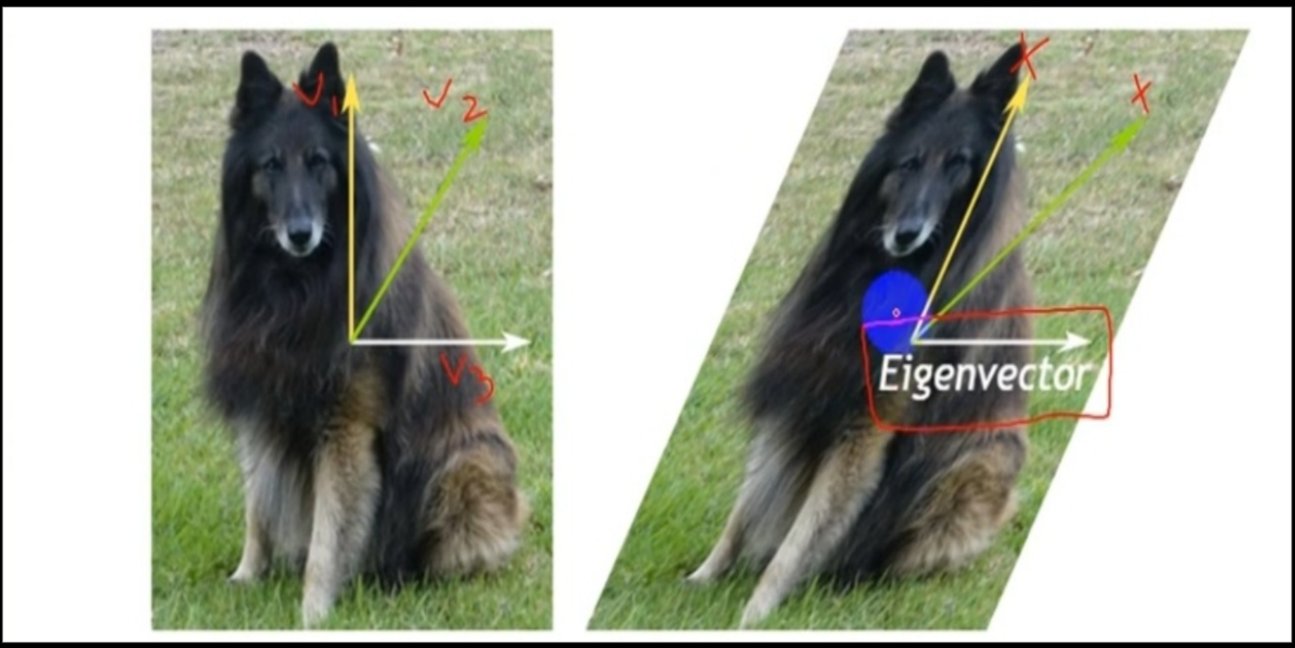
$$x + y = 8$$

$$Ax = b$$

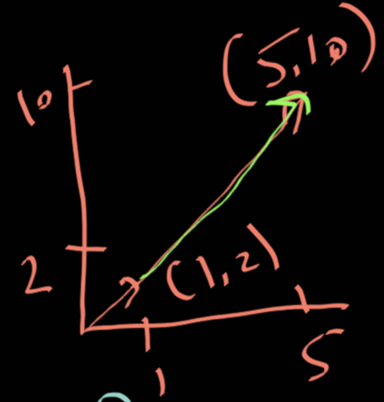
$$\begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 19 \\ 8 \end{bmatrix}$$

Eigenvector & Eigen Value

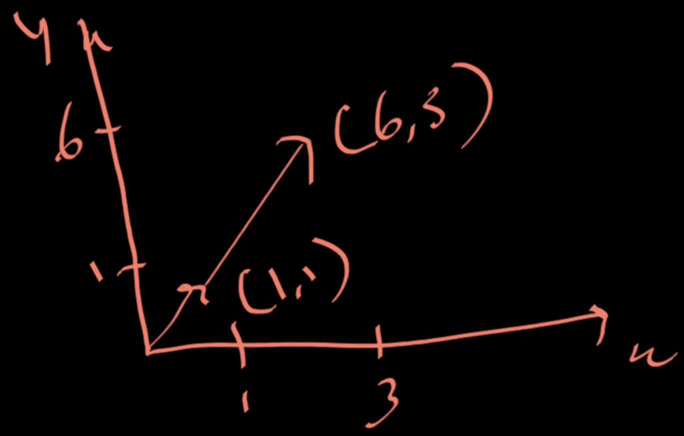
→ Eigenvector does not change direction in a transformation.



$$\overset{A}{\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}} \overset{u}{\begin{bmatrix} 1 \\ 2 \end{bmatrix}} = \begin{bmatrix} 5 \\ 10 \end{bmatrix}$$



$$\overset{A}{\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}} \overset{u}{\begin{bmatrix} 1 \\ 1 \end{bmatrix}} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$$



in first graph no rotation only scale

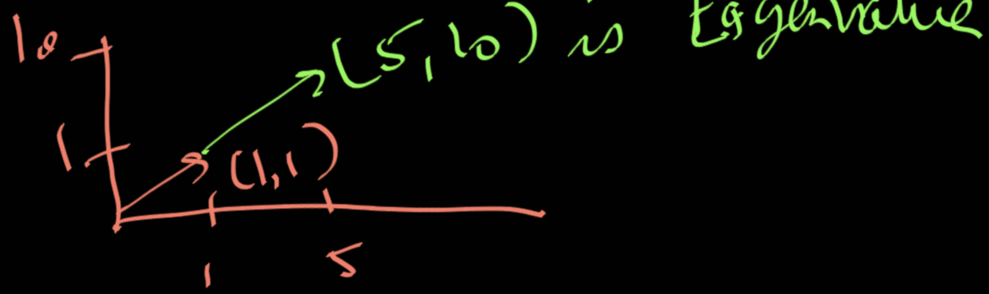
in 2nd one scaled & rotation also

$$S = E \cdot V.$$

if only scaled not rotate then we can say its eigen vector.

- An Eigenvector is a vector that maintains its direction after undergoing a linear transformation.

- An Eigenvalue is the scalar value that the eigenvector was multiplied by during the linear transformation



$$\boxed{A \cdot v = \lambda \cdot v}$$

matrix

Eigenvector

Eigen Value.

How to finden Eigen value?

$$|A - \lambda I| = 0$$

$$\left|\begin{bmatrix}-6 & 3\\ 4 & 5\end{bmatrix}-\lambda\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\right| = 0$$

$$\begin{bmatrix}-6-\lambda & 3\\ 4 & 5-\lambda\end{bmatrix} = 0$$

$$\underbrace{(-6-\lambda)(5-\lambda)} - \underbrace{3\times 4}_{12} = 0$$

$$= (-6)(5) + (-6)(-\lambda) +$$
$$(-\lambda)(5) + (-\lambda)(-\lambda)$$

$$= -30 + 6\lambda - \lambda 5 + \lambda^2$$

$$= \lambda^2 - \lambda - 30$$

$$\lambda^2 - \lambda - 30 - 12 = 0$$

$$\lambda^2 - \lambda - 42 = 0$$

$$\lambda = 7 \text{ or } -6$$

So 7 or -6 will be the Eigenvalues, so we will get Eigenvector using those value.

first try with 6

$$Av = \lambda v$$

$$\begin{bmatrix} -6 & 3 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 6 \begin{bmatrix} x \\ y \end{bmatrix}$$

$$-6x + 3y = 6x$$

$$4x + 5y = 6y$$

$$-6x - 6x + 3y = 0$$

$$4x + 5y - 6y = 0$$

$$-12x + 3y = 0$$

$$4x - 1y = 0$$

$$3y = 12x$$

$$y = 4x$$

$$\frac{\cancel{3}y}{\cancel{3}} = \frac{\cancel{12}^{4}x}{\cancel{3}_{1}}$$

$$y = 4x$$

Either equation reveals that $y = 4x$, so the eigenvector is any non-zero multiple of this:

$$\begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

$$\begin{bmatrix} -6 & 3 \\ 4 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 6 \\ 24 \end{bmatrix}$$

$$6\begin{bmatrix} 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 6 \\ 24 \end{bmatrix}$$

Check Tutorial 21, 22, 23 later

Factorization / Decomposition

$12 = 3 \cdot 4$

$12 = 3 \cdot 2 \cdot 2$

$12 = 6 \cdot 2$

Factorization / Decomposition

Many complex matrix operations cannot be solved efficiently or with stability using the limited precision of computers.

Matrix decompositions are methods that reduce a matrix into constituent parts that make it easier to calculate more complex matrix operations. Matrix decomposition methods, also called matrix factorization methods, are a foundation of linear algebra in computers, even for basic operations such as solving systems of linear equations, calculating the inverse, and calculating the determinant of a matrix.

Eigenvalue Decomposition

Eigenvalue Decomposition

Eigendecomposition is the factorization of a matrix into a canonical form, whereby the matrix is represented in terms of its eigenvalues and eigenvectors. Only diagonalizable matrices can be factorized in this way. When the matrix being factorized is a normal or real symmetric matrix, the decomposition is called "spectral decomposition", derived from the spectral theorem.

Square

3*3 2*1

$$\mathbf{A} = \mathbf{Q}\,\Lambda\,\mathbf{Q}^{-1} = I$$

$$\begin{bmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{bmatrix} = \underbrace{\begin{bmatrix} | & | & | \\ \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \\ | & | & | \end{bmatrix}}_{\text{Eigen vectors of } \mathbf{A}} \underbrace{\begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}}_{\text{Eigen values of } \mathbf{A}} \underbrace{\begin{bmatrix} | & | & | \\ \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \\ | & | & | \end{bmatrix}^{-1}}_{\text{Eigen vectors of } \mathbf{A}}$$

Matrix Norm
L1, L-infinity, Frobenius norm

A matrix norm is a number defined in terms of the entries of the matrix. The norm is a useful quantity which can give important information about a matrix.

To estimate how big a vector/tensor is
To estimate how close one tensor to another

The 1-norm

$$\|A\|_1 = \max_{1 \le j \le n} \left(\sum_{i=1}^{n} |a_{ij}| \right)$$

the maximum absolute Column sum.

- We Sum the absolute values down each Column & take the biggest answer.

ex:- Calculate 1-norm of $A = \begin{bmatrix} 1 & -7 \\ -2 & -3 \end{bmatrix}$

Columnwise Sums A

$\hookrightarrow$ $1 + |-2| = 1 + 2 = 3$

& $|-7| + |-3| = 7 + 3 = 10$

max value.

The infinity norm

$$\|A\|_\infty = \max_{1 \le i \le n} \left(\sum_{j=1}^{n} |a_{ij}| \right)$$

(the maximum absolute row sum).

- We Sum the absolute values

along each row & then take the Biggest answer

$$A = \begin{bmatrix} 1 & -7 \\ -2 & -3 \end{bmatrix}$$

$$|1| + |-7| = 1 + 7 = 8$$

$$|-2| + |-3| = 2 + 3 = 5$$

$$||A||_\infty = 8$$

The Euclidean norm

$$||A||_E = \sqrt{\sum_{i=1}^{n} \sum_{j=1}^{n} (a_{ij})^2}$$

(the square roots of sum of all the squares)

$$\begin{bmatrix} 1 & -7 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & -7 \\ -2 & -3 \end{bmatrix}$$

$$\|A\|_E = \sqrt{1^2 + (-7)^2 + (-2)^2 + (-3)^2}$$

$$= \sqrt{1 + 49 + 4 + 9}$$

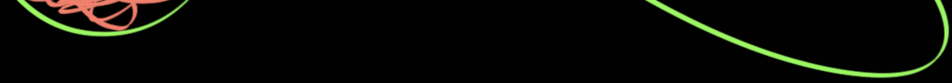
$$= \sqrt{63} \approx \underline{7.937}$$

$$\|A\|_E = 8$$

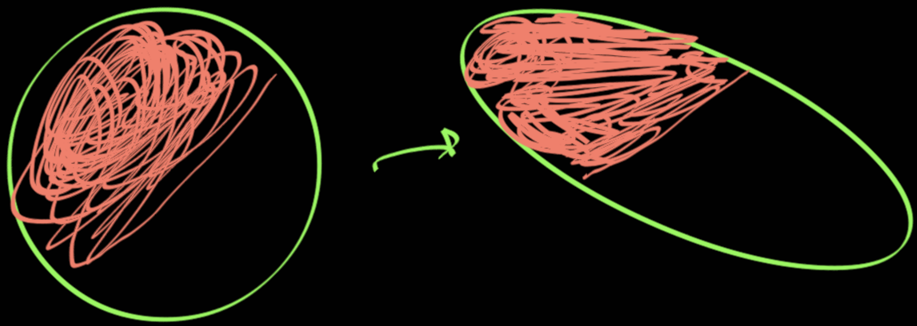
Singular Value Decomposition

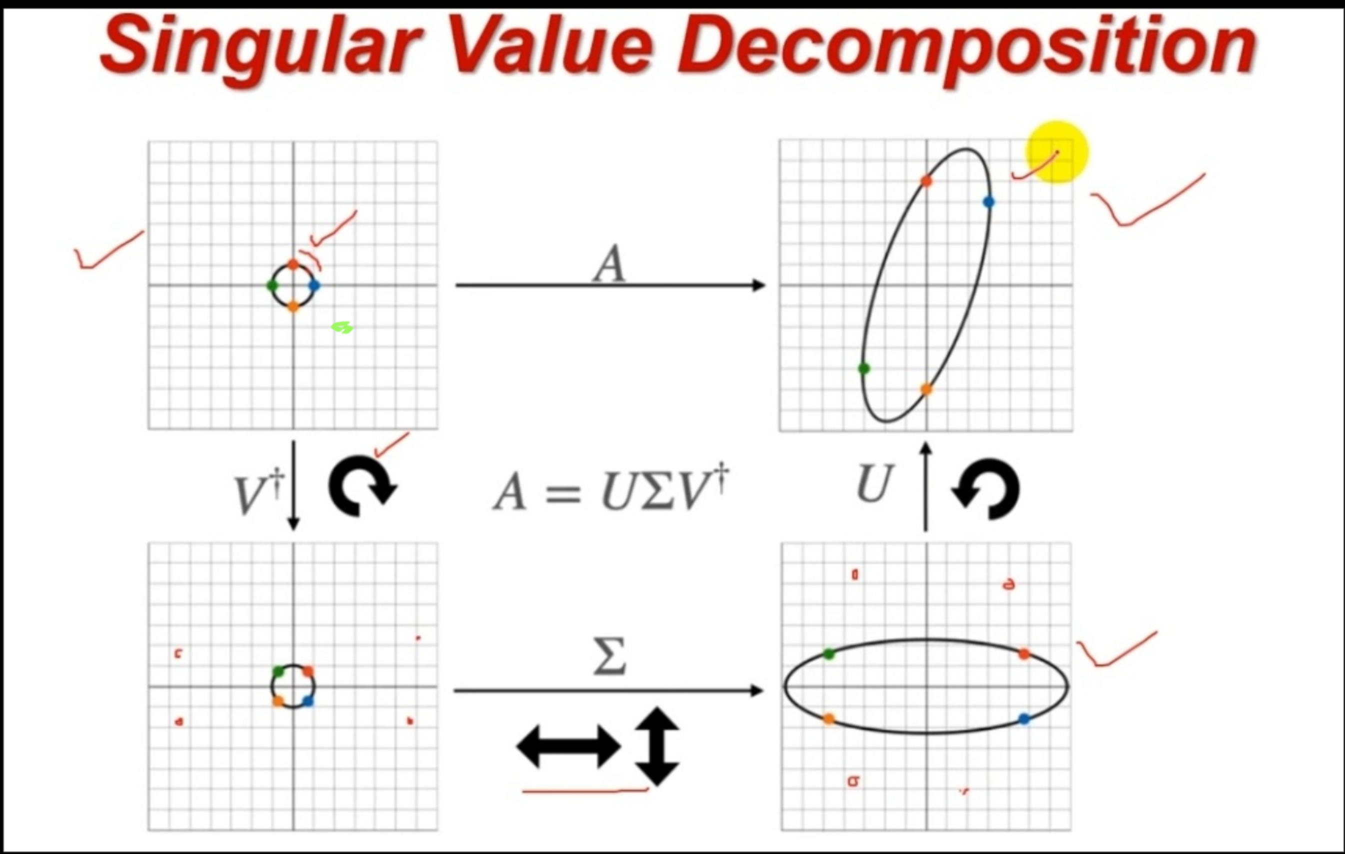
to convert

we need to stretch and Rotate

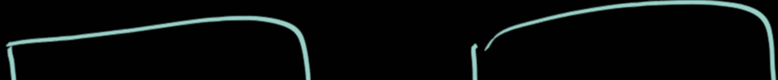


need to rotate first & then Stretch.





Singular Value Decomposition is used when we don't have rows & Columns same



$$C_{m\times n} = U_{m\times r} \times \Sigma_{r\times r} \times V_{r\times n}$$

$$A = UWV^T$$

where

U: $m\times n$ matrix of orthonormal eigenvectors of AA^T

V^T: transpose of a $n\times n$ matrix containing the orthonormal eigenvectors of $A^{\wedge}\{T\}A$.

W: a $n\times n$ diagonal matrix of the singular values which are the square roots of eigenvalues of A^TA

SVD of matrix $A = \begin{bmatrix} 1 & 1 \\ 7 & 7 \end{bmatrix}$

$A^T = \begin{bmatrix} 1 & 7 \\ \end{bmatrix}$

$$\begin{bmatrix} 1 & 7 \end{bmatrix}$$

Calculate $A \times A^T$

$$= \begin{bmatrix} 1 & 1 \\ 7 & 7 \end{bmatrix} \times \begin{bmatrix} 1 & 7 \\ 1 & 7 \end{bmatrix}$$

$$= \begin{bmatrix} 1\times1 + 1\times1 & 1\times7 + 1\times7 \\ 7\times1 + 7\times1 & 7\times7 + 7\times7 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 14 \\ 14 & 98 \end{bmatrix}$$

Calculate $A^T \times A$

$$= \begin{bmatrix} 1 & 7 \\ 1 & 7 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 7 & 7 \end{bmatrix}$$

$$= \begin{bmatrix} 50 & 50 \\ 50 & 50 \end{bmatrix}$$

To find the value of Left
Singular vector U,

Lets find the eigen value & eigenvector
of $A \times A^T$

$$AA^T - \lambda I = 0$$

$$\begin{bmatrix} 2-\lambda & 14 \\ 14 & 98-\lambda \end{bmatrix} = 0$$

$$(2-\lambda)(98-\lambda) - 14 \times 14 = 0$$

$$(2)(98) + (2)(-\lambda) + (-\lambda)(98) + (-\lambda)(-\lambda) - 196 = 0$$

$$196 - 2\lambda - 98\lambda + \lambda^2 - 196 = 0$$

$$\lambda^2 - 100\lambda = 0$$

take common

$$\lambda(\lambda - 100) = 0$$

eigenvalues

Hence $\boxed{\lambda = 0}$ or $\boxed{\lambda = 100}$

one time one time

Lets find the 1^{st} eigenvector corresponding to $\lambda = 0$ -

Lets call it $v = \begin{bmatrix} x \\ y \end{bmatrix}$

$$AA^T \cdot v = \lambda \cdot v$$

$$\begin{bmatrix} 2 & 14 \\ 14 & 98 \end{bmatrix} \times \begin{bmatrix} x \\ y \end{bmatrix} = 0 \times \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{bmatrix} 2x + 14y \\ 14x + 98y \end{bmatrix} = 0$$

taken any 1 equation

$$2x + 14y = 0$$

$$14y$$

$$2x = -14y$$

Cancel out 2

$$\frac{\cancel{2}x}{\cancel{2}} = -\frac{\overset{7}{\cancel{14}}y}{\cancel{2}}$$

$$x = -7y$$

What can be put to get $x = -7$?

if we put $y = 1$

then

$$x = -7(1)$$

$$\boxed{x = 7}$$

$$\boxed{y = 1}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -7 \\ 1 \end{bmatrix}$$

Eigenvector.

Now we need to convert this eigenvector into unit eigenvector

By dividing the eigenvector by its length.

How to calculate length?

$$\sqrt{(-7)^2 + (1)^2} = \sqrt{50}$$

Length $= 7.07$

we need to divide vector by length

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -7/7.07 \\ 1/7.07 \end{bmatrix}$$

$$= \begin{bmatrix} -0.99 \\ 0.14 \end{bmatrix}$$

Lets find the 2^{nd} eigenvector

Corresponding to $\lambda = 100$.

Lets call it $v = \begin{bmatrix} x \\ y \end{bmatrix}$

$$\begin{bmatrix} 2 & 14 \\ 14 & 98 \end{bmatrix} \times \begin{bmatrix} x \\ y \end{bmatrix} = 100 \times \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{bmatrix} 2x + 14y \\ 14x + 98y \end{bmatrix} = \begin{bmatrix} 100x \\ 100y \end{bmatrix}$$

$$\begin{bmatrix} -100x + 2x + 14y = 0 \\ 14x + 98 - 100y = 0 \end{bmatrix}$$

$$\begin{bmatrix} -98x + 14y \\ 14x - 2y \end{bmatrix} = 0$$

$$14x - 2y = 0$$

$$-2y = -14x$$

$$\frac{-\cancel{2}y}{-\cancel{2}} = \frac{-\cancel{14}^{7}x}{-\cancel{2}}$$

$$y = 7x$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 7 \end{bmatrix}$$

$$\text{Length} = \sqrt{(1)^2 + (7)^2} = \sqrt{50}$$

$$= 7.07$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1/7.07 \\ 7/7.07 \end{bmatrix}$$

$$\text{Eigen vector} = \begin{bmatrix} 0.14 \\ 0.99 \end{bmatrix}$$

Sort the eigen vector according to eigen values.

$$\begin{array}{ccc} & 100 & 0 \\ u = & \begin{bmatrix} 0.14 & -0.99 \\ 0.99 & 0.14 \end{bmatrix} \end{array}$$

To find the value of right singular vector V^T

$$A^T \times A - \lambda I = 0$$

Same process we get V^T

$$\begin{bmatrix} 0.70 & -0.70 \\ 0.70 & 0.70 \end{bmatrix}$$

Σ (sigma) Eigen value

$$= \begin{bmatrix} \sqrt{100} & 0 \\ 0 & \sqrt{0} \end{bmatrix}$$

$$\Sigma = \begin{bmatrix} 10 & 0 \\ 0 & 0 \end{bmatrix}$$

$$U \Sigma V^T = \begin{bmatrix} 0.14 & -0.99 \\ 0.99 & 0.14 \end{bmatrix} \times \begin{bmatrix} 10 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\times \begin{bmatrix} 0.70 & -0.70 \\ 0.70 & 0.70 \end{bmatrix}$$

$$= \begin{bmatrix} 0.98 & -0.98 \\ 6.93 & -6.93 \end{bmatrix}$$