Correlation and Covariance Matrix and Wiener Filter

Sunday, October 6, 2024 8:06 AM

I/P data $\underline{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_p \end{bmatrix}$ 1st $\begin{bmatrix} x_1 \\ \vdots \\ x_p \end{bmatrix}$ 2nd $\quad \underline{\mu_x} = E[\underline{x}] = \begin{bmatrix} \mu_{x_1} \\ \mu_{x_2} \\ \vdots \\ \mu_{x_p} \end{bmatrix}$

$\underline{x}^T = [x_1 \; x_2 \; \dots \; x_p]$

auto-Covariance Matrix $\underline{C}_{p\times p}(\underline{x}) = E\left[(\underline{x} - \mu_{\underline{x}}) (\underline{x} - \underline{\mu_x})^H \right] = E\left[\|(x-\mu_x)\|^2 \right]$

A matrix C is said to be Hermition matrix if $\boxed{C^H = C}$

$y = a + jb$

$y^* = a - jb$

$H \to$ Operator $\to$ Conjugate + Transpose

$B = \begin{bmatrix} a+jb & 2+3j \\ 4-9j & 5 \end{bmatrix} \Rightarrow B^* = \begin{bmatrix} a-jb & 2-3j \\ 4+9j & 5 \end{bmatrix}$

$B^{*T} = \begin{bmatrix} a-jb & 4+9j \\ 2-3j & 5 \end{bmatrix}$

$$\underline{C}_{p\times p} = E\left[\underbrace{\begin{bmatrix} x_1-\mu_{x_1} \\ x_2-\mu_{x_2} \\ \vdots \\ x_p-\mu_{x_p} \end{bmatrix}}_{p\times 1} \underbrace{\left[(x_1-\mu_{x_1}), (x_2-\mu_{x_2}), \dots (x_p-\mu_{x_p}) \right]^*}_{1\times p} \right]$$

$$= \begin{bmatrix} C_{x_1 x_1} & C_{x_1 x_2} & \cdots & C_{x_1 x_p} \\ C_{x_2 x_1} & C_{x_2 x_2} & \cdots & C_{x_2 x_p} \\ \vdots & & & \\ C_{x_p x_1} & \cdots & & C_{x_p x_p} \end{bmatrix}$$

(Indep $\to$ uncorrelated
uncorr $\not\to$ Indep, under the condit r.v follows Gaussian jointly)

Covariance matrix is always Conjugate symmetry / Hermition

$$\underline{C} = E\left[(\underline{x} - \mu_{\underline{x}}) (\underline{x} - \underline{\mu_x})^H \right]$$

$$\boxed{\underline{C}^H = \underline{C}}$$

$A - m\times n$

$(AB)^H = \left[(AB)^t \right]^* = (B^t A^t)^* = B^H A^H$

$$\underline{C}^H = E\left[\left(\underbrace{(\underline{x} - \underline{\mu_x})}_{A} \underbrace{(\underline{x} - \underline{\mu_x})^H}_{B} \right)^H \right]$$

$$= E\left[(\underline{x} - \underline{\mu_x}) (\underline{x} - \underline{\mu_x})^H \right]$$

$$= \underline{C}$$

$\mu_x = E[x]$ $\quad \sigma_x^2 = E[\,|(x-\mu_x)|^2\,]$ ⇒ variance = (auto covariance)

Covariance

$C_{\underline{x}\,\underline{y}} = E\left[(\underline{x}-\underline{\mu}_x)(\underline{y}-\underline{\mu}_y)^H\right]$

$1^T = 1$

$\underline{x} = [x_1] \quad \underline{y} = [y_1]$

$C_{x_1 y_1} = E\left[(\underset{1\times1}{x_1} - \underset{1\times1}{\mu_{x_1}})(\overset{1\times1}{y_1} - \overset{1\times1}{\mu_{y_1}})^H\right]$

$= E\left[(x_1 - \mu_{x_1})(y_1 - \mu_{y_1})^*\right]$

$= \underbrace{E[x_1 y_1^*]} - \underbrace{E[\mu_{x_1} y_1^*]} - \underbrace{E[x_1 \mu_{y_1}^*]} + E[\mu_{x_1}\mu_{y_1}^*]$

$= E[x_1 y_1^*] - \mu_{x_1} E[y_1^*] - \mu_{y_1}^* E[x_1] + \mu_{x_1}\mu_{y_1}^*$

$C_{x_1 y_1} = \underbrace{E[x_1 y_1^*]} - \underbrace{\mu_{x_1}\mu_{y_1}^*} - \cancel{\mu_{y_1}^*\mu_{x_1}} + \cancel{\mu_{x_1}\mu_{y_1}^*}$

if

$C_{x_1 y_1} = 0 \rightarrow$ uncorrelated.

$\boxed{E[x_1 y_1^*] = \mu_{x_1}\mu_{y_1}^*}$

↓ correlation

Correlation ⇒ $E[(x-0)(y-0)^*] = \mu_x \cdot \mu_y^*$

Stationarity

First order stationarity

$\underbrace{E[x(n)]} = \mu_n = \underline{\mu}$

1 μ 100 300 400 μ

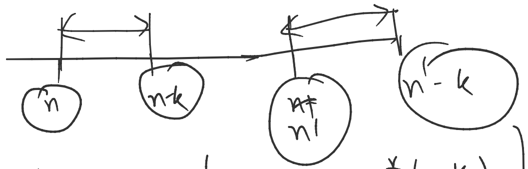
Second order stationarity / wide sense stationarity (WSS)

Second Order Stationarity | Wide Sense Stationarity (WSS)

K: lag | gap

1) $E[x(n)] = \mu$

2) $\underline{E[x(n)\, x^*(n-k)]} = r(k)$



$r^*(-k) = r(k)$

$r(-k) = r^*(k)$

$\rightarrow r(k) = E[x(n)\, x^*(n-k)]$ $\quad n = m-k$

$r^*(-k) = E[\underbrace{x(n+k)}_{m}\, x^*(n)]$

$= E[x(m)\, x^*(m-k)] = r(k)$

Conjugate symmetry

1 2 5 7

$P = \dfrac{1^2+2^2+5^2+7^2}{4}$

$k=0$

$r(0) = E[|x(n)|^2] \geq 0$

$\hookrightarrow$ Average power

$\rightarrow \underline{A}$: Hermition matrix $\quad \underline{A}^H = \underline{A}$

$$\underline{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_p \end{bmatrix} \qquad \mu_{\underline{x}} = E[\underline{x}] = \begin{bmatrix} E(x_1) \\ E(x_2) \\ \vdots \\ E(x_p) \end{bmatrix} = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_p \end{bmatrix}$$

Auto covariance

$\hat{\underline{x}} = \underline{x} - \mu_x \qquad \underline{C}_{p\times p} = E[\hat{\underline{x}}\, \hat{\underline{x}}^H]$

Auto correlation $\rightarrow \underline{R}_{p\times p} = E[\underline{x}\, \underline{x}^H]$

$x(n)$: Discrete time Random process

$$\underline{x} = \begin{bmatrix} x(n) \\ x(n-1) \\ \vdots \\ x(n-p) \end{bmatrix} \rightarrow n\text{th}$$

$E[\underline{x}\, \underline{x}^H]$

$E[|x(n)|^2] \quad E[x(n)\, x^*(n-1)] \cdots \quad E[x(n)\, x^*(n-p)]$

$$\underline{x} = \begin{bmatrix} x(n-1) \\ \vdots \\ x(n-p) \end{bmatrix} \downarrow \qquad E[\underline{x}\,\underline{x}^{\ldots}]$$

$$= \begin{bmatrix} E[|x(n)|^2] & E[x(n)x^*(n-1)] & \cdots & E[x(n)\,x^*(n-p)] \\ E[x(n-1)\,x^*(n)] & E[|x(n-1)|^2] & \cdots & E[x(n-1)\,x^*(n-p)] \\ \vdots & & & \\ E[x(n-p)\,x^*(n)] & E[x(n-p)\,x^*(n-1)] & \cdots & E[|x(n-p)|^2] \end{bmatrix}$$

if WSS holds $\qquad r(k) = E[\,x(n)\,x^*(n-k)\,] \qquad n-(n-k)$

$$= \begin{bmatrix} r(0) & r(1) & \cdots & r(p) \\ (r^*(1)) & & & \\ & & & \\ r(-p) = r^*(p) & r(1-p) & \cdots & r(0) \end{bmatrix} \qquad n-p-(n+1)$$

Toeplitz matrix (Hermition)