DEC01

Statistical Foundation

Quiz 3

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Q2. let's

R be the real autocorrelation Matrix.

λ eigen values.

x Eigen vector.

Then:

$\underline{R}\,\underline{x} - \lambda \underline{x} = 0$ ...(1) We know that R is a Hermitian Matrix, then $R = R^H$

$\Rightarrow \underline{R}\,\underline{x} = \lambda \underline{x}$

$\Rightarrow (\underline{R}\,\underline{x})^H = (\lambda \underline{x})^H$

$\Rightarrow \underline{x}^H \underline{R}^H = \lambda^* \underline{x}^H$ As λ is a scalar $\lambda^H = \lambda^*$

$\Rightarrow \underline{x}^H \underline{R} = \lambda^* \underline{x}^H$ (As $R = R^H$)

Multiplying both sides by $\underline{x}$

$\Rightarrow \underline{x}^H \underline{R}\,\underline{x} = \lambda^* \underline{x}^H \underline{x}$

$\Rightarrow \underline{x}^H \lambda \underline{x} = \lambda^* \underline{x}^H \underline{x}$

$\Rightarrow \lambda \cdot \underline{x}^H \underline{x} = \lambda^* \underline{x}^H \underline{x}$

$\Rightarrow (\lambda - \lambda^*) \cdot \underline{x}^H \underline{x} = 0$

As $\underline{x} \neq 0$, Hence $x^H x \neq 0$

$\lambda - \lambda^* = 0$

$\therefore \boxed{\lambda = \lambda^*}$ → It implies that all eigen values of a real auto correlation matrix are also real.

\# Prove Eigenvectors corresponding to distinct Eigen values of R are mutually orthogonal.

Let's assume λ_1 and λ_2 are two distinct Eigen values of R and $\underline{x}_1$ and $\underline{x}_2$ are the ~~Then~~ corresponding Eigen vectors.

Here,

$\lambda_1 \neq \lambda_2$

$\Rightarrow (\lambda_1 - \lambda_2) \neq 0$

Then.

$$R\underline{x}_1 - \lambda_1 \underline{x}_1 = 0$$

$$\Rightarrow R\underline{x}_1 = \lambda_1 \underline{x}_1 \quad \ldots (1)$$

$$R\underline{x}_2 - \lambda_2 \underline{x}_2 = 0$$

$$\Rightarrow R\underline{x}_2 = \lambda_2 \underline{x}_2 \quad \ldots (2)$$

Taking Hermiltian of both sides of eq. 2

$$(R x_2)^H = (\lambda_2 x_2)^H$$

$$\Rightarrow \underline{x}_2^H \underline{R}^H = \lambda_2^* \underline{x}_2^H$$

we know that for R real autocorrelation matrix $\lambda = \lambda^*$ (all the eigen values are real)

So, $\lambda_1 = \lambda_1^*$, $\lambda_2 = \lambda_2^*$

$$\underline{x}_2^H R^H = \lambda_2^* \underline{x}_2^H$$

$$\Rightarrow \underline{x}_2^H R = \lambda_2 \underline{x}_2^H \quad [\text{As}, R = R^H \text{ and } \lambda_2 = \lambda_2^*]$$

Multiplying both sides by $\underline{x}_1$

$$\underline{x}_2^H R \underline{x}_1 = \lambda_2 \underline{x}_2^H \underline{x}_1$$

$$\Rightarrow \underline{x}_2^H \lambda_1 \underline{x}_1 = \lambda_2 \underline{x}_2^H \underline{x}_1 \quad [\text{Replacing } R\underline{x}_1 = \lambda_1 \underline{x}_1]$$

$$\Rightarrow \lambda_1 \underline{x}_2^H \underline{x}_1 = \lambda_2 \underline{x}_2^H \underline{x}_1$$

$$\Rightarrow (\lambda_1 - \lambda_2) \underline{x}_2^H \underline{x}_1 = 0$$

As $(\lambda_1 - \lambda_2) \neq 0$

$$\boxed{\underline{x}_2^H \underline{x}_1 = 0}$$

This implies that $\underline{x}_1$ and $\underline{x}_2$ are mutually orthogonal to each other.

Q3. If X and Y are independent Random variable

a) then Prove $H(XY) = H(X) + H(Y)$

Ans. we know that

$$H(x) = -\sum_{x} P(x) \log P(x)$$

$$H(y) = -\sum_{y} P(y) \log P(y)$$

$$H(x,y) = -\sum_{x,y} P(x,y) \log P(x,y)$$

As, x, y are independent,

$$P(x,y) = P(x) \cdot P(y)$$

$$\sum_{x} P(x,y) = P(y)$$

$$\sum_{y} P(x,y) = P(x)$$

$$= -\sum_{x,y} P(x,y) \left[\log [P(x) \cdot P(y)]\right]$$

$$= -\sum_{x,y} P(x,y) \left[\log P(x) + \log P(y)\right]$$

$$= -\sum_{x}\sum_{y} P(x,y) \log P(x) - \sum_{y}\sum_{x} P(x,y) \log P(y)$$

$$= -\sum_{x} P(x) \log P(x) - \sum_{y} P(y) \log P(y)$$

$$\boxed{H(x,y) = H(x) + H(y)}$$

Hence Proved

b) In the given distribution.

For x distribution.

$$P(1) = \frac{5}{9} \quad \text{and } P(0) = \frac{4}{9}$$

$$H(x) = -\sum_{x} P(x) \log P(x)$$

$H(x)$

$$= -\left[5 \times P(1) \times \log P(1) + 4 \times P(0) \times \log P(0)\right]$$

$$= -\left[5 \times \frac{5}{9} \times \log_2 \frac{5}{9} + 4 \times \frac{4}{9} \times \log_2 \frac{4}{9}\right]$$

$$= -\left[\frac{25}{9} \times (-0.848) + \frac{16}{9} \times (-1.17)\right]$$

$$= \frac{25 \times 0.848}{9} + \frac{16 \times 1.17}{9}$$

$$= \frac{21.2 + 18.72}{9} = \frac{39.92}{9} = 4.43$$

$$\boxed{H(x) = 4.43}$$

For y distribution.

$P(1) = \frac{1}{9}$

$P(a) = \frac{2}{9}$, $P(b) = \frac{2}{9}$

$P(c) = \frac{2}{9}$, $P(d) = \frac{2}{9}$

$H(y)$

$= -\sum_{y} P(y) \log P(y)$

$= -\left[P(1) \times \log P(1) + 2 \times P(a) \times \log P(a) + 2 \times P(b) \times \log P(b) + 2 \times P(c) \times \log P(c) + 2 \times P(d) \times \log P(d) \right]$

$= -\left[\frac{1}{9} \times \log \frac{1}{9} + 2 \times \frac{2}{9} \times \log \frac{2}{9} + 2 \times \frac{2}{9} \log \frac{2}{9} + 2 \times \frac{2}{9} \times \log \frac{2}{9} + 2 \times \frac{2}{9} \times \log \frac{2}{9} \right]$

$= -\left[\frac{1}{9} \log (0.11) + 4 \times 2 \times \frac{2}{9} \log (0.22) \right]$

$= +\left[\frac{3.18}{9} + \frac{16}{9} \times 2.18 \right] = \cancel{\frac{3.18 + 32.18}{9}}$

$$= \frac{3.18 + 34.88}{9}$$

$$= \frac{38.06}{9} = 4.22$$

$$\boxed{H(y) = 4.22}$$

$H(xy)$

$$= -\sum_{x,y} P(x,y) \log P(x,y)$$

$$= -\left[\frac{1}{2} \times \log \frac{1}{2} + 8 \times \frac{1}{16} \times \log \frac{1}{16}\right]$$

$$= +\left[\frac{1}{2} \times 1 + 8 \times \frac{1}{16} \times 4\right]$$

$$= \frac{1}{2} + \frac{4 \times 8}{16} = \frac{2}{2} = \frac{1}{2} + 2$$

$$= \frac{1+4}{2} = \frac{5}{2}$$

$$= 2.5$$

$$\boxed{H(xy) = 2.5}$$

Q1.

$$w(n+1) = w(n) + \mu\left[P - Rw(n)\right]$$

Cross correlation vector $P = \begin{bmatrix} 0.8 \\ 0.3 \end{bmatrix}$

auto correlation matrix $R = \begin{bmatrix} 1 & 0.5 \\ 0.5 & 1 \end{bmatrix}$

$\mu = 1.0$

Optimal weight $W = R^{-1} P$

$$R^{-1} = \frac{1}{|\det R|} \begin{bmatrix} 1 & -0.5 \\ -0.5 & 1 \end{bmatrix}$$

$$|\det R| = 1 - 0.25 = 0.75$$

$$R^{-1} = \frac{1}{0.75} \begin{bmatrix} 1 & -0.5 \\ -0.5 & 1 \end{bmatrix} = \begin{bmatrix} 1.33 & -0.66 \\ -0.66 & 1.33 \end{bmatrix}$$

$$W = R^{-1} P$$

$$= \underset{2\times2}{\begin{bmatrix} 1.33 & -0.66 \\ -0.66 & 1.33 \end{bmatrix}} \underset{2\times1}{\begin{bmatrix} 0.8 \\ 0.3 \end{bmatrix}}$$

$$= \begin{bmatrix} 1.064 - 0.198 \\ -0.528 + 0.399 \end{bmatrix} = \begin{bmatrix} 0.866 \\ -0.129 \end{bmatrix}$$

Hence,

$$W_{opt} = \begin{bmatrix} 0.866 \\ -0.129 \end{bmatrix}$$