

Welcome Class  
to the course

*Computer Vision*



Course Instructor  
Dr. Harkeerat Kaur

# The History of Vision

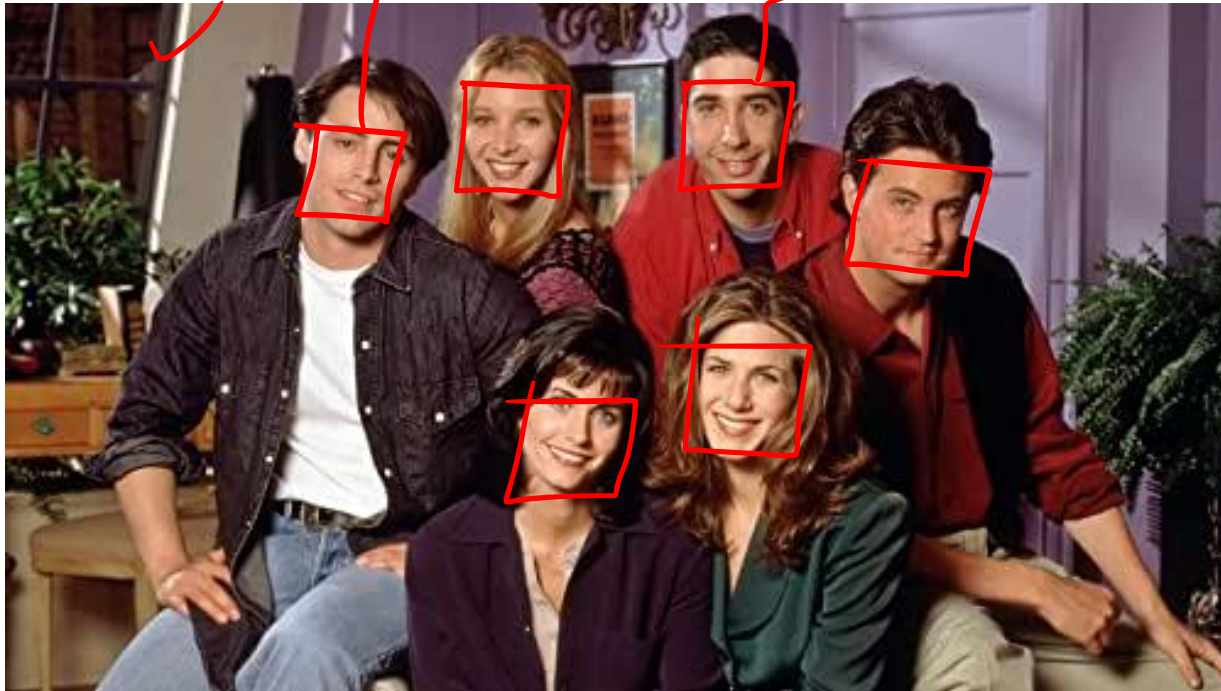
- <https://www.youtube.com/watch?v=2W5hOJaFjxU>

# What is computer vision



**Figure 1.1** The human visual system has no problem interpreting the subtle variations in translucency and shading in this photograph and correctly segmenting the object from its background.

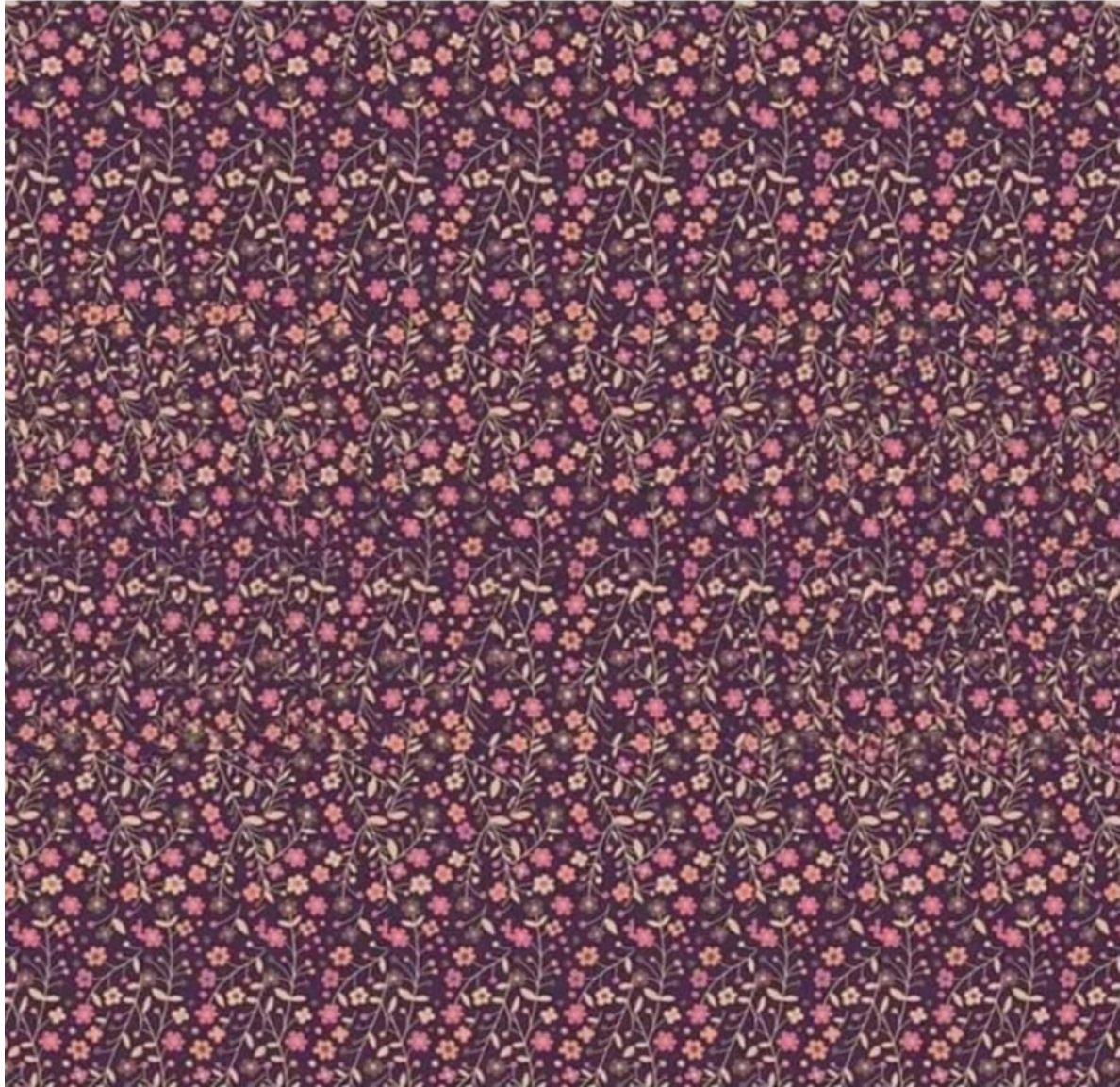
# What is computer vision



*face detection & re-  
cognition of  
males & females*

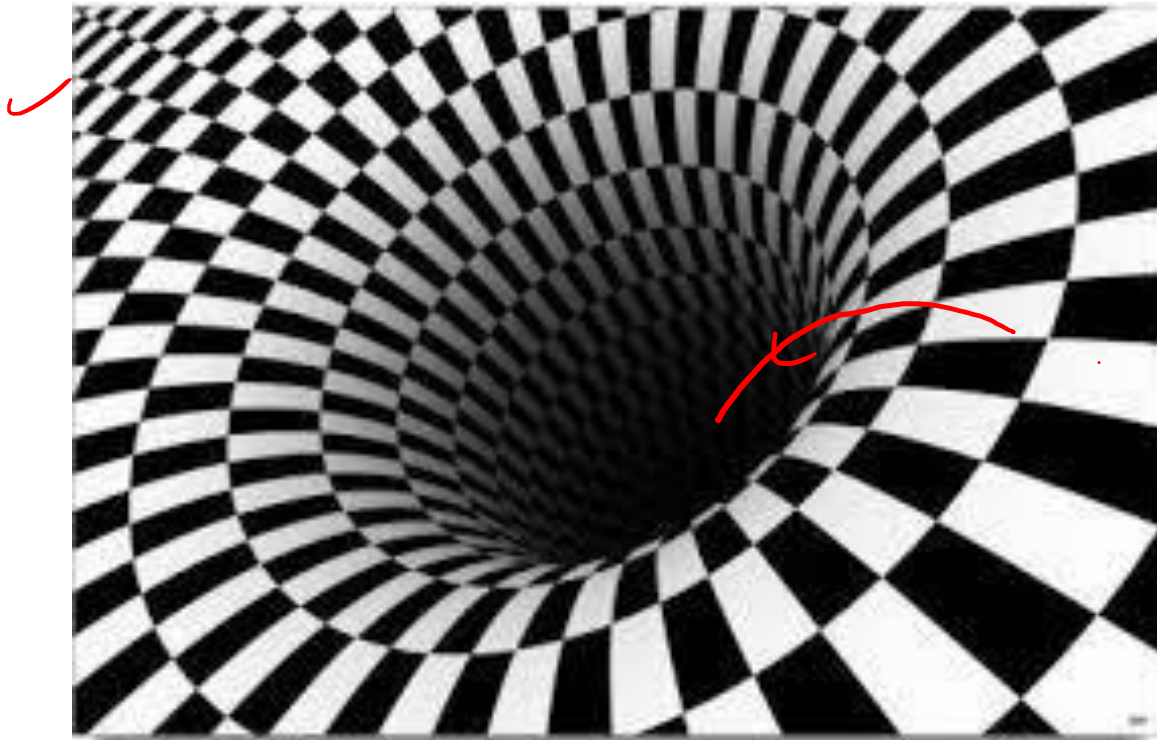


**Put your nose to the center of the screen.  
Focus on the center of the screen for 10 seconds.  
Then pull away from the screen, slowly.**



# What is computer vision

2D  $\rightarrow$  effect of vision 3D

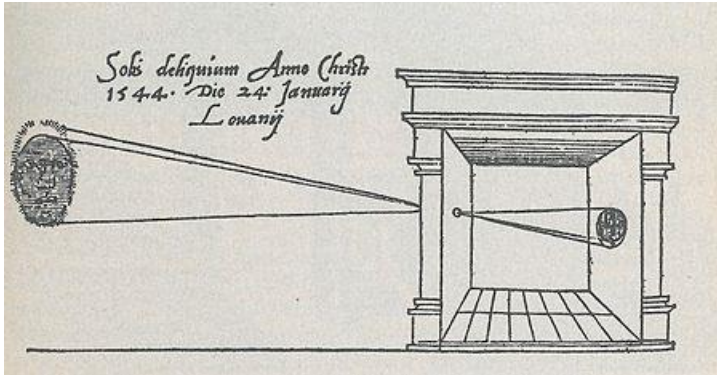




# Picture is worth a thousand words

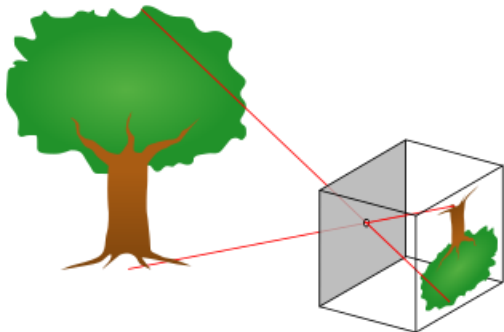
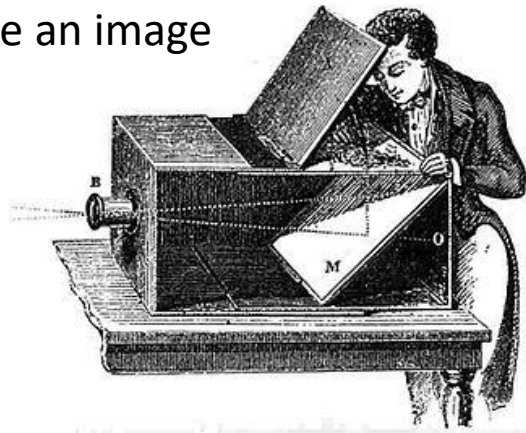


# History of computer vision



First published picture of a [camera obscura](#) in Gemma Frisius' 1545 book *De Radio Astronomica et Geometrica*

An artist using an 18th-century camera obscura to trace an image



[Pinhole camera](#). Light enters a dark box through a small hole and creates an inverted image on the wall opposite the hole. [\[1\]](#)



# History of computer vision



[View from the Window at Le Gras](#) (1825), the earliest surviving photograph



Late 19th-century studio camera

# History of computer vision

Around 1990s.....



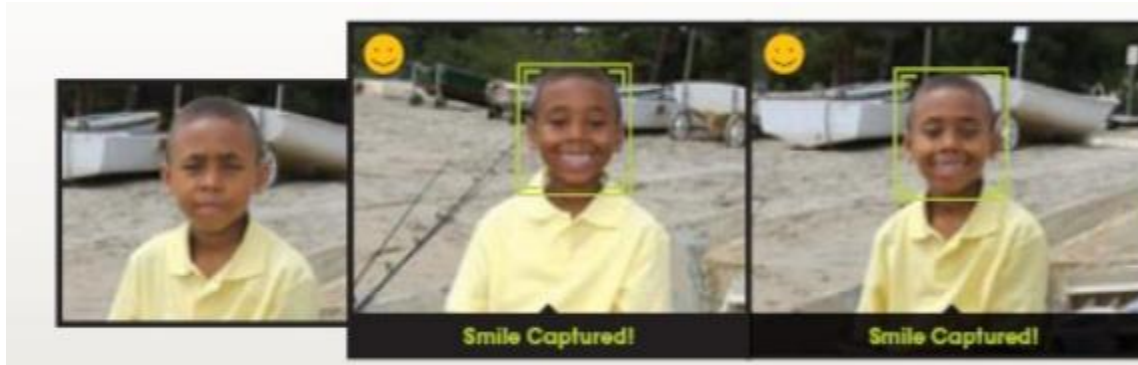
# Evolution of computer vision



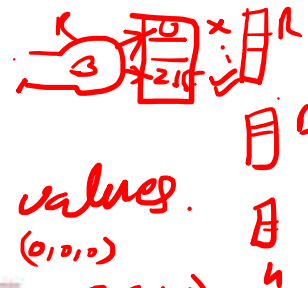
Voila Jones Face  
detector 2001

Many new digital cameras now detect faces  
– Canon, Sony, Fuji, ...

# Evolution of computer vision

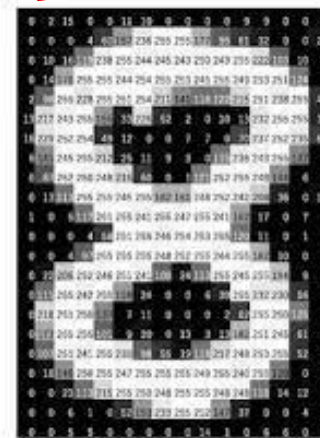
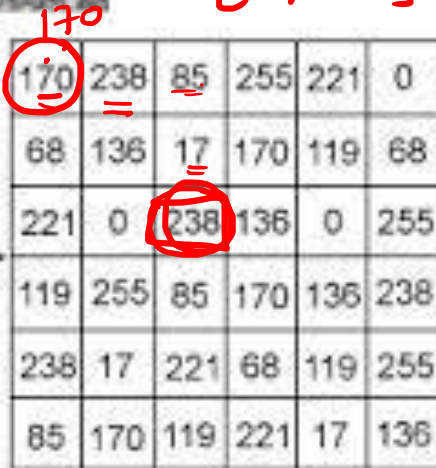






$(0, 0, 0)$   
 $(2, 2, 2)$   
 $(1, 1, 1)$   
 $(3, 3, 3)$   
 $(4, 4, 4)$   
 $(5, 5, 5)$   
 $(6, 6, 6)$   
 $(7, 7, 7)$   
 $(8, 8, 8)$   
 $(9, 9, 9)$

(-greyscale values  
[0, 255] -



# The goal of computer vision

- Design algorithms to bridge the gap between pixels and “meaning”

2D - value



La Gare Montparnasse, 1895

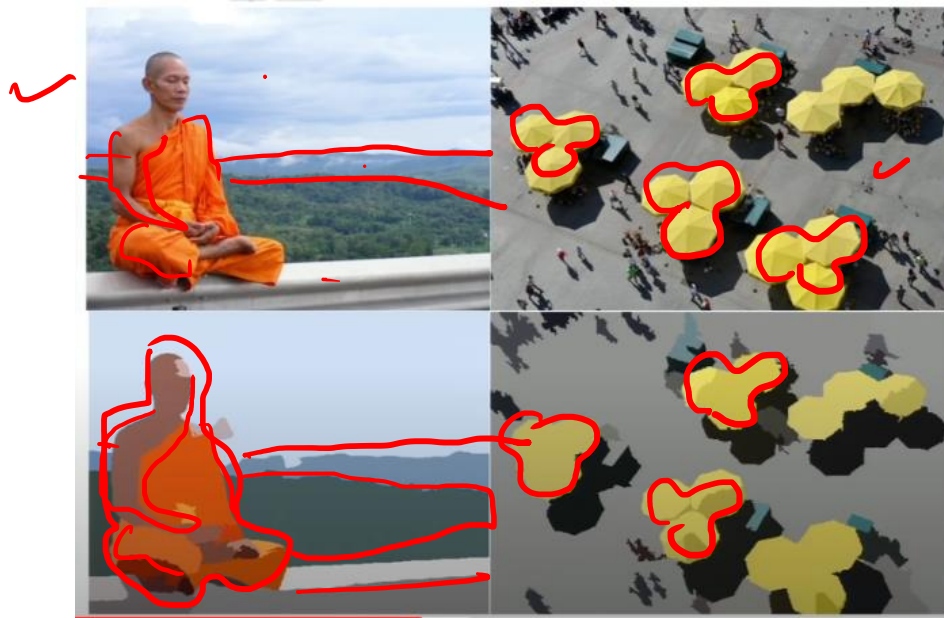
What we see

6	3	2	5	4	7	6	9	8
3	0	1	2	3	4	5	6	7
2	1	0	3	2	5	4	7	6
5	2	3	0	1	2	3	4	5
4	3	2	1	0	3	2	5	4
7	4	5	2	3	0	1	2	3
6	5	4	3	2	1	0	3	2
9	6	7	4	5	2	3	0	1
8	7	6	5	4	3	2	1	0

What a computer sees

# How do we interpret from images

- Derive meaning full information from intensity values



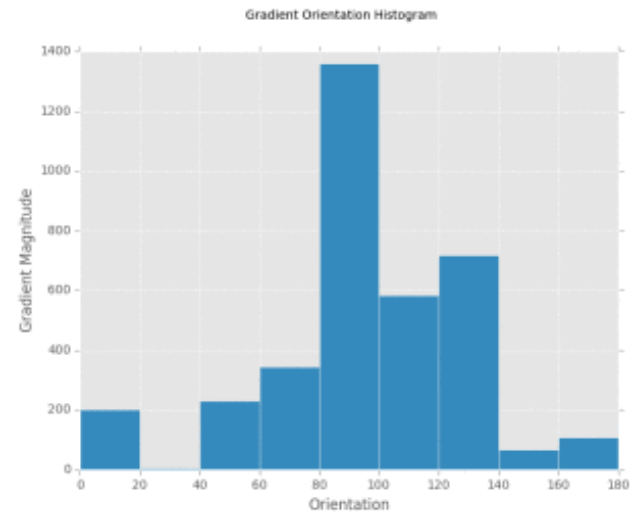
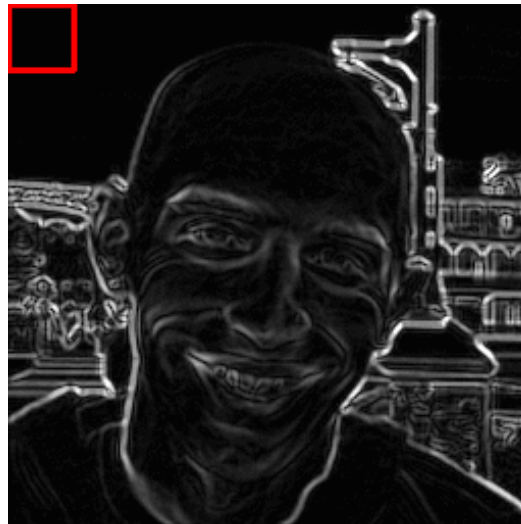
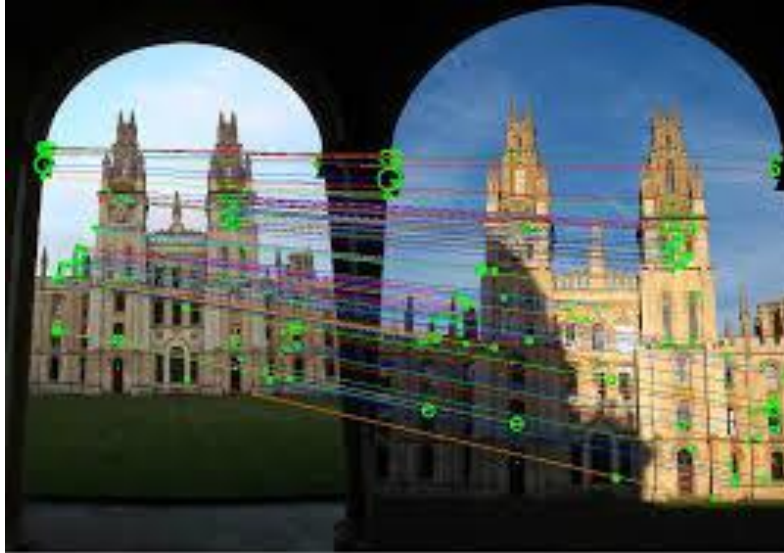
Segmentation.

# Learning Motion from Images





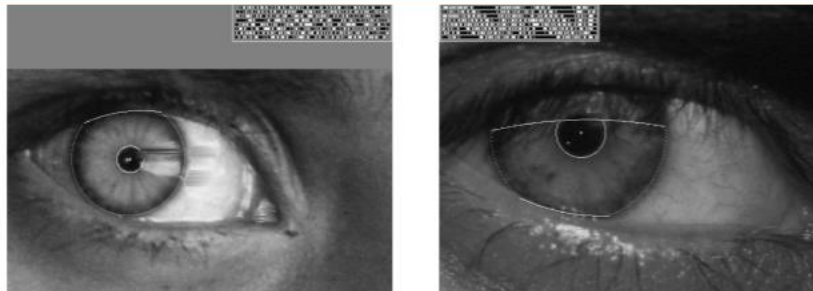
# Feature Extraction



# Biometrics



## How the Afghan Girl was Identified by Her Iris Patterns



Source: S. Seitz

# Convolution Neural networks



- CNNL+RHN : a black and white cat looking at itself in a mirror
- CNNL+RNN : a black and white cat sitting in front of a mirror
- GRU : a black and white cat standing next to a mirror
- LSTM : a black and white cat sitting in a bathroom sink
- RNN : a cat sitting on the floor in a bathroom

- there is a black tuxedo cat looking in the mirror
- two cats sitting on top of a wooden floor
- a cat looking at itself in the mirror next to a tripod
- a cat and a tripod sitting in front of a mirror
- a close up of a cat in a mirror



- CNNL+RHN : a man standing next to a child on a snow covered slope
- CNNL+RNN : a man and a woman standing on a snow covered slope
- GRU : a man and a child standing on a snow covered slope
- LSTM : a man and a child are standing in the snow
- RNN : a man and a woman are skiing on the snow

- a woman and child in ski gear next to a lodge
- a man and a child are smiling while standing on skis
- a young man poses with a little kid in the snow
- an adult and a small child dressed for skiing
- a man and a little girl in skis stand in front of a mountain lodge

Joining the dots.....



# The contents

Course Contents:	Topics	No. of Hours
1	<u>Introduction</u> and Digital Image Fundamentals	3
2	Image <u>Enhancement</u> and <u>Restoration</u>	5
3	Image Enhancement in the <u>Frequency domain</u>	5
4	Image <u>Segmentation</u> , Representation and Description	3
5	Harris Detector, <u>Sift</u> , <u>Point</u> Matching, Ransac, Local Binary Pattern	3
6	Projective <u>geometry</u> for Computer Vision:	3

# The contents

Course Contents:	Topics	No. of Hours
7	Multiple views geometry: <i>stew</i>	3
8	Simultaneous Localization and Mapping	3
9	Optical Flow & Motion Analysis	3
10	3D reconstruction with a calibrated camera	3
11	Object Tracking	3
12	Object Recognition	3
13	Advanced Topics	2

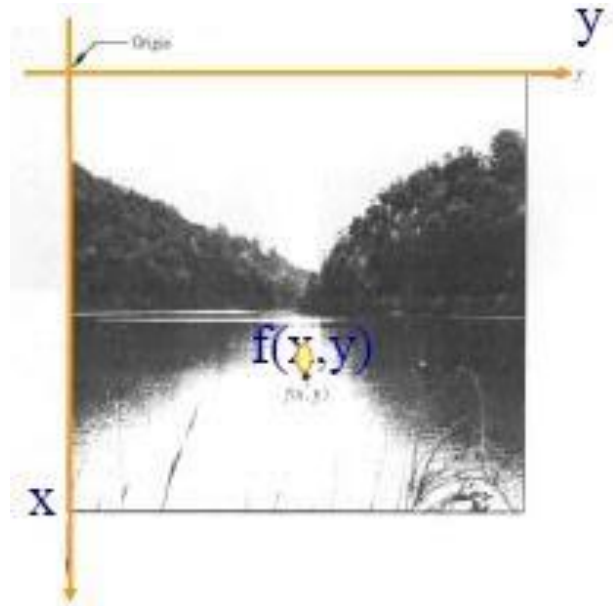
Text Book:	<ol style="list-style-type: none"> <li>1. Forsyth &amp; Ponce, <u>Computer Vision: A Modern Approach</u>, Pearson, 2002, ISBN 0130851981</li> <li>2. <u>Gonzalez R. C., Woods R. E., Digital Image Processing</u>, Pearson Education (2007).</li> </ol>
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70  
CT  
(ms | 20) ( 56.1 )  
end sem)

# Image Processing Fundamentals

# Digital Image Representation

- A digital image is an image  $f(x,y)$  that has been digitized both in spatial coordinates and brightness
- The value of  $f$  at any point  $(x,y)$  is proportional to the brightness (or gray level) of the image at that point



Pixel values in highlighted region

99	71	61	51	49	40	35	53	86	99
93	74	53	56	48	46	48	72	85	102
101	69	57	53	54	52	64	82	88	101
107	82	64	63	59	60	81	90	99	100
114	93	76	60	72	85	91	99	95	99
117	108	94	92	97	101	100	108	105	99
116	114	109	106	105	108	108	102	107	110
115	113	109	114	111	111	113	108	111	115
110	113	111	109	106	108	110	115	120	122
103	107	106	108	109	114	120	124	124	132

CAMERA



DIGITIZER



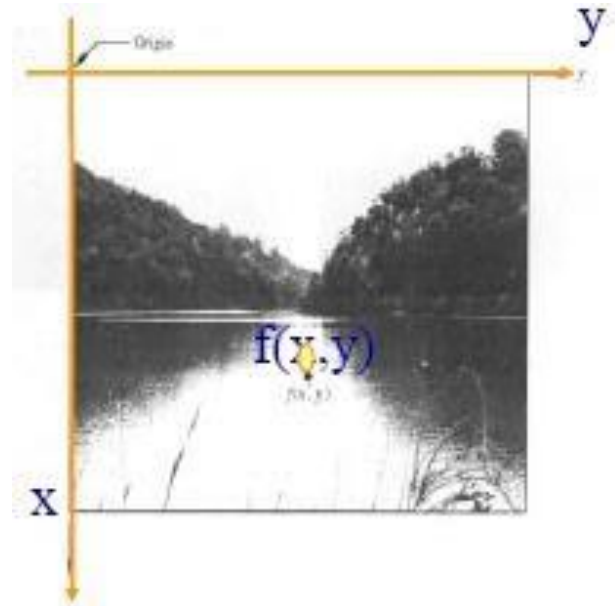
A set of number  
in 2D grid

Samples the analog data and digitizes it.



# Digital Image Representation

A digital image is a representation of a two dimensional signal as a finite set of digital values, called picture elements or pixels



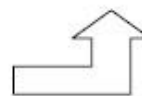
Pixel values in highlighted region

99	71	61	51	49	40	35	53	86	99
93	74	53	56	48	46	48	72	85	102
101	69	57	53	54	52	64	82	88	101
107	82	64	63	59	60	81	90	99	100
114	93	76	60	72	85	91	99	95	99
117	108	94	92	97	101	100	108	105	99
116	114	109	106	105	108	108	102	107	110
115	113	109	114	111	111	113	108	111	115
110	113	111	109	106	108	110	115	120	122
103	107	106	108	109	114	120	124	124	132

CAMERA

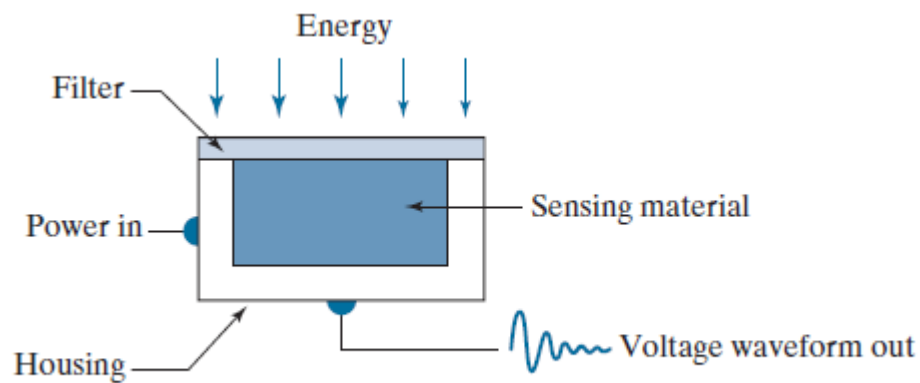
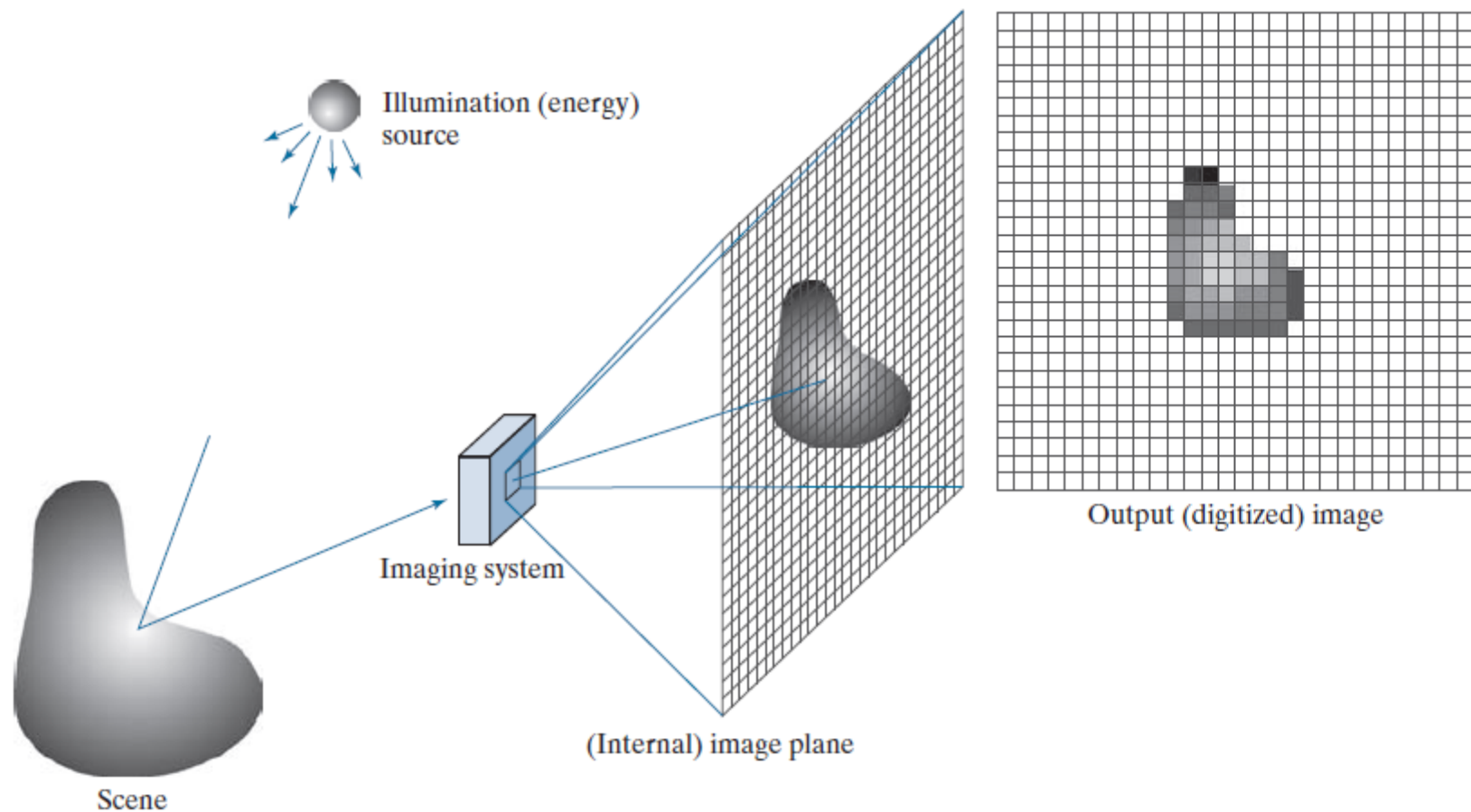


DIGITIZER



A set of number  
in 2D grid

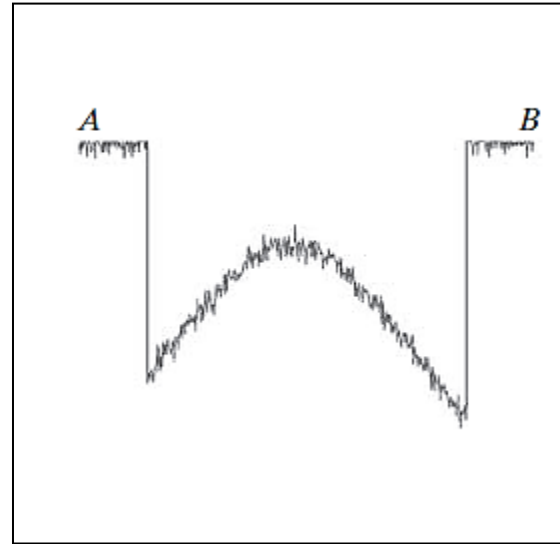
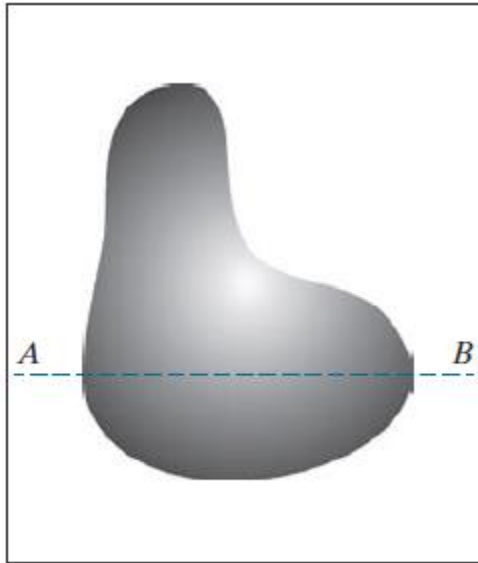
Samples the analog data and digitizes it.



# Image Sampling and Quantization

- The output of most sensors is a continuous voltage waveform whose amplitude and spatial behavior are related to the physical phenomenon being sensed.
- To create a digital image, we need to convert the continuous sensed data into a digital format:
  - **Sampling:** digitizing the coordinate values
  - **Quantization:** digitizing the amplitude values
    - Usually thought of as the number of bits per sample of the signal (e.g., 1 bit/pixel b/w images, 8 bit/pixel grayscale images 16-bit audio, 24-bit color images etc.)

# Image Sampling and Quantization



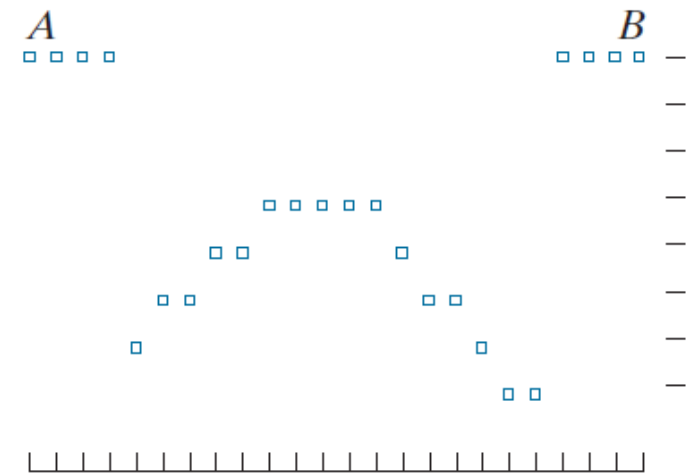
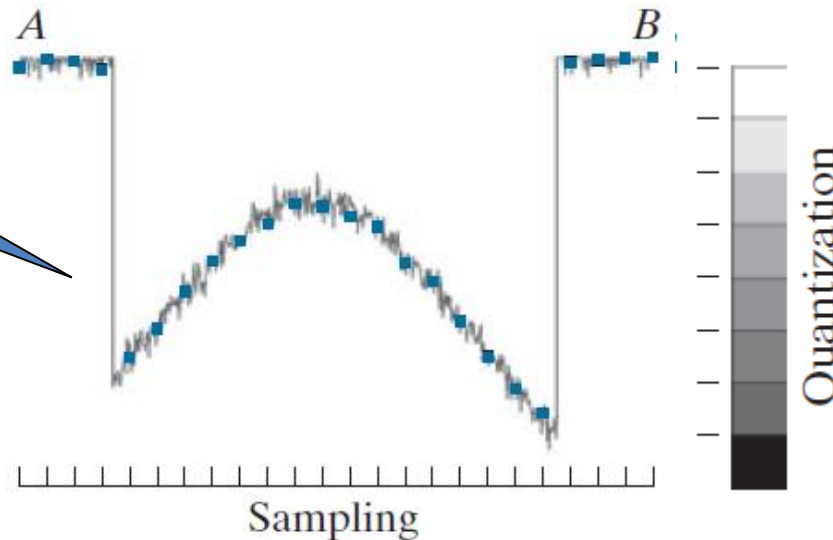
a	b
c	d

**FIGURE 2.16**

Generating a digital image.

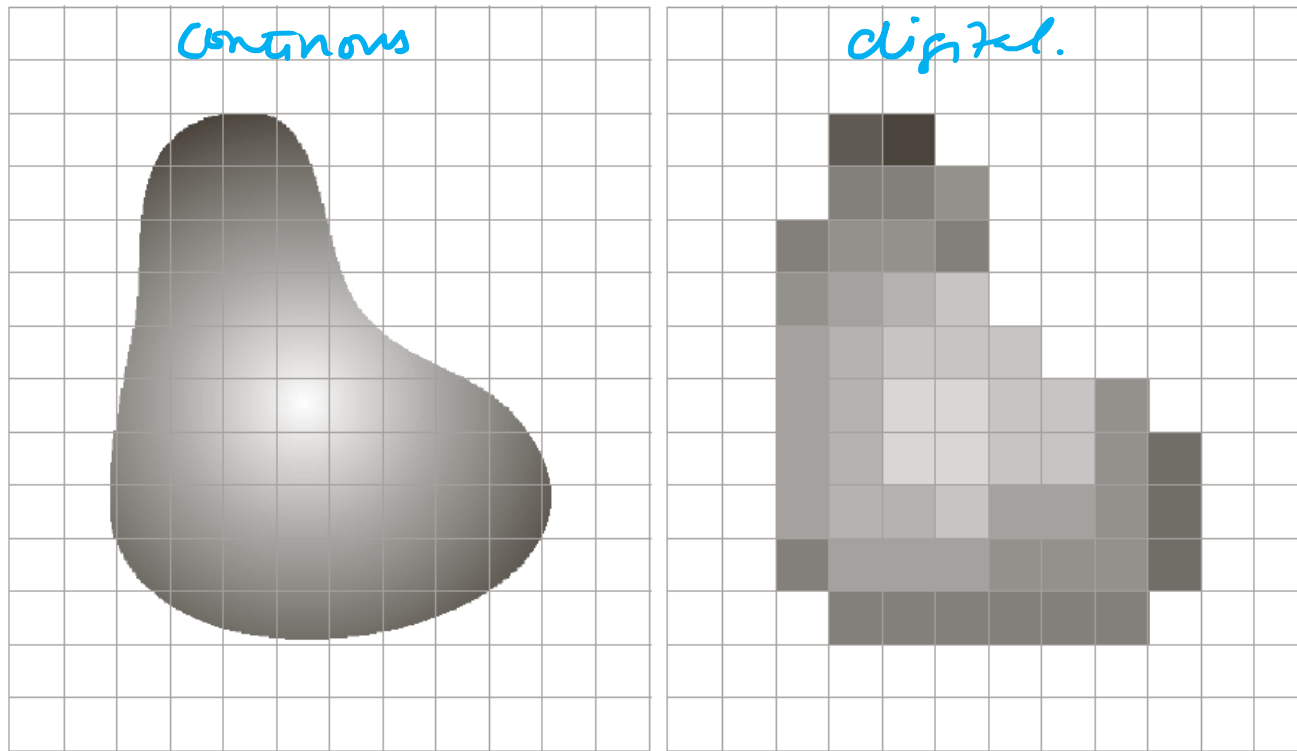
(a) Continuous image. (b) A scan line from *A* to *B* in the continuous image, used to illustrate the concepts of sampling and quantization. (c) Sampled and quantized scan line.

Digitizing the amplitude



Digitizing the coordinate values

# Image Sampling and Quantization



a b

**FIGURE 2.17** (a) Continuous image projected onto a sensor array. (b) Result of image sampling and quantization.

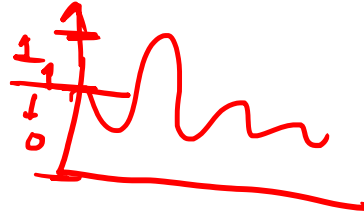


# The uniform quantizer

- The input brightness range  $u \in [Lmin, Lmax]$
- No. of bits of the quantizer =  $B$
- Reconstruction levels =  $L = 2^b$
- The expressions of the decision levels:
- $T_1 = Lmin, T_{L+1} = Lmax$
- $t_k - t_{k-1} = q = (Lmax - Lmin) / L$
- Computation of the quantization error:  
for a given image of size  $M \times N$  pixels,
  - $U$  – non-quantized, and  $U'$  – quantized
  - estimate the MSE:  $err = \frac{1}{MN} \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} (u(m, n) - u'(m, n))^2$

# Results

- B=1  $\Rightarrow$   $L=2$   <sup>$2^1=2$</sup>



Non-quantized image

$\hookrightarrow u(m,n)$



Quantized image

$\hookrightarrow$  2 bits per pixel  
 $u'(m,n)$



Quantization error; MSE=36.2

$\sum_{i=0}^M \sum_{j=0}^N (u(m,n) - u'(m,n))^2$

# Some Results

- bits per pixel
- $\underline{B=2} \Rightarrow L=4$

$(m,n) \rightarrow \text{pixel}$

$(0,0)$



Non-quantized image

$(u)$

$u(m,n)$   
0 0

$(0,0)$



Quantized image

$(u')$

$u'(m,n)$   
0 0



Quantization error; MSE=15

$$\frac{1}{(m \times n)} \sum \left( u'(m,n) - u(m,n) \right)^2$$

# Some Basics

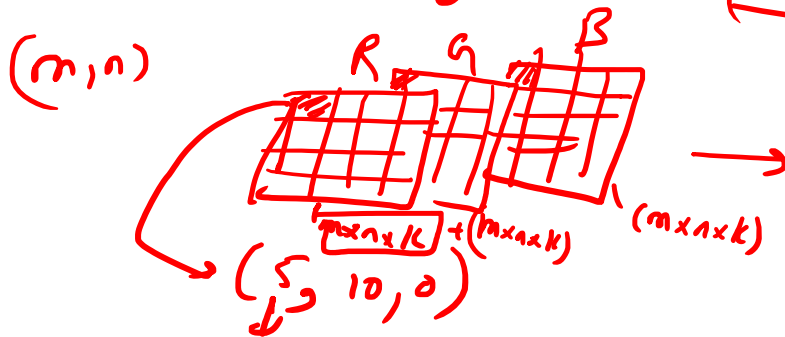
- ✓ • Dynamic Range:
- ✓ • Image Contrast:
- ✓ • Contrast Ratio:
  - appreciable number of pixels in image have a high dynamic range -> high contrast.
  - Image low dynamic range -> dull, low contrast

# Some Basics

- No. of bits required to store an image  $\times 3$
- Size: M x N
- Intensity levels =  $2^K$  ?  $\rightarrow$  16-bits per pixel



Size of the image =  $m \times n \times k \times 3$  - bits.





# Spatial and Intensity Resolution

- Spatial resolution
- Intensity resolution

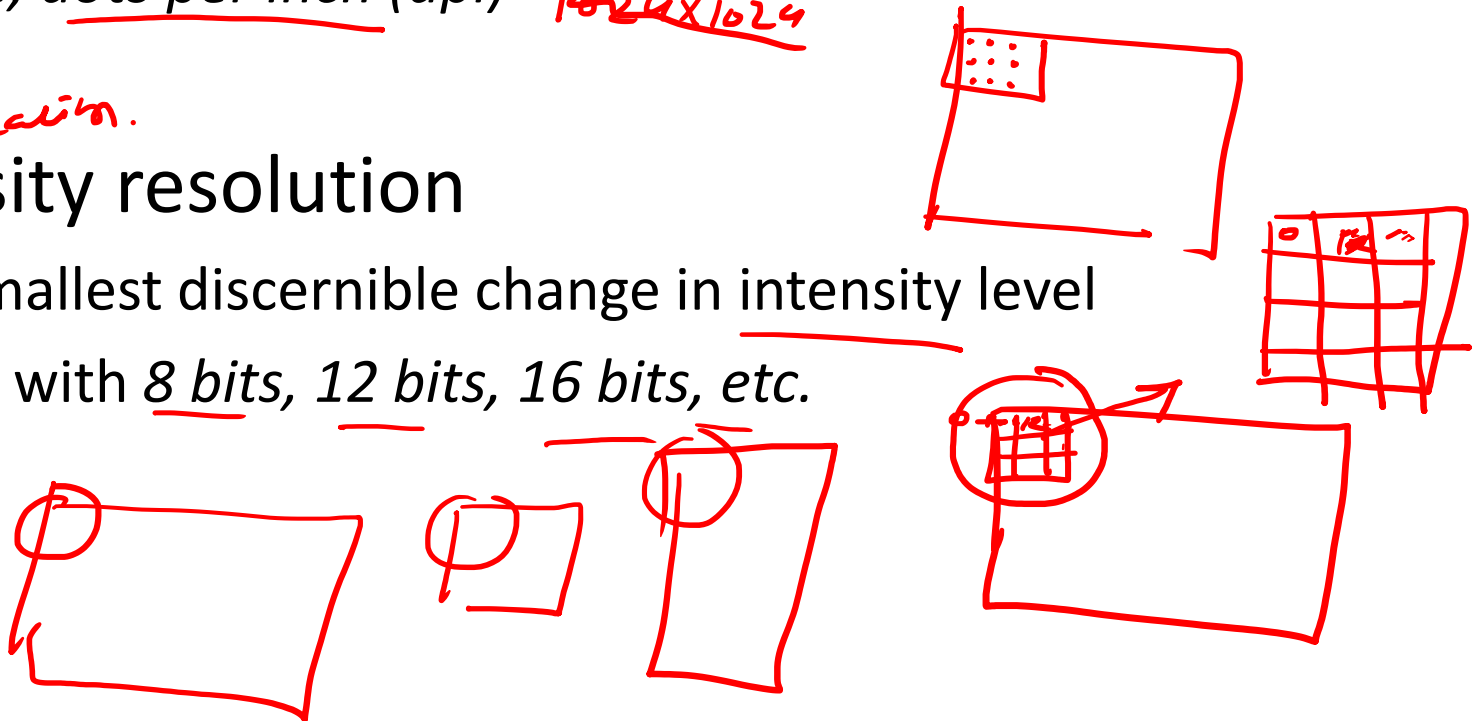
Image  $\rightarrow$  Interpolation  $\rightarrow$  Tex  $\rightarrow$  NN  
B-linear  
B-bilinear

# Spatial and Intensity Resolution

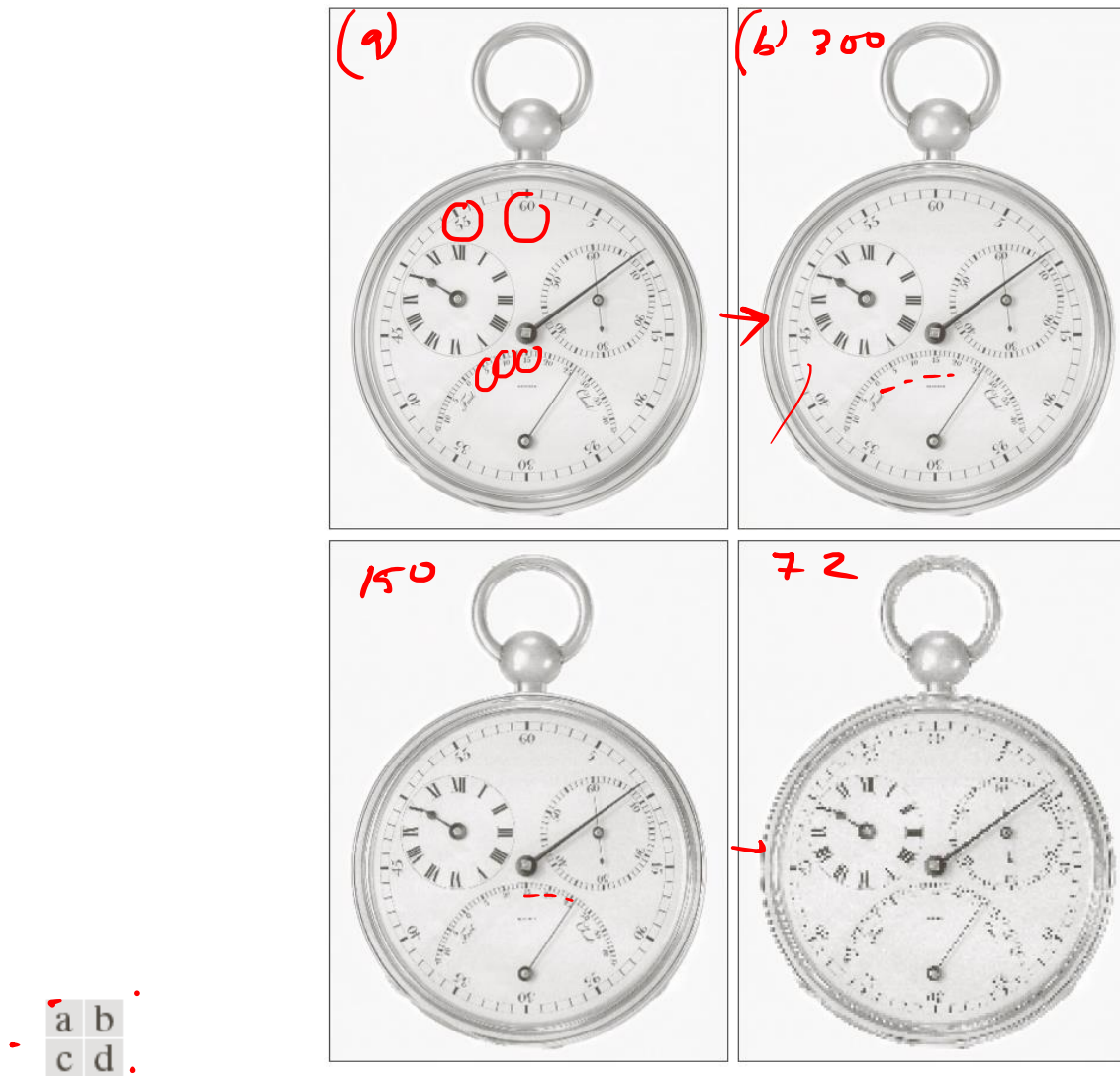
- $\overset{\text{Sampling}}{=}$  Spatial resolution  $\overset{\text{Sampling} \rightarrow \text{per unit-inch}}{=}$ 
  - A measure of the smallest discernible detail in an image
  - stated with line pairs per unit distance, dots (pixels) per unit distance, dots per inch (dpi)

~~1024x1024~~

- $\overset{\text{Quantization}}{=}$  Intensity resolution
  - The smallest discernible change in intensity level
  - stated with 8 bits, 12 bits, 16 bits, etc.

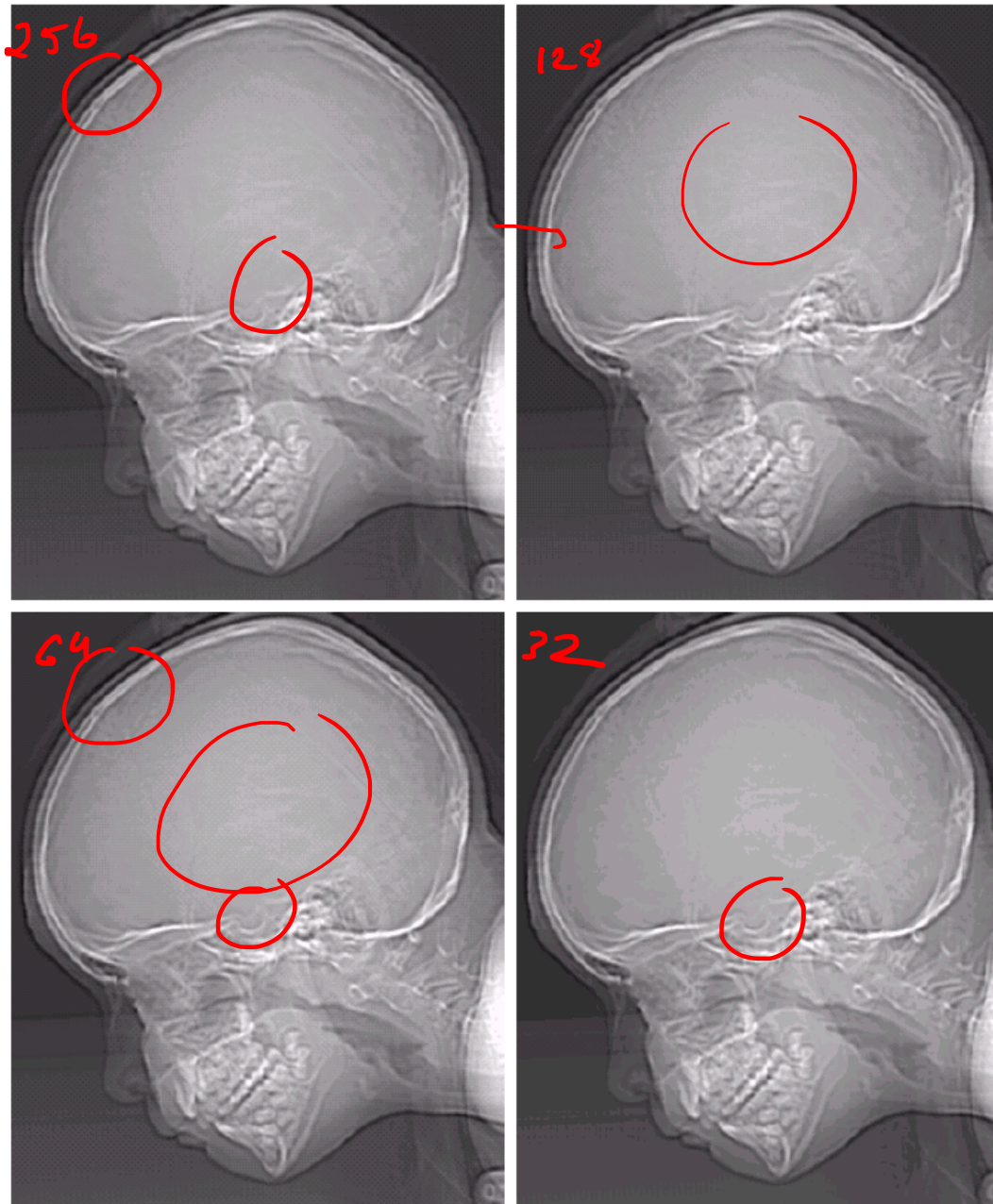


# Spatial and Intensity Resolution



**FIGURE 2.20** Typical effects of reducing spatial resolution. Images shown at: (a) 1250 dpi, (b) 300 dpi, (c) 150 dpi, and (d) 72 dpi. The thin black borders were added for clarity. They are not part of the data.

# Spatial and Intensity Resolution



a	b
c	d

**FIGURE 2.21**

(a)  $452 \times 374$ , 256-level image. (b)–(d) Image displayed in 128, 64, and 32 gray levels, while keeping the spatial resolution constant.

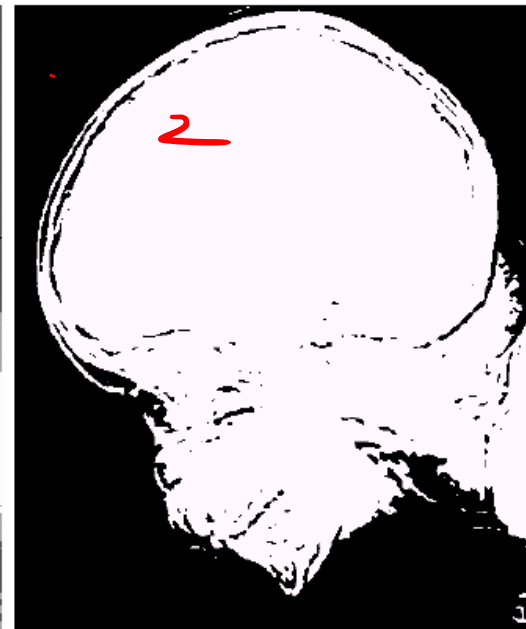
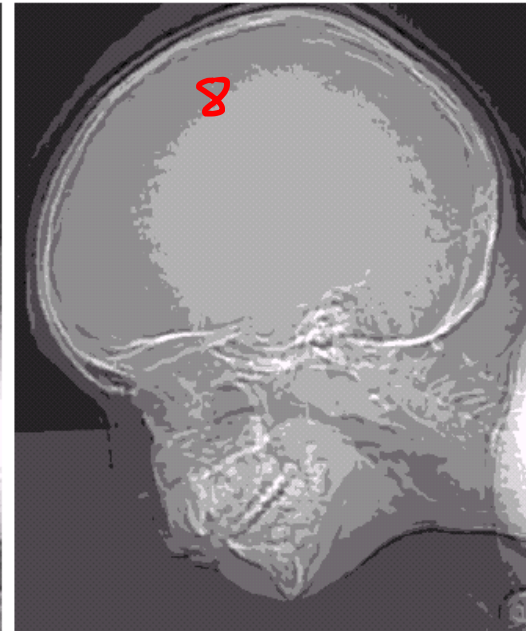
# Spatial and Intensity Resolution

e f  
g h

**FIGURE 2.21**

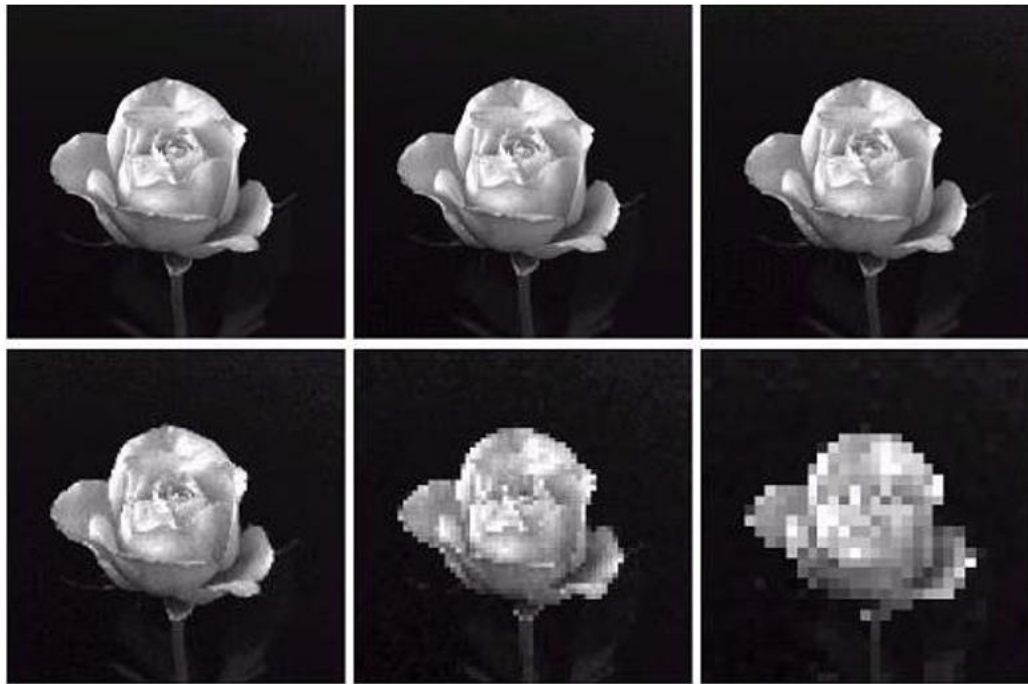
*(Continued)*

(e)–(h) Image displayed in 16, 8, 4, and 2 gray levels. (Original courtesy of Dr. David R. Pickens, Department of Radiology & Radiological Sciences, Vanderbilt University Medical Center.)





# Checkerboard Effect-Spatial Resolution



a	b	c
d	e	f

(a) 1024x1024

(b) 512x512

(c) 256x256

(d) 128x128

(e) 64x64

(f) 32x32

All images are of 8 bits

- If the resolution is decreased too much, the checkerboard effect can occur

# False Contouring –Intensity Resolution



(a)



(b)



(c)



(d)

(a) Gray level = 16

(b) Gray level = 8

(c) Gray level = 4

(d) Gray level = 2

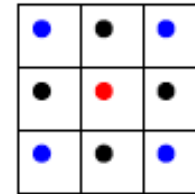
- If the gray scale is not enough, the smooth area will be effected
- False contouring can occur on the smooth area which has fine gray scales

# Basic Relationship b/w Pixels

- Neighbors of a pixel
- Adjacency, and its types
- What is meant by Connectivity
- How to define Path, Boundary
- Region
- Labeling of connected components
- Distance Measures

# Basic Relationship b/w Pixels

- Neighbors of a Pixel: A pixel  $p$  at coordinate  $(x,y)$  has
  - $N_4(p)$ : 4 neighbors of  $p$ 
    - $(x-1,y)$ ,  $(x+1,y)$ ,  $(x,y-1)$ ,  $(x,y+1)$
  - $N_D(p)$ : 4 diagonal neighbors of  $p$ 
    - $(x+1,y+1)$ ,  $(x+1,y-1)$ ,  $(x-1,y+1)$ ,  $(x-1,y-1)$
  - $N_8(p)$ : 8 neighbors of  $p$ 
    - a combination of  $N_4(p)$  and  $N_D(p)$



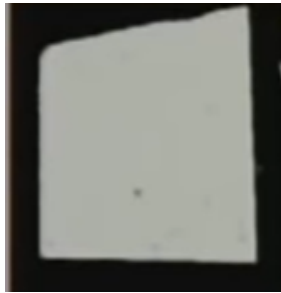
$x$   
 $x$   $p$   $x$   
 $x$

$x$       $x$   
            $p$   
 $x$       $x$

$x$     $x$     $x$   
 $x$     $p$     $x$   
 $x$     $x$     $x$

# Connectivity

- Useful for define object boundaries
- Image components or regions



If  $f(x,y) > Th$   
 $\Rightarrow f(x,y)$  is object or fore ground  
Else  
 $f(x,y)$  is Background

- Two pixels are connected if they are adjacent in some sense.
- Adjacent
  - A pixel **p** is adjacent to a pixel **q** if they are connected



# Connectivity

- Two pixels are connected iff
  - They are neighbors  $N_4$ ,  $N_8$ ,  $N_D$
  - They have similar intensity values
- Let  $V$  be the set of gray-level values used to defined connectivity
  - 4-connectivity: 2 pixels  $p$  and  $q$  with values from  $V$  are 4-connected if  $q$  is in the set  $N_4(p)$
  - 8-connectivity: 2 pixels  $p$  and  $q$  with values from  $V$  are 8-connected if  $q$  is in the set  $N_8(p)$

# Connectivity

- m-connectivity (mixed connectivity)
  - 2 pixels  $p$  and  $q$  having value belonging to set  $V\{\}$  are m-connected if
    - $q$  is in the set  $N_4(p)$  or
    - $q$  is in the set  $N_D(p)$  and the set  $N_4(p) \cap N_4(q)$  has no pixels whose values are from  $V$
    - (the set of pixels that are 4-neighbors of both  $p$  and  $q$  whose values are from  $V$ )

• Example:

0	1	1
0	1	0
0	0	1

Pixel arrangement  
for  $V = \{1\}$

0	1	1
0	1	0
0	0	1

Pixels that are 8-  
adjacent to the  
center pixel  
(shown dashed)

0	1	1
0	1	0
0	0	1

m-adjacency

m-connectivity eliminates the multiple path connections that arise in 8-connectivity

# Path

- Path

- A path from pixel **p** with coordinates  $(x,y)$  to pixel **q** with coordinates  $(s,t)$  is a sequence of distinct pixels with coordinates  $(x_0, y_0), (x_1, y_1), \dots, (x_n, y_n)$  where  $(x_0, y_0) = (x, y)$ ,  $(x_n, y_n) = (s, t)$  and pixels  $(x_i, y_i), (x_{i-1}, y_{i-1})$  are adjacent for  $1 \leq i \leq n$
- $n$  is the length of the path
- We can define 4-, 8-, or  $m$ -paths depending on type of adjacency specified

# Adjacency

- Two pixels  $p, q$  are adjacent iff
  - 4 connected
  - 8 connected
  - M connected
- Two image area subsets  $S1$  and  $S2$  are adjacent if some pixel in  $S1$  is adjacent to some pixel  $S2$

# Examples: Adjacency and Path

$$V = \{1, 2\}$$

$0_{1,1}$   $1_{1,2}$   $1_{1,3}$

$0_{2,1}$   $2_{2,2}$   $0_{2,3}$

$0_{3,1}$   $0_{3,2}$   $1_{3,3}$

0 1 1

0 2 0

0 0 1

**8-adjacent**

0 1 1

0 2 0

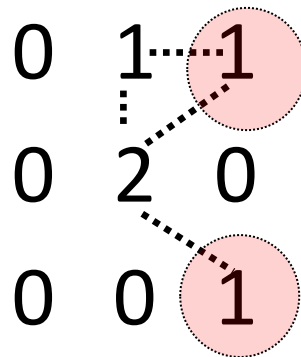
0 0 1

**m-adjacent**

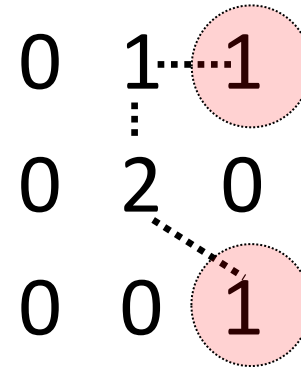
# Examples: Adjacency and Path

$$V = \{1, 2\}$$

0 <sub>1,1</sub>	1 <sub>1,2</sub>	1 <sub>1,3</sub>
0 <sub>2,1</sub>	2 <sub>2,2</sub>	0 <sub>2,3</sub>
0 <sub>3,1</sub>	0 <sub>3,2</sub>	1 <sub>3,3</sub>



**8-adjacent**



**m-adjacent**

The 8-path from (1,3) to (3,3):

- (i) (1,3), (1,2), (2,2), (3,3)
- (ii) (1,3), (2,2), (3,3)

The m-path from (1,3) to (3,3):

(1,3), (1,2), (2,2), (3,3)

m-connectivity eliminates the multiple path connections that arise in 8-connectivity



# Basic Relationship b/w Pixels



- Connected Components

- S: a subset of pixels in an image
- *Two pixels  $p$  and  $q$  are said to be connected in S* if there exists a path between them consisting entirely of pixels in S
- For any pixel  $p$  in S, the set of pixels that are connected to it in S is called a connected component of S

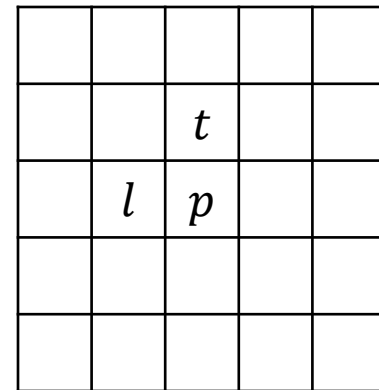
- Region

- R: a subset of pixels in an image
- R is called a region if every pixel in R is connected to any other pixel in R
- Boundary (border or contour) of a region: set of pixels in the region that have one or more neighbors that are not in R

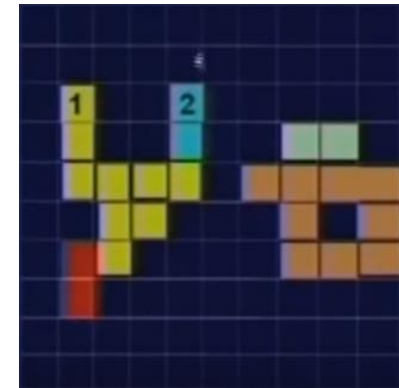
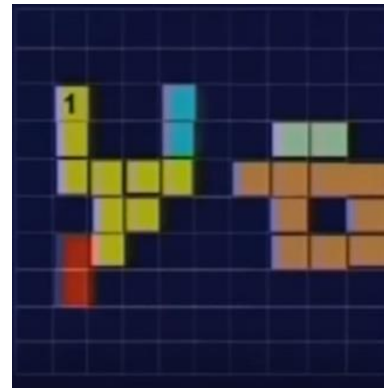
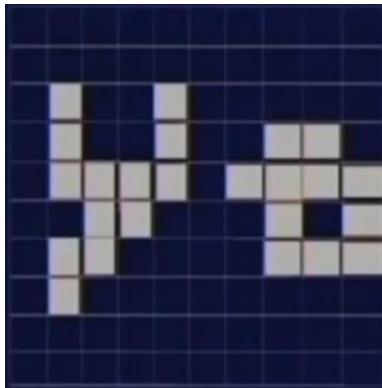


# Labeling of Connected Components

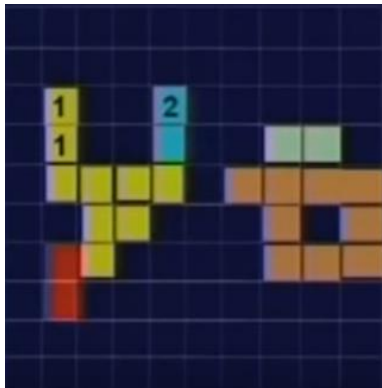
- Scan the image from left to right, assume 4 -connectivity
- Let  $p$  denote the pixel at any step in the scanning process
- Let  $t$  denote the upper neighbor of  $p$
- Let  $l$  denote the left-hand neighbors of  $p$ , respectively
- When we get to  $p$ , points  $t$  and  $l$  have already been encountered and labeled if they were 1's
- If the value of  $p = 0$ , move on
- If the value of  $p = 1$ , examine  $l$  and  $t$ 
  - If they are both 0, assign a new label to  $p$
  - If only one of them is 1, assign it's label to  $p$
  - If they are both 1
    - If they have same label, assign that label to  $p$
    - If not, assign one of the labels to  $p$  and make a note that the two labels are equivalent ( $r$  and  $t$  are connected through  $p$ )
- At the end of the scan, all points with value 1 have been labeled



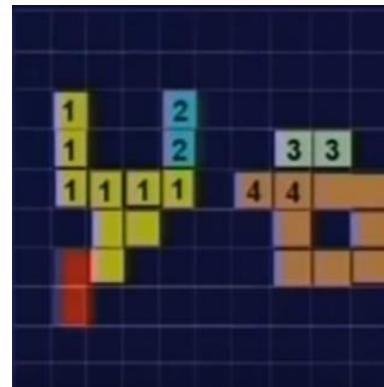
# Example: Labeling of Connected Components



$\{1\}\{2\}$



$\{1,2\}, \{3\}$



$\{1,2\} \{3,4\}$



$\{1,2,5\} \{3,4\}$

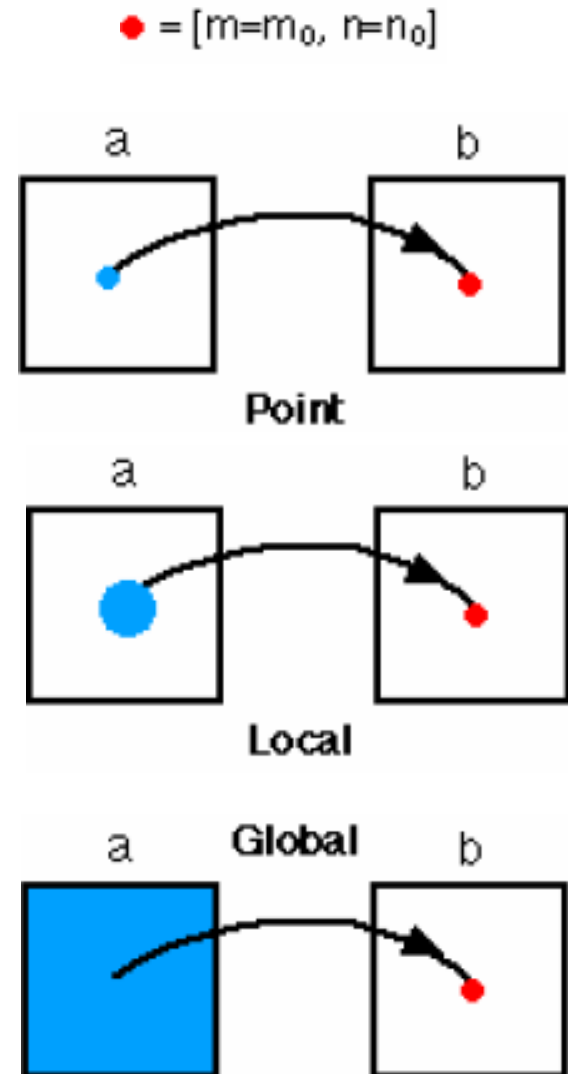
After completing the scan, do the second round and introduce a unique label to each equivalent class

# Labeling of Connected Components

- What shall we do with 8-connected components?
- Do the same way but examine also the upper diagonal neighbors of  $p$ 
  - If  $p = 0$ , move on
  - If  $p$  is 1
    - If all four neighbors are 0, assign a new label to  $p$
    - If only one of the neighbors is 1, assign its label to  $p$
    - If two or more neighbors are 1, assign one of the label to  $p$  and make a note of equivalent classes
    - After completing the scan, do the second round and introduce a unique label to each equivalent class

# Types of Operations

- Operations that can be applied to digital images to transform an input image  $a[m,n]$  in to an output image  $b[m,n]$
- Point
  - The output value at a specific coordinate is dependent only on the input value at that same coordinate
- Local
  - The output value at a specific coordinate is dependent on the input values in the neighborhood of that same coordinate
- Global
  - The output value at a specific coordinate is dependent on all the input values in the input image



# Types of Operations

- Point Operation
  - Contrast Stretching
  - Binarization
    - Reduction of gray scale image to binary
- Local Operations
  - Convolution, Filtering
- Global Operations
  - Image Profiling
    - Histogram of ON-valued pixels summed across rows or columns of a binary image
  - Frequency Analysis
    - Images are often characterized not by particular objects but by the global configuration of all image objects and resulting patterns or texture



# Arithmetic Operations on Images

- Used extensively, Carried out pixel by pixel
- Let  $I_1$  and  $I_2$  are two images of same dimension
  - **Addition**: used in image average to reduce noise
    - $I(x,y) = \min(I_1(x,y) + I_2(x,y), I_{\max})$
  - **Subtraction**: basic tool in medical imaging
    - $I(x,y) = \max(I_1(x,y) - I_2(x,y), I_{\min})$
  - Possibility of overflow and underflow are taken care by clipping
  - Clipping can be avoided by rescaling the range of intensity values encountered in both input images
  - **Modified addition**:
    - $I(x,y) = (I_1(x,y) + I_2(x,y)) * 255 / (I_1 + I_2)_{\max}$
  - **Modified Subtraction** :
    - $I(x,y) = (I_1(x,y) - I_2(x,y)) * 255 / (I_1 - I_2)_{\min}$

# An Example of Image Multiplication and Division for shading correction



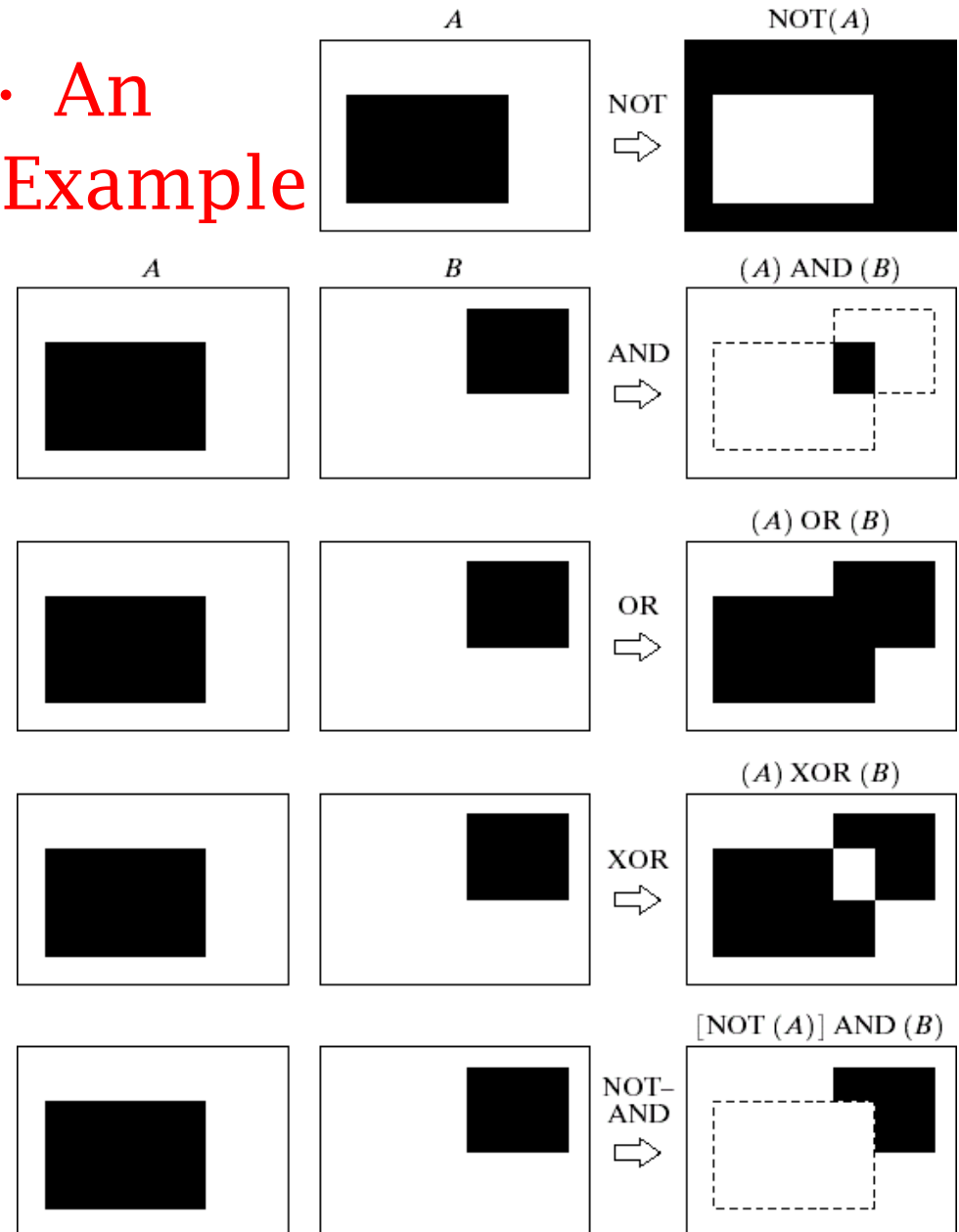
a b c

**FIGURE 2.34** (a) Digital dental X-ray image. (b) ROI mask for isolating teeth with fillings (white corresponds to 1 and black corresponds to 0). (c) Product of (a) and (b).

# Logical Operations on Images

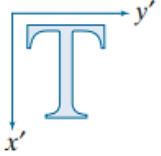
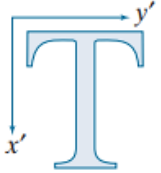
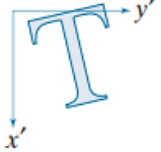
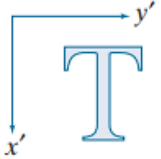
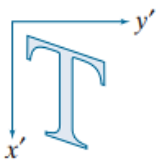
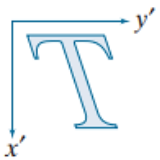
- Carried out pixel by pixel
- Logical operations apply only to binary images, while arithmetic operations apply to multivalued pixels
- Used for tasks such as masking, feature detection, and shape analysis

## • An Example



# Image Geometry

- Basic Transformations:  
expressed in 3D Cartesian coordinate system (x,y,z)
  - Translation
  - Scaling
  - Rotation
  - Concatenation and Inverse Transformation

Transformation Name	Affine Matrix, A	Coordinate Equations	Example
Identity	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{aligned} x' &= x \\ y' &= y \end{aligned}$	
Scaling/Reflection (For reflection, set one scaling factor to -1 and the other to 0)	$\begin{bmatrix} c_x & 0 & 0 \\ 0 & c_y & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{aligned} x' &= c_x x \\ y' &= c_y y \end{aligned}$	
Rotation (about the origin)	$\begin{bmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{aligned} x' &= x \cos \theta - y \sin \theta \\ y' &= x \sin \theta + y \cos \theta \end{aligned}$	
Translation	$\begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{aligned} x' &= x + t_x \\ y' &= y + t_y \end{aligned}$	
Shear (vertical)	$\begin{bmatrix} 1 & s_v & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{aligned} x' &= x + s_v y \\ y' &= y \end{aligned}$	
Shear (horizontal)	$\begin{bmatrix} 1 & 0 & 0 \\ s_h & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{aligned} x' &= x \\ y' &= s_h x + y \end{aligned}$	

# Distance Measure

- The distance between two pixels ( $p(x_1, y_1)$  and  $q(x_2, y_2)$ ) in a given image can be given by three different types of measures
- $D_e$  (Euclidian) distance
  - $D_e(p, q) = ((x_1 - x_2)^2 + (y_1 - y_2)^2)^{1/2}$
- $D_4$  distance
  - Also called city-blocking distance
  - $D_4(p, q) = |(x_1 - x_2)| + |(y_1 - y_2)|$
- $D_8$  distance
  - Also called chessboard distance
  - $D_8(p, q) = \max(|(x_1 - x_2)|, |(y_1 - y_2)|)$

# Distance Metric

- The distance function is called metric if the following properties are satisfied:
  - $D(p,q)$  is well-defined and finite for all  $p$  and  $q$ .
  - $D(p,q) \geq 0$ ; if  $p=q$ , then  $D(p,q)=0$ .
  - The distance  $D(p,q) = D(q,p)$ .
  - $D(p,q)+D(q,t) = D(p,t)$ . (triangular inequality)