Lecture 5 Dr. karan. 08/09/24.

Statistical Foundation

Q2) Eigen vectors and eigen values ; SVD

$$\begin{bmatrix} 4 & 0 \\ 3 & -5 \end{bmatrix}$$

Soln A^T, $A = U \Sigma V^T$

$\boxed{\text{Step 1 Find } V^T}$

$$A^T \cdot A = \begin{bmatrix} 4 & 3 \\ 0 & -5 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 3 & -5 \end{bmatrix} = \begin{bmatrix} 16+9 & 0-15 \\ 0-15 & 0+25 \end{bmatrix}$$

$$= \begin{bmatrix} 25 & -15 \\ -15 & 25 \end{bmatrix}$$

$|A^T A - \lambda I| = 0$ → characteristic equation.

$$\begin{vmatrix} 25-\lambda & -15 \\ -15 & 25-\lambda \end{vmatrix} = 0$$

$\boxed{\lambda^2 - S1\lambda + S2 = 0}$ → C.E

$S1 = \text{Trace}(A^T \cdot A)$
$= 25 + 25$

$S2 = \begin{vmatrix} 25 & -15 \\ -15 & 25 \end{vmatrix}$
$= 625 - 225$
$= 400$

$\lambda^2 - 50\lambda + 400 = 0$

$\lambda^2 - 40\lambda - 10\lambda + 400 = 0$

$\lambda(\lambda - 40) - 10(\lambda - 40) = 0$

$(\lambda - 10)(\lambda - 40) = 0$

$\boxed{\lambda = 10, \quad \lambda = 40}$

$\lambda = 40$, $\lambda = 10$ $\quad A^T \cdot A = \begin{bmatrix} 25 & -15 \\ -15 & 25 \end{bmatrix}$

Eigen vectors at $\lambda = 40$

$$A^T \cdot A \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = 40 \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$$

$$\begin{bmatrix} 25 & -15 \\ -15 & 25 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} 40 V_1 \\ 40 V_2 \end{bmatrix}$$

$25 V_1 - 15 V_2 = 40 V_1$ — ①

$-15 V_1 + 25 V_2 = 40 V_2$ — ②

Consider ①

$-15 V_2 = 15 V_1$ or $V_1 = -V_2$

$V_1 = 1$, $V_2 = -1$

$$\boxed{L_2 \text{ Norm} = \sqrt{(1)^2 + (-1)^2} = \sqrt{2}}$$

$$V_1 = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$$

Eigen vector at $\lambda = \cancel{40}\ 10$

$$\begin{bmatrix} 25 & -15 \\ -15 & 25 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} 10 V_1 \\ 10 V_2 \end{bmatrix}$$

$25 V_1 - 15 V_2 = 10 V_1$ — ①

$-15 V_1 + 25 V_2 = 10 V_2$ — ②

considering (2)

$$-15V_1 = -15V_2$$

$$\therefore V_1 = V_2 = 1$$

$\therefore$ eigen vector $V_2 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$

$$V = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

Step 2 Calculate Σ which is diagonal matrix in desending order of eigen values

$$\Sigma = A^T \cdot A = \|A\|^2 = \Sigma^2$$

$$\Sigma = \begin{bmatrix} \sqrt{40} & 0 \\ 0 & \sqrt{10} \end{bmatrix}$$

Step 3 Calculate U

$$A \cdot A^T = \begin{bmatrix} 4 & 0 \\ 3 & -5 \end{bmatrix} \begin{bmatrix} 4 & 3 \\ 0 & -5 \end{bmatrix}$$

$$= \begin{bmatrix} 16+0 & 12+0 \\ 12+0 & 9+25 \end{bmatrix} = \begin{bmatrix} 16 & 12 \\ 12 & 34 \end{bmatrix}$$

$$A \cdot A^T = \begin{bmatrix} 16 & 12 \\ 12 & 34 \end{bmatrix}$$

$$C \cdot E = \left| A \cdot A^T - \lambda I \right| = 0$$

$$\begin{vmatrix} 16-\lambda & 12 \\ 12 & 34-\lambda \end{vmatrix} = 0$$

$$\lambda^2 - S1\lambda + S2 = 0$$

$$S1 = 16 + 34 = 50$$

$$S2 = \begin{vmatrix} 16 & 12 \\ 12 & 34 \end{vmatrix} = 544 - 144 = 400$$

$$\lambda^2 - 50\lambda + 400 = 0$$

Same as

$$\lambda^2 - 40 - 10\lambda + 400 = 0$$

$$(\lambda - 40)(\lambda - 10) = 0$$

$$\boxed{\lambda = 40, \quad \lambda = 10}$$

eigen vector at $\lambda = 40$

$$A \cdot A^T \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \lambda \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$$

$$\begin{bmatrix} 16 & 12 \\ 12 & 34 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} 40V_1 \\ 40V_2 \end{bmatrix}$$

$$16V_1 + 12V_2 = 40V_1 \quad - (1)$$

$$12V_1 + 34V_2 = 40V_2 \quad - (2)$$

considering 1

$$12V_2 = 24V_1$$

$$V_2 = 2V_1$$

$\cancel{V_2} = 1 \quad \cancel{V_1 = 2}$

$V_1 = 1 \quad V_2 = 2$

$$V = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

eigen vector at $\lambda = 10$.

$$\begin{bmatrix} 16 & 12 \\ 12 & 34 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} 10V_1 \\ 10V_2 \end{bmatrix}$$

$$16V_1 + 12V_2 = 10V_1 \quad \text{—} 1$$

$$12V_1 + 34V_2 = 10V_2 \quad \text{—} 2$$

considering 2

$\cancel{34V_2} = \cancel{2} \qquad 12V_1 = -24V_2$

$$V_1 = -2V_2$$

$$V = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

$\therefore \quad V_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \qquad V_2 \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

L2 Norm $= \sqrt{1^2 + 2^2}$
$= \sqrt{5}$

L2 Norm $= \sqrt{(-2)^2 + 1}$
$= \sqrt{5}$

$$v_1 = \begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix} \qquad v_2 = \begin{bmatrix} -2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix}$$

$$U = \begin{bmatrix} 1/\sqrt{5} & -2\sqrt{5} \\ 2\sqrt{5} & 1/\sqrt{5} \end{bmatrix}$$

$$A = U \Sigma V^T$$

$$= \begin{bmatrix} 1/\sqrt{5} & -2\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix} \begin{bmatrix} \sqrt{40} & 0 \\ 0 & \sqrt{10} \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

Q3) $\|A\|_\infty$, $\|A\|_1$ for $\begin{bmatrix} 100 & 300 & -50 \\ -30 & 20 & -70 \\ -100 & 50 & 10 \end{bmatrix}$

max of obsulute sume of row

$$\|A\|_\infty = \max\big(|100| + |300| + |-50|,$$
$$|-30| + |20| + |-70|$$
$$|-100| + |50| + |10| \big)$$

$$\boxed{\|A\|_\infty = \max(450, 120, 160) = 450}$$

max of obsulute sum of column.

$$\|A\|_1 = \max\big(|100| + |-30| + |-100|,$$
$$|300| + |20| + |50|$$
$$|-50| + |-70| + |10| \big)$$

$$= \max(230, 370, 130)$$

$$\boxed{\|A\|_1 = 370}$$