Filters

Sunday, November 3, 2024 8:06 AM

$T_{p\times p}$: A Hermition Matrix $\qquad \underline{T}^H = \underline{T}$

H: Conjugate + Transpose

EVD $\qquad \underline{T}\,\underline{x} = \lambda\,\underline{x} \quad (\underline{x} \neq 0)$

$\lambda \rightarrow$ eigenvalue

$$(\underline{T}\,\underline{x})^H = \lambda^* \underline{x}^H$$

$$(\lambda\,\underline{x})^H = \underline{x}^H \lambda^H_{1\times 1} = \lambda^* \underline{x}^H$$

$$\underline{x}^H \underline{T}^H = \lambda^* \underline{x}^H$$

$$\underline{x}^H \underline{T} = \lambda^* \underline{x}^H$$

$$\underline{x}^H (\underline{T}\,\underline{x}) = \lambda^* \underline{x}^H \underline{x}$$

$$\underline{x}^H \lambda \underline{x} = \lambda^* \underline{x}^H \underline{x}$$

$$\lambda\, \underline{x}^H \underline{x} = \lambda^* \underline{x}^H \underline{x}$$

$$\underline{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_p \end{bmatrix}$$

$$\underline{x}^H \underline{x} = \sum_{i=1}^{p} |x_i|^2$$

$$\underbrace{(\lambda - \lambda^*)}_{\rightarrow 0} \underbrace{\underline{x}^H \underline{x}}_{\neq 0} = 0$$

$$\boxed{\lambda = \lambda^*}$$

(2)

$$\underline{T}\,\underline{x}_1 = \lambda_1 \underline{x}_1 \quad \text{--- (1)} \qquad \lambda_1 \neq \lambda_2$$

$$\underline{T}\,\underline{x}_2 = \lambda_2 \underline{x}_2 \quad \text{--- (2)}$$

Apply "H" on (1)

$$(\underline{T}\,\underline{x}_1)^H = (\lambda_1\, x_1)^H$$

$$(\underline{T}\,\underline{x}_1)^H = (\lambda_1\, x_1)$$

$$\underline{x}_1^H\, \underline{T}^H = \lambda_1^*\, x_1^H$$

$$\underline{x}_1^H\, \underline{T}^H = \lambda_1\, x_1^H \quad \text{— } (3)$$

multiply $\underline{x}_2$ both side of (3)

$$\underline{x}_1^H\, \underline{T}^H\, x_2 = \lambda_1\, \underline{x}_1^H\, \underline{x}_2$$

$$\underline{x}_1^H\, \underline{T}\,\underline{x}_2 = \lambda_1\, x_1^H\, \underline{x}_2$$

$$\underline{x}_1^H\, \lambda_2\, \underline{x}_2 = \lambda_1\, x_1^H\, \underline{x}_2$$

$$(\lambda_2 - \lambda_1)\, \underbrace{\underline{x}_1^H\, \underline{x}_2}_{=0} = 0$$

orthogonal

Ex:

$$A = \begin{bmatrix} 1+2j & 2 \\ 3 & 4+2j \end{bmatrix} \qquad A^H = \begin{bmatrix} (1+2j)^* & 3^* \\ 2^* & (4+2j)^* \end{bmatrix}$$

$$\underline{A}^H A$$

$$= \begin{bmatrix} 1-2j & 3 \\ 2 & 4-2j \end{bmatrix} \begin{bmatrix} 1+2j & 2 \\ 3 & 4+2j \end{bmatrix}$$

$$= \begin{bmatrix} 1^2+4+9 & 2-2j+12+6j \\ 2+4j+12-6j & 4+16+4 \end{bmatrix}$$

$$= \begin{bmatrix} 14 & 14+4j \\ 14-2j & 24 \end{bmatrix}$$

$$T[e_1\ e_2\ e_3\ \cdots e_p] = [\lambda_1 e_1\ \ \lambda_2 e_2 \ \cdots \lambda e_p]$$

$$\underline{T}\underbrace{\begin{bmatrix} \underline{e}_1 & \underline{e}_2 & \underline{e}_3 & \cdots & \underline{e}_p \end{bmatrix}}_{\underline{E}} = \underbrace{\begin{bmatrix} \lambda_1 \underline{e}_1 & \lambda_2 \underline{e}_2 & \cdots & \lambda_p \underline{e}_p \end{bmatrix}}$$

$T\underline{x}_1 = \lambda_1 x_1$

$T\underline{x}_2 = \lambda_2 \underline{x}_2$

$$\underline{T}\,\underline{E} = \underbrace{\begin{pmatrix} \underline{e}_1 & \underline{e}_2 & \cdots & \underline{e}_p \end{pmatrix}}_{\underline{E}} \underbrace{\begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & & & \\ 0 & \cdots & & \lambda_p \end{bmatrix}}_{\underline{D}}$$

$$\underline{T}\,\underline{E} = \underline{E}\,\underline{D} \quad \text{— (4)}$$

$$\underline{E}^H \underline{E} = \begin{bmatrix} \underline{e}_1^H \\ \underline{e}_2^H \\ \vdots \\ \underline{e}_p^H \end{bmatrix} \begin{bmatrix} e_1 & e_2 & \cdots & e_p \end{bmatrix}$$

$AA^H = I$

A is orthogonal

$T\underline{E} = \underline{E}\,\underline{D}$

$\underline{E}\,E^H = I$

$\underline{T}\,\underline{E}\,\underline{E}^H = \underline{E}\,\underline{D}\,\underline{E}^H$

$$\boxed{\underline{T} = \underline{E}\,\underline{D}\,\underline{E}^H} \Rightarrow$$

Decomposition of Toeplitz matrix

$$\boxed{E^{-1} = E^H}$$

$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \rightarrow$ Not a square

SVD

$\det|A| = 0$

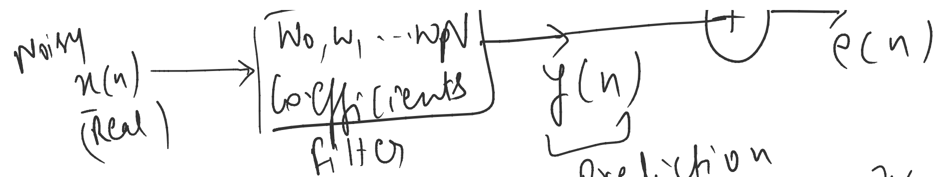
H: conj transpose

Wiener filter (Optimization)

n: time index

noisy $x(n) \longrightarrow$ [$w_0, w_1 \cdots w_N$ coefficients] $\longrightarrow y(n)$ $\longrightarrow$ (+) $\longrightarrow e(n)$

$d(n)$ actual



$$\underline{x}(n) = \begin{bmatrix} x(n) \\ x(n-1) \\ \vdots \\ x(n-N) \end{bmatrix}_{N+1 \times 1}$$

$$y(n) = W_0 x(n) + W_1 x(n-1) + \cdots W_N x(n-N)$$

$$\underline{W} = \begin{bmatrix} W_0 \\ W_1 \\ \vdots \\ W_N \end{bmatrix}_{N+1 \times 1}$$

$$y(n) = \begin{bmatrix} W_0 & W_1 & \cdots & W_N \end{bmatrix} \begin{bmatrix} x(n) \\ x(n-1) \\ \vdots \\ x(n-N) \end{bmatrix} \rightarrow \underline{x}(n)$$

$$\underbrace{y(n)}_{1 \times 1} = \underbrace{\underline{W}^t}_{(1 \times N+1)} \underbrace{\underline{x}(n)}_{(N+1 \times 1)}$$

Assumption $\rightarrow$ WSS $\rightarrow$ $x(n) \rightarrow$ Real

auto correlation $= E[\underline{x}(n)\, \underline{x}^H(n)]$

$R(k) = E[\underline{x}(n)\, \underline{x}^t(n)]$ $\rightarrow$ auto-Correlation matrix

WSS : Cross correlation holds b/w $\underline{x}(n)$ & $d(n)$ $\rightarrow p(k) = E[x(n)\, d(n-k)]$

$$\underline{x}(n) = \begin{bmatrix} x(n) \\ x(n-1) \\ \vdots \\ x(n-N) \end{bmatrix} \qquad \underline{p} = \begin{bmatrix} p(0) \\ p(1) \\ \vdots \\ p(N) \end{bmatrix}$$

$$= E[\underline{x}(n)\, d(n)]$$

$$e(n) = d(n) - y(n)$$

$$= E[e^2(n)] \rightarrow \text{mean square error}$$

$e(n) =$

$$\epsilon^2 = E\left[\underbrace{e^2(n)}\right] \rightarrow \text{mean square error}$$

$$= E\left[e(n)e^H(n)\right]$$

$$= E\left[e(n)\,e^t(n)\right] \quad (w^t\,x(n)\,x^t(n)\,w)$$

$$= E\left[\left(d(n) - \underline{w}^t\,\underline{x}(n)\right)\left(d(n) - w^t\,\underline{x}(n)\right)^t\right]$$

$$= E\left[d^2(n) - \underline{w}^t\,\underline{x}(n)\,d(n) - \underline{w}^t\,\underline{x}(n)\,d(n) + \underline{w}^t\,\underline{x}(n)\,x^t(n)\,\underline{w}\right]$$

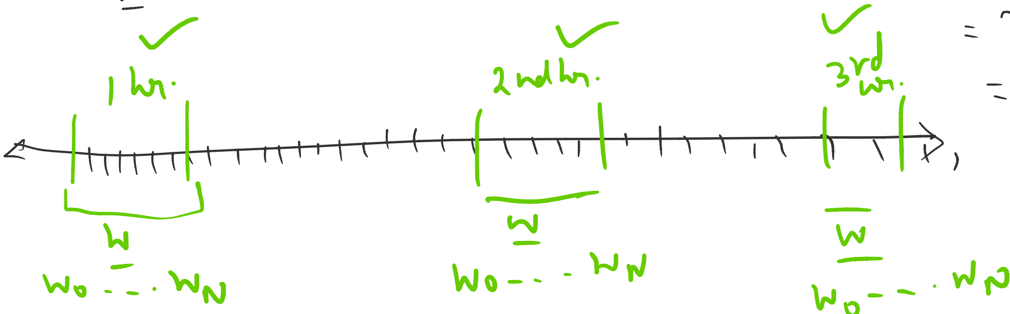
$$= \sigma_d^2 - 2E\left[\underline{w}^t\,\underline{x}(n)\,d(n)\right] + E\left[\underline{w}^t\,\underline{x}(n)\,x^t(n)\,\underline{w}\right]$$

$$w^t\,x(n) \qquad \underline{w} = \begin{bmatrix} w_0 \\ w_1 \end{bmatrix}$$

$$\text{Same } x^t(n)\,\underline{w} \qquad \underline{x}(n) = \begin{bmatrix} x(n) \\ x(n-1) \end{bmatrix}$$

$$(\underline{w}^t\,\underline{x}(n))^t = x^t(n)\,w = w^t\,x(n)$$

$=$



$$= \frac{\cancel{3}(w_0, w_1 \cdots w_N)}{\cancel{3}}$$

$$= \sigma_d^2 - 2w^t\,\underbrace{E\left[\underline{x}(n)\,d(n)\right]} + \underline{w}^t\,\underbrace{E\left[\underline{x}(n)\,x^t(n)\right]}\,\underline{w}$$

$$\boxed{\epsilon^2 = \sigma_d^2 - 2\underline{w}^t\,\underline{p} + \underline{w}^t\,\underline{R}\,\underline{w}} \quad \text{—— (1)}$$

Gradient Descent Optimization $\rightarrow$

Gradient Descent Optimization →

10/11/24

first derivative of (1) w.r.t "w"

$$\nabla_w \epsilon^2 = \underline{0} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = 0$$

$$[w_0 \; w_1 \cdots w_N] \begin{bmatrix} p(0) \\ p(1) \\ \vdots \\ p(N) \end{bmatrix}$$

$$A = \underline{w}^t \underline{p} = w_0 p(0) + w_1 p(1) + w_2 p(2) + \cdots + w_k p(k) \cdots + w_N p(N) = \sum_{i=0}^{N} w_i p(i)$$

$$\frac{\partial A}{\partial w_k} = p(k)$$

$$\nabla_w A = \begin{bmatrix} \frac{\partial A}{\partial w_0} \\ \frac{\partial A}{\partial w_1} \\ \vdots \\ \frac{\partial A}{\partial w_N} \end{bmatrix} = \begin{bmatrix} p(0) \\ p(1) \\ \vdots \\ p(N) \end{bmatrix} = \underline{p}$$

$$B = \underline{w}^t \underline{R}\, \underline{w} \rightarrow$$

$$B = \sum_{i=0}^{N} w_i \sum_{j=0}^{N} R_{ij} w_j$$

$$\nabla_w B = \begin{bmatrix} \frac{\partial B}{\partial w_0} \\ \frac{\partial B}{\partial w_1} \\ \vdots \\ \frac{\partial B}{\partial w_k} \\ \vdots \\ \frac{\partial B}{\partial w_N} \end{bmatrix} =$$

$$\Rightarrow B = \sum_{\substack{i=0 \\ i \neq k}}^{N} w_i \sum_{j=0}^{N} R_{ij} w_j + w_k \sum_{j=0}^{N} R_{kj} w_j$$

$$\frac{\partial B}{\partial w_k} = \sum(\;)\left[R_{10} w_0 + R_{11} w_1 + \cdots R_{1k} w_k + \cdots R_{1N} w_N\right]$$

$$= \sum_{\substack{i=0 \\ i \neq k}}^{N} w_i R_{ik} + \sum_{j=0}^{N} R_{kj} w_j +$$

(*) $w_k \left[R_{k0} w_0 + R_{k1} w_1 + \cdots + R_{kk} w_k + \cdots R_{kN} w_N \right]$

$+R_{kk}W_k + \dots R_{kN}W_N]$ $\sum_{i=0, i\neq k}$ $\sum_{j=0}$ $+W_k R_{kk}$

$$\frac{\partial B}{\partial W_k} = \sum_{i=0}^{N} W_i R_{ik} + \sum_{j=0}^{N} R_{kj} W_j$$

$\underline{R}^t = \underline{R}$

$R_{ij} = R_{ji}$

$$= \underbrace{\sum_{i=0}^{N} W_i R_{ki}} + \underbrace{\sum_{j=0}^{N} R_{kj} W_j}$$

$$\frac{\partial B}{\partial W_k} = 2\sum_{i=0}^{N} W_i R_{ki}$$

$$\frac{\partial B}{\partial W_0} = 2\sum_{i=0}^{N} W_i R_{0i} = 2\left[W_0 R_{00} + W_1 R_{01} + \dots + W_N R_{0N}\right]$$

$$\frac{\partial B}{\partial W_1} = 2\sum_{i=0}^{N} W_i R_{1i} =$$

$$\begin{pmatrix} \frac{\partial B}{\partial W_0} \\ \frac{\partial B}{\partial W_1} \\ \vdots \\ \frac{\partial B}{\partial W_N} \end{pmatrix} = \underline{\underline{2\,\underset{(N+1\times N+1)}{\underline{R}}\,\underset{(N+1\times 1)}{\underline{W}}}}$$

$$\nabla_W \epsilon^2 = 0 - 2\underline{p} + 2\underline{R}\,\underline{W} = \underline{0}$$

$$\underline{R}\,\underline{W} = \underline{p}$$

Optimal weights $\rightarrow$ $\boxed{\underline{W} = \underline{R}^{-1}\underline{p}}$

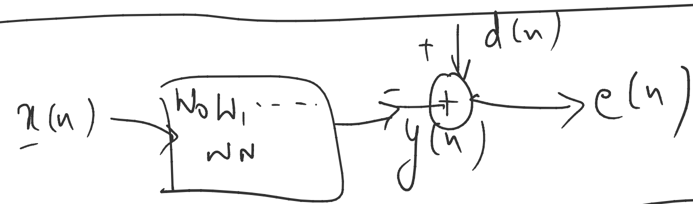
(10 marks)

$R = \begin{bmatrix} 1 & 0.9 \end{bmatrix}$ $\quad p = \begin{bmatrix} 0.2 \end{bmatrix}$

(10)

$$\underline{\underline{R}} = \begin{bmatrix} 1 & 0.9 \\ 0.9 & 1 \end{bmatrix}, \quad \underline{p} = \begin{bmatrix} 0.2 \\ 0.1 \end{bmatrix}$$

$$\underline{\underline{w_{opt}}} = ?$$



$$\epsilon^2 = E\left[e^2(n)\right] \Rightarrow \boxed{w_{opt} = R^{-1} p}$$

Optimal filter

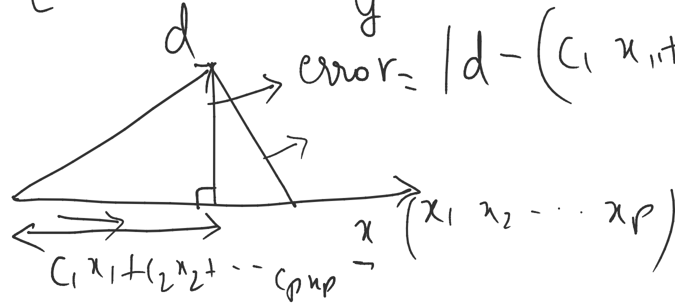
Suppose, we have a set of Random variable

$x_1 \; x_2 \; \dots \; x_p$.

we want to estimate another Random variable "d" as a linear combination of $x_1, x_2, \dots x_p$ i.e it is Called as a linear estimation.

$$d = c_1 x_1 + c_2 x_2 + \dots + c_p x_p$$

$$E\left[\,\Big|d - \underbrace{(c_1 x_1 + c_2 x_2 + \dots + c_p x_p)}_{y}\Big|^2\,\right]$$



error $= |d - (c_1 x_1 + c_2 x_2 \dots c_p x_p)|$

polynomial (scaler) x

$$= w_0 x_0^0 + w_1 x_1^2 + \dots + w_M x_0^M + \dots$$

Too much complicated

$W_{10} \quad x_0 x_1$

W_{00}

$W_{012} \quad x_0 \; x_1 \; x_2$

W_{10}

w_{00} W_{012} x_0 x_1 x_2