Filters

Sunday, November 3, 2024 8:06 AM

$\underline{T}_{p\times p}$: A Hermitian Matrix $\qquad \underline{T}^H = \underline{T}$

H: Conjugate + Transpose

EVD

$$\underline{T}\,\underline{x} = \lambda\,\underline{x} \quad (\underline{x} \neq 0)$$

$\lambda \rightarrow$ eigenvalue

$$(\underline{T}\,\underline{x})^H = \lambda^* \underline{x}^H$$

$$\underline{x}^H \underline{T}^H = \lambda^* \underline{x}^H$$

$$\underline{x}^H \underline{T} = \lambda^* \underline{x}^H$$

$$\underline{x}^H (\underline{T}\,\underline{x}) = \lambda^* \underline{x}^H \underline{x}$$

$$\underline{x}^H \lambda \underline{x} = \lambda^* \underline{x}^H \underline{x}$$

$$\lambda\, \underline{x}^H \underline{x} = \lambda^* \underline{x}^H \underline{x}$$

$$\underbrace{(\lambda - \lambda^*)}_{\rightarrow 0}\, \underbrace{\underline{x}^H \underline{x}}_{\neq 0} = 0$$

$$\boxed{\lambda = \lambda^*}$$

$$(\lambda\, \underline{x})^H = \underline{x}^H \lambda^H_{1\times 1} = \lambda^* \underline{x}^H$$

$$\underline{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_p \end{bmatrix}$$

$$\underline{x}^H \underline{x} = \sum_{i=1}^{p} |x_i|^2$$

(2)

$$\underline{T}\,\underline{x}_1 = \lambda_1 \underline{x}_1 \quad \text{— (1)} \qquad \lambda_1 \neq \lambda_2$$

$$\underline{T}\,\underline{x}_2 = \lambda_2 \underline{x}_2 \quad \text{— (2)}$$

Apply "H" on (1)

$$(\underline{T}\,\underline{x}_1)^H = (\lambda_1\, x_1)^H$$

$$(\underline{T}\,\underline{x}_1)^H = (\lambda_1 \, x_1)^H$$

$$\underline{x}_1^H \, \underline{T}^H = \lambda_1^* \, x_1^H$$

$$\underline{x}_1^H \, \underline{T}^H = \lambda_1 \, x_1^H \quad \text{—} \; (3)$$

multiply $\underline{x}_2$ both side of (3)

$$\underline{x}_1^H \, \underline{T}^H x_2 = \lambda_1 \, \underline{x}_1^H \, \underline{x}_2$$

$$\underline{x}_1^H \, \underline{T}\,\underline{x}_2 = \lambda_1 \, x_1^H \, \underline{x}_2$$

$$\underline{x}_1^H \, \lambda_2 \, \underline{x}_2 = \lambda_1 \, x_1^H \, \underline{x}_2$$

$$(\lambda_2 - \lambda_1) \underbrace{\underline{x}_1^H \, \underline{x}_2}_{=0} = 0$$

orthogonal

Ex:

$$A = \begin{bmatrix} 1+2j & 2 \\ 3 & 4+2j \end{bmatrix} \qquad A^H = \begin{bmatrix} (1+2j)^* & 3^* \\ 2^* & (4+2j)^* \end{bmatrix}$$

$$\underline{A}^H A$$

$$= \begin{bmatrix} 1-2j & 3 \\ 2 & 4-2j \end{bmatrix} \begin{bmatrix} 1+2j & 2 \\ 3 & 4+2j \end{bmatrix}$$

$$= \begin{bmatrix} 1^2+4+9 & 2-2j+12+6j \\ 2+4j+12-6j & 4+16+4 \end{bmatrix}$$

$$= \begin{bmatrix} 14 & 14+4j \\ 14-2j & 24 \end{bmatrix}$$

$$\underline{T}\underbrace{\begin{bmatrix}\underline{e}_1 & \underline{e}_2 & \underline{e}_3 & \cdots & \underline{e}_p\end{bmatrix}}_{\underline{E}} = \underbrace{\begin{bmatrix}\lambda_1 \underline{e}_1 & \lambda_2 \underline{e}_2 & \cdots & \lambda_p \underline{e}_p\end{bmatrix}}$$

$$\underline{T}\,\underline{x}_1 = \lambda_1 \underline{x}_1$$

$$\underline{T}\,\underline{x}_2 = \lambda_2 \underline{x}_2$$

$$\underline{T}\,\underline{E} = \underbrace{\begin{pmatrix}\underline{e}_1 & \underline{e}_2 & \cdots & \underline{e}_p\end{pmatrix}}_{\underline{E}} \underbrace{\begin{bmatrix}\lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & & & \\ 0 & \cdots & \cdots & \lambda_p\end{bmatrix}}_{\underline{D}}$$

$$\underline{T}\,\underline{E} = \underline{E}\,\underline{D} \quad \text{— } 4$$

$$\underline{E}^H \underline{E} = \begin{bmatrix}\underline{e}_1^H \\ \underline{e}_2^H \\ \underline{e}_p^H\end{bmatrix}\begin{bmatrix}e_1 & e_2 & \cdots & e_p\end{bmatrix}$$

$A A^H = I$

A is orthogonal

$$\underline{T}\,\underline{E} = \underline{E}\,\underline{D}$$

$$\underline{T}\,\underline{E}\,\underline{E}^H = \underline{E}\,\underline{D}\,\underline{E}^H$$

$$\boxed{\underline{T} = \underline{E}\,\underline{D}\,\underline{E}^H} \implies$$

$\underline{E}\,\underline{E}^H = \underline{I}$

Decomposition of Toeplitz matrix

$$\boxed{\underline{E}^{-1} = \underline{E}^H}$$

$$A = \begin{bmatrix}1 & 2 \\ 3 & 4 \\ 5 & 6\end{bmatrix} \longrightarrow \text{Not a square}$$

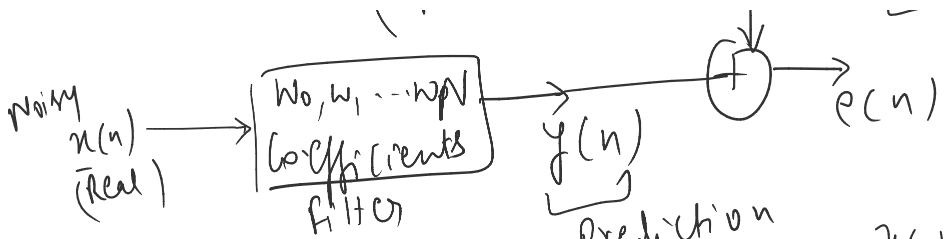
SVD

$\text{Det}|A| = 0$

H: conj + transpose

Wiener filter (Optimization) $d(n)$ actual

$[W_0, W_1 \cdots W_N] \longrightarrow (+) \longrightarrow e(n)$



$$\underline{x}(n) = \begin{bmatrix} x(n) \\ x(n-1) \\ \vdots \\ x(n-N) \end{bmatrix} \to N+1 \times 1$$

$$y(n) = W_0 x(n) + W_1 x(n-1) + \cdots W_N x(n-N)$$

$$y(n) = [W_0 W_1 \cdots W_N] \begin{bmatrix} x(n) \\ x(n-1) \\ \vdots \\ x(n-N) \end{bmatrix} \qquad \underline{W} = \begin{bmatrix} W_0 \\ W_1 \\ \vdots \\ W_N \end{bmatrix} \to N+1 \times 1$$

$$\underbrace{y(n)}_{1\times 1} = \underline{W}^t \, \underline{x}(n)$$

$$(1 \times N+1) \; (N+1 \times 1)$$

Assumption → WSS → $\underline{x}(n)$ → Real

auto correlation

$$= E[\underline{x}(n) \, \underline{x}^H(n)]$$

$$R(k) = E[\underline{x}(n) \, \underline{x}^t(n)]$$

↳ auto-Correlation matrix

WSS : Cross correlation holds b/w $\underline{x}(n)$ & $d(n)$ → $p(k) = E[x(n) \, d(n-k)]$

$$\underline{x}(n) = \begin{bmatrix} x(n) \\ x(n-1) \\ \vdots \\ x(n-N) \end{bmatrix} \qquad \underline{p} = \begin{bmatrix} p(0) \\ p(1) \\ \vdots \\ p(N) \end{bmatrix}$$

$$= E[\underline{x}(n) \, \underline{d(n)}]$$

$$e(n) = d(n) - y(n)$$

$$e(n) = d(n) - y(n)$$

$$\epsilon^2 = E\left[\underbrace{e^2(n)}\right] \rightarrow \text{mean square error}$$

$$= E\left[e(n)\,e^H(n)\right]$$

$$= E\left[e(n)\,e^t(n)\right] \qquad \left(w^t x(n)\, x^t(n)\, w\right)$$

$$= E\left[\left(d(n) - \underbrace{\underline{w}^t \underline{x}(n)}\right)\left(d(n) - \underbrace{\underline{w}^t \underline{x}(n)}\right)^t\right]$$

$$= E\left[d^2(n) - \underline{w}^t \underline{x}(n)\, d(n) - \underline{w}^t \underline{x}(n)\, d(n) + \underbrace{\underline{w}^t \underline{x}(n)\, \underline{x}^t(n)\, \underline{w}}\right]$$

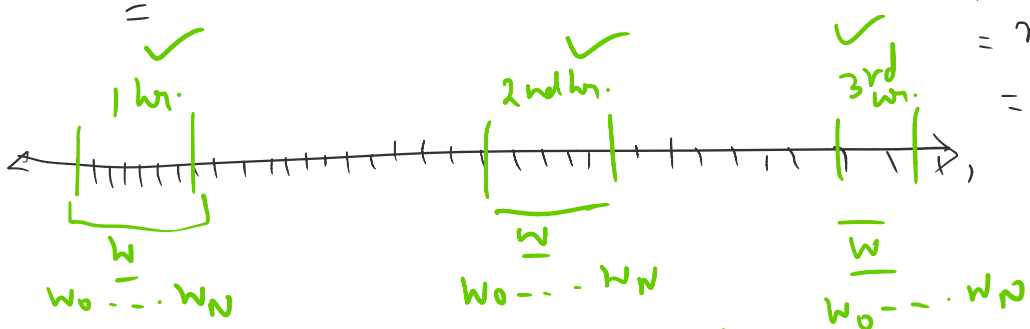
$$= \sigma_d^2 - 2E\left[\underline{w}^t \underline{x}(n)\, d(n)\right] + E\left[\underline{w}^t \underline{x}(n)\, \underline{x}^t(n)\, \underline{w}\right]$$

$\underline{w}^t \underline{x}(n)$ $\quad \underline{w} = \begin{bmatrix} w_0 \\ w_1 \end{bmatrix}$

$\underline{x}^t(n)\, \underline{w}$ $\quad \underline{x}(n) = \begin{bmatrix} x(n) \\ x(n-1) \end{bmatrix}$

$$\left(\underline{w}^t \underline{x}(n)\right)^t = \underline{x}^t(n)\, w = w^t x(n)$$

$=$



$$= \frac{\cancel{3}\left(w_0, w_1, \ldots w_N\right)}{\cancel{3}}$$

$$= \sigma_d^2 - 2\,\underline{w}^t \underbrace{E\left[\underline{x}(n)\, d(n)\right]} + \underline{w}^t \underbrace{E\left[\underline{x}(n)\, x^t(n)\right]}\, \underline{w}$$

$$\boxed{\epsilon^2 = \sigma_d^2 - 2w^t \underline{p} + \underline{w}^t \underline{R} \underline{w}}$$

Gradient Descent Optimization