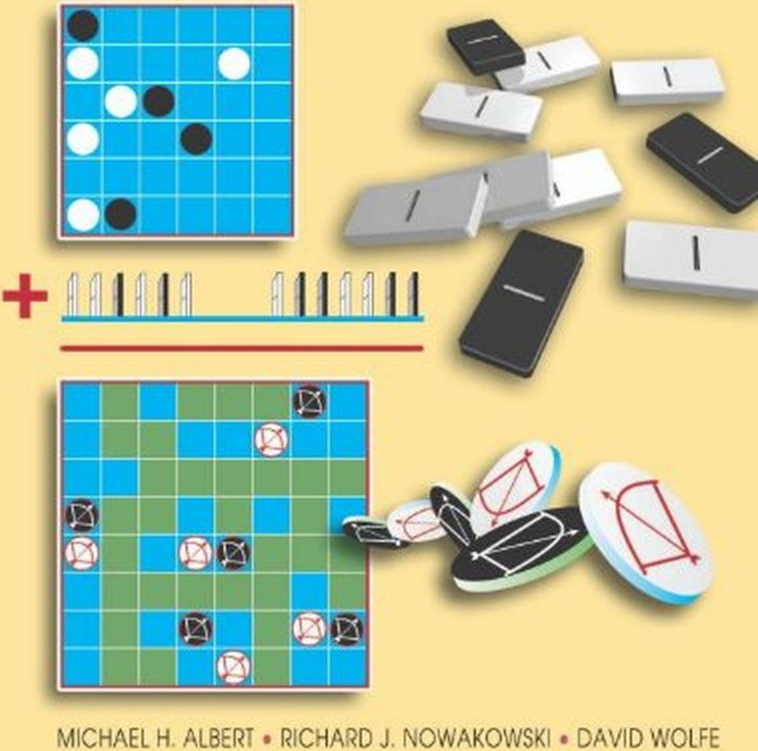
LESSONS IN PLAY

An Introduction to Combinatorial Game Theory



MICHAEL H. ALBERT • RICHARD J. NOWAKOWSKI • DAVID WOLFE

Left	Right
Louise	Richard
Positive	Negative
bLack	White
bLue	Red
Vertical	Horizontal
Female	Male
Green	
Gray	

Symbol	Description	Page
$G = \{\mathcal{G}^L \mid \mathcal{G}^R\}$	Definition of a game	36, 66
G^L and G^R	Typical left option and right option	36
$\{A \parallel B \mid C, D\}$	Notation for a game	36
$G + H$	$\{\mathcal{G}^L + H,\ G + \mathcal{H}^L \mid \mathcal{G}^R + H,\ G + \mathcal{H}^R\}$	68
$-G$	$\{-\mathcal{G}^R \mid -\mathcal{G}^L\}$ (negative)	69
$G \cong H$	identical game trees (congruence)	66
$G = H$	game equivalence	70
$\geq, \leq, >, <$	comparing games	73
$G \parallel H$	$G \not\geq H$ and $H \not\geq G$ (incomparable, confused with)	74
$G \,\|\!\!\triangleright\, H$	$G \not\leq H$ (greater than or incomparable)	74
$G \,\triangleleft\!\!\|\, H$	$G \not\geq H$ (less than or incomparable)	74
$G \sim_{\star} H$	far-star equivalence	195
n	integers	88
$m/2^j$	numbers or dyadic rationals	91
$\uparrow$ and $\downarrow$	$\{0 \mid *\}$ "up" and $\{* \mid 0\}$ "down"	100
$\Uparrow$ and $\Downarrow$	$\uparrow + \uparrow$ "double-up" and $\downarrow + \downarrow$ "double-down"	101
$*$	$\{0 \mid 0\}$ "star"	66
$*n$	$\{0, *, *2, \ldots \mid 0, *, *2, \ldots\}$ "star-n" (nimbers)	136
$+_G$; $-_G$	$0 \parallel 0 \mid -G$ "tiny-G" and its negative "miny-G"	108
☽	loony	22
$\star$	far-star	196
$\mathbf{LS}(G)$, $\mathbf{RS}(G)$	(adorned) left and right stops	(161) 123
$\mathcal{G}_n$ and g_n	games born by day n and their number	117
$\vee$ and $\wedge$	join and meet	127
$n \cdot G$	$\overbrace{G + G + G + \cdots + G}^{n}$	103
$G \cdot U$	Norton product	175
G^n	"G-nth"	188
.213	uptimal notation	188
$G^{\rightarrow n}$	$G^1 + G^2 + \cdots + G^n$	189
AW(G)	atomic weight	198

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Editorial, Sales, and Customer Service Office

A K Peters, Ltd.
888 Worcester Street, Suite 230
Wellesley, MA 02482
www.akpeters.com

Library of Congress Cataloging-in-Publication Data

Albert, Michael H.
Lessons in play : an introduction to combinatorial game theory / Michael H. Albert, Richard J. Nowakowski, David Wolfe
p. cm.
Includes bibliographical references and index.
ISBN 13: 978-1-56881-277-9 (alk. paper)
ISBN 10: 1-56881-277-9 (alk. paper)
1. Game theory--Textbooks. 2. Combinatorial analysis--Textbooks. I. Nowakowski, Richard J. II. Wolfe, David. III. Title.

QA269.A425 2007
519.3--dc22

2006036279

About the cover: The sum of the values of the illustrated KONANE and TOPPLING DOMINOES positions equals the value of the AMAZONS position. Our readers are invited to verify this after having read the text through Chapter 5.

Printed in India
11 10 09 08 07 10 9 8 7 6 5 4 3 2 1

To Richard K. Guy, a gentleman and a mathematician

Contents

Preface

It should be noted that children's games are not merely games. One should regard them as their most serious activities.

Michel Eyquem de Montaigne

Herein we study games of pure strategy, in which there are only two players[1] who alternate moves, without using dice, cards or other random devices and where the players have perfect information about the current state of the game. Familiar games of this type include: TIC TAC TOE, DOTS & BOXES, CHECKERS and CHESS. Obviously, card games such as GIN RUMMY, and dice games such as BACKGAMMON are not of this type. The game of BATTLESHIP has alternate play, and no chance elements, but fails to include perfect information — in fact that's rather the point of BATTLESHIP. The games we study have been dubbed *combinatorial games* to distinguish them from the games usually found under the heading of *game theory,* which are games that arise in economics and biology.

For most of history, the mathematical study of games consisted largely of separate analyses of extremely simple games. This was true up until the 1930s when the Sprague-Grundy theory provided the beginnings of a mathematical foundation for a more general study of games. In the 1970s, the twin tomes *On Numbers and Games* by Conway and *Winning Ways* by Berlekamp, Conway, and Guy established and publicized a complete and deep theory, which can be deployed to analyze countless games. One cornerstone of the theory is the notion of a disjunctive sum of games, introduced by John Conway for normal-play games. This scheme is particularly useful for games that split naturally into components. *On Numbers and Games* describes these mathematical ideas at a sophisticated level. *Winning Ways* develops these ideas, and many more, through playing games with the aid of many a pun and witticism. Both books

[1] In 1972, Conway's first words to one of the authors, who was an undergraduate at the time, was "What's $1 + 1 + 1$?" alluding to three-player games. This question has still not been satisfactorily answered.

have a tremendous number of ideas and we acknowledge our debt to the books and to the authors for their kind words and teachings throughout our careers.

The aim of our book is less grand in scale than either of the two tomes. We aim to provide a guide to the evaluation scheme for normal-play, two-player, finite games. The guide has two threads, the theory and the applications.

The theory is accessible to any student who has a smattering of general algebra and discrete math. Generally, a third year college student, but any good high school student should be able to follow the development with a little help. We have attempted to be as complete as possible, though some proofs in Chapters 8 and 9 have been omitted, because the theory is more complex or is still in the process of being developed. Indeed, in the last few months of writing, Conway prevailed on us to change some notation for a class of all-small games. This *uptimal* notation turned out to be very useful and it makes its debut in this book.

We have liberally laced the theory with examples of actual games, exercises and problems. One way to understand a game is to have someone explain it to you; a better way is to muse while pushing some pieces around; and the best way is to play it against an opponent. Completely solving a game is generally hard, so we often present solutions to only some of the positions that occur within a game. The authors invented more games than they solved during the writing of this book. While many found their way into the book, most of these games never made it to the rulesets found at the end. A challenge for you, the reader of our missive, and as a test of your understanding, is to create and solve your own games as you progress through the chapters.

Since the first appearance of *On Numbers and Games* and *Winning Ways* there have been several conferences specifically on combinatorial games. The subject has moved forward and we present some of these developments. However, the interested reader will need to read further afield to find the theories of loopy games, misère-play games, other (non-disjunctive) sums of games, and the computer science approach to games. The proceedings of these conferences [Guy91, Now96, Now02, FN04] would be good places to start.

Organization of the Book

The main idea of this part of the theory of combinatorial games is the assigning of values to games, values that can be used to replace the actual games when deciding who wins and what the winning strategies might be.

Each chapter has a prelude which includes problems for the student to use as a warm-up for the mathematics to be found in the following chapter. The prelude also contains guidance to the instructor for how one can wisely deviate from the material covered in the chapter.

Exercises are sprinkled throughout each chapter. These are intended to reinforce, and check the understanding of, the preceding material. Ideally then, a student should try every exercise as it is encountered. However, there should be no shame associated with consulting the solutions to the exercises found at the back of the book if one or more of them should prove to be intractable. If that still fails to clear matters up satisfactorily, then it may be time to consult a *games guru.*

Chapter 0 introduces basic definitions and loosely defines that portion of game theory which we will address in the book. Chapter 1 covers some general strategies for playing or analyzing games and is recommended for those who have not played many games. Others can safely skim the chapter and review sections on an as-needed basis while reading the body of the work. Chapters 2, 4, and 5 contain the core of the general mathematical theory. Chapter 2 introduces the first main goal of the theory, that being to determine a game's *outcome class* or who should win from any position. Curiously, a great deal of the structure of some games can be understood solely by looking at outcome classes. Chapter 3 motivates the direction the theory takes next. Chapters 4, 5, and 6 then develop this theory (i.e., assigning values and the consequences of these values.)

Chapters 7, 8, and 9 look at specific parts of the universe of combinatorial games and as a result, these are a little more challenging but also more concrete since they are tied more closely to actual games. Chapter 7 takes an in-depth look at *impartial* games. The study of these games pre-dates the full theory. We place them in the new context and show some of the new classes of games under present study. Chapter 8 addresses hot games, games such as GO and AMAZONS in which there is a great incentive to move first. There is much research in this area and we can only give an introduction to this material. Chapter 9 looks at the analysis of *all-small* games. Most of the research emphasis has been on impartial and hot games. Only recently have there been developments in this area and we present the original and latest results in light of all the new developments in combinatorial game theory.

Chapter $\boldsymbol{\omega}$ is a brief listing of other areas of active research that we could not fit into an introductory text.

In Appendix A, we present top-down induction, an approach that we use often in the text. While the student need not read the appendix in its entirety, the first few sections will help ground the format and foundation of the inductive proofs found in the text.

Appendix B is a brief introduction to CGSuite, a powerful programming toolkit written by Aaron Siegel in Java for performing algebraic manipulations on games. CGSuite is to the combinatorial game theorist what Maple or Mathematica is to a mathematician or physicist. While the reader need not use CGSuite while working through the text, the program does help to build intuition, double-check work done by hand, develop hypotheses, and handle some of the drudgery of rote calculations.

Appendix D contains the rules to any game in the text that either appears multiple times or is found in the literature. We do not always state the rules to a game within the text, so the reader will want to refer to this appendix often.

The supporting website for the book is located at www.lessonsinplay.com. Look there for links, programs, and addenda, as well as instructions for accessing the online solutions manual for instructors.

Acknowledgments

While we are listed as the *authors* of this text, we do not claim to be the main contributors. The textbook emerged from a mathematically rich environment created by others. We got to choose the words and consequently, despite the best efforts of friends and colleagues, all the errors are ours.

Many of the contributors to this environment are cited within the book. There were many that also contributed to and improved the contents of the text itself and who deserve special thanks. We are especially grateful to Elwyn Berlekamp, John Conway, and Richard Guy who encouraged — and, at times, hounded — us to complete the text, and we hope it helps spawn a new generation of active aficionados.

Naturally, much of the core material and development is a reframing of material in *Winning Ways* and *On Number and Games.* We have adopted some of the proofs of J. P. Grossman, particularly that of the *Number-Avoidance Theorem.* Aviezri Fraenkel contributed the *Fundamental Theorem of Combinatorial Games,* which makes its appearance at the start of Chapter 2. Dean Hickerson helped us to prove Theorem 6.19 on page 126, that a game with negative incentives must be a number. Conway repeatedly encouraged us to adopt the *uptimal* notation in Chapter 9, and it took us some time to see the wisdom of his suggestions. Elwyn Berlekamp and David Molnar contributed some fine problems. Paul Ottaway, Angela Siegel, Meghan Allen, Fraser Stewart, and Neil McKay were students who pretested portions of the book and provided useful feedback, corrections, and clarifications. Elwyn Berlekamp, Richard Guy, Aviezri Fraenkel, and Aaron Siegel edited various chapters of our work for technical content, while Christine Aikenhead edited for language. Brett Stevens and Chris Lewis read and commented on parts of the book. Susan Hirshberg contributed the title of our book.

In this age of large international publishers, A K Peters is a fantastic and refreshing publishing house to work with. While they appreciate and under-

stand the business of publishing, we are convinced they care more about the dissemination of fine works than about the bottom line.

We thank our spice[2] for their loving support, and Lila and Tovia, who are the real *Lessons in Play.*

[2]affectionate plural of spouse

Preparation for Chapter 0

Before each chapter are several quick prep problems which are worth tackling in preparation to reading the chapter.

Prep Problem 0.1. List all the two-player games you know which do not involve chance (dice or coin flips).

Prep Problem 0.2. Locate the textbook website, www.lessonsinplay.com, and determine whether it might be of use to you.

To the instructor: Before each chapter, we will include a few suggestions to the instructor. Usually these will be examples which do not appear in the book, but which may be worth covering in lecture. The student unsatisfied by the text may be equally interested in seeking out these examples.

We highly recommend that the instructor and the student read Appendix A on top-down induction. We present induction in a way that will be unfamiliar to most, but which leads to more natural proofs, particularly those found in combinatorial game theory.

The textbook website, www.lessonsinplay.com, has instructions for how instructors can obtain a solution manual.

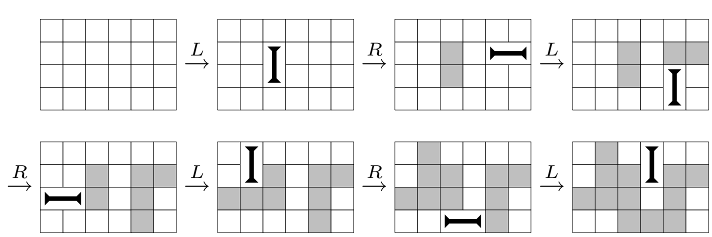
Chapter 0

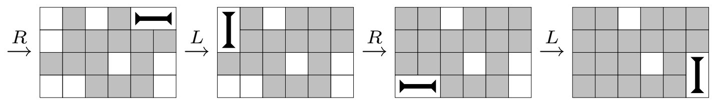
Combinatorial Games

We don't stop playing because we grow
old; we grow old because we stop playing.

George Bernard Shaw

This book is all about *combinatorial games* and the mathematical techniques that can be used to analyze them. One of the reasons for thinking about games is so that you can be more skillful and have more fun playing them; so let's begin with an example called DOMINEERING. To play you will need a chessboard and a set of dominoes. The domino pieces should be big enough to cover or partially cover two squares of the chessboard but no more. You can make do with a chessboard and some slips of paper of the right size or even play with pen or pencil on graph paper (but the problem there is that it will be hard to undo moves when you make a mistake!). The rules of DOMINEERING are simple. Two players alternately place dominoes on the chessboard. A domino can only be placed so that it covers two adjacent squares. One player, Louise, places her dominoes so that they cover vertically adjacent squares. The other player, Richard, places his dominoes so that they cover horizontally adjacent squares. The game ends when one of the players is unable to place a domino, and that player then loses. Here is a sample game on a 4×6 board with Louise moving first:

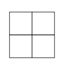
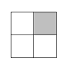




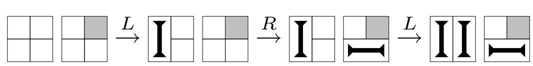
Since Louise placed the last domino, she has won.

Exercise 0.1. Stop reading! Find a friend and play some games of DOMINEERING. A game on a full chessboard can last a while so you might want to play on a 6×6 square to start with.

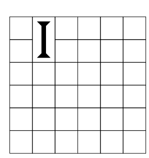
If you did the exercise then you probably made some observations and learned a few tactical tricks in DOMINEERING. One observation is that after a number of dominoes have been placed the board *falls apart* into disconnected regions of empty squares. When you make a move you need to decide what region to play in and how. Suppose that you are the vertical player and that there are two regions of the form:

 and 

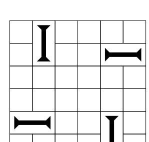
Obviously you could move in either region. However, if you move in the hook-shaped region then your opponent will move in the square. You will have no more moves left so you will lose. If instead you move in the square, then your opponent's only remaining move is in the hook. Now you still have a move in the square to make, and so your opponent will lose. If you are L and your opponent is R play should proceed



This is also why an opening move such as



is good since it *reserves* the two squares in the upper left for you later. In fact, if you play seriously for a while it is quite possible that the board after the first four moves will look something like:



Simply put, the aim of combinatorial game theory is to understand in a more detailed way the principles underlying the sort of observations we have just made about DOMINEERING. We will learn about games in general and how to understand them but, as a bonus, how to play them well!

0.1 Basic Terminology

In this section we will provide an informal introduction to some of the basic concepts and terminology that will be used and a description of how combinatorial games differ from some other types of games.

Combinatorial games

In a *combinatorial game* there are two players who take turns moving alternately. Play continues until the player whose turn it is to move has no legal moves available. No chance devices such as dice, spinners, or card deals are involved, and each player is aware of all the details of the game position (or game state) at all times. In this text, the rules of each game we study will ensure that it will end after a finite sequence of moves, and the winner is often determined on the basis of who made the last move. Under *normal play* the last player to move wins. In *misère* play the last player loses.

In fact, combinatorial game theory can be used to analyze some games that do not quite fit the above description. For instance, in DOTS & BOXES, players may make two moves in a row. Most CHECKERS positions are *loopy* and can lead to infinitely long sequences of moves. In GO and CHESS the last mover does not determine the winner. Nonetheless, combinatorial game theory has been applied to analyze positions in each of these games.

By contrast, the classical mathematical theory of games is concerned with *economic games.* In such games the players often play simultaneously and the outcome is determined by a payoff matrix. Each player's objective is to guarantee the best possible payoff against any strategy of the opponent. For a taste of economic game theory, see Problem 5.

The challenge in analyzing economic games stems from simultaneous decisions: each player must decide on a move without knowing the move choice(s) of her opponent(s). The challenge of combinatorial games stems from the sheer quantity of possible move sequences available from a given position.

Combinatorial game theory is most straightforward when we restrict our attention to *short games.* In the play of a short game, a position may never be repeated, and there are only a finite number of other positions that can be reached. We implicity (and sometimes explicitly) assume all games are short in this text.

Introducing the players

The two players of a combinatorial game are traditionally called *Left* (or just L) and *Right* (R). Various conventional rules will help you to recognize who is playing, even without a program:

Left	**Right**
Louise	Richard
Positive	Negative
bLack	White
bLue	Red
Vertical	Horizontal
Female	Male
Green	
Gray	

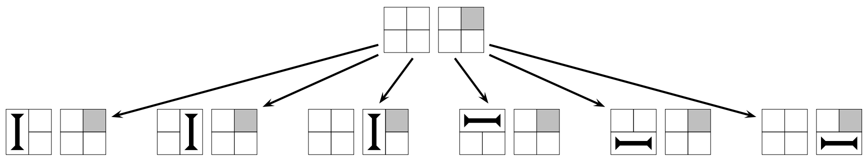
Alice and *Bob* will also make an appearance when the first player is important. To help remember all these conventions, note that despite the fact that they were introduced as long ago as the early 1980s in *WW* [BCG01], the chosen dichotomies reflect a relatively modern "politically correct" viewpoint.

Often, particularly in games involving pieces or in pen and paper games we will need a neutral color. If the game is between blue and red then this neutral color is green (because green is good for everyone!) while if it is between black and white then the neutral color is gray (because gray is neither black nor white!). Of course, this book is printed in black and white, so blue becomes black, red becomes white, and green becomes gray. That is,

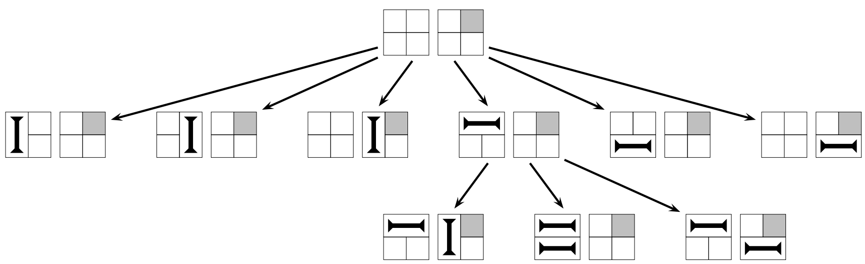
$$\begin{aligned} \text{blue} &= \text{black}, \\ \text{red} &= \text{white}, \\ \text{green} &= \text{gray}. \end{aligned}$$

Options

If a position in a combinatorial game is given and it happens to be Left's turn to move she will have the opportunity to choose from a certain set of moves determined by the rules of the game. For instance in DOMINEERING, where Left plays the vertical dominoes, she may place such a domino on any pair of vertically adjacent empty squares. The positions that arise from exercising these choices are called the *left options* of the original position. Similarly, the *right options* of a position are those which can arise after a move made by Right. The *options* of a position are simply the elements of the union of these two sets. We can draw a game tree of a position by diagrammatically listing its left and right options, with left options appearing below and to the left of the game:

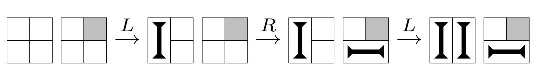


We can show as many or as few *game trees* of options as we wish:

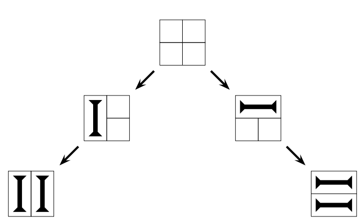


It may seem odd that we are showing two consecutive right moves in a game tree, but much of the theory of combinatorial games is based on analyzing situations where games *decompose* into several subgames. It may well be the case that in some of the subgames of such a decomposition the players do not alternate moves.

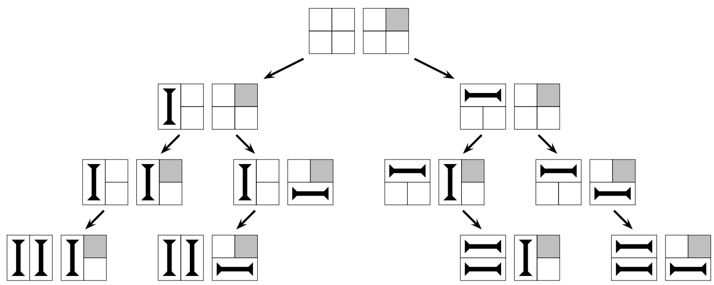
We saw this already in the DOMINEERING "square and hook" example. Left, if she wants to win, winds up making two moves in a row in the square:



Thus, we show the game tree for a square with Left and/or Right moving twice in a row:



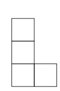
As we will see later, *dominated options* are often omitted from the game tree, when an option shown is at least as good:



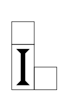
In some games the left options and the right options of a position are always the same. Such games are called *impartial*. The study of impartial combinatorial games is the oldest part of combinatorial game theory and dates back to the early twentieth century. On the other hand the more general study of non-impartial games was pioneered by John H. Conway in *ONAG* [Con01] and by Elwyn Berlekamp, John H. Conway, and Richard K. Guy in *WW* [BCG01]. Since "non-impartial" hardly trips off the tongue, and "partial" has a rather ambiguous interpretation it has become commonplace to refer to non-impartial games as *partizan games.*

To illustrate the difference between these concepts, consider a variation of DOMINEERING called CRAM. CRAM is just like DOMINEERING except that each player can play a domino in either orientation. Thus, it becomes an impartial game since there is now no distinction between legal moves for one player and legal moves for the other.

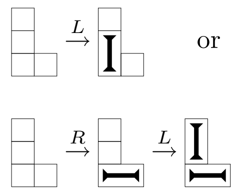
Let's look at a position in which there are only four remaining vacant squares in the shape of an L:



In CRAM the next player to play can force a win by playing a vertical domino at the bottom of the vertical strip, leaving



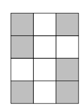
which contains only two non-adjacent empty squares and hence allows no further moves. In DOMINEERING if Left (playing vertically) is the next player, she can win in exactly this way. However, if Right is the next player his only legal move is to cover the two horizontally adjacent squares, which still leaves a move available to Left. So (assuming solid play) Left will win regardless of who plays first:



Much of the theory that we will discuss is devoted to finding methods to determine who will win a combinatorial game assuming sensible play by both sides. In fact, the eventual loser has no really *sensible* play[1] so a *winning strategy* in a combinatorial game is one that will guarantee a win for the player employing it no matter how his or her opponent chooses to play. Of course such a strategy is allowed to take into account the choices actually made by the opponent — to demand a uniform strategy would be far too restrictive!

Problems

1. Consider the position:



   (a) Draw the complete game trees for both CRAM and DOMINEERING. The leaves (bottoms) of the tree should all be positions in which neither player can move. If two left (or right) options are symmetrically identical, you may omit one.

   (b) Who wins at DOMINEERING if Vertical plays first? Who wins if Horizontal plays first? Who wins at CRAM?

2. Suppose that you play DOMINEERING (or CRAM) on *two* 8×8 chessboards. At your turn you can move on either chessboard (but not both!). Show that the second player can win.

3. Take the ace through five of a suit from a deck of cards and place them face up on the table. Play a game with these as follows. Players alternately pick a card and add it to the righthand end of a row. If the row ever contains a sequence of three cards in increasing order of rank (ace is low), or in decreasing order of rank, then the game ends and the player who formed that sequence is the winner. Note that the sequence need not be

[1] Unless he has some ulterior motive not directly related to the game such as trying to make it last as long as possible so that the bar closes before he has to buy the next round of drinks.

consecutive either in position or value, so for instance, if the play goes $4, 5, 2, 1$ then the $4, 2, 1$ is a decreasing sequence.

(a) Show that this is a proper combinatorial game (the main issue is to show that draws are impossible).

(b) Show that the first player can always win.

4. Start with a heap of counters. As a move from a heap of n counters, you may either:

   - assuming n is not a power of 2, remove the largest power of 2 less than n; or
   - assuming n is even, remove half the counters.

   Under normal play, who wins? How about misère play?

5. The goal of this problem is to give the reader a taste of what is *not* covered in this book. Two players play a 2×2 *zero-sum matrix game.* (Zero sum means that whatever one person loses, the other gains.) The players are shown a 2×2 matrix of positive numbers. Player A chooses a row of the matrix, and player B simultaneously chooses a column. Their choice determines one matrix entry, that being the number of dollars B must pay A. For example, suppose the matrix is

$$\begin{pmatrix} 1 & 4 \\ 3 & 2 \end{pmatrix}.$$

   If player A chooses the first row with probability $1/4$, then no matter what player B's strategy is, player A is *guaranteed* to get an average of \$2.50. If, on the other hand, player B chooses the columns with 50-50 odds, then no matter what player A does, player B is *guaranteed* to have to pay an average of \$2.50. Further, neither player can guarantee a better outcome, and so B should pay player A the fair price of \$2.50 to play this game.

   In general, if the entries of the matrix game are

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

   as a function of a, b, c, and d, what is the fair price which B should pay A to play? (Your answer will have several cases.)

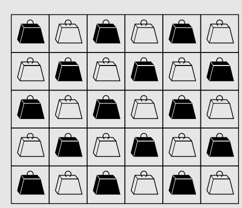
Preparation for Chapter 1

Prep Problem 1.1. Play DOTS & BOXES with a friend or classmate. The rules are found on page 267 of Appendix D. You should start with a 5×6 grid of dots. You should end up with a 4×5 grid of 20 boxes, so the game might end in a tie.

When playing a game for the first time, feel free to move quickly to familiarize yourself with the rules and to get a sense for what can happen in the game.

After a few games of DOTS & BOXES, write a few sentences describing any observations you have made about the game. Perhaps you found a juncture in the game when the nature of play changes? Did you have a strategy? (It need not be a good strategy.)

Prep Problem 1.2. Play CLOBBER with a friend or classmate. The rules are found on page 265 of Appendix D. (Note that if the Winner is not specified in a ruleset, you should assume normal play, that the last legal move wins.) You should start with a 5×6 grid of boxes:



Jot down any observation you have about the game.

Prep Problem 1.3. Play NIM (rules on page 271) with a friend or classmate. Begin with the three heap position with heaps of sizes 3, 5, and 7.

To the instructor: While DOTS & BOXES is a popular topic among students, it also takes quite a bit of time to appreciate. View the topic as optional. If you do cover it, allow time for students to play practice games. Another option is to cover it later in the term before a holiday break.

Chapter 1
Basic Techniques

If an enemy is annoying you by playing
well, consider adopting his strategy.

Chinese proverb

There are some players who seem to be able to play a game well immediately after learning the rules. Such gamesters have a number of tricks up their sleeves that work well in many games without much forethought. In this chapter we will teach you some of these tricks or, to use a less emotive word, *heuristics*.

Of course, the most interesting games are those to which none of the heuristics apply directly, but knowing them is still an important part of getting started with the analysis of more complex games. Often, you will have the opportunity to consider moves that lead to simple positions in which one or more of the heuristics apply. Those positions are then easily understood, and the moves can accordingly be taken or discarded.

1.1 Greedy

The simplest of the heuristic rules or strategies is called the *greedy strategy.* A player who is playing a greedy strategy grabs as much as possible whenever possible. Games which can be won by playing greedily are not terribly interesting at all — but most games have some aspects of greedy play in them. For instance, in CHESS it is almost always correct to capture your opponent's queen with a piece of lesser value (taking a greedy view of "getting as much material advantage as possible") but not if doing so allows your opponent to capture your queen, or extra material, or especially not if it sets up a checkmate for the opponent. Similarly, the basic strategy for drawing in TIC TAC TOE is a greedy one based on the idea "always threaten to make at least one line, or block any threat of your opponent."

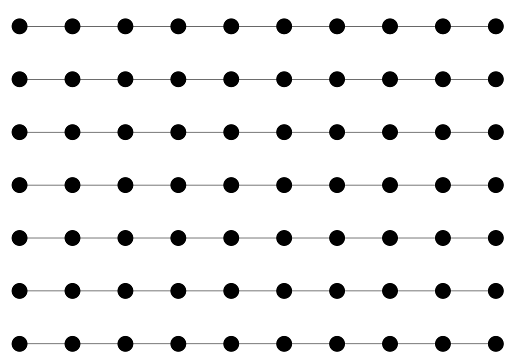
Definition 1.1. A player following a *greedy strategy* always chooses the move that maximizes or minimizes some quantity related to the game position after the move has been made.

Naturally, the quantity on which a greedy strategy is based should be easy enough to calculate that it does not take too long to figure out a move. If players accumulate a *score* as they play (where the winner is the one who finishes with the higher score), then that score is a natural quantity to try to maximize at each turn.

Does a greedy strategy always work? Of course not, or you wouldn't have a book in front of you to read. But in some very simple games it does. In the game GRAB THE SMARTIES[1] each player can take at his or her move any number of Smarties from the box, provided that they are all the same color. Of course, assuming that each player wants to collect as many Smarties as possible, the greedy strategy is ideal for this sort of game. You just grab all the Smarties of some color, and the color you choose is the one for which your grab will be biggest.

Sometimes though, a little subtlety goes a long way.

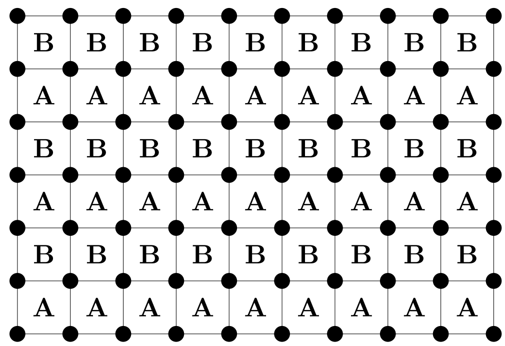
Example 1.2. Below is the board after the first moves in a very boring game of DOTS & BOXES:



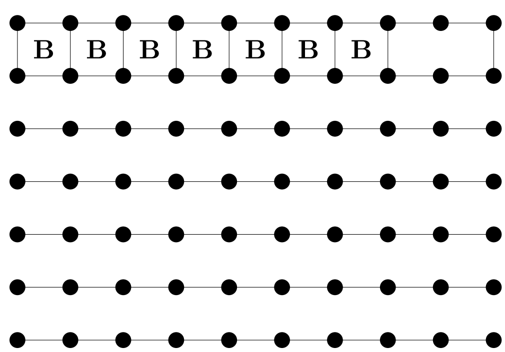
Suppose that it is now Alice's turn. No matter where Alice moves, Bob can take all the squares in that row. If he does so, he then has to move in another

[1] An American player might play the less tasty variant, GRAB THE M&MS.

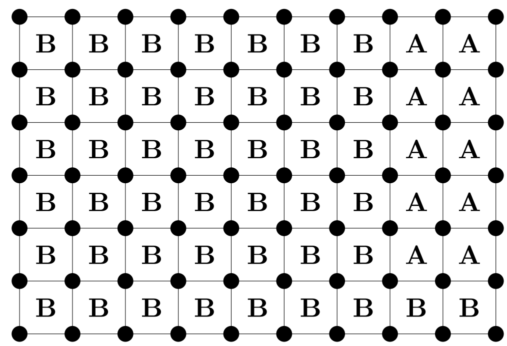
row, and Alice can take all the squares in this row. They trade off in this way until they both have 27 boxes and the game is tied:



But Bob is being too greedy. Instead of taking all the squares in the row that Alice opens up, he should take all but two of them, then make the *double-dealing move,* which gives away the last two boxes to Alice. For example, if Alice moves somewhere towards the left-hand end of the first row, Bob replies to:



Alice now has a problem. Regardless of whether she takes the two boxes Bob has left for her, she still has to move first in another row. So she might as well take the last two in a *double-cross* (since otherwise Bob will get them on his next turn) but she then has to give seven boxes of some other row to Bob in his next turn. By repeating this strategy for each but the last row, where he takes all the boxes, Bob finishes with $5 \times 7 + 9 = 44$ boxes while Alice gets only $5 \times 2 = 10$ boxes:



Again, a move that makes two boxes with one stroke is called a *double-cross.* (A person who makes such a play might feel double-crossed for having to reply to a *double-dealing move.*)

It is not a bad idea to use a greedy strategy on your first attempt at playing a new game, particularly against an expert. It is easy to play, so you won't waste time trying to figure out good moves when you don't know enough about the game to do so. And when the expert makes the moves that refute your strategy (i.e., exposes the traps hidden in the greedy strategy) then you can begin to understand the subtleties of the game.

Exercise 1.3. Now that you have learned a bit of strategy, play DOTS & BOXES against a friend. (If you did the prep problems for this chapter, you already played once. Now, you may be able to play better.)

1.2 Symmetry

A famous CHESS wager goes as follows: An unknown CHESS player, Jane Pawnpusher, offers to play two games, playing the white pieces against Garry Kasparov and black against Anatoly Karpov simultaneously. She wagers $1 million dollars that she can win or draw against one of them. Curiously, she can win the wager without knowing much about CHESS. How?

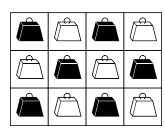
What she does is simply wait for Karpov to make a move (white moves first in CHESS), and whatever Karpov does, she makes the same move as her first move against Kasparov. Once Kasparov replies, she plays Kasparov's reply against Karpov. If Kasparov beats her, she will beat Karpov the same way.[2] A strategy that maintains a simple symmetry is called *Tweedledum-Tweedledee.*

[2]Readers familiar with cryptography may observe Jane is making a man-in-the-middle attack.

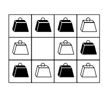
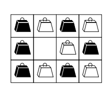
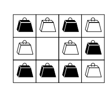
Example 1.4. The Tweedledum-Tweedledee strategy is effective in two-heap NIM. If the two heaps are of the same size then you should invite your opponent to move first. She must choose a heap and remove some counters. You choose the other heap and take away the same number of counters leaving two equal sized heaps again. On the other hand, if the game begins with two heaps of different sizes, you should rush to make the first move, taking just enough counters from the larger heap to make them equal. Thereafter, you adopt the Tweedledum-Tweedledee approach.

Symmetry is an intuitively obvious strategy. Whenever your opponent does something on one part of the board you should mimic this move in another part. Deciding how this mimicry should happen is the key. To be played successfully, you should not leave a move open to your opponent that allows him to eliminate your mimicking move.

Example 1.5. If Black moves from the 3×4 CLOBBER game



to any of the three positions

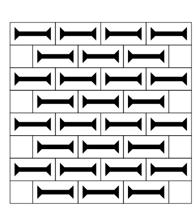
 ,  , or 

then White can play the remainder of the game using a 180 degree symmetry strategy. This establishes that each of these three moves for Black was a poor choice on her first turn. In fact, from this position it happens to be the case that she simply has no good first moves, but the rest of her initial moves cannot be ruled out so easily due to symmetry.

Sometimes, symmetry can exist which is not apparent in the raw description of a game.

Example 1.6. Two players take turns putting checkers down on a checkerboard. One player plays black, one plays white. A player who completes a 2×2 square with four checkers of one color wins.

This game should end in a draw. First, tile most of the checkerboard with dominoes using a brickwork pattern:



If your opponent plays in a domino, you respond in the same domino. If you cannot (because you move first, or the domino already is filled, or your opponent fails to play in a domino) play randomly. Since every 2×2 square contains a domino, your opponent cannot win.

Exercise 1.7. Two players play $m \times n$ CRAM.

1. If m and n are even, who should win? The first player or the second player? Explain your answer.

2. If m is even and n is odd, who should win? Explain.

(When m and n are odd, the game remains interesting.)

1.3 Change the Game!

Sometimes a game is just another game in disguise. Sometimes one view can be more, or less, intuitive than the other.

Example 1.8. In 3-TO-15, there are nine cards, face up, labeled with the digits $\{1, 2, 3, \ldots, 9\}$. Players take turns selecting one card from the remaining cards. The first player who has three cards adding up to 15 wins.

This game should end in a draw. Surprisingly, this is simply TIC TAC TOE in disguise! To see this, construct a *magic square* where each row, column, and diagonal add up to 15:

4	9	2
3	5	7
8	1	6

You can confirm that three numbers add up to 15 if and only if they are in the same TIC TAC TOE line. Thus, you can treat a play of 3-TO-15 as play of TIC TAC TOE. Suppose that you are moving first. Choose your TIC TAC TOE move, note the number on the corresponding square, and select the corresponding card. When your opponent replies by choosing another card, mark the TIC TAC TOE board appropriately, choose your TIC TAC TOE response, and again take the corresponding card. Proceed in this fashion until the game is over. So if you can play TIC TAC TOE, you can play 3-TO-15 just as well.

Exercise 1.9. Play 3-TO-15 against a friend. As you and your friend move, mark the magic square TIC TAC TOE board with Xs and Os to convince yourself the game really is the same.

1.4 Parity

Parity is a critical concept in understanding and analyzing combinatorial games. A number's *parity* is whether the number is odd or even. In lots of games, only the parity of a certain quantity is relevant — the trick is to figure out just what quantity! With the normal play convention that the last player with a legal move wins, it is always the objective of the first player to play to ensure that the game lasts an odd number of moves, while the original second player is trying to ensure that it lasts an even number of moves.

This is part of the reason why symmetry as we mentioned earlier is also important — it allows the second player (typically) to think of moves as being blocked out in pairs, ensuring that she has a response to any move her opponent might make.

The simplest game for which parity is important is called SHE LOVES ME SHE LOVES ME NOT. This game is played with a single daisy. The players alternately remove exactly one petal from the daisy and the last player to remove a petal wins. Obviously, all that matters is the original parity of the number of petals on the daisy. If it is odd then the first player will win; if it is even, the second player will win.

More usually SHE LOVES ME SHE LOVES ME NOT is delivered in some sort of disguise.

Example 1.10. Take a heap of 29 counters. A move is to choose a heap (at the start there is only one) and split it into two non-empty heaps. Who wins?

Imagine the counters arranged in a line. A move effectively is to put a bar between two counters. This corresponds to splitting a heap into two: the counters to the left and those to the right up to the next bar or end of the row. There are exactly 28 moves in the game. The game played with a heap of n counters has exactly $n - 1$ moves! The winner is the first player if n is even and the second player if n is odd.

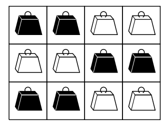
Exercise 1.11. A chocolate bar is scored into smaller squares or rectangles. (Lindt's Swiss Classic, for example, is 5×6.) Players take turns picking up one piece (initially the whole bar), breaking the piece into two along a scored line, and setting the pieces back down. The player who moves last wins. Our goal is to determine *all* winning moves.

1.5 Give Them Enough Rope!

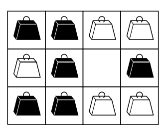
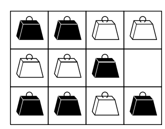
The previous strategies are all explicit and when they work, you can win the game. This section is about confounding your opponent in order to gain time for analysis.

If you are in a losing position, it pays to follow the *Enough Rope Principle:* Make the position as complicated as you can with your next move.[3] Hopefully, your opponent will tie himself up in knots while trying to analyze the situation.

For example, suppose you are black and are about to move from the following CLOBBER position:



If you are more astute than the authors, you could conclude you have no winning moves. However, you should probably not throw in the towel just yet. But you also should not make any moves for which your opponent has a simple strategy for winning. If your opponent has read this chapter, you should avoid capturing an edge piece with your center piece, for then White can play a rotationally symmetric strategy. However, there are several losing responses from either of the positions


or


and so these moves, while losing, are reasonable.

The *Enough Rope Principle* has other implications as well. If you are confused about how best to play, do not simplify the position to the point where your opponent will not be confused, especially if you are the better player.

The converse applies as well. If you are winning, play moves which simplify. Do not give you opponent opportunities to complicate the position, lest you be hoist by your own petard.

Don't give them any rope

This is contrary to the advice in the rest of the section. If you do not know you are losing the game and you are playing against someone of equal or less game-playing ability then a very good strategy is to move so as to restrict the number of options your opponent has and increase the number of your own options. This is a heuristic that is often employed in first attempts to produce programs that will play games reasonably well. This has been used in AMAZONS, CONNECT-4, and OTHELLO.

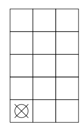
[3]At least one of the authors feels compelled to add, *except if you are playing against a small child.*

1.6 Strategy Stealing

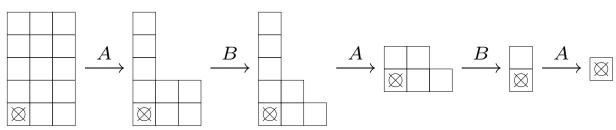
Strategy stealing is a technique whereby one player steals another player's strategy. Why would you want to steal a strategy? Let's see....

CHOMP

The usual starting position of a game of CHOMP consists of a rectangle with one poison square in the lower-left corner:



A move in CHOMP is to choose a square and to remove it and all other squares above or to the right of it. A game between players Alice and Bob might progress:



And Bob loses for he must take the poison square.

Theorem 1.12. CHOMP, *when played on a rectangular board larger than* 1×1, *is a win for the first player.*

Proof: Suppose the first player chomps *only* the upper-right square of the board. If this move wins, then it is a first-player win. If, on the other hand, this move loses, then the second player has a winning response chomping all squares aboves or to the right of some square, x. But move x was available to the first player on move one, and removes the top right square, so the first player has move x as a winning first move. □

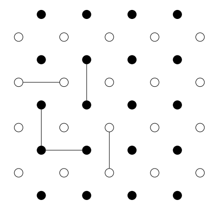
This is a *non-constructive proof* in that the proof gives no information about what the winning move is. The proof can be rephrased as a *guru argument,* echoing the CHESS wager of Section 1.2:

Alternate proof: Suppose, to the contrary, that CHOMP can be won by the second player. Jane Pawnpusher can arrange to play first against both Garry Kasparov and Anatoly Karpov simultaneously, both *gurus* in the arena of games.

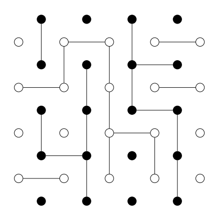
Jane takes the upper-right square on board 1 and then waits for Kasparov's response. Whatever Kasparov does, she plays the same move on board 2. She can now maintain both boards in the same position, making sure it is her move on one board and her opponent's on the other. Hence, she will win on one of the two boards, contradicting the supposition. □

BRIDG-IT

The game of BRIDG-IT is played on two offset grids of black and white dots. Here is a position after 5 moves; Black played first:

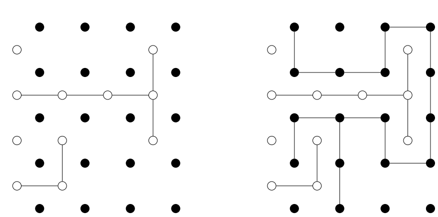


The players, Black and White, alternate drawing horizontal or vertical lines joining adjacent dots of the player's chosen color. Black wins by connecting any dot in the top row to any dot in the bottom row by a path. White is trying to connect the left side to the right side. In the following position, Black has won, with a path near the right side of the board:



Lemma 1.13. BRIDG-IT *cannot end in a draw.*

Sketch of proof: The game is unaffected if we consider the top and bottom rows of black dots as connected. Suppose the game has ended, and neither player can win. Let S be the set of nodes that White can reach from the left side of the board. Then, starting from the upper-left black dot, Black can go from the top to the bottom edge by following the boundary of the set S. As an example, set S consists of the white dots below, and the black path following the boundary is shown on the right:



All the edges connecting black dots must be present, for otherwise set S could be extended. □

Theorem 1.14. *The first player wins at* BRIDG-IT*, where the starting board is an $n \times n+1$ grid of white dots overlapping an $n+1 \times n$ grid of black dots.*

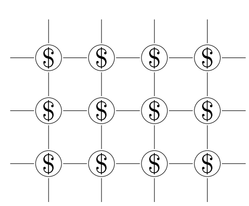
Proof: Note that the board is symmetric when reflected about the main diagonal. If the second player has a winning strategy, the first player can adopt it. In particular, before her first move, the first player pretends that some random invisible move x has been made by the opponent, and then responds as the second player would have. If the opponent's n^{th} move is move x, then the first player pretends that the opponent actually played the n^{th} move at some other location x'. Continuing in this fashion, the first player has stolen the winning second-player strategy, with the only difference that the opponent always has one fewer line on the board. This missing move can be no worse for the first player, and so she will win. □

There are explicit winning strategies for BRIDG-IT, avoiding the need for a non-constructive strategy stealing argument. So, not only can you find out you *should* win, but by utilizing a bit of elementary graph theory, you can also find out *how* to win! See, for example, [BCG01, volume 3, pp. 744–746] or [Wes01, pp. 73–74].

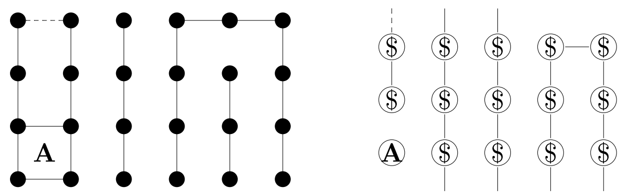
1.7 Case Study: Long Chains in Dots & Boxes

We already observed in Example 1.2 on page 12 that the first player to play on a long chain in a DOTS & BOXES game typically loses. In this section, we will investigate how that can help a player win against any first grader.

First, consider a dual form of DOTS & BOXES called STRINGS & COINS. Here is a typical starting position:

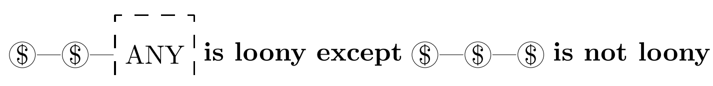


A move in STRINGS & COINS consists of cutting a string. If a player severs the last of the four strings attached to a coin, the player gets to pocket the coin and must move again. This game is the same as DOTS & BOXES but disguised: A coin is a box, and cutting a string corresponds to drawing the line between two boxes. For example, here is a DOTS & BOXES position and its dual STRINGS & COINS position:

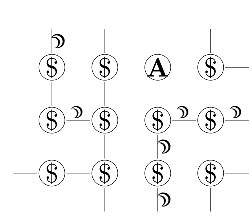


Drawing the dotted line in DOTS & BOXES corresponds to cutting the dotted string in STRINGS & COINS. We will investigate STRINGS & COINS positions, since people tend to find that important properties of the positions (such as long chains) are easier to visualize and identify in this game than in DOTS & BOXES.

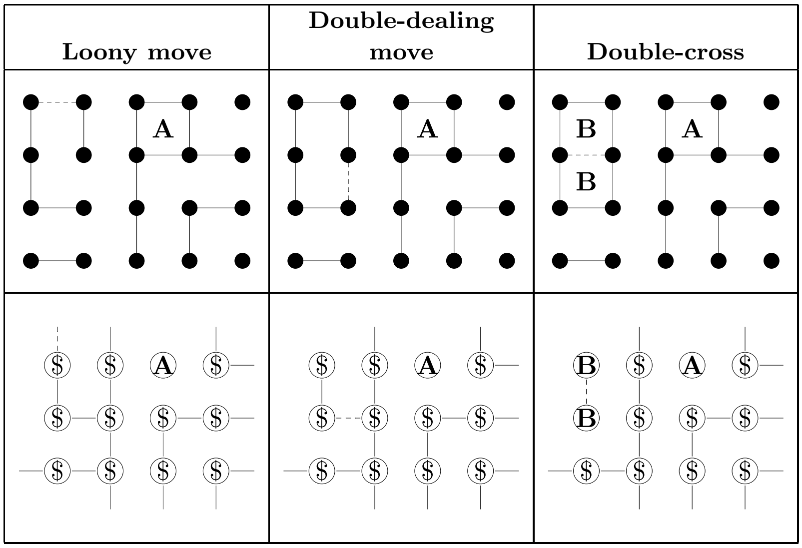
Positions of the following form are termed *loony:*



The hidden portion of the position (in the box marked ANY) can be any position except a single coin. The defining characteristic shared by all loony positions is that the next player to move has a choice of whether or not to make a double-dealing move. A *loony move,* denoted by a ☽, is any move to a loony position. All loony moves are labeled in the following DOTS & BOXES and equivalent STRINGS & COINS positions:



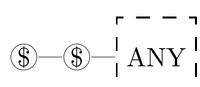
To summarize, if Bob makes a loony move (a move to a loony position), Alice may (or may not) reply with a double-dealing move. From there, Bob might as well double-cross before moving elsewhere.



(Bob must move again after the double-cross.)

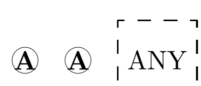
Theorem 1.15. *Under optimal play from a loony position, the player to move next can get at least half the remaining coins.*

Proof: Suppose that player Alice is to move next from the loony position:

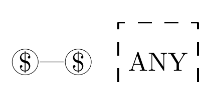


Consider playing only on the hidden position in the box marked ANY (without the two extra coins). Suppose the player moving next from ANY can guarantee pocketing n coins. In the full position, Alice has at least two choices:

- Pocket the two coins (making two cuts) and move on ANY, pocketing n more coins for a total of $n + 2$:



- Sacrifice the two coins, cutting off the pair in one move. Whether or not the opponent chooses to pick up the two coins, he must move next on ANY, and so the most he can pocket is $n + 2$ coins:



Thus, Alice can collect $n+2$ coins, or all but $n+2$ coins. One of these two numbers is at least half the total number of coins! □

In practice, the player about to move often wins decisively, especially if there are very long chains.

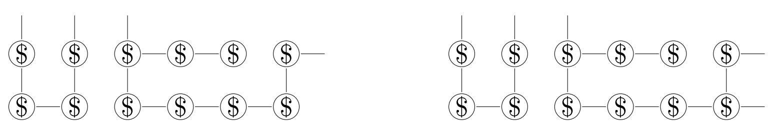
Hence, from most positions, a ☽ move (i.e., a move to a ☽ position) is a losing move and might as well be illegal when making a first pass at understanding a position.

A *long chain* consists of $k \geq 3$ coins and **exactly** $k+1$ strings connected in a line:

Notice that any move on a long chain is ☽.

Exercise 1.16. Find the non-☽ move(s) on a (short) chain of length 2:

Exercise 1.17. Here are two separate STRINGS & COINS positions. Alice is about to play in each game:



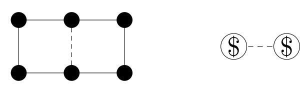
1. Both positions are loony. Explain why.
2. In one of the positions, Alice should make a double-dealing move. Which one? Why?
3. Estimate the score of a well-played game from each of the two positions. (Alice should be able to win either game.)

So it is crucial to know whose move it is if only long chains remain, for in such a position all moves are loony and Theorem 1.15 tells us the player about to move will likely lose. To this end, consider a position with only long chains. We distinguish a move (drawing a line or cutting a string) from a turn, which may consist of several moves.

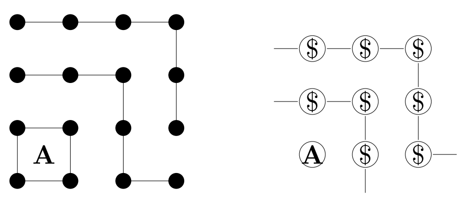
Define

$$\begin{array}{lcl}
M^- & = & \text{Number of moves played so far;} \\
M^+ & = & \text{Number of moves remaining to be played;} \\
M & = & M^+ + M^- = \text{Possible moves from the start position;} \\
T & = & \text{Number of turn transfers so far;} \\
B^- & = & \text{Number of boxes (or coins) taken already;} \\
B^+ & = & \text{Number of boxes (or coins) left to be taken;} \\
B & = & B^+ + B^- = \text{Total number of boxes in the start position;} \\
C & = & \text{Number of long chains;} \\
D & = & \text{Number of double-crosses so far.}
\end{array}$$

Recall that double-crosses are single moves that take two coins in one cut (or complete two boxes in one stroke):



We can compute the above quantities for the following position:



$$\begin{array}{lcl}
M^- & = & 14; \\
M^+ & = & 10; \\
M & = & 24; \\
T & = & 13; \\
B^- & = & 1; \\
B^+ & = & 8; \\
B & = & 9; \\
C & = & 2; \\
D & = & 0.
\end{array}$$

(If there were two adjacent boxes taken by the same player, we would not know if $D = 0$ or $D = 1$ without having watched the game in progress.)

We will now proceed to set up equations describing (as best we can) the state of affairs when we are down to just long chains.

- Since every long chain has one more string than coin, we know

$$M^+ = C + B^+.$$

- Since every move either completes a turn, completes a box, or completes two boxes,
$$M^- = T + B^- - D.$$

Adding these equations, we conclude

$$M = C + T + B - D.$$

Whose turn it is depends only on whether T is even or odd. M and B are fixed at the start of the game, so whose turn it is is determined by the number of long chains and the number of double crosses. We have all but proved:

Theorem 1.18. *If a* STRINGS & COINS *(or* DOTS & BOXES*) position is reduced to just long chains, player P can earn most of the remaining boxes, where*

$$P \equiv M + C + B + D \pmod 2,$$

where the first player to move is player $P = 1$, and her opponent is player $P = 2$ (or, if you like, $P = 0$).

Proof: By the discussion preceding the theorem,

$$M = C + T + B - D.$$

If all that remains are long chains, whoever is on move must make a loony move, which by Theorem 1.15 guarantees the last player can take at least half the remaining coins. If you are player 1, say, then your opponent is on move if an odd number of turns have gone by since the start of the game; i.e., T is odd. T is odd if and only if $M - C - B + D$ is odd; i.e., if and only if $P \equiv M + C + B + D \pmod 2$. (Replace odd by even for $P = 2$.) □

In a particular game, viewed from a particular player's perspective, P, M, and B are all constant, and D is nearly always 0 (until someone makes a loony move.) So the parity of T depends only on C, a quantity that depends on the actual moves made by the players.

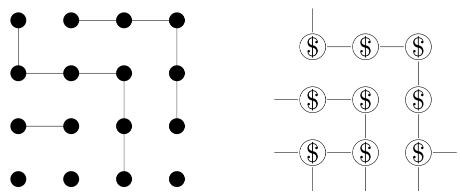
In summary, when you sit down to a game of DOTS & BOXES, count

$$P + M + B,$$

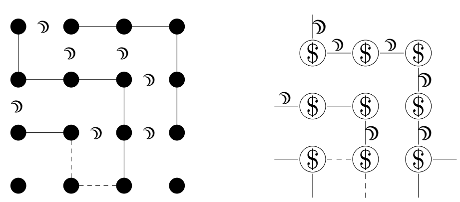
where P is your player number. You seek to ensure that the parity of C, the number of long chains matches this quantity.[4] That is, you want to arrange that $C \equiv P + M + B \pmod 2$. If you can play to make the parity of the long chains come out in your favor, you will usually win.

[4]In rectangular DOTS & BOXES boards, the number of dots is $1 + M + B \pmod 2$, and some players prefer to count the dots. This is the view taken in [Ber00].

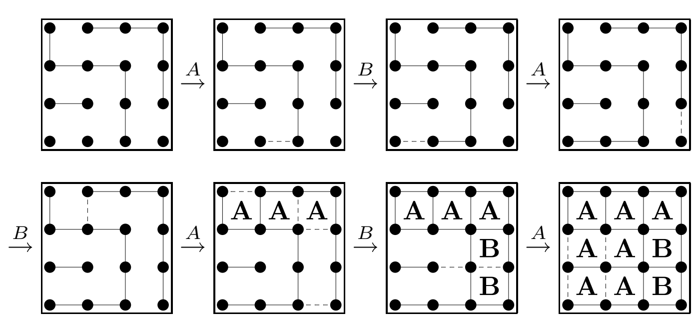
An example is in order. Alice played first against Bob, and they reach the following position with Alice to play:



At the start of the game, Alice computed $P + M + B = 1 + 24 + 9$, an even quantity, and therefore knows that she wishes for an even number of long chains. Having identified all loony moves, she knows that the chain going around the upper and right sides will end in one long chain *unless* someone makes a loony (losing) move. So, she hopes that the lower-left portion of the board ends in a long chain. Two moves will guarantee that end, those marked below with dashed lines:

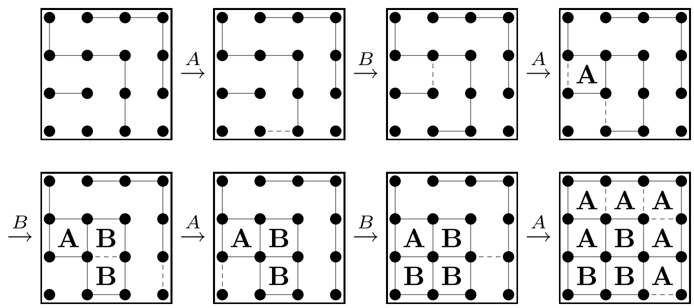


Of the two moves, Alice prefers the horizontal move, since that lengthens the chain; since she expects to win most of the long chains, longer chains favor her. If Bob is a beginner, a typical game might proceed:



with Alice winning 7 to 2.

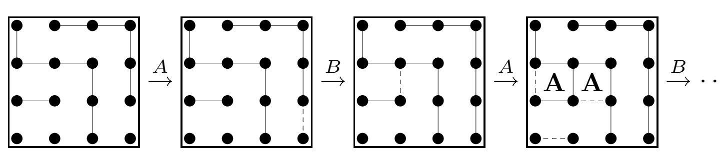
A more sophisticated Bob might recognize that he has lost the game of long chains and might try to finagle a win by playing loony moves earlier. This has the advantage of giving Alice fewer points for her long chains. A sample game between sophisticated players might go:



Not only does Bob lose by only 6-3, Bob might win if Alice hastily takes all three boxes in the first long chain!

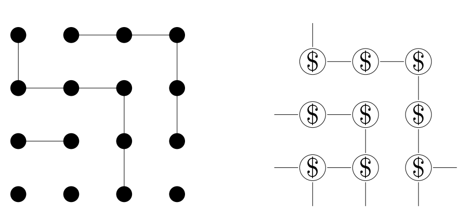
Exercise 1.19. What is the final score if Alice takes all three boxes instead of her first double-dealing move in the last game? Assume both players play their best thereafter.

Suppose Alice fails to make a proper first move. Bob can then steal control by sacrificing two boxes (without making a loony move), breaking up the second long chain. For example, play might proceed:



In this game, Alice should get the lower-left four boxes, but Bob will get the entire upper-right chain, winning 5 to 4.

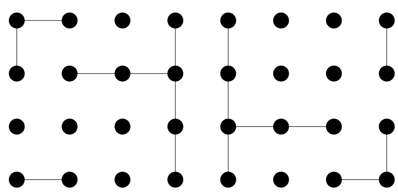
Lastly, note that nowhere in the discussion leading up to or in the proof of Theorem 1.18 did we use any information about what the start position is. Consequently, if you come into a game already in play, you can treat the current position as the start position! In our example game between Alice and Bob repeated below, since M, the number of moves available from this start position, is four and B, the number of boxes still available, is nine, the player on move (who we dub player 1 from this start position) wants an even number of long chains:

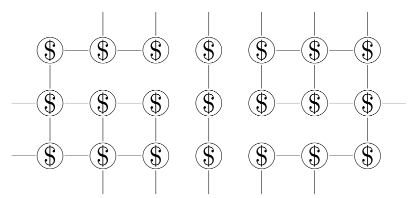


Warning: Not all games reach an endgame consisting of long chains. There are other positions in which all moves are loony. See, for example, Problem 18. These sorts of positions come up more often the larger the board size.

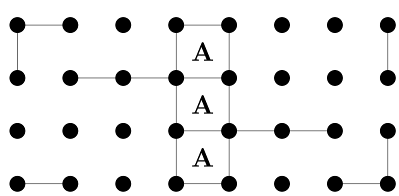
Enough Rope revisited

In the following DOTS & BOXES (or the equivalent STRINGS & COINS position), Bob has stumbled into Alice's trap, and Alice is now playing a symmetry strategy. Note that the upper-right box and the lower-left box are, in fact, equivalent. If this is not obvious from the DOTS & BOXES position, try looking at the corresponding STRINGS & COINS position, where the upper-right dangling string could extend toward the right without changing the position:

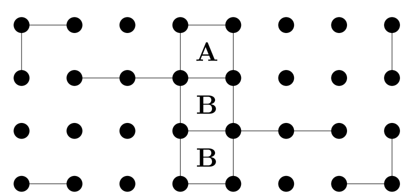




If Bob allows Alice to keep symmetry to the end of the game, Alice will succeed in getting an odd number of long chains and win. So Bob should make a loony move now on the long chain, forcing Alice to choose between taking the whole chain or making one box and a double-dealing move:



or



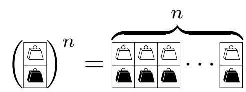
While Theorem 1.15 guarantees that Alice can win from one of the two positions (Alice first takes one box and then uses the theorem to assure half the remainder), the proof of the theorem gives no guidance about *how* to win.

If Alice chooses the first option, it is now her move on the rest of the board; she cannot play symmetry. If, on the other hand, she chooses the second option, Bob has gained a one-point advantage, which he may be able to parley into a win.

Problems

Note that a few problems require some familiarity with graph theory. In particular, Euler's Formula, Theorem A.7 on page 223, will come in handy.

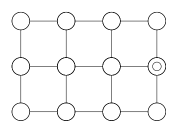
1. Consider the $2 \times n$ CLOBBER position



Show that if n is even then

is a second-player win. (By the way, the first player wins when $n \leq 13$ is odd and, we conjecture, for all n odd.)

2. Prove that Left to move can win in the COL position



3. Suppose two players play STRINGS & COINS with the additional rule that a player, on her turn, can spend a coin to end her turn. The last player to play wins. (Spending a coin means discarding a coin that she has won earlier in the game.)

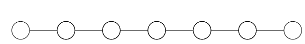
   (a) Prove that the first player to take any coin wins.

   (b) Suppose the players play on an m-coin by n-coin board with the usual starting position. Prove that if $m + n$ is even, the second player can guarantee a win.

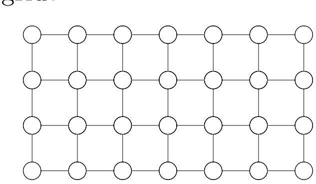
   (c) Prove that if $m + n$ is odd, the first player can guarantee a win.

4. Two players play the following game on a round tabletop of radius R. Players take turns placing pennies (of unit radius) on the tabletop, but no penny is allowed to touch another or to project beyond the edge of the table. The first player who cannot legally play loses. Determine who should win as a function of R.[5]

5. Who wins SNORT when played on a path of length n?



How about an $m \times n$ grid?



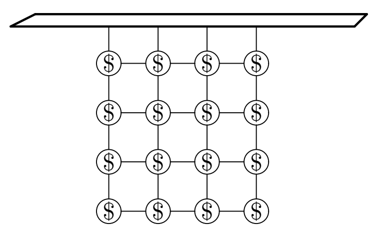
[5]The players are assumed to have perfect fine motor control!

6. The game of ADD-TO-15 is the same as 3-TO-15 (page 16) except that the first player to get *any* number of cards adding to 15 wins. Under perfect play, is ADD-TO-15 a first-player win, second-player win, or draw?

7. The following vertex deletion game is played on a directed graph. A player's turn consists of removing any single vertex with even indegree (and any edges into or out of that vertex.) Determine the winner if the start position is a directed tree, with all edges pointing toward the root.

8. Two players play a vertex deletion game on an undirected graph. A turn consists of removing exactly one vertex of even degree (and all edges incident to it.) Determine the winner.

9. A bunch of coins is dangling from the ceiling. The coins are tied to one another and to the ceiling by strings as pictured below. Players alternately cut strings, and a player whose cut causes any coins to drop to the ground loses. If both players play well, who wins?



10. The game of SPROUTLETTES is played like SPROUTS, only the maximum allowed degree of a node is only 2. Who wins in SPROUTLETTES?

11. Find a winning strategy for BRUSSELS SPROUTS. (*Hint*: Describe how end positions must look and deduce how many moves the game lasts.)

12. How many moves does a game of SPROUTS last as a function of both the number of initial dots and the number of isolated degree two nodes at the end of the game? Give a rule for playing SPROUTS analogous to the number of long chains in DOTS & BOXES.

13. Prove that the first player wins at HEX. You are free to find and present a proof you find in the literature, but be sure to cite your source and rephrase the argument in your own words.

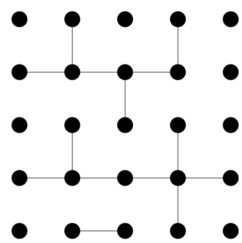
14. SQUEX is a game like HEX but is played on a square board. A player makes a turn by placing a checker of her own color on the board. Squares on the board are *adjacent* if they share a side. Black's goal is to connect the top and bottom edges with a path of black checkers, while White wishes to connect the left and right edges with white checkers.

(a) Prove that the first player should win or draw an $n \times n$ SQUEX position.

(b) For what values of n is $n \times n$ SQUEX a win for the first player, and when is it a draw? Prove your answer by giving an explicit strategy for the first player to win or the second player to draw as appropriate.

(c) How about $m \times n$ SQUEX?

15. Alice is about to make a move in the following DOTS & BOXES position:

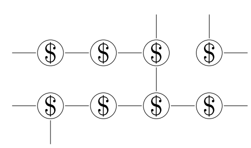


(a) Construct the equivalent STRINGS & COINS position.

(b) Determine if Alice wants an even or odd number of long chains.

(c) Determine all of Alice's winning first moves from this position.

16. You are about to make a move from the following STRINGS & COINS position:

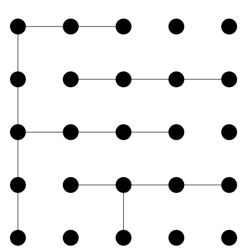


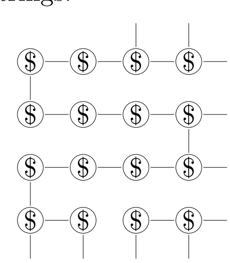
(a) Determine all winning moves.

(b) Draw the corresponding DOTS & BOXES position.

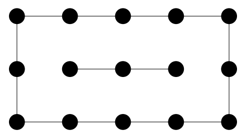
(c) How many boxes should you get in a well-played game?

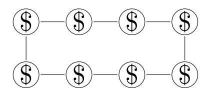
17. Determine all winning moves in the following DOTS & BOXES position. For your convenience, the matching STRINGS & COINS is shown on the right below which has 16 coins and 24 strings.



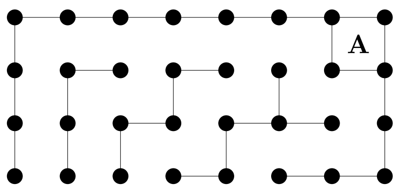


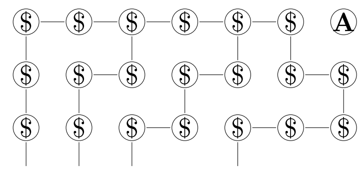
18. There are other DOTS & BOXES (or STRINGS & COINS) positions where every move is loony. For example, there can be cycles:





and many-legged *spiders*:

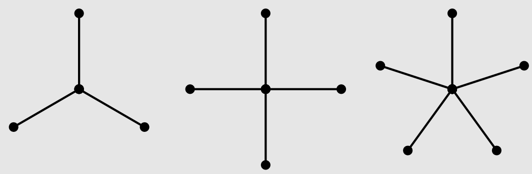




How can you adapt Theorem 1.18 and its proof to account for these positions? Note that much of the proof is found in the paragraphs preceding the theorem.

Preparation for Chapter 2

Prep Problem 2.1. Recruit an opponent and play impartial CUTTHROAT on star graphs. You can choose any start position you wish; here is one possibility:



Prep Problem 2.2. Play PARTIZAN ENDNIM. You can roll a six-sided die five times to create a random five-heap position, or start from a particular position such as the six-heap position 254653.

To the instructor: In place of Section 2.3, consider covering theorems about the outcome classes of *partizan subtraction games* from [FK87]. In particular, Theorem 4 and perhaps Theorems 5 and 6 from their paper are appropriate.

Chapter 2

Outcome Classes

There's just one thing I've got to know.
Can you tell me please, who won?

Crosby, Stills, and Nash in Wooden Ships

In this book, we are usually concerned with who wins if both players *play perfectly.* What does playing perfectly mean? One aspect of playing perfectly is clear: if a player can force a win, then she makes a move that allows her to force a win. What if there is no such move available? In real games it is often then good play to make life as difficult as possible for your opponent, what we called the *Enough Rope Principle.* In theory, since your opponent is assumed to be playing perfectly, such moves cannot help. So, if a player cannot force a win, then playing perfectly simply means "making a move." We formalize this discussion in the *Fundamental Theorem of Combinatorial Games* for partizan, finite, acyclic games:

Theorem 2.1. (Fundamental Theorem of Combinatorial Games) *Fix a game G played between Albert and Bertha, with Albert moving first. Either Albert can force a win moving first, or Bertha can force a win moving second, but not both.*

Proof: Each of Albert's moves is to a position which, by induction, is either a win for Bertha playing first or a win for Albert playing second. If any of his moves belong to the latter category, then by choosing one of them Albert can force a win. On the other hand, if all of his moves belong to the first category, then Bertha can force a win by using her winning strategy in the position resulting from any of Albert's moves. □

We will use this theorem implicitly many times throughout the book.

Suppose that we fix a position G, and see what happens when the first player is Left and when the first player is Right. The *Fundamental Theorem* allows four possibilities, which we can use to categorize this (or any other) position into one of four *outcome classes:*

Class	Name	Definition
$\mathcal{N}$	fuzzy	The $\mathcal{N}$ext player to play whether it be Left or Right (i.e., the 1st to play) can force a win
$\mathcal{P}$	zero	The $\mathcal{P}$revious player who played (or 2nd to play) can force a win
$\mathcal{L}$	positive	$\mathcal{L}$eft can force a win regardless of who moves first
$\mathcal{R}$	negative	$\mathcal{R}$ight can force a win regardless of who moves first

Games in $\mathcal{P}$ and $\mathcal{N}$ are referred to as $\mathcal{P}$*-positions* and $\mathcal{N}$*-positions.* At first glance, the possibility exists that not all four of these outcome classes arise in actual games. However, it is not too difficult to come up with examples of each type. For instance, in DOMINEERING, Left wins on an $n \times 1$ strip for $n > 1$ whether Left or Right moves first. So this game is in $\mathcal{L}$ and is termed *positive.* On the other hand, the $1 \times n$ game is in $\mathcal{R}$ and is *negative.* In two-heap NIM with equal heap sizes, the second player wins by the Tweedledum-Tweedledee strategy. The second (or $\mathcal{P}$revious) player has the advantage and so the game is in $\mathcal{P}$. Two-heap NIM with unequal heap sizes is a win for the first (or $\mathcal{N}$ext) player to move. This game is in the class $\mathcal{N}$.

Exercise 2.2. Find examples of simple DOMINEERING positions that are in $\mathcal{P}$ and in $\mathcal{N}$.

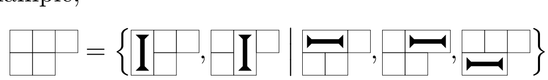
Another way to view the different outcome classes is shown in the next table:

Outcome classes		*When Right moves first*: Right wins	*When Right moves first*: Left wins
When Left moves first	Left wins	$\mathcal{N}$	$\mathcal{L}$
	Right wins	$\mathcal{R}$	$\mathcal{P}$

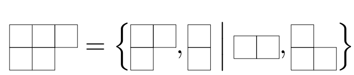
2.1 Game Positions and Options

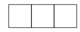
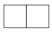
Definition 2.3. A game (position) G is defined by its options, $G = \{\mathcal{G}^L \mid \mathcal{G}^R\}$, where $\mathcal{G}^L$ and $\mathcal{G}^R$ are the set of left and right options, respectively.

We usually omit the braces around sets $\mathcal{G}^L$ and $\mathcal{G}^R$ for brevity, so, in DOMINEERING for example,



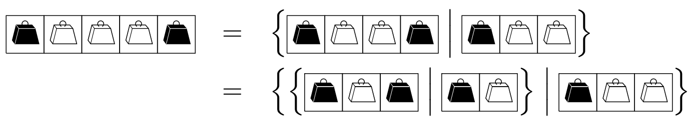
We will often abbreviate a set of positions by simplifying and observing symmetries:



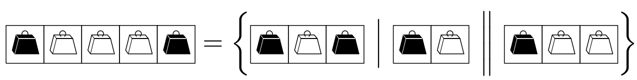
(In the last equation, we omitted  since it has the same game tree as .)

The elements of $\mathcal{G}^L$ and $\mathcal{G}^R$ are called the left and right *options* of G, and we often write G^L or G^R to denote typical representatives of $\mathcal{G}^L$ and $\mathcal{G}^R$.

One, some, or all of the options of a game may be listed as games themselves:



More iteration down the game tree could lead to many levels of braces. Another way to represent this game is to drop the internal braces and introduce a hierarchy of bars instead. Specifically, we might write:



The outcome class of a game may be determined from those of its options instead of having to play out the whole game. This recursive approach is very useful when analyzing games as we will see in the next section.

Observation 2.4. The outcome class of a game, G, can be determined from the outcome classes of its options as shown in the following table:

	some $G^R \in \mathcal{R} \cup \mathcal{P}$	**all $G^R \in \mathcal{L} \cup \mathcal{N}$**
some $G^L \in \mathcal{L} \cup \mathcal{P}$	$\mathcal{N}$	$\mathcal{L}$
all $G^L \in \mathcal{R} \cup \mathcal{N}$	$\mathcal{R}$	$\mathcal{P}$

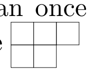
Proof: We will first confirm the upper-right entry in the table.

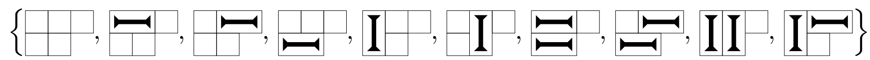
If $G \in \mathcal{L}$ then Left can win playing first. Therefore, she must have an option in $\mathcal{G}^L$ from which she wins playing second. That is, she has an option in $\mathcal{L} \cup \mathcal{P}$. Left can also win playing second, in which case Right has no good move and all Right's options must be in $\mathcal{L} \cup \mathcal{N}$.

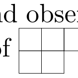
Conversely, if Left has an option in $\mathcal{L} \cup \mathcal{P}$ then Left has a winning move. If Right has only options in $\mathcal{L} \cup \mathcal{N}$ then Right has no winning move. Therefore, $G \in \mathcal{L}$.

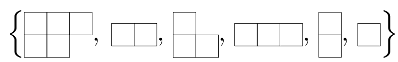
The verifications of the other three entries in the table are similar and left as exercises for the reader. $\square$

Note that for a leaf of a tree the statements "all $G^R \in \mathcal{L} \cup \mathcal{N}$" and "all $G^L \in \mathcal{R} \cup \mathcal{N}$" are both vacuously satisfied. All zero of the options are in the sets. So leaves are in $\mathcal{P}$.

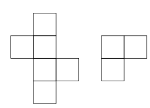
We can define G and the *positions* of G recursively to include G itself and the positions of the options of G.[1] In other words, the positions of G are those games one can reach from G, allowing for the possibility that one player may move more than once consecutively. For example, the positions of the DOMINEERING game  are



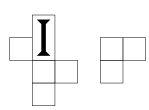
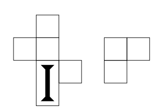
Again, we will often abbreviate a set of positions by simplifying and observing symmetries. For instance, we could choose to write the positions of  as



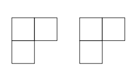
Example 2.5. The following game tree shows followers reachable from

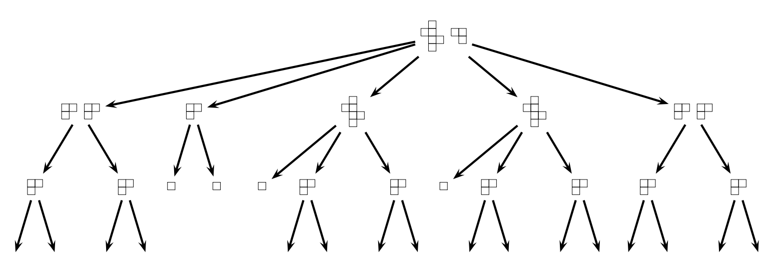


but if two moves are symmetric, say

 and 

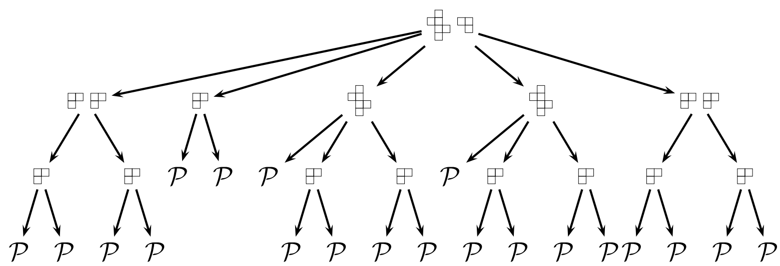
only one is included. Also, in this diagram, we removed any covered or singleton squares, writing these last positions as



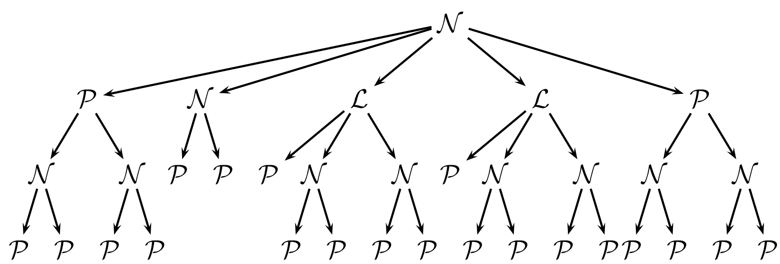


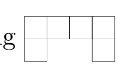
[1]Even though the positions of G include G itself, we will redundantly write "G and its positions" to be perfectly clear.

We can now identify each position's outcome class by referring only to the shape of the game tree using Observation 2.4. First, the leaves of the tree are $\mathcal{P}$-positions, for all options (though there are none) are in $\mathcal{N}$.[2]

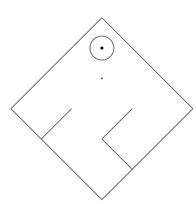


Now, continuing to work upwards from the leaves using Observation 2.4, all the nodes of the tree can be classified according to their outcome classes:

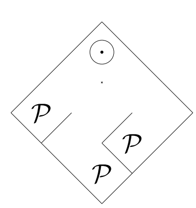


Exercise 2.6. Redo Example 2.5 using  .

Example 2.7. To find the outcome of the following game of MAIZE (page 270)

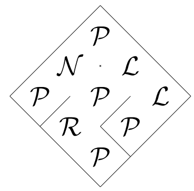


you have to determine the *leaves* first:



[2]This, like the statement "All flying pigs can whistle," is vacuously satisfied and therefore true.

and working backwards we find



Exercise 2.8. Find the outcome of the previous example if the game is MAZE.

Example 2.9. The partizan subtraction game SUBTRACTION$(1,2\,|\,2,3)$ is played with a single heap of n counters. A move by Left is to remove one or two counters, while Right can remove two or three counters. Here, we wish to determine the outcome classes for every possible value of n.

Let G_n denote the game with n counters. Neither player can move from a heap of size 0, so $G_0 \in \mathcal{P}$. From G_1, Left has a move, while Right has none, so $G_1 \in \mathcal{L}$. $G_2 \in \mathcal{N}$ for either player can remove two counters and win. $G_3 \in \mathcal{N}$ as well, for Left removes two counters and wins, while Right removes three counters and wins. $G_4 \in \mathcal{P}$, for if the first player removes k, the second player can legally remove $4-k$, since $1+3=2+2=4$.

One can quickly identify a pattern:

$$G_n \in \begin{cases} \mathcal{P} & \text{if } n \equiv 0 \pmod 4, \\ \mathcal{L} & \text{if } n \equiv 1 \pmod 4, \\ \mathcal{N} & \text{if } n \equiv 2 \text{ or } n \equiv 3 \pmod 4. \end{cases}$$

Proof:

- If $n \equiv 0$, if the first player removes k, then the second can remove $4-k$, leaving a heap of size $n-4 \equiv 0$, which by induction the second player wins. So $G_n \in \mathcal{P}$ if $n \equiv 0 \pmod 4$.
- If $n \equiv 1$, then Left playing first can remove one (winning by induction), but Right playing first only has moves to $n-2 \equiv 3$ and $n-3 \equiv 2$, both of which leave a winning position for Left playing first. So $G_n \in \mathcal{L}$ if $n \equiv 1 \pmod 4$.
- If $n \equiv 2$, then either player can remove two leaving $n-2 \equiv 0 \in \mathcal{P}$. So $G_n \in \mathcal{N}$ if $n \equiv 2 \pmod 4$.
- Finally, if $n \equiv 3$, then Left can remove two, leaving $n-2 \equiv 1 \in \mathcal{L}$, or Right can remove three leaving $n-3 \equiv 0 \in \mathcal{P}$. So $G_n \in \mathcal{N}$ if $n \equiv 3 \pmod 4$. □

Problem 6 identifies a more general reason why this example was periodic with period 4.

2.2 Impartial Games: Minding Your $\mathcal{P}$s and $\mathcal{N}$s

Definition 2.10. A game is *impartial* if both players have the same options from any position.

For example, GEOGRAPHY, NIM, and most subtraction games are impartial. CLOBBER, DOMINEERING, and CHESS are not impartial, since Left cannot move (or, in DOMINEERING, place) Right's pieces.

In the development of the theory, impartial games were studied first and there are many impartial games that are currently unsolved despite many attempts over many years. NIM was analyzed by Bouton [Bou02] in 1902. In the 1930s and 1940s, Sprague and Grundy extended Bouton's analysis and showed that it applied to all impartial games. Guy continued the work throughout the 50s, 60s, and 70s. So prevalent are impartial games that games which are not impartial have been dubbed *partizan* to distinguish them.

Impartial games form a nice subclass of games and we will explore them more in Chapter 7. One property that makes this subclass interesting is that they only belong to two outcome classes.

Theorem 2.11. *If G is an impartial game then G is in either $\mathcal{N}$ or $\mathcal{P}$.*

Proof (*by Strategy Stealing*): If G were in $\mathcal{L}$ then Left could win going first but then Right going first can use Left's strategy and win. □

Exercise 2.12. Give an inductive proof of Theorem 2.11 using Observation 2.4.

To find the outcome of an impartial game, the following observation is very useful.

Theorem 2.13. *Suppose the positions of a finite impartial game can be partitioned into mutually exclusive sets A and B with the properties:*

- *every option of a position in A is in B; and*
- *every position in B has at least one option in A.*

Then A is the set of $\mathcal{P}$-positions and B is the set of $\mathcal{N}$-positions.

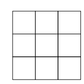
Proof: See Problem 3. □

Exercise 2.14. Explain why Theorem 2.13 correctly addresses the end-positions of the game.

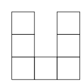
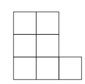
The mechanics of finding the outcome of an impartial position is easier than that of a partizan one. We again draw the game tree then label the vertices recursively from the end of the tree to the root, only this time the only labels are $\mathcal{P}$ and $\mathcal{N}$. (Whether or not this is easy, of course, depends on the depth and width of the tree.) For a normal play, impartial game, the end positions are all $\mathcal{P}$ and thereafter a position is labeled $\mathcal{P}$ or $\mathcal{N}$ according to the following rules:

- It is an $\mathcal{N}$-position if at least one of its options is a $\mathcal{P}$-position.
- It is a $\mathcal{P}$-position if all of its options are $\mathcal{N}$-positions.

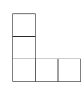
Example 2.15. Consider the CRAM position:



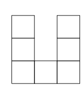
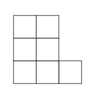
Considering symmetries, there are only two distinct first moves:

 and 

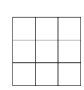
From either, the second player can reach the $\mathcal{P}$-position:



Hence,

 and 

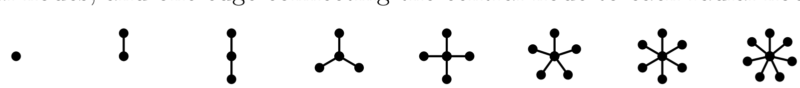
are both $\mathcal{N}$-positions, and so



is a $\mathcal{P}$-position.

Of course, our goal is not to classify positions one by one, but rather to identify patterns that allows us to quickly determine whether any position of a game is in $\mathcal{P}$ or $\mathcal{N}$. Our opponent would have long since lost interest in playing us if we drew the game tree for every game we played.

Example 2.16. (CUTTHROAT stars) A move in impartial CUTTHROAT consists of deleting a vertex and any incident edges from a graph with the proviso that at least one edge must be deleted on every move. We will play CUTTHROAT on star graphs. A *star graph* is a graph with one central node, any number of radial nodes, and one edge connecting the central node to each radial node:



$K_{1,0}$ $K_{1,1}$ $K_{1,2}$ $K_{1,3}$ $K_{1,4}$ $K_{1,5}$ $K_{1,6}$ $K_{1,7}$

In graph theory, a star graph with n radial nodes is denoted $K_{1,n}$. We will refer to n as the size of the star.

On a star of size 1 only a single move is available. On all larger stars there are two types of move: the *supernova,* which removes the central vertex, leaving a set of isolated vertices on which no further moves are possible; and the *shrink,* which removes an outer vertex, thus decreasing the size of the star by 1. From now on, all star sizes will be at least 1 — any isolated vertices left by a supernova move are not of any interest to us and are discarded.

A single star is always an $\mathcal{N}$-position, since the first player can simply use the supernova move. In multiple star positions, the $\mathcal{P}$-positions are precisely those where the number of even stars and the number of odd stars are both even.

To see this, we show how the partition of positions implicit in the above satisfies the conditions of Theorem 2.13:

- If there is an even number both of even stars and of odd stars then a supernova move changes the parity of either the odd or the even stars, while a shrink move changes the parity of both. Note that a supernova move on an even star does not leave an even star since we ignore the resulting isolated vertices.

- If one type of star occurs with odd parity, then a supernova move on a star of that type makes the parities of both types even. If both types of star occur with odd parity then a shrink move on an even star leaves an even number of both types.

If we wish to apply Theorem 2.13, then how do we find the right partition in the first place? A good starting point is to catalog a few small positions and identify a pattern. Then use Theorem 2.13 (or an inductive argument, or both) to confirm the pattern. In Chapter 7 we present much more powerful tools for analyzing games that are disjunctive sums.

Let S be a set of positive integers. The game SUBTRACTION(S) is played with a heap of n counters, and a play is to select some $s \in S$ less than or equal to n and remove s counters, leaving a heap of size $n - s$.

Exercise 2.17. Denote by G_n the game SUBTRACTION$(1, 3, 4)$ played on a heap of n counters. Determine which G_n are $\mathcal{P}$-positions and prove your answer. (*Hint*: You should find that the game is periodic with period 7.)

2.3 Case Study: Partizan Endnim

Louise (Left) and Richard (Right) are fork-lift operators, with a penchant for combinatorial games. Many of the warehouses, from which they need to remove boxes, have the boxes in stacks, with the stacks arranged in a row. Only boxes belonging to the stacks at the end of a row are accessible, but the fork-lifts are sufficiently powerful that they can move an entire stack of boxes if necessary. The game which Louise and Richard play most often is won by the player who removes the last box from a row of stacks. In this, the partizan version analyzed in [AN01] and [DKW07], Louise must remove boxes from the leftmost stack, and Richard is restricted to removing from the rightmost one.

For example, Left's legal options from 23341 are to 13341, or 3341, while Right has only one legal move, to 2334.

We use boldface Latin characters to stand for strings of positive integers, and non-bold characters for single positive integers. Concatenation of strings is denoted by their juxtaposition. So, for instance, we might write 23341 as $a\mathbf{w}b$ where $a = 2$, $\mathbf{w} = 334$, and $b = 1$.

Exercise 2.18. Build a table of outcome classes of $a\mathbf{w}$ for $a \in \{0, 1, 2, \ldots\}$ when

1. $\mathbf{w} = 22$;
2. $\mathbf{w} = 23$.

Exercise 2.19. Prove the following: If Left has a winning move from $a\mathbf{w}b$, then one of:

- removing a single box;
- removing the entire stack;

is a winning move.

Note that for a fixed, non-empty string $\mathbf{w}$, if a is large enough, Left can win from $a\mathbf{w}$. For example, if the left heap has more boxes than all the remaining heaps combined, then all Left need do is remove one box at a time until Right has removed all the boxes in $\mathbf{w}$, and then Left can remove the remaining stack. So we can define

$$\begin{aligned} L(\mathbf{w}) &= \text{the minimum } a \geq 0 \text{ such that Left wins } a\mathbf{w} \text{ moving second, and} \\ R(\mathbf{w}) &= \text{the minimum } b \geq 0 \text{ such that Right wins } \mathbf{w}b \text{ moving second} \end{aligned}$$

Observation 2.20. We can use $L(\mathbf{w}b)$ and $R(a\mathbf{w})$ to determine the outcome class of $a\mathbf{w}b$. In particular, the outcome class of $a\mathbf{w}b$ for $a, b > 0$ is given by:

$$a\mathbf{w}b \in \begin{cases} \mathcal{L} & \text{if } a > L(\mathbf{w}b) \text{ and } b \leq R(a\mathbf{w}), \\ \mathcal{R} & \text{if } a \leq L(\mathbf{w}b) \text{ and } b > R(a\mathbf{w}), \\ \mathcal{N} & \text{if } a > L(\mathbf{w}b) \text{ and } b > R(a\mathbf{w}), \\ \mathcal{P} & \text{if } a \leq L(\mathbf{w}b) \text{ and } b \leq R(a\mathbf{w}). \end{cases}$$

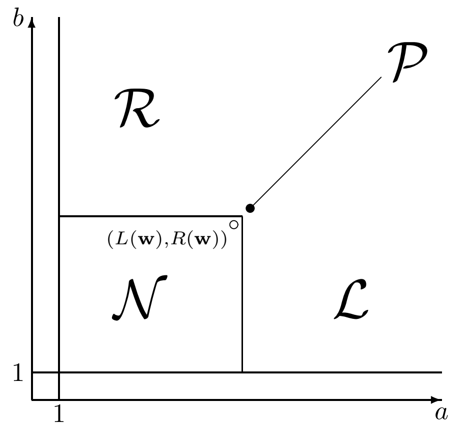


Figure 2.1. This figure illustrates the outcome classes of the game $a\mathbf{w}b$ for all $a, b > 0$. The unfilled circle represents the triple point, $(L(\mathbf{w}), R(\mathbf{w}))$, which has an outcome class of $\mathcal{N}$. The filled circle is the point $(L(\mathbf{w}) + 1, R(\mathbf{w}) + 1)$ and the games on the line originating from the filled circle, which are of the form $(L(\mathbf{w}) + c, R(\mathbf{w}) + c)$, where $c > 0$, have an outcome class of $\mathcal{P}$.

Exercise 2.21. Prove the last observation.

In fact, we can do one better by determining the outcome class of $a\mathbf{w}b$ using just $L(\mathbf{w})$ and $R(\mathbf{w})$. For fixed $\mathbf{w}$, there is one critical point of note, and that determines the outcome class of any $a\mathbf{w}b$!

Definition 2.22. The *triple point* of $\mathbf{w}$ is $(L(\mathbf{w}), R(\mathbf{w}))$.

Theorem 2.23. *The outcome class of $a\mathbf{w}b$ for $a, b > 0$ is given by:*

$$a\mathbf{w}b \in \begin{cases} \mathcal{N} & \text{if} \quad a \leq L(\mathbf{w}) \text{ and } b \leq R(\mathbf{w}), \\ \mathcal{P} & \text{if} \quad a = L(\mathbf{w}) + c \text{ and } b = R(\mathbf{w}) + c \quad \text{for some} \quad c > 0, \\ \mathcal{L} & \text{if} \quad a = L(\mathbf{w}) + c \text{ and } b < R(\mathbf{w}) + c \quad \text{for some} \quad c > 0, \\ \mathcal{R} & \text{if} \quad a < L(\mathbf{w}) + c \text{ and } b = R(\mathbf{w}) + c \quad \text{for some} \quad c > 0. \end{cases}$$

This theorem is visually rendered in Figure 2.1.

Proof: First, assume that $a \leq L(\mathbf{w})$ and $b \leq R(\mathbf{w})$. If Left removes all of a, Right cannot win since any move is to $\mathbf{w}b'$ where $b' < b$ and b was the least value such that Right wins moving second on $\mathbf{w}b$. Symmetrically, Right can also win moving first by removing all of b. Thus, $a\mathbf{w}b \in \mathcal{N}$.

Next, assume that $a = L(\mathbf{w}) + c$ and $b = R(\mathbf{w}) + c$ for some $c > 0$. Also, assume that Left moves first. If Left changes the size of a to $a' = L(\mathbf{w}) + c'$

where $0 < c' < c$, Right simply responds by moving on b to $b' = R(\mathbf{w}) + c'$. By induction, this position is in $\mathcal{P}$. On the other hand, if Left changes the size of a to a' where $a' \leq L(\mathbf{w})$, Right can win by removing b as shown in the previous case. Thus, Left loses moving first. Symmetrically, Right also loses moving first. So, $a\mathbf{w}b \in \mathcal{P}$.

Finally, assume $a = L(\mathbf{w})+c$ and $b < R(\mathbf{w})+c$ for some $c > 0$. If $b \leq R(\mathbf{w})$, Left can win moving first by removing all of a as shown in the first case. If $b > R(\mathbf{w})$, Left wins moving first by changing the size of a to $L(\mathbf{w}) + c'$, where c' is defined by $b = R(\mathbf{w})+c'$, which is in $\mathcal{P}$ by induction. Left can win moving second from $a\mathbf{w}b$ by making the same responses as in the previous case. Thus, $a\mathbf{w}b \in \mathcal{L}$. □

So, if we can find the triple point, we can analyze the whole position. The functions $R(\cdot)$ and $L(\cdot)$ can be easily computed recursively using

Theorem 2.24.

$$
\begin{aligned}
R(a\mathbf{w}) &= \begin{cases} 0 & \textit{if } a \leq L(\mathbf{w}), \\ R(\mathbf{w}) - L(\mathbf{w}) + a & \textit{if } a > L(\mathbf{w}); \end{cases} \\
L(\mathbf{w}b) &= \begin{cases} 0 & \textit{if } b \leq R(\mathbf{w}), \\ L(\mathbf{w}) - R(\mathbf{w}) + b & \textit{if } b > R(\mathbf{w}). \end{cases}
\end{aligned}
$$

Proof: As already shown in Theorem 2.23, Left loses moving first on $a\mathbf{w}$ where $a \leq L(\mathbf{w})$, so $R(a\mathbf{w}) = 0$. On the other hand, assume that $a > L(\mathbf{w})$. Again by Theorem 2.23, the least value of b that lets Right win moving second on $a\mathbf{w}b$ is $b = R(\mathbf{w}) + c = R(\mathbf{w}) + (a - L(\mathbf{w}))$. □

An algorithm to compute $R(\mathbf{w})$ and $L(\mathbf{w})$ using the above recurrence can be written to take $\Theta(n^2)$ time, where n is the number of heaps in $\mathbf{w}$.

Example 2.25. As an example, we will determine who wins from

$$3\ \ 5\ \ 2\ \ 3\ \ 3\ \ 1\ \ 9$$

when Left moves first and when Right moves first. Fix $\mathbf{w} = 52331$. We wish to compute $L(\mathbf{w})$ and $R(\mathbf{w})$ using Theorem 2.24. For single-heap positions, $L(a) = R(a) = a$, because $R(a) = R() + (a - L()) = 0 + (a - 0) = a$. For two-heap positions, we have

$$R(ab) = \begin{cases} 0 & \text{if } a \leq b, \\ a & \text{if } a > b; \end{cases} \qquad L(ab) = \begin{cases} 0 & \text{if } a \geq b, \\ b & \text{if } a < b. \end{cases}$$

We can compute $L(52331)$ by first calculating $L(\mathbf{w})$ and $R(\mathbf{w})$ for each shorter substring of heaps:

$\mathbf{w}$	$L(\mathbf{w})$	$R(\mathbf{w})$
523	0	2
233	6	2
331	1	6
5233	1	0
2331	0	7
52331	2	12

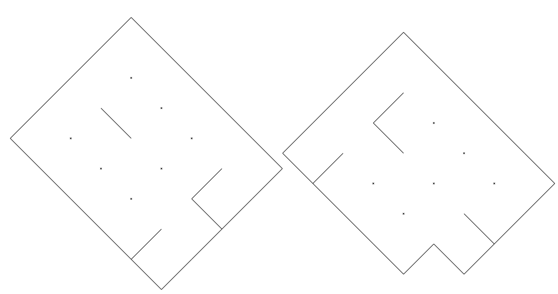
For instance, $R(523) = R(23) + (5 - L(23)) = 0 + (5 - 3) = 2$.

For the original position $a\mathbf{w}b = 3\mathbf{w}9$, we have $3 > L(\mathbf{w}) = 2$ and $9 \leq R(\mathbf{w}) = 12$, and hence, by Theorem 2.23, 3523319 is an $\mathcal{L}$-position.

A small change to an individual heap size can have a large effect on the triple-point; see Problem 8.

Problems

1. Find the outcomes of playing MAIZE and MAZE in each of the two boards. You should determine the outcome class for every possible starting square for the piece.



2. Find the sets A and B of Theorem 2.13 for

   (a) SUBTRACTION$(2, 3, 4)$;

   (b) SUBTRACTION$(1, 3, 6)$;

   (c) SUBTRACTION$(2, 3, 6)$;

   (d) SUBTRACTION$(2^n : n = 0, 1, 2, \ldots)$;

   (e) SUBTRACTION$(2^n : n = 1, 2, \ldots)$.

3. Give an inductive proof of Theorem 2.13 using Observation 2.4.

4. Consider the following two-player subtraction game. There is a heap of n counters. A move consists of removing any proper factor of n counters from the heap. (For example, if there are $n = 12$ counters, you can leave a heap with 11, 10, 9, 8, or 6 counters.) The player to leave a heap with one counter wins.

   (a) Determine a winning strategy from those positions in which you can win. In particular, you will need to determine which positions are $\mathcal{P}$-positions and which are $\mathcal{N}$-positions. Prove your answer by induction.

   (b) How about the *misère* version? In misère, the player to leave a heap with one counter loses.

5. GREEDY NIM is just like NIM, but you must always take from the (or a) largest heap. Identify the $\mathcal{P}$- and $\mathcal{N}$-positions in GREEDY NIM.

6. (This problem generalizes Example 2.9.) Fix a number p and a partizan subtraction game SUBTRACTION$(L \mid R)$, where $|L| = |R|$ and for each $x \in L$ there is a matching $y \in R$ such that $x + y = p$. (For instance, the game might be SUBTRACTION$(1, 2, 4, 7 \,|\, 2, 5, 7, 8)$ with $p = 9$.) Denote this game played on a heap of size n by G_n. Prove that the outcome classes of G_n are strictly periodic with period p. That is, prove that for all $n \geq p$, G_n has the same outcome class as G_{n-p}.

7. (Modified CUTTHROAT stars) Consider a collection of stars $K_{1,n}$. A move consists of deleting a vertex and the incident edges. Any vertex may be deleted so long as at least one star, somewhere, contains an edge before the move. In other words, players can play on isolated vertices unless all edges are gone. An isolated vertex will be considered a trivial star, other stars will be called real stars.

   Show that the $\mathcal{P}$-positions are those in which there are at least two real stars and the number of even stars is even.

8. (a) For $\mathbf{w} = 35551$, compute $L(\mathbf{w})$ and $R(\mathbf{w})$.

   (b) Repeat with $\mathbf{w} = 35451$.

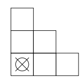
9. The COMMON DIVISOR game is played with heaps of counters. A move is to choose a heap and take away a common divisor of all the heap sizes. For this game, $\gcd(0, a) = a$. An example game would be

$$(2, 6) \to (2, 4) \to (2, 3) \to (1, 3) \to (0, 3) \to (0, 0).$$

   Find the $\mathcal{P}$-positions in the two-heap game. (*Hint*: binary.)

10. Prove that in PARTIZAN ENDNIM, there are no $\mathcal{N}$-positions of even length (meaning an even number of non-empty stacks).

11. Determine some of the $\mathcal{P}$-positions in two-dimensional CHOMP. In particular, determine *all* the $\mathcal{P}$-positions for width 1 and width 2 boards, and find at least two $\mathcal{P}$-positions for boards that include the following six squares:



12. The following extended investigation derives the triple point in PARTIZAN ENDNIM in a more intuitive, though less terse, manner. (In fact, this is how the triple point was discovered.) Complete the exercise without using any of the theorems proved in Section 2.3.

    As in Exercise 2.18, we may consider the sequence of types of the positions $a\mathbf{w}$ for $a \geq 1$. (We think of this as the *left phase diagram* of $\mathbf{w}$.)

    (a) For fixed $\mathbf{w}$, prove that for some value of a, $a\mathbf{w} \in \mathcal{L}$.

    (b) Prove that the left phase diagram of $\mathbf{w}$ consists either of:

    - a string (possibly empty) of $\mathcal{N}$s followed by $\mathcal{L}$s, or,
    - a string of $\mathcal{R}$s (again possibly empty) followed by a single $\mathcal{P}$, and then $\mathcal{L}$s.

    (Similarly, the right phase diagram is $\mathcal{N}$s–$\mathcal{R}$s or $\mathcal{L}$s–$\mathcal{P}$–$\mathcal{R}$s.)

    (c) Show that if $a\mathbf{w}b \in \mathcal{P}$ then $(a+1)\mathbf{w}(b+1) \in \mathcal{P}$, where $a, b \geq 0$.

    (d) Use the above observations to show that the picture in Figure 2.1 is accurate. (You do not need to justify the identifier $L(x), R(x)$.) In particular, show that there is a lower-left square (possibly empty) of $\mathcal{N}$-positions, a diagonal coming off the square of $\mathcal{P}$-positions, and the remaining positions are in $\mathcal{L}$ and $\mathcal{R}$.

Preparation for Chapter 3

To the instructor: This chapter and the prep problems before the next chapter, motivate abstract algebraic thinking. Feel free to remind students of other groups or partial orders they might have seen in your curriculum.

Chapter 3

Motivational Interlude: Sums of Games

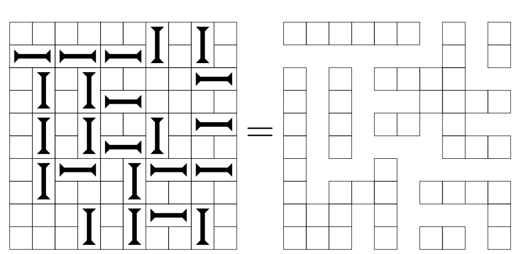
It is a mistake to try to look too far ahead.

Sir Winston Churchill

3.1 Sums

So far, we have seen a collection of ad hoc techniques for analyzing games. Historically, that was the state of the art until Sprague [Spr35] and Grundy [Gru39] proved that Bouton's analysis [Bou02] of NIM could be applied to all impartial games.

Berlekamp, Conway, and Guy then set out to develop a unified theory of partizan games. They observed that many games have positions that are made up of independent components. Sometimes, as in NIM, the components are simply part of the game. In NIM, a player may play on any one of a number of independent heaps. Other games naturally split into components as play proceeds. Most famously, the endgame of the ancient and popular Asian game of GO has this property. But we have seen this phenomenon in other games as well; in DOMINEERING, for instance, the playing field is often separated into different regions of play as in the following game:

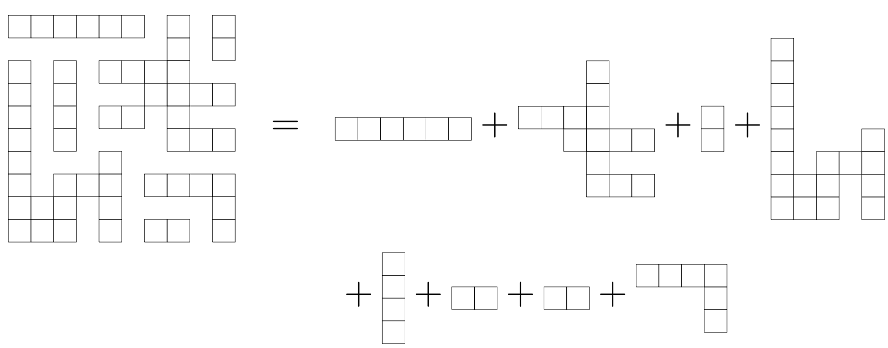


Given the fact that many games separate into independent regions, the natural question a mathematician asks is, "How can we exploit the decomposition?" In other words, what is the minimum amount of information we need to know about an individual component to know how it behaves in the context of other components?

This naturally leads to a definition of *game sum:*

> A sum of two or more game positions is the position obtained by placing the game positions side by side. When it is your move, you can make a single move in a summand of your choice. As usual, the last person to move wins.

For example, in DOMINEERING, we have:



John Conway had a brilliant insight. Since game sums appear to be central to so many games, he chose to distill a few game axioms which encapsulate the following natural notions:

- A move is from a position to a sub-position; the last player to play wins.
- In a sum of games, a player can move on any summand.
- In the negative of a game, the players' roles are reversed.
- Position A is at least as good as B (for Left) if Left is always satisfied when B is replaced by A in any sum.

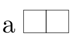
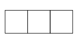
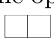
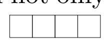
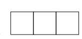
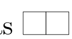
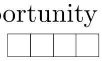
We will list the formal axioms in the next chapter. The goal here is to motivate these axioms and explore some of their consequences.

First, note what is *not* part of Conway's axioms. For example, games like GO and DOTS & BOXES appear to be excluded. Although they break up into sums, it is the person who accumulates the highest score who wins, not the last player to play. Conway chose to use a minimal number of axioms in order to develop a powerful and general mathematical foundation. What is remarkable is that, as we will see, a notion of score naturally develops from these axioms, making the theory applicable to games with scores.

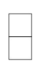
Conway's axioms also appear to exclude misère games in which the last player to play *loses*. There are many misère games for which this proves to be only a minor inconvenience, but most misère games seem to be much harder than their non-misère counterparts. In just the last couple of years, tremendous progress has been made in understanding misère games; we briefly discuss this progress in Chapter $\boldsymbol{\omega}$.

The description of the axioms is suggestive of both an algebraic structure and a partial order. It includes references to addition, negation, and comparison. For the rest of the book we will have two goals: to understand the structure implied by Conway's axioms, and to learn how to apply what we know about the structure to a wide variety of games.

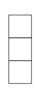
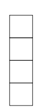
3.2 Comparisons

In a sum of DOMINEERING games, Right will be satisfied if a  summand is replaced by  since he gets one move in either of the summands and Left has the same options as before. Right will not only be satisfied but will be very happy if a  summand is replaced by  since now he will have the opportunity to make an extra move. The position  is as good as  but  is better than either of these from Right's perspective.

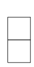
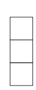
Like Right, Left will be satisfied if a



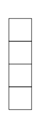
summand is replaced by

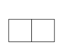
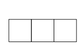
 or 

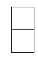
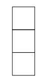
It makes sense to say that both the

 and 

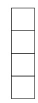
games are worth 1 move to Left; that

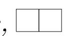
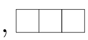
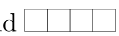


is worth 2 moves to Left; that  and  are worth 1 to Right; etc. It becomes too unwieldy to keep referring to the player; instead we arbitrarily make Left the positive player. So, we can say that the

 and 

are worth 1 move,



is worth 2, while their *negatives,* , , and , are worth -1, -1, and -2 moves, respectively.

In arithmetic, adding 0 to anything does not change the value. Do the two summands 2×1 and 1×3 really cancel out just as $1 - 1 = 0$? To see whether or not this is true consider how the players might respond if both are added to a game G. The goal is to show that whoever could have won G will also win in the new sum.

If Left could win G then she can play one of her winning strategies in G ignoring the two new summands unless Right makes a move in one of them. At some point in the play, even as late as after the play in G has been exhausted, Right will make a move outside of G. His only possible move of this type is to play in 1×3. Left can answer this move in 2×1 and Right must continue back in G as if this exchange had never happened. So the outcome of the sum will still be a win for Left.

Similarly, if Right can win in G then likewise he can win in G plus these two new summands. Adding this particular 0 to a game does not change the outcome of the game! This argument, extended, also works for any second-player-win game H. Though we're being informal in this chapter it's worth recording this fact formally.

Proposition 3.1. *Let G be any game and let $Z \in \mathcal{P}$ be any game that is a second-player win. The outcome classes of G and of $G + Z$ are the same.*

Proof: Either player can ensure an outcome in $G + Z$ at least as favorable as the outcome of G by simply playing in G *unless* replying to an opponent's move in Z, and in that case playing the second player winning strategy in Z. Since $G + Z$ is *no worse* than G for either player under any circumstances, it must be the same as G under any circumstances. $\square$

Furthermore, if Z is any game which has the property that the outcome classes of G and $G + Z$ are always the same, then in particular the outcome classes of the empty game and the sum of the empty game and Z (which is just Z, obviously) are the same, and so Z must be a second-player win. So it makes sense to say:

Any, and only a, second-player-win game has value 0.

A more careful reading of the proof of Proposition 3.1 shows that if Left wants to win in $G+Z$ she need only exploit the fact that she wins G and wins moving second on Z. Consequently,

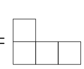
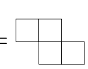
Proposition 3.2. *Let G be any game and let $X \in \mathcal{P} \cup L$ be any game which Left wins moving second. The outcome class of $G+X$ is at least as favorable for Left as that of G.*

Of course there is also a dual version of this proposition, which applies to Right. Using Proposition 3.2 and its dual, we can determine an addition table, which shows the possible outcome class of a sum given the outcome classes of the two summands:

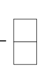
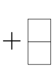
	$\mathcal{L}$	$\mathcal{P}$	$\mathcal{R}$	$\mathcal{N}$
$\mathcal{L}$	$\mathcal{L}$	$\mathcal{L}$	?	$\mathcal{L}$ or $\mathcal{N}$
$\mathcal{P}$	$\mathcal{L}$	$\mathcal{P}$	$\mathcal{R}$	$\mathcal{N}$
$\mathcal{R}$	?	$\mathcal{R}$	$\mathcal{R}$	$\mathcal{R}$ or $\mathcal{N}$
$\mathcal{N}$	$\mathcal{L}$ or $\mathcal{N}$	$\mathcal{N}$	$\mathcal{R}$ or $\mathcal{N}$	?

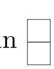
That 4×1 is actually better for Left than 2×1 is clear, but how would we test it? One way is to give the advantage of the 2×1 game to Right and play the two games together. That is, play 4×1 plus 1×2. Left going first moves to 2×1 plus 1×2, which is a second-player win with Left as the second player. If Right moves first in the original game she moves to 4×1, again a Left win. So $(4 \times 1) + (1 \times 2) \in \mathcal{L}$. Not only intuitively, but also by the play of the game, the advantage to Left in 4×1 is greater than that in 2×1.

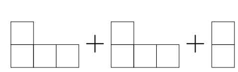
Can positions be worth a non-integer number of moves? There are certainly some candidates. Let

$G =$  and $H =$ 

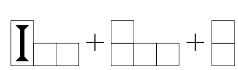
and consider the games

$G + G +$  and $H + H +$ 

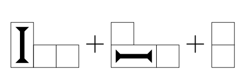
It seems clear that Left will not want to take the move available in  unless she has to. In



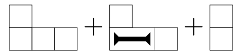
Left going first moves to



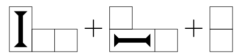
and Right responds with



and both Left and Right have exactly one move each remaining with Left to play, so she loses. Right going first will move to

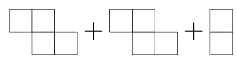


and Left will respond with

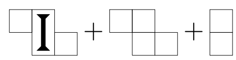


Both players have one move left but it is Right's turn to play and now he will lose. So, apparently, $G + G + 1 = 0$.

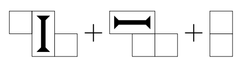
Similarly, in



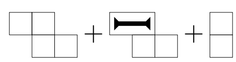
Left going first moves to



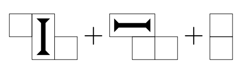
Right responds with



and both Left and Right have exactly one move each remaining with Left to play so he loses. Right going first will move to

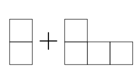


and Left will respond with

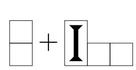


Both players have one move left but it is Right's turn to play and now she will lose. So, likewise, $H + H + 1 = 0$.

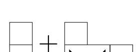
It seems that a good guess would be that both G and H are worth $-\frac{1}{2}$ moves. However, the arithmetic of games turns out to be more complicated than that. For one thing, G and H belong to different outcome classes, for G is a Right win and H is a first-player win. By our convention only G should be negative. Right would be happy to have G replace a second-player-win game (i.e., a value 0 game). So $G < 0$. Right *could* be unhappy if 2×1 were replaced by G. To show this, let's hand the 2×1 over to Left and compare. In



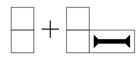
Left, going first, moves to



and wins. Right, going first, moves to



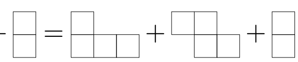
which Left wins, or he moves to



which is even worse since Left has two free moves. Therefore, $-1 < G < 0$, which together with $G + G = -1$ suggests that G is a good candidate for a half-move advantage to Right.

We will build techniques for doing this algebra of games more efficiently in the next chapter, and we will then be able to better formalize what it means for a position to be worth, say, a non-integer number of moves in Chapter 5, which introduces all sorts of game *values,* not merely numbers.

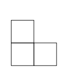
Exercise 3.3. Determine the outcome class of

$G + H +$ 

3.3 Equality and Identity

When are two games equal? It seems a silly question — and perhaps it is if you interpret "equal" as "identical." Two CHESS games in which the same first four moves have been played are certainly equal (at this point) under this definition. But what if we reach identical CHESS positions by different sequences of moves (called a *transposition*)? Certainly, from the standpoint of the game as it will proceed from here the two games must still be considered equal. On the human side, though, they may not be equal at all — there may be important psychological and tactical messages which can be read from the actual sequence of moves chosen. However, as theorists of games, we will not even pretend to address such messages, so for us these two games would still be identical (and hence certainly equal).

Can two different games ever be equal? Consider the DOMINEERING position

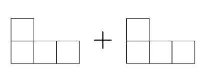


and the NIM position consisting of a single counter. In both these positions each player has precisely one legal move and after making that move his or her opponent will have no move available. So these two games have identical (and rather trivial) game trees — there is an exact correspondence between any sequence of moves from one game and any sequence of moves from the other. So, from anything except an aesthetic standpoint, the two games are the same and can be considered equal. Moreover, the game tree captures completely all the possible sequences of play in the game — the ebbs and flows of fortune, the tricks that might be tried, the seemingly advantageous moves which are really terrible traps. So the idealist might well choose to argue that two games should be considered equal if and only if they have identical game trees.

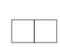
The utter pragmatist looks at games differently. To him, when faced with a game, the only question of interest is: "Who will win?" That is, the sole significant feature of a game is its outcome class. Beyond the four possible outcome classes everything is just window dressing, a way to hide the essential nature of the underlying position. Such a person might well claim that two games are equal if (and only if) they have the same outcome class.

Combinatorial game theory steers a middle course between these two positions. Though the ultimate goal in practice may well be to determine the outcome type of a particular game, as we do for some instances of DOMINEERING rectangles in the next section, the aim of the subject is to produce practical and theoretical tools, which enhance our ability to make such determinations. The problem with the purely pragmatic view is that it treats each game in isolation — it becomes a matter of brute force search, possibly guided by some good heuristic techniques, to actually work out the relevant outcome type. What we would like to be able to do is to use theory to simplify this process.

A key insight in this simplification is the recognition that two games might as well be considered equal if they can be freely substituted for one another in any context (which we will take, without terribly strong justification, to mean *sum*) without changing the outcome type. This provides us with the outline of a strategy for analyzing complex games — decompose them into sums, and replace complicated summands by simpler ones that they are equal to. For instance, we have already seen that *any* second-player win can be ignored in a sum — it has no effect on the outcome class. Likewise, we have seen that the domineering position



behaves *just like*



If we had a reasonably large catalog of simplest forms of various DOMINEERING positions then we would be well on our way to understanding how to play even quite complex DOMINEERING games perfectly.

In the next chapter we will pursue the goal of defining appropriate notions of sum, equality, and comparison for games in a way that takes advantage of the observations we have made in this chapter. However, to demonstrate that even the pragmatist can draw some benefit from an understanding of the importance of decomposition in games, the next section shows how a player could use imagined or virtual decompositions as a guide to playing well in some (admittedly rather simple) games of DOMINEERING.

3.4 Case Study: Domineering Rectangles

A complete analysis of the game of DOMINEERING is not yet feasible, though there are techniques (many covered in this book) to help analyze a great many positions.

However, if we only wish to know the outcome classes then DOMINEERING can be solved completely on various rectangular boards. In this section we will consider an analysis of some $2 \times n$ positions, and all the $3 \times n$ positions. If we think of n as being large (after all, most positive integers are!) then it is intuitively clear that the $3 \times n$ positions should be advantageous to Right who plays the horizontal dominoes as the board can contain $3\lfloor n/2 \rfloor$ horizontal dominoes but only n vertical ones.

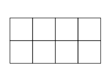
Our analysis of these positions uses a powerful technique which simplifies many arguments. Here it is:

> If it is true that Right wins some game G when he promises not to make certain types of moves, then he has a winning strategy in G itself.

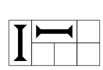
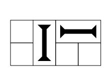
The point of this technique is that by enforcing a promise not to make certain kinds of moves it may well be the case that the game becomes much simpler — but if this does not harm Right's winning chances then it is at no cost to him. We call this the *One-Hand-Tied Principle* since Right is promising that he can win even with one hand tied behind his back.

In considering DOMINEERING played on rectangular boards the types of promises we will make on Right's behalf are not to play any move that crosses certain vertical dividing lines. That is, Right will promise to play as if the board were split along certain vertical lines into a sum of games. If this promise does not harm his winning strategy, then we can be sure that the original game is also a win for Right.

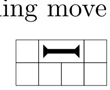
We first consider some $2 \times n$ cases:

 $\in \mathcal{R}$.

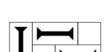
To see this we first illustrate winning replies to each of Left's two possible first moves (up to symmetry):

We must also demonstrate a winning move for Right as first player:



This wins because, up to symmetry, Left has only one move available and then Right can end the game in the position



In the remaining arguments we will mostly leave the last step or two of the verification to you!

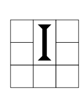
Now the *One-Hand-Tied Principle* implies that for all $k \geq 1$, DOMINEERING played on a $2 \times 4k$ rectangle is in outcome class $\mathcal{R}$. Right will simply promise to pretend that the game is being played on k copies of 2×4 and use his winning strategy in each of them. A simple analysis shows that the 2×3 game is in $\mathcal{N}$. Then applying the *One-Hand-Tied Principle* again it follows that Right can always win the $2 \times (4k+3)$ games playing first (by making his winning move in the 2×3 game to begin with), so the $2 \times (4k+3)$ game is either in $\mathcal{N}$ or in $\mathcal{R}$. In fact, though we will not show it here, all the games $2 \times \{7, 11, 15, 19, 23, 27\}$ are in $\mathcal{N}$ while $2 \times (31 + 4t)$ is in $\mathcal{R}$ for any $t \geq 0$.

Suppose that a trustworthy mathematician told us that the 2×13 game was in $\mathcal{P}$ (this is, in fact, true). Could we make use of this information? Tying one of his hands, Right can afford to ignore any 2×13 parts of the game (planning to play the winning strategy as second player there if Left makes a move). Now observe:

$$\begin{aligned}
36 &= 9 \times 4; \\
37 &= 6 \times 4 + 13; \\
38 &= 3 \times 4 + 2 \times 13; \\
39 &= 3 \times 13.
\end{aligned}$$

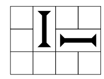
It follows that $2 \times \{36, 37, 38\} \in \mathcal{R}$ and that 2×39 is either in $\mathcal{R}$ or $\mathcal{P}$. But as $39 = 9 \times 4 + 3$ we already knew that 2×39 is in $\mathcal{R}$ or $\mathcal{N}$. Therefore, it is in $\mathcal{R}$. But now we have four consecutive values of k such that the $2 \times k$ game is in $\mathcal{R}$. By adding suitable copies of 2×4 this is also the case for all larger values of k. In fact it turns out that the 2×27 game, which is in $\mathcal{N}$, is the last $2 \times n$ instance of DOMINEERING that does not belong to $\mathcal{R}$, but the detailed arguments required to show this are a bit too complex for us here.

Somewhat surprisingly it turns out that it is easier to analyze the $3 \times n$ positions. Clearly $3 \times 1 \in \mathcal{L}$, and we observed above that $2 \times 3 \in \mathcal{N}$ from which it follows that $3 \times 2 \in \mathcal{N}$ also (by symmetry). The 3×3 game is symmetrical and so must be in $\mathcal{N}$ or $\mathcal{P}$. However, it is clear that

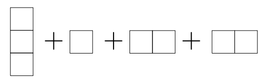
 $\in \mathcal{L}$

and so the 3×3 game is in $\mathcal{N}$. We will show that $3 \times \{4, 5, 6, 7\} \in \mathcal{R}$ and then it follows as above that $3 \times k \in \mathcal{R}$ for all $k \geq 4$ using the *One-Hand-Tied Principle.* In analyzing these four specific cases we will frequently make use of symmetry to limit the number of moves that we need to consider, without explicitly mentioning it. Also, Right generally plays naively anywhere he can in the middle row unless we specify otherwise.

Consider first, the 3×4 case. Suppose that the first pair of moves leads to

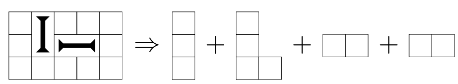


Right can tie one of his hands by vertically separating the board in two places leaving

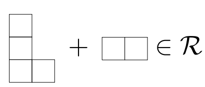


which is in $\mathcal{R}$; Right will win whether he moved first or second. If, on the other hand, Left's first move is on an edge of the board it is also easy to construct a winning strategy for Right.

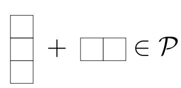
On 3×5, Left's first move on an edge is bad by the previous argument, so again no matter what the first two moves, Right can then tie his hands and separate the board into



which is also a Right win, for we argued in Section 3.2 that

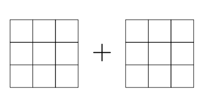


and, of course,



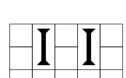
For 3×6 and 3×7 an initial move by Right in the middle row of the rightmost two columns leaves him with two more moves there, together with a position which he wins as second (or first) player, so that can't be bad!

So for these boards we only need to demonstrate (and check) winning replies to Left's possible first moves (and moves by Left in the first column are known to be bad). For 3×6, Right ties one hand to

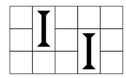


and proceeds to play to maintain $90°$ rotational symmetry between the two boards.

For 3×7, Right's response to Left's first move is in the middle row at one end of the board. Even with the first two moves on the remaining 3×5, Left cannot prevent Right from getting two more moves there. For example, from the 3×5,



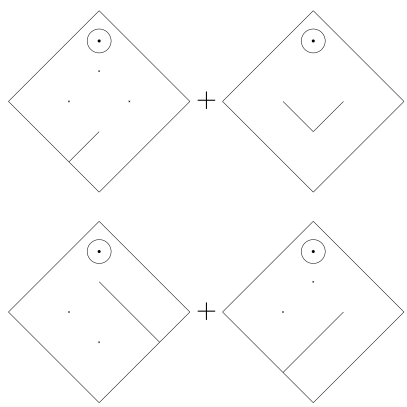
Right plays the bottom left and will get another move, and from



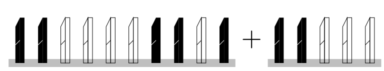
Right will get a move on both top and bottom. In total, Right is guaranteed five moves throughout the game, enough to lock in a victory, for Left has at most five columns available after Right's initial move.

Problems

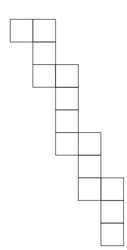
1. Who wins in each of these two sums of MAIZE positions?



2. To which outcome class does the following sum of TOPPLING DOMINOES positions belong?



3. Prove that Left can win from the following DOMINEERING position moving first or second:



Preparation for Chapter 4

Prep Problem 4.1. Show that if Left wins moving second on G_1 and G_2, Left can win moving second on $G_1 + G_2$.

Prep Problem 4.2. Give an example of games G_1 and G_2 so that Left can win moving first on G_1 and on G_2, but cannot win on $G_1 + G_2$, even if she can choose whether to move first or second on the sum.

Prep Problem 4.3. List the properties (or axioms) you usually associate with the symbols $=$, $+$, $-$, and $\geq$. One property might be, "If $a \geq b$ and $b \geq c$ then $a \geq c$"; that is, transitivity of $\geq$. After spending about 10 to 15 minutes listing as many as you can think of, compare your notes with a classmate.

To the instructor: In an undergraduate class, we recommend covering material from this chapter interleaved with examples from the next chapter to motivate this more theoretical chapter. In particular, this chapter provides axioms for combinatorial game theory and important theorems that can be derived reasonably directly from the axioms. The next chapter, however, places the axioms in context. (We chose this approach in order to maintain logical flow.) For a more concrete development, present the games of HACKENBUSH and CUTCAKE as in Chapter 2 of *WW* [BCG01] through their section entitled, *Comparing Hackenbush Positions,* and assign problems primarily from the next chapter.

Chapter 4

The Algebra of Games

The grand aim of all science is to cover the greatest number of empirical facts by logical deduction from the smallest number of hypotheses or axioms.

Albert Einstein

4.1 The Fundamental Definitions

We will state John Conway's definitions of the principal concepts of combinatorial game theory forthwith and then discuss and explain each of them in sequence.[1] You should not expect to understand what these definitions mean until you read the following subsections that discuss each one in greater detail. The five concepts to be defined are: what a game is, what the sum of two games is, what the negative of a game is, when two games are equal, and when one game is preferred over another by Left.

$$
\begin{aligned}
G \quad &\textbf{is} \quad \left\{\mathcal{G}^L \mid \mathcal{G}^R\right\}, \text{ where } \mathcal{G}^L \text{ and } \mathcal{G}^R \text{ are sets of games.} && (4.1)\\
G+H \quad &\stackrel{\text{def}}{=} \quad \left\{\mathcal{G}^L+H,\ G+\mathcal{H}^L \mid \mathcal{G}^R+H,\ G+\mathcal{H}^R\right\}. && (4.2)\\
-G \quad &\stackrel{\text{def}}{=} \quad \left\{-\mathcal{G}^R \mid -\mathcal{G}^L\right\}. && (4.3)\\
G=H \quad &\text{if} \quad (\forall X)\ G+X \text{ has the same outcome as } H+X. && (4.4)\\
G\geq H \quad &\text{if} \quad (\forall X) \text{ Left wins } G+X \text{ whenever Left wins } H+X. && (4.5)
\end{aligned}
$$

In (4.4) and (4.5) the "X" in question ranges over all games. Also, in (4.5) we write "whenever" to mean that if Left wins moving *first* on $H+X$ then Left wins moving *first* on $G+X$ **and** if Left wins moving *second* on $H+X$ then Left wins moving *second* on $G+X$.

[1] Conway actually defines $=$ and $\geq$ a bit differently, but these definitions are equivalent.

Recall the discussion of Section 3.3 and remember that at this point we need to be particularly careful to not think "$G = H$" (in the sense defined above) means that "G is the same as H." In fact, beginning on page 70, we will prove a number of theorems that tell us just how safe it is to make this identification. We say G is *isomorphic* to H, $G \cong H$, if G and H have identical game trees. As discussed in Section 3.3, if $G \cong H$, then to all intents and purposes G and H are the same game, and we can replace G by H (or vice versa) wherever it might occur.

Definition of a game

Effectively, we defined a game to be a pair of sets of games when we said that G was $\{\mathcal{G}^L \mid \mathcal{G}^R\}$. The intention of this definition is just what we saw previously in Section 2.1 — $\mathcal{G}^L$ represents the set of left options of G, and $\mathcal{G}^R$ its right options. The definition itself is rather terse and recursive. The recursive part might — rather, *should* — trouble you, since we neglected to give a base case for the recursion. Thus, defining a game as a pair of sets of games sounds circular. But don't panic. We can *bootstrap* the definition since the sets could be empty. In particular, the definition tells us that $G \stackrel{\text{def}}{=} \{\mathcal{G}^L \mid \mathcal{G}^R\}$ where $\mathcal{G}^L = \mathcal{G}^R = \emptyset$ is a game, a game which we now dub *zero.* One advantage of a recursive definition without explicit base cases is that many inductive proofs based on the definition will not need explicit base cases either. (For a proof without a base case — but with an explanation for why it is not needed — see Theorem A.3 on page 220.)

Definition 4.1. The *birthday* of a game $G = \{\mathcal{G}^L \mid \mathcal{G}^R\}$ is defined recursively as 1 plus the maximum birthday of any game in $\mathcal{G}^L \cup \mathcal{G}^R$. For the base case, if $\mathcal{G}^L = \mathcal{G}^R = \emptyset$, then the birthday of G is 0.

In other words, the birthday of a game is the height of its game tree. A game is *born by* day n if its birthday is less than or equal to n.

The game $0 \stackrel{\text{def}}{=} \{\,\mid\,\}$ is the only game *born on day* 0.[2] We can proceed to implement the recursive definition of a game to list the four games *born by day* 1:

$$\begin{aligned} 0 &\stackrel{\text{def}}{=} \{\,\mid\,\}; \\ 1 &\stackrel{\text{def}}{=} \{0 \mid\,\}; \\ -1 &\stackrel{\text{def}}{=} \{\,\mid 0\}; \\ * &\stackrel{\text{def}}{=} \{0 \mid 0\}. \end{aligned}$$

[2]Recall that $\{\,\mid\,\} = \{\emptyset \mid \emptyset\}$ is the game where $\mathcal{G}^L$ and $\mathcal{G}^R$ are both empty. This is not the same as $\{0 \mid 0\}$, where either play has an option to move to 0.

Recall that $\{\,|\,\}$ means the game in which neither player has a legal move; the sets of left and right options are empty.

The games born by day 2 must have all of their left and right options born by day 1. Since there are 16 possible subsets of the four games born by day 1, there appear to be 256 games born on day 2. In fact, as we will see in Chapter 6, many of these 256 games turn out to be equal in the sense suggested by the definition given in (4.4), and there are in fact only 22 distinct games born by day 2.

Conway observed that you can define games with transfinite birthdays. For example, one can play NIM with an infinite ordinal number of counters in a heap.[3] We will barely touch on such transfinite games in this book, but the curious reader who wants more than a taste should refer to *ONAG* [Con01] or [Knu74].

If G is allowed to be an element of $\mathcal{G}^L$ or $\mathcal{G}^R$ (or, more generally, if G appears anywhere in its own game graph), we arrive at *loopy* games; loopy games are discussed in detail in *ONAG* [Con01] and *WW* [BCG01]. More recently, Aaron Siegel has obtained significant results in this area [Sie05].

For this text, the reader should assume that each game has a finite birthday and so each game has a finite number of options. The following lemma and corollary justify the "and so" in the last sentence:

Lemma 4.2. *The number of games born by day n, call it $g(n)$, is finite.*

Proof: Since a game born by day n is given by a pair of sets of games born by day $n-1$, the number of distinct games born by day n is at most $2^{g(n-1)} \cdot 2^{g(n-1)}$. (Keep in mind that $\mathcal{G}^L$ and $\mathcal{G}^R$ are sets, not multi-sets, so each distinct option can appear only once.) By induction $g(n-1)$ is finite, and so we can conclude $g(n)$ is as well. $\square$

Corollary 4.3. *If G has a finite birthday, then G must also have a finite number of distinct options.*

Proof: Game G with finite birthday n must have options born by day $n-1$, and there are only $g(n-1)$ options to choose from. So $\mathcal{G}^L$ and $\mathcal{G}^R$ are finite. $\square$

If you are *still* worried about the recursive definition of a game you are now in a position to replace it by an inductive one. Namely, the "games born on day n" can be defined inductively in terms of "games born before day n"

[3]The simplest version of this game allows heaps of undetermined size where the first move is to simply name a (finite) size for the heap — these correspond to heaps of size ω. An alternative version is to have counters in place of heaps. These are placed on the grid of non-negative integers, and a legal move for a counter from position (x, y) is to any position (a, b) with $a \le x$ and either $a < x$ or $b < y$. Such counters correspond to heaps of size $x \cdot \omega + y$.

(either using $\{\ |\ \}$ as an explicit base case on day 0 or noting that no base case is actually necessary), and then a "game" is simply a "game born on day n" for some n. However, the recursive, or top-down, view of games is a much more useful one than the inductive, or bottom-up, view when proving theorems or carrying out analysis.

Definition of addition

Conway defines addition by

$$G + H \stackrel{\text{def}}{=} \left\{\mathcal{G}^L + H,\ G + \mathcal{H}^L \;\middle|\; \mathcal{G}^R + H,\ G + \mathcal{H}^R\right\}.$$

This introduces a couple of abuses of notation requiring explanation. G and H are games, while $\mathcal{G}^L$, $\mathcal{G}^R$, $\mathcal{H}^L$, and $\mathcal{H}^R$ are sets of games. So what is meant by $\mathcal{G}^L + H$? We define the addition of a single game, G, to a set of games, $\mathcal{S}$, as the set of games obtained by adding G to each element of $\mathcal{S}$:

$$G + \mathcal{S} = \{G + X\}_{X \in \mathcal{S}}.$$

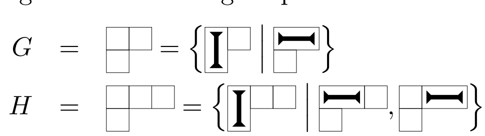
The other abuse of notation is the use of the comma between two sets of games. This comma is intended to mean set union. This notation turns out to be more intuitive and less cumbersome than, say,

$$G + H = \left\{(\mathcal{G}^L + H) \cup (G + \mathcal{H}^L) \;\middle|\; (\mathcal{G}^R + H) \cup (G + \mathcal{H}^R)\right\},$$

for once we have decided to remove braces around sets, treating them essentially as lists, a comma is the natural way to join two lists.

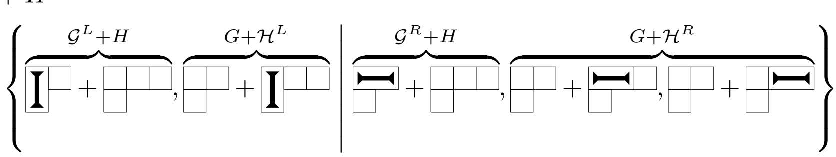
Let's confirm that this notion of addition matches up with that explained by the DOMINEERING position on page 52. We said that in the sum of two games G and H, a player can move on either summand. Left, for instance, could move on the first summand, moving G to some $G^L \in \mathcal{G}^L$, leaving H alone. A typical left option from the sum $G + H$ is therefore $G^L + H$, as in the definition.

As an example, here are two positions, shown expanded (as in the definition of a game) by listing their left and right options:

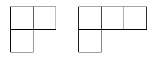


Now, if we mechanically follow the scheme for adding $G + H$, we get

$G + H =$



and this corresponds exactly to what we would expect for the left and right options from the position



Theorem 4.4. $G + 0 = G$.

Proof: $G + 0 = \left\{\mathcal{G}^L \mid \mathcal{G}^R\right\} + \{ \mid \} = \left\{\mathcal{G}^L + 0, G + \emptyset \mid \mathcal{G}^R + 0, G + \emptyset\right\}.$

But for any set $\mathcal{S}$, $\mathcal{S} + \emptyset = \emptyset$,[4] and by induction, $\mathcal{G}^L + 0 = \mathcal{G}^L$ and $\mathcal{G}^R + 0 = \mathcal{G}^R$, so the last expression simplifies to $\left\{\mathcal{G}^L \mid \mathcal{G}^R\right\} = G$. □

Theorem 4.5. *Addition is commutative and associative. That is,*

(1) $G + H = H + G$, *and*

(2) $(G + H) + J = G + (H + J)$.

Exercise 4.6. Prove, by induction, Theorem 4.5.

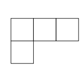
Definition of negative

The definition of negative,

$$-G \stackrel{\text{def}}{=} \left\{-\mathcal{G}^R \mid -\mathcal{G}^L\right\},$$

corresponds exactly to reversing the roles of the two players. (As you might expect, taking the negative of a set negates all the elements of the set; i.e., $-\mathcal{G}^R = \{-G^R\}_{G^R \in \mathcal{G}^R}$.) We swap the left and right options, and recursively swap the roles in all the options.

Exercise 4.7. Negate the DOMINEERING position



using the formal definition. Confirm that the resulting game has exactly the same game tree as the DOMINEERING position obtained when the roles of the players are reversed by rotating the position through 90 degrees.

We can now define

$$G - H = G + (-H).$$

Exercise 4.8. Prove that $-(-G) = G$.

Exercise 4.9. Prove that $-(G + H) = (-G) + (-H)$.

[4] To see why, a game is in $\mathcal{S} + \emptyset$ if it is of the form $A + B$ for $A \in S$ and $B \in \emptyset$. But there is no $B \in \emptyset$.

Definition of game equivalence

We defined

$$G = H \text{ if } (\forall X)\ G + X \text{ has the same outcome as } H + X.$$

In essence, $G = H$ if G acts like H in any sum of games.

Exercise 4.10. Confirm that $=$ is an equivalence relation; that is, it is reflexive, symmetric, and transitive.

On the face of it this definition appears natural, but not very useful. It captures the notion of "being able to substitute G for H" but in order to show $G = H$ it seems that we need to try out *every* possible context X. In this section, we will develop the machinery for an equivalent definition which is much more useful in practice. As a side effect, we will prove that when $G = H$ you can generally substitute G for H wherever you wish, thereby justifying the use of the symbol $=$.

We will begin by spelling out in a little more detail the proof of Proposition 3.1 in terms of our newly defined notion of equality.

Theorem 4.11. *$G = 0$ if and only if G is a $\mathcal{P}$-position (i.e., G is a win for the second player).*

Proof:

$\Rightarrow$ If $G = 0$, then $G + X$ has the same outcome as $0 + X = X$ for all X. In particular, when $X = 0$, this says G has the same outcome as 0, which is a $\mathcal{P}$-position.

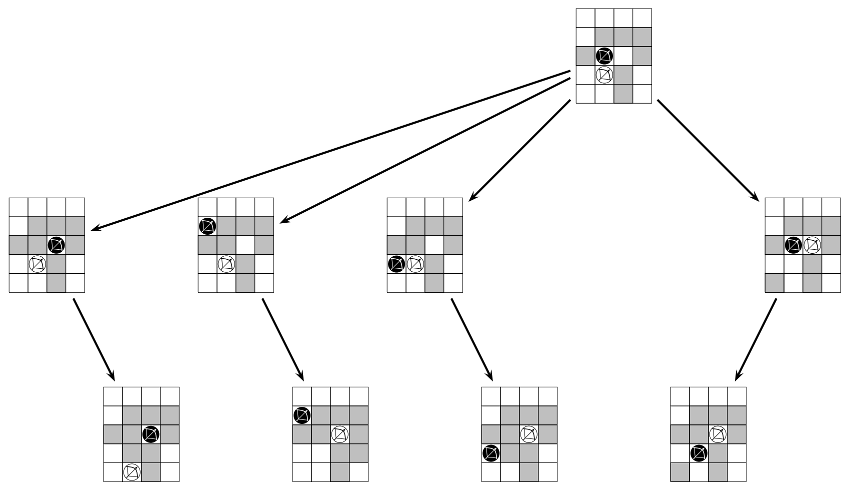
$\Leftarrow$ Let G be a $\mathcal{P}$-position. Fix any position X; we wish to show that $G + X$ has the same outcome as X.

If Left can win moving second on X, Left can also win moving second on $G+X$ by using almost the same strategy. When playing $G+X$, whenever Right plays on the first component, Left responds on that component, pretending she is playing G alone. Similarly, if Right plays on the second component, Left responds locally pretending she is playing X alone. Since Left can get the last move on G (since it is a $\mathcal{P}$-position), and on X, she will get the last move on $G + X$.

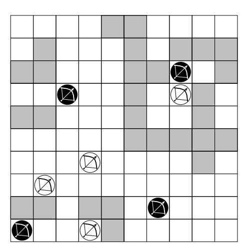
If Left can win moving first on X, she can win moving first on $G + X$ by making the same move on the second component of $G+X$, and proceeding as in the last paragraph.

We can argue symmetrically that if Right wins X, Right wins $G + X$. Hence, X and $G + X$ have the same outcome. □

Example 4.12. The following diagram shows part of the game tree for an AMAZONS position with value 0. In particular, the diagram shows winning responses to some typical first moves:



As a consequence, both players can safely ignore that portion of the board when searching for guaranteed winning moves in the following position:



Exercise 4.13. Convince yourself that the original AMAZONS position has value 0. First, confirm that the second player wins in the lines shown. Second, rule out other promising first moves omitted from the game tree.

Corollary 4.14. *$G - G = 0$.*

Proof: The second player wins on $G-G$ by playing the Tweedledum-Tweedledee strategy. In other words, if the first player moves from $G - G$ to, say, $G - H$, the second player can match the option in the other component, moving to $H - H$, and can proceed to win by induction. $\square$

Theorem 4.15. *Fix games G, H, and J. Then $G = H$ if and only if $G + J = H + J$.*

Proof:

- $\Rightarrow$ Suppose $G = H$. We wish to show that for all X, $(G + J) + X$ has the same outcome as $(H + J) + X$. If we choose $X' = (J + X)$ and apply the definition of $G = H$: $G + X' = H + X'$; i.e., $G + (J + X) = H + (J + X)$ and associativity (Theorem 4.5) yields $(G + J) + X = (H + J) + X$.

- $\Leftarrow$ Suppose $G + J = H + J$. We can use the $\Rightarrow$ direction just proved to conclude $(G + J) + (-J) = (H + J) + (-J)$. By associativity and noting $J - J = 0$, we have
$$\begin{aligned} G &= G + 0 \\ &= G + (J - J) \\ &= (G + J) - J \\ &= (H + J) - J \\ &= H + (J - J) \\ &= H + 0 \\ &= H. \end{aligned}$$
□

Exercise 4.16. Identify which theorem or definition justifies each = in the last expression.

Exercise 4.17. Prove that if $G = G'$ and $H = H'$ then $G + H = G' + H'$.

Corollary 4.18. *$G = H$ if and only if $G - H = 0$; that is, $G - H$ is a $\mathcal{P}$-position.*

Proof: Simply add $-H$ to both sides and use associativity. □

This corollary is important, for it gives a clear constructive way to determine whether $G = H$: simply play the game $G - H$ and see if the second player always wins. In practice, this is the easiest and most common way of testing whether two games are equal. This completes a circle begun on page 57. We've provided a "middle road" definition of equality in order to allow for later gains in simplifying positions to determine their outcome classes. On the other hand, at this point, in order to determine whether or not two games G and H are equal we must return to the pragmatic position and answer the question of whether or not $G - H \in \mathcal{P}$. However, once we have this information it allows us to replace G by H freely in any context — which, if say H is much simpler than G, will be of tremendous value in later analysis. Furthermore, in Section 4.3, we will provide *automatic* means for finding a unique simplest form for any game G sparing us the guesswork of trying to find suitably simple games H for which $G = H$.

Definition of greater or equal

$$G \geq H \quad \text{if} \quad (\forall X) \text{ Left wins } G + X \text{ whenever Left wins } H + X$$
$$G \leq H \quad \text{if} \quad (\forall X) \text{ Right wins } G + X \text{ whenever Right wins } H + X$$

In short, this says that $G \geq H$ if replacing H by G can *never* hurt Left, no matter what the context, assuming she plays optimally.

Exercise 4.19. Confirm that $G \geq H$ if and only if $H \leq G$. Also confirm that $G = H$ if and only if $G \geq H$ and $G \leq H$.

As with our original definition of $G = H$, while this definition is reasonably intuitive, it is not very constructive. In this section, we will give a more constructive definition for how to test if $G \geq 0$ and generalize that notion to test if $G \geq H$.

Theorem 4.20. *The following are equivalent:*

(1) $G \geq 0$.

(2) *Left wins moving second on G; that is, $G \in \mathcal{P}$ or $G \in \mathcal{L}$.*

(3) *For all games X, if Left wins moving second on X, then Left wins moving second on $G + X$.*

(4) *For all games X, if Left wins moving first on X, then Left wins moving first on $G + X$.*

Proof:

$1 \Leftrightarrow$ both 3 and 4 hold: The outcome class of a game is determined by whether Left wins moving first and whether Left wins moving second.

$2 \Rightarrow 3$: Suppose Left can win moving second on G and on X. To win on $G+X$ playing second, she uses the *One-Hand-Tied Principle,* choosing, when Right moves in G to play the move dictated by her second-player-winning strategy for G and when Right moves in X to play the move dictated by her second-player-winning strategy for X.

$2 \Rightarrow 4$: The proof is analogous to $(2 \Rightarrow 3)$.

$3 \Rightarrow 2$: Choose $X = 0$.

$4 \Rightarrow 2$: Consider the equivalent contrapositive to the fourth condition: "If Left does not win moving first on $G + X$ then Left does not win moving first on X." Take $X = -G$. Since $G - G = 0$, Left does not win moving first on $G - G$. Therefore, Left does not win moving first on $-G$. Equivalently, Right does not win moving first on G; that is, Left wins moving second on G. □

Exercise 4.21. Prove $2 \Rightarrow 4$ in Theorem 4.20.

Theorem 4.22. *$G \geq H$ if and only if $G+J \geq H+J$, for all games G, H, and J.*

Proof: The proof parallels that of Theorem 4.15. □

Theorem 4.23. *$G \geq H$ if and only if Left wins moving second on $G - H$.*

Proof: $G \geq H$ if and only if $G - H \geq H - H = 0$ (i.e., $G - H \geq 0$). □

This last theorem is how, in practice, one should compare G with H. In particular, when comparing G with H, determine the outcome of $G - H$. If $G - H$ is a $\mathcal{P}$-position, then $G = H$. If $G - H$ is an $\mathcal{L}$-position, then $G \geq H$, but $G \neq H$, that is, $G > H$. If $G - H \in \mathcal{R}$, we have $G < H$. Lastly, if $G - H$ is in $\mathcal{N}$, then it is neither the case that $G \geq H$ nor that $G \leq H$; G and H are incomparable! When two games are incomparable, we say G is *confused with* or *incomparable* with H, and denote this by $G \parallel H$.

To recap,

$G > 0$ when L wins G	$G > H$ when L wins $G - H$
$G = 0$ when 2^{nd} wins G	$G = H$ when 2^{nd} wins $G - H$
$G < 0$ when R wins G	$G < H$ when R wins $G - H$
$G \parallel 0$ when 1^{st} wins G	$G \parallel H$ when 1^{st} wins $G - H$

Lastly, just as we write $G \geq H$ to mean $G = H$ or $G > H$, we will use the symbol $G \mathrel{|\!\!\triangleright} H$ to mean $G > H$ or $G \parallel H$; that is, G is greater than or incomparable to H. Note that $G \mathrel{|\!\!\triangleright} H$ is equivalent to $G \not\leq H$, but is somewhat more intuitive. Similarly, $G \mathrel{\triangleleft\!\!|} H$ means $G < H$ or $G \parallel H$.

4.2 Games Form a Group with a Partial Order

It is easy to prove that $\geq$ is a partial order so long as we use game-equality (which we have written as $=$) for equality. To be a partial order, a relation must be transitive, reflexive, and antisymmetric:

Transitive: If $G \geq H$ and $H \geq J$ then $G \geq J$.

Reflexive: For all games G, $G \geq G$.

Antisymmetric: If $G \geq H$ and $H \geq G$ then $G = H$.

Theorem 4.24. *The relation $\geq$ is a partial order on games.*

Proof:

Transitive: Suppose $G \geq H$ and $H \geq J$. Then Left wins moving second on $(G - H)$ and on $(H - J)$, and so she can win moving second on the sum $(G - H) + (H - J)$ by following her second-player-winning strategy in each summand. However, $(G-H)+(H-J) = (G-J)+(H-H)$, which has the same outcome as $(G - J)$. Hence, $G \geq J$.

Reflexive: Left wins moving second on $G - G = 0$.

Antisymmetric: Given by Exercise 4.19. $\square$

Games also form an *abelian (or commutative) group.* A *group* is a collection of elements (games) along with a binary operation (here, $G + H$), an inverse $(-G)$, and an identity (the game 0), which satisfy:

Closure: If G and H are elements of the group, so is $G + H$;

Associativity: For all games G, H, and J, we have $(G+H)+J = G+(H+J)$;

Identity: For all games G, $G + 0 = G$;

Inverse: For all games G, $G + (-G) = 0$.

The group is *abelian* if, in addition, the operation is commutative; that is, $G + H = H + G$.

The main subtlety here is that we are really talking about the group structure of games under the equivalence relation $=$. In particular, there is not just one zero game $0 = \{\,|\,\}$, but rather any game equal to 0 (i.e., any $\mathcal{P}$-position) is an identity.

Theorem 4.25. *Games form an abelian group.*

Proof:

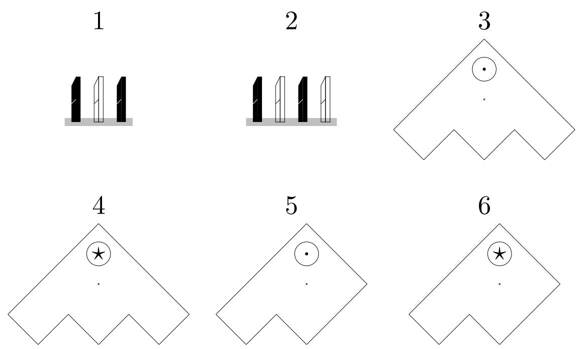
Closure: $G + H \overset{\text{def}}{=} \left\{\mathcal{G}^L + H,\ G + \mathcal{H}^L \,\middle|\, \mathcal{G}^R + H,\ G + \mathcal{H}^R\right\}$ is a pair of sets of games (by induction), and so $G + H$ is a game.

Associativity and commutativity: Theorem 4.5.

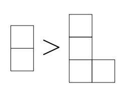
Identity: Theorems 4.4 and 4.11.

Inverse: Corollary 4.14. $\square$

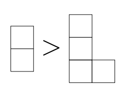
Exercise 4.26. For each of the following six separate positions — two TOPPLING DOMINOES, two MAIZE (shown with $\odot$), and two MAZE ($\circledast$) — determine the outcome class for each position, and whether it equals a game born by day 1 (either 0, 1, -1, or $*$).



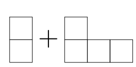
Example 4.27. Show that in DOMINEERING,



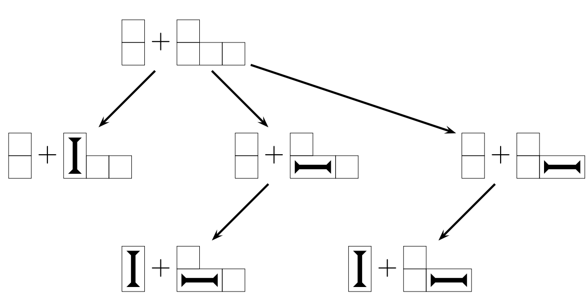
Proof: To show



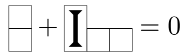
it suffices to show that Left wins



whether she moves first or second. All the relevant lines of play are summarized below:



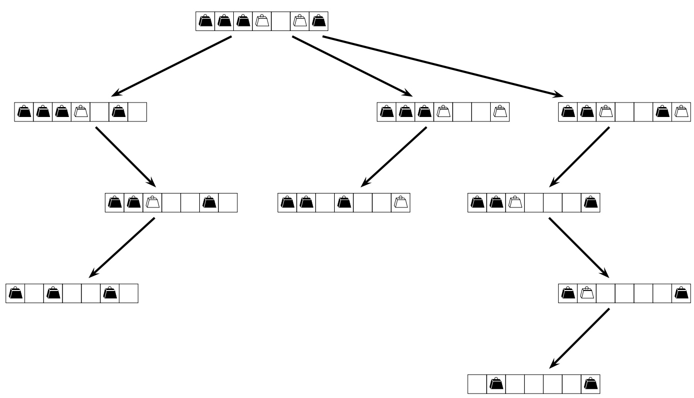
In particular, if she moves first, she wins by moving on the second component to



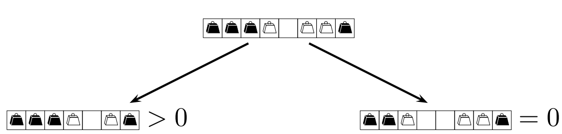
Moving second, Right can only make one move no matter what Left does and is doomed. □

Example 4.28. Show that, in CLOBBER, $>$ but that is confused with .

First, consider the difference game $-$ $=$ $+$. Left wins moving first or second by playing in the second component (if possible) at her first opportunity, and then finishing the game by taking Right's piece in the first component. Hence, $>$. Left's winning lines are shown below:

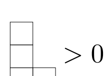


Now consider $-$ $=$ $+$. Left moving first plays to $+$, and we just showed this wins. If Right moves first, however, he can play to $+$ $= 0$ and wins as well. Hence, $\Vert$:

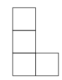


Exercise 4.29.

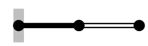
1. In DOMINEERING, show that



2. Show that the DOMINEERING position

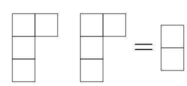


is equal to the HACKENBUSH position



4.3 Canonical Form

As we have seen, two games can be equal despite having very different game trees. For example, careful analysis will show that

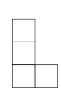


The surprising, but extremely useful, fact is that every game G has a unique *smallest* game which is equal to it. This game is called G's *canonical form,* and there is a clear, methodical procedure to simplify G to reach its canonical form. So, one way to check if G and H are equal is to see whether their canonical forms are identical.

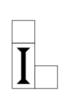
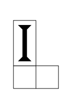
There are two mechanisms for simplifying a game given in the next two subsections. What is remarkable is that these are the only two simplifications one has to perform to reduce a game to its unique canonical form!

Dominated options

Consider the DOMINEERING position



Left, who plays the vertical dominoes, has two options:

 and 

The first of these leaves a position of value 0 since neither player has a move available. The second leaves a position in which Right, but not Left, has a move, a position of value -1. Unless Left is playing the game against a small child who shares a significant portion of her DNA there are no circumstances when she should choose to play the second move. We say it is *dominated* by the first move, since she would always prefer the situation arising after her first move (whatever the remaining context might be) to that arising from her second move. This is really a version of the *One-Hand-Tied Principle* — Left promises not to make the second move, knowing that it can never be in her interests to do so. We now generalize and formalize this idea.

Theorem 4.30. *If*

$$G = \{A, B, C, \dots \mid H, I, J \dots\}$$

and $B \geq A$, then $G = G'$ where

$$G' = \{B, C, \dots \mid H, I, J \dots\}\,.$$

Similarly for Right, if $I \geq H$, then option I can be removed. Here, option B is said to *dominate* option A for Left and H dominates I for Right, and the theorem allows us to *remove a dominated option.*

Proof: To see that $G = G'$, we will confirm that $G - G' = 0$; that is, that either player wins moving second on $G - G'$. On this difference game, if Left moves G to B or C, Right responds by moving $-G'$ to $-B$ or $-C$, respectively. In fact, all moves by Left or Right on G pair up with a move by the opponent on $-G'$, except the move to A. The second player can respond to any first player move on $-G'$ by making the corresponding move on G. So, the only way the first player might hope to avoid a simple Tweedledum-Tweedledee strategy is to play Left and move G to A. But then Right responds by moving G' to $-B$, leaving $A - B$ which, since $B \geq A$, Right wins as the second player.

This argument is summarized in the following diagram. We have paired up moves with obvious responses with $\surd$s. For example,

if Left moves to $\overset{\checkmark}{B}$,

Right responds on the other component to $-\overset{\checkmark}{B}$

and vice-versa.

$$\underbrace{\left\{ A, \overset{\checkmark}{B}, \overset{\checkmark}{C}, \dots \;\middle|\; \overset{\checkmark}{H}, \overset{\checkmark}{I}, \overset{\checkmark}{J} \dots \right\}}_{G} - \underbrace{\left\{ \overset{\checkmark}{B}, \overset{\checkmark}{C}, \dots \;\middle|\; \overset{\checkmark}{H}, \overset{\checkmark}{I}, \overset{\checkmark}{J} \dots \right\}}_{G'}$$

$$\begin{array}{c} \swarrow \\ A - G' \\ \searrow \\ A - B \\ \text{R wins} \\ \text{since} \\ B \geq A \end{array}$$

□

Reversible options

A *reversible option* by Left is one that Right can promise to respond to immediately in such a way that his prospects after this exchange of moves are at least as good as they were before. That is, a left option A of G can be considered to be reversible if A has a right option A^R which, from Right's point of view, is just as good or better than G; that is, $A^R \le G$. In any context containing G, Right can make a *One-Hand-Tied* promise, "if you ever choose option A of G then I will immediately move to A^R." When Right makes such a promise it doesn't make sense for Left to choose option A *unless* she intends to follow up Right's move to A^R with an immediate response to one of A^R's left options. If she plans some other move elsewhere, she might just as well start with that. So Left can consider the left options of A^R as being *immediately* available to her from G instead of worrying about the interchange "first I choose A, then Right chooses A^R, then I choose one of its left options." Reversibility is our second simplification principle that leads to canonical forms for games.

This simplification rule is not at all intuitive, and may take some getting used to. To repeat,

Theorem 4.31. *Fix a game*

$$G = \{A, B, C, \ldots \mid H, I, J, \ldots\}$$

and suppose that for some right option of A, call it A^R, $G \ge A^R$. If we denote the left options of A^R by $\{W, X, Y, \ldots\}$:

$$A^R = \{W, X, Y, \ldots \mid \ldots\}$$

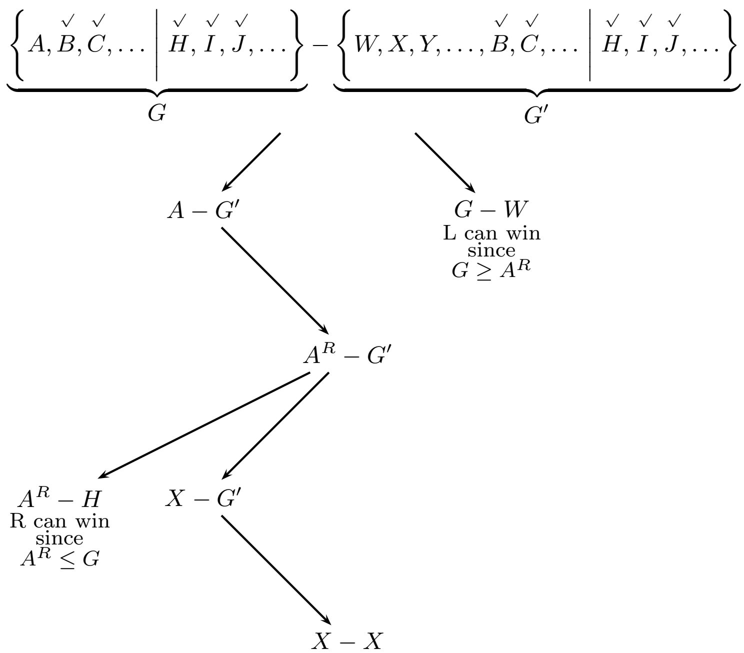
and define the new game

$$G' = \{W, X, Y, \ldots, B, C, \ldots \mid H, I, J, \ldots\},$$

then $G = G'$.

Where the theorem applies, we say that Left's move to A is a *reversible option.* One can also say the move to A is *reversible* and the move to A *reverses through* A^R to the left options of A^R. (If there are no left options of A^R, the move to A is said to *reverse out* through A^R.) The process of replacing A with the left options of A^R is called *bypassing the reversible option* A.

Proof: Assume all the hypotheses of the theorem. To show that $G = G'$, we will show that the second player wins the difference game $G - G'$. Unlike the last proof, we summarize the proof with a diagram:



The heart of the proof lies in the two assertions justified in the words "...can win since...". To address the first, to find a winning move $A^R - H$, Right uses the fact that $A^R \leq G$ (by the definition of reversible option). In particular, Right wins moving second on $A^R - G$, and so Right has a winning response to *every* Left option from $A^R - G$, and $A^R - H$ is one such left option. □

Exercise 4.32. Summarize the diagrammed proof in words. As you do so, convince yourself that every case is handled. Be sure to understand the second case in which the diagram has the words, "...can win since...".

We will see examples of how to simplify a game by bypassing reversible options in the next chapter. We are now ready to prove the main theorem of the section.

Reduction to canonical form

We say that G is in *canonical form* if G and all of G's positions have no dominated or reversible options. The following theorem justifies the term *canonical*:

Theorem 4.33. *If G and H are in canonical form and $G = H$, then $G \cong H$.*

In other words, if you start with a game G and bypass all its reversible options and remove all its dominated options until no more simplification is possible, then you arrive at the unique smallest game, called G's canonical form, which is equivalent to G. There are two important things to note about this process. First of all, both the removal of a dominated option and the reversal of a reversible option produce a game whose game tree has fewer nodes than the original tree did. Therefore, the process of removing dominated options and bypassing reversible ones must terminate. Second, the order in which we choose to carry out this procedure turns out to be irrelevant — there may well be choices, but they will all lead to the same canonical form.

This important theorem is surprisingly easy to prove.

Proof: Suppose that G, H, and all their positions have no reversible or dominated options. By induction, their left and right options are in canonical form, and it suffices to show that G's left options match up with H's.

Since $G = H$, Left wins moving second on $G - H$. In particular, Left has a winning response to $G^R - H$. That winning response cannot be in G^R, for then we would have that some $G^{RL} - H \geq 0$, and so $G^{RL} \geq H = G$, and so G would have a reversible option. So the winning response to $G^R - H$ must be on the second component to some $G^R - H^R$, and we have that $G^R \geq H^R$ for some H^R.

We can construct the same argument for any initial move on G or H. In particular, for each G^R there exists an H^R such that $G^R \geq H^R$. A parallel argument proves that for each H^R there exists a $G^{R'}$ such that $H^R \geq G^{R'}$. Since $G^R \geq H^R \geq G^{R'}$, G^R and $G^{R'}$ must be identical (for otherwise G^R is dominated.) So, every right option of G equals some right option of H; i.e., $\mathcal{G}^R \subseteq \mathcal{H}^R$. By symmetric arguments, $\mathcal{H}^R \subseteq \mathcal{G}^R$, so $\mathcal{G}^R = \mathcal{H}^R$, and similarly $\mathcal{G}^L = \mathcal{H}^L$. Hence, $G \cong H$. □

We close with a useful related lemma, which we state here since the proof is similar to that of the last theorem.

Lemma 4.34. *If $G = H$ and G is in canonical form, then each option of G is dominated by an option of H; i.e.,*

$$(\forall G^L)(\exists H^L) \quad \textit{such that} \quad H^L \geq G^L, \ \textit{and}$$
$$(\forall G^R)(\exists H^R) \quad \textit{such that} \quad H^R \leq G^R.$$

Proof: The proof is similar to, but simpler than, that of Theorem 4.33, and we leave it as an exercise. □

Exercise 4.35. Prove Lemma 4.34. (Note that H need not be canonical.)

4.4 Incentives

It is often helpful, when playing games, to focus on the amount gained or lost by a move rather than on the resulting position.

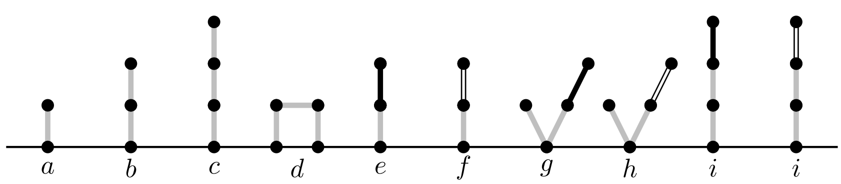
Definition 4.36. The *left incentives* and *right incentives* of a game are the sets $\mathcal{G}^L - G$ and $G - \mathcal{G}^R$, respectively. The *incentives* of a game are the union of the left and right incentives. Similarly, the left incentive (or, respectively, right incentive) of a particular move G^L is $G^L - G$ (or $G - G^R$).

Note, as you compare the definitions of left and right incentives, that the order of terms in the difference has switched. Consequently, both players should favor moves of higher incentive. This makes it meaningful, for example, to compare the left incentives of G against its right incentives. In particular, when playing in a sum of games $A+B+C+D+E+F+G$, Left should choose to play in the component that has largest left incentive (if such a component exists).

We will see calculations of incentives early in the next chapter.

Problems

1. Draw the partial order of the following HACKENBUSH positions:

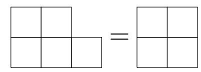


2. Fix a particular (partial) HACKENBUSH position G, with one edge e unspecified. Four different positions can be obtained depending on whether edge e is blue, green, red, or missing. (If missing, note that other edges might be disconnected from the ground and therefore removed as well.) Give the partial order on these four games, and prove that the partial order is independent of the position. As always, induction is preferred over arguments like, "and then right continues to . . . ".

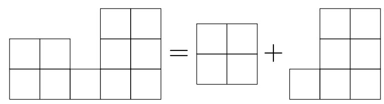
3. The following beautiful DOMINEERING decomposition theorem appears in *ONAG* [Con01] and *WW* [BCG01]:

Here, G and H are arbitrary domineering positions.

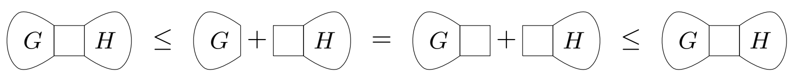
For an example application of the theorem, it is not hard to convince yourself that



So by the theorem

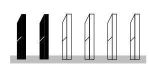


John Conway's proof in *ONAG* [Con01] is summarized by



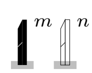
Justify each inequality or equality in the last expression to complete the proof of the DOMINEERING decomposition theorem.

4. (a) Prove that Left's incentive from the TOPPLING DOMINOES position

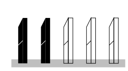


equals Right's incentive.

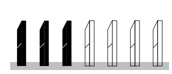
(b) State and prove a theorem about how to compare the incentive of two games of the form



for $m, n \geq 1$. For example, using your theorem it should be possible to compare the incentive from



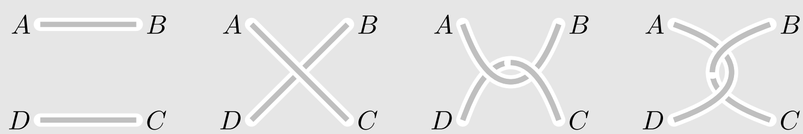
with those of



Preparation for Chapter 5

The purpose of this (rather fun) exercise is to assure the reader that a number is just a symbol, and context determines its usefulness. Conway's *rational tangles* are a way to describe the various ways in which two strands of rope can be entwined. Place two ends of rope on a table, and label the corners of the table A, B, C, and D in a clockwise fashion. Now, place the ropes on the table, one going from point A to B and one from D to C:

A *twist* is performed by swapping the ends at B and C, passing the end which started at B over that which started at C. A rotate consists of rotating the ends 90° clockwise, $A \to B \to C \to D$. Here are two twists followed by a rotate:



Conway associates with each two-rope tangle a rational number. The starting point is 0. Each twist adds 1, and each rotate takes the negative reciprocal. So two twists and a rotate yield

$$0 \xrightarrow{T} 1 \xrightarrow{T} 2 \xrightarrow{R} -\frac{1}{2}.$$

He proved that tangles are in one-to-one correspondence with rational numbers. Consequently, you can untangle the above tangle by doing another twist, a rotate, and then two more twists:

$$-\frac{1}{2} \xrightarrow{T} \frac{1}{2} \xrightarrow{R} -2 \xrightarrow{T} -1 \xrightarrow{T} 0.$$

(By the way, rotating 0 yields $-\frac{1}{0} = \frac{1}{0}$.)

Prep Problem 5.1. Determine the sequence of twists and rotates to construct the *tangle* $\frac{3}{7}$. Next, how do you untangle it with more twists and rotates? Check your answer using two strings or shoelaces.

To the instructor: As alluded to in the previous chapter's suggestions, CUTCAKE and HACKENBUSH from [BCG01, Ch. 2] provide good examples for numbers. Noam Elkies constructs CHESS endgame positions in [Elk96] which include $\uparrow$, $\frac{1}{2}$, $\frac{1}{4}$, ± 1, and $+_1$. If the class has CHESS players, we recommend covering his material as a case study toward the end of this chapter.

From here on, students should be encouraged to use CGSuite (Appendix B) to help attack problems (some of which are quite challenging to do entirely by hand) and to translate their observations into clear proofs.

Chapter 5

Values of Games

Y'see it's sort of a game with me. Its whole object is to prove that two plus two equals four. That seems to make sense, but you'd be surprised at the number of people who try to stretch it to five.

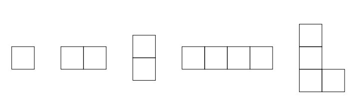
Dalton Trumbo in The Remarkable Andrew

In this chapter, we begin the process of naming games. On page 66, we assigned names to the four games born by day 1, those being 1, -1, 0, and $*$. Since there are already 1474 distinct games born by day 3, we do not intend to name all games. We will focus on naming the most *important* games, where importance of a game is determined by informal criteria: Does the game appear naturally as a position in played games? Have we proved theorems about the game? Do we know its partial order structure with other named games? Do we know how to add the game with other named games?

The reason for assigning specific names to games is not just to provide a shorthand method of referring to them. So, when we choose a name for a game it should be one that is related to the central properties of that game. Further, the names and notation we choose should be consistent and easy to remember.

5.1 Numbers

We have already argued that there are good reasons to associate numbers with various DOMINEERING positions (and hence with any other games having the same structure). Specifically:



$0 \quad -1 \quad 1 \quad -2 \quad 1/2$

There are other DOMINEERING positions that we could also easily assign numerical values to — for instance a vertical strip of $2n$ squares represents n moves available to Left and none to Right, so it would seem sensible to declare that it has value n. We will now formalize and continue this development. It turns out that the only numbers that we need to associate with games whose game tree is finite are the *dyadic rationals,* those rational numbers whose denominator is a power of 2. Other numbers do occur as the values of games, but require games whose length is not bounded in advance. Since we only study short games in this book, all rational numbers we encounter will be dyadic.

Integers

It is natural to define a game in which Left has n free moves available as having *value* n. Similarly, a game in which Right has n free moves is $-n$. In this way, we define all integers. More formally, for n a positive integer, we define the games

$$\begin{aligned} 0 &\stackrel{\text{def}}{=} \{\,\mid\,\} \text{ and} \\ n &\stackrel{\text{def}}{=} \{n-1 \mid \,\}\,. \end{aligned}$$

The game $-n$ will be defined as the negative in the sense of the definition given in (4.3) on page 65 of the game n. Remember, in this definition, n and $n-1$ really stand for "the game whose value is n" and "the game whose value is $n-1$," respectively. So, as a game,

$$2 = \{1 \mid \,\} = \{\{1 \mid \,\} \mid \,\} = \{\{\{0 \mid \,\} \mid \,\} \mid \,\} = \{\{\{\{\,\mid\,\} \mid \,\} \mid \,\} \mid \,\}\,.$$

You can see why it might be preferable just to write 2.

Exercise 5.1. Use the definition of negative, page 69, and confirm that

$$-n = \{\,\mid 1-n\}\,.$$

Exercise 5.2. Confirm that the DOMINEERING position that is a vertical strip of four empty squares is the game whose value is 2.

Observation 5.3. Let n be an integer and N a game whose value is n.

- If $n = 0$, neither player has a move available in N.
- If $n > 0$, then Left has a move in N to the game whose value is $n-1$, and Right has no move available in N.
- If $n < 0$, then Right has a move in N to the game whose value is $n+1$, and Left has no move available in N.

The previous definition already puts us on thin ice — dangerous and slippery. It is slippery in the sense that when we write 10 do we mean the number 10, or the game whose value is 10? It is dangerous in that we have at present no evidence that the games whose values are integers share any significant properties with the integers. Since we can compare and add games, and we can compare and add integers, we would be in serious trouble if the properties of these games and the corresponding integers differed. As you may well have guessed — they don't. But for peace of mind we had better confirm that now. We will let you start that process with an exercise:

Exercise 5.4. In this exercise only, write "n" to mean the game whose value is n. Show that "$n+1$" equals the game sum of "n" and "1". Confirm that the similar property holds for "$n-1$".

For the remainder of this section, we will try to limit the notational confusion between games and integers by referring to games with capital letters, and integers with lowercase ones. So the games A, B, and C will have integer values a, b, and c, respectively.

Lemma 5.5. *The integer sum $a+b+c \geq 0$ if and only if $A+B+C \geq 0$ as games. (Symmetrically, $a+b+c \leq 0$ if and only if $A+B+C \leq 0$.)*

(Recall that $A+B+C \geq 0$ means Left wins moving second in $A+B+C$.)

Proof: First, suppose that $a+b+c \geq 0$. We wish to show that $A+B+C \geq 0$.

We will show $A+B+C \geq 0$ by showing that Left wins moving second on $A+B+C$. Any move by Right (on C, say) must be on a negative game. But then at least one of a, b, and c must also be positive since $a+b+c \geq 0$; without loss of generality $a > 0$. So Right has moved in the game C to $C+1$. Then Left can move on A to $A-1$. By induction, $(A-1)+B+(C+1) \geq 0$. (The induction is valid since the sum of the absolute values of a, b, and c is guaranteed to decrease.) So, after Left's move, the position is a second-player win for Left, and Left thus wins the original game as the second player.

For the converse, suppose $A+B+C \geq 0$. This means that Left wins moving second from $A+B+C$.

If Right has no move at all, then we have $A, B, C \geq 0$ and $a+b+c \geq 0$ trivially. If, on the other hand, Right has a move available in $A+B+C$, say to $A+B+(C+1)$, then Left must have a winning countermove. This can be taken to be to the game $(A-1)+B+(C+1)$. The sum $(A-1)+B+(C+1)$ is a simpler game than $A+B+C$ (since two of the summands are simpler with smaller game trees) and so by induction, $(a-1)+b+(c+1) \geq 0$. But now by the associativity and commutativity of integer addition, $a+b+c \geq 0$ as we had hoped. $\square$

Note that the forward direction in the above induction proof requires no base case, since the proof handles all cases as written. Were you to want to explicitly include $a = b = c = 0$ as a separate case, you would notice that the first player has no move in $A + B + C$ and so loses trivially.

Theorem 5.6. *$A + B = C$ if and only if $a + b = c$.*

Proof: From the previous result, $A+B+(-C) = 0$ if and only if $a+b+(-c) = 0$. The stated result now follows by adding C (respectively c) to both sides of each equality. □

In addition to their additive structure, integers are also ordered. As part of justifying the ambiguous notation we ask the reader to confirm that integer games maintain the same ordering:

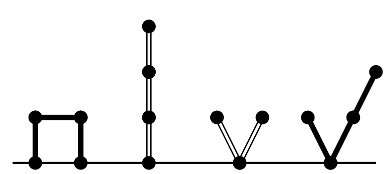
Exercise 5.7. Prove that $A \geq B$ if and only if $a \geq b$.

We now feel comfortable enough with our definition to declare that from now on when we write 17 we might be referring to "the game whose value is 17" (or any other game equal to that specific game) or to the integer 17.

Exercise 5.8. Find the incentives from 0 and from integers $n \neq 0$. (Incentives are defined in Section 4.4 on page 83.) By the way, incentives are never positive or zero.

Example 5.9. A HACKENBUSH position that has only black edges has an integer value equal to the number of edges in the position.

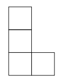
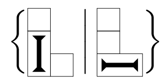
Exercise 5.10. What is the value of the following HACKENBUSH position?



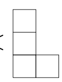
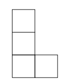
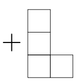
Numbers

Now we will continue identifying games with numbers, and in particular to the dyadic rational numbers; that is, fractions whose denominators are a power of 2, such as 1/2, 3/8, and 19/32. When the definition of numbers is naturally extended beyond short games, we get the *surreal numbers,* which include the reals, the ordinals, and more [Con01, Knu74].

We can let our previous investigation of DOMINEERING guide the definition:

 =  $= \{0 \mid 1\}$

Here we eliminated a dominated option (Left playing in the top two vertical squares) to produce the canonical form of this position. We argued previously that this value was a pretty good candidate for $1/2$ since

$0 <$  < 1 and  $+$  $= 1$

Exercise 5.11. Try to come up with "reasonable" candidates for $1/4$ and $1/8$ (as abstract games, not as DOMINEERING positions).

We will now give the formal definition of numbers, and then do our ice dancing trick again....

Definition 5.12. For $j > 0$ and m odd, we define the *number*[1]

$$\frac{m}{2^j} = \left\{ \frac{m-1}{2^j} \;\middle|\; \frac{m+1}{2^j} \right\}.$$

So, for example, $19/32 = \{18/32 \mid 20/32\} = \{9/16 \mid 5/8\}$.

Problem 7 on page 112 asks you to prove that the games given by Definition 5.12 are in canonical form.

Example 5.13. Prove that $1/2 + 1/2 = 1$.

Proof: It suffices to show that either player wins moving second on the difference game $1/2+1/2-1$. If either player moves on a $1/2$, the other will respond on the other $1/2$, leaving $1 - 1 = 0$, and so the first player loses. On the other hand, if Right moves -1 to 0, this will leave $1/2 + 1/2$ from which Left can move to $1/2$, Right can respond to 1, and Left takes the 1 to 0. $\square$

Exercise 5.14. Confirm a few more complicated equalities, such as $\frac{15}{16} + \frac{1}{4} = 1\frac{3}{16}$, until you are comfortable with the definition of numbers. As you do so, assume inductively that simpler values behave as one would expect.

It is important that moves on a number change it to a worse number for the mover. (Left moves make it smaller, Right moves make it larger.) In particular, the *incentive* for either player to move on $\frac{m}{2^j}$ (for m odd and $j > 1$) is $-\frac{1}{2^j}$.

[1]A *better* definition of number is any game x such that all $x^L < x < x^R$ and x^L and x^R are numbers. This naturally generalizes to non-short games. That our definition is equivalent for short games will be a consequence of Theorem 5.21.

Exercise 5.15. Confirm the last assertion. Note that it is important that $\frac{m}{2^j}$ is in canonical form. Incentives are defined in Section 4.4 on page 83.

We will later strengthen this observation concerning the negative incentives of numbers in two ways. First, the *Number-Avoidance Theorem* (Theorem 6.17 on page 125) states that one should only move on numbers as a last resort. Second, the *Negative-Incentives Theorem* (Theorem 6.19 on page 126) proves that if for a game G, all of G's incentives are negative, then G is a number.

As with integers, by adopting these definitions of numbers, we are suggesting that they add and are ordered as one would expect. Suppose games A, B, and C have values which are numbers (dyadic rationals) a, b, and c, respectively. As we did for integers, we wish to show that $A + B = C$ *if and only if* $a + b = c$ and that $A > B$ *if and only if* $a > b$. Echoing the proof for integers would be tedious, however, both because of the more complicated definition of numbers, and because we want to mix non-integers and integers. This lemma obtains the results more quickly. Again we adopt the convention of using uppercase letters to stand for "games whose names are numbers" and the corresponding lowercase letters to stand for the numbers themselves.

Lemma 5.16.
$$\begin{aligned} a+b+c=0 &\iff A+B+C=0;\\ a+b+c>0 &\iff A+B+C>0;\\ a+b+c<0 &\iff A+B+C<0. \end{aligned}$$

Proof: We will prove these three assertions collectively. Note that since the three conditions are mutually exclusive, we need only prove the forward implication of each of the three assertions, "if $a + b + c = 0$ then $A + B + C = 0$," and so forth, for together they imply the $\iff$.

Suppose $a + b + c \geq 0$. If Right moves to some $A^R + B + C$, then the definitions of number and integer dictate that $A^R > A$. Hence, $a^R + b + c > 0$, and by induction $A^R+B+C > 0$, and so Right loses. So we have $A+B+C \geq 0$. A symmetric argument made for $\leq$ gives the first forward implication.

If $a+b+c > 0$, we already argued Left wins moving second from $A+B+C$, so it suffices to show Left wins moving first. Now, $A + B + C > 0$ must be a dyadic rational, say $\frac{i}{2^j}$ where either i is odd or $j = 0$. Then one of the following two cases must hold:

- One of a, b, or c, say a, is of the form $\frac{i'}{2^{j'}}$ for $j' \geq j$ and $j' > 0$ and $a + b + c - \frac{1}{2^{j'}} \geq 0$. In this case, $A^L + B + C \geq 0$ by induction.
- All of a, b, and c are integers, one of which (say a) exceeds 0. Then $(a - 1) + b + c \geq 0$, and by induction, $A^L + B + C \geq 0$.

In both cases, Left has a winning first move, yielding the second forward implication. The third implication is symmetric. □

Note here that there is an implicit induction on the dyadic rationals, a, b, and c. To confirm the induction terminates, either note that the sum of the denominators of a, b, and c always decrease, or note that the birthdays of A, B, and/or C decrease.

Exercise 5.17. Does the proof properly handle the trivial case when $a = b = c = 0$ as written, or should that case have been included as an easy base case?

Theorem 5.18.

- *$A + B = C$ if and only if $a + b = c$.*
- *$A \geq B$ if and only if $a \geq b$.*

Proof: The proof is immediate from the lemma and is left as an exercise. □

Exercise 5.19. Prove Theorem 5.18.

When playing a sum of games, it is generally wisest to avoid playing on numbers. This assertion is codified in two theorems, the *Weak Number-Avoidance Theorem* proved here, and the *strong* one proved later.

Theorem 5.20. (Weak Number Avoidance) *Suppose that x is a number and G is not. If Left can win moving first on $G + x$, then Left can do so with a move on G.*

Proof: We may assume that x is in canonical form. Rephrased, this says that if some $G + x^L \geq 0$ then some $G^L + x \geq 0$. Assume $G + x^L \geq 0$. Since G is not a number (in particular, $G \neq -x^L$) we know $G + x^L > 0$. So Left wins moving first on $G + x^L$, and by induction some $G^L + x^L \geq 0$. Since $x > x^L$, we have $G^L + x \geq 0$. □

The simplest number

The definition of numbers, Definition 5.12, can be generalized to recognize when G is a number even if it is not in canonical form.

Theorem 5.21. *If all options of a game G are numbers and all $\mathcal{G}^L < \mathcal{G}^R$, then G is also a number. In particular, G is the simplest number x satisfying $\mathcal{G}^L < x < \mathcal{G}^R$. (Simplest number is defined below.)*

Definition 5.22. For numbers $x^L < x^R$, the *simplest number* x between x^L and x^R is defined by the unique number with the smallest birthday strictly between x^L and x^R.

Definition 5.23. (Alternate definition) For $x^L < x^R$, the *simplest number* x between x^L and x^R is given by the following:

- If there are integer(s) n such that $x^L < n < x^R$, x is the one that is smallest in absolute value.
- Otherwise, x is the number of the form $\frac{i}{2^j}$ between x^L and x^R for which j is minimal. (The reader might recognize this as the longest ruler mark between x^L and x^R.)

Theorem 5.24. *Both definitions of simplest-number are well-defined and equivalent.*

We will first state two lemmas for the reader to prove, and then prove the result.

Lemma 5.25. *If $x_1 < x_2$ both have the same birthday, then some number x, with $x_1 < x < x_2$ has a smaller birthday.*

Proof: Problem 8 on page 112 asks you to prove this. □

Lemma 5.26. *The integers $\pm n$, $n \geq 0$ have birthday(s) $n+1$. The two numbers $\pm\left(n + \frac{i}{2^j}\right)$, where $n \geq 0$, i odd and $0 < i < 2^j$, have birthdays $n + j + 1$.*

Exercise 5.27. Prove Lemma 5.26 by induction using the definitions of integer and number.

Proof (of Theorem 5.24): To prove the definitions are well defined, we need to prove that each produces a *unique* x. For the first definition, if two numbers x_1 and x_2 have the same birthday, then by the last lemma, some x between x_1 and x_2 has a smaller birthday. For the second definition, if n_1 and n_2 are candidate integers with the same (smallest) absolute value, then $n_1 = -n_2$, and so 0 (which is a simplest number between the two) is also a candidate integer; while if $\frac{i_1}{2^j}$ and $\frac{i_2}{2^j}$ are candidate numbers, so is $x = \frac{i}{2^j}$ where i is an even number between the two odd numbers i_1 and i_2, and the fraction x reduces to one with a smaller denominator.

To prove the definitions are equivalent, if an integer lies between x^L and x^R, then the integer with smallest absolute value (by Lemma 5.26) has the smallest birthday. If no integer lies between, then they both lie between two consecutive $n < x^L, x^R < n + 1$, and by Lemma 5.26, both definitions will minimize j. □

Proof (of Theorem 5.21): Fix $G = \left\{\mathcal{G}^L \mid \mathcal{G}^R\right\}$ where all $\mathcal{G}^L < \mathcal{G}^R$ are numbers, and let x be the simplest number satisfying $\mathcal{G}^L < x < \mathcal{G}^R$. It suffices to show that the second player wins playing $x - G$. Right playing first can play to some

$x - G^L$ or to $x^R - G$. The former is positive since we chose $\mathcal{G}^L < x < \mathcal{G}^R$. For the latter, it cannot be that $x^R < \mathcal{G}^R$, for then $\mathcal{G}^L < x^R < \mathcal{G}^R$, but x^R is simpler than x, and x was the simplest number in that range. Hence, some $x^R \geq G^R$, and Left's move to $x^R - G^R$ wins.

By a symmetric argument, Right moving second wins, and so $x = G$. □

Exercise 5.28. Determine the values of

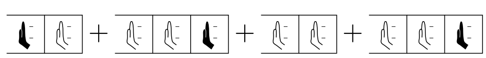
1. $\{\frac{1}{2} \mid 2\}$;
2. $\{\frac{1}{8} \mid \frac{5}{8}\}$;
3. $\{-1\frac{27}{64} \mid -1\frac{9}{32}\}$.

Theorem 5.21 can be generalized to handle some cases when the options are not numbers:

Theorem 5.29. *If there is some number x such that $\mathcal{G}^L \lhd\!\!| \; x \lhd\!\!| \; \mathcal{G}^R$, then G is the simplest such x.*

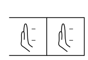
Proof: The proof is nearly identical to that of Theorem 5.21 and is left for the reader as Problem 14. □

Example 5.30. In PUSH, who should win



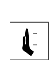
and how?

First note that the negative of a PUSH position is obtained by changing the color of each black and white piece. For Right, moving in

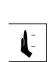


seems bad. Saving a counter that can be pushed or pushing a left counter looks better but how to decide? Left's best move would appear to be in the last game where she can push two right pieces but is it a winning move? If the games are all numbers, and at least one contains a fraction, then by the previous exercise, the best move will be in the game with the highest denominator.

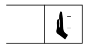
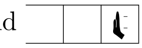
The position



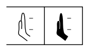
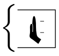
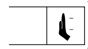
is clearly worth one move to Left:

 $= \{0 \mid \} = 1.$

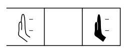
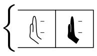
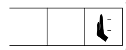
Also,

 $= \{1 \mid \ \} = 2,$ and  $= \{2 \mid \ \} = 3.$

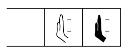
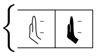
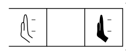
With a right piece, we have

 $= \Big\{$  $\Big|$  $\Big\} = \{1 \mid 2\} = \frac{3}{2}.$

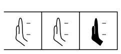
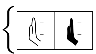
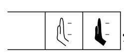
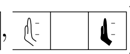
Also,

 $= \Big\{$  $\Big|$  $\Big\} = \left\{\frac{3}{2} \,\middle|\, 3\right\} = 2$

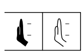
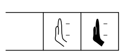
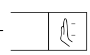
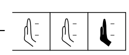
and

 $= \Big\{$  $\Big|$  $\Big\} = \left\{\frac{3}{2} \,\middle|\, 2\right\} = \frac{7}{4}.$

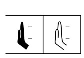
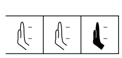
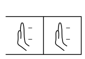
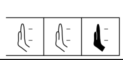
Lastly,

 $= \Big\{$  $\Big|$  $,$  $\Big\} = \left\{\frac{3}{2} \,\middle|\, \frac{7}{4}, 2\right\} = \left\{\frac{3}{2} \,\middle|\, \frac{7}{4}\right\} = \frac{13}{8}.$

Therefore, the value of the disjunctive sum of these games is

 $+$  $+$  $+$  $= -\frac{3}{2} + \frac{7}{4} + (-2) + \frac{13}{8} = -\frac{1}{8},$

which is a Right win. Since these values appeared pretty much in canonical form,

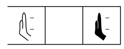
Position	**Left incentive**	**Right incentive**
 $= -\frac{3}{2} = \{-2 \mid -1\}$	$-\frac{1}{2}$	$-\frac{1}{2}$
 $= \frac{7}{4} = \{\frac{3}{2} \mid 2\}$	$-\frac{1}{4}$	$-\frac{1}{4}$
 $= -2 = \{\ \mid -1\}$		-1
 $= \frac{13}{8} = \{\frac{3}{2} \mid \frac{7}{4}\}$	$-\frac{1}{8}$	$-\frac{1}{8}$

Left can only change the sum of the games to a more negative number, but the best of a bad lot is to move in the last summand where the sum only decreases by $\frac{1}{8}$. Right has only one winning move, to move from

 to 

which changes the sum to 0 — every other move leaves a positive game that Left will win.

When a number appears with a reversible option (and is not in canonical form) then the incentive could be different. For example, the left and right incentives from

 $= \left\{ \frac{3}{2} \,\middle|\, 3 \right\} = 2$

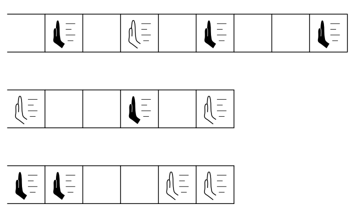
are $-\frac{1}{2}$ and -1, respectively. However, if all you seek is *some* component with a winning move (if it exists), you may assume each component is in canonical form.

While we do not yet have a formula to easily compute the value of any PUSH position, a close variant called SHOVE has been solved.

SHOVE

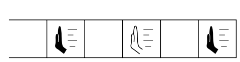
The game of SHOVE is played on one or more strips of squares. Each square can be empty or it might contain a black or white counter. Left moves by selecting a black counter and moving it, along with all the counters to its left on the same strip, left one square. Counters can fall off the left-hand end of a strip. Right moves by selecting a white counter and moves it, along with all the counters to its left, left one square. Note that counters always move leftward.

Exercise 5.31. Play SHOVE[2] with a classmate or friend from the following start position. (You might want to use coins as counters.)



In order to analyze this game, we need only consider positions consisting of a single strip, as the multiple strip positions are sums of such games. Though not at all obvious from the rules, there is a simple recipe for calculating the value of any SHOVE position.

First, number the squares $\{1, 2, \ldots\}$ from the left. Assume that the rightmost piece is black and is on square n. If the piece 2^{nd} from the right is also black, or if there are no white pieces, then the value of G is n plus the value of the position achieved by removing the piece on square n. For example,

 $=$  $+ 9$

The justification for this is that, faced with a choice between moving either her rightmost piece or the piece second from the right, Left always prefers to move

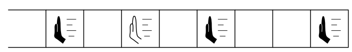
[2] In this text, SHOVE pieces look almost like PUSH pieces, only with more lines behind the hand.

the piece second from the right. You can verify this by a direct comparison of the games resulting from each of the possible two moves, or more elegantly observing that the latter move “preserves future options” for Left. This general technique (which amounts to eliminating from consideration a certain type of dominated option) might well be called the *Don’t Burn Your Bridges Principle.*

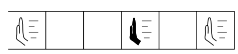
After repeated application of this rule, we will either obtain an empty position (and know that the value of the position was an integer), or reach a position in which the two rightmost counters are of opposite colors.

In a position in which the rightmost piece is black, and the second from the right is white, a piece at position n with c counters to its right contributes $\frac{n}{2^c}$ multiplied by the appropriate sign.

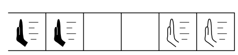
Each counter below is labeled with its value as given by the recipe:



$\frac{2}{4}$ $\quad -\frac{4}{2}$ $\quad 6$ $\quad 9$



$-\frac{1}{4}$ $\quad \frac{4}{2}$ $\quad -6$



$\frac{1}{4}$ $\frac{2}{2}$ $\quad -5$ -6

More formally, any n-piece SHOVE position may be defined by two functions p and c, where

$$\begin{aligned} p(i) &= \text{position of } i^{\text{th}} \text{ piece from the left;} \\ c(i) &= \begin{cases} 1 & \text{if the } i^{\text{th}} \text{ piece is black,} \\ -1 & \text{if the } i^{\text{th}} \text{ piece is white.} \end{cases} \end{aligned}$$

Additionally, define $r(i)$ to be the number of pieces to i’s right up to and including the piece after the last color alternation. (So $r(i) = 0$ if it and all pieces to its right are the same color.)

Theorem 5.32. *Adopting the preceding notation, the value of a* SHOVE *position is*

$$x = \sum_{1 \le i \le n} c(i) \frac{p(i)}{2^{r(i)}}.$$

Proof: One of Left's options (as we will see, her best option) is to move her leftmost piece j at position $p(j)$. By induction, that move is to

$$\begin{aligned} & \sum_{1\le i\le j} c(i)\frac{p(i)-1}{2^{r(i)}} + \sum_{j<i\le n} c(i)\frac{p(i)}{2^{r(i)}} \\ = \quad & x + \sum_{1\le i\le j} c(i)\frac{-1}{2^{r(i)}} \\ = \quad & x - \frac{1}{2^{r(1)}}. \end{aligned}$$

For the last equality, since j is the leftmost black piece, $c(j) = 1$, while $c(i) = -1$ for $i < j$. Left's options to move any other piece, say $j' > j$, is inferior, for then the last summation evaluates to a quantity $< x - \frac{1}{2^{r(1)}}$ since $c(j') = c(j) = 1$.

By a symmetric argument, Right's best option is to $x + \frac{1}{2^{r(1)}}$. Hence, the value of the shove position, is given by

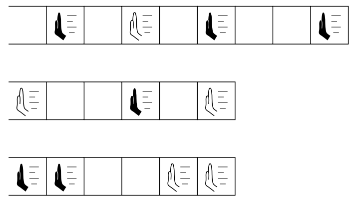
$$\left\{ x - \frac{1}{2^{r(1)}} \;\middle|\; x + \frac{1}{2^{r(1)}} \right\} = x$$

since x is the simplest number between.

Since the formula for x is unaffected by pretending that a piece off the board is actually on the board at position 0 (either way, the piece contributes 0 to the summation), the proof remains sound when $p(1) = 1$. □

You, gentle reader, might well wonder, "Where on earth did that formula come from?" The answer is fairly simple. When we begin to analyze a new game we always start by looking at simple positions. In SHOVE the simplest positions are those in which there is only one counter — their values are rather trivial. Next we might well observe that positions in which all the counters are the same color have equally trivial values. So the next case to consider is positions with two counters, one of each color. Starting with the case of a white counter on square 1, and a black counter somewhere to its right we'd find that white counter to be worth half a move to Right. When the white counter is on square 2, we'd find it was worth a full move. And so on. At this point we might well introduce CGSuite to consider slightly more complex positions. We'd soon observe that the values of SHOVE positions always seemed to be numbers, and make the observation that the pieces to the right of the final color alternation could be removed (with suitable compensation). Then we might look at the effect on the value of removing the leftmost (or rightmost) counter — and eventually come up with the formula above, along with a fair degree of confidence in its correctness.

Exercise 5.33. For the following SHOVE position from the start of this section, find the incentive for each move and then identify all the best first moves for each player:



5.2 A Few All-Smalls: Up, Down, and Star

We have already named one game that is not a number, that being $* = \{0 \mid 0\}$. We saw this game as one of the four games born on day 1. There are many important games that are nearly 0, but are not numbers.

Definition 5.34. A game G is *infinitesimal* if $-x < G < x$ for all positive numbers x.

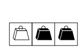
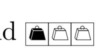
Example 5.35. We will show that $*$ is infinitesimal. Since $*$ is incomparable with 0 (the first player wins from $*$), $*$ cannot be a number. Note that $* = -*$, so by symmetry it suffices to show $* < x$ for positive numbers x (i.e., Left wins moving first or second on $x - * = x + *$). From $x + *$, Left wins moving first by moving to x. On the other hand, Right's choices for his first move from $x + *$ are not terribly palatable. By Theorem 5.20, the *Weak Number-Avoidance Theorem,* if Right had a winning first move it would have to be in $*$. However, his only available move there leaves $x > 0$, which Left will now win moving first. Since Right has no winning first move, but Left does, $x + * > 0$.

We now proceed to define two more infinitesimals:

Definition 5.36. The games *up* and its negative, *down*, are given by

$$\uparrow \stackrel{\text{def}}{=} \{0 \mid *\};$$

$$\downarrow \stackrel{\text{def}}{=} \{* \mid 0\}.$$

For example, in CLOBBER,  $= \uparrow$ and  $= \downarrow$.

Observation 5.37. The game $\uparrow$ is a positive infinitesimal. Correspondingly, $\downarrow$ is a negative infinitesimal.

Proof: We wish to show that $0 < \uparrow < x$ for numbers $x > 0$. Left wins moving first or second on $\uparrow$, so $\uparrow > 0$. Assume $x > 0$ is in canonical form. From $x - \uparrow$, Right's move to x loses immediately and if Right has a move to $x^R - \uparrow$, it loses by induction since $x^R > x$ is also a number. Hence, $x \geq \uparrow$. Since Left can win moving first from $x - \uparrow$ to $x - *$ (which is positive since $*$ is infinitesimal), $x > \uparrow$. □

Definition 5.38. G is *all-small* if every position H in G has the property that Left can move from H if and only if Right can.

The games $*$ and $\uparrow$ are both all-small games, while the only all-small number is 0.

Observation 5.39. G is all-small if and only if either

1. $G = 0$, or
2. $\mathcal{G}^L$ and $\mathcal{G}^R$ are non-empty and are all all-small.

Theorem 5.40. *Every all-small game is infinitesimal.*

Proof: Problem 12 asks you to prove this. □

Sums made from $\uparrow$ and $*$

We now know how these games compare with all numbers. How do they compare with one another? How about sums of these games?

Observation 5.41.
$$\downarrow < 0 < \uparrow$$
$$\downarrow \parallel * \parallel \uparrow$$
$$\downarrow + \downarrow < * < \uparrow + \uparrow$$

We leave the proof as a short exercise:

Exercise 5.42. Prove Observation 5.41. We have already shown the first inequality in Observation 5.37. Confirm the other two by playing the games $\uparrow - *$ (which equals $\uparrow + *$) and $\uparrow + \uparrow - *$.

The game $\uparrow + \uparrow$ comes up so frequently, that we abbreviate it $\Uparrow$. Similarly, we have $\Uparrow\!\!\!\uparrow$, $\Downarrow$, and so forth. Figure 5.1 summarizes the relative ordering of numbers, ups, and star. In Combinatorial Game Theory, we draw the number line backwards, with the positive numbers to the left so as to reinforce the convention that Left is positive.

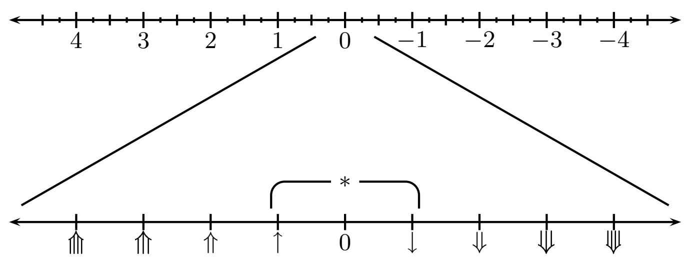


Figure 5.1. Multiples of $\uparrow$ are all infinitesimal (i.e., they lie between $+x$ and $-x$ for any positive number x). The game $*$ is incomparable with $\uparrow$, 0, and $\downarrow$, but lies strictly between $\Uparrow$ and $\Downarrow$.

On notation: When addition is implicit

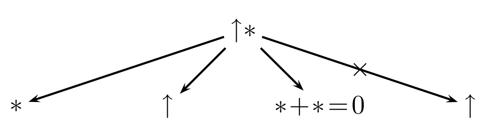
In grade school we learned that $1\frac{1}{2}$ is shorthand for $1 + \frac{1}{2}$. We will adopt the same convention, that when concatenating named games, we mean to add them. We always list numbers, then $\uparrow$s then $*$. So,

$$2{\Uparrow}* = 2 + {\Uparrow} + *,$$

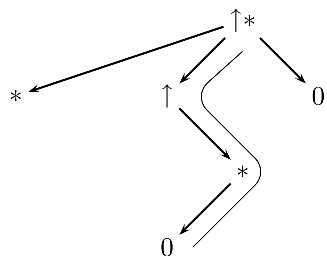
but we would *not* write $*\uparrow$. We will later define a game called $*2$, and it will not equal $2* = 2 + *$.

Canonical forms of $n\cdot\uparrow$ and $n\cdot\uparrow*$

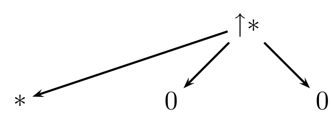
We will first compute the canonical forms of $\uparrow*$ and $\Uparrow$. Since $\uparrow*$ is an option of $\Uparrow$, it is best to work on $\uparrow*$ first. Here is the game tree for $\uparrow*$:



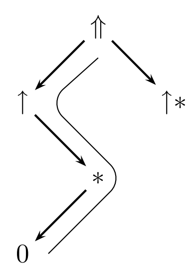
Right's move to $\uparrow$ is dominated by 0, and there are no other dominated options. As we check for reversible options, note that $\uparrow*$ is incomparable with 0, but that $* < \uparrow*$ since $\uparrow > 0$. So the move from $\uparrow*$ to $\uparrow$ reverses through $*$ to 0:



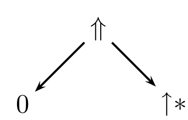
There are no further dominated or reversible options, and we arrive at the canonical form of $\uparrow\!* = \{0, * \mid 0\}$:



We next compute the canonical form of $\Uparrow$. Either player can move on one of the $\uparrow$s, in Left's case moving the $\uparrow$ to 0, and in Right's case moving $\uparrow$ to $*$. So, $\Uparrow = \{\uparrow \mid \uparrow\!*\}$. Clearly, there are no dominated options, for each player has only one option. Right's move to $\uparrow\!* = \{0, * \mid 0\}$ does not reverse, for $\Uparrow > 0$ and $\Uparrow > *$. However, Left's move to $\uparrow$ reverses through $*$ to 0, for $\Uparrow > *$:



We arrive at the canonical form of $\Uparrow$:



In similar fashion, we can mechanically compute the canonical forms of $\Uparrow\!*$, ⤊, $\text{⤊}*$, and so forth. A pattern quickly emerges:

$$\begin{array}{rcl@{\qquad\qquad}rcl}
\uparrow & = & \{0 \mid *\} & \uparrow\!* & = & \{0, * \mid 0\} \\
\Uparrow & = & \{0 \mid \uparrow\!*\} & \Uparrow\!* & = & \{0 \mid \uparrow\} \\
\text{⤊} & = & \{0 \mid \Uparrow\!*\} & \text{⤊}* & = & \{0 \mid \Uparrow\} \\
\text{⟰} & = & \{0 \mid \text{⤊}*\} & \text{⟰}* & = & \{0 \mid \text{⤊}\}
\end{array}$$

We denote

$$n\cdot g = \begin{cases} 0 & \text{if } n = 0, \\ \overbrace{g + g + \cdots + g}^{n \text{ times}} & \text{if } n > 0, \\ (-n)\cdot(-g) & \text{if } n < 0. \end{cases}$$

So, for example, $3\cdot\uparrow = \text{⤊}$ and $-3\cdot\uparrow = \text{⤋}$.

Theorem 5.43. *For $n \geq 1$, the canonical forms of $n{\cdot}\uparrow$ and $n{\cdot}\uparrow *$ (parsed as $(n{\cdot}\uparrow)*$) are given by*

$$n{\cdot}\uparrow \;=\; \{0 \mid (n-1){\cdot}\uparrow *\}\,;$$

$$n{\cdot}\uparrow * \;=\; \begin{cases} \{0 \mid (n-1){\cdot}\uparrow\} & \text{if } n > 1, \\ \{0, * \mid 0\} & \text{if } n = 1. \end{cases}$$

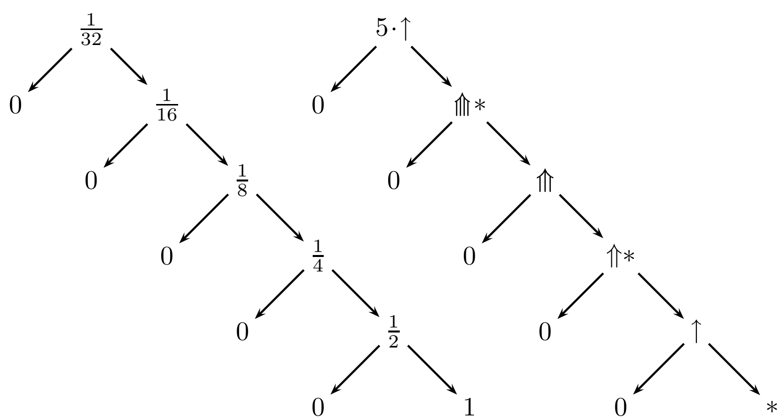
Symmetrically,

$$n{\cdot}\downarrow \;=\; \{(n-1){\cdot}\downarrow * \mid 0\}\,,$$

$$n{\cdot}\downarrow * \;=\; \begin{cases} \{(n-1){\cdot}\downarrow \mid 0\} & \text{if } n > 1, \\ \{0 \mid 0, *\} & \text{if } n = 1. \end{cases}$$

Proof: Problem 16 asks you to prove this. □

It might be instructive to compare the game trees for $n{\cdot}\uparrow$ and 2^{-n}:



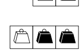
Values with similar game trees can differ dramatically in their algebraic behaviors.

Ups in Clobber

Let us investigate some of the simplest CLOBBER positions, those with a single white stone and a row of black stones, . We can build a short list of positions and a pattern quickly emerges:

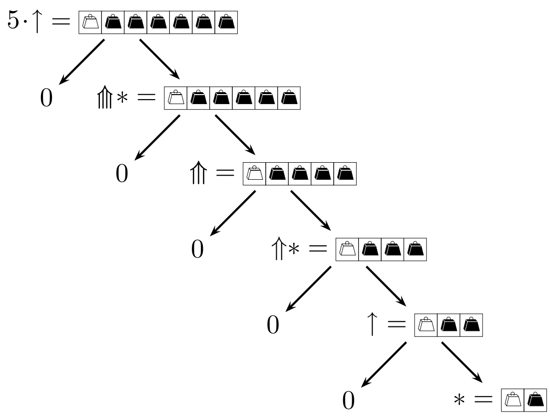
 $= 0$

 $= \{0 \mid 0\} = *$

 $= \{0 \mid *\} = \uparrow$

 $= \{0 \mid \uparrow\} = \Uparrow *$

Pictorially,



Exercise 5.44. Formalize the pattern by giving an expression for the CLOBBER position  with n black stones. Inductively prove your expression holds true for all n.

5.3 Switches

The positions $\{y \mid z\}$ for y and z numbers with $y > z$ are called *switch games* or *switches.* We can normalize any such game by rewriting

$$\{y \mid z\} = a + \{x \mid -x\} = a \pm x,$$

where $a = (y+z)/2$ and $x = (y-z)/2$ and $\pm G$ is shorthand for $\{G \mid -G\}$.

Exercise 5.45. Prove that $\{y \mid z\} = a + \{x \mid -x\} = a \pm x$ as advertised above. (We will later see that this is a special case of the *Number Translation Theorem* on page 126.)

In the switch $\pm x$, the person who plays gets x points. In contrast with numbers, players are eager[3] to play on switches, making them *hot* games.

Naturally, $a \pm x \pm y \pm z$ is shorthand for $a + \pm x + \pm y + \pm z$.

Exercise 5.46. Evaluate $\pm x \pm x$ where $x > 0$ is a number.

Exercise 5.47. Consider the game

$$G = a \pm x \pm y \pm z,$$

where $x > y > z > 0$ and a are all numbers.

[3]Well, pretty eager. A game can never have positive incentive.

1. Play the game (perhaps with a classmate) with various values of a, x, y, and z.

2. For what values of a does Left win moving first? Express your answer as an inequality in x, y, and z. How about moving second?

3. Generalize the last answer to

$$G = a \pm x_1 \pm x_2 \pm x_3 \cdots \pm x_n.$$

While it is easy to evaluate the winner in sums of switches, their canonical forms grow large. For instance,

$$5 \pm 3 \pm 2 \pm 1 = \left\{ 11 \mid 9 \,\Big\|\, 7 \mid 5 \,\Big|\Big\|\, 5 \mid 3 \,\Big\|\, 1 \mid -1 \right\}.$$

Exercise 5.48. The TOPPLING DOMINOES position with m black dominoes has value m; if there are m black dominoes followed by n white dominoes then the value is $\{m-1 \mid -(n-1)\}$. What is the value of the game if there are m black dominoes followed by n white dominoes and then p black where $m \geq p$?

Hard switch-like games

One can generalize $\pm x$ to obtain games of the form $\{x \,\|\, y \mid z\}$ where $x \geq y \geq z$ are numbers. Best play on sums and differences of games of this form is very hard, indeed. Computer science has techniques for formally proving a problem is *computationally hard,* and the following theorem is for those readers who can appreciate such results.

Theorem 5.49. *Determining whether Left can win playing first on sums and differences of games of the form* $\{x \,\|\, y \mid z\}$ *is PSPACE-complete.*

If you know about PSPACE-completeness, read on a bit longer. The reader who does not can safely skip to the next section.

Theorem 5.49 can be used to prove other games are PSPACE-hard, but some care is required. In general, one wishes to show that $\{x \,\|\, y \mid z\}$ appears in the game. Unfortunately, most of the time, if the game does appear, it requires a board size which is polynomial in x (not merely polynomial in the number of bits in x), and hence requires an exponential time reduction. It is still an important open question to prove Theorem 5.49 when x, y, and z are integers specified in unary.

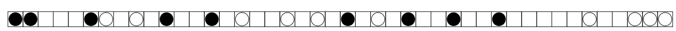
Not all is lost, however. Let β exceed the number of bits in x, y, and z. Although the game tree for integers x has a number of positions which is exponential in β, one can move the decimal point, for the game $\frac{1}{2^\beta}$ has only a polynomial number of positions in β! If you can then exploit the fact that $\frac{x}{2^\beta}$ can be decomposed into a sum of positions of the form $\frac{1}{2^{\beta'}}$, you are done.

In summary, if you can construct $\left\{\frac{x}{2^\beta} \,\middle\|\, \frac{y}{2^\beta} \,\middle|\, \frac{z}{2^\beta}\right\}$ on a game board polynomial in size in β, you can prove the game is hard.

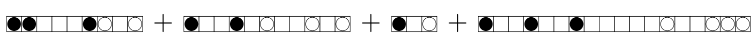
For a proof of Theorem 5.49, and an application to proving GO endgames are PSPACE-hard, see [Wol02].

ELEPHANTS & RHINOS

Who wins in the following game of ELEPHANTS & RHINOS?



The first observation to make is that the board breaks up into smaller boards: reading from left to right if there is a ○ followed by ● then pieces to the left of and including this ○ never interact with the pieces to the right. The given game breaks up into the disjunctive sum of four boards:



If any summands consists only of ●s (or of ○s), then computing their values is simply a matter of counting moves. The only positions remaining that require analysis are those in which ●s are all to the left of ○s. Some are easy: on the first board, ● has 6 moves and ○ has 1, and the value is $6-1=5$. On the third board, both players have a move to 0 so its value is $*$.

Fix a position G in which the two *central pieces,* one elephant and one rhino, are separated by s spaces. To the left of these central pieces lie $n_\bullet$ additional elephants, and to the right are $n_\circ$ rhinos. (The total number of pieces is $n_\bullet + n_\circ + 2$.) Let $m_\bullet$ be the maximum number of moves that the elephants can make without moving the center elephant and likewise $m_\circ$ for the rhinos.

Moving the center elephant gains $n_\bullet$ moves (each of the other elephants can move one space extra), while also restricting the rhinos. So it seems clear that moving the center piece dominates all other moves. In fact, the value of this game can be found by assuming that the central pieces take turns approaching one another until they come to a standstill. We will proceed to formalize this.

Define

$$x = m_\bullet - m_\circ + \lfloor \frac{s}{2} \rfloor (n_\bullet - n_\circ).$$

Exercise 5.50. Suppose the two players each make $\lfloor \frac{s}{2} \rfloor$ moves with their central beasts so that they are separated by 0 or 1 square. Convince yourself that x represents the remaining difference between Left's and Right's available moves by non-central beasts.

Claim 5.51. *Using the above notation,*

$$G = \begin{cases} x & \textit{if } s \textit{ is even,} \\ \{x + n_{\bullet} \mid x - n_{\circ}\} & \textit{if } s \textit{ is odd.} \end{cases}$$

Proof: Suppose $s = 0$, the result is trivial. Using induction, if $s > 0$ is even, Left can either move the central ● to $\{x + n_{\bullet} + n_{\circ} \mid x\}$ or a non-central ● to $x - 1$. The former move dominates, and

$$\Big\{(x + n_{\bullet} + n_{\circ}) \mid x \,\Big\|\, x \mid (x - n_{\bullet} - n_{\circ})\Big\} = x.$$

If s is odd, Left can move a central ● to $x + n_{\bullet}$, or (possibly) a non-central ● to $\{x + n_{\bullet} - 1 \mid x - n_{\circ} - 1\}$. The former moves dominates. □

In the second component of our motivating example, ●□□●□○□□○□○ , we have $s = 1$, $m_{\bullet} = 2$, $m_{\circ} = 5$, $n_{\bullet} = 1$, $n_{\circ} = 2$ so $x = 2 - 5 + \lfloor\frac{1}{2}\rfloor(-1) = -3$ and the value is $\{-2 \mid -5\}$. In the fourth component ●□□●□□●□□□□□○□□○○○ , $s = 5$, $m_{\bullet} = m_{\circ} = 6$, $n_{\bullet} = 2$, $n_{\circ} = 3$, and so $x = 6 - 6 + \lfloor\frac{5}{2}\rfloor(2 - 3) = -2$, and the game has value $\{0 \mid -5\}$. Therefore,

●●□□□●○□○ + ●□□●□○□□○□○ + ●□○ + ●□□●□□●□□□□□○□□○○○

$$\begin{aligned} &= 5 + \{-2 \mid -5\} + * + \{0 \mid -5\} \\ &= \Big\{3 \mid 0 \,\Big\|\, -2 \mid -4\Big\} + *. \end{aligned}$$

Right's only losing initial moves are to move a non-central ○. Left, however, can only win by moving the central ● in the last component.

If you do not wish to calculate the value of each position, an optimal strategy is to select that component, among all the components in which the central beasts can move, which contains the largest number of beasts, and move your central beast in that component. If there are no central beasts, move any legal piece.

Exercise 5.52. Convince yourself of the last assertion. In particular, what is the incentive for each central and non-central move?

5.4 Tiny and Miny

There are two more infinitesimals of note, *tiny-G* and its negative *miny-G*, which are denoted $⧾_G$ and $⧿_G$, respectively:

$$\begin{aligned} ⧾_G &= \Big\{0 \,\Big\|\, 0 \mid -G\Big\}; \\ ⧿_G &= \Big\{G \mid 0 \,\Big\|\, 0\Big\}. \end{aligned}$$

Usually, G is a positive number, but the properties of these infinitesimals remain unchanged when G exceeds some positive number.

Exercise 5.53. Let G be a game which exceeds all negative numbers. Confirm that $+_G$ is a positive infinitesimal. (What happens when G is equal to a negative number?)

A game $g > 0$ is *infinitesimal with respect to* $h > 0$, if for all integers n, $n \cdot g < h$. So, for example, a positive game is infinitesimal if and only if it is infinitesimal with respect to 1.

Theorem 5.54. *$+_x$ is infinitesimal with respect to $\uparrow$ for any positive number x.*

Proof: It suffices to show that $\uparrow + n \cdot -_x \geq 0$ for all n. Whatever Right's first move, Left "attacks" any remaining $-_x$ moving it to $\{x \mid 0\}$. If Right responds locally, Left continues. If Right fails to respond locally, Left's move to x dominates all other positions. (Left plays arbitrarily on the other positions, ignoring x until it is the only game remaining. If, in the meantime, Right has played on x, that play only increases x.) Once all $-_x$s are gone, it is Left's move on either $\uparrow$ or $*$ (if Right's first move was on $\uparrow$), and Left moves to 0. $\square$

Exercise 5.55. Confirm that if $x > y > 0$ are numbers, then $+_x$ is infinitesimal with respect to $+_y$.

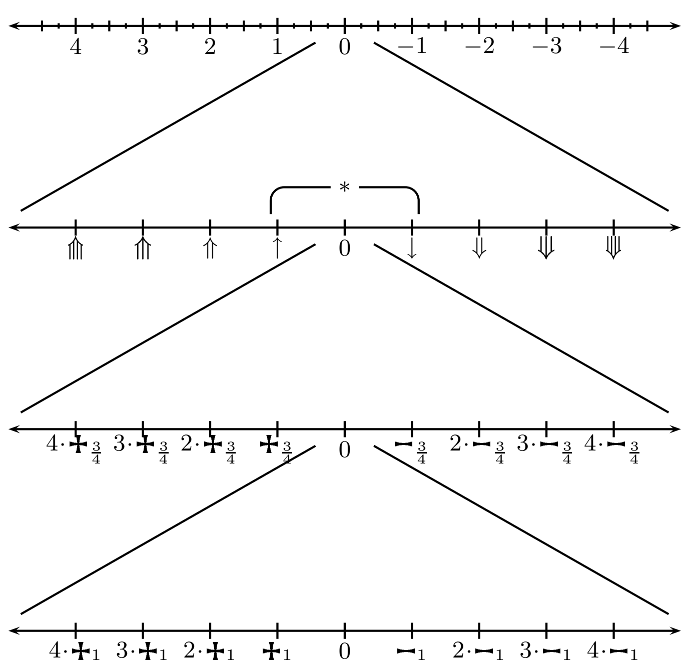


Figure 5.2. While $\uparrow$ is infinitesimal, $+_x$ is infinitesimal with respect to $\uparrow$, and $+_y$ is infinitesimal with respect to $+_x$, for numbers $y > x > 0$.

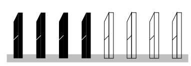
Tinies and minies are the purest examples of threats: If Right is allowed two moves in a row from $⧾_x$, he can cash in on x “points.” Despite Right’s saved up threat, Left still wins moving first or second, but the threat makes Left’s advantage minuscule. In the game of GO (which is not a pure combinatorial game due to the possibility of loops), this threat becomes more significant because these threats can act as *ko threats,* which can be significant for White.

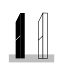
To emphasize the relationships between important infinitesimals so far, Figure 5.2 on page 109 expands on Figure 5.1.

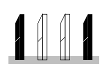
5.5 Case Study: Toppling Dominoes

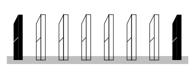
While playing sums of TOPPLING DOMINOES positions can be quite challenging, most of our favorite named values appear:

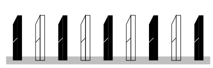
 $= 8$

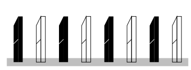
 $= \pm 3$

 $= *$

 $= \uparrow$

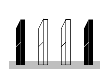
 $= ⧾_4$

 $= \frac{1}{16}$

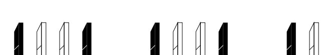
 $= *4$ (Chapter 7 introduces $*n$)

In particular:

- Integer n is n consecutive black dominoes ($n \geq 0$).
- Switch $\{m \mid -n\}$ is $m+1$ black dominoes followed by $n+1$ white dominoes ($m, n \geq 0$).
- While

  is $\uparrow$, we do not know how to create $\uparrow*$ or $\Uparrow$ other than as a sum. For instance,

   $= \uparrow + \uparrow + * = \Uparrow*.$

- $+_n$ is $n+2$ white dominoes sandwiched between 2 black ones ($n \geq 0$).
- $\frac{1}{2^n}$ is an alternating sequence of $n+1$ black dominoes with n white ones.
- $*n$ (see Chapter 7) is an alternating sequence of n black and white dominoes.

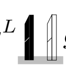
All of the above are easily confirmed by induction. While Left can move from $\frac{1}{2^n}$ to games in the last case, you need not know that the latter games have value $*n$; you need only know they are infinitesimal and incomparable with 0 to prove $\frac{1}{2^n}$ is the correct value. Right's move to $\frac{1}{2^{n-1}}$ is the dominant one (toppling a white and black domino), while Left's moves that fail to topple all the dominoes reverse out.

Exercise 5.56. Construct a TOPPLING DOMINOES position with value $\{a \parallel b \mid -c\}$ for $a \geq b \geq 0$ and $c \geq 0$.[4]

Shortly before going to print, the authors discovered a fascinating construction for any number (without merely adding up copies of $\frac{1}{2^n}$.) First of all, 0 is the empty position and

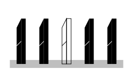
1 is 

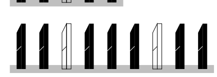
Conjecture 5.57. (Surely true) *Let x be a rational number, $x \geq 0$. If x is an integer, it is n consecutive black dominoes. Otherwise,*

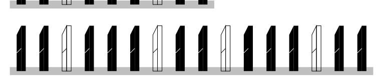
$x = g^L$  g^R

where $\{x^L \mid x^R\}$ is x's canonical form and g^L and g^R are the TOPPLING DOMINOES *positions for x^L and x^R constructed recursively.*

For example, assuming we have already figured out 3/2 and 7/4 recursively, we can find $13/8 = \{3/2 \mid 7/4\}$:

3/2 = 

7/4 = 

13/8 = 

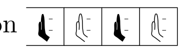
Exercise 5.58. Construct 3/8 in TOPPLING DOMINOES.

Surprisingly, the construction yields only palindromes! (A palindrome reads the same forwards and backwards.) Furthermore, we have found no numbers other than these leading us to conjecture:

[4]Note that although we have constructed $\{a \parallel b \mid -c\}$, this does not constitute a proof that TOPPLING DOMINOES is PSPACE-complete, for the construction is exponential in the *number of bits* in a, b, and c.

Conjecture 5.59. (Surely speculative) *All* TOPPLING DOMINOES *numbers are palindromes.*

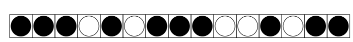
Problems

1. What is the value of the PUSH position  ?

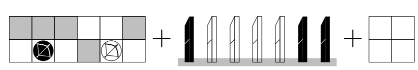
2. From a heap of n counters: if $n = 3k$ then both Left and Right can remove one or two counters; if $n = 3k + 1$ then Left can remove one or two counters; and if $n = 3k + 2$ then Right can remove one or two counters. Find the values for all n.

3. From a heap of n counters: if n is even then Left can remove two counters and Right can remove one; if n is odd then Left can remove one counter and Right can remove two. Find the values for all n.

4. Give a fast way to compute the value of any RED-BLUE CHERRIES path. You need not prove your method correct. Using your method, in a few seconds you should be able to find the value of



5. Who wins in the following sum of positions from AMAZONS, TOPPLING DOMINOES, and DOMINEERING?



6. Let $g(l, r)$ be the value of EROSION heap (l, r). Use induction to prove

$$g(l,r) = \begin{cases} \left\lceil \frac{l}{r} - \phi \right\rceil & \text{if } l \geq r, \\ -\left\lceil \frac{r}{l} - \phi \right\rceil & \text{if } l \leq r, \end{cases}$$

where $\phi = \frac{1+\sqrt{5}}{2}$ is the Golden Ratio and satisfies $\phi = \frac{1}{\phi - 1}$.

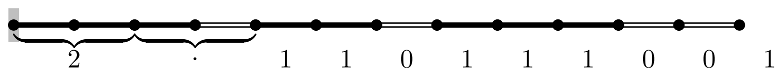
7. Confirm that a number x given by Definition 5.12 is in canonical form.

8. Prove Lemma 5.25 on page 94.

9. Use the result you proved in Problem 2 on page 83 to show that LR-HACKENBUSH consists only of numbers.

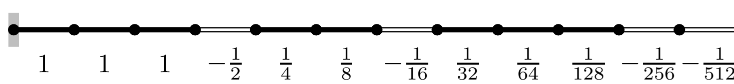
10. Elwyn Berlekamp found a simple rule for computing the value of an LR-HACKENBUSH string. Assume the grounded edge is Left's. If all edges in the string are Left's, the value is clearly an integer equal to the number

of edges. Otherwise identify the first left-right alternation. Left's edges before the alternation contribute 1 each. Replace the two alternating edges by a decimal point and replace each subsequent left (respectively, right) edge by a 1 (respectively, 0) and append a 1. You can now read off the fractional value in binary. For example,



$$= 2 + .110111001 = 2 + \frac{1}{2} + \frac{1}{4} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \frac{1}{512} = 2\frac{441}{512}.$$

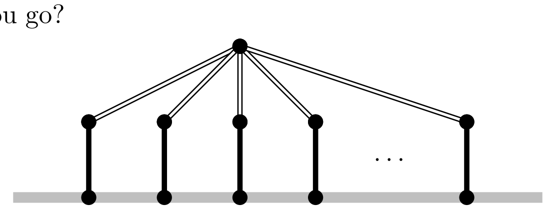
Thea van Roode has another way of assigning values to Hackenbush strings. Assign value 1 to edges until the first color change. Thereafter, divide by 2 at each new edge. The sign of each edge depends on its color. For example,



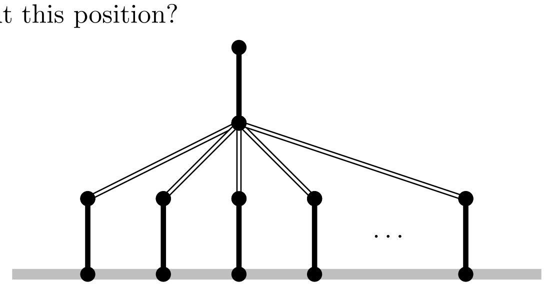
$$= 1 + 1 + 1 - \frac{1}{2} + \frac{1}{4} + \frac{1}{8} - \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \frac{1}{128} - \frac{1}{256} - \frac{1}{512} = 2\frac{441}{512}.$$

Prove that either (or both) of these methods work.

11. (a) Call f_n the value of the following HACKENBUSH position as a function of n, the number of legs. Determine f_n for small values of n. How far can you go?

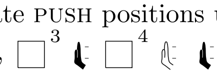


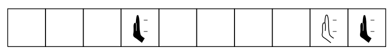
(b) How about this position?



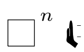
(c) Come up with an infinite series of HACKENBUSH positions of your own. See if you can find the first few values.

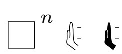
12. Prove that any all-small is infinitesimal. That is, if G is all-small, then $-x < G < x$ for all positive numbers x.

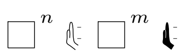
13. In this problem, abbreviate PUSH positions using superscripts for repetition of blank squares. So,  is the position



Prove the following:

(a)  has value $n + 1$.

(b)  has value $2 - \frac{1}{2^{n+1}}$.

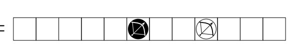
(c)  (where $m > 0$) has value $m + 1$.

14. Prove Theorem 5.29 on page 95.

15. Prove that if G either has no right options (or has no left options), then G is an integer.

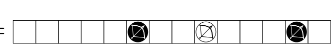
16. Prove Theorem 5.43, which gives the canonical form of $n{\cdot}\uparrow$ and $n{\cdot}\uparrow*$. You should use diagrams (like those preceding the statement of the theorem) to indicate reversible and dominated options.

17. Let $g(a, b, c)$ be the one-dimensional AMAZONS position with a blank squares, then a black amazon, then b blank squares, a white amazon, and c blank squares. For example,

$g(5, 2, 3) =$ 

What is the value of $g(a, b, c)$?

18. Define $g(a, b, c, d)$ similar to the last problem but with one additional black amazon and one additional gap. For example,

$g(5, 2, 3, 1) =$ 

What is the value of $g(a, b, c, d)$? (Be sure to check whether and when special case(s) are required if $abcd = 0$.)

19. One expects $g+g$ to have a more appealing canonical form than $g+g+g$. This problem explores a counterexample. In particular, let $x > 0$ be a number; then $+_x+_x+_x$ can be rewritten as $+_G$, where G is infinitesimally close to x.

(a) Compare $\boldsymbol{+}_x$ and $\{0 \mid -x\}$.

(b) Compare $\boldsymbol{+}_x\boldsymbol{+}_x$ and $\{0 \mid -x\}$.

(c) Compute the canonical form of $\boldsymbol{+}_x\boldsymbol{+}_x$. (One position should have either multiple left options or multiple right options.)

(d) Compute the canonical form of $\boldsymbol{+}_x\boldsymbol{+}_x\boldsymbol{+}_x$, and write the result in the form $\boldsymbol{+}_G$.

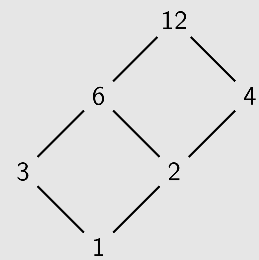
Determine G by hand, showing all your work. (You may use software to help you find mistakes as you work.)

20. John Conway [Con01, p. 215] observes that, "It is amusing to verify that for *any* game G, we have $\boldsymbol{+}_{\boldsymbol{+}_{\boldsymbol{+}_G}} = \uparrow$, so that, in particular, $\uparrow$ is the unique solution of $G = \boldsymbol{+}_G$." Prove all of these observations.

21. For integers a and c, what is the canonical form of $g = \{a \,\|\, 0 \mid -c\}$? Naturally, you will need several cases depending, in part, on the order of a, 0, and c. If a game (or a position of the game) has a value that appeared in this chapter, be sure to name it.

Preparation for Chapter 6

Prep Problem 6.1. One can define the partial order on positive integers where $a \leq b$ if a exactly divides b. Here is a diagram (called a Hasse diagram) of the factors of 12 under this partial order:



Note that in the diagram, a line is drawn from a upward toward b if $a < b$ and no other c fits in between (i.e., any comparisons can be inferred from the diagram).

1. Draw two Hasse diagrams, one each for the factors of 36 and of 30.
2. How would you determine the greatest common divisor (gcd) and the least common multiple (lcm) of two elements by looking at the Hasse diagram?

Prep Problem 6.2.

1. List all games each with a single left option and a single right option each chosen from {1,0,-1}. Draw the partial order of these nine games. (The diagram should look like that of the factors of 18 from Prep Problem 6.1.)
2. How many additional games would there be if we allowed the left or right options to be empty? List them.

To the instructor: While we make use of the material in Sections 6.3 and 6.4 in Chapter 8, the material in Sections 6.5 and 6.6 can be skipped without loss of continuity.

Chapter 6

Structure

. . . a chess player may offer the sacrifice of a pawn or even a piece, but a mathematician offers the game.

Godfrey Hardy in A Mathematician's Apology

In this chapter we will prove theorems that say something about all games, or about games born on or before a given day. The sorts of questions which we hope to answer are:

- In what ways can we classify games in an informative or interesting way?
- What are some extremal games? Biggest, smallest, etc.?
- We know that games form a group with a partial order. Are there other underlying algebraic structures in classes of games?

6.1 Games Born by Day 2

One way of classifying games is by their birthday. Recall from Definition 4.1 on page 66 that the birthday of a game is the height of its game tree. A game is *born by* day n if its birthday is less than or equal to n. Define

$$\mathcal{G}_n = \text{the set of games born by day } n,$$

and let

$$g_n = |\mathcal{G}_n| = \text{the number of games born by day } n.$$

There is only one game born on day 0, and that is the game 0. So, in our notation, $\mathcal{G}_0 = \{0\}$ and $g_0 = 1$. There are $g_1 = 4$ games born by day 1, those being $\mathcal{G}_1 = \{1, *, 0, -1\}$. The left and right options of games from $\mathcal{G}_n$ are subsets of $\mathcal{G}_{n-1}$. Since the number of subsets of a set of n elements is 2^n, we have the following observation.

Observation 6.1. $$g_n \leq 2^{g_{n-1}} \cdot 2^{g_{n-1}} = 2^{2g_{n-1}}.$$

Many of the games constructed in this way will not be in canonical form, so we expect that the actual value of g_n will be much less than that provided by this estimate. On day 2, the observation states there are at most $2^8 = 256$ games. But note that if two comparable options are available to a player, one of the two will be dominated. Consequently, candidate sets of left or right options are *antichains* of games born by day $n-1$. An *antichain* is a set consisting only of incomparable elements. There are six antichains of games born by day 1:

$$\{1\}, \{0, *\}, \{0\}, \{*\}, \{-1\}, \{\,\}.$$

That reduces the potential number of games born by day 2 to 36 as shown in the following table. These six antichains are arranged, in some sense, with Left's preferred left options listed first. (We have also dropped the brackets.) In particular, Left wins moving first if Left's options are any of the first three. The same six antichains are sorted for Right in a similar fashion.

<table>
<tr><th colspan="8">Right</th></tr>
<tr><td rowspan="7">Left</td><td></td><td>-1</td><td>$0, *$</td><td>0</td><td>$*$</td><td>1</td><td>$\emptyset$</td></tr>
<tr><td>1</td><td colspan="3" rowspan="3">$\mathcal{N}$</td><td colspan="3" rowspan="3">$\mathcal{R}$</td></tr>
<tr><td>$0, *$</td></tr>
<tr><td>0</td></tr>
<tr><td>$*$</td><td colspan="3" rowspan="3">$\mathcal{L}$</td><td colspan="3" rowspan="3">$\mathcal{P}$</td></tr>
<tr><td>-1</td></tr>
<tr><td>$\emptyset$</td></tr>
</table>

The lower-right quadrant consists entirely of games equal to 0. Three other pairs of positive positions (and, symmetrically, negative positions) also turn out to be equal, leading to only 22 games born by day 2:

<table>
<tr><th colspan="8">Right</th></tr>
<tr><td rowspan="7">Left</td><td></td><td>-1</td><td>$0, *$</td><td>0</td><td>$*$</td><td>1</td><td>$\emptyset$</td></tr>
<tr><td>1</td><td>± 1</td><td>$1 \mid 0, *$</td><td>$1 \mid 0$</td><td>$1 \mid *$</td><td>$1*$</td><td>2</td></tr>
<tr><td>$0, *$</td><td>$0,* \mid -1$</td><td>$*2$</td><td>$\uparrow *$</td><td rowspan="2">$\uparrow$</td><td rowspan="2">$\frac{1}{2}$</td><td rowspan="2">1</td></tr>
<tr><td>0</td><td>$0 \mid -1$</td><td>$\downarrow *$</td><td>$*$</td></tr>
<tr><td>$*$</td><td>$* \mid -1$</td><td colspan="2">$\downarrow$</td><td colspan="3" rowspan="3">0</td></tr>
<tr><td>-1</td><td>$-1*$</td><td colspan="2">$-\frac{1}{2}$</td></tr>
<tr><td>$\emptyset$</td><td>-2</td><td colspan="2">-1</td></tr>
</table>

Exercise 6.2. Confirm that the three non-canonical positive values, $\{0, * \mid *\}$, $\{0, * \mid 1\}$, and $\{0, * \mid \}$ are equal to their purported values, $\uparrow$, $\frac{1}{2}$, and 1, respectively, by converting each to canonical form. Clearly identify any dominated and/or reversible options.

We can investigate the partial order of just those games born by day n. Figure 6.1 shows *Hasse diagrams* of the partial orders of games born by day 1 and by day 2. In a Hasse diagram, two games $G > H$ are joined by a line if one game is greater than the other, but for no J is $G > J > H$. The bigger game is always above the smaller one. For example, $\uparrow* > *2$, and they are joined by a line. However, $\uparrow > \downarrow*$, but they are not joined by a line because $\uparrow > *2 > \downarrow*$. Since there is no monotonic path from $\downarrow$ to $\{1 \mid -1\}$, the diagram shows these two games as incomparable.

There are exactly 1474 games born by day 3. Using bounds from [WF04], the number of games born by day 4 is somewhere between 3 trillion (3×10^{12}) and 10^{434}.

6.2 Extremal Games Born By Day n

In this section, we will describe the largest and smallest positive games born by day n, as well as the largest infinitesimals.

Theorem 6.3. *The largest game born by day n is n.*

Proof: We wish to show that Left wins moving second from $n - G$ whenever G is born by day n. Left's strategy is simply to move on n for n consecutive turns. Since Right's only legal moves are on G, and since G is born by day n, Right will run out of moves before Left does. □

An inductive proof, while neither more revealing nor more clear, may help to motivate other proofs in this section.

Alternate Proof of Theorem 6.3: We wish to show that Left wins moving second from $n - G$ whenever G is born by day n. Right's only legal moves are to some $n - G^L$, and Left can respond to $(n-1) - G^L$. By the definition of birthday, G^L must be born by day $n-1$, and so by induction Left wins moving second on $(n-1) - G^L$. □

Theorem 6.4. *The smallest positive number born by day $n+1$ is 2^{-n}.*

Proof: Suppose $x > 0$ is a number born by day $n+1$. Without loss of generality, x is in canonical form, for conversion to canonical form can only reduce birthday. So, x is of the form $\{y \mid z\}$ for $0 \le y < z$. For x to be minimal, y and z are chosen as small as possible; i.e., $y = 0$ and z is the minimum positive number

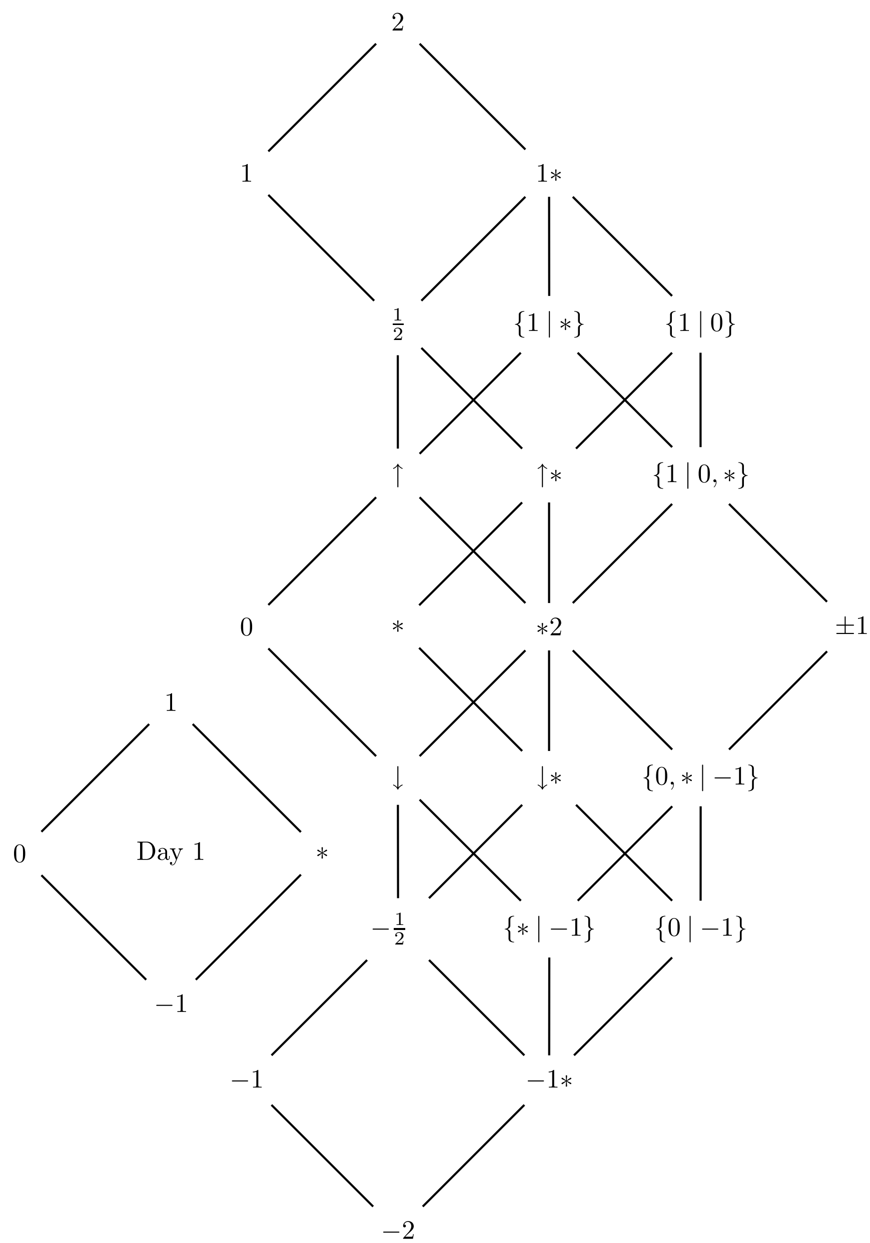


Figure 6.1. The partial order of games born by days 1 and 2.

born by day n which, by induction, is 2^{1-n}. Since $\{0 \mid 2^{1-n}\} = 2^{-n}$, we have that 2^{-n} is the minimum.

For the base case of $n = 0$, $1 = 2^0$ is the only positive game born on day 1. □

Theorem 6.5. *The smallest positive game born by day $n+2$ is $⧾_n$.*

Proof: Let $G > 0$ be born by day $n+2$. We wish to show that Left wins moving second on $G - ⧾_n = G + ⧿_n = G + \{n \mid 0 \| 0\}$.

$$\begin{tikzcd}
G+⧿_n \arrow[dr] \arrow[drr] & & \\
& G^R+⧿_n \arrow[dl] & G>0 \\
G^R+\{n \mid 0\} \arrow[dr] \arrow[drr] & & \\
& G^{RR}+\{n \mid 0\} \arrow[dl] & G^R \mathrel{|\!\rhd} 0 \\
G^{RR}+n \geq 0 & &
\end{tikzcd}$$

If Right moves the second component to 0, Left wins since G is positive. If, on the other hand, Right starts on G, Left responds on the second component, leaving $G^R + \{n \mid 0\}$. From here, if Right moves the second component to 0, then Left has a winning move from G^R since G was positive. If, on the other hand, Right moves on the first component, Left moves on the second leaving $G^{RR} + n$. Since G was born by day $n+2$, G^{RR} was born by day n, and hence, by Theorem 6.3, Left wins $G^{RR} + n$ moving second. □

The last theorem along these lines is a bit more complicated. There are two maximal infinitesimals born by day $n+1$, those being $n{\cdot}\uparrow$ and $n{\cdot}\uparrow*$. For example, Figure 6.2 shows just the infinitesimals born by day 2.

Theorem 6.6. *For any infinitesimal G born by day $n+1$, either $G \leq n{\cdot}\uparrow$ or $G \leq n{\cdot}\uparrow*$.*

Proof: Lemma 6.13 at the end of the next section generalizes this result. □

To motivate additional machinery required to prove this theorem, we will first prove a special case of the theorem that applies only to all-small games.

Lemma 6.7. *For any all-small game G born by day $n+1$, either $G \leq n{\cdot}\uparrow$ or $G \leq n{\cdot}\uparrow*$.*

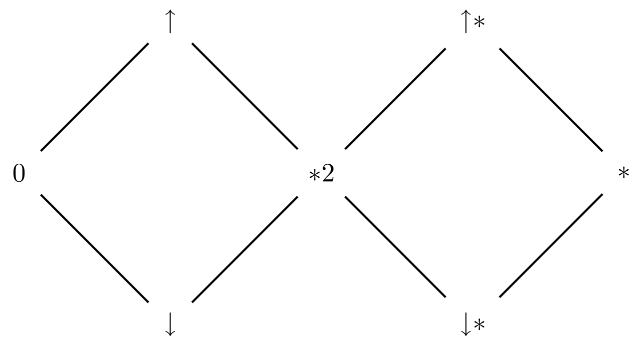


Figure 6.2. The partial order of infinitesimals born by day 2.

Proof: Assume $n \geq 2$ and let G be an all-small game born by day $n+1$. We wish to show Left wins moving second on $n\cdot\uparrow - G$ or on $n\cdot\uparrow* - G$, so suppose both games are played simultaneously, and Right moves first in both. To prove the theorem, Left need only win one of the two games.

Suppose Right moves $-G$ to $-G^L$ in either game. If G^L is a number, it must be 0, and the resulting difference game is positive. If G^L is not a number, Left replies by moving to $-G^{LR}$, which is born by day $n-2$. Left then wins by induction, for both $n\cdot\uparrow$ and $n\cdot\uparrow*$ exceed both of the games $(n-2)\cdot\uparrow$ and $(n-2)\cdot\uparrow*$.

Suppose Right moves to the pair of games $(n-1)\cdot\uparrow* - G$ and $(n-1)\cdot\uparrow - G$. Then Left can move both copies of $-G$ to the same $-G^R$. Left, moving second, can proceed to win either $(n-1)\cdot\uparrow* - G^R$ or $(n-1)\cdot\uparrow - G^R$ by induction.

The base cases are when $n = 0$ and $n = 1$. On day 1, the games 0 and $*$ are the only infinitesimals. Day 2 can be confirmed by verifying that Figure 6.2 is correct. □

The reason the same proof fails to prove Theorem 6.6 is that in the two places induction is used, we have no guarantee that G^{LR} and, respectively, G^R are infinitesimal games. For example, if $G = \{⧾_2 \mid 0\}$, then $G^{LR} = \{0 \mid -2\}$ is not infinitesimal. Similarly, if $G = ⧾_2$, $G^R = \{0 \mid -2\}$, which is not infinitesimal.

Exercise 6.8. Take a moment to make sure you understand and appreciate the first sentence of the preceding paragraph.

In both of these examples the appearance of negative numbers in G (meaning positive numbers in $-G$) should make it *easier* for Left to win $n\cdot\uparrow - G$. Put differently, in the induction argument, Right is never given the opportunity to move twice in a row in $-G$, and should therefore never be able to reach a negative number in $-G$, for that would contradict the fact that G is infinitesimal. Left, however, may move twice in a row, so while she may reach a positive number in $-G$, that fails to hurt the argument, since Left wins even more decisively.

To formalize the line of reasoning in the last paragraph, we introduce the notion of left and right stops.[1]

6.3 Stops

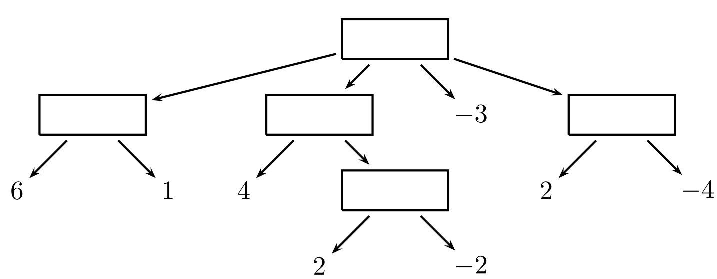
Suppose two players play a game and stop as soon as the game is a number. Left attempts to have this stopping position be as large as possible, and Right wants it to be as small as possible. The number arrived at when Left (or, respectively, Right) moves first is called the left (or right) *stop* of the position. More formally,

Definition 6.9. Denote the *left stop* and *right stop* of a game G by $\mathbf{LS}(G)$ and $\mathbf{RS}(G)$, respectively. They are defined in a mutually recursive fashion:

$$\mathbf{LS}(G) = \begin{cases} G & \text{if } G \text{ is a number,} \\ \max(\mathbf{RS}(G^L)) & \text{if } G \text{ is not a number;} \end{cases} \tag{6.1}$$

$$\mathbf{RS}(G) = \begin{cases} G & \text{if } G \text{ is a number,} \\ \min(\mathbf{LS}(G^R)) & \text{if } G \text{ is not a number.} \end{cases} \tag{6.2}$$

Exercise 6.10. Compute the left and right stops of each position marked with a box below. Rest assured that the game is in canonical form, so none of the interior nodes are numbers in disguise:



Theorem 6.11. *For any game G, $\mathbf{LS}(G) \geq \mathbf{RS}(G)$.*

Proof: If G is a number, then $\mathbf{LS}(G) = \mathbf{RS}(G)$. If G is not a number, suppose the theorem were false, and that $\mathbf{LS}(G) < \mathbf{RS}(G)$. Then there is some number x, $\mathbf{LS}(G) < x < \mathbf{RS}(G)$. To complete the proof, we will show that $G = x$, contradicting our assumption that G is not a number. When playing $G - x$, the

[1] In truth, the notion of stops will have many other applications, so read this section with care.

Weak Number-Avoidance Theorem asserts that if there is a winning move, it is on G. So, without loss of generality, both players play on G until it reaches a number. When Left moves first, the maximum she can achieve is $\mathbf{LS}(G)$, and when Right moves first, the minimum he can achieve is $\mathbf{RS}(G)$. Neither are good enough for the first player to achieve a win on $G - x$. □

Theorem 6.12. *G is infinitesimal if and only if $\mathbf{LS}(G) = 0 = \mathbf{RS}(G)$.*

Proof: See Problem 3. □

We are now ready to prove Theorem 6.6. Recall from the discussion following Lemma 6.7 that the challenge is in accounting for the possibility that G^{LR} and G^R might not be infinitesimal, making the induction argument fail. So we will strengthen the induction hypothesis to allow G to take on some non-infinitesimal values:

Lemma 6.13. *If G is born by day $n+1$ with $\mathbf{LS}(G) \le 0$, then either $G \le n\cdot\uparrow$ or $G \le n\cdot\uparrow*$.*

Theorem 6.6 is an immediate corollary since all infinitesimals have zero stops.

Proof: Fix $n \ge 2$, and fix G with $\mathbf{LS}(G) \le 0$ and born by day $n+1$. We wish to show Left wins moving second on $n\cdot\uparrow - G$ or on $n\cdot\uparrow* - G$, so suppose both games are played simultaneously, and Right moves first in both. We will show that Left can win one of the two games.

If G is already a number, $G \le 0$, and the result is trivial.

Suppose Right moves $-G$ to $-G^L$ in either game. If G^L is a number, it must be less than or equal to 0, and the resulting difference game is positive. If G^L is not a number, since $\mathbf{LS}(G) \le 0$, Left can reply locally to some $-G^{LR}$ (which is born on day $n-2$) with $\mathbf{LS}(G^{LR}) \le \mathbf{LS}(G) \le 0$. Left then wins by induction, for $n\cdot\uparrow$ and $n\cdot\uparrow*$ exceed both of the games $(n-2)\cdot\uparrow$ and $(n-2)\cdot\uparrow*$.

Suppose Right moves to the pair of games $(n-1)\cdot\uparrow* - G$ and $(n-1)\cdot\uparrow - G$. Then Left can move both copies of $-G$ to the same $-G^R$, where G^R is chosen with minimal left stop. Then, $\mathbf{RS}(G^R) \le \mathbf{LS}(G^R) = \mathbf{RS}(G) \le \mathbf{LS}(G) \le 0$. Left, moving second, can proceed to win either $(n-1)\cdot\uparrow* - G^R$ or $(n-1)\cdot\uparrow - G^R$ by induction.

The base cases of $n = 1$ and $n = 2$ can be confirmed by checking Figure 6.1. □

Exercise 6.14. We tacitly assumed that if $\mathbf{LS}(G) \le 0$ and G^L is not a number, then some G^{LR} has $\mathbf{LS}(G^{LR}) \le \mathbf{LS}(G)$. Confirm this fact.

Exercise 6.15. Justify each of the relations appearing in the second to last paragraph of the above proof, those being

$$\mathbf{RS}(G^R) \le \mathbf{LS}(G^R) = \mathbf{RS}(G) \le \mathbf{LS}(G) \le 0.$$

One can adopt other definitions of when to stop a game. For example, one could define the *integer stops* of a game by replacing the word "number" with "integer" in Definition 6.9. The analogy to Theorem 6.11 is the following:

Corollary 6.16. *The difference between the integer left stop and the integer right stop of any game G is at least -1. Further, when the difference is -1, the person who moves first on G also moves last on G to reach the stop.*[2]

Proof: Observe that the difference in integer stops can be -1 only if $\mathbf{LS}(G)$ and $\mathbf{RS}(G)$ are numbers strictly between the same consecutive integers, n and $n+1$ and Left is forced to move first on $\mathbf{LS}(G)$, while Right moves first on $\mathbf{RS}(G)$. □

Examples of when the difference in integer stops are -1 include $G = 1/4$ and $\left\{\frac{1}{2}* \,\middle|\, \frac{1}{4}*\right\}$. In both cases, the left integer stop is 0 and the right integer stop is 1.

6.4 More About Numbers

We are now prepared to prove a few more theorems about numbers. To review, we know from page 92 that in canonical form, numbers have negative incentives. The *Weak Number-Avoidance Theorem* states that if x is a number and G is not, and if Left can win $x + G$ moving first, then she can do so with a move on G. We now state the strong version of this theorem:

Theorem 6.17. (Number Avoidance) *Suppose that x is a number in canonical form with a left option and that G is not a number. Then, there exists a G^L such that $G^L + x > G + x^L$.*

Proof: Equivalently, we wish to show that some $G^L - G$ exceeds $x^L - x$. It is clearly sufficient to show that some left incentive exceeds all negative numbers. In order to establish this, it suffices to prove that for some G^L, $\mathbf{RS}(G^L - G) \ge 0$, for then Lemma 6.13 tells us that $G^L - G \ge n\cdot\downarrow$ or $G^L - G \ge n\cdot\downarrow*$ for some value of n, where $n+2$ is $G^L - G$'s birthday. This implies that $G^L - G$ exceeds all negative numbers.

To see that for some G^L, $\mathbf{RS}(G^L - G) \ge 0$, choose G^L to be a left option with maximum right stop so that $\mathbf{RS}(G^L) = \mathbf{LS}(G)$. Left's strategy on

[2] The last sentence of the corollary will be relevant when we prove Theorem 9.32 several chapters hence.

$G^L - G$ to achieve the left stop is to respond locally to each of Right's moves, achieving at least $\mathbf{RS}(G^L) + \mathbf{RS}(-G) = 0$. If Right's move reaches a number in one component, Left can move twice in a row on the other, in which case Theorem 6.11 says she could do no worse.

There is one subtlety — the difference of the two components might be a number (yielding a stop). Allow both players to play on as above except allow Right to pass, which (since optional) can only help Right. If, in fact, the difference is a number (having negative incentive) Left's consecutive moves can only hurt her. Despite this change, the above argument still leaves Left at a belated stop which is at least 0. □

Theorem 6.18. (Number Translation) *If x is a number and G is not, then* $G+x = \left\{\mathcal{G}^L + x \mid \mathcal{G}^R + x\right\}$.

Proof: $G+x = \left\{\mathcal{G}^L + x, G + x^L \mid \mathcal{G}^R + x, G + x^R\right\}$. By the *Number-Avoidance Theorem,* the last option on each side, if it exists, is dominated by one of the previous options. □

Theorem 6.19. (Negative Incentives) *If all of G's incentives are negative, then G is a number.*

Proof: Suppose to the contrary, that G is not a number and all its incentives are negative.

If $G \parallel x$, where x is a number, then Left wins moving first on $G - x$ and (by *Number Translation*) some $G^L - x \geq 0$. But then, since G's incentives are negative, we have $G - x > G^L - x \geq 0$, contradicting $G \parallel x$. So G is comparable with all numbers, so $\mathbf{LS}(G) = \mathbf{RS}(G)$ and G must be a number plus an infinitesimal comparable with 0, say greater than 0. If G's incentives are negative, so are those of $G-x$ and $-G$. So we can translate our counterexample to make the number portion 0.

So, without loss of generality, G is a positive infinitesimal.

Next, we will show (by induction) that if s is all-small, then $G + s > 0$. Since $G + s^L$ and $G + s^R$ are positive (by induction) we only need to confirm that Left wins moving second if Right plays to some $G^R + s$. Left responds to any $G^R + s^L$, and we have $G^R + s^L > G + s^L > 0$. (The first inequality is because G's incentives are negative, and the second is by induction.)

So, $G + n \cdot \downarrow > 0$ for all positive integers n, but Theorem 6.6 then provides a contradiction, since no infinitesimal has this property. □

Exercise 6.20. The preceding proof concludes from $\mathbf{LS}(G) = \mathbf{RS}(G)$ that G must be a number plus an infinitesimal. Use Theorems 6.12 and 6.18 to prove this statement.

The stops give another common way of classifying games into categories:

Definition 6.21. A game is dubbed *cold* if it is a number, *tepid* if $\mathbf{LS}(G) = \mathbf{RS}(G)$ but G is not a number, and *hot* if $\mathbf{LS}(G) > \mathbf{RS}(G)$.

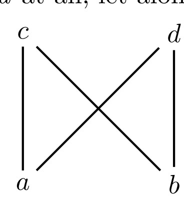
Observation 6.22. Every game is either cold, tepid, or hot. Every tepid game is a number plus a non-zero infinitesimal.

6.5 The Distributive Lattice of Games Born by Day n

We can investigate the partial order of games (how they compare) as distinct from the group of games (how they add). Refer back to Figure 6.1 on page 120, which shows the partial order of games born by days 1 and 2. This is not merely *any* partial order. It has quite a bit of structure.

Definition 6.23. A *lattice* is a partial order with the additional property that any pair of elements a and b in the partial order has a *least upper bound* or *join*, denoted $a \vee b$, and a *greatest lower bound* or *meet*, denoted $a \wedge b$.

For example, the following partial order is not a lattice for a number of reasons. For one, a and b have no join, since the only two elements greater or equal to both, c and d, are incomparable. So a and b have no *least* upper bound. Secondly, there are no elements greater than or equal to both c and d, so c and d have no *upper bound* at all, let alone a least upper bound.



Common lattices in mathematics include the integers under the ordinary $\geq$ and the positive integers where we define $b \geq a$ if a is a divisor of b.

Theorem 6.24. *The games born by day n form a lattice.*

Before embarking on the proof, it is worthwhile to explore the theorem's meaning.

Exercise 6.25. Referring to Figure 6.1, identify the joins and meets of the following pairs of elements among the games born by day 2:

1. $* \vee 0$;

2. $* \vee \frac{1}{2}$;

3. $\uparrow \vee \pm 1$;

4. $\uparrow \wedge \pm 1$.

Recompute your answers using the definitions of $\vee$ and $\wedge$ given in the proof below.

Proof: We will prove Theorem 6.24 by explicitly constructing the join operation (and, symmetrically, the meet), and then confirming that it is, indeed, a least upper bound (respectively, greatest lower bound).

$$
\begin{aligned}
\lceil G \rceil &\stackrel{\text{def}}{=} \{H \in \mathcal{G}[n-1] : \; H \mathrel{|\!\triangleright} G\}; \\
\lfloor G \rfloor &\stackrel{\text{def}}{=} \{H \in \mathcal{G}[n-1] : \; H \mathrel{\triangleleft\!|} G\};
\end{aligned}
$$

$$
\begin{aligned}
G_1 \vee G_2 &\stackrel{\text{def}}{=} \left\{ G_1^L \cup G_2^L \;\middle|\; \lceil G_1 \rceil \cap \lceil G_2 \rceil \right\}; \\
G_1 \wedge G_2 &\stackrel{\text{def}}{=} \left\{ \lfloor G_1 \rfloor \cap \lfloor G_2 \rfloor \;\middle|\; G_1^R \cup G_2^R \right\}.
\end{aligned}
$$

To show that $G_1 \vee G_2$ is a least upper bound, we need to show that

$$
\begin{aligned}
&G_1 \vee G_2 \geq G_i \text{ (for } i = 1, 2\text{), and} && (6.3) \\
&\text{if } G \geq G_1 \text{ and } G \geq G_2 \text{ then } G \geq G_1 \vee G_2. && (6.4)
\end{aligned}
$$

Referring to the definition of join above, we will prove each of these two assertions in sequence using diagrams:

- Left wins the following game moving second:

$$
\begin{tikzcd}
(G_1 \vee G_2) - G_i \arrow[dr] \arrow[drr] & & \\
& (G_1 \vee G_2) - G_i^L \arrow[dl] & H - G_i \\
G_i^L - G_i^L & &
\end{tikzcd}
$$

  where (in the second right option) $H \in \lceil G_1 \rceil \cap \lceil G_2 \rceil$. In particular, $H \mathrel{|\!\triangleright} G_i$, and so Left wins moving first from $H - G_i$.

- If $G \geq G_1$ and $G \geq G_2$, then Left wins moving second from

$$\begin{tikzcd}
G-(G_1 \vee G_2) \arrow[dr] \arrow[drrr] & & & \\
& G^R-(G_1 \vee G_2) \arrow[dl] & & G-G_i^L \\
G^R-G^R & & &
\end{tikzcd}$$

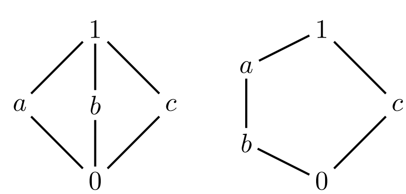
  In the first case, observe that $G^R \in \lceil G_1 \rceil \cap \lceil G_2 \rceil$. In the second case, $G \geq G_i$, so Left will win. □

Exercise 6.26. Confirm the second to last sentence of the proof, that $G^R \in \lceil G_1 \rceil \cap \lceil G_2 \rceil$.

Definition 6.27. A *distributive lattice* is a lattice in which meet distributes over join (or, equivalently, join distributes over meet). That is,

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c).$$

The parenthetical remark is not intended to be obvious, but has a short proof easily found in textbooks covering lattice theory; see, for example, [Grä71, p. 35] or [CD73, p. 19]. Not all lattices are distributive; the lattices in the following two diagrams are not:



Exercise 6.28. Compute $a \wedge (b \vee c)$ and $(a \wedge b) \vee (a \wedge c)$ for each of the above lattices to confirm they are not distributive.

In fact, in some sense, these are the *only* non-distributive lattices. Garrett Birkhoff showed that a lattice is distributive *if and only if* it fails to contain both of the above as a sublattice.

Theorem 6.29. *The games born on day n form a distributive lattice.*

Proof: See Problem 5. □

6.6 Group Structure

In the last section, we investigated the partial order structure of games born by day n, as distinct from the group structure. That is, we focused on the $\geq$ symbol rather than the $+$ sign. We can also do the reverse by looking at the group generated by games born by day n. In other words, what are the possible sums of games born by day n?

On day 0, only the game 0 is born. Not much to do there.

On day 1, we get 4 games, $\{1, *, 0, -1\}$. Since $*$ is of order 2 (meaning $*+* = 0$), sums of games born by day 1 are either an integer or an integer plus $*$. In group-speak, the group generated by games born by day 2 is just $\mathcal{Z} \times \mathcal{Z}_2$.

In [Moe91], David Moews does a similar analysis on all games born by day 2 and on games born by day 3 ignoring infinitesimal shifts. For example, the group of games generated by those born by day 2 has an independent generating set:

$$1/2, *2, A, \uparrow, \alpha, \pm\frac{1}{2}, \pm 1,$$

where

$$\begin{aligned} A &= \{1 \mid 0\} - \{1 \mid *\}, \text{ and} \\ \alpha &= \{1 \mid 0\} - \{1 \mid 0, *\} = \{1* \mid * \,\|\, 0 \mid -1\}. \end{aligned}$$

A has order 4 since $A + A = *$, while $\alpha > 0$ but $n \cdot \alpha \mathrel{\lhd\!\!|} \uparrow$ for any $n > 0$ and is therefore additively independent of $\uparrow$. So, the group of games generated by those born by day 2 is isomorphic to $\mathcal{Z}^3 \times \mathcal{Z}_4 \times \mathcal{Z}_2^3$.

Exercise 6.30. Confirm that $n \cdot \alpha \mathrel{\lhd\!\!|} \uparrow$ for all $n > 0$.

Problems

1. Curiously, each of the 22 games born by day 2 appears as a KONANE position with four stones, two of each color, on (say) an 8×8 board. Demonstrate this is possible by constructing each position (or its negative).

2. How many statements of the form $G > H$ and $G \parallel H$ must be confirmed to check that Figure 6.1 (day 2 diagram) is accurate? As you count, try to exploit symmetries or other space saving techniques. Naturally, you should explain your higher-level reasoning used to shorten your list. (For example, if $G > H$, then $-H > -G$, so no need to explicitly confirm both.)

   Do not prove each of the statements; that would get old very fast.

3. Prove Theorem 6.12 on page 124, that G is infinitesimal *if and only if* its stops are 0. Do not use theorems proved after Theorem 6.12.

4. (a) Show that the sum of a tepid game and a hot game is hot.
   (b) Show that the sum of two hot games can be tepid or even cold.

5. Prove Theorem 6.29 by working through the following exercises. We wish to show that $S_1 = S_2$, where

$$\begin{aligned} S_1 &= H \wedge (G_1 \vee G_2) \quad \text{and} \\ S_2 &= (H \wedge G_1) \vee (H \wedge G_2). \end{aligned}$$

   (a) Confirm that

$$\begin{aligned} \lfloor G_1 \vee G_2 \rfloor &= \lfloor G_1 \rfloor \cup \lfloor G_2 \rfloor \quad \text{and} \\ \lceil G_1 \wedge G_2 \rceil &= \lceil G_1 \rceil \cup \lceil G_2 \rceil. \end{aligned}$$

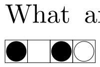
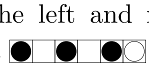
   (b) Show that

$$\begin{aligned} S_1 &= \{\, \lfloor H \rfloor \cap \lfloor G_1 \vee G_2 \rfloor \mid H^R,\ \lceil G_1 \rceil \cap \lceil G_2 \rceil \,\} \\ S_2 &= \{\, \lfloor H \rfloor \cap \lfloor G_1 \vee G_2 \rfloor \mid \lceil H \rceil,\ \lceil G_1 \rceil \cap \lceil G_2 \rceil \,\}. \end{aligned}$$

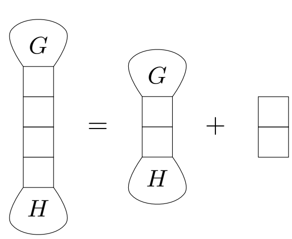
   (c) Prove $S_1 = S_2$ by playing $S_1 - S_2$.

6. Consider the following conjecture: If G is in canonical form, all its left stops (not just the maximum one) exceed all of its right stops. More precisely, if G is canonical and not a number then for every G^L and G^R, $\mathbf{RS}(G^L) \geq \mathbf{LS}(G^R)$.

   Prove the conjecture false by providing a counterexample.

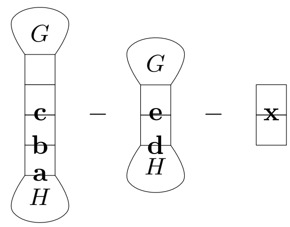
7. What are the left and right stops for CLOBBER THE POND positions  and  ?

8. In the paper, *Snakes in Domineering Games,* one of the authors (Wolfe) claimed that

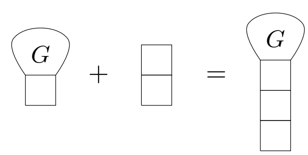


   While the fact may be true, it remains conjecture.

Explain what is wrong with the following "proof": We wish to show the second player wins the difference game



If either Left or Right plays entirely in one of the G or H pieces, the second player can respond in the other, leaving (by induction) 0. If Left moves covering the squares adjacent to b or c, Right responds at x, leaving a position ≤ 0. If Left moves covering the squares adjacent to a, Right responds at d, leaving 0 since



(This latter statement is an application of Problem 3 of Chapter 4 on page 83.) If Right moves covering the squares adjacent to d or e, Left responds at a or b (respectively), leaving a game ≥ 0 by the *One-Hand-Tied Principle.* Lastly, Right's move at x need not be considered by the *Number-Avoidance Theorem.*

Preparation for Chapter 7

Prep Problem 7.1. Review binary (base 2) arithmetic. If you need a refresher, type `base 2` into wikipedia.org. Complete the missing entries in the following table:

Decimal fractional notation	Binary positional notation
$-2\frac{1}{2}$	-10.1
23	
	10101
$\frac{1}{4}$	
$\frac{5}{8}$	
	$.01101$
$-12\frac{9}{16}$	
	-11.10101

(In truth, you will not need to deal with binary fractions in this chapter, but they appear frequently in combinatorial game theory. See, for example, Problem 10 on page 112.)

To the instructor: GREEN HACKENBUSH from [BCG01, pp. 189–196] is completely solved, and the proofs are just hard enough to be a nice challenge for the solid undergraduate student. You could cover the material immediately after Section 7.3.

Chapter 7

Impartial Games

The real excitement is playing the game.

Donald Trump

If for a game and all its options, the left options equal the right options, then the game is dubbed *impartial.* The rulesets for legal moves in impartial games make no distinction between the players. For example, in a popular impartial game among children, there is a heap of n counters, and a move consists of removing one, two, or three counters. We have also seen the impartial variant of DOMINEERING known as CRAM in Chapter 0. Any partizan game, with a slight change of the rules, can be made impartial. DOTS & BOXES is made impartial by eliminating the score, where if it is your turn, and you have no legal move, then you lose. That is, the player completing the last box loses since he has to play again. (When this impartial variant is played on a STRINGS & COINS board it is known as NIMSTRING.)

One important feature of an impartial game is that, at least intuitively, neither Left nor Right can accumulate an advantage, since if Left has a legal move then Right also has that move available.[1] In addition, recall Theorem 2.11 from page 41, that any impartial game is either an $\mathcal{N}$-position or a $\mathcal{P}$-position.

Exercise 7.1. Every impartial game is its own negative. That is, if G is impartial, then $G + G = 0$.

By definition the impartial games are a subset of the all-small games and therefore the value of an impartial game will be an infinitesimal. However, the set of values that can occur for impartial games is a very limited subset even of the infinitesimals. For instance, no impartial game can have value $\uparrow$ since $\uparrow > 0$; that is, $\uparrow \in \mathcal{L}$. Also, while $\uparrow* \in \mathcal{N}$, $\uparrow* + \uparrow* = \Uparrow \neq 0$. So, by the previous exercise, no impartial game can have value $\uparrow*$. An alternative argument would

[1]The reader experienced with playing NIMSTRING might reasonably question this assertion.

be to consider the canonical form of $\uparrow*$, which is $\{0 \mid *\}$. This has different left and right options and is therefore not impartial.

Exercise 7.2. Confirm the implicit claim made in the last paragraph that the *canonical form* of an impartial game has the same left and right options.

Finding the value of an impartial game is typically much easier than finding the value of an arbitrary partizan game. Because both left and right options are the same, we only have to list, say, Left's options. This also allows us to introduce later, in Section 7.4, a more concise notation.

When an impartial game does not naturally decompose into a disjunctive sum, it is usually easiest and most useful to restrict attention to outcome classes alone as we did in Section 2.2. However, when the game does naturally decompose, identifying values is usually the more powerful method for describing and analyzing the game.

NIM was the first combinatorial game to have its full strategy published in 1901 [Bou02]. In the 1930s, 40s, and 50s Roland Sprague [Spr35], Patrick Grundy [Gru39], and Richard Guy [GS56] showed that the theory for NIM could be extended to all impartial games and analyzed several games using the theory. We follow this development, but with a modern treatment.

7.1 A Star-Studded Game

There is a class of infinitesimals that do not behave like any values that we have yet encountered. They occur naturally as the values of the impartial game NIM and, as we shall see, they are the only values that impartial games take on!

NIM is played with heaps of counters (or, if you prefer, a multi-set of non-negative integers). A move is to choose a heap and remove any number of counters from that heap. So a p-heap NIM game is a disjunctive sum of p one-heap games. If we can determine the canonical form of the one-heap game then we hope, by learning how to add these games, to be able to solve the sum.

The value of the heap of size 0 is $\{\ \mid\ \} = 0$ and the heap of size 1 is $\{0 \mid 0\} = *$. What about the heap of size 2? Well, it's of the form $\{0, * \mid 0, *\}$. Neither option for Left (or Right) is dominated or reversible so this is a new value that we now dub $*2$ (pronounced "star two"). A heap of size 3 is $\{0, *, *2 \mid 0, *, *2\}$ and again there are no dominated or reversible options. This game is $*3$. In general, the values are called *nimbers* (or, sometimes more informally, *stars*).

Definition 7.3. The value of a nim-heap of size n, $n \geq 0$, is the *nimber* $*n$.

Note that $*0 = 0$ and $*1 = *$, and we will usually use the shorter names for these two games.

Theorem 7.4. *For $k > 0$, the canonical form of $*k$ is*

$$*k = \{0, *, *2, \ldots, *(k-1) \mid 0, *, *2, \ldots, *(k-1)\}\,.$$

Proof: See the next exercise. □

Exercise 7.5. Prove Theorem 7.4 by showing there are no dominated or reversible options. (*Hint*: First show that $*i$ and $*j$ are incomparable whenever $i \neq j$.)

We will now see that if the left and right options of a game G are equal and consist of just nimbers then there is an easy way to find the canonical form and the value of G.

Definition 7.6. The *minimum excluded value* or *mex* of a set of non-negative integers is the least non-negative integer which does not occur in the set. This is denoted by $\operatorname{mex}\{a, b, c, \ldots, k\}$.

For example, $\operatorname{mex}\{0, 1, 2, 4, 5, 7\} = 3$, $\operatorname{mex}\{1, 2, 4, 5, 7\} = 0$ and $\operatorname{mex}\{\,\} = 0$.

Theorem 7.7. *Let $G = \{*l_1, *l_2, \ldots, *l_k \mid *r_1, *r_2, \ldots, *r_j\}$ and suppose that $\operatorname{mex}\{l_1, l_2, \ldots l_k\} = \operatorname{mex}\{r_1, r_2, \ldots, r_j\} = n$, then $G = *n$ and consequently, G has the canonical form $\{0, *, *2, \ldots, *(n-1) \mid 0, *, *2, \ldots, *(n-1)\}$.*

Proof: We will show that $G - *n = 0$.

If either player moves either component to $*k$ for $k < n$, there is a matching move in the other component. In particular, since the mex of the $\{l_i\}$ and $\{r_i\}$ are both n, we have that $*k$ is both a left and a right option from G (and is also an option from $*n$). Hence, the second player can respond to $*k - *k = 0$.

The only other moves are from $G - *n$ to $*k - *n$ for $k > n$. In this case, $*n$ is an option from $*k$, so the second player responds locally to $*n - *n = 0$. □

Though Theorem 7.7 may appear mundane, its implications are truly profound:

Corollary 7.8. *Every impartial game is equivalent to a nim-heap. That is, for every impartial game G there is a non-negative integer n such that $G = *n$.*

Proof: Let G be impartial. By induction, the options of G are equivalent to nim-heaps. Theorem 7.7 then gives the recipe for finding the equivalent nim-heap for G. □

Exercise 7.9.

- Find k so that $G = \{0, *3, *4, *8, *2 \mid 0, *6, *4\} = *k$ using Theorem 7.7.

- Confirm the theorem gave the correct k by playing the difference game $\{0, *3, *4, *8, *2 \mid 0, *6, *4\} - *k$.

Adding nimbers will be unusual. Already, we know that $* + * = 0$ and now from Exercise 7.1, in general, $*k + *k = 0$ for all $k \geq 0$. To find the *correct* way to add nimbers we need to complete the analysis of NIM. We do this first with no mention of the values of the game or its summands.

7.2 The Analysis of Nim

A game of NIM played with heaps of size $a, b, \ldots, k$ will be denoted $\text{NIM}(a, b, \ldots, k)$. The *nim-sum* of numbers a, b, …, k, written $a \oplus b \oplus \cdots \oplus k$, is obtained by adding the numbers in binary without carrying.[2] For example, to compute the nim-sum of $12 \oplus 13 \oplus 7$:

$$\begin{array}{r|cccc} 12 & 1 & 1 & 0 & 0 \\ 13 & 1 & 1 & 0 & 1 \\ 7 & & 1 & 1 & 1 \\ \hline & 0 & 1 & 1 & 0 \end{array}$$

and 0110 is binary for 6. For each column of bits (binary digits), if the column has an even number of 1s, the column sum is 0; if odd, the column sum is 1.

Theorem 7.10. *Let a, b, and c be non-negative integers. Nim-sum satisfies the properties:*

- *commutativity:* $a \oplus b = b \oplus a$*;*
- *associativity:* $(a \oplus b) \oplus c = a \oplus (b \oplus c)$*;*
- $a \oplus a = 0$*; and*
- $a \oplus b \oplus c = 0$ *if and only if* $a \oplus b = c$*.*

Exercise 7.11. Prove Theorem 7.10.

We are now ready to reveal the secret behind winning NIM.

Theorem 7.12. *Let $G = \text{NIM}(a, b, \ldots, k)$. Then G is a $\mathcal{P}$-position if and only if*

$$a \oplus b \oplus \cdots \oplus k = 0.$$

Proof: The proof of this result is an application of Theorem 2.13. That is, it suffices to show two things:

[2] In computer science, the nim-sum operation is called the *exclusive-or* or *xor* for short.

- If $a \oplus b \oplus \cdots \oplus k = 0$, then every move is to a position whose nim-sum is not zero, and
- if $a \oplus b \oplus \cdots \oplus k \neq 0$, then some move exists to a position whose nim-sum is zero.

Suppose that $a \oplus b \oplus \cdots \oplus k = 0$. Without loss of generality, suppose that the first player removes r counters from heap a. Since the binary expansion of $a - r$ is not the same as a then $(a-r) \oplus b \oplus \cdots \oplus k \neq 0$.

Now suppose that $q = a \oplus b \oplus \cdots \oplus k \neq 0$. Let $q_j q_{j-1} \ldots q_0$ be the binary expansion of q; that is, each bit q_i is either 1 or 0 and $q_j = 1$. Then one of the heaps, again without loss of generality say a, must have a 1 in position j in its binary expansion.

We will show that there is a move from this heap of size a which produces a position whose nim-sum is 0; in particular, reducing a to $x = q \oplus a$.

Move is legal: In changing a to $q \oplus a$ the leftmost bit in a that is changed is a 1 (to a 0) and therefore, $q \oplus a < a$, and so the move is legal. (The move *reduces* the size of the heap.)

Nim-sum 0: The resulting position is

$$\begin{aligned}(q \oplus a) \oplus b \oplus \cdots \oplus k &= \big((a \oplus b \oplus \cdots \oplus k) \oplus a\big) \oplus b \oplus \cdots \oplus k \\ &= (a \oplus a) \oplus (b \oplus b) \oplus \cdots \oplus (k \oplus k) \\ &= 0. \qquad \square\end{aligned}$$

There are some observations that follow from this proof.

- If there is only a single heap left, the strategy of the theorem and common sense agree — the winning move is to take everything.
- If there are two heaps then the winning move, if there is one, is to make the two heaps the same size then, afterwards, play Tweedledum-Tweedledee. The theorem gives the same strategy since $n \oplus n = 0$.
- The strategy applies even if the heap sizes are allowed to be transfinite. See Problem 7.

Example 7.13. Suppose we are playing $\textsc{nim}(7, 12, 13)$:

$$\begin{array}{r|cccc} 12 & 1 & 1 & 0 & 0 \\ 13 & 1 & 1 & 0 & 1 \\ 7 & & 1 & 1 & 1 \\ \hline 6 & 0 & 1 & 1 & 0 \end{array}$$

Since the nim-sum is not 0, this is an $\mathcal{N}$-position, and the person on move can win. In particular, above the leftmost 1 in the sum there is a 1 in *every* heap, and so there is a winning move from every heap. The player on move can reduce

- the heap of size 12 to one of size $12 \oplus 6 = 10$, or
- the heap of size 13 to one of size $13 \oplus 6 = 11$, or
- the heap of size 7 to one of size $7 \oplus 6 = 1$.

Exercise 7.14. Find all winning moves, if any, in the following.

1. NIM$(3, 5, 7)$;
2. NIM$(2, 3, 5, 7)$;
3. NIM$(2, 4, 8, 32)$;
4. NIM$(2, 4, 10, 12)$.

7.3 Adding Stars

All the hard work was done in the last section. We can now wrap everything up.

Theorem 7.15. *For non-negative integers k and j, $*k + *j = *(k \oplus j)$.*

Proof: From Theorem 7.12, we have that NIM with heaps of size k, j, and $k \oplus j$ is a $\mathcal{P}$-position. The values of the individual nim-heaps are, respectively, $*k$, $*j$ and $*(k \oplus j)$. That they form a $\mathcal{P}$-position means that

$$*k + *j + *(k \oplus j) = 0, \text{ or } *k + *j = *(k \oplus j).$$

□

Corollary 7.16. $*a + *b + \cdots + *n = *(a \oplus b \oplus \cdots \oplus n)$.

Exercise 7.17. The game of POKER NIM is played with a collection of nim-heaps, with one additional bag with a finite number of counters. As a turn, a player can either make a normal NIM play, removing (and discarding) counters from one heap, *or* the player can increase the size of one heap by drawing counters from the bag.

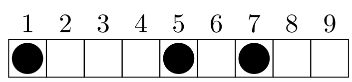
Show that a winning strategy for POKER NIM is the same as in NIM. That is, if a player can win at NIM, the player can also win at POKER NIM by making the identical winning move(s).

Sometimes even a loopy, partizan game is really a finite, impartial game in disguise:

Exercise 7.18. Consider a second variant of POKER NIM from Exercise 7.17 in which each player has her own bag of counters. You can move either by removing counters from a single heap and placing them in your bag, or by taking counters from your bag and placing them on a single heap.

1. As in Exercise 7.17, prove that a winning strategy for NIM suffices for playing this game.
2. Explain why the game is *loopy.* Also explain why, nonetheless, a game played between talented players should end.

Example 7.19. SLIDING is played on a strip of squares numbered 1, 2, 3, …, n. There are several coins on the squares, at most one per square. A move is to slide a coin from its square to a lower numbered square. The coin is not allowed to jump over or land on a square already occupied by a coin. Coins are not allowed to slide off the end of the strip.



For example, in the position above, the coin on square 7 can only move to 6, the coin on 5 can be moved to square 4, 3, or 2. The coin on 1 cannot move.

While this game, too, is equivalent to NIM, the heap sizes are not the positions of the coins. Rather, starting from the highest numbered occupied square, pair up the coins. The nim-heap sizes are then the number of empty squares between each adjacent pair. If the number of coins is odd, then the leftmost coin at position i is tantamount to a heap of size $i - 1$.

For example, if the coins were on 15, 8, 7, 5 then pair up 15 with 8 and 7 with 5. We claim this is equivalent to NIM with heaps of size 6 (gap between 15 and 8) and 1 (gap between 7 and 5), and the player on move can win by moving 15 to 10 (leaving gaps of 1 and 1). If the coins were on 15, 13, 12, 5, and 3, then we would pair 15-13, 12-5 and leave 3 unpaired; the heaps would be of sizes 1, 6, and 2. Since $*6 + * + *2 = *5$, there is at least one winning move. In NIM, one would replace $*6$ by $*3$, which is accomplished by moving the coin on square 12 to square 9.

Clearly, any heap can be decreased to any smaller size by moving the coin on the higher numbered square without changing the size of any other heap. A heap may be increased in size if the coin on the lower square slides. However, there are only a finite number of moves that increase the size of a heap and, as in POKER NIM, these moves are of no consequence.

Exercise 7.20. Consider a variant of SLIDING in which the players are allowed to slide coins off the strip. Find the winning strategy of this game.

The nimbers have many other interesting properties. For one, they can be extended to a field, see *ONAG* [Con01] for example, but such a development is beyond the scope of this book. One feature, though, is that in the nimber $*k$, k does not have to be finite. Problem 7 shows such nimbers in action.

7.4 A More Succinct Notation

The original notation for impartial games was developed before the theory for partizan games was known. If only impartial games are under consideration, the old notation is terser and more convenient. For the remainder of the chapter, we will therefore adopt it.

In this notation, since the left and right options are the same, they are not repeated. A nimber is abbreviated by writing k rather than $*k$. Specifically,

Definition 7.21. Let G be an impartial game. Then the *nim-value* (or *Grundy-value*), denoted by $\mathcal{G}(G)$, is k *if and only if* $G = *k$.

Observation 7.22. Let G be an impartial game and suppose that it has exactly n options. Then $\mathcal{G}(G) \leq n$.

Exercise 7.23. Prove this observation.

The term Grundy-value was the original term for $\mathcal{G}(G)$. More recently, it has been called the nim-value since if $G = *k$ then $\mathcal{G}(G) = k$ is the size of the nim-heap to which G is equivalent. We restate the central results for impartial games given in the last sections using this new notation:

Theorem 7.24.

- $\mathcal{G}(G) = \text{mex}\{\mathcal{G}(H) \mid H \textit{ is an option of } G\}$.
- *If G, H, and J are impartial games then $G = H + J$ if and only if $\mathcal{G}(G) = \mathcal{G}(H) \oplus \mathcal{G}(J)$.*

Example 7.25. The game SUBTRACTION$(1, 2, 4)$ is played with heaps of counters. A move consists of choosing a heap and removing either one, two, or four counters from that heap.

Table 7.1 shows how the first few nim-values of this game are calculated; a pattern seems apparent. Proving that such patterns hold is typically a rote induction argument:

n	options	nim-values	mex	$\mathcal{G}(n)$
0	$\{\}$	$\text{mex}\{\}$		0
1	$\{0\}$	$\text{mex}\{\mathcal{G}(0)\}$	$\text{mex}\{0\}$	1
2	$\{1,0\}$	$\text{mex}\{\mathcal{G}(1),\mathcal{G}(0)\}$	$\text{mex}\{0,1\}$	2
3	$\{2,1\}$	$\text{mex}\{\mathcal{G}(2),\mathcal{G}(1)\}$	$\text{mex}\{1,2\}$	0
4	$\{3,2,0\}$	$\text{mex}\{\mathcal{G}(3),\mathcal{G}(2),\mathcal{G}(0)\}$	$\text{mex}\{0,2,0\}$	1
5	$\{4,3,1\}$	$\text{mex}\{\mathcal{G}(4),\mathcal{G}(3),\mathcal{G}(1)\}$	$\text{mex}\{1,0,1\}$	2
6	$\{5,4,2\}$	$\text{mex}\{\mathcal{G}(5),\mathcal{G}(4),\mathcal{G}(2)\}$	$\text{mex}\{2,1,2\}$	0

Table 7.1. Calculating the first few nim-values $\mathcal{G}(n)$ of a heap of size n appearing in SUBTRACTION$(1,2,4)$.

Claim 7.26. *In* SUBTRACTION$(1,2,4)$*, the nim-value of a heap of size* n *is given by* $\mathcal{G}(n) = (n \bmod 3)$.

Proof:
$$\begin{aligned}
\mathcal{G}(n) &= \text{mex}\{\mathcal{G}(n-1),\mathcal{G}(n-2),\mathcal{G}(n-4)\} \\
&= \text{mex}\begin{cases} n-1 \bmod 3 \\ n-2 \bmod 3 \\ n-4 \bmod 3 \end{cases} \qquad \text{(by induction)} \\
&= \text{mex}\begin{cases} n-1 \bmod 3 \\ n-2 \bmod 3 \end{cases} \qquad \text{(since } n-1 \equiv n-4 \pmod 3\text{)} \\
&= n \bmod 3.
\end{aligned}$$

For this last step, the three possible cases are $\text{mex}\{0,1\} = 2$, $\text{mex}\{1,2\} = 0$, and $\text{mex}\{2,0\} = 1$. The base cases for $n \le 3$ are confirmed in Table 7.1. □

Let G be the game with heaps of size 3, 4, 15, and 122. Now $\mathcal{G}(3) = 0$, $\mathcal{G}(4) = 1$, $\mathcal{G}(15) = 0$, and $\mathcal{G}(122) = 2$. Consequently, $\mathcal{G}(G) = 0 \oplus 1 \oplus 0 \oplus 2 = 3$ and so G is in $\mathcal{N}$. A winning move is one that reduces the nim-value of the game to 0 and surely such a move exists from 122, changing its nimber from 2 to 1. Remove one from the heap leaving a heap of size 121.

In general, other winning moves might exist, similar to increasing a heap size in POKER NIM. Here, one might also change a nimber 1 to 2 or a nimber 0 to 3. The latter is impossible, for 3 does not appear as a nimber in the game. But another winning move is to remove two from the heap of size 4 leaving 2.

Exercise 7.27. Find the nim-values for SUBTRACTION$(1,4)$.

7.5 Taking-and-Breaking Games

There are many natural variations on NIM obtained by modifying the legal moves. For example, sometimes a player, in addition to taking counters, might also be permitted to split the remaining heap into two (or sometimes more) heaps. These rule variants yield a rich collection of Taking-and-Breaking games that are discussed in *WW* [BCG01].

After NIM, Taking-and-Breaking games are among the earliest and most studied impartial games; however, by no means is everything known. To the contrary, much of the field remains wide open. Values in games such as GRUNDY'S GAME (choose a heap and split it into two different sized heaps) and Conway's COUPLES ARE FOREVER (choose a heap and split it but heaps of size 2 are not allowed to be split) have been computed up to heaps of size 11×10^9 and 5×10^7, respectively, yet there is no complete analysis for these games.

In Taking-and-Breaking variants, the legal moves may vary with the size of the heap and the history of the game. For example, the legal moves might be, "a player must take at least one-quarter of the heap and no more than one-half," or "a player must take between $n/2$ and $n+3$, where n is the number taken on the last move." For an example, see Problem 11. Games whose allowed moves depend on the history of the game are typically more difficult to analyze, but when the legal moves are independent of the history (and of moves in other heaps), then the game is a disjunctive sum and we only need analyze games that have a single heap!

Definition 7.28. For a given Taking-and-Breaking game G, let $\mathcal{G}(n)$ be the nim-value of the game played with a heap of size n. The *nim-sequence* for the game is $\mathcal{G}(0)$, $\mathcal{G}(1)$, $\mathcal{G}(2)$, ...

In order to automate the process for finding (and proving) the nim-sequences of selected Taking-and-Breaking games, we need to address two main questions:

1. What types of regularities occur in nim-sequences?
2. When do we know that some regularity observed in a nim-sequence will repeat for eternity?

There are three types of regularities that have been observed in many nim-sequences to which we can answer the second question. These are listed in the next definition but we only consider two of the three, those that are periodic and arithmetic periodic, in this book. The reader interested in split-periodicity should read [HN03, HN04].

Definition 7.29. A nim-sequence is

- *periodic* if there is some $l \geq 0$ and $p > 0$ so that $\mathcal{G}(n+p) = \mathcal{G}(n)$ for all $n \geq l$;

- *arithmetic periodic* if there is some $l \geq 0$, $p > 0$, and $s > 0$ so that $\mathcal{G}(n+p) = \mathcal{G}(n) + s$ for all $n \geq l$;[3] and
- *sapp regular* (or *split arithmetic periodic/periodic*) if there exist integers $l \geq 0$, $s > 0$, $p > 0$, and a set $S \subseteq \{0, 1, 2, \ldots, p-1\}$ such that for all $n \geq l$,

$$\mathcal{G}(n+p) = \begin{cases} \mathcal{G}(n) & \text{if } (n \bmod p) \in S, \\ \mathcal{G}(n) + s & \text{if } (n \bmod p) \notin S. \end{cases}$$

The subsequence $\mathcal{G}(0), \mathcal{G}(1), \ldots, \mathcal{G}(l-1)$ is called the *pre-period* and its elements are the *exceptional values.* When l and p are chosen to be as small as possible, subject to meeting the conditions of the definition, we say that l is the *pre-period length* and p is the *period length*, while s is the *saltus.* If there is no pre-period the nim-sequence is called *purely periodic, purely arithmetic periodic, purely sapp regular,* respectively.

Exercise 7.30. Match each sequence on the left one-to-one to a category on the right:

1231451671...	periodic
1123123123...	purely periodic
1122334455...	sapp regular
0123252729...	arithmetic periodic
0120120120...	purely sapp regular
1112233445...	purely arithmetic periodic

In each case, identify the period and (when non-zero) the saltus and pre-period.

7.6 Subtraction Games

Definition 7.31. A *subtraction game* denoted SUBTRACTION(S), is played with heaps of counters and a set S of positive integers. A move is to choose a heap and remove any number of counters provided that number is in S.

- If $S = \{a_1, a_2, \ldots, a_k\}$ is finite, we have a *finite subtraction game*, which we denote SUBTRACTION($a_1, a_2, \ldots, a_k$).
- If, on the other hand, $S = \{1, 2, 3, \ldots\} \setminus \{a_1, a_2, \ldots, a_k\}$ consists of all the positive integers except a finite set, we have an *all-but subtraction game*, denoted ALLBUT($a_1, a_2, \ldots, a_k$).

In Example 7.25, SUBTRACTION($1, 2, 4$) was shown to be periodic. On the other hand, ALLBUT() is another name for NIM and is arithmetic periodic with saltus 1.

[3] Reminder: + means normal, not nimber, addition!

Finite subtraction games

The next table gives the first 15 values of the nim-sequence for several subtraction games SUBTRACTION(S).

S	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**
{1,2,3}	0	1	2	3	0	1	2	3	0	1	2	3	0	1	2
{2,3,4}	0	0	1	1	2	2	0	0	1	1	2	2	0	0	1
{3,4,5}	0	0	0	1	1	1	2	2	0	0	0	1	1	1	2
{3,4,6,10}	0	0	0	1	1	1	2	2	2	0	3	3	1	4	0

It is not surprising that the nim-sequence for SUBTRACTION$(1, 2, 3)$ is purely periodic with values $\dot{0}12\dot{3}$. (*Note*: we use dots to indicate the first and last values in the period.) The nim-sequence for SUBTRACTION$(2, 3, 4)$ is $\dot{0}0112\dot{2}$ and $\dot{0}001112\dot{2}$ for SUBTRACTION$(3, 4, 5)$. The pattern is not yet evident for SUBTRACTION$(3, 4, 6, 10)$, but if we pushed on we would eventually find that the nim-sequence is $00011122203314\dot{0}020131\dot{2}$.

Calculating nim-sequences with a Grundy scale

The calculation of nim-sequences for subtraction games can be done by a computer (CGSuite, Maple or Mathematica, for example, or any spreadsheet program) but there is an easy, hand technique as well. Use two pieces of graph paper to construct what is known as a *Grundy scale*. Here, we will work with the example SUBTRACTION$(3, 4, 6, 10)$.

To begin, take two sheets of lined or graph paper. On each sheet make a scale, with the numbers marked in opposite directions. One of the scales is marked with △s to indicate the positions of numbers in the subtraction set; you can put a ▲ to indicate 0. The other sheet will be used to record the nim-values:

0	1	2	3	4	5	6	7

	△				△		△	△			▲
11	10	9	8	7	6	5	4	3	2	1	0

As you fill in nim-values on the top scale, slide the bottom scale to the right. Shown below is the calculation $\mathcal{G}(9)$; take the mex of the positions marked by arrows. In this case, $\text{mex}(1, 1, 2) = 0$, so we will fill in a 0 for $\mathcal{G}(9)$, now indicated by the ▲:

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
0	0	0	1	1	1	2	2	2	**0**					

	△				△		△	△			▲
11	10	9	8	7	6	5	4	3	2	1	0

Similarly, $\mathcal{G}(13) = \text{mex}(1, 2, 0, 3) = 4$:

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
0	0	0	1	1	1	2	2	2	0	3	3	1	**4**						

	△				△		△	△			▲
11	10	9	8	7	6	5	4	3	2	1	0

Exercise 7.32. Make a photocopy of the last Grundy scale, make a cut to separate the bottom and top portions, and use it to calculate more of the nim-sequence of SUBTRACTION$(3, 4, 6, 10)$. For more practice, try using Grundy scales for the examples in problem 2 until you are comfortable.

Periodicity of finite subtraction games

After working out a few finite subtraction games, it will come as no surprise that their nim-sequences are always periodic.

Theorem 7.33. *The nim-sequences of finite subtraction games are periodic.*

Proof: Consider the finite subtraction game SUBTRACTION$(a_1, a_2, \ldots, a_k)$ and its nim-sequence. From any position there are at most k legal moves. So, using Observation 7.22, $\mathcal{G}(n) \leq k$ for all n.

Define $a = \max\{a_i\}$. Since $\mathcal{G}(n) \leq k$ for all n there are only finitely many possible blocks of a consecutive values that can arise in the nim-sequence. So we can find positive integers q and r with $a \leq q < r$ such that the a values in the nim-sequence immediately preceding q are the same as those immediately preceding r. Then $\mathcal{G}(q) = \mathcal{G}(r)$ since

$$\mathcal{G}(q) = \text{mex}\{\mathcal{G}(q - a_i) \mid 1 \leq i \leq k\} = \text{mex}\{\mathcal{G}(r - a_i) \mid 1 \leq i \leq k\} = \mathcal{G}(r).$$

In fact, for such q and r and all $t \geq 0$, $\mathcal{G}(q + t) = \mathcal{G}(r + t)$. This is easily shown by induction — we have just seen the base case, and the inductive step is really just an instance of the base case translated t steps forwards. Now set $l = q$ and $p = r - q$ and we see that the above says that for all $n \geq l$, $\mathcal{G}(n + p) = \mathcal{G}(n)$; that is, that the nim-sequence is periodic. □

This proof shows that the pre-period and period lengths are at most $(k+1)^a$. However, this is generally a wild overestimate, and using the following corollary the values of the period and pre-period lengths can usually be determined by computer:

Corollary 7.34. *Let* $G = \textsc{subtraction}(a_1, a_2, \ldots, a_k)$ *and let* $a = \max\{a_i\}$*. If* l *and* p *are positive integers such that* $\mathcal{G}(n) = \mathcal{G}(n+p)$ *for* $l \le n < l + a$*, then the nim-sequence for* G *is periodic with period length* p *and pre-period length* l*.*

Proof: See Problem 5. □

That is, given conjectured values of l and p, it suffices to inspect the values of $\mathcal{G}(n)$ for $n \in \{l, l+1, \ldots, l+p+a-1\}$ to confirm the periodicity! It is then rote for a computer to identify the smallest pre-period and period given by the corollary.

Applying the corollary to the games in Table 7.6 we see that for:

$\textsc{subtraction}(1, 2, 3)$ we have $l = 0$, $p = 4$ and $a = 3$, and these values can be confirmed by inspection of $\mathcal{G}(n)$ for $n \in \{0, \ldots, 6\}$;

$\textsc{subtraction}(2, 3, 4)$ $l = 0$, $p = 6$ and $a = 4$, inspect $\mathcal{G}(n)$ for $n \in \{0, \ldots, 9\}$;

$\textsc{subtraction}(3, 4, 5)$ $l = 0$, $p = 8$ and $a = 5$, inspect $\mathcal{G}(n)$, $n \in \{0, \ldots, 12\}$; and

$\textsc{subtraction}(3, 4, 6, 10)$ $l = 14$, $p = 7$ and $a = 10$, inspect $\mathcal{G}(n)$, $n \in \{14, \ldots, 30\}$.

Exercise 7.35. Use a Grundy scale to compute values of $\textsc{subtraction}(2, 4, 7)$ until you have enough to apply Corollary 7.34. For more practice, see Problem 2.

Two of the main questions, which still attract researchers, are:

1. As a function of the a_i, how long can the period of $\textsc{subtraction}(a_1, a_2, \ldots, a_k)$ be?
2. Find general forms for the nim-sequence for $\textsc{subtraction}(a_1, a_2, a_3)$.

Many games with different subtraction sets are actually the same in the sense that they have the same nim-sequence. For example, in the game $\textsc{subtraction}(1)$, the first player wins precisely when there is an odd number of counters in the heap. The odd-sized heaps only have moves to even-sized heaps; even-sized heaps only have moves to odd-sized heaps and the end position consist of heaps of size 0; that is, even. Therefore, a heap with n-counters is in $\mathcal{N}$ if n is odd, otherwise it is in $\mathcal{P}$. Adjoining any odd numbers to the subtraction set does not change this argument and it is easy to show that the nim-sequence of any game $\textsc{subtraction}(1, \text{odds})$ is $\dot{0}\dot{1}0101$. A more general version of this analysis is sufficient to prove:

Theorem 7.36. *Let* $G = \textsc{subtraction}(a_1, a_2, \ldots, a_k)$ *be purely periodic with period* p. *Let* $H = \textsc{subtraction}(a_1, a_2, \ldots, a_k, a_1 + mp)$ *for* $m \geq 0$, *then* G *and* H *have the same nim-sequence.*

Proof: See Problem 6. □

All-but-finite subtraction games

The next table gives the first 15 values of the nim-sequence for $\textsc{allbut}(S)$ for several sets S:

S	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**
{1,2,3}	0	0	0	0	1	1	1	1	2	2	2	2	3	3	3
{2,3,4}	0	1	0	1	0	1	2	3	2	3	2	3	4	5	4
{1,2,8,9,10}	0	0	0	1	1	1	2	2	2	3	0	3	4	1	4

Grundy scales can still be used for the all-but subtraction games. This time, the small arrows mark the heaps that are not options.

Exercise 7.37. Use a Grundy scale to find the first 20 terms in the nim-sequence of $\textsc{allbut}(1, 3, 4)$.

These values certainly do not look periodic, for $\mathcal{G}(n)$ appears to steadily increase with n, as confirmed by the following lemma:

Lemma 7.38. *Let* $G = \textsc{allbut}(a_1, a_2, \ldots, a_k)$ *and* $a = \max\{a_i\}$. *Then* $\mathcal{G}(n+t) > \mathcal{G}(n)$ *for all* $t > a$.

Proof: Since any option from n is also an option from $n+t$, $\mathcal{G}(n+t) \geq \mathcal{G}(n)$. Additionally, n is an option from $n+t$, so $\mathcal{G}(n)$ occurs as the nim-value of an option from $n+t$ and thus we do not have equality. □

However, as we will see, all-but subtraction games are arithmetic periodic. From the table above, we might conjecture the following (we denote the saltus information in parentheses with a + sign):

S	**nim-sequence**
{1,2,3}	$\dot{0}00\dot{0}(+1)$
{2,3,4}	$\dot{0}1010\dot{1}(+2)$
{1,2,8,9,10}	$00011122\dot{2}3\dot{0}(+1)$

Proving a nim-sequence is arithmetic periodic is typically more difficult than proving periodicity. Nonetheless, we can parallel the work done for finite subtraction games, proving first that $\textsc{allbut}$ subtraction games are arithmetic periodic, and then identifying how one (a person or a computer) can automatically confirm the arithmetic periodicity.

Theorem 7.39. *Let $G = \text{ALLBUT}(a_1, a_2, \ldots, a_k)$. Then the nim-sequence for G is arithmetic periodic.*

Proof: Lemmas 7.41 and 7.43, motivated and then proved below, yield the theorem. □

The proof of this theorem is more technical than that of Theorem 7.33, but in broad strokes is similar. In the proof of Theorem 7.33, we first argued that nim-values cannot get too large, and that therefore some sufficiently long sequence of nimbers must repeat. Once a sufficiently long sequence appears identically twice, an induction argument establishes that those two sequences remain in lock-step ad infinitum.

With arithmetic periodicity, the repetition we seek is in the *shape* of a sequence rather than its values as shown in Figure 7.1 on page 152. As you read through the proofs, keep in mind that two subsequences of nim-values, call them $(\mathcal{G}(n_0), \ldots, \mathcal{G}(n_0 + c))$ and $(G(n'_0), \ldots, \mathcal{G}(n'_0) + c)$ have the same *shape* if

1. the two subsequences differ by a constant:
$$\mathcal{G}(n'_0) - \mathcal{G}(n_0) = \mathcal{G}(n'_0 + 1) - \mathcal{G}(n_0 + 1) = \cdots = \mathcal{G}(n'_0 + c) - \mathcal{G}(n_0 + c),$$
2. or equivalently, both subsequences move up and down the same way: for all $0 \le i < c$,
$$\mathcal{G}(n_0 + i + 1) - \mathcal{G}(n_0 + i) = \mathcal{G}(n'_0 + i + 1) - \mathcal{G}(n'_0 + i).$$

It will turn out that the base case for our inductive proof will require a repetition of length about $2a$, where $a = \max\{a_i\}$. We show that such a repetition exists in the next two lemmas.

Lemma 7.40. *Let $G = \text{ALLBUT}(a_1, a_2, \ldots, a_k)$, and define $a = \max\{a_i\}$. For all $n \ge a$,*
$$k - a \le \mathcal{G}(n + 1) - \mathcal{G}(n) \le a - k + 1.$$

Proof: Fix $n > a$ and let $X \subseteq \{\mathcal{G}(0), \mathcal{G}(1), \mathcal{G}(2), \ldots, \mathcal{G}(n-1)\}$ be the nim-values of the options of n.[4] Now, since $\mathcal{G}(n)$ is the mex of X, we know that $\{0, 1, 2, \ldots, \mathcal{G}(n) - 1\} \subseteq X$. Further, play in $G = \text{ALLBUT}(a_1, a_2, \ldots, a_k)$ prohibits moves to k of the top a heap sizes. Hence, one of $\{\mathcal{G}(0), \ldots, \mathcal{G}(n-a-1)\}$, say $\mathcal{G}(m)$, must be at least $\mathcal{G}(n) - 1 - (a - k)$, for only $a - k$ of the terms from X can appear among $\{\mathcal{G}(n-a), \ldots, \mathcal{G}(n) - 1\}$. Further, m and all the options from m are also options from $n + 1$. So we have

$$\begin{aligned} \mathcal{G}(n+1) &> \mathcal{G}(m), \text{ and so} \\ \mathcal{G}(n+1) &\ge \mathcal{G}(n) - (a - k). \end{aligned}$$

[4]In other words, $X = \{\mathcal{G}(n - \alpha) \mid \alpha \notin \{a_1, \ldots, a_k\}\}$.

Similarly, for the second inequality, one of $\{\mathcal{G}(0), \ldots, \mathcal{G}(n-a-1)\}$ is at least $\mathcal{G}(n+1) - 2 - (a-k)$, and it and its options are options of n. So,

$$\mathcal{G}(n) \quad \geq \quad \mathcal{G}(n+1) - (a-k) - 1. \qquad \square$$

Lemma 7.41. *Let $G = \text{ALLBUT}(a_1, a_2, \ldots, a_k)$ and $a = \max\{a_i\}$. There exist n_0, n_0', s, and $p = n_0' - n_0 > 0$ such that $\mathcal{G}(n+p) - \mathcal{G}(n) = s$ for $n_0 \leq n \leq n_0 + 2a$.*

Proof: By Lemma 7.40, for all n, $\mathcal{G}(n+1) - \mathcal{G}(n)$ must lie between $k - a$ and $a - k + 1$. But there are only $2(a-k)+2$ values in that range. Hence, setting $c = 2(a-k)+2$, there are at most c^{2a} possible sequences of the form

$$\begin{array}{c} (\qquad \mathcal{G}(n+1) - \mathcal{G}(n), \qquad\quad \\ \mathcal{G}(n+2) - \mathcal{G}(n+1), \\ \ldots, \\ \qquad \mathcal{G}(n+2a) - \mathcal{G}(n+2a-1) \quad) \end{array} \tag{7.1}$$

and so eventually, for two values $n = n_0$ and $n = n_0'$, the two corresponding sequences are identical. The lemma follows. $\square$

Exercise 7.42. Complete the algebra to confirm, "The lemma follows." In particular, given the two identical sequences satisfying (7.1), one with n_0', and one with n_0, you need to define p and explain why $\mathcal{G}(n+p) - \mathcal{G}(n)$ is a constant (call it $s = \mathcal{G}(n_0') - \mathcal{G}(n_0)$) for $n_0 \leq n \leq n_0 + 2a$. (The matching shapes in Figure 7.1 provide some intuition.)

The next lemma completes the inductive step of the proof of the theorem showing that once two sufficiently long sequences have the same shape, they are fated to continue in lock step.

Lemma 7.43. *Let $G = \text{ALLBUT}(a_1, a_2, \ldots, a_k)$ and $a = \max\{a_i\}$ and suppose $\mathcal{G}(n+p) - \mathcal{G}(n) = s$ for $n_0 \leq n \leq n_0 + 2a$. Then $\mathcal{G}(n+p) - \mathcal{G}(n) = s$ for all $n \geq n_0$.*

In particular, $\mathcal{G}(n)$ has pre-period length $l = n_0$, period p, and saltus s, and one can identify and confirm the period and saltus by only inspecting the first $l + 2a + p + 1$ values; i.e., $\mathcal{G}(n)$ for $0 \leq n \leq l + 2a + p$.

Proof: Using induction, it suffices to prove $\mathcal{G}(n+p) - \mathcal{G}(n) = s$ for $n = n_0 + 2a + 1$. Define the following quantities as shown in Figure 7.1: $n_0' = n_0 + p$, $n_1 = n_0 + a$, $n_2 = n_0 + 2a$, $n_1' = n_0' + a$, and $n_2' = n_0' + 2a$.

As we compute $\mathcal{G}(n') = \mathcal{G}(n+p)$ as the mex of the nim-values of its options, by Lemma 7.38, $\mathcal{G}(n')$ exceeds all $\mathcal{G}(m)$ for $m < n' - a - 1 = n_1'$. And we

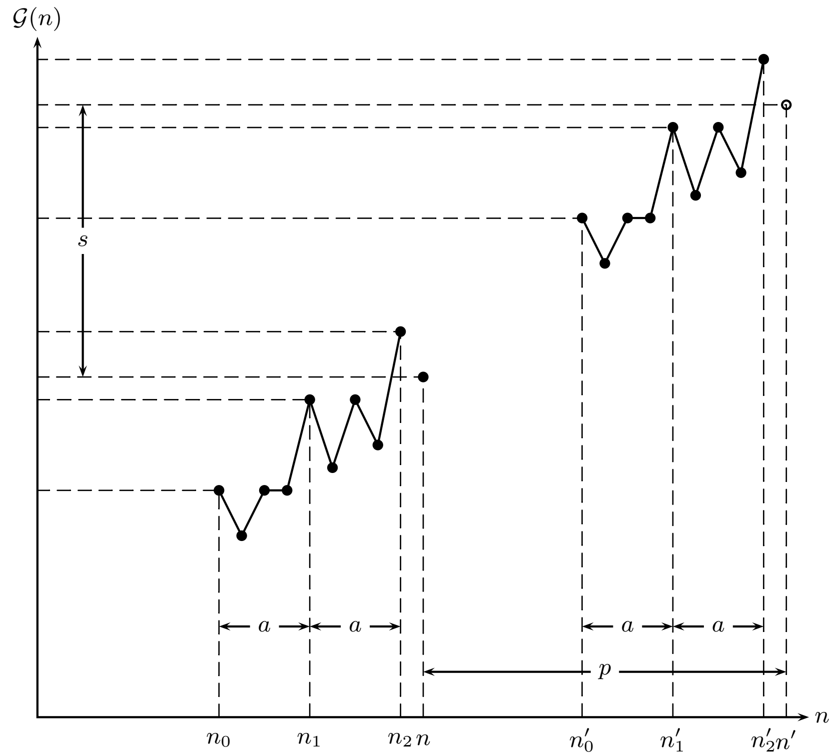


Figure 7.1. A diagram sketching the proof of arithmetic periodicity in ALLBUT subtraction games. For the game ALLBUT$(a_1, \ldots, a_k)$, $a = \max\{a_i\}$, s is the proposed saltus, p is the proposed period. We first find two sufficiently long sequences of nim-values with the same shape (but translated upward), and inductively prove that those sequences remain in lock-step.

know $\mathcal{G}(n') > \mathcal{G}(n'_1)$ in any event, so we can safely ignore $\mathcal{G}(m)$ for $m < n'_0$. In other words, $\mathcal{G}(n')$ is the minimum excluded value from $\{\mathcal{G}(n'_0), \ldots, \mathcal{G}(n'_2)\}$ *that exceeds* $\mathcal{G}(n'_1)$. Since the assignment of $\mathcal{G}(n')$ is unaffected by linear translation of those nim-values, $\mathcal{G}(n') - \mathcal{G}(n) = s$. □

This last lemma gives an automated method for testing when an all-but subtraction game nim-sequence has become arithmetic periodic. Although Figure 7.1 shows the two subsequences non-overlapping (suggesting that $p > 2a$), the proof is unaffected by overlap.

Exercise 7.44. Apply Lemma 7.43 to find the period length p and the saltus s of the game from Exercise 7.37. In particular, how many values of $\mathcal{G}(n)$ need computing to confirm the period and saltus? (*Hint*: The game is purely arithmetic periodic with period between 10 and 15.)

Exercise 7.45. We asserted at the start of this section that the nim-sequence for ALLBUT$(1, 2, 8, 9, 10)$ is given by $0001112\dot{2}3\dot{0}(+1)$. To apply Lemma 7.43, which values of $\mathcal{G}(n)$ need be confirmed to be confident of the nim-sequence?

As was seen in the table on page 149, some ALLBUT subtraction games have pre-periods. If you do Problems 12, 13, and 14. then you will have shown that the nim-values of ALLBUT games where s has cardinality 1 and 2 are purely arithmetic-periodic.

Frequently, the ALLBUT subtraction set can be reduced. While most such reductions remain specific to individual games, we do have one general reduction theorem.

Theorem 7.46. *Let $a_1 < a_2 < \cdots < a_k$ be positive integers, and let $b > 2a_k$. Then, the nim-sequences of* ALLBUT$(a_1, a_2, \ldots, a_k)$ *and* ALLBUT$(a_1, a_2, \ldots, a_k, b)$ *are equal.*

Proof: Let $\mathcal{G}(n)$ denote the nim-sequence of ALLBUT$(a_1, a_2, \ldots, a_k)$ and $\mathcal{G}'(n)$ that of ALLBUT$(a_1, a_2, \ldots, a_k, b)$. Certainly, $\mathcal{G}(n) = \mathcal{G}'(n)$ for $n < b$ since the options in the two games are identical to that point. Suppose inductively that the two nim-sequences agree through $n - 1$, and consider $\mathcal{G}(n)$ and $\mathcal{G}'(n)$. Since the options of ALLBUT$(a_1, a_2, \ldots, a_k, b)$ are a subset of those of ALLBUT$(a_1, a_2, \ldots, a_k)$, the only possible way to have $\mathcal{G}(n) \neq \mathcal{G}'(n)$ would be if $\mathcal{G}'(n) = \mathcal{G}(n - b)$, since the latter is the only possible value that does not occur among the options of ALLBUT$(a_1, a_2, \ldots, a_k, b)$ but does occur among the options of ALLBUT$(a_1, a_2, \ldots, a_k)$. In order for this to be true it would also be the case that no option of n in ALLBUT$(a_1, a_2, \ldots, a_k, b)$ had value $\mathcal{G}(n - b)$.

However, consider $m = n - b + a_k$. By the inductive hypothesis $\mathcal{G}(m) = \mathcal{G}'(m)$. Moreover, all values smaller than $n - b$ are options from a heap of size m in ALLBUT$(a_1, a_2, \ldots, a_k)$, so $\mathcal{G}(m) \geq \mathcal{G}(n - b)$. As m is an option of n in ALLBUT$(a_1, a_2, \ldots, a_k, b)$, in order to avoid a contradiction we would require that $\mathcal{G}(m) > \mathcal{G}(n - b)$. But then, m would have an option m' in ALLBUT$(a_1, a_2, \ldots, a_k)$ with $\mathcal{G}(m') = \mathcal{G}(n - b)$. Since $m' \neq n - b$ and $m' < n - a_k$, m' is also an option of n in ALLBUT$(a_1, a_2, \ldots, a_k, b)$. This contradiction establishes the desired result. □

For more on all-but-finite subtraction games see [Sieg06].

Kayles and kin

KAYLES is played with a row of pins standing in a row. The players throw balls at the pins. The balls are only wide enough to knock down one or two adjacent pins.

† † †   † † † † † †   †

Above is a game in progress. Would you like to take over for the next player?

KAYLES can also be thought of as a game played with heaps of counters and a player is allowed to take one or two counters and possibly split the remaining heap into two. KAYLES is clearly a Taking-and-Breaking game but maybe it is too hard right now. We will work up to it with a couple of simpler variants.

First, suppose we play a game where a player is only allowed to split-the-heap into two non-empty heaps; no taking. We saw in Example 1.10 of Chapter 1, that this is SHE LOVES ME SHE LOVES ME NOT disguised. That is,

$$\mathcal{G}(n) = \begin{cases} 0 & \text{if } n = 0, \\ 0 & \text{if } n > 0 \text{ is odd}, \\ 1 & \text{if } n > 0 \text{ is even}. \end{cases}$$

In particular, moves from an even heap are to even + even $= * + * = 0$ or odd + odd $= 0$, while moves from an odd heap are to even + odd $= * + 0 = *$.

The version where you are allowed to split-*or*-take-1-and-must-split (into non-empty heaps) needs a bit more care. For a heap of n, the splitting move gives the options $n - i$ and i for $i = 1, 2, \ldots, n-1$ and the take-1-and-split move leaves the positions $n - 1 - j$ and j, $j = 1, 2, \ldots, n-2$. To find the value of the game, we need to find the mex of

$$\begin{aligned} & \{\mathcal{G}(i) \oplus \mathcal{G}(n-i) \mid i = 1, 2, \ldots, n-1\} \\ \cup \quad & \{\mathcal{G}(j) \oplus \mathcal{G}(n-1-j) \mid j = 1, 2, \ldots, n-2\}. \end{aligned}$$

The first few values are $\mathcal{G}(0) = 0$, $\mathcal{G}(1) = 0$, $\mathcal{G}(2) = \text{mex}\{\mathcal{G}(1) + \mathcal{G}(1)\} = 1$, $\mathcal{G}(3) = \text{mex}\{\mathcal{G}(2) \oplus \mathcal{G}(1)\} \cup \{\mathcal{G}(1) \oplus \mathcal{G}(1)\} = 2$. This becomes tedious very quickly and a machine will be very useful. If one is not available then a Grundy scale can help. This time both pieces of paper have the nim-values, one from left-to-right as before, the other has them from right-to-left. However, we have to use them twice and record the intermediate values. The next diagram shows the calculation of $\mathcal{G}(9)$. First, line up the scales for the split move, $8+1$, $7+2$, …, $1+8$. Clearly we only need go to $4+5$ since we repeat the calculations in reverse. Nim-sum the pairs of numbers and record:

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
0	0	1	2	3	1	4	3	2						

			2	3	4	1	3	2	1	0	0
11	10	9	8	7	6	5	4	3	2	1	0

This gives $\{2, 2, 6, 2, 2, 6, 2, 2\}$ as the set of values for these options. Now line them up for the take-one-and-split move.

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
0	0	1	2	3	1	4	3	2						

			2	3	4	1	3	2	1	0	0
11	10	9	8	7	6	5	4	3	2	1	0

This gives the set $\{3, 5, 3, 0, 3, 5, 3\}$ for these options. The least non-negative number that does not appear in either set is 1, so record $\mathcal{G}(9) = 1$ on both papers:

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
0	0	1	2	3	1	4	3	2	**1**					

		1	2	3	4	1	3	2	1	0	0
11	10	9	8	7	6	5	4	3	2	1	0

Exercise 7.47. Continue using the Grundy scale to compute five more nim-values of the last game where you are allowed to split-*or*-split-and-take-one-but-always-leave-two-heaps.

For a fixed (finite) set S of positive integers, we can define the variant SPLITTLES(S) where a player is allowed to take away s, for some $s \in S$, from a heap of size at least s and possibly split the remaining heap.

In particular, KAYLES is just SPLITTLES($1, 2$).

What sort of regularities could the nim-sequences of SPLITTLES(S) have? In the few examples so far, the nim-values do not grow very quickly at all suggesting that the nim-sequences are periodic *but Nobody Knows.* All we know is that the nim-sequences are *not* arithmetic periodic [Aus76, BCG01]. It is believed that SPLITTLES(S) is periodic when S is finite, and for periodicity we do have an automatic check.

Theorem 7.48. *Fix* SPLITTLES(S) *with* $m = \max S$. *If there exists integers,* $l \geq 0$ *and* $p > 0$ *such that* $\mathcal{G}(n+p) = \mathcal{G}(n)$ *for* $l \leq n \leq 2l + 2p + s$, *then* $\mathcal{G}(n+p) = \mathcal{G}(n)$ *for all* $n \geq l$. *That is, the period persists forever.*

Proof: See Problem 15. □

Problems

1. Find the nim-sequences for SUBTRACTION(S), where $|S| = 2$ and $S \subseteq \{1, 2, 3, 4\}$.

2. Use a Grundy scale and Corollary 7.34 to compute the nim-sequences of

   (a) SUBTRACTION$(2, 3, 5)$;

   (b) SUBTRACTION$(3, 5, 8)$;

   (c) SUBTRACTION$(1, 3, 4, 7, 8)$.

3. Find the period for

   (a) ALLBUT$(1, 2, 3)$;

   (b) ALLBUT$(5, 6, 7)$;

   (c) ALLBUT$(3, 4, 6, 10)$.

4. A game is played like KAYLES, only you cannot bowl the end of a row of pins. In NIM language, you can take one or two counters from a heap and you must split that heap into two *non-empty* heaps. Using a Grundy scale, calculate the first 15 nim-values for this game.

5. Prove Corollary 7.34 on page 148.

6. Prove Theorem 7.36 on page 149.

7. (This is a generalization of NIM.) POLYNIM is played on polynomials with non-negative coefficients. A move is to choose a single polynomial and reduce one coefficient and arbitrarily change or leave alone the coefficients on the smaller powers in this polynomial — $3x^2 + 15x + 3$ can be reduced to $0x^2 + 19156x + 2345678 = 19156x + 2345678$. Analyze POLYNIM. In particular, identify a strategy for determining when a position is a $\mathcal{P}$-position analogous to Theorem 7.12.

8. Find the nim-sequences for SUBTRACTION$(1, 2q)$ for $q = 1, 2, 3$. Find the form of the nim-sequence of SUBTRACTION$(1, 2q)$ for arbitrary q.

9. Show that the nim-sequence for SUBTRACTION$(q, q+1, q+2)$ is $\dot{0}0^{q-1}1^q2\dot{2}$ if $q > 1$. (As usual, x^b is x repeated b times.)

10. Analyze SUBTRACTION$(1, 2, 4, 8, 16, \ldots, 2^i, \ldots)$.

11. Analyze this variant of NIM: On any move a player must remove at least half the number of counters from the heap.[5]

12. Find the periods for ALLBUT(1), ALLBUT(2), and ALLBUT(3). Conjecture and prove your conjecture for the period of ALLBUT(q).

[5] The nim-sequence for the game in which no more than half can be removed has a remarkable self-similarity property: If you remove the first occurrence of each number in the nim-sequence, then the resulting sequence is the same as the original! See [Lev06].

13. Find the periods for ALLBUT$(1,2)$, ALLBUT$(2,3)$, and ALLBUT$(3,4)$. Conjecture and prove your conjecture for the period of ALLBUT$(q, q+1)$.

14. Find the nim-sequence for ALLBUT(q,r), $q < r$. (*Hint*: There are two cases $r = 2q$ and $r \neq 2q$.)

15. Prove Theorem 7.48 on page 155.

16. The rules of the game TURN-A-BLOCK are at the textbook website, www.lessonsinplay.com, and you can play the game against the computer. Determine a winning strategy for TURN-A-BLOCK. You should be able to consistently beat the computer on the *hard* setting at 3×3 and 5×3 turn-a-block (and even bigger boards!). You should be able to determine who should win from any position up to 5×5.

Preparation for Chapter 8

Prep Problem 8.1. Redo Exercise 6.10 on page 123 to remind yourself of the definition of left and right stops.

To the instructor: We follow the non-constructive proof of the *Mean Value Theorem* of games, Theorem 8.6 on page 164. Our proof mirrors that of *ONAG* [Con01]. As an alternative, you may wish to present the constructive proof that appears in [BCG01, pp. 152–155]. Their approach uses thermographs, so you should present the proof after covering material from Section 8.3 on drawing thermographs.

Chapter 8

Hot Games

Who, ...
....
...through the heat of conflict, keeps the
law
In calmness made, and sees what he foresaw;
....
This is the happy Warrior ....

William Wordsworth in Character of the
Happy Warrior

In many classical games, the issue of whose turn it is to move is critical either in terms of determining the outcome, or in terms of its influence on the final score (assuming error-free play). The terminology of these games reflects the importance of this concept: one of the fundamental strategic concepts in CHESS is *tempo,* while GO distinguishes moves which keep *sente* (i.e., demand a response and so preserve the initiative) from those that are *gote* (valuable, perhaps, but not requiring a local reply).

In a limited sense, these concepts relate only to the outcome class of the game in question. Specifically, if a game is of type $\mathcal{N}$ then both players would like to have the opportunity to move next. However, there is certainly more to it than that — players recognize these situations as being important exactly when the value of the next move is such that it establishes a decisive advantage, usually in terms of material gain in CHESS, or gain of territory or a successful capture in GO. The positions that are important in these analyses are more like switches, such as $\pm 5 = \{5 \mid -5\}$, than stars.

Thermography is a technique that allows an understanding of some of the complex issues which arise in playing sums of hot games, or games with several hot options. Thermography is useful in the exact analysis of endgame positions in GO. Thermostrat and Sentestrat, modifications of the greedy strategy based on thermography, show promise as tools in building effective computer programs for playing combinatorial games.

8.1 Comparing Games and Numbers

The way in which a game G compares to an arbitrary number provides a certain amount of information about it — sometimes enough to establish the winner in more complicated positions. For example, if we know that $G \geq 1$, then Left will be happy about playing the game $H = G - 1 + \uparrow$ since she can see that H is positive. Roughly speaking, Left is interested in the answer to the question "For which numbers x is $G \geq x$?" while Right would like to know "For which numbers y is $G \leq y$?"

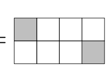
Theorem 8.1. *Let G be a short game. There exists a positive integer n such that*

$$-n < G < n.$$

Proof: This is an immediate corollary to Theorem 6.3, which states that the largest game born by day n is n. Let n be one plus the birthday of G. □

If Left and Right are playing a single short game G, then neither will be particularly keen to carry on when a position is reached whose value is a number. For at this point it is clear who has won based simply on the sign of the number and, if the number is 0, whose turn it is. If we think of this number as the score of the game (a concept which we have been avoiding until now!) then Left will want to play to maximize the score, and Right to minimize it. Note also that when the game reaches its stopping value, Left would rather that it were Right's turn to move (since a move by Right would then further improve the position for Left) and vice versa. This notion of score (or rather scores, since it depends on the initial player) has already been seen in Section 6.3 under the guise of the left and right stops of a game. Since this concept is central to our development here we will recapitulate and extend it a bit, as well as consider some examples.

Consider the DOMINEERING position

$G =$ 

Either of Left's moves leave the value $*$. Right can choose to leave $*$ or -1 and clearly prefers the latter. So $G = \{* \mid -1\}$. If Right plays first the stopping value of G is -1 and when G stops it will be Left's turn to move. So the *right stop* of G is -1_R. Notice that we have adorned the *stopping value* -1 with a subscript to indicate who moved last. Similarly, if Left moves first, the stopping value will be 0 (after Right plays in the $*$) and again it will be Left's turn to move. So the *left stop* of G is 0_R.

Now consider a game $H = \{0, * \mid -1\}$. The right stop of H is clearly -1_R. What about the left stop? Well, regardless of which move Left chooses the stopping value will be 0. However, if she moves to $*$ it will be her turn to play when the stop is reached, while if she moves to 0 it will be Right's turn. As we noted above, she prefers the latter alternative so the left stop of H is 0_L.

Definition 8.2. An *adorned stop* (or just stop) at x is a symbol of the form x_R or x_L, where x is a number. Stops are totally ordered, larger numbers being larger and $\Box_L > \Box_R$ used for tie-breaks; i.e., if $x > y$, then

$$x_L > x_R > y_L > y_R.$$

Context will often determine whether we mean the adorned or unadorned stop, as the word stop and the notation $\mathbf{RS}(G)$ are ambiguous. If we want to emphasize the unadorned stop, we can write the *value* of the stop x_L (or of x_R) as x.

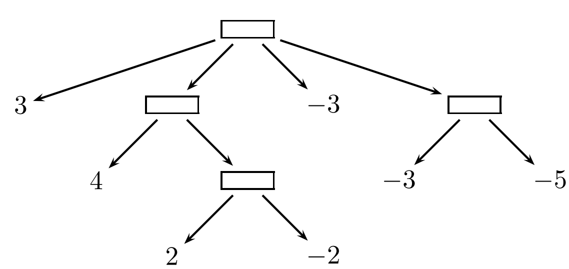
We repeat the definition of left and right stops from page 123, which now can include adornment:

Definition 8.3. Denote the *left stop* and *right stop* of a game G by $\mathbf{LS}(G)$ and $\mathbf{RS}(G)$, respectively. They are defined in a mutually recursive fashion:

$$\mathbf{LS}(G) = \begin{cases} x_R & \text{if } G \text{ is the number } x, \\ \max(\mathbf{RS}(G^L)) & \text{if } G \text{ is not a number;} \end{cases} \tag{8.1}$$

$$\mathbf{RS}(G) = \begin{cases} x_L & \text{if } G \text{ is the number } x, \\ \min(\mathbf{LS}(G^R)) & \text{if } G \text{ is not a number.} \end{cases} \tag{8.2}$$

Exercise 8.4. Compute the left and right stops of each position marked with a box below. Rest assured that the game is in canonical form, so none of the interior nodes are numbers in disguise:



The main importance in knowing the adorned left and right stops of a game is that it allows us to determine precisely how G compares to any number x.

Theorem 8.5. *If G is a short game and x is a number, then*

- *If the value of the left stop of G is smaller than x, then $G < x$.*

- *If the value of the left stop of G is equal to x, then $G \leq x$ when* $\mathbf{LS}(G) = x_R$ *while G is confused with or greater than x when* $\mathbf{LS}(G) = x_L$.
- *If the value of the left stop of G is greater than x then G is confused with or greater than x.*

Proof: Suppose that the value of the left stop of G is $y < x$ and consider the game $G - x$. Since the value of the right stop of G is at most the value of its left stop (as was shown in Theorem 6.11), Right has an option G^R with $\mathbf{LS}(G^R) \leq y$. By induction $G^R - x < 0$ so the option $G^R - x$ is a winning one for Right from $G - x$. Suppose that Left were to move first in $G - x$. A move in $-x$ is simply bad, so consider options of the form $G^L - x$. By the definition of the left stop, either $G^L - x = y - x < 0$ or the right stopping value of G^L is less than or equal to y. In that case, Right has an option to $G^{LR} - x$, where G^{LR} has left stopping value less than or equal to y and so by again by induction $G^{LR} - x < 0$. So we see that Right has a winning strategy in $G - x$ moving either first or second, and hence $G - x < 0$.

Now suppose that $\mathbf{LS}(G) = x_R$ and consider again the game $G - x$. We must show that Right has a winning strategy moving second. Again, we need only consider options of the form $G^L - x$. If the right stopping value of G^L is smaller than x we continue as above. The only other possible value is x itself, and in this case it must also be true that $\mathbf{RS}(G^L) = x_R$ (since otherwise, $\mathbf{LS}(G) = x_L$, not x_R). So Right can make a move to a position whose left stop is x_R and induction applies.

Now suppose that $\mathbf{LS}(G) \geq x_L$ (this includes the case where the left stopping value of G is greater than x). Then, moving first, Left can win in $G - x$ by choosing a left option G^L with $\mathbf{RS}(G^L) \geq x_L$ and then using the already proved results (with left and right interchanged). □

Combining this proposition with the dual one in which the roles of Left and Right are interchanged does indeed allow us to find how any short game G compares to any number. Specifically, associated to every game H there is a *confusion interval*, the set of numbers x with which H is confused. This confusion interval can be computed directly from the stops of the game as follows. The confusion interval of H is

$$\begin{array}{ll} [a,b] & \text{exactly if } \mathbf{LS}(H) = b_L \text{ and } \mathbf{RS}(H) = a_R; \\ (a,b] & \text{exactly if } \mathbf{LS}(H) = b_L \text{ and } \mathbf{RS}(H) = a_L; \\ [a,b) & \text{exactly if } \mathbf{LS}(H) = b_R \text{ and } \mathbf{RS}(H) = a_R; \\ (a,b) & \text{exactly if } \mathbf{LS}(H) = b_R \text{ and } \mathbf{RS}(H) = a_L. \end{array}$$

Notice that there is one case where the confusion interval is empty; that is, if $a = b$, $\mathbf{LS}(H) = a_R$ and $\mathbf{RS}(H) = a_L$. In that case, $H = a$.

Let's see these principles in action and check them with an example.

Consider the game $G = \{* \mid -1\}$ for which we know $\mathbf{LS}(G) = 0_R$ and $\mathbf{RS}(G) = -1_R$. Certainly, if $x > 0$, $G < x$ by the above, and if $x < -1$, $G > x$ by the dual form. For $-1 < x < 0$ the proposition above (and its dual) guarantees that G is both confused with or greater than x and confused with or less than x. So for such x, G must be confused with x. What about $x = -1$? Again, the two conditions "confused with or greater than" and "confused with or less than" both apply, so G is confused with -1. For safety we can certainly check this directly. In $G + 1$ if Right moves first he moves to $-1 + 1 = 0$ and wins, while if Left moves first she moves to $* + 1$ and wins. Finally, how does G compare with 0, its left stop? This time we know that $G \leq 0$ and also that G is confused with or less than 0. So the only possibility is that $G < 0$. To summarize, G has a confusion interval of $[-1, 0)$ and is greater than all numbers below its confusion interval and less than all numbers greater than its confusion interval.

8.2 Coping with Confusion

A game whose confusion interval has positive length is in a state of some excitement. Each player is eager to make the first move as this guarantees a better outcome for her or him than if his or her opponent moves first. The value of the game can be thought of as an indeterminate cloud, covering the confusion interval. In such a situation it is certainly natural to ask — is there a fair settlement about the value of the game? Certainly if such a settlement, let us call it the *mean value* of the game, exists then it should lie in the confusion interval. In some cases it is easy to propose the value of such a settlement.

Consider first the simple switch $G = \{b \mid a\}$ (with $b > a$). If Left moves first the value is b while if Right moves first the value is a. In fact the sum

$$\{b \mid a\} + \{b \mid a\} = a + b.$$

So playing G twice (in some sense) gives a value of $a + b$. In fact, $2k$ copies of G is exactly $k(a + b)$, while $2k + 1$ copies of G equal $k(a + b) + G$ whose confusion interval has width only $b - a$. So clearly the mean value of G should be $(a + b)/2$.

Is it then the case that the mean value of a game should simply be the midpoint of its confusion interval? Unfortunately (?) not. Consider the game

$$H = \left\{19 \mid 1 \,\middle\|\, -1\right\} = \{10 + \{9 \mid -9\} \mid -1\}.$$

Certainly $\mathbf{LS}(H) = 1_R$ and $\mathbf{RS}(H) = -1_R$ so the confusion interval is $[-1, 1)$. Is the mean value 0? No. Consider first $H + H$ and let us work out the stops of this game. Left's move is to

$$10 + \{9 \mid -9\} + H.$$

Right could now choose to move in H (to -1) and Left would collect 9 more stopping at 18. It would be better for Right to move in $\{9 \mid -9\}$ leaving $1+H$ with Left to move, and eventually a left stop of 2_L. What about the right stop? Right's first move is to $H-1$ and from there things proceed automatically to 0 with Left to move, so the right stop is 0_R. So the confusion interval of $H+H$ is $[0,2)$. Continuing this analysis we see that whenever Left makes a move in a copy of H, Right's reply (to minimize the stopping value) will be in the $\{9 \mid -9\}$. So basically, once Left gets a move in she picks up 1 point in each remaining copy of H. If we play n copies of H the right stopping value will be $n-2$ and the left stopping value n. So the mean value of H must be 1 — the very edge of the confusion interval.

Left's move in H (or sums of copies of H) to $10+\{9 \mid -9\}$ is an embodiment of the concept of *a point in sente*. It is a move that creates a threat to gain a large amount (by following up with a move to 9). This threat dominates all other considerations and must be answered by Right. As a result in the exchange of these two moves Left gains a "point" while maintaining the initiative.

We are left with two problems. First of all, is there a well-defined notion of "mean value"? Secondly, how can we compute it in practice?

We can provide an affirmative answer to the first question now, while answering the second will occupy us for the rest of the chapter. Let $n \cdot G$ denote the sum of n copies of G (if n is a non-negative integer) or of $-n$ copies of $-G$ (if n is a negative integer).[1]

Theorem 8.6. (Mean Value) *For every short game G there is a number $m(G)$ (the* mean value *of G) and a number t such that*

$$n{\cdot}m(G) - t \le n{\cdot}G \le n{\cdot}m(G) + t$$

for all integers n.

Proof: We begin with an observation about stopping values. Suppose that G and H are any two short games. Then[2]

$$\begin{aligned}\mathbf{RS}(G) + \mathbf{RS}(H) &\le \mathbf{RS}(G+H)\\ &\le \mathbf{RS}(G) + \mathbf{LS}(H)\\ &\le \mathbf{LS}(G+H)\\ &\le \mathbf{LS}(G) + \mathbf{LS}(H).\end{aligned}$$

[1]The probabilist might compare this theorem with the notions of mean and standard deviation from probability. Whereas the standard deviation of a sum of n independent, identically distributed random variables grows by a factor of $\sqrt{n}$, the temperature (defined on page 167) of a sum of games remains bounded.

[2]If $\mathbf{LS}(G) = 2_L$ and $\mathbf{RS}(G) = 4_R$, then $\mathbf{LS}(G+H)$ could be 6_L or 6_R, so we are not adorning $\mathbf{LS}(G+H) = 6$.

All of these inequalities follow from the first one after suitable changes of sign. And the first one is easy, since Left, playing second in $G+H$ can guarantee a stopping value greater than or equal to $\mathbf{RS}(G)+\mathbf{RS}(H)$ simply by answering Right's move in whichever game he happens to play.

Now consider $\mathbf{RS}(n\cdot G)$ (for simplicity assume that n is a positive integer). Certainly $\mathbf{RS}(n\cdot G)\le \mathbf{LS}(n\cdot G)$. Of course, they are equal if $n\cdot G$ happens to be a number. Otherwise, $\mathbf{LS}(n\cdot G)=\mathbf{RS}((n-1)\cdot G+G^L)$ for some left option G^L of G. Since $(n-1)\cdot G+G^L=n\cdot G+(G^L-G)$, our observation above shows that

$$\mathbf{RS}((n-1)\cdot G+G^L)\le \mathbf{RS}(n\cdot G)+\mathbf{LS}(G-G^L).$$

So $\mathbf{LS}(n\cdot G)$ is sandwiched between $\mathbf{RS}(n\cdot G)$ and $\mathbf{RS}(n\cdot G)+\mathbf{LS}(G-G^L)$. In particular,

$$\mathbf{LS}(n\cdot G)-\mathbf{RS}(n\cdot G)\le \mathbf{LS}(G-G^L).$$

That is, the width of the confusion interval of $n\cdot G$ is bounded. We also know that

$$\mathbf{LS}((k+m)\cdot G)\le \mathbf{LS}(k\cdot G)+\mathbf{LS}(m\cdot G).$$

This means that the sequence $\mathbf{LS}(n\cdot G)$ is a *subadditive* sequence. It is well known that for any subadditive sequence a_n, the limit as n tends to infinity of a_n/n exists and is either $-\infty$ or the lower bound of a_n/n. But dually in our case the sequence $\mathbf{RS}(n\cdot G)$ is *superadditive* and $\mathbf{RS}(n\cdot G)/n$ has a limit that is either ∞ or the upper bound of this sequence. However, $\mathbf{RS}(n\cdot G)\le \mathbf{LS}(n\cdot G)$ and their difference is bounded, so both the limits must actually exist and be a single common number, which we dub $m(G)$.

Now since the width of the confusion intervals for $n\cdot G$ are bounded, and $\mathbf{RS}(n\cdot G)\le nm(G)\le \mathbf{LS}(n\cdot G)$ for all n, there is a t such that $nm(G)-t<\mathbf{RS}(n\cdot G)\le \mathbf{LS}(n\cdot G)<nm(G)+t$ for all n, and so in particular,

$$n\,m(G)-t\le n\cdot G\le n\,m(G)+t,$$

as claimed. □

If we believe that the mean value $m(G)$ represents a fair value for G as a number, then we would hope that $m(G+H)=m(G)+m(H)$. Indeed this is the case.

Theorem 8.7. *For any two games G and H, $m(G+H)=m(G)+m(H)$.*

Proof: We know that

$$\begin{aligned}\mathbf{RS}(n\cdot G)+\mathbf{RS}(n\cdot H)\le \mathbf{RS}(n\cdot(G+H))&\le \mathbf{LS}(n\cdot(G+H))\\ &\le \mathbf{LS}(n\cdot G)+\mathbf{LS}(n\cdot H).\end{aligned}$$

If we divide by n and then take a limit as n tends to infinity we get

$$m(G) + m(H) \leq m(G + H) \leq m(G) + m(H),$$

exactly as we wanted. □

Exercise 8.8. In SUBTRACTION$(1, 2 \mid 2, 3)$, what is the approximate value of the disjunctive sum of 6 heaps of size 10, 20 of size 11, and 132 of size 31?

8.3 Cooling Things Down

The next problem is how to compute the value $m(G)$, as well as possibly obtain extra information about G, which can be useful in play. The proofs in the preceding section give us a method in principle for computing $m(G)$ but in practice they are not necessarily so useful. Unless we are in luck and can see some obvious pattern in the canonical forms of the games $n \cdot G$ (for example, if $4 \cdot G$ turns out to be a number as was the case in one of our preceding examples), then computing the limits required to evaluate $m(G)$ will be impracticable. The technique we require, which allows an efficient computation of $m(G)$, is called *cooling*, and to introduce it we can do no better than quote verbatim from *ONAG* [Con01, p. 102]:

> We can regard the game G as vibrating between its Right and Left values in such a way that on average its center of mass is at $m(G)$. So in order to compute $m(G)$ we must find some way of cooling it down so as to quench these vibrations, and perhaps if we cool it sufficiently far, it will cease to vibrate at all, and *freeze* at $m(G)$.

Consider the simplest sort of hot game, a switch such as $\pm 2 = \{2 \mid -2\}$. As both players are keen to get the opportunity to move first, they will (presumably) not object if we charge them some fraction of a move for the privilege. Suppose, for example, that we asked Left for a donation of one move to Right in exchange for the privilege of moving first. She would certainly agree to accept the offer since her move would still be to $2 - 1 = 1$, a position favorable to her. If we made the corresponding offer to Right, he too would be happy to accept. However, suppose that we asked for a donation of 3 moves to Right in exchange for the first move. Now Left would scornfully decline, since we are giving her the opportunity to move to $2 - 3 = -1$.

The matched pair of offers "you can move first in G if you donate t moves to your opponent" represents an attempt to reduce the heat of battle. For the moment we will say this represents cooling G by t (we will see shortly that to define this operation precisely we need to be a little more careful). For example, if we play the sum of ± 2 and the same game cooled by $1/2$ then, because of the reduced incentive to play in the latter component, the first move will occur in ± 2.

Now instead of ± 2 let's look at the game $\{4 \mid 2\}$. If we adopt the naive definition of cooling suggested above, then what happens when we cool the game by say, 10? We would get $\{-6 \mid 12\} = 0$. However, if we cool by exactly 1 we would get $3*$, and if we cool by anything just slightly larger than 1 we would get 3 (and two players who were unhappy about accepting the offer). Once a game has been cooled enough so that it is infinitesimally close to a number then neither player will be willing to play if we continue to cool it further — we recognize this by saying that the game has been *frozen*, and agree to declare at this point that cooling it further has no effect. We also define the *temperature* of G to be the smallest amount of cooling required to freeze it (in other words, we think of all games having the same "freezing point"). We denote the result of cooling a game G by t as G_t. So

$$(\{4 \mid 2\})_t = \begin{cases} \{4 - t \mid 2 + t\} & \text{if } t < 1, \\ \{3 \mid 3\} = 3* & \text{if } t = 1, \\ 3 & \text{if } t > 1. \end{cases}$$

Consider now the game $g = \{\{4 \mid 2\} \mid -2\}$. In isolation, g does not look very different from ± 2. After all, if Left moves first, then after Right's reply she'll be two moves ahead, while if Right moves first he will wind up two moves ahead. However, the fact that Right needs to make an extra move in order to restrict Left's gains (in other words, the fact that Left's option was to a hot game rather than a number) has a significant effect on matters when we play sums of g with other games. For example, while $\pm 2 + \pm 2 = 0$, $g + g = \{2 \mid 0\}$, and $g + g + g + g = 2$. This last equation in particular implies that the mean value of g is $1/2$. Can we capture the mean value by looking at the results of cooling g by various amounts? Since cooling is a universal process when we cool g we must also cool any of its unfrozen options. So, for small values of t (we will see how small in a moment)

$$g_t = \{-t + \{4 - t \mid 2 + t\} \mid -2 + t\} = \{\{4 - 2t \mid 2\} \mid -2 + t\}\,.$$

For $t < 1$ the left option is still hot and the right option is certainly smaller, so this equation holds for such t. For $t = 1$ the left option becomes $2*$ and thereafter freezes. So for $t = 1 + s$ (and s reasonably small)

$$g_t = \{-s + 2 \mid -1 + s\}\,.$$

Now things finally freeze at $s = 3/2$ with

$$g_{5/2} = (1/2)*.$$

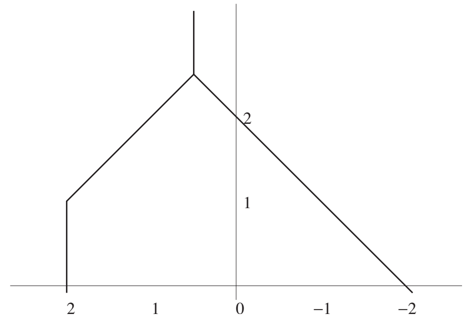


Figure 8.1. Thermograph of $g = \{4 \mid 2 \parallel -2\}$.

So the frozen value of g is $1/2$ exactly matching its mean value. Furthermore, the temperature of g is $5/2$.

The results of cooling g by various amounts are conveniently plotted on its *thermograph.* In the thermograph, at height t we plot both $\mathbf{LS}(g_t)$ and $\mathbf{RS}(g_t)$. For historical reasons, the horizontal axis of a thermograph is reversed from the standard orientation. That is, values more favorable to Left (i.e., larger values) are placed at the left-hand side, rather than the right. The thermograph of g is shown in Figure 8.1.

In order to prove properties about cooling, it turns out to be simplest (arguably!) to first define the thermograph of a game inductively, and then to derive properties about cooling from that. The thermograph of any game will be a set of points in the plane. For each value of $t \geq 0$ the set will contain two points, which may coincide: $(\mathbf{LS}(G_t), t)$ and $(\mathbf{RS}(G_t), t)$. The notation we use here for these points will be justified *post facto*. The set of points $(\mathbf{LS}(G_t), t)$ for $t \geq 0$ will form a path whose segments are either vertical or slanting on a $45°$ angle upwards and to the right finishing in a vertical mast. Likewise, the set of points $(\mathbf{RS}(G_t), t)$ for $t \geq 0$ will form a path whose segments are either vertical or slanting on a $45°$ angle upwards and to the left finishing in the same vertical mast.

So, let G be a game, and suppose that we have access to the thermographs of all the options of G, and that these have the characteristics we have mentioned. We define the thermograph of G as follows:

- If G is a number x, then $(\mathbf{LS}(G_t), t) = (\mathbf{RS}(G_t), t) = (x, t)$ for all t.

- Otherwise, take $\mathbf{LS}(G_t)$ to be the maximum of $\mathbf{RS}(G^L_t) - t$ and $\mathbf{RS}(G_t)$ to be the minimum of $\mathbf{LS}(G^R_t) - t$ *unless* this would require $\mathbf{LS}(G_t) < \mathbf{RS}(G_t)$. In that case, $\mathbf{LS}(G_t) = \mathbf{RS}(G_t) = \mathbf{LS}(G_u)$ where u is the smallest value for which $\mathbf{LS}(G_u) = \mathbf{RS}(G_u)$.

Such an involved definition needs to be checked out in order to ensure that it works properly. The first part is simple enough (and clearly satisfies the characteristics we have specified). So, suppose that the second part of the definition is in force. Under the assumptions made, all the segments $\mathbf{RS}(G_t^L)$ are vertical or upwards left. So when we subtract t from them, we get segments that are upwards right or vertical (remember the axis is reversed from its conventional orientation). This property is preserved by taking the maximum. The same remark applies to the $\mathbf{LS}(G_t^R)+t$. So that gives us two segments, each finishing in a diagonal. These segments intersect for the first time at some height and the remainder of the definition says that we form the thermograph of G at and above that height, by finishing with a vertical mast. The horizontal coordinate defined by this mast is called the *mast value* of G.

Now let us try and define how to cool a game G by t in order to obtain G_t. We set inductively

$$G_t = \left\{G_t^L - t \;\middle|\; G_t^R + t\right\}$$

unless there is some $u < t$ where G_u has the property that

$$\mathbf{LS}(G_u) = \mathbf{RS}(G_u) = x$$

in which case we define $G_t = x$.

This definition works, precisely because our consideration of the thermograph (which we can see plots the left and right stops of G_t) guarantees that there will be a smallest such u.

We will try to consider these two definitions in parallel with respect to the following example:

$$G = \{2, \{4 \mid 1\} \mid \{-1 \mid -2\}\}.$$

First, we superimpose the thermographs of 2 and $\{4 \mid 1\}$ and highlight the *leftmost right boundary* (see Figure 8.2).

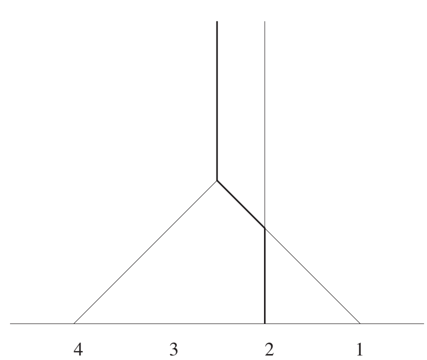


Figure 8.2.

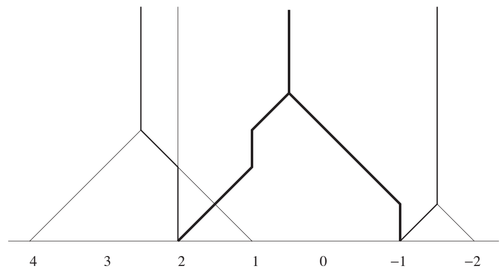


Figure 8.3.

That boundary, shifted rightwards by t units at height t, will form the left boundary of the final thermograph, up to the point where it intersects with the right boundary. The right boundary is simply obtained from the left boundary of thermograph of $\{-1 \mid -2\}$ shifted leftwards by t units at height t. The finished thermograph is shown as the highlighted central part in Figure 8.3.

Now let us consider the inductive definition of cooling G by t. For sufficiently small t,

$$G_t = \{2-t, \{4-t \mid 1+t\} - t \mid \{-1-t \mid -2+t\} + t\}\,.$$

How small is "sufficiently small"? Exactly until some pair of left and right stops of G_t, or one of its subgames, coincides. By inspection, this occurs when $t = 1/2$. For $t \leq 1/2$, $\mathbf{LS}(G_t) = 2 - t$ and $\mathbf{RS}(G_t) = -1$. When we cool by more than $1/2$, the right-hand option of G freezes and the right stopping value becomes $-3/2 + t$. The next important temperature is $t = 1$. At this point, the two left options of G_t are 1 and $\{3 \mid 1\}$. Up to this point, in computing $\mathbf{LS}(G_t)$, Left has preferred the move in the first option. However, for $t > 1$, Left will prefer to move in the second option as its right stop remains at 1, whereas the right stop of $2-t$ becomes smaller than 1. At $t = 3/2$ the hot left option freezes. Thereafter, the left stopping value becomes $5/2-t$. Finally, the left and right boundaries collide when $5/2-t = -3/2+t$ (that is, at $t = 2$) and the game freezes entirely at this point. Collecting this information we obtain

$$G_t = \begin{cases} \{2-t, \{4-2t \mid 1\} \mid \{-1 \mid -2+2t\}\} & \text{for } 0 \leq t \leq 1/2, \\ \{2-t, \{4-2t \mid 1\} \mid -3/2+t\} & \text{for } 1/2 < t \leq 1, \\ \{\{4-2t \mid 1\} \mid -3/2+t\} & \text{for } 1 < t \leq 3/2, \\ \{5/2-t \mid -3/2+t\} & \text{for } 3/2 < t \leq 2, \\ 1/2 & \text{for } t > 2. \end{cases}$$

We see that the thermograph reflects exactly the left and right stopping values of G_t (as indeed it must!).

To continue with the development of the theory of thermography, we introduce one more notion. Any game G is really just a tree, whose root is the game itself, and which has leftward pointing branches to the left options of G and rightward pointing branches to the right options. The leaves of such a tree are the terminal positions in which neither player can move. Ordinary induction on games can be viewed as an induction on these trees — based on an inductive hypothesis that the result holds for all subtrees of a given tree. Another tree that we can associate with a game is the *stopped tree.* In the stopped tree we recognize that when play has proceeded to a point where the value is a number, neither player will be terribly interested in continuing. So, instead of completing the tree we simply label such a vertex with the number that is its value. So now the leaves of the tree simply carry numbers.

Given a stopped tree it is trivial to work out the left and right stops of any of its internal vertices as the left and right stops of a number are the number itself, and otherwise we can use the usual inductive rule. Likewise, the stopped tree of G provides a simple means of computing (the stopped tree of) G_t. At each leaf v, count the excess of left branchings over right branchings when we follow the path from the root to that leaf. Call this number e_v. Now replace the label n_v on that leaf by $n_v - e_v x$. We will think of x as a variable temperature which we will allow to run from 0 through t. In the simplest case, if we simply replaced x by t this new tree would represent the value obtained when G_t is played out to this leaf. However, we know that cooling is not quite so simple and that for actual concrete values of t we must not allow overcooling.

So we think of the cooling process as taking place in a series of phases. One type of phase will be called *congealing.* In a congealing phase, any internal vertex whose left and right stops are now equal is replaced by the number equal to those stops (note that this number might be an expression like $2-x$). The other type of phase will simply be allowing the value of x to increase by some amount (actual cooling). A *critical temperature* is a value of x at which congealing can take place.

It is easy to check, though there is a *little* thought required, that any critical temperature is a dyadic rational.

Now, the point is that G_t is just the game that we get by alternately congealing and cooling, pausing for a congealing step each time we reach (and intend to exceed) a critical temperature. This is essentially exactly the description of the cooling process that we followed in the example where

$$G = \{2, \{4 \mid 1\} \mid \{-1 \mid -2\}\}.$$

From the description of the congealing process it is clear that the stopped tree of G_t is always a subtree of the stopped tree of G for any temperature $t \geq 0$. This gives us an inductive tool to prove the following critical results about cooling.

Lemma 8.9. *Let X and Y be games, and suppose that t is a sufficiently small real number that all of X_t, Y_t, and $(X+Y)_t$ can be computed by the inductive definition. Then $X_t + Y_t = (X+Y)_t$. If any of X, Y, or $X+Y$ is a number, then this result is also true.*

Exercise 8.10. Prove the lemma in the case that none of X, Y, and $X+Y$ is a number. You are asked to prove the other cases in Problem 10.

Theorem 8.11. *Cooling is an additive function on games. That is, for any games G and H, and any $t \geq 0$, $G_t + H_t = (G+H)_t$.*

Proof: Contrary to our usual practice we will give this inductive proof in terms of a minimal counterexample argument. That is, suppose that the result were false. Then there would be a counterexample $(G, H, G+H)$ with the sum of the sizes of the stopped trees of G, H, and $G+H$ minimized.

In this counterexample, none of the three parts can be a number by the lemma above.

So consider starting the cooling process. If a congealing phase is needed, to begin with we obtain G', H', and $(G+H)'$. Do we know that $G' + H' = (G+H)'$. Yes we do, because congealing can be thought of as "cooling by an infinitesimal." For such cooling the inductive definition of the cooling operation is always applied and the lemma above shows that this is additive. However, we now know that $(G', H', G'+H')$ is another counterexample to the theorem, and the total size of the stopped trees has decreased.

Thus, we can cool at least until the smallest critical temperature of G, H, and $G+H$ without violating the theorem. However, this leaves us in exactly the same situation as above. That is, after cooling by this amount we will congeal. At least one of the trees will become smaller, and we will still have a counterexample. $\square$

Finally, we are in a position to illustrate the connection between cooling and the mean value of a game.

Theorem 8.12. *The mast value and the mean value of a game G are the same.*

Proof: Let $M(G)$ denote the mast value of G. Then, for all sufficiently large t, $M(G) = G_t$. By the additivity of cooling, for all positive integers n and all sufficiently large t, $nM(G) = (n \cdot G)_t = M(n \cdot G)$. However, the mast value of any game lies between its left and right stops, so we have

$$\mathbf{LS}(n \cdot G) \leq nM(G) \leq \mathbf{RS}(n \cdot G).$$

Dividing this inequality by n and taking the limit as n tends to infinity gives

$$m(G) \leq M(G) \leq m(G);$$

that is, $m(G) = M(G)$ as claimed. $\square$

The significance of this theorem is that the mast value of G can be easily computed simply by computing the thermograph of G, and the description above provides a simple and efficient algorithm for performing that computation.

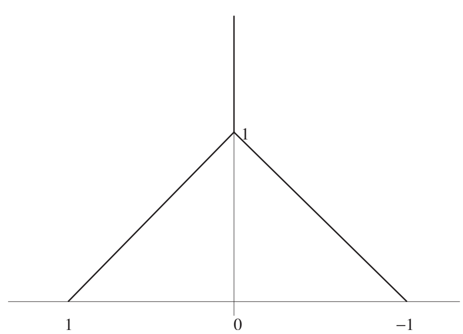
Proposition 8.13. *Let G and H be games. The temperature of $G + H$ is less than or equal to the maximum of the temperatures of G and H.*

Proof: The temperature $t(X)$ of a game X is the least real number that has the property that for all $t > t(X)$, X_t is a number. Suppose that $t > \max(t(G), t(H))$. Then $(G + H)_t = G_t + H_t$, which is the sum of two numbers and so is a number. Thus, $t(G + H) \leq \max(t(G), t(H))$ as claimed. □

Note that in the situation described above it is certainly possible that $t(G + H) < \max(t(G), t(H))$. The most obvious case is when $G = -H$. Problem 9 asks you to show that if $t(G) \neq t(H)$ then $t(G + H) = \max(t(G), t(H))$.

8.4 Strategies for Playing Hot Games

Thermographs are quite helpful for playing sums of hot games. Be warned, however, that different games can have the same thermograph. For example, $X = \pm 1$, $Y = \{1, 1 \pm 1 \mid -1, -1 \pm 1\}$ (and even $X + Y$) all have the thermograph given by



And yet, on $-1 + X + Y$, the only winning move by Left is on Y. Therefore, the thermograph alone cannot dictate optimal play. Phrased differently, canonical form is *the* information required to know how to play a game in any context; thermographs lose some of this information.

We already know from Chapter 5 that it is NP-hard to decide the value of the sum of games of the form $\{a \mid \{b \mid c\}\}$. Yet, if we know the thermographs of all components, they can be used to guide play. In other hot games (such as AMAZONS and GO), it is quite difficult, indeed, to compute the full thermograph of each component.[3] However, even in those cases, we can often estimate the thermograph or calculate the temperature (but not the thermograph). The central question is how should we now interpret the information given by temperature and thermography?

Here, we only give a brief description, a manual of "how-to" with appropriate warnings. For more on the real "how?" and "why?" the reader should consult [BCG01, ML06, Ber96].

Three main strategies that use temperature have been identified.[4] One of these strategies, Thermostrat, requires a little additional machinery:

Definition 8.14. Given the sum $A + B + C + \ldots$, let

$$W_t = \max\{W_t(A), W_t(B), W_t(C), \ldots\},$$

where $W_t(X) = \mathbf{LS}(X_t) - \mathbf{RS}(X_t)$ is the width of the thermograph of X at temperature t. The *compound thermograph* of the sum has right boundary $\mathbf{RS}(A_t) + \mathbf{RS}(B_t) + \mathbf{RS}(C_t) + \cdots$ and left boundary $\mathbf{RS}(A_t) + \mathbf{RS}(B_t) + \mathbf{RS}(C_t) + \cdots + W_t$. The *ambient temperature* of $A + B + C + \ldots$ is the least T for which $\mathbf{RS}(A_T) + \mathbf{RS}(B_T) + \mathbf{RS}(C_T) + \ldots + W_T$ is maximal. That is, the height of the lowest horizontal line whose leftmost intersection with the compound thermograph is farthest left.

While the underlying thermographs have slopes of 1, 0, and -1, compound thermographs can have integral slopes. (Slopes are viewed with the thermograph turned 90 degrees, for temperature is the independent variable.)

There are three standard strategies based on temperature:

Hotstrat: Move in a component whose temperature is maximal. While the strategy is simple and intuitive, there are no theorems guaranteeing good performance.

Thermostrat: Find the ambient temperature T and move in the component whose thermograph is widest at T. This strategy is guaranteed to achieve an outcome within the temperature of the mean; in fact *WW* [BCG01] uses Thermostrat to give a constructive proof of Theorem 8.12. This strategy is used when the thermographs of each component can be easily calculated.

Sentestrat: This time, define the *ambient temperature* as the lowest temperature of any of your previous moves; it starts out infinite. If your opponent

[3]In computer analysis of games, a more urgent question, one we do not even mention, is to first identify the components of the sum and this is not necessarily easy.

[4]Computer scientists have many other strategies at their disposal for identifying reasonable moves. Here, we are concerned with provably good strategies for playing in disjoint sums of games.

has just moved in a component leaving it at a temperature above the ambient, respond in that component; otherwise play Hotstrat. Sentestrat is guaranteed to do well in the context of enough switch games $\pm x$ for numbers x. Since Sentestrat requires only the temperature of each component (and not its full thermograph), it is widely applicable and should most often be the preferred strategy.

Exercise 8.15. Consider play on two components $P + Q$, where $P = \{\{20 \mid 0\} \mid -1\}$ and $Q = \{2 \mid -2\}$.

1. Does Hotstrat advise optimal play when Left moves first? When Right moves first?

2. Answer the same question for Thermostrat.

Exercise 8.16. There is asymmetry in the description of Sentestrat, for it defines the ambient to be the lowest temperature of any of *your* previous moves (and not your opponent's). Why does this make sense?

8.5 Norton Products

We understand numbers and many infinitesimals, but do not understand many hot games. Hence, we often attempt to associate the hot positions from a particular game with numbers or tepid games in a one-to-one fashion in the hopes of gaining insight into the hot game. (Recall from Definition 6.21 on page 127 that a tepid game is the sum of a number and a non-zero infinitesimal.)

But cooling by t is a many-to-one function. For instance, all infinitesimals cool to 0 when cooled by any $x > 0$. So there are many ways to add heat to a game in a way that inverts cooling. We will focus on one such way, the Norton product, since it is guaranteed to be linear and order-preserving. The serious practitioner will also want to study more general ways to invert cooling such as the *heating* and *overheating* operators defined in *WW* [BCG01, pp. 167–178]. Like Norton products, these can also provide a terse representation of an otherwise complex game. Although well-behaved in practice, heating and overheating are not a priori guaranteed to be linear.

Definition 8.17. Let G be any game, and let *unit* U be any positive game in canonical form and let

$$\tau = U + \mathcal{I},$$

where $\mathcal{I}$ are the incentives from U. Then, the *Norton product* of G by U is given by

$$G \cdot U = \begin{cases} \overbrace{U + U + U + \cdots + U}^{n} & \text{if } G = n \text{ is a positive integer,} \\ 0 & \text{if } G = 0, \\ \overbrace{-U - U - U - \cdots - U}^{n} & \text{if } G = -n \text{ is a negative integer,} \\ \left\{\mathcal{G}^L \cdot U + \tau \;\middle|\; \mathcal{G}^R \cdot U - \tau\right\} & \text{otherwise.} \end{cases}$$

Note that while $\mathcal{I}$ and therefore τ are both sets of games, in most applications τ is a singleton. We will refer to a typical element of τ by t.

Example 8.18. Define the unit $U = x*$, where x is a number. In this case, the left and right incentives are both $*$, and there is only one game $t \in \tau$, that being $t = x* + * = x$. So we get

$$\begin{aligned} n \cdot x* &= \begin{cases} nx & \text{if } n \text{ is even,} \\ nx* & \text{if } n \text{ is odd;} \end{cases} \\ * \cdot x* &= \{0 + t \mid 0 - t\} = \{x \mid -x\}\,; \\ \uparrow \cdot x* &= \{0 + t \mid \{t \mid -t\} - t\} = \{x \,\|\, 0 \mid -2x\}; \\ \frac{1}{2} \cdot x* &= \{0 + t \mid t* - t\} = \{x \mid *\}\,. \end{aligned}$$

Norton product by $1*$ appears prominently in one-point GO endgames [BW94]. Other cases of Norton products by $U = x*$ also appear. Norton product by $U = \uparrow$ is of special importance, and we will explore its use in Chapter 9.

Exercise 8.19. List $\frac{1}{2^n} \cdot 1*$ for integers $n \geq 0$ until the pattern is clear.

Ignoring base cases, when cooling by t each player is taxed t for moving. In a Norton product, each player benefits by t for each move. To show that Norton products are well-behaved, we will echo the proof technique we last saw in Lemma 5.16 on page 92.

Lemma 8.20. *For unit $U > 0$,*

$$\begin{aligned} A + B + C \geq 0 \quad &\Longleftrightarrow \quad A \cdot U + B \cdot U + C \cdot U \geq 0 \textit{ and} \\ A + B + C \leq 0 \quad &\Longleftrightarrow \quad A \cdot U + B \cdot U + C \cdot U \leq 0. \end{aligned}$$

Proof: In essence, play on $A+B+C$ mirrors play on $A \cdot U + B \cdot U + C \cdot U$ except that on each play, a quantity $t \geq 0$ changes sides (or, more exactly, quantities τ change sides.) Problem 12 asks you to prove that $t \geq 0$.

It suffices to prove the following stronger pair of assertions:

$$\text{If } A+B+C \mathrel{|\!\!\triangleright} 0 \quad \text{then} \quad A\cdot U + B\cdot U + C\cdot U - t \mathrel{|\!\!\triangleright} 0 \tag{8.3}$$

$$\text{If } A+B+C \geq 0 \quad \text{then} \quad A\cdot U + B\cdot U + C\cdot U \geq 0 \tag{8.4}$$

For we could equally well have also proved the symmetric pair of assertions with $\mathrel{|\!\!\triangleright}$ replaced by $\mathrel{\triangleleft\!\!|}$ and $\geq$ replaced by $\leq$, and together the four resulting "If... then..." statements imply the lemma.

We mean for (8.3) to hold for every $t \in \tau$. Now (8.3) and (8.4) do yield the first " $\iff$ ", since every $t \geq 0$, (8.3) implies, "if $A+B+C \mathrel{|\!\!\triangleright} 0$ then $A\cdot U + B\cdot U + C\cdot U \mathrel{|\!\!\triangleright} 0$." By symmetry, we can change $\mathrel{|\!\!\triangleright}$ to $\mathrel{\triangleleft\!\!|}$ in both places; then taking the contrapositive yields, "if $A\cdot U + B\cdot U + C\cdot U \geq 0$ then $A+B+C \geq 0$."

We will now proceed to prove both (8.3) and (8.4) in tandem by induction.

To prove the assertion given in (8.3), assume $A+B+C \mathrel{|\!\!\triangleright} 0$ and so some $A^L + B + C \geq 0$. By the *Number-Avoidance Theorem,* we may assume that players move on non-integers when available. So, without loss of generality, either A is a non-integer, or A, B, and C are all integers. If A is a non-integer, then Left has the same move available from $A\cdot U + B\cdot U + C\cdot U - t$ to $A^L\cdot U + B\cdot U + C\cdot U$, which she wins moving second by induction. If, however, all three are integers, then $A+B+C$ is some integer $n \geq 1$, and Left wins moving first on n copies of the positive game U.

For (8.4), if A is a non-integer then Right's move from $A\cdot U + B\cdot U + C\cdot U$ to $A^R\cdot U + B\cdot U + C\cdot U - t$ loses by induction since $A^R + B + C \geq 0$. If, on the other hand, A is an integer, then if $A < 0$, Right's move is to some

$$(A-1)\cdot U + U^R + B\cdot U + C\cdot U = (U^R - U) + A\cdot U + B\cdot U + C\cdot U.$$

If, on the other hand, $A > 0$, Right's move is to some

$$(A+1)\cdot U - U^L + B\cdot U + C\cdot U = (U - U^L) + A\cdot U + B\cdot U + C\cdot U.$$

Since $U^R - U$ and $U - U^L$ are negative incentives, these each equal some

$$(A+1)\cdot U + B\cdot U + C\cdot U - t$$

for some $t \in \tau$. Now, $(A+1)+B+C > A+B+C \geq 0$, and so by induction $(A+1)\cdot U + B\cdot U + C\cdot U - t \mathrel{|\!\!\triangleright} 0$. □

Exercise 8.21. The last inductive step appears backwards since it uses a statement about $A+1$ to prove one about A. Explain why the proof is just fine.

Theorem 8.22. *The Norton product satisfies:*

Independence of form: *If $A = B$ then $A\cdot U = B\cdot U$.*

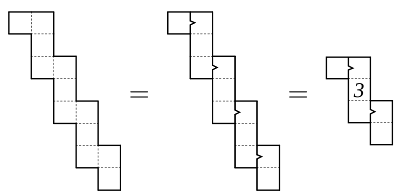
Monotonicity: $A \geq B$ *if and only if* $A \cdot U \geq B \cdot U$.

Distributivity: $(A + B) \cdot U = A \cdot U + B \cdot U$.

Proof: All three facts follow directly from Lemma 8.20. □

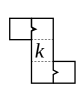
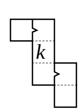
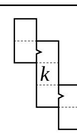
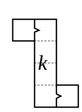
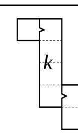
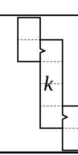
Domineering snakes

Some snaky DOMINEERING repeating patterns are easily described by the Norton product by $\frac{1}{2}*$ [Wol93]. First, we introduce some handy pictorial notation:



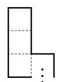
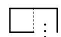
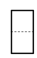
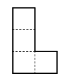
The little notches show how pieces fit together, and the repeated piece is shown with an integer indicating how many copies of the piece are required to form the snake.

The following table summarizes the values that appear when playing on these two snakes:

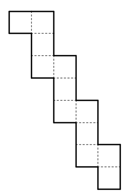
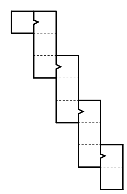
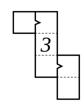
	$\boldsymbol{\int g \stackrel{\text{def}}{=} g \cdot \frac{1}{2}*}$					
$\boldsymbol{k}$						
0	$\int -2$	$*$	$1\|0$	$\int -2$	$*$	$1\|0$
1	0	$\int \frac{1}{2}$	$\int 2$	0	$\int \frac{3}{2}$	$\int 2$
2	$*$	$*\int \frac{5}{4}$	$*\int 2$	$*\int 2$	$*\int \frac{11}{4}$	$*\int 4$
3	$\int 1$	$\int 2$	$\int \frac{5}{2}$	$\int 3$	$\int 4$	$\int \frac{11}{2}$
4	$*\int \frac{3}{2}$	$*\int \frac{5}{2}$	$*\int \frac{13}{4}$	$*\int \frac{9}{2}$	$*\int \frac{11}{2}$	$*\int \frac{27}{4}$
5	$\int 2$	$\int 3$	$\int 4$	$\int 6$	$\int 7$	$\int 8$
6	$*\int 3$	$*\int \frac{7}{2}$	$*\int \frac{9}{2}$	$*\int 7$	$*\int \frac{17}{2}$	$*\int \frac{19}{2}$
	period 5, saltus $*\int 3 = \frac{3}{2}$			period 5, saltus $*\int 7 = \frac{7}{2}$		

When reading the table, exceptions to the period information are shown above a horizontal line. A few entries are omitted, since appropriately tacking on

tends to add 1 or $\int 2 = 2 \cdot \frac{1}{2}*$:

 $=$  $+$  *except*  $= \dfrac{1}{2}.$

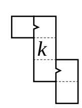
As an example of how to use the table, consider the position we saw at the start of the section:

 $=$  $=$ 

This position appears in the 2$^{\text{nd}}$ column with $k = 3$, and so has value $2 \cdot \frac{1}{2}* = 1$. Were it the case that $k = 8$, the value would be

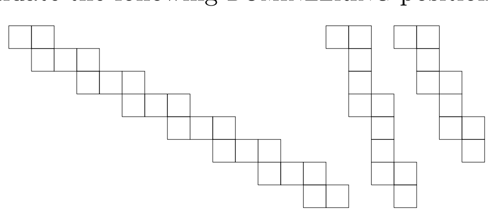
$$\int 2 + * \int 3 = * \int 5 = * + 5 \cdot \frac{1}{2}* = \frac{5}{2}.$$

Proving that the table is correct is mechanical but quite tedious. For example, suppose we wish to confirm the entry



for $k \equiv 3 \pmod 5$. So, suppose that $k = 5(j+1)+3$ for some integer $j \geq 0$. We need to describe all the possible left and right options from this position, which are typically sums of pairs of entries from the table plus a multiple of the saltus. Assuming that the values provided for the options are correct, by induction, we can confirm that the value provided for the original position is correct. Perhaps the proof is best left to the computer.

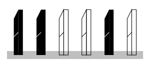
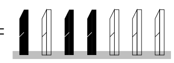
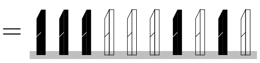
Exercise 8.23. Evaluate the following DOMINEERING position:



Problem 15 asks you to find all winning moves.

Problems

1. In TOPPLING DOMINOES find the canonical forms, means, and temperatures for

   $A =$  , $B =$  , and $C =$ 

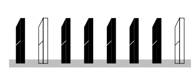
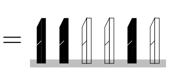
   What are the left and right stops for $A + B + C$?

2. The canonical form of $A + B + C$ from Problem 1, is given by

$$\left\{ \begin{array}{c} \{\frac{7}{2} \mid 2 \parallel 1 \mid -1/2\}, \\ \{\frac{7}{2} \mid 2 \mid\mid\mid 1 \mid \frac{1}{2} \parallel -\frac{1}{2} \mid -1\} \end{array} \middle| \begin{array}{c} \{1 \mid -1/2 \mid\mid\mid -\frac{3}{2} \mid -2 \parallel -3 \mid -\frac{7}{2}\}, \\ \{1 \mid \frac{1}{2} \parallel -\frac{1}{2} \mid -1 \mid\mid\mid -\frac{3}{2} \mid -2 \parallel -3 \mid -\frac{7}{2}\} \end{array} \right\}.$$

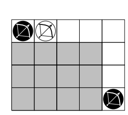
   Find the thermograph of $A + B + C$. (Do not be put off by the size of the canonical form; some positions appear more than once.) While you should show your work, feel free to check your answer in CGSuite using, for instance, `Plot(Thermograph({1|-1/2}))`.

3. In TOPPLING DOMINOES, with

   $G =$  and $H =$ 

   find the canonical forms, stops, and thermographs for G, H, and $G+H$.

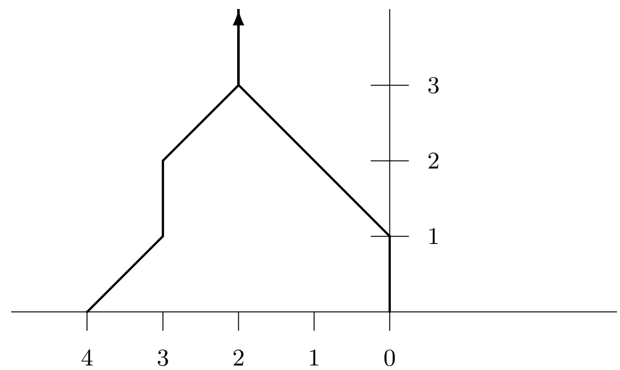
4. Find the stops, confusion interval, and the mean value and temperature for the AMAZONS positions



5. (Open ended) Practice calculating means and temperatures of hot games until you become proficient. Begin with games of the form $a \parallel b \mid c$. See what happens when you add an option. Use CGSuite to check your work. Try to find games of temperature 1/2, 1/4, 3/8, etc.

   Select two or three computations and write them up carefully so it is clear that you understand how to properly construct thermographs.

6. (a) Find a game with the following thermograph:



(b) How can *any* such game G, with the above thermograph, relate to zero? Be specific and explain your answer (i.e., can a game G with the above thermograph be greater than zero? Can another be less than zero? Equal? Incomparable?).

7. Find the stops and the confusion interval for AMAZONS positions of the form  and show that the mean value is always $-\frac{1}{4}$.

8. The subtraction game SUBTRACTION$(1, 4, 10 \,|\, 4, 10, 13)$ is purely periodic with period 14. The first period is: 0, 1, 2, 3, $\{3 \,|\, 0\}$, $1*$, $2*$, $3*$, 3, $\{3 \,|\, 1*\}$, $\{2*, \{3 \,|\, 1*\} \,|\, 0\}$, $\{3* \,|\, 1\}$, $\{3 \,|\, 2\}$, $\{3, \{3 \,|\, 2\} \,|\, 0\}$. Suppose you play a game G consisting of 50 heaps, 10 each of of sizes 10, 20, 30, 40, and 50. Determine numbers x and ϵ so that $x - \epsilon \le G \le x + \epsilon$ and prove your answer. How small can you make ϵ?

9. Show that if $t(G) \ne t(H)$ then $t(G + H) = \max(t(G), t(H))$.

10. Prove Lemma 8.9 in the case that at least one of X, Y, and $X + Y$ is a number.

11. Let U be a Norton unit; i.e., a positive game in canonical form.

    (a) For all games G, prove that if U is all-small, so is $G \cdot U$.

    (b) Prove that if U is infinitesimal, so is $G \cdot U$.

12. Let U be a Norton unit; i.e., a positive game in canonical form. Let $\tau = U + \mathcal{I}$ as in Definition 8.17 on page 175. Prove that all the values of τ are at least 0. (*Hint*: A typical $t \in \tau$ is either $U + U^L - U$ or $U + U - U^R$. You can use the fact that U has no reversible nor dominated options to show each $t \ge 0$.)

13. In Problem 12, if $t = 0$, find, with proof, all possible values for U?

14. In the definition of Norton product, confirm that $U > 0$ is a necessary condition for all three assertions in Theorem 8.22. In particular, pretend that $U = *$. For each of the three assertions of the theorem, find games A and B violating the assertion. For your monotonicity and distributivity counterexamples, A and B should be in canonical form.

15. Find several winning moves for Left from the position in Exercise 8.23 on page 179. (There are seven winning moves in total.)

Preparation for Chapter 9

Prep Problem 9.1. Review reversibility and rederive the canonical form of $\Uparrow = \uparrow + \uparrow$ and of $\uparrow * = \uparrow + *$ from scratch. If you find yourself having to refer to Section 5.2, you can also try $\Uparrow\!\!\uparrow$.

To the instructor: The game of HACKENBUSH is ideal for covering integer atomic weights. Start with flower gardens from *WW* [BCG01, Ch. 7]. An advanced group of students will be unsatisfied without seeing proofs of atomic weights from *WW* [BCG01, Ch. 8].

Chapter 9

All-Small Games

It has long been an axiom of mine that the little things are infinitely the most important.

Sir Arthur Conan Doyle in The Adventures of Sherlock Holmes: A Case of Identity

If a game and all of its positions have the property that either both players have a move or neither one does, then the game is called *all-small* (see page 101). Typical all-small games are those, such as ALL-SMALL CLEAR THE POND, where players are attempting to race their pieces off the board, and which end when one player has successfully done so, or games such as CLOBBER and CUTTHROAT STARS where the objective is to eliminate all of your opponent's resources, and each of your moves must contribute to that elimination.

All-small games form a subset of the infinitesimal games and arise frequently as parts of positions in larger games. In practice, all-small games are often quite subtle and challenging in part because complex canonical forms abound, and it is difficult for the amateur to understand who has the advantage and by how much in each local position.

In such all-small games, it is often the case that many of the options are reasonably close in value. By carefully establishing what we mean by *close*, we can define an equivalence relation on the class of all-small games and assign to each equivalence class a descriptive value called its *atomic weight.* The atomic weight may be used as a close approximation of the game's real value and, in many instances, determines the outcome class of the game and provides a good move for the winner. While the results in this chapter are motivated by the analysis of all-small games, many also extend to other infinitesimals.

9.1 Cast of Characters

Let us repeat the definition of an all-small game:

Definition 9.1. A game G is *all-small* if and only if either $G = \{\,|\,\}$ or both $\mathcal{G}^L$ and $\mathcal{G}^R$ are non-empty and all the elements of $\mathcal{G}^L$ and $\mathcal{G}^R$ are all-small. Equivalently, every position of G is either terminal or has both left and right options.

As stated, whether or not a game is all-small depends on its form and not just its value. For instance, while $\{\,|\,\}$ whose value is 0 is all-small, the game $\{-1 \mid 1\}$, which has the same value, is not all-small. In practice, this distinction is spurious since we are interested in playing games well — and playing games well depends only on knowing their values, not their forms. So we will not pay any attention to this distinction in the sequel. Further justification for this decision can be found in Problem 13.

Any all-small game, G, is infinitesimal, since if x is a positive number, Left can win $G+x$ by playing blindly in G so long as she has a move there. At some point, either Right will have lost, or Left will have no further options in G; but the remaining game will be some number $x' \geq x$ (where inequality might occur if Right has foolishly ignored the *Number-Avoidance Theorem* at some point in the play). So, for any positive real number x, $G+x > 0$, and dually $G-x < 0$. But not all infinitesimals are all-small; for example, $+_2 = \{0 \mid \{0 \mid -2\}\}$ is not all-small, since from -2 Right has a move but Left does not. Nor, according to our comment above, is $+_2$ equal to any all-small game since its canonical form is not all-small. We have seen that $\uparrow$, $*$, and $\uparrow *$ are all-small, and so are all impartial games.

The game $\uparrow$ and the nimbers play central roles in the theory surrounding all-small games, so before attacking that theory headlong we will do some warming up by considering properties of various games defined from them.

The game $\uparrow * n$

Our first warm-up routine will be to compute the canonical forms of sums of $\uparrow$ and nimbers.

Lemma 9.2. *Let n be a non-negative integer. Then $\uparrow * n > 0$ if and only if $n \neq 1$.*

Proof: We have long since dealt with the cases $n = 0$ and $n = 1$. Suppose that $n > 1$ and consider $G = \uparrow * n$. Moving first, Left can move to $\uparrow$ and win. Right's options in G are: $\uparrow * n'$ for some $n' < n$ which is in either $\mathcal{L}$ or $\mathcal{N}$, and $*(n \oplus 1)$ which is also in $\mathcal{N}$. So, Left wins moving either first or second, and thus $\uparrow * n > 0$. $\square$

Corollary 9.3. *Let n and m be non-negative integers. Then $\uparrow{*n} > {*m}$ if and only if $m \neq n \oplus 1$.*

Exercise 9.4. Prove the corollary.

We already know the canonical forms:

$$\begin{aligned} \uparrow &= \{0 \mid *\}\,; \\ \uparrow* &= \{0, * \mid 0\}\,. \end{aligned}$$

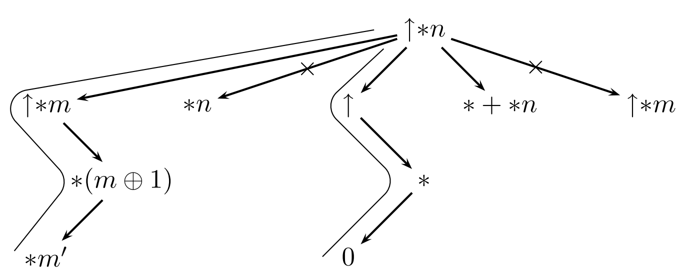
The following lemma completes the catalog of canonical forms for the games $\uparrow{*n}$.

Lemma 9.5. *The canonical form of $\uparrow{*n}$ for $n > 1$ is $\{0 \mid *(n \oplus 1)\}$.*

Proof: To find the canonical form, we begin by listing the options:

$$\uparrow{*n} = \{{*n}, \uparrow, \uparrow*, \uparrow{*2}, \ldots, \uparrow{*(n-1)} \mid * + {*n}, \uparrow, \uparrow*, \uparrow{*2}, \ldots, \uparrow{*(n-1)}\}$$

That is, Left's options are $\uparrow{*m}$ for $0 \leq m < n$, and $*n$; Right's options are $* + {*n}$ and $\uparrow{*m}$ for $0 \leq m < n$. The options are summarized by



Note that for $m < n$, $m \oplus 1 \neq n \oplus 1$ so, by Lemma 9.2, $*(m \oplus 1) < \uparrow{*n}$. Hence, Left's first option reverses to $*m'$, where each $m' \neq n \oplus 1$. Once bypassed, these options, as well as Left's second option to $*n$, are all dominated by the third option to $\uparrow$. This move to $\uparrow$, however, also reverses through $*$ to 0. This leaves the single left option from $\uparrow{*n}$ of 0.

For Right, all options $\uparrow{*m}$, are dominated by $* + {*n} = *(n \oplus 1)$. Once the reader confirms that there are no further reversible options, we are done. □

Example 9.6. In FORKLIFT, a single heap has value 0 since neither player has a move. If there are two heaps, $(1, a)$, then both Left and Right can only move to the position with one heap of size $a + 1$. Therefore, $(1, a)$ has value $*$. The position $(2, 2 + i)$ for $i \geq 0$ has options $\{*, 0 \mid *, 0\} = *2$.

Exercise 9.7. Show that $(a, a + i)$, for $i \geq 0$, has value $*a$.

The values of three heap positions are more complicated to describe. The position (p, q, p) for $p, q > 0$ has value 0 by the Tweedledum-Tweedledee strategy.

Exercise 9.8. Show that $(a, a+i, 1) = \uparrow*(a \oplus 1)$ for $a \geq 2$ and $i \geq 1$.

Exercise 9.9. Show that $(a+i, a, 1) = \uparrow*((a+1) \oplus 1))$ for $a \geq 2$ and $i \geq 1$.

G^n and sums of G^n

Definition 9.10. Let $\{0 \mid \mathcal{G}^R\}$ be a positive infinitesimal in canonical form. For a positive integer n define G^n by[1]

$$G^n \stackrel{\text{def}}{=} \left\{0 \mid \mathcal{G}^R - G^1 - G^2 - \cdots - G^{n-1}\right\}.$$

We will soon see that this defines a sequence of positive games, each infinitesimal with respect to the previous one; that is, for all positive integers m, $m \cdot G^{n+1} < G^n$.

Exercise 9.11. Find the canonical form for $\uparrow^2$.

For a fixed G, John Conway and Alex Ryba suggest the *uptimal notation:*[2]

$$.i_1 i_2 i_3 \ldots = i_1 G + i_2 G^2 + i_3 G^3 + \cdots$$

For example, for the most common case when $G = \uparrow$, we have

$$.2013 = \Uparrow + \uparrow^3 + \uparrow^4 + \uparrow^4 + \uparrow^4.$$

For negative coefficients, place a bar over the integer:

$$.2\bar{1}1\bar{2} = \Uparrow - \uparrow^2 + \uparrow^3 - \uparrow^4 - \uparrow^4.$$

[1]The definition generalizes nicely if you use the *ordinal sum*, $G : H$, from [BCG01, p. 219]:

$$\begin{aligned} G : H &\stackrel{\text{def}}{=} \left\{\mathcal{G}^L, G : \mathcal{H}^L \,\middle|\, \mathcal{G}^R, G : \mathcal{H}^R\right\}, \\ G^x &\stackrel{\text{def}}{=} \left\{0 \,\middle|\, G^R : (1-x)\right\}, \end{aligned}$$

for any number $x \geq 1$. When $x = n$ is a positive integer, the two definitions are equivalent. (Be warned that the definition of $G : H$ depends on the form of G, so be sure to use canonical form in G^x.)

[2]For generalized G^x, Conway and Ryba propose *uppity notation*, which uses raised digits as on a ruler to indicate fractional power terms. For instance,

$$.1^{2^3}0^{4}0 = .1^{2^{3}0}0^{0^{4}0}0 = G^1 + 2 \cdot G^{1+1/2} + 3 \cdot G^{1+1/2+1/4} + 4 \cdot G^{2+1/4}$$

Two such values are easily compared by inspecting the leftmost ruler mark on which they differ.

If the base G, say $G = +_2$, is not clear from context, you can specify it parenthetically:

$$.i_1 i_2 i_3 \ldots \quad (\text{base } +_2).$$

An older notation, where

$$G^{\rightarrow n} = .\underbrace{11\ldots1}_{n} = G + G^2 + \cdots G^n,$$

is often convenient and appears in CGSuite. Since the time when *ONAG* [Con01] and *WW* [BCG01] were written, many more all-small games have been characterized and found to be important: Conway's uptimal notation is quite intuitive and is more general; see, for example, Problem 15. Not every all-small game corresponds to an uptimal expression though; see Problem 16.

Theorem 9.12. *Let base $G = \left\{0 \mid \mathcal{G}^R\right\}$ be a positive infinitesimal. Then, for all positive integers n and all non-negative integers m, $G^n > m \cdot G^{n+1}$.*

Proof: We will first prove that $G^n > 0$, by demonstrating that Left wins moving first or second from G^n. When $n = 1$, $G^1 > 0$ as asserted by the theorem. When $n > 1$, as first player she, perforce, moves to 0, which fortunately wins. As the second player, she can reply to Right's move to some $G^R - G^1 - \cdots - G^{n-1}$, by playing on the G^{n-1} leaving a total which is 0:

$$\begin{tikzcd}
 & G^n \arrow[dl] \arrow[dr] & \\
0 & & G^R - G^1 - \cdots - G^{n-1} \arrow[dl] \\
 & \begin{array}{c} G^R - G^1 - \cdots - G^{n-2} \\ -G^R + G^1 + \cdots + G^{n-2} \end{array} &
\end{tikzcd}$$

We will now show that $G^n - m \cdot G^{n+1} > 0$ by giving a winning strategy for Left as either the first or second player. As the first player she selects some option $G^R \in \mathcal{G}^R$ such that $G^R \mathrel{\lhd\!\!|} G$. One such G^R must exist, for otherwise G was a number by Theorem 6.19 on page 126. She then moves in one of the $-G^{n+1}$ components to $-G^R + G^1 + G^2 + \cdots + G^n$ leaving Right to move in

$$(G - G^R) + G^2 + G^3 + \cdots + G^{n-1} + 2 \cdot G^n = (G - G^R) + X.$$

We will now dispense with Right's possible responses. Suppose first that he makes a move in $G - G^R$. This is either to 0, to some $G^{R'} - G^R$, or to some $G - G^{RL}$. By induction, the first of these leaves a positive remainder, which Left wins easily. The second leaves a first player win (since all the right options of G must be incomparable with one another) plus a positive remainder, and Left can win as first player. For the final case $G - G^{RL}$ cannot be negative, lest G^R be reversible, so again Left wins the position as first player.

If Right moves in some G^k that is part of X to $G^{R'} - G^1 - G^2 - \cdots - G^{k-1}$, then Left will be faced with $G^{R'} - G^R$ plus a positive remainder — and even if $R' = R$ will be able to win.

As second player, after Right's move, Left is faced either with

$$G^R - G^1 - G^2 - \cdots - G^{n-1} - m \cdot G^{n+1}$$

or with

$$G^n - (m-1) \cdot G^{n+1}.$$

The second case she wins by induction. In the first case Left can move in $-G^{n+1}$ to

$$-G^R + G^1 + G^2 + \cdots + G^{n-1} + G^n$$

leaving

$$G^n - (m-1) \cdot G^{n+1},$$

which again is an adequate reply by induction. $\square$

Corollary 9.13. *The sign of any game represented in uptimal notation is the same as the sign of the first non-zero digit.*

Exercise 9.14. Prove Corollary 9.13.

Theorem 9.15. *Fix $n > 0$ and let base $G = \{0 \mid \mathcal{G}^R\}$ be a positive infinitesimal in canonical form. The canonical form of $.\underbrace{11\ldots1}_{n}$ is given by*

$$.\underbrace{11\ldots1}_{n} = \left\{ .\underbrace{11\ldots1}_{n-1} \,\middle|\, \mathcal{G}^R \right\}$$

.

Proof: See Problem 10. $\square$

In Section 5.2, we found that on the infinitesimal scale marked by $\uparrow$, the confusion interval of $*$ is between about $\downarrow$ and $\uparrow$. In particular, $* \parallel \uparrow$ but $* < \Uparrow$. We can now pin that confusion interval down more exactly. The following theorem tells us that when working base $\uparrow$,

$$.1111111 \parallel *, \text{ but}$$

$$.1111112 > *;$$

and so on the infinitesimal scale, the confusion interval of $*$ is about $(.\bar{1}\bar{1}\bar{1}\ldots, .111\ldots)$.

Theorem 9.16. *Fix an infinitesimal canonical base* $G = \{0 \mid \mathcal{G}^R\} > 0$, *and fix* $G^R \in \mathcal{G}^R$. *Then,*

$$.111\ldots11 \lhd\!\!| \; G^R, \text{ but}$$

$$.111\ldots12 > G^R.$$

More precisely, if $G \parallel G^R$, *then* $.111\ldots11 \parallel G^R$.

For example, $\uparrow + \uparrow^2 + \uparrow^3$ is incomparable with $*$, but $\uparrow + \uparrow^2 + \uparrow^3 + \uparrow^3 > *$.

Proof: Right moving first from $.111\ldots11 - G^R$ can move to 0 by moving on the $.000\ldots01$ and so win. But Right's dominant moves from $.111\ldots12 - G^R$ are to $.000\ldots01 + (G^{R'} - G^R)$ from which Left wins. What remains is the sum of a positive game $.000\ldots01$ and a game $G^{R'} - G^R$ which, since G is in canonical form and has no dominated options, is either 0 or incomparable with 0.

When $G \parallel G^R$, Left moving first rewrites $.111\ldots11 - G^R = .011\ldots11 + (G - G^R)$, and wins by some move on $G - G^R$ since $.011\ldots11 > 0$. □

Corollary 9.17. *Working base* $\uparrow$, *fix* $S = .i_1i_2i_3\ldots i_n$, *where every* $i_k \geq 1$ *and at least one* $i_k > 1$. *Then the canonical form of* $S + *$ *is given by* $\{0 \mid .j_1j_2j_3\ldots j_n\}$, *where each* $j_k = i_k - 1$.

For example, $.11214* = \{0 \mid .00103\}$. As usual, we write $.11214*$ to mean $.11214 + *$.

Proof: See Problem 11. □

Exercise 9.18. In base $\uparrow$, show that $.\underbrace{11\ldots1}_{n+1}* = \left\{0, .\underbrace{11\ldots1}_{n}* \;\middle|\; 0\right\}$.

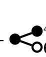
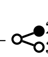
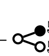
Example 9.19. We are now prepared to give a complete analysis of CUTTHROAT STARS. In particular, for $j > 0$ we have,

Theorem 9.20.

$$^{i}_{j} = \begin{cases} .\underbrace{jjj\ldots j}_{i} + * & \text{if } i > 0, \text{ and} \\ .j - .1 + j\cdot * = (j-1)\cdot{\Uparrow}* + * & \text{if } i = 0. \end{cases}$$

Proof: See Problem 14. □

For example,

 $^{3}_{2}$ +  $^{4}_{6}$ +  $^{2}_{3}$ +  $^{5}_{5}$ $= .222* + .6666* + .\bar{3}\bar{3}* + .\bar{5}\bar{5}\bar{5}\bar{5}\bar{5}* = .0031\bar{5}$,

which is a win for Left since the sign is positive.

ALL-SMALL CLEAR THE POND

In ALL-SMALL CLEAR THE POND, one greedy strategy which might be worth trying is, “always move a piece that goes furthest.” However, this is not the optimal strategy. In $\boxed{\bullet\bigcirc\square\bullet\bullet\bigcirc}$, Left has three options,

$$b = \boxed{\square\bigcirc\bullet\bullet\bullet\bigcirc},\quad c = \boxed{\bullet\bigcirc\square\square\bullet\bigcirc},\ \text{and}\ d = \boxed{\bullet\bigcirc\square\bullet\square\bigcirc},$$

where $b > d > c$ so the longest jump is not necessarily the best move.

While the game in general appears tough, we can find all the values for a game of ALL-SMALL CLEAR THE POND with only two pieces. After a few minutes thought, it becomes apparent that there seem to be several local situations on the board that affect the value. Clearly, jumping off the board is a move to zero. If the two pieces are adjacent and can jump, this seems to be hot (well, hot on the $\uparrow$ scale); if there are empty squares between them then the parity of the distance between them could be a factor.

Claim 9.21. $$\square^m\bigcirc\square^n\bullet\square^{m+i} = i\cdot{\downarrow}{*}+*$$

by which we mean $i \cdot ({\downarrow}*) + *$.

Proof: Freely using induction and the symmetric claim about $\square^{m+i}\bigcirc\square^n\bullet\square^m$,

$$\begin{aligned}
\bigcirc\square^n\bullet &= \{0 \mid 0\} = * = 0\cdot{\downarrow}{*}+*\,;\\
\square^m\bigcirc\square^n\bullet &= \{0 \mid (m-1)\cdot{\uparrow}{*}+*\} = m\cdot{\uparrow}{*}+*\,;\\
\square^m\bigcirc\square^n\bullet\square^{m+i} &= \{(i-1)\cdot{\downarrow}{*}+* \mid (i+1)\cdot{\downarrow}{*}+*\}\\
&= \{(i-1)\cdot{\downarrow}{*}+* \mid 0\}\\
&= i\cdot{\downarrow}{*}+*\,.
\end{aligned}$$

In the second to last equality, Right’s options reverse all the way to 0. $\square$

Claim 9.22.
$$\begin{aligned}
\bullet\bigcirc\square &= *\,;\\
\bullet\bigcirc\square^i &= (i-1)\cdot({\downarrow}*)+* \qquad \textit{for } i>1\,.
\end{aligned}$$

Proof:
$$\begin{aligned}
\bullet\bigcirc\square &= \{\uparrow \mid 0\} = *\,;\\
\bullet\bigcirc\square^i &= \{(i-2)\cdot{\downarrow}{*}+* \mid 0\}\\
&= (i-1)\cdot{\downarrow}{*}+*\,.
\end{aligned}$$
$\square$

Claim 9.23. *For $m > 0$ and $i \geq 0$,*

$$\square^m \bullet \circ \square^{m+i} = \{(i-2) \cdot {\downarrow}{*} + * \mid (i+2) \cdot {\downarrow}{*} + *\} .$$

Equivalently, when $m, n > 0$,

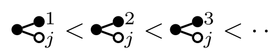
$$\square^m \bullet \circ \square^n = \{\Uparrow \mid \Downarrow\} + (m-n) \cdot {\uparrow}{*} + *.$$

Proof: Writing out the options, we have

$$\begin{aligned} \square^m \bullet \circ \square^{m+i} &= \left\{\square^{m+1} \circ \bullet \square^{m+i-1} \,\middle|\, \square^{m-1} \circ \bullet \square^{m+i+1}\right\} \\ &= \{(i-2) \cdot {\downarrow}{*} + * \mid (i+2) \cdot {\downarrow}{*} + *\} . \end{aligned}$$ $\square$

9.2 Motivation: The Scale of Ups

The canonical forms of infinitesimal games that appear in practice are frequently quite complicated. In this sense, many of the infinitesimal positions we have seen so far are, in fact, anomalous in their simplicity. However, it is often the case that an infinitesimal game consists of a sum of a simple dominant (or biggest) infinitesimal and a more complicated error term. For example, from Theorem 9.20 we know that



form an increasing sequence, but all are very nearly $j \cdot \uparrow$. If we know the dominant part of a number of infinitesimal games, then this is frequently sufficient information to determine the outcome of their sum.

Recall from page 70 that we defined $G = H$ if for all games X, $G + X$ has the same outcome as $H + X$. When games are particularly complicated, it is often helpful to relax this definition. We could, for instance, define an equivalence relation in which two games are equivalent if they differ by only an infinitesimal.

Exercise 9.24. Review the definition of an equivalence relation. Define a relation $G \sim_{\text{inf}} H$ if $G - H$ is an infinitesimal and quickly confirm that $\sim_{\text{inf}}$ is an equivalence relation.

If we are interested in understanding all-small games (or other infinitesimals), then this equivalence is too coarse, for all infinitesimals are then equivalent to 0. In the next section, we will define an equivalence that captures a great deal of what is important about all-small games. This equivalence relation will help us to describe any all-small game (and many infinitesimals that are not all-small) according to about how many $\uparrow$s the games are worth. We foreshadowed this when we showed a number line marked by multiples of $\uparrow$ in Figure 5.1 on page 102 and when we proved that $n \cdot \uparrow$ is larger than all infinitesimals born by day $n - 3$ (an easy consequence of Theorem 6.6 on page 121). In short, multiples of the game $\uparrow$ are a natural scale with which to measure infinitesimals.

9.3 Equivalence Under ☆

Somewhat surprisingly, it turns out that the equivalence relation we seek on infinitesimal games, which is intended to identify their dominant infinitesimal part, is most easily defined in terms of games played in the presence of a large nim-heap of indeterminate size. To prepare for this surprising definition, we first require the following theorem.

Theorem 9.25. *If $*m$ and $*n$ are not equal to any position of G (including G itself), then $G + *m$ has the same outcome as $G + *n$.*

Proof: It suffices to show that if Left wins moving first on $G + *m$ then Left wins moving first on $G + *n$.

If Left wins moving first on $G + *m$ by moving to $G^L + *m$, Left also wins on $G^L + *n$ by induction. So suppose Left wins by moving $G + *m$ to $G + *p$ for some $p < m$. If $*p$ is a position of G, then necessarily $p < n$ since $*n$ is *not* a position of G. So, in this case, the move to $G + *p$ is also available from $G + *n$. If, on the other hand, $*p$ is not a position of G, then, since $G + *p \geq 0$ we also have

$$G = G + *p + *p \geq 0 + *p = *p.$$

But, $G \neq *p$, and so $G > *p$ is strict. Hence, Left wins moving first on $G + *p$, and thus also on $G + *n$, which has the same outcome as $G + *p$ by induction. □

Note that in particular the theorem applies if both m and n are larger than the birthday of G. This theorem justifies the following definition:

Definition 9.26. $G \sim_{☆} H$ if for all games X, both $G + X + *n$ and $H + X + *n$ have the same outcome for n sufficiently large.

Again, by sufficiently large, we mean that $*n$ is not the value of any position of G or H. It suffices, for example, to select n to be greater than the birthday of both G and H.

Repeating the phrase, "$*n$ for n sufficiently large" is a bit of a nuisance. It becomes even more of a nuisance when adding two games each of which has a "$*n$ for n sufficiently large." It is therefore convenient (and natural) to introduce a symbol, ☆, to mean "$*n$ for sufficiently large n" and describe the algebraic properties of ☆. For example, when we add $G +$ ☆ $+$ ☆, each ☆ could, in turn, be set to be $*n_1$ and $*n_2$, respectively, from left to right, so that n_1 is sufficiently large in the context of G, and n_2 is sufficiently large in the context of $G + *n_1$. But then $*n_1 + *n_2$ can be replaced by a sufficiently large $*n$ without affecting the outcome, and so $G +$ ☆ $+$ ☆ should be the same as $G +$ ☆, i.e., ☆ $+$ ☆ $=$ ☆.

Let us pause right now to remark that ☆ *is not a short game.* In fact, it is not really a game at all. It serves very much the same notational purpose as the symbol ∞ serves in certain arguments about real numbers. We will use it in certain contexts as if it were a game, but the rules for working with ☆ will reflect its intended meaning — it stands for a nim-heap which is *large enough*, where the precise meaning of large enough depends on the surrounding context.

In mathematics, any quantity with the property that $a + a = a$ is termed an *idempotent*. For instance, considering the reals under addition, 0 is an idempotent, while under multiplication both 0 and 1 are idempotents. We have just argued that ☆ should be an idempotent.

In summary,

Definition 9.27. The symbol ☆ (*far star*) is defined to have the following properties in additive expressions involving ☆ and games:

- ☆ is an idempotent: $\text{☆} + \text{☆} = \text{☆}$.
- From ☆, a player may move to $*n$ for any value of $n \geq 0$.

The second property reflects the fact that playing $G + \text{☆}$, either player should, at least, be able to move to $G + *n$ whenever $*n$ is a position of G. The ability to move to even larger nimbers has no effect on the outcome of $G + \text{☆}$ by Theorem 9.25.

We now restate Definition 9.26.

Definition 9.28. $G \sim_{\text{☆}} H$ if $G + X + \text{☆}$ and $H + X + \text{☆}$ have the same outcome for all games X.

So, two games are equivalent in this sense if ☆ blurs them enough so they behave the same in any context X.

Theorem 9.29. *The relation $\sim_{\text{☆}}$ is an equivalence relation that respects addition.*

Proof: The fact that $\sim_{\text{☆}}$ is reflexive, symmetric, and transitive is clear from the symmetries in the definition. To show that it respects addition, we need to show that if $G_1 \sim_{\text{☆}} H_1$ and $G_2 \sim_{\text{☆}} H_2$ then $G_1 + H_1 \sim_{\text{☆}} G_2 + H_2$. For the latter, using $G_1 \sim_{\text{☆}} H_1$, $G_1 + G_2 + X + \text{☆}$ has the same outcomes as $H_1 + G_2 + X + \text{☆}$, and using $G_2 \sim_{\text{☆}} H_2$ has the same outcomes as $H_1 + H_2 + X + \text{☆}$. $\square$

Neither definition of $\sim_{\text{☆}}$ is constructive. It would take a long time, indeed, to play G and H in every possible context $X + \text{☆}$. The following theorem comes to the rescue and also begins to suggest that $\sim_{\text{☆}}$ will provide the $\uparrow$ scale that we seek.

Theorem 9.30. *$G \sim_{\text{☆}} H$ if and only if $\downarrow\text{☆} < G - H < \uparrow\text{☆}$.*

We will use one trick in the proof that is sufficiently subtle to warrant stating clearly in advance. According to Theorem 9.25, when playing a game of the form $J + \text{☆}$, Left can, at any time, insist that a ☆ be converted to any $*n$ as long as n is sufficiently large. This change will not affect the outcome of the game. In particular, Left can choose any n exceeding the birthday of J. This change can be safely made at any time and does not use up a turn. We will call this process, *fixing the* ☆.[3]

Proof: For the $\Rightarrow$ direction, suppose $G \sim_{☆} H$. Choosing $X = -G$ in the definition of $\sim_{☆}$, $(G - H) + {\downarrow}☆$ has the same outcomes as $(G - G) + {\downarrow}☆ = {\downarrow}☆ < 0$. Hence, $(G - H) + {\downarrow}☆ < 0$ and $G - H < {\uparrow}☆$. A symmetric argument shows ${\downarrow}☆ < G - H$.

The $\Leftarrow$ direction is more difficult. Suppose by way of contradiction that ${\downarrow}☆ < G - H < {\uparrow}☆$ and yet it is *not* the case that $G \sim_{☆} H$. From the latter we may assume that for some game X, Left wins $G + X + ☆$ moving first but cannot win $H + X + ☆$ moving first, and so $H + X + ☆ \leq 0$. We will subscript the X's to help track the games as they change; so define $X_1 = X_2 = X$. We now have:

1. Right wins $-(G + X_1) + ☆$ moving first.

2. Right wins $(G + X_1) - (H + X_2) + {\downarrow} + ☆$ moving second.

3. Right wins $(H + X_2) + ☆$ moving second.

(The second item is a consequence of $G - H < {\uparrow}☆$ and $X_1 - X_2 = 0$.)

So suppose Left plays all three of these games against three gurus. Left moves first in games two and three, but moves second in game one. If Left wins any of these three games, we reach a contradiction.

Left's strategy is to let the gurus dictate play. At each point in the play, there will always be one guru whose move it is, and Left waits for that guru to make a move. If the guru cannot move, Left has won that game and we have established our contradiction. Based on the guru's move, Left then elects to move in one of the three games and waits for the response of the opposing guru on that board.

If a guru moves on $(G + X_1)$ in the second game, then Left can copy the move in $-(G + X_1)$ and vice versa. Similarly, moves on $(H + X_2)$ are matched to moves in $-(H + X_2)$. For the more challenging cases, the guru could move on a ☆ or move $\downarrow$ to 0. In response, respectively, Left will either move the remaining $\downarrow$ to $*$ or move on a ☆ (to anything). As described in the paragraph preceding this proof, Left will now *fix* the remaining two ☆s. In particular, Left can do so in such a way that the three positions are now in the form

$$-G' - X_1' + *n_1 \qquad (G' + X_1') - (H' + X_2') + *n_2 \qquad H' + X_2' + *n_3,$$

[3]This is really a version of the *One-Hand-Tied Principle.* In fixing the ☆, Left agrees not to use some of the options available to her from ☆ and, moreover, promises that if Right makes a move in ☆ to $*m$ for some $m > n$ then she will respond immediately to $*n$.

where Left ensures that $*n_1+*n_2+*n_3=0$. In particular, since a guru chooses at most one of the n_i's, Left is free to fix the remaining two to be "sufficiently large" while maintaining that the sum $*n_1+*n_2+*n_3$ is 0.

After the ☆s are fixed in this way, the sum of the resulting games is 0, so Left has no trouble finding a natural response to any opposing move on an outer board by moving on the symmetric term in the inner board, and vice versa. Throughout play, Left can always find a move in one of the three components after a guru has moved, and so never runs out of moves until after one of the gurus has run out and lost. □

9.4 Atomic Weight

Definition 9.31. If $g \sim_{☆} G\cdot\uparrow$ then we say that g has *atomic weight* G and write $G = \mathrm{AW}(g)$.

In this section, we will use an uppercase letter to represent the atomic weight of a game written with a lowercase letter, since g is infinitesimal, while G typically is not.

To implement this definition, we can *guess* the atomic weight G of g, compute $G\cdot\uparrow$, and use Theorem 9.30 to test whether $g \sim_{☆} G\cdot\uparrow$. When using the unit $U=\uparrow$, the definition of Norton product requires U's incentives. The left incentive from $\uparrow$ is $\downarrow$, and the right incentive is $\uparrow*$. The latter is dominant, and we have $\tau = \{\Uparrow*\}$. So, the Norton product reduces to

$$G\cdot\uparrow = \begin{cases} \overbrace{\uparrow+\uparrow+\uparrow+\cdots+\uparrow}^{n} & \text{if } G=n \text{ is a positive integer,}\\ 0 & \text{if } G=0,\\ \underbrace{\downarrow+\downarrow+\downarrow+\cdots+\downarrow}_{n} & \text{if } G=-n \text{ is a negative integer,}\\ \{\mathcal{G}^L\cdot\uparrow+\Uparrow* \mid \mathcal{G}^R\cdot\uparrow+\Downarrow*\} & \text{otherwise.}\end{cases}$$

Theorem 9.32. *The atomic weight G of g is well-defined. That is, if G_1 and G_2 are both atomic weights of g, then $G_1=G_2$.*

Proof: We will show that if $G_1\cdot\uparrow \sim_{☆} G_2\cdot\uparrow$, then $G_1=G_2$. Since the Norton product is additive, it suffices to show that if $G\cdot\uparrow \sim_{☆} 0$ for G in canonical form then $G=0$.

Assume, by way of contradiction, that $G\neq 0$; then one player, say Left, wins moving first on G. This yields a winning strategy on $G\cdot\uparrow+\downarrow☆$: Left pretends she is playing G alone and plays the corresponding moves in $G\cdot\uparrow$ until it becomes an integer multiple of $\uparrow$.

In essence, using this strategy, players benefit by the amount of $t = \Uparrow *$ when playing on the Norton product (that is, Left gains t while Right gains $-t$), while Right can benefit by at most $\uparrow *$ by playing on $\downarrow$ (or less by playing on ☆). With careful accounting (see below), one can then confirm that once G reaches an integer, that either Right is to move from a game of value > 0, or Left is to move from a game of value $\mathrel{|\!\!\triangleright} 0$. Either way, Left wins.

Now, for the accounting. The key is that each move on $G \cdot \uparrow$ picks up $\Uparrow *$ from the definition of Norton product.

- If Right always responds locally on $G \cdot \uparrow$, since Left wins on G, when G first becomes an integer, say n, it is with Left having moved last and $n \geq 0$ or Right having moved last and $n \geq 1$. So Left achieves a game $\geq \Uparrow * + \downarrow\text{☆} = \uparrow\text{☆}$ moving second or a game $\geq \uparrow + \downarrow\text{☆} = \text{☆}$ moving first. Either way she wins.

- If Right ever moves on $\downarrow\text{☆}$, Left gets two moves in a row in the $G \cdot \uparrow$ component. By Corollary 6.16, the integer left stop minus the integer right stop from G is at most -1 (and is only -1 when Left moves last from the left integer stop). Hence, each time Left plays twice in a row in the Norton product, she loses no more than $\downarrow$ from the stops, but picks up $\Uparrow *$ from the Norton product definition. Right, however, picks up at most $\downarrow *$ for his move in the second component. In total, Left does no worse than the analysis in the preceding paragraph. □

Theorem 9.33. *Atomic weights are additive:* $\mathrm{AW}(g) + \mathrm{AW}(h) = \mathrm{AW}(g + h)$.

Proof: This is an immediate consequence of the fact that $\sim_{\text{☆}}$ respects addition. □

Again, we can use Theorem 9.30 along with the definition of atomic weights to confirm that a conjectured atomic weight is correct.

Example 9.34.

- It is trivially the case that $\mathrm{AW}(G \cdot \uparrow) = G$.
- $\mathrm{AW}(*) = 0$ for $\downarrow\text{☆} < * < \uparrow\text{☆}$.
- $\mathrm{AW}(\uparrow *) = \mathrm{AW}(\uparrow) + \mathrm{AW}(*) = 1 + 0 = 1$.
- $\mathrm{AW}(\pm\Uparrow) = \mathrm{AW}(\pm\Uparrow *) + \mathrm{AW}(*) = * + 0 = *$, because $* \cdot \uparrow = \pm\Uparrow *$.

Exercise 9.35. Determine $\mathrm{AW}(+_2)$ and $\mathrm{AW}(\{0 \mid +_2\})$. (*Hint*: Both are in $\{-1, 0, 1\}$.)

Exercise 9.36. Show that $\mathrm{AW}(.pq) = p$, where $.pq$ is in uptimal (base $\uparrow$).

It is important to note that not all infinitesimal games have an atomic weight.

Example 9.37. If $g = \{1 \mid \Uparrow \| \Downarrow \mid -1\}$ there is no G such that $G = \mathrm{AW}(g)$; for suppose there were a G such that

$$\downarrow\!* < g - G \cdot \uparrow < \uparrow\!*.$$

When Left wins moving second from $g - G \cdot \uparrow - \downarrow\!*$, Left must reply to a move on g (for -1 is less than any option of the all-small game $G \cdot \uparrow$). This leaves, $\Downarrow - G \cdot \uparrow + \uparrow\!*$ from which Left should win moving second. So, $G \cdot \uparrow \geq \uparrow\!*$. A symmetric argument for Right yields

$$\downarrow\!* \geq G \cdot \uparrow \geq \uparrow\!*,$$

which is impossible.

Knowing the atomic weight of a game does not necessarily determine the outcome. For example, all four games $g \in \{0, *, \uparrow^2, -\uparrow^2\}$ have atomic weight 0, but have different outcomes. However, a sufficient advantage in atomic weight is decisive.

Theorem 9.38. (The Two-Ahead Rule)

$$\begin{array}{llll} \textit{If} & \mathrm{AW}(g) \geq +2 & \textit{then} & g > 0; \\ \textit{If} & \mathrm{AW}(g) \geq +1 & \textit{then} & g \mathrel{|\!\!\!>} 0; \\ \textit{If} & \mathrm{AW}(g) \leq -1 & \textit{then} & g \mathrel{<\!\!\!|} 0; \\ \textit{If} & \mathrm{AW}(g) \leq -2 & \textit{then} & g < 0. \end{array}$$

Proof: If $G = \mathrm{AW}(g) \geq 2$, then $g - G \cdot \uparrow \geq g - \Uparrow > \downarrow\!\star$, and so $g > \uparrow\!\star \geq 0$. When $\mathrm{AW}(g) \geq 1$, the same line of reasoning yields $g > \star \mathrel{|\!\!\!>} 0$. The third and fourth assertions are symmetric. $\square$

While Theorem 9.30 gives a way to test whether g has atomic weight G, it gives no guidance how to find this atomic weight G, or even whether it exists. Example 9.37 gave an infinitesimal with no atomic weight, but every all-small game does, in fact, have an atomic weight and [BCG01] proves that the following mechanical method computes it!

Theorem 9.39. *If $g = \left\{g^L \mid g^R\right\}$ and g^L and g^R have atomic weights, then the atomic weight of g is given by*

$$\left\{\mathrm{AW}(g^L) - 2 \mid \mathrm{AW}(g^R) + 2\right\}$$

unless this is an integer. In that case, let x be the least integer such that $\mathrm{AW}(g^L) - 2 \mathrel{<\!\!\!|} x$, and y the greatest integer such that $y \mathrel{<\!\!\!|} \mathrm{AW}(g^R) + 2$.

- *If* $g \parallel ☆$, $\mathrm{AW}(g) = 0$.
- *If* $g > ☆$, $\mathrm{AW}(g) = y$.
- *If* $g < ☆$, $\mathrm{AW}(g) = x$.

In practice, atomic weights are often integers, and the exception is invoked.

Proof: We refer you to the proof in [BCG01, pp. 248–251]. If you work through their proof, be aware that they use the symbol $\doteqdot$ where we use $\sim_{☆}$ and that their ☆ is our ☆. □

To give credibility to this theorem, let's try a few examples:

- $\mathrm{AW}(0)$: $\{-2 \mid 2\} = 0$ and since $0 \parallel ☆$ then $\mathrm{AW}(0) = 0$.
- $\mathrm{AW}(*)$: $\{0-2 \mid 0+2\} = 0$ and since $* \parallel ☆$ then $\mathrm{AW}(*) = 0$. By identical reasoning (now employing induction) $\mathrm{AW}(*n) = 0$.
- $\mathrm{AW}(\uparrow)$: $\{0-2 \mid 0+2\} = 0$ but $\uparrow > ☆$. So $\mathrm{AW}(\uparrow) = 1$, the largest integer less than or incomparable with 2.
- $\mathrm{AW}(\uparrow *n)$: $\{0-2 \mid 0+2, 1+2\} = 0$ but since $\uparrow *n > ☆$, $\mathrm{AW}(\uparrow *) = 1$ the largest number less than or incomparable with both 2 and 3.
- $\mathrm{AW}(\Uparrow)$: Recall $\Uparrow = \{0 \mid \uparrow *\}$, so the purported atomic weight is $\{0-2 \mid 1+2\} = 0$ and since $\Uparrow > ☆$ we have $\mathrm{AW}(\uparrow) = 2$.
- $\mathrm{AW}(.11)$: Working base $\uparrow$, $\mathrm{AW}(.11) = \mathrm{AW}(\{\uparrow \mid *\})$. So we get $\{1-2 \mid 0+2\} = 0$ again, and since $.11 > \uparrow > ☆$ then $\mathrm{AW}(.11) = 1$.

Exercise 9.40. Use Theorem 9.39 to determine $\mathrm{AW}(\{(n+1)\cdot\uparrow \mid n\cdot\uparrow\})$ for all integers n. In other words, compute the atomic weights of

$$\ldots, \{\downarrow \mid \Downarrow\}, \{0 \mid \downarrow\}, \{\uparrow \mid 0\}, \{\Uparrow \mid \uparrow\}, \{⤊ \mid \Uparrow\}, \ldots$$

Exercise 9.41. Let

$$g = \{0, *, *2, \ldots, *m \mid 0, *, *2, \ldots, *n\}.$$

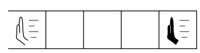
Use Theorem 9.39 to show that if $m > n$ then $\mathrm{AW}(g) = 1$.

ALL-SMALL CLEAR THE POND had some hot values (on the infinitesimal scale). The atomic weights of these games are interesting and we show how to interpret (use) them later. In the following examples, the exceptional case of Theorem 9.39 fails to occur, and so

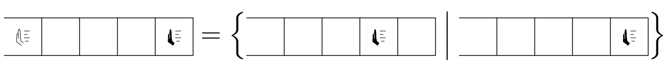
$$\begin{aligned} \mathrm{AW}(\{\Uparrow * \mid \Downarrow *\}) &= \{2-2 \mid -2+2\} = *; \\ \mathrm{AW}(\{\uparrow \mid \Downarrow\!\!\downarrow\}) &= \{1-2 \mid -3+2\} = -1*. \end{aligned}$$

9.5 All-Small Shove

This variant of SHOVE has the same rules except that the game is deemed to be over when the last of either player's pieces disappear from the board. For example, in SHOVE, the position

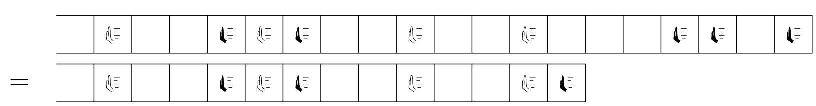


has value $4\frac{1}{2}$, but in ALL-SMALL SHOVE



$= \{0 \mid 0\} = *.$

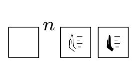
Unlike SHOVE, in this game the rightmost piece does not determine the winner of a single component. Any position in which the rightmost squares contain at least one black piece and no white pieces, the squares can be replaced by a single square with a single black piece:



The game will end when the rightmost white piece leaves the board, and any move to the right of this piece has the same effect. So without loss of generality, we need only consider positions in which the rightmost two pieces are adjacent and opposite in color.

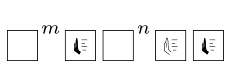
As usual, let $\square^n$ denote a row of n empty squares.

Exercise 9.42. Show that in ALL-SMALL SHOVE



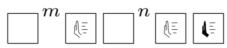
$= (n+1) \cdot *.$

Exercise 9.43. Show that in ALL-SMALL SHOVE

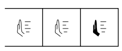
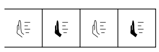
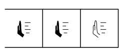
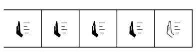
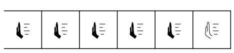
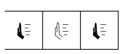
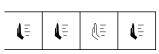
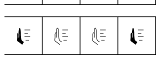
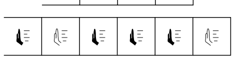


$= (m+1) \cdot \uparrow + (n+m) \cdot *.$

Exercise 9.44. Show that in ALL-SMALL SHOVE

 $= (m+1)\cdot\downarrow + (n+m)\cdot *.$

While some positions in this game are easily described using uptimal notation, others appear quite complex for current techniques:

Position	**Value**	**Atomic Weight**
	$\downarrow$	-1
	$\{0 \mid \uparrow, \Uparrow *\}$	2
	$.21*$	2
	$.321$	3
	$.4321*$	4
	$\uparrow$	1
	$.21*$	2
	$\{\downarrow, \Downarrow * \mid 0\}$	-2
	$0 \,\|\, 0 \mid .21*, \{0 \mid .21*\} \,\|\|\,\, 0 \mid .21*, \{0 \mid .21*\}$	4

9.6 More Toppling Dominoes

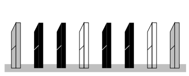
Consider TOPPLING DOMINOES positions with black, white, and gray dominoes. (The gray dominoes can be tipped by either player.)

Exercise 9.45. The outcome of a single component in TOPPLING DOMINOES is completely determined by the colors of the two ends. Show how.

We have the following astonishingly simple result:

Theorem 9.46. *Any* TOPPLING DOMINOES *component in which both ends are gray has atomic weight* 1, 0, *or* -1.

What makes this theorem surprising is that the values of even small TOPPLING DOMINOES positions can have quite complicated canonical forms. For instance,

 $$= \left\{ \begin{array}{c} 0,\ \{1 \mid 0\},\ \{1* \mid 0\}, \\ \{1, 1*, \{1, 1* \mid 0, \{1 \mid 0\}\} \mid 0, *, \{1, \{1 \mid 0\} \mid 0, *\}\}, \\ \{0, \{1 \mid 0\}, \{1, 1* \mid 0, \{1 \mid 0\}\} \mid 0\} \end{array} \middle|\ 0, * \right\}.$$

Before we prove the theorem, we start with another lemma. In particular, were we to pretend that TOPPLING DOMINOES were all-small, we would be led to proving:

Lemma 9.47. *Any all-small game of the form*

$$\{0, \{? \mid 0\}, \ldots \mid 0, \{0 \mid ?\}, \ldots\}$$

has atomic weight 1, 0, *or* −1.

Note that the TOPPLING DOMINOES under consideration are of this form *except* that the question marks are not all-small.

Proof: The preliminary atomic weight calculation given by Theorem 9.39 yields

$$\text{“}\{-2, \text{‘}\{? \mid 0\}\text{’}, \ldots \mid 2, \text{‘}\{0 \mid ?\}\text{’}, \ldots\}\text{”},$$

where these question marks are the atomic weights of the previous question marks shifted by −4 and 4. Now the actual atomic weights of the inner quotation marks either have right-option 0, or (in the exceptional case) are integers. Either way, the game in double-quotes is a second player win and has value 0. So we are in the exceptional case. But the only integers that can possibly fit between the left options and right options are 1, 0, and −1. □

Proof of Theorem 9.46: Let g be any gray-ended TOPPLING DOMINOES position, and let g' be the same game with every position that is a positive number replaced by ⤊ and every negative integer replaced by ⤋. According to the lemma, the atomic weight of g' is 1, 0, or −1. Observe that $g - \Uparrow\star$ has the same outcome as $g' - \Uparrow\star$, and that the same statement can be made of $\uparrow\star$, $\star$, $\downarrow\star$, and $\Downarrow\star$. Therefore, by Theorem 9.30, $g \sim_{\star} g'$ and has the same atomic weight. □

Note that the proof gives no efficient way to determine which atomic weight is correct, 1, 0, or −1. The best method we know to date is to compare the game to $\star$ and find out.

9.7 Clobber

Generally, by about halfway through a game of CLOBBER the board has decomposed into a disjunctive sum. Our techniques should be of great use when trying to play this game, but CLOBBER is a hard game! As we will see, even the atomic weights can be bad. We present some results and conjectures about CLOBBER then show how to find the atomic weight of a position that has exactly one white piece.

The one-dimensional version decomposes almost from the first move. Remember our usual convention: if X is a pattern of black, white, and empty spaces we denote a string consisting of n copies of this pattern by X^n, so

$$(\circ\bullet^2)^3 = \circ\bullet\bullet\circ\bullet\bullet\circ\bullet\bullet$$

Consider the one-dimensional version with just one white piece at the end of a string of black pieces. The first few values are easy to calculate:

$$\begin{aligned}
\circ &= \bullet = 0;\\
\circ\bullet &= *;\\
\circ\bullet\bullet &= \{0 \mid \circ\bullet\} = \{0 \mid *\} = \uparrow;\\
\circ\bullet\bullet\bullet &= \{0 \mid \circ\bullet\bullet\} = \{0 \mid \uparrow\} = \Uparrow*.
\end{aligned}$$

We have seen this sequence before. In general,

Lemma 9.48.

$$\circ\bullet^n = \left\{0 \,\middle|\, \circ\bullet^{n-1}\right\} = \{0 \mid (n-2)\cdot\uparrow + (n-1)\cdot *\} = (n-1)\cdot\uparrow*+*.$$

Exercise 9.49. Show that if $m > 1$ and $n > 1$ then $\circ^m\bullet^n = 0$.

Note, in particular, that $\circ\bullet^n$ can be much bigger than $\circ\circ\bullet^n$, so a single stone can make a dramatic difference. Another natural sequence to investigate is alternating black and white stones:

$$\begin{aligned}
\bullet\circ\bullet &= *;\\
\bullet\circ\bullet\circ &= \pm(*,\uparrow);\\
\bullet\circ\bullet\circ\bullet &= .\bar{1}\bar{1};\\
\bullet\circ\bullet\circ\bullet\circ &= 0;\\
\bullet\circ\bullet\circ\bullet\circ\bullet &= \{\downarrow, \pm(*,\uparrow) \mid 0\};\\
\bullet\circ\bullet\circ\bullet\circ\bullet\circ &= \pm(.11, \{\uparrow*, \Uparrow \mid 0, \uparrow*, \pm(0,\uparrow*)\}).
\end{aligned}$$

The canonical forms get worse as the length of the board grows. Do the atomic weights help? Starting with $\bullet\circ\bullet$ the atomic weights are 0, 0, -1, 0, -1, 0, -1, 0, -1, 0, -1, 0, -1, 0, -1, 0, and -1. Even though a one-dimensional board with a repeating pattern of pieces looks set up for induction, at the time of writing, none of the values, atomic weights, and outcomes of an alternating board are known.

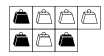
The reader may wish to try proving either or both of:

Conjecture 9.50. $\bullet\circ^n$ *is a first-player win for* $n \neq 3$.

Conjecture 9.51. $\bullet\bullet\circ^n = \lfloor (n+1)/2 \rfloor \cdot\uparrow$.

The first conjecture has been verified by computer up to $n = 19$ and, except for $n = 3$, the first player has few losing moves. The second conjecture has been verified up to $n = 17$.

For $n > 4$, the game

 ^{n}

has atomic weight $\{0 \mid \{0 \mid 4-n\}\} = \text{+}_{n-4}$. In addition, positions with atomic weights of 1/2, 1/4, 1/8, $\uparrow$, $*$ and $\text{+}_{1/2}$ can fit on a 6×3 board. The game

 ^{n}

has atomic weight $\{n - 3 \mid 0\}$ showing that arbitrarily hot atomic weights can occur.

In the rest of this section, we will consider graphs with only one white counter. Consider a graph G and the corresponding game G_v in which all but vertex v is occupied by black counters and v is occupied by a white counter.

A *blocked v-path* in G is any maximal path $P = \{v_0 = v, v_1, v_2, \ldots, v_r\}$ starting at v. That is, every vertex adjacent to v_r is in P so that the path cannot be extended from v_r. A *blocked split* v-path, or split path for short, is a blocked path $P = \{v_0 = v, v_1, v_2, \ldots, v_r\}$ for which there exists vertices x, y such that $Q = \{v_0 = v, v_1, v_2, \ldots, v_{r-1}, x, y\}$ is also a blocked path. The *length*, $l(P)$, of a blocked path $P = \{v_0 = v, v_1, v_2, \ldots, v_r\}$ depends on whether the path is split:

$$l(P) = \begin{cases} r & \text{if } P \text{ is not a split path,} \\ r-1 & \text{if } P \text{ is a split path.} \end{cases}$$

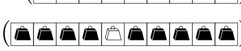
It may be that the two paths in a split path only have v in common but then $l(P) = 1$; for example, . Let

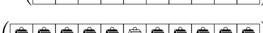
$$l(G_v) = \min\{l(P) \mid P \text{ is a blocked path or a split path of } G_v\}.$$

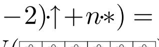
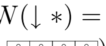
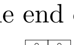
For example,

$l($$) = 3;$

$l($$) = 4;$

$l($$) = 5;$

$l($$) = 6.$

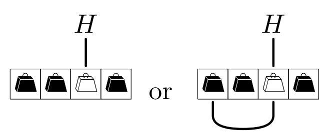
Note that if G is a path of $n \geq 2$ vertices with v at one end then Lemma 9.48 gives that $\mathrm{AW}(G) = \mathrm{AW}((n-2)\cdot\uparrow + n\cdot*) = n-2$. Also, note that AW() $=$ $\mathrm{AW}(\downarrow *) = -1$, whereas AW() $= 1$. At the end of a split path (i.e., at ) Right has the option of moving to 0 or to  $= *$, both of which have atomic weight 0. This choice between two apparently small games is actually the difference of playing to a first-player-win and a second-player-win game. This gives Right a local advantage that is reflected in the atomic weights.

Lemma 9.52. *For a graph G with $v \in V(G)$ having at least one neighbor,*

$$\mathrm{AW}(G_v) = l(G_v) - 2.$$

Proof: We prove this by induction on the number of vertices, and assume throughout that G is connected and v has a neighbor. Note that Left's options are all identically 0.

First, suppose $l(G_v) = 1$. In particular, G_v is of the form



where H is arbitrary and possibly empty. Right has moves to $*$ and to 0. A move to a vertex in H by induction has atomic weight ≥ 0. In the preliminary atomic weight calculation of G_v (given by Theorem 9.39 on page 200),

$$G_0 = \{0 - 2 \mid 0 + 2, \ldots\},$$

and the atomic weight of G_v is either -1, 0, or 1 depending on how G_v compares with ☆. But

$$G_v = \{0 \mid 0, *, \ldots\} \leq \{0 \mid 0, *\} = {\downarrow}*,$$

which has atomic weight -1. So $\mathrm{AW}(G_v) = -1$.

Henceforth, we assume that $l(G_v) \geq 2$. We know that

$$G_v = \{0 \mid (G - v)_w, w \text{ adjacent to } v\}.$$

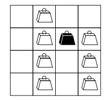
There is at least one vertex b that is the next vertex on a shortest blocked path from v and $l((G-v)_b) = l(G_v) - 1$. So, the preliminary atomic weight of G_v is

$$G_0 = \{0 - 2 \mid l((G-v)_b - 2 + 2\} = \{-2 \mid l(G_v) - 1)\},$$

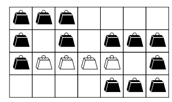
and $\mathrm{AW}\, G_v$ is -1, 0, or $l(G_v) - 2$, depending on how G_v compares with ☆. If $l(G_v) = 2$, then the $\mathrm{AW}\, G_v$ is restricted to just -1 and 0. Since Left can win moving first on $G_v + ☆$ by moving to G_v, $\mathrm{AW}(G_v) = 0 = l(G_v) - 2$. Left wins moving first or second when $l(G_v) > 2$ by removing ☆ at the earliest opportunity. Hence, in this case, $\mathrm{AW}(G_v) = l(G_v) - 2$. □

Problems

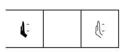
1. Find the value of

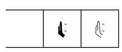


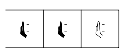
2. Find the value of



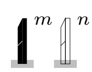
3. Find the values of the following ALL-SMALL PUSH positions. Express them using uptimal notation (base $\uparrow$) from page 188.

   (a) 

   (b) 

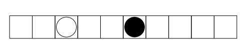
   (c) 

4. Determine the left and right incentives from TOPPLING DOMINOES positions of the form

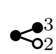


   Give specific guidance on how you should play on sums and differences of these games. (This is a follow-up to Problem 4 from Chapter 4.)

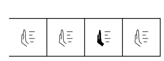
5. Who wins the disjunctive sum of the ALL-SMALL CLEAR THE POND position,



   the CUTTHROAT STARS position,



   and the ALL-SMALL SHOVE position,

 ?

6. In ALL-SMALL RUN OVER show that

$$\bigcirc\bullet^n = .\underbrace{\bar{1}\bar{1}\ldots\bar{1}}_{n}*.$$

7. In ALL-SMALL DRAG OVER show that

$$\bigcirc^m \bullet^n = .\underbrace{mm\ldots m}_{n-1} + m\cdot * = m \cdot (.\underbrace{11\ldots 1}_{n-1} *).$$

8. In the partizan version of FORKLIFT find the values for all positions of the form $(1, 1, \ldots, 1)$.

9. In PARTIZAN ENDNIM find the value for the one-pile position a and the two-pile positions of the form $b1$.

10. Prove Theorem 9.15 on page 190.

11. Prove Corollary 9.17 on page 191. If your proof is overly confusing, you might want to also demonstrate your proof using .11214 as an example.

12. Let $X = .\underbrace{11\ldots 1}0Y$ be a string of 0s and 1s and let k be a positive integer. Express each of the following as an uptimal (base $\uparrow$), perhaps plus $*$:

    (a) $\{0\,,\ .X* \mid 0\}$;

    (b) $\left\{0\,,\ .X\bar{k}* \mid 0\right\}$.

    These are generalizations of Exercise 9.18.

13. Prove that a game in canonical form that is not all-small is not equal to any all-small game. (*Hint*: Prove that if G is all-small, then it remains all-small when converted to canonical form.)

14. Prove Theorem 9.20.

15. Consider CUTTHROAT played on a path. Show

    (a) $RL^n = .\underbrace{11\ldots 1}_{n-1}*$;

    (b) $LRL^n = .\underbrace{00\ldots 0}_{n-1}\bar{1}$.

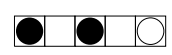
16. Show that $\{0 \mid *, *2\}$ cannot be expressed as a (base $\uparrow$) uptimal plus a nimber by showing that for all non-negative j and k,

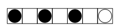
    (a) $\{0 \mid *, *2\} \mathrel{|\!\!\!>} .0k + *j$, and

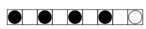
    (b) $\{0 \mid *, *2\} \mathrel{<\!\!\!|} .1\bar{k} + *j$.

17. Show that the values, in uptimal notation, of the ALL-SMALL BOXCARS positions are:

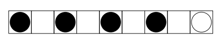
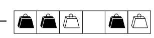
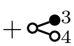
    (a)  $= *$,

    (b)  $= .1$,

    (c)  $= .2*$,

    (d)  $= .31$.

18. In the following disjunctive sum of an ALL-SMALL BOXCARS, a CLOBBER, and a CUTTHROAT STAR position, is there a winning move for Left? For Right?

 $+$  $+$ 

Chapter ω

Further Directions

An optimist, in the atomic age, is a person
who thinks the future is uncertain.

Howard Lindsay and Russell Crouse in
State of the Union

After having read this text, we hope you are eager to learn more. The first resource you should use remains the gem *Winning Ways* [BCG01], which (as you might have guessed from the number of times we referenced it) remains authoritative and thorough.

READ IT!

Having said that, combinatorial game theory is an active field of current research, and the most recent innovations are not covered in detail in *WW* [BCG01]. In this chapter, we indicate some other directions for further reading, with an emphasis on newer results. While making no attempt to be complete, we hope to provide a flavor for what we could not cover in this text.

ω.1 Transfinite Games

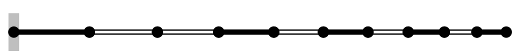
Games with ordinal birthdays are quite natural. For example, play the following game on polynomials with non-negative coefficients. (You are asked to analyze this game in Chapter 7, Problem 7.) A move is to choose a single polynomial and reduce one coefficient and arbitrarily change or leave alone the coefficients on the smaller powers — $3x^2 + 15x + 3$ can be reduced to $0x^2 + 19156x + 2345678 = 19156x + 2345678$. If there are no powers of x^i for $i > 0$ then this is just NIM; if there are powers of x then this is NIM with heaps of ordinal heights. Thus, $1x^1 = \{0, *, *2, *3, \ldots \mid 0, *, *2, *3, \ldots\}$, which is born on day ω.

With the correct notion of ordinal powers of 2, the theory for finite NIM extends to heaps of transfinite ordinals. Similarly, it is not at all hard to imagine playing HACKENBUSH on infinite graphs. For example, a string of blue edges might be

 $\cdots = \{0, 1, 2, \ldots \mid \} = \omega,$

the first infinite ordinal; adding a single blue edge after the $\cdots$ would give $\{0, 1, 2, \ldots, \omega \mid \} = \omega + 1$; changing the added edge to red gives $\{0, 1, 2, \ldots \mid \omega\}$, which behaves like $\omega - 1$, a value that does not occur in the standard set-theoretical development of ordinal arithmetic. Taking a single blue edge and adding an infinite string of red edges gives $\left\{0 \mid 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots\right\} = \frac{1}{\omega}$. Note that in these examples, *the elimination of all dominated options is no longer a safe operation*; in the latter case each and every option $\frac{1}{2^n}$ is dominated by another option $\frac{1}{2^{n+1}}$, but the original game is not equivalent to $\{0 \mid \} = 1$.

The HACKENBUSH string

 $\cdots = \dfrac{1}{3},$

simply because in binary $\frac{1}{3} = .010101\ldots$. Similarly, all real numbers can be expressed in binary (with up to ω bits to the right of the decimal point) and are born by day ω. With a little imagination, it is not too difficult to find a game that should have value $\frac{\omega}{2}$ but more exotic values such as $\omega^{\frac{1}{2}}$ and $\omega^{\frac{1}{\omega}}$ also exist. More can be found in *On Numbers and Games* [Con01] and [Con02], or in the more light-hearted exposition *Surreal Numbers* [Knu74].

ω.2 Algorithms and Complexity

In PARTIZAN ENDNIM, Section 2.3, it seemed that one might have to analyze most of the game tree to actually find a good move. But, in that instance we were able to provide a relatively simple algorithm to determine the outcome class of a position, and hence a good move. On the other hand, in Theorem 5.49, we saw that the disjunctive sum of relatively simple looking games could be hard to calculate. Determining the value of a CLOBBER position with just one white piece (on an arbitrary graph) [AGNW05] and also determining whether a player can jump and finish the game PHUTBALL in a single move [DDE02] are both NP-complete problems. In fact, there is a general, but completely imprecise, observation that any game that is actually interesting to play throws up algorithmically difficult problems at every turn.

In general, one would expect strategies and values to be complicated to find since we are looking for the resolution of a satisfiability problem for a formula that contains a long string of alternating quantifiers; for example, Left might want

$$\exists G^L \forall G^{LR} \exists G^{LRL} \forall G^{LRLR} \exists G^{LRLRL} \forall G^{LRLRLR} \ldots \exists G^{LR\ldots L} \geq 0.$$

In words, this simply says: there exists a left move such that, no matter what Right does there exists a left move, such that, no matter what Right does there exists a left move, such that, ..., such that Left wins. A (possibly) interesting question is whether or not there are simple conditions that determine classes of games whose values are easy to determine.

Combinatorial game theory starts at the end of the game and works back to the beginning, finding exact values when possible or at least outcome classes. Computer scientists are interested in algorithms and heuristics that allow play to start at the beginning and guide the player through to the end of the game. If we have a complete theory then both should coincide. However, even then, if the evaluation takes too long to compute, then how does one proceed? This leads directly to the problem of finding strategies and heuristics for play, which may not be optimal, but are rarely bad. We mentioned a few such in Chapter 1 and other based on temperature considerations in Chapter 8.

If you are interested in this area then good websites to start from are:

- The Games Group at the department of Computer Science, University of Alberta, http://www.cs.ualberta.ca/~games/;
- Erik Demaine's games pages, http://theory.csail.mit.edu/~edemaine/games/; and
- Aviezri Fraenkel's home page, http://www.wisdom.weizmann.ac.il/~fraenkel/ and in particular the papers [Fra00, Fra04].

ω.3 Loopy Games

The theory we have presented assumes that no position may be repeated; otherwise, the game may not terminate, violating one of the restrictions we have placed on the games. However, there are many loopy games, so called because their game graphs may contain cycles, for example, FOX & GEESE, HARE & HOUNDS, BACKSLIDING TOADS & FROGS, and CHECKERS. Their theory is more complicated and difficult. There is not, as yet, a good general definition of canonical form. However, in some loopy games, even though there is the potential for infinite play, one player has a winning strategy, one that has the game terminating in a finite number of moves, so that he never needs to avail himself of the repetitions. For these games, much of the *finite* theory can be mimicked. A game G is called a *stopper* if there is no infinite alternating sequence of play from any follower of G. That is, the game graph of G contains no alternating cycles. Stoppers, when played in isolation, are guaranteed to terminate, but infinite play might be possible if G is part of a disjunctive sum. In the 1970s, Conway, Bach, and Norton showed that every stopper has a unique canonical form. Aaron Siegel has extended the theory, finding a general method to determine whether an arbitrary loopy game is equivalent to a stopper [Sie07, Sie05]. However, loopy games in general, are not well understood.

ω.4 Kos: Repeated Local Positions

GO is a peculiar game: the *ko* rule forbids most repeated positions so the game is not inherently loopy, but local positions my repeat arbitrarily often. Even the game of WOODPUSH (played on a finite strip of squares with blue and red pieces; a piece retreats to the next empty square, Left to the left and Right to the right, and eventually moves off the strip, except if there is a contiguous string to an opponent's piece then it can move in the opposite direction pushing the string ahead of it) has many of the same ko situations that occur in GO:

□□□●○○□ $\xrightarrow{L}$ □□□□●○○ $\xrightarrow{R}$ □□□●○□○ $\xrightarrow{L}$ ko-threat

$\xrightarrow{R}$ answers ko-threat $\xrightarrow{L}$ □□□□●○○ $\xrightarrow{R}$ □□□●○○□

Note that in the global game, the final position is not the same as the original since there have been two moves, the ko-threat and the answer, played elsewhere. The position □●○□ is hot, but how hot? The thermographs for such situations have to take into account that one opponent might get two consecutive moves, since a player may ignore a ko-threat somewhere else. The thermographs presented in this book only have slopes of ± 1, or ∞. The *Extended Thermographs* [Spi01] required for GO and WOODPUSH can have slopes of $\pm\frac{1}{n}$ for any $n \neq 1$.

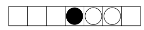
ω.5 Top-Down Thermography

Building a thermograph from the bottom up is time consuming. In practice, the shape of the thermograph near the temperature of the game is most important. Top-down thermography is the art (at the time of writing, it is not yet a science) of finding the slopes just before the thermograph turns into a mast.

For example, for the WOODPUSH position □●○□ we would have

$-4 =$	○□□□	$\leftrightarrow$	●○□□	$\leftrightarrow$	□●○□	$\leftrightarrow$	□□●○	$\leftrightarrow$	□□□●	$= 4,$
	$(-4, 0)$		$(-2, 2)$		$(0, 2)$		$(2, 2)$		$(4, 0)$	

where the means and temperatures are written underneath. In this example, the temperature is the difference in numbers divided by the number of steps;

i.e., $(4-(-4))/4$ and the change in the mean is obtained by moving in the adjacent positions and gaining the temperature. This is consistent across this example. The situation for  is much less clear.

ω.6 Enriched Environments

In cooling games, we have the concept of *taxing*. To use Berlekamp's words, "No one likes paying taxes." He avoided taxation by introducing bonuses, or *coupons* [Ber96]. In ENVIRONMENTAL GO, at the side of the board, there is a stack of coupons starting at 20 points and decreasing by $1/2$ until at the end there are three coupons each worth -1. On his turn, a player may move on the board or take a coupon — taking a coupon is also an answer to a ko-threat, and, at the end of the game, the value of the coupon is added to the points won on the board. The game is over when the last coupon is taken (assuming no resignation). Jiang Zhujiu and Rui Naiwei played the first Environmental Go game in April 1998, see [Spi02].

The coupons reflect the temperature of the game on the board. If the temperature is low but the coupon value is high, both players will take coupons until finally, a play on the board will gain more points than taking the next coupon.

This approach redefines temperature for numbers: the temperature essentially becomes the incentive. The temperature of 1 is -1 and the temperature of $3/2$ is $-1/2$.

ω.7 Idempotents

A game G is an idempotent if $G+G=G$. Idempotents have the effect of hiding from view games of smaller values. If there is an idempotent of the right size in play then the players can restrict their attention to the positions with only the largest temperatures and values. For example, ☆ is an idempotent that essentially hides the nimbers and those infinitesimals that are infinitesimal with $\uparrow$ and leads directly to the concept of atomic weights. Is there any equivalent notion for $\boldsymbol{+}_2$? Berlekamp [Ber02] lists and discusses other useful idempotents.

ω.8 Misère Play

In misère play, the last player to play loses. This seemingly innocuous change drastically affects the underlying theory. While *WW* [BCG01, Ch. 13] provides some rather helpful guidance, their chapter evidences how *hard* misère games are in general. As an exercise in the peculiarities of misère analysis, find two left-win games in misère play whose disjunctive sum is a right-win game in misère play. In misère play, if G has the property that $G + X$ has the same outcome as X for all games X then G *can only be* the game $\{ \mid \}$. By contrast, in normal play, all second-player-win games are equivalent. In normal play, the equivalence classes were large, thereby providing a significant simplification in the analysis of games. If we try the same approach for misère games, it turns out that the equivalence classes are rather small.

Recently, however, Thane Plambeck had a brilliant insight that leads to a coherent theory for impartial misère games. In normal play, the values encountered are the stars $0, *, *2$, etc., and these values give all the information needed to play in any disjunctive sum. The stars can also be regarded as elements of $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \cdots$. Plambeck's insight is to construct semi-groups that replace $\mathbb{Z}_2$. One small limitation with this approach is that the information provided is only sufficient to play the game as a disjunctive sum with other positions of the same game.

At the time of writing, Thane Plambeck and Aaron Siegel have made great progress, and we refer you to [SP06] or to the website http://www.miseregames.org/.

ω.9 Dynamical Systems

The work of Eric Friedman and Adam Landsberg, appearing in [FL07] and featured in *Science News* [Pet06], takes a novel view toward trying to understand otherwise impenetrable games. They use experimental techniques from dynamical systems (or chaos theory) to identify self-similarity patterns in "instant winner" positions in CHOMP. Their techniques provide strong evidence that CHOMP is very sensitive to the initial conditions (which small positions are deemed $\mathcal{P}$-positions) and therefore demonstrate why finding winning moves in CHOMP remains challenging.

Preparation for Appendix A

To the instructor: This appendix may be a bit of a stretch for the typical undergraduate. We highly recommend covering material up through Example A.5 on page 221. The remainder of the chapter provides additional motivation for a top-down view of induction and can safely be left for the stronger undergraduate to read at her leisure.

Appendix A
Top-Down Induction

Miss one step, and you will always be one
step behind.

Chinese proverb

A.1 Top-Down Induction

The type of induction most commonly used in combinatorial game theory (and indeed in most of discrete mathematics) is somewhat different from that traditionally taught in beginning mathematics courses. One version of it is summarized below:

1. To prove a statement $P(s)$ for all $s \in S$, feel free at any time (i.e., whenever convenient) to assume $P(r)$ is true for all $r < s$. Whenever such an assumption is invoked write, "by induction."
2. Carefully review your proof, and prove any case not handled — "base case."

A few notes are in order:

- You may be concerned that this form of induction seems to be going "backwards." Rather than saying, "Assume $P(n)$ and prove $P(n+1)$" as you may have first learned, this says, "Prove $P(s)$ by using $P(r)$ for $r < s$." The two views are totally consistent; both use the smaller cases to help prove larger cases.
- You never have to decide in advance that you plan to use induction. You use it whenever you wish in *any* proof, and later worry about any base cases you may have missed.

- *Most importantly*, S can be any set and $<$ can be any partial ordering which has the property that any non-empty subset of S has at least one minimal element.[1]

  One of the advantages of this form of induction is that it often turns out that no base cases are necessary! Though it is true that the set S will contain minimal elements, these are often covered in the "by induction" step of the argument. For the argument that shows "$P(s)$ is true provided that $P(r)$ is true for all $r < s$" may well not depend on there being any such r. Indeed, when such r do not exist it may well be that $P(s)$ follows automatically without any further discussion — such statements are often referred to as being *vacuously true*.

  The most elementary example of a partial ordering satisfying this minimality requirement is the non-negative integers under the usual $<$. In most proofs in this book, we usually have in mind the ordering of game positions, where $H < G$ if H is an option of G. When proving a fact about games G, we will assume, by induction, that the fact holds true for all options of G. Usually, for the trivial game with no options, the fact will vacuously hold, and no explicit base case will be required!

A.2 Examples

Example A.1. Show that the sum of the first n positive integers is $n(n+1)/2$.

Proof:

$$\begin{aligned} 1+2+\cdots+n &= (1+2+\cdots+(n-1))+n && \text{(A.1)}\\ &= (n-1)n/2+n, \text{ (by induction)} && \text{(A.2)}\\ &= n(n+1)/2 && \text{(A.3)}\end{aligned}$$

The base case is $n = 0$, and $n(n+1)/2 = 0$ as it should. □

Note that the base case of $n = 0$ is the *only* case not handled by (A.1). In particular, when $n = 0$, there are no terms in $1 + 2 + \cdots + n$, and so there is no n to separate out.[2]

Exercise A.2. Prove that the sum of the first n *odd* positive integers is n^2.

Example A.3. (Two-heap NIM) The game of two-heap NIM is played with two heaps of counters of sizes m and n. A move consists of removing any number of

[1] Such orders are called *well quasi-orders* and logicians sometimes call this technique ϵ-induction. See problem 1 for more formalism.

[2] This example is presented *only* because it is almost always the first example of a proof by induction and so will be familiar! A *much better* argument in this case is to use Gauss's trick — note that the average value of a summand is $(n+1)/2$ and hence the sum must be $n(n+1)/2$.

counters from either heap. The person who takes the last counter wins. Prove that the player on move can guarantee a win *if and only if* $m \neq n$.

Proof: Denote the position by the ordered pair (m, n), where without loss of generality $m \leq n$. If $m < n$, the player on move can move to (m, m), which, by induction, wins. If, on the other hand, $m = n$, every move leaves heaps of differing sizes, which, by induction, allows the opponent to guarantee a win; since all moves lose, the position is losing. □

Note that no base case is required in the last theorem. When, for example, $m = n = 0$, the last sentence still holds: There is no legal move, and so every move really does leave heaps of differing sizes.

Exercise A.4. The following variant of NIM is played with a single heap starting with n counters. A move consists of removing between one and ten counters, and the player who takes the last counter wins.

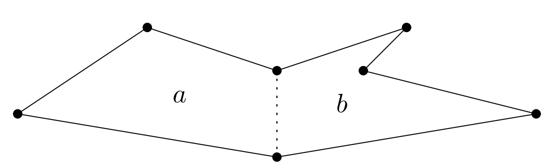
1. Prove that the second player wins *if and only if* $n \equiv 0 \pmod{11}$.

2. Explain why your proof requires no base case.

When trying to solve any problem, first try to break up your problem into smaller pieces. In Equation (A.1) above, there was one very natural way to do this, and induction followed naturally. The next two examples illustrate this even more clearly.

Example A.5. Show that the sum of the interior angles of an n-sided polygon is $180 \cdot (n - 2)$.

Proof: Let an n-sided polygon be given. Find an interior diagonal of this polygon; that is, a line connecting two of its vertices that passes through the interior of the polygon and meets it only at its endpoints.[3]



[3]Showing that a diagonal exists is surprisingly difficult! Choose three consecutive vertices A, B, and C such that the interior angle $\angle ABC$ is less than $180°$. If AC is a diagonal then we are done. Otherwise, there are points of the polygon on the boundary or interior of the triangle ABC. Among these choose a vertex of the polygon D such that the line through D parallel to AC is as close to B as possible. Then BD is a diagonal.

The sum of the angles of the original polygon is equal to the sum of the angles of the two parts of the polygon formed by the diagonal. Let these parts have a and b sides, respectively. The sum $a+b$ is equal to $n+2$ since each original side of the polygon is a side of one of the parts, and the diagonal is a side of both parts. By induction, the sum of the angles of the polygon is

$$180 \cdot (a-2) + 180 \cdot (b-2) = 180 \cdot (a+b-4) = 180 \cdot (n-2) \text{ degrees.}$$

The base case of the induction occurs when we cannot find such a diagonal; that is, if $n = 3$, but the result is known in that case. □

Example A.6. Let the Fibonacci numbers a_n be given by

$$a_n = \begin{cases} 0 & \text{if } n = 0, \\ 1 & \text{if } n = 1, \\ a_{n-1} + a_{n-2} & \text{if } n \geq 2. \end{cases}$$

Show that $a_n = a_r a_{n-r-1} + a_{r+1} a_{n-r}$ for $0 \leq r < n$.

Proof:

$$\begin{aligned} a_n &= a_{n-1} + a_{n-2} && \text{(A.4)} \\ &= (a_r a_{n-r-2} + a_{r+1} a_{n-r-1}) + (a_r a_{n-r-3} + a_{r+1} a_{n-r-2}) && \text{(A.5)} \\ &= a_r(a_{n-r-2} + a_{n-r-3}) + a_{r+1}(a_{n-r-1} + a_{n-r-2}) && \text{(A.6)} \\ &= a_r a_{n-r-1} + a_{r+1} a_{n-r}. && \text{(A.7)} \end{aligned}$$

Here, Equation (A.5) is obtained by induction and Equation (A.7) by the definition of the Fibonacci numbers. Now, to figure out the base cases we need to see how the argument above can fail. Equation (A.4) fails when $n = 0$ or $n = 1$. Equation (A.5) fails when $r = n-1$ or $r = n-2$, since the statement of the theorem insists that $r < n$. In other words, when invoking the induction hypothesis, we need $r < n-1$ to expand a_{n-1} and $r < n-2$ to expand a_{n-2}. Equations (A.6) and (A.7) introduce no further problematic cases. So the base cases are:

- $n = 0$: No $0 \leq r < n$ exists, so the theorem is vacuously true;
- $n = 1$: Then $r = 0$ and $a_0 a_0 + a_1 a_1 = 0 + 1 = 1 = a_1$;
- $r = n-1$: $a_{n-1} a_0 + a_n a_1 = a_n$;
- $r = n-2$: $a_{n-2} a_1 + a_{n-1} a_2 = a_{n-2} + a_{n-1} = a_n$. □

Note that there is really no need to differentiate between "base cases" of an induction and "special cases" of any proof. One could rewrite the whole proof without any "base case," and the choice is a matter of style:

Alternate proof sketch: If $n = 0$ or $n = 1$, we can verify that the theorem holds. Otherwise,

$$\begin{aligned} a_n &= a_{n-1} + a_{n-2},\ n \geq 2 \\ &= \begin{cases} a_{n-1}a_0 + a_n a_1 = a_n & \text{if } r = n-1, \\ a_{n-2}a_1 + a_{n-1}a_2 = a_{n-2} + a_{n-1} = a_n & \text{if } r = n-2, \\ (a_r a_{n-r-2} + a_{r+1}a_{n-r-1}) + \\ \quad (a_r a_{n-r-3} + a_{r+1}a_{n-r-2}) \\ \ = a_r(a_{n-r-2} + a_{n-r-3}) + & \text{otherwise (by induction).} \\ \quad a_{r+1}(a_{n-r-1} + a_{n-r-2}) \\ \ = a_r a_{n-r-1} + a_{r+1}a_{n-r} \end{cases} \end{aligned}$$

□

There is another proof, which is arguably even better — because it leads to a deeper understanding of *why* this identity holds, and also to methods for generating similar identities. Consider the matrix

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} a_0 & a_1 \\ a_1 & a_2 \end{pmatrix}.$$

Then it is easy to check (by induction!) that

$$A^n = \begin{pmatrix} a_{n-1} & a_n \\ a_n & a_{n+1} \end{pmatrix}.$$

The desired identity is then just a consequence of $A^n = A^r A^{n-r}$.

Theorem A.7. (Euler's Formula) *For any planar embedding of a graph with E edges, V vertices, C connected components, and R regions, we have*

$$E - V - R + C + 1 = 0.$$

Proof: Fix a planar embedding as above. Removing an edge either merges two regions into one or increases the number of components by 1. So, by induction, either,

$$\begin{aligned} (E-1) - V - (R-1) + C + 1 &= 0, \text{ or} \\ (E-1) - V - R + (C+1) + 1 &= 0. \end{aligned}$$

It trivially follows that $E - V - R + C + 1 = 0$. For the base case, if $E = 0$, then $V = C$ and $R = 1$, and Euler's Formula again holds. □

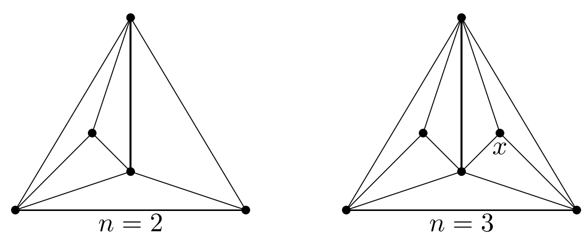
A.3 Why is Top-Down Induction Better?

The usual axiomatic presentation of mathematical induction suggests that to prove some statement $P(n)$ for $n \geq 0$,

1. Prove the base case, $P(0)$.
2. Induction step: Assume $P(n)$ and prove $P(n+1)$.

There are many reasons why top-down induction is better to use than the axiomatic approach. The most important is that in complicated proofs using axiomatic induction it is easy to overlook some cases of size $n+1$. This is a popular pitfall among even the best of mathematicians! One such false proof will exemplify the pitfall.

Example A.8. Prove that n internal points triangulate a triangle into $2n+1$ smaller triangles. In other words, n points are placed in the interior of a triangle indicated by three points. If non-crossing line segments (*edges*) are introduced connecting the $n+3$ points until no more can be added, then the triangle will contain $2n+1$ regions:



False proof: For a base case, if there are $n = 0$ added points, there are $2n+1 = 1$ triangles. For the inductive step, assume that if there are n points, there must be $2n+1$ triangles. To prove the case for $n+1$, add one more point, say x, to the triangle. This removes one region, but creates three, leaving a total of $2n+1-1+3 = 2(n+1)+1$ regions. □

This argument is very seductive; in fact, most are taken in by it unless warned it is fallacious. The reason the proof is incomplete is that we have not argued about *all* possible triangulations — only those that can be arrived at from smaller triangulations. You may be tempted to patch the proof by permitting the addition of points along previous edges, but that still would not be good enough! With some work, you can come up with a triangulation on $n = 11$ interior points where each point is incident to at least five edges. This triangulation could not have been arrived at from a smaller triangulation with the addition of one point — the last point added would only be incident to three edges (or four edges in the proposed "patch").

Correct proof of Example A.8: Begin with a triangulation on n interior points. Remove one of the points. If the point had k incident edges, then k triangles will be removed, leaving a k-sided polygonal region. A k-sided polygon can be divided into $k-2$ triangles (by an argument parallel to Example A.5). By induction, this new triangulation has $2n-1$ triangles, but it has $k-(k-2)$ or two fewer triangles than the original. So the original triangulation had $2n+1$ triangles. The base case of $n=0$ is trivial. □

Being able to write consistently correct proofs is the most important advantage with top-down induction. But a few more advantages are worth noting:

- The axiomatic model's assumption that you are always trying to prove a statement $P(n)$ for $n \geq 0$ is simplistic. First, in actual proofs, the parameter n might take on many different forms. Further, determining what in your proof *should* serve the role of n is not always obvious. Of course, those investigating the fundamentals of mathematics would consider this simplistic view a benefit, but most students are interested in understanding how to apply induction in the widest possible context.

- In a complicated proof, it is unclear what is the *right* base case to choose, particularly if the induction step has not been proved yet. In Example A.6, it is hard to foresee what base case is needed before doing the induction argument; so why do the base case first? Further, top-down induction gives guidelines for how to choose the strongest possible, minimal-sized base case.

- Students often wonder, "How do I know when to use induction?" This is a surprisingly hard question to answer. The top-down induction paradigm sidesteps this issue somewhat, since there is no need for the prover to know in advance that induction is required. Through the process of breaking up a problem or expanding or massaging an expression, if a smaller case ever appears, induction can be used freely.

- Induction on several variables is completely demystified. The prover need not know in advance what parameters induction will be used on, nor whether induction will be used on several variables.

- The axiomatic induction suggests that to prove $S(n)$, first assume (by way of induction) $S(n)$. This *sounds* circular. Of course it is not, since you would then proceed to prove $S(n+1)$. But since you are trying to prove $S(n)$ doesn't it make more sense to assume $S(n-1)$ and prove $S(n)$? Top-down induction, like strong induction, goes one step further, allowing you to assume $S(m)$ for $m < n$.

A.4 Strengthening the Induction Hypothesis

Strengthening the induction hypothesis is a technique of trying to prove more than you need to. It may seem counterintuitive that this could make life easier; but if you try to prove more, you also get to *assume* more (by induction). Here is an example:

Example A.9. Define the Fibonacci numbers a_n as in Example A.6. Suppose $n = 2k+1$ is odd. Prove that $a_n = a_{k+1}^2 + a_k^2$.

The natural first step is to write

$$\begin{aligned} a_{2k+1} &= a_{2k} + a_{2k-1} \\ &= a_{2k} + a_k^2 + a_{k-1}^2, \text{ by induction.} \end{aligned}$$

It is hard to go further, since we have no formula for a_n when n is even. Let's be optimistic and try to see what a_{2k} would need to complete the proof. Well, if a_{2k} were equal to $a_{k+1}^2 - a_{k-1}^2$, that would do it easily! This suggests *strengthening the induction hypothesis*. So we will aim to prove a statement that deals with both cases. Rather than using the form above for a_{2k} experience suggests that something more similar to the form for a_{2k+1} might make the algebra simpler. Such a form is easily found:

$$\begin{aligned} a_{k+1}^2 - a_{k-1}^2 &= (a_{k+1} - a_{k-1})(a_{k+1} + a_{k-1}) \\ &= a_k(a_{k+1} + a_{k-1}) \\ &= a_k a_{k+1} + a_k a_{k-1}. \end{aligned}$$

So, we need to prove that

$$\begin{aligned} a_{2k+1} &= a_{k+1}^2 + a_k^2, \text{ and} \\ a_{2k} &= a_k a_{k+1} + a_k a_{k-1}. \end{aligned}$$

Proof:

$$\begin{aligned} a_{2k+1} &= a_{2k} + a_{2k-1} \\ &= a_k a_{k+1} + a_k a_{k-1} + a_k^2 + a_{k-1}^2, \text{ by induction} \\ &= a_k a_{k+1} + (a_k + a_{k-1})a_{k-1} + a_k^2 \\ &= a_{k+1}(a_k + a_{k-1}) + a_k^2 \\ &= a_{k+1}^2 + a_k^2. \end{aligned}$$

$$\begin{aligned} a_{2k} &= a_{2k-1} + a_{2k-2} \\ &= a_k^2 + a_{k-1}^2 + a_{k-1}a_k + a_{k-1}a_{k-2}, \text{ by induction} \\ &= a_k(a_k + a_{k-1}) + a_{k-1}(a_{k-1} + a_{k-2}) \\ &= a_k a_{k+1} + a_k a_{k-1}. \end{aligned}$$

We omit the base case. □

A.5 Inductive Reasoning

Example A.10. (Pick's Theorem) The *lattice points* in the plane are the points with integer coordinates. Let P be a polygon that does not cross itself (i.e., a *simple* polygon) such that all of its vertices are lattice points. Let p be the number of lattice points that are on the boundary of the polygon (including its vertices), and let q be the number of lattice points that are inside. Prove that the area of the polygon is $p/2 + q - 1$.

The key to the proof of *Pick's Theorem* is that when using induction, we can assume the statement we are trying to prove is true for *simpler* polygons rather than *smaller* polygons. As in Example A.5, we can divide the polygon into smaller polygons and use induction. But proving *Pick's Theorem* for a triangle still looks challenging. It's not hard to prove *Pick's Theorem* for right triangles (and rectangles) aligned with the x- and y-axes, though, since it is easier to count lattice points on the boundary and interior of these right triangles. So how do we reduce the problem to just right triangles? Read on....

The following lemma is a simple exercise.

Lemma A.11. *Let P_1 and P_2 be lattice polygons whose union, P, is also a simple lattice polygon. Assume that P_1 and P_2 intersect only along a set (possibly only one) of edges. Then*

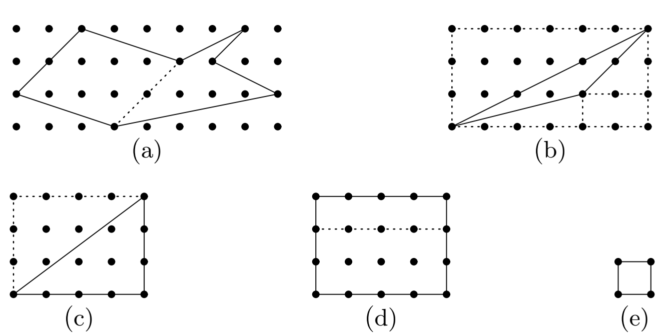
Addition Lemma: If P_1 and P_2 satisfy Pick's formula, then so does P.

Subtraction Lemma: If P and P_1 satisfy Pick's formula, then so does P_2.

Halving Lemma: If P_1 and P_2 are isomorphic, then P satisfies Pick's formula if and only if P_1 does.

The areas of P_1 and P_2 add to P. So, to prove the lemma, it suffices to show that Pick's formula is also additive; i.e., $(p_1/2+q_1-1)+(p_2/2+q_2-1) = (p/2+q-1)$.

Now for a picture proof of *Pick's Theorem*:



A polygon (a) with at least four sides can be separated into two regions, each which, by induction, satisfy *Pick's Theorem,* and so by the Addition Lemma the original polygon satisfies *Pick's Theorem* as well.

A triangle (b) can be formed by the difference between a large rectangle, and smaller rectangles and right triangles all aligned with the x- and y- axes. Each of the rectangles and right triangles satisfy *Pick's Theorem* (by induction), and so by the Subtraction Lemma, so does an arbitrary triangle.

For a right-triangle (c), adjoin another right triangle to form a rectangle. The rectangle satisfies *Pick's Theorem* by induction, and so by the Halving Lemma, so does the right triangle.

A rectangle (d) can be divided into two rectangles by a horizontal or vertical line; apply induction.

Finally, for the base case of a unit rectangle, Pick's formula yields $(p/2 + q - 1) = 4/2 + 0 - 1 = 1$, which is indeed the unit rectangle's area.

Problems

1. Formally, the principle of top-down induction can be stated as follows. Let universe U be a partially ordered set with the property that every non-empty subset of U has at least one minimal element. If

$$\forall n : [\forall m < n : P(m)] \;\Rightarrow\; P(n),$$

   then

$$\forall n : P(n).$$

   (Although not relevant to this problem, note that if n is minimal, $\forall m < n : P(m)$ is vacuously true. This is usually called a *base case.*)

   Prove the principle of top-down induction by contradiction. (*Hint*: Let $T = \{n : \text{P(n) is false}\}$. You will prove $T = \emptyset$.)

2. To highlight the importance of the assumption "every non-empty subset of U has at least one minimal element," show that the statement $x = 0$ about real numbers $x \in [0, 1]$ satisfies the inductive condition above, but is *not* true of every real number in $[0, 1]$. In other words,

   (a) Sketch a "good-looking" top-down induction proof that $x = 0$ for all $0 \leq x \leq 1$.

   (b) Explain how every subset of reals in $[0, 1]$ may not have a minimal element by giving a specific counterexample.

3. Prove that any triangulation of an n-sided polygon yields $n - 2$ triangles. This should complete the correct proof to Example A.8.

4. Complete the proof of *Pick's Theorem* by proving Lemma A.11.

5. Prove that any rational number between 0 and 1 can be written as a sum of distinct fractions with numerator 1. As an example, $7/11 = 1/2 + 1/11 + 1/22$.

6. Prove the *Arithmetic-Mean-Geometric-Mean Inequality for n Numbers.* For any n non-negative numbers $a_1, a_2, \ldots, a_n$,
$$\frac{a_1 + a_2 + \cdots + a_n}{n} \geq \sqrt[n]{a_1 a_2 \cdots a_n}.$$
Let $P(n)$ be the theorem for fixed n.
   (a) For a warm-up, prove $P(1)$ and $P(2)$.
   (b) If $n = 2m$, prove $P(m)$ implies $P(n)$.
   (c) Prove $P(n+1)$ implies $P(n)$ (this is *backwards induction*).
   (d) Explain how using induction you have proved the theorem for all n.

Appendix B
CGSuite

Besides black art, there is only automation
and mechanization.

Federico García Lorca

The Combinatorial Game Suite [Sie03] is an open-source program authored by Aaron Siegel that does all of the algebraic manipulations of games. Written in Java, the program is platform independent. Its features include:

- a great user interface;
- ability to manipulate short games and loopy games;
- lots of built-in games;
- for the Java programmer, ability to add plug-ins to, for example, further augment the built-in games;
- a graphical explorer to navigate a position's game tree;
- a Maple-like programming language for writing scripts.

The goal of this appendix is to give you a brief introduction to the software. Much of this appendix was excerpted, with permission, from the CGSuite tutorial at www.cgsuite.org.

B.1 Installing CGSuite

Installing the software on most modern systems is easy. Go to www.cgsuite.org and follow the directions. In particular, you will need to be sure to have a sufficiently recent version of Java running; download and unpack the software, and you should be ready to go.

B.2 Worksheet Basics

The most fundamental part of CGSuite is the worksheet, where you can type commands and perform calculations directly. If you are new to CGSuite, you should launch CGSuite and begin experimenting by typing commands into the worksheet.

Entering games

You can type games directly into the worksheet. For example, click on the window labeled "Worksheet 1" and enter the following:

```
1/2vv*2
```

This represents the game $\frac{1}{2} + \Downarrow + {*2}$. In general, you can enter combinations of numbers, ups and downs, and nimbers just as you would expect. The symbol `^` is used for $\uparrow$, `v` for $\downarrow$, and `*` for nimbers. To enter, say, ⟰, you could type either `^^^^` or `^4`. More complicated games can be entered using braces and slashes. For example:

```
{2||1|0,*}
```

Expressions containing slashes must be enclosed in braces. That is, to enter the game `1|0`, you must type `{1|0}`. Ambiguous expressions, such as `{1|0|-1}`, will be rejected; you would need to enter `{1||0|-1}` (or `{1|{0|-1}}`) or `{1|0||-1}` (or `{{1|0}|-1}`) instead.

CGSuite recognizes a wide variety of common games and displays them using standard shorthand notation (such as $+_2$ for tiny-two). Here are a few to try (make sure to enter each on a separate line):

```
{0||0|-2}
{0|v*}
{1,1*|-1,-1*}
```

The game `{0||0|-2}` could also be entered as `Tiny(2)`.

Operations on games

You can enter sums and products directly. A period (`.`) is used for Norton product. For example:

```
{3|2} + {2|0}
4.{2||1|0}
1/8.^
```

There are nine comparison operators:

```
==   <=   >=   <   >   !=   <|   |>   <>
```

(Respectively: equals, less than or equal to, greater than or equal to, less than, greater than, not equal to, less than or confused with, greater than or confused with, confused with.) For example, try entering:

```
^^ > *
0 <| ^*
```

Other common operations can be entered as method calls. Some examples to try:

```
Mean({3||2|1})
Temperature({3||2|1})
Plot(Thermograph({3||2*|1}))
AtomicWeight({^^|vv})
Cool({3||2|1},1/2)
Freeze({3||2|1})
Heat(*,3)
Overheat(1/2,1,2)
OrdinalSum(*7,1)
```

Many more are available as well; see the glossary of methods in CGSuite's tutorial for a complete list. Variable assignments are permitted, e.g.,

```
G := {3||2|1}
Freeze(G)
```

And multiple statements can be strung together as a single command using semicolons, so the above could be rewritten on a single line as

```
G := {3||2|1}; Freeze(G)
```

Note that output is generated only for the last command. If a command ends with a semicolon, no output will be generated at all.

Types and canonical forms

Every object in CGSuite has a type associated with it. Objects such as `^^`, `{3||2|1}`, etc., are canonical games, but CGSuite also supports games that are not in canonical form. In this way, positions in a game such as DOMINEERING are distinguished from their canonical forms.

For example, try entering the following:

```
H := DomineeringRectangle(4,4)
```

This assigns to H the DOMINEERING position represented by an empty 4×4 grid. Note that CGSuite does not yet try to calculate the canonical form of H. The canonical form of H can be obtained by explicit user request, using the `Canonicalize` method:

```
Canonicalize(H)
```

Alternatively, you can use the shorthand `C(H)`. Of course we could have just typed:

```
C(DomineeringRectangle(4,4))
```

Several other games are included as examples, including AMAZONS, CLOBBER, FOX & GEESE, KONANE, and TOADS & FROGS. Typically, positions can be entered as a sequence of strings, separated by commas; each string corresponds to a single row in the grid. For example, try entering:

```
A := Amazons("L..X.","R....")
```

As always, `C(A)` calculates the canonical form of A.

Exercise B.1. Let G be the game $\{2 \,\|\, 1 \mid *\}$. Calculate the canonical forms of $G + G$ and $G + G + G + G$. Compute the mean and temperature of G. How does G compare with 1 and $1*$?

Exercise B.2. Use the `LeftOptions` and `RightOptions` commands to determine the canonical form of $\Uparrow$.

Exercise B.3. Let H be the canonical form of the 4×4 DOMINEERING rectangle that we calculated above. Try comparing H with small positive and negative numbers. Make a conjecture as to whether or not H is an infinitesimal, and then use `IsInfinitesimal(H)` to test your conjecture. How does H compare with various tinies?

Exercise B.4. Heat $*6$ by 1. Note that only the first few lines of the canonical form are displayed. Click on the “More . . . ” button to display the full canonical form.

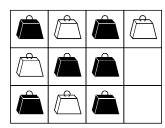
Exercise B.5. Try to find constraints on positive numbers a, b, c $(a > b)$ such that $\{a \mid b\} + ⧾_c = \{a⧾_c \mid b⧾_c\}$.

Graphically exploring positions

A more intuitive interface is provided by the graphical explorer. After reviewing the online tutorial section entitled "Explorer" at www.cgsuite.org, try the exercises.

Exercise B.6. Find the best moves from a 4×4 DOMINEERING rectangle. Then find the "sensible lines of play," observing that one move for each player reverses out.

Exercise B.7. Find the sensible lines of play for the following 3×4 CLOBBER position:



Then calculate its atomic weight. If you have the patience, find the sensible lines of play for each of its sensible options, and observe that CLOBBER is not an easy game!

Exercise B.8. Calculate the canonical form and atomic weight of the following 3×7 CLOBBER position from [AGNW05]:

Its canonical form is surprisingly complicated given that Right has just one piece.

Exercise B.9. Load the Additional Games plug-in (you can find it by selecting "Plug-in Manager" from the Tools menu) and experiment with Samson de Jager's Cherry Tree editor.

B.3 Programming in CGSuite's Language

You can write your own programs in the worksheet in CGSuite's built-in language, a language similar to Maple or Visual Basic. Here is an example script that finds the values of TOPPLING DOMINOES positions; the script is followed by commentary explaining each line. For more examples, see the CGSuite's tutorial or the scripts in directory "examples" that comes with the software distribution.

```
Topple := proc(pos)
  local i,l,r,g1,g2;
  option remember;
  l := [];
  r := [];
  for i from 1 to Length(pos) do
    g1 := Topple(seq(pos[p],p=1..i-1));
    g2 := Topple(seq(pos[p],p=i+1..Length(pos)));
    if (pos[i] == 'L' or pos[i] == 'G') then
      Add(l, g1);
      Add(l, g2);
    fi;
    if (pos[i] == 'R' or pos[i]=='G') then
      Add(r, g1);
      Add(r, g2);
    fi;
  od;
  return {l | r};
end;
```

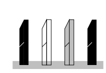
To enter the script, it's generally easiest to use a text editor and either

1. copy and paste the script into a worksheet and hit return to evaluate, or
2. select the menu item under "File" called "Run script" and enter the full filename of your script.

Once you have evaluated the above script, you should be able to type

```
Topple(['L','R','G','L'])
```

to compute the value of the position



The first line defines `Topple` to be a procedure which takes one argument called `pos`. `Topple` expects `pos` to be a *sequence,* each element of which is a character, `'L'`, `'R'`, or `'G'`. (Sequences are comma-separated and surrounded by brackets.)

The next command, `local`, declares that the variables `i`, `l`, `r`, `g1`, and `g2` have scope only within `Topple`. If this were not here, these would be *global* variables visible outside of the procedure. Perhaps more importantly, specifying `local` assures that each recursive invocation of `Topple` will get its own copy of the three local variables.

The command `option remember` tells CGSuite to store results of every call to `Topple` in a table so that if `Topple` is ever called with the same argument, it can return the same answer rather than recomputing it. If using this feature before having fully debugged your script, you will want to type `clear(Topple)` between changes to tell CGSuite to forget old (presumably buggy) results.

The next two lines initialize `l` and `r` to empty sequences. The main body of the program will now build up the sequences of left and right options from `pos`.

To build up the options, the `for` loop is used to loop the index `i` across the sequence `pos`. First, `g1` and `g2` are the values of the games when the i^{th} domino is tipped toward the right and to the left, respectively. In particular, `Topple(seq(pos[p],p=1..i-1))` builds a new sequence consisting of the first $i-1$ dominoes of `pos`, and then recursively evaluates this sequence as a TOPPLING DOMINOES position. Then `pos[p],p=i+1..Length(pos)` finds the value of the position consisting of only the rightmost dominoes after position i.

Next, `Add(l, g1)` and `Add(l, g2)` append these games to the left- and/or right-options as appropriate depending on the color of the i^{th} domino.

Lastly, `return {l | r}` constructs the game consisting of the left option sequence versus the right option sequence.

B.4 Inserting a Newline in CGSuite

If you wish to edit programs or complex expressions within CGSuite, you can *hold the shift key while pressing enter.* This suppresses evaluation and simply inserts a newline. Use this technique to enter, for example

```
lst:=['G'];
for i from 1 to 4 do
Add(lst,'L');
[lst,Topple(lst)]>>out;
Add(lst,'R');
[lst,Topple(lst)]>>out;
Add(lst,'G');
[lst,Topple(lst)]>>out;
od;
```

B.5 Programming in Java for CGSuite

If you have programmed in Java, you can extend CGSuite's commands by adding plug-ins, software extensions which work in a uniform way to piggyback on CGSuite's versatile user interface.

To learn to write plug-ins, again we refer you to the tutorial, and also to the examples that come in the `plugins` directory of CGSuite's source distribution.

Appendix C

Solutions to Exercises

Chapter 1 Solutions

1.7: (page 16) If m, n are both even, then the second player should play a symmetry strategy — play the same move but rotate the board by 180°.

If one of m, n is even and the other odd, then the first player should take the central two squares and then play a symmetry strategy — play the same move but rotate the board by 180°.

1.11: (page 17) In this game, the number of moves in the game is fixed! Each move creates one more piece, at the start of the game there is one piece, and at the end there are 30. So the game lasts 29 moves, and the first player wins no matter how she plays.

1.16: (page 24) The middle move is the only non-loony move.

1.17: (page 24)

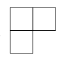
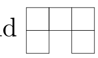
(a) Both positions are loony since the next player can take the fourth and fifth coins in the top row and play first in the rest of the game; or can cut the string joining them and move second in the rest of the game.

(b) Alice should double-deal in the left-hand position because then Bob has to move first in the long chains. In the other position, Alice should take both coins and then cut at the bottom-right string.

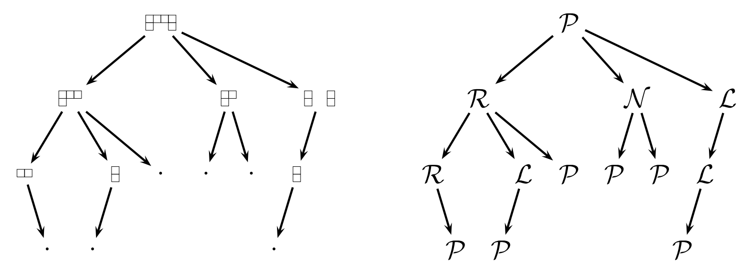
(c) Left-hand game: Alice 6, Bob 4; right-hand game: Alice 8, Bob 2.

1.19: (page 28) Alice 4, Bob 5.

Chapter 2 Solutions

2.2: (page 36) The position  is an $\mathcal{N}$-position, and  and  are both $\mathcal{P}$-positions.

2.6: (page 39)



2.8: (page 40) The only difference is that from the top square Left can move to a $\mathcal{P}$-position, and so the top square has outcome class $\mathcal{L}$.

2.12: (page 41) Fix an impartial game G. By induction, all options are either in $\mathcal{N}$ or in $\mathcal{P}$, and since the left and right options are identical, some $G^R \in \mathcal{P}$ *if and only if* some $G^L \in \mathcal{P}$. Hence, $G \in \mathcal{P}$ or $G \in \mathcal{N}$. (In other words, we can only be in the upper-left or lower-right corner of the table shown in Observation 2.4.)

2.14: (page 41) The end-positions have no options, and so the statement "every option is in B" is vacuously satisfied. So end-positions must be assigned to set A.

2.17: (page 43) Doing all arithmetic mod 7, the $\mathcal{P}$-positions are exactly those where $n \equiv 0$ or $n \equiv 2$. From such a position, all moves are in $\{0-1, 0-3, 0-4, 2-1, 2-3, 2-4\}$, which are equivalent to $\{6, 4, 3, 1, 6, 5\}$, all $\mathcal{N}$-positions. On the other hand, if $n \in \{1, 3, 4, 5, 6\}$, there are winning moves to $\{0, 0, 0, 2, 2\}$, respectively.

2.18: (page 44)

$\boldsymbol{a}$	$\boldsymbol{a22}$	$\boldsymbol{a23}$
0	$\mathcal{P}$	$\mathcal{R}$
1	$\mathcal{N}$	$\mathcal{R}$
2	$\mathcal{N}$	$\mathcal{R}$
3	$\mathcal{L}$	$\mathcal{P}$
4	$\mathcal{L}$	$\mathcal{L}$
5	$\mathcal{L}$	$\mathcal{L}$
$\vdots$	$\mathcal{L}$	$\mathcal{L}$

2.19: (page 44) If Left can win by taking a then there is nothing to prove. (This covers the case when $\mathbf{w}b$ is empty.) If Left has a winning move to $a'\mathbf{w}b$, $0 < a' < a-1$, after Right's move to $a'\mathbf{w}b'$ then Left has a winning move to $a''\mathbf{w}b'$. However, Left has this winning move from $(a-1)\mathbf{w}b'$.

The point is that the options from leftmost-heap size a', where $0 < a' < a$ form a subset of those from $a-1$. So the move to $a-1$ is at least as good as the move to a', for it burns fewer bridges.

2.21: (page 45) If $a > L(\mathbf{w}b)$, then Left can win moving first by decreasing pile a to exactly $L(\mathbf{w}b)$, from which she wins moving second by the definition of $L(\cdot)$. If, however, $a \leq L(\mathbf{w}b)$, then every move by Left loses, for it leaves a position of the form $a'\mathbf{w}b$ for $0 \leq a' < L(\mathbf{w}b)$. Similarly, Right wins moving first *if and only if* $b > R(a\mathbf{w})$. The observation follows.

Chapter 3 Solutions

3.3: (page 57) $G + H +$  is in $\mathcal{N}$. For both players, the winning first move is in H.

Chapter 4 Solutions

Prep 4.1: (page 64) Left simply responds locally. When playing $G_1 + G_2$, if Right moves on G_1, Left makes a response on G_1 which wins moving second on it. Similarly, Left responds to moves in G_2 by moving on G_2. Eventually, Right will run out of moves in both G_1 and G_2 and so will lose $G_1 + G_2$.

Prep 4.2: (page 64) One example is $G_1 = G_2 = \{0 \mid -1\}$.

4.6: (page 69)

(a) By definition, $G + H = \left\{\mathcal{G}^L + H, G + \mathcal{H}^L \,\middle|\, \mathcal{G}^R + H, G + \mathcal{H}^R\right\}$ and by induction, all the simpler games are commutative; for example, $G^L + H = H + G^L$ for $G^L \in \mathcal{G}^L$. So,

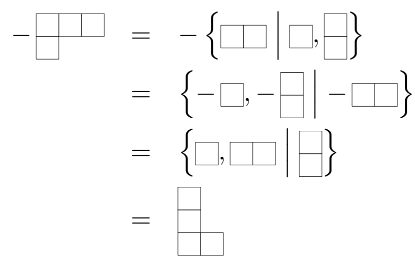
$$\begin{aligned} G + H &= \left\{\mathcal{G}^L + H, G + \mathcal{H}^L \,\middle|\, \mathcal{G}^R + H, G + \mathcal{H}^R\right\} \\ &= \left\{H + \mathcal{G}^L, \mathcal{H}^L + G \,\middle|\, H + \mathcal{G}^R, \mathcal{H}^R + G\right\} \qquad \text{(by induction)} \\ &= H + G. \end{aligned}$$

(b) To save page width, we will focus on the left options only:

$$\begin{aligned}
&[(G+H)+J]^L \\
&\quad = \{(G+H)^L + J, (G+H)+\mathcal{J}^L\} \\
&\quad = \{(\mathcal{G}^L+H)+J, (G+\mathcal{H}^L)+J, (G+H)+\mathcal{J}^L\} \\
&\quad = \{\mathcal{G}^L+(H+J), G+(\mathcal{H}^L+J), G+(H+\mathcal{J}^L)\} \text{ (by induction)} \\
&\quad = [G+(H+J)]^L,
\end{aligned}$$

and similarly for the right options, so $(G+H)+J = G+(H+J)$.

4.7: (page 69)



4.8: (page 69)

$$\begin{aligned}
-(-G) &= -\left\{-\mathcal{G}^R \mid -\mathcal{G}^L\right\} \\
&= \left\{-(-\mathcal{G}^L) \mid -(-\mathcal{G}^R)\right\} \\
&= \left\{\mathcal{G}^L \mid \mathcal{G}^R\right\}.
\end{aligned}$$

4.9: (page 69)

$$\begin{aligned}
-(G+H) &= -\left\{\mathcal{G}^L + H, G+\mathcal{H}^L \mid \mathcal{G}^R + H, G+\mathcal{H}^R\right\} \\
&= \left\{-\mathcal{G}^R - H, -G-\mathcal{H}^R \mid -\mathcal{G}^L - H, -G-\mathcal{H}^L\right\} \\
&= \left\{(-\mathcal{G})^L + (-H), (-G)+(-\mathcal{H})^L \mid (-\mathcal{G})^R + (-H), (-G)+(-\mathcal{H})^R\right\} \\
&= (-G)+(-H).
\end{aligned}$$

4.10: (page 70) Reflexivity: $G+X$ does have the same outcome as $G+X$ for all games X.

Symmetry: if $G+X$ has the same outcome as $H+X$ for all games X then $H+X$ has the same outcome as $G+X$ for all games X.

Transitivity: if for any game X, $G+X$ has the same outcome as $H+X$ and $H+X$ has the same outcome as $K+X$ then $G+X$ has the same outcome as $K+X$.

4.13: (page 71) If Left moves northwest and fires upwards then Right moves up and fires to the right: Left has at most three moves remaining whereas Right has four. If Left moves right and fires south-east, Right moves up and blocks Left. If Left moves south-west and fires anywhere but the squares just vacated then Right occupies this and fires anywhere but northwest and Right has more moves remaining.

If Right makes any move but the one shown, Left moves right and fires back. Left has at least as many moves as Right. (*Note*: Left may have better moves, but this wins.)

4.16: (page 72)

$$\begin{array}{rcll} G & = & G+0 & \text{(Theorem 4.4)} \\ & = & G+(J-J) & \text{(Corollary 4.14)} \\ & = & (G+J)-J & \text{(Theorem 4.5)} \\ & = & (H+J)-J & \text{(Assumed)} \\ & = & H+(J-J) & \text{(Theorem 4.5)} \\ & = & H+0 & \text{(Corollary 4.14)} \\ & = & H & \text{(Theorem 4.4)} \end{array}$$

4.17: (page 72) By two applications of Theorem 4.15 (and, as always, commutativity of $+$),

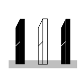
$$G+H=G'+H=G'+H'.$$

4.19: (page 73) $G \geq H$ means that Left wins $G+X$ whenever Left wins $H+X$; that is, whenever Right wins $G+X$ then Right wins $H+X$ and therefore $H \leq G$.

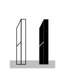
Now, $G=H$ *if and only if* Left wins $G+X$ whenever Left wins $H+X$ and Right wins $G+X$ whenever Right wins $H+X$, that is, $G \geq H$ and $G \leq H$.

4.21: (page 74) $2 \Rightarrow 4$: Suppose Left can win moving second on G and can win moving first on X. To win on $G+X$ playing first, she plays a winning move in X (treating X as an isolated game), and then replies locally as in $2 \Rightarrow 3$.

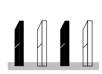
4.26: (page 76)

1.  $= \{0 \mid 1\} = \frac{1}{2}$

   which is in $\mathcal{L}$.

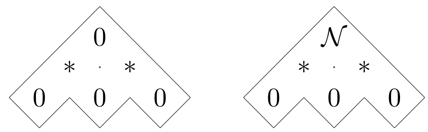
2.  $= \{0 \mid 0\} = *$

   and so

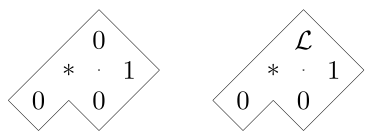
    $= \{0, *, -1 \mid 0, *, 1\} = \{0, * \mid 0, *\}$

   which happens to be $*2$ (not yet defined) and is not born until day 2.

3–4. The MAIZE position is a $\mathcal{P}$-position with value 0. In MAZE, we get an $\mathcal{N}$-position; its value is $*2$, born on day 2.

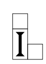


5–6. The MAIZE position is in $\mathcal{P}$ and has value 0. The MAZE position is in $\mathcal{L}$ and has value $\frac{1}{2}$, which is born on day 2.

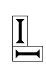


4.29: (page 78)

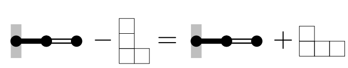
(a) Left moving first moves to

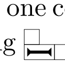
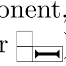


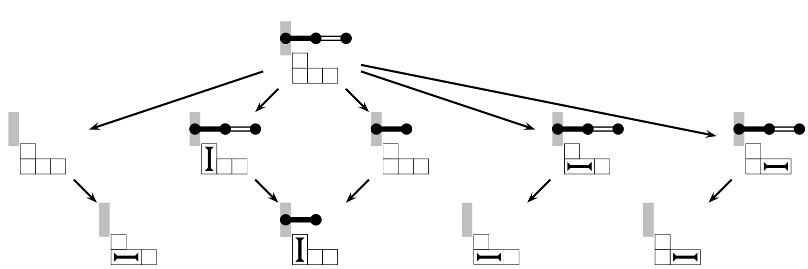
Moving second, play is forced, and Left again gets the last move to



(b) The second player wins the difference game



In particular, if one player plays in one component, the second player plays in the other (with Right preferring  over ). Diagrammatically,



4.32: (page 81) If Left or Right play any of the ticked options then the opponent plays the same (but negative) version (e.g., to $B - B$), reducing the game to 0.

The only first move for Left not covered is her move to $A - G'$. Now Right wins by moving to $A^R - G'$. To see this, note that if Left moves to $X - G'$ then Right plays to $X - X = 0$ and wins. If Left moves to $A^R - H$ then this is an option from $A^R - G$. However, $A^R - G \le 0$ so that Right wins going second regardless of Left's move so he wins if Left moves to $A^R - H$.

The moves not covered for Right are those where he moves in G' to one of the new options (such as W) which came from A^{RL}; that is, to some $G - W$. However, since $G - A^R \ge 0$ then Left can win playing second, specifically, she can win if Right moves to $G - W$.

4.35: (page 82) We will show the first assertion, that for any G^L, there exists an $H^L \ge G^L$. A symmetric argument shows the second assertion.

Since $G = H$, Left wins moving second on $H - G$. Left's winning response to $H - G^L$ cannot be to $H - G^{LR}$, for then $H - G^{LR} \ge 0$, and since $G = H$ we have $G^{LR} \le G$ and G had a reversible option. Hence, the winning move must be to some $H^L - G^L \ge 0$, and we have $H^L \ge G^L$.

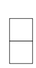
Chapter 5 Solutions

Prep 5.1: (page 86)

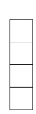
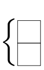
$$
\begin{aligned}
0 \xrightarrow{4T} 4 \xrightarrow{R} -\tfrac{1}{4} \xrightarrow{2T} \tfrac{7}{4} \xrightarrow{R} -\tfrac{4}{7} \xrightarrow{T} \tfrac{3}{7} \xrightarrow{R} -\tfrac{7}{3} \\
\xrightarrow{3T} \tfrac{2}{3} \xrightarrow{R} -\tfrac{3}{2} \xrightarrow{2T} \tfrac{1}{2} \xrightarrow{R} -2 \xrightarrow{2T} 0
\end{aligned}
$$

5.1: (page 88) Since $n = \{n-1 \mid \}$ then $-n = \{ \mid -(n-1)\}$.

5.2: (page 88)

 $= \{0 \mid \} = 1;$

and

 $= \Big\{$  $\Big| \Big\} = \{1 \mid \} = 2.$

5.4: (page 89) We will do the two examples in two different ways:

$$
\begin{array}{rcll}
\text{“}n+1\text{”} & = & \{\text{“}n-1\text{”} \mid \} + \{0 \mid \} & \text{(definition of games } n \text{ and 1)} \\
 & = & \{\text{“}n-1\text{”} + \text{“}1\text{”}, \text{“}n\text{”} + \text{“}0\text{”} \mid \} & \text{(definition of +)} \\
 & = & \{\text{“}n\text{”}, \text{“}n\text{”} + 0 \mid \} & \text{(by induction} \\
 & & & \text{ and definition of game 0)} \\
 & = & \{\text{“}n\text{”} \mid \}\,. &
\end{array}
$$

For the second, we can show "n" $-$ "1" $=$ "$n-1$" by confirming that the second player wins the difference game "$n-1$" $-$ "n" $+$ "1."

5.7: (page 90) By Theorem 5.5, $A+(-B)+0 \geq 0$ *if and only if* $a+(-b)+0 \geq 0$.

5.8: (page 90) There is no incentive moving from 0. From $n = \{n-1 \mid \}$ Left's incentive is $(n-1)-n = -1$ and Right has no move and so no incentive.

5.10: (page 90) Each component is of one color, and the value is $3-3-2+3 = 1$.

5.11: (page 91) $\frac{1}{4} = \{0 \mid \frac{1}{2}\}$ and $\frac{1}{8} = \{0 \mid \frac{1}{4}\}$.

5.14: (page 91)

$$\begin{aligned}
\frac{15}{16}+\frac{1}{4} &= \left\{\frac{7}{8} \,\middle|\, 1\right\} + \left\{0 \,\middle|\, \frac{1}{2}\right\} \\
&= \left\{\frac{7}{8}+\frac{1}{4}, \frac{15}{16}+0 \,\middle|\, 1+\frac{1}{4}, \frac{15}{16}+\frac{1}{2}\right\} \\
&= \left\{\frac{9}{8}, \frac{15}{16} \,\middle|\, \frac{5}{4}, \frac{23}{16}\right\} \\
&= \left\{\frac{9}{8} \,\middle|\, \frac{5}{4}\right\} \\
&= \left\{\frac{9}{8} \,\middle|\, \frac{10}{8}\right\} \\
&= \frac{19}{16}.
\end{aligned}$$

Or confirm that the second player wins on $\frac{15}{16}+\frac{1}{4}-1\frac{3}{16}$. Moves on $\frac{15}{16}$ are matched by moves on $1\frac{3}{16}$ leaving 0. A move on $\frac{1}{4}$ is worse and a response in either of the other components leaves the second player in a favorable position.

5.15: (page 92) Since $\frac{m}{2^j} = \left\{\frac{m-1}{2^j} \,\middle|\, \frac{m+1}{2^j}\right\}$, the left incentive is $\frac{m-1}{2^j} - \frac{m}{2^j} = -\frac{1}{2^j}$ and the right incentive is $\frac{m}{2^j} - \frac{m+1}{2^j} = -\frac{1}{2^j}$.

5.17: (page 93) This is covered by the induction. In the second paragraph, we only needed to show how Left wins in response to Right's legal moves. (When $a=b=c=0$, there are no legal moves, so no worries.)

5.19: (page 93) From Lemma 5.16, $A+B-C=0$ if and only if $a+b-c=0$. Similarly, $A-B+0 \geq 0$ if and only if $a-b+0 \geq 0$.

5.27: (page 94) $\pm n = \{n \mid -n\}$ is in canonical form and both n and $-n$ have birthdays n. Therefore $\pm n$ has birthday $n+1$ by Definition 4.1.

$$\pm\left(n+\frac{i}{2^j}\right) = \left\{n+\frac{i}{2^j} \,\middle|\, -n-\frac{i}{2^j}\right\}$$

and

$$n + \frac{i}{2^j} = \left\{ n + \frac{2i-1}{2^{j-1}} \;\middle|\; n + \frac{2i+1}{2^{j-1}} \right\}.$$

n has birthday n and, by induction, $n + \frac{i}{2^j}$ has birthday $n + j$. Therefore, $\pm\left(n + \frac{i}{2^j}\right)$ has birthday $n + j + 1$.

5.28: (page 95)

(a) $\left\{\frac{1}{2} \mid 2\right\} = 1$.

(b) $\left\{\frac{1}{8} \mid \frac{5}{8}\right\} = \frac{1}{2}$.

(c) $\left\{-1\frac{27}{64} \mid -1\frac{9}{32}\right\} = -1\frac{3}{8}$.

5.33: (page 100) The incentives are all equal to $-\frac{1}{4}$, and either player would move the leftmost piece possible in any summand to achieve this incentive.

5.42: (page 101) The first player wins on $\uparrow + *$, Left moves on the second component leaving $\uparrow + 0$, while Right moves on the first component leaving $* + * = 0$.

$\uparrow + \uparrow + * > \uparrow + *$, so Left still wins moving first. Right, however, has two moves both of which lose, one to $\uparrow + \uparrow + 0 > 0$ and one to $* + \uparrow + * = \uparrow > 0$.

5.44: (page 105) Let ○●^{n} denote the CLOBBER position ○●●●●···● with n black pieces. Then

$$○●^{2n} = \left\{0 \;\middle|\; ○●^{2n-1}\right\} = \{0 \mid (2n-2)\cdot\uparrow *\} = (2n-1)\cdot\uparrow *$$

and

$$○●^{2n+1} = \left\{0 \;\middle|\; ○●^{2n}\right\} = \{0 \mid (2n-1)\cdot\uparrow *\} = 2n\cdot\uparrow *.$$

In other words,

$$○●^{n} = n \cdot \uparrow\!* + *,$$

where the latter should be parsed as $n \cdot (\uparrow *) + *$.

5.45: (page 105) On $a + \{x \mid x\} - \{y \mid z\}$, if the first player moves on one switch, the opponent moves on the other switch leaving either $a - x - z = 0$ or $a + x - y = 0$. If the first player moves on a, the second player plays on switches until both are resolved, leaving a favorable number.

5.46: (page 105) $\pm x \pm x = 0$ since it is a second-player win.

5.47: (page 106) Left moving first gets a maximum total of $a + x - y + z$ so she wins if $a > -x + y - z$. Left moving second gets a maximum total of $a - x + y - z$

and so wins if $a \geq x - y + z$. Left, moving first, wins $G = a \pm x_1 \pm x_2 \pm x_3 \cdots \pm x_n$ if $a + x_1 - x_2 + x_3 \cdots \pm x_n > 0$ and wins moving second if $a - x_1 + x_2 - x_3 \cdots \pm x_n \geq 0$.

5.48: (page 106) Denote the position by (m, n, p). Claim: $(m, n, p) = \{m - 1 \mid \{p - 1 \mid -(n - 2)\}\}$.

If Left moves to eliminate the white dominoes then the best option is $m - 1$. Her other options are to positions of values $\{m' - 1 \mid \{p - 1 \mid -(n - 1)\}\}$ or $\{m - 1 \mid \{p' - 1 \mid -(n - 2)\}\}$. However, Left, playing second, wins $(m - 1) - \{m' - 1 \mid \{p - 1 \mid -(n - 2)\}\}$ so this option is dominated. Left's option $\{m - 1 \mid \{p' - 1 \mid -(n - 1)\}\}$ reverses to $p' - 1$, which is also dominated. Right's options are to $\{m - 1 \mid -(n' - 1\}$ and $\{p - 1 \mid -(n' - 1)\}$ for $n' \leq n$. All these are dominated by $\{p - 1 \mid -(n - 1)\}$. Therefore,

$$(m, n, p) = \{m - 1 \mid \{p - 1 \mid -(n - 2)\}\}.$$

5.50: (page 107) Each of the $\lfloor \frac{s}{2} \rfloor$ moves by Left gains $n_{\mathcal{L}}$ moves for the remaining elephants. (Similarly for Right.) In total, that changes the difference in available non-central moves by $\lfloor \frac{s}{2} \rfloor (n_{\mathcal{L}} - n_{\mathcal{R}})$.

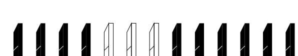
5.52: (page 108) The left and right incentives of moving a central beast on a component with $n + 2$ beasts is $\{n \mid 0\}$. The incentive of moving a non-central beast is -1. The incentive $\{n \mid 0\}$ with maximal n dominates.

5.53: (page 109) Let x be a positive number and consider $x - ⧾_G = x + ⧿_G$. Left wins by playing to $x + \{G \mid 0\}$, for Right's response to $x > 0$ fails immediately, and his response to $x^R + \{G \mid 0\}$ fails when Left counters to $x^R + G$, which is also positive since $x^R > x$ is also a positive number.

Right moving first in $x + ⧿_G$ can move to $x > 0$ or to $x^R + ⧿_G$, which is also positive by induction since $x^R > x$ and is simpler than x.

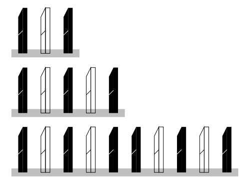
5.55: (page 109) Consider $⧾_y - k \cdot ⧾_x = ⧾_y + k \cdot ⧿_x$. Left playing always moves in one of the $⧿_x$ to $\{x \mid 0\}$. If Right fails to respond then Left's move to x dominates all other positions. (As in the proof of the previous theorem, Left plays arbitrarily on the other positions, ignoring x until it is the only game remaining. If, in the meanwhile, Right has played on x, that play only increases x.) After all of the $⧿_x$ summands have been played out, it is Left's move in either $⧾_y$ or $\{0 \mid -y\}$, both of which she wins.

5.56: (page 111) The position is $c + 2$ white dominoes sandwiched between $a + 1$ dominoes at one end and $b + 1$ dominoes at the other end. For example,

 $= \{5 \parallel 3 \mid -1\}$

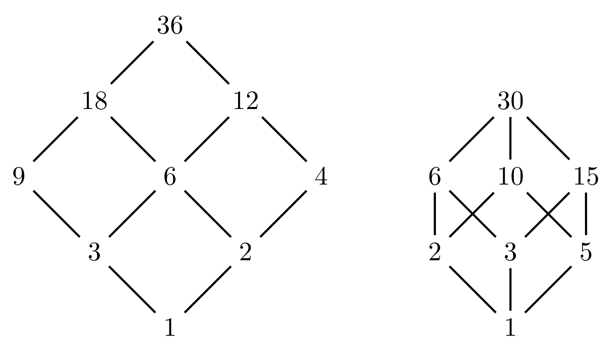
5.58: (page 111)

$$1/2 = \{0 \mid 1\} =$$
$$1/4 = \{0 \mid 1/2\} =$$
$$3/8 = \{1/4 \mid 1/2\} =$$



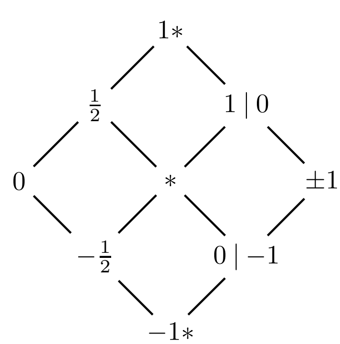
Chapter 6 Solutions

Prep 6.1: (page 116)



The gcd of a and b is found by looking at the lowest element that has a path down to both a and b. (Reverse this for lcm.)

Prep 6.2: (page 116)

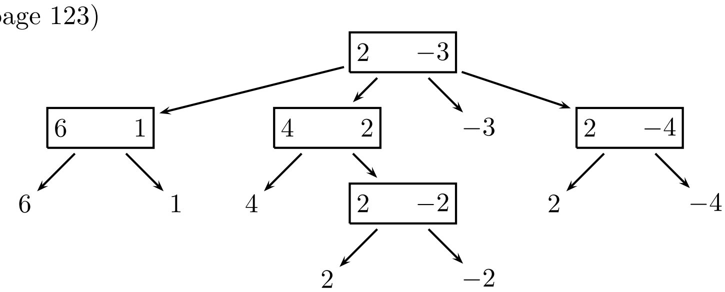


If one (or both) sets of option can be empty, we would also include the games 2, 1, -1, and -2.

6.2: (page 119) Since $\{0, * \mid *\} > 0$, the left option to $*$ reverses out through 0. Since $\{0, * \mid 1\} > 0$ and $\{0, * \mid \} > 0$, the same is true in the other two examples.

6.8: (page 122) The statement we are trying to prove is of the form, "For any infinitesimal $G \ldots$". Inductively, we can only assume the statement is true for smaller *infinitesimal* games.

6.10: (page 123)



6.14: (page 124) To the contrary, suppose $\mathbf{LS}(G^{LR}) > \mathbf{LS}(G)$ for all G^{LR}. By the definition of stops, $\mathbf{RS}(G^L)$ is defined as the maximum $\mathbf{LS}(G^{LR})$, and so $\mathbf{RS}(G^L) > \mathbf{LS}(G)$. But $\mathbf{LS}(G)$ is the maximum among $\mathbf{RS}(G^{L'})$ of which $\mathbf{RS}(G^L)$ is one. So, $\mathbf{LS}(G) \geq \mathbf{RS}(G^L) > \mathbf{LS}(G)$, which is a contradiction.

6.15: (page 125) $\mathbf{RS}(G^R) \leq \mathbf{LS}(G^R)$ and $\mathbf{RS}(G) \leq \mathbf{LS}(G)$ by Theorem 6.11. $\mathbf{LS}(G^R) = \mathbf{RS}(G)$ since G^R was chosen to be the right option from G with minimal left stop which, by definition, is $\mathbf{RS}(G)$. Lastly, $\mathbf{LS}(G) \leq 0$ was assumed in the statement of the theorem.

6.20: (page 126) Suppose $\mathbf{LS}(G) = x = \mathbf{RS}(G)$. By repeated application of Theorem 6.18, $\mathbf{LS}(G - x) = 0 = \mathbf{RS}(G - x)$, and so $G = x + (G - x)$ is a number plus an infinitesimal.

6.25: (page 128) We use the definitions given in the proof below; the partial order diagram confirms the results:

$$
\begin{aligned}
* \vee 0 = &\quad \{0 \mid \lceil * \rceil \cap \lceil 0 \rceil\} &=&\quad \{0 \mid \{0,1\} \cap \{*,1\}\} &= \tfrac{1}{2};\\
* \vee \tfrac{1}{2} = &\quad \left\{0,0 \mid \lceil * \rceil \cap \left\lceil \tfrac{1}{2} \right\rceil\right\} &=&\quad \{0 \mid \{0,1\} \cap \{1\}\} &= \tfrac{1}{2};\\
\uparrow \vee \pm 1 = &\quad \{0,1 \mid \lceil \uparrow \rceil \cap \lceil \pm 1 \rceil\} &=&\quad \{1 \mid \{*,1\} \cap \{-1,0,*,1\}\} &= \{1 \mid *\};\\
\uparrow \wedge \pm 1 = &\quad \{\lfloor \uparrow \rfloor \cap \lfloor \pm 1 \rfloor \mid *, -1\} &=&\quad \{\{0,*,-1\} \cap \{1,0,*,-1\} \mid -1\} &= \{0,* \mid -1\}.
\end{aligned}
$$

6.26: (page 129) For each i, $G \geq G_i$, so Left wins moving second on $G - G_i$, and so Left wins moving first on $G^R - G_i$, and we have $G^R \in \lceil G_i \rceil$.

6.28: (page 129) For the diagram on the left,

$$
\begin{aligned}
a \wedge (b \vee c) &= a \wedge 1 = a;\\
(a \wedge b) \vee (a \wedge c) &= 0 \vee 0 = 0.
\end{aligned}
$$

For the right,

$$\begin{aligned} a \wedge (b \vee c) &= a \wedge 1 = a; \\ (a \wedge b) \vee (a \wedge c) &= b \vee 0 = b. \end{aligned}$$

6.30: (page 130) On $\uparrow - n \cdot \alpha$, Left moving first attacks the αs one by one. Right must either reply locally or on another α; after two moves, one or two αs have been converted to $* = \{1* \mid \{0 \mid -1\}\}$. (If Right moves on $\uparrow$, Left grabs $1*$, an overwhelming advantage.) After all αs are gone, Left wins moving first on $\uparrow$ or $\uparrow*$.

Chapter 7 Solutions

Prep 7.1: (page 134)

Decimal	**Binary**
$-2\frac{1}{2}$	-10.1
23	10111
21	10101
$\frac{1}{4}$	.01
$\frac{5}{8}$	.101
$\frac{13}{32}$	.01101
$-12\frac{9}{16}$	-1100.1001
$-3\frac{21}{32}$	-11.10101

7.1: (page 135) The second player can play the Tweedledum-Tweedledee strategy and apply induction. That is, whatever option the first player chooses in one component, the second player makes the same move in the other component.

7.2: (page 136) If G and H are impartial games and $G \leq H$, then $G = H$ since $H - G$ is impartial and cannot be positive. Therefore, the dominated or reversible left options of an impartial game are likewise dominated or reversible as right options. So, by induction, the canonical form has the same left and right options.

7.5: (page 137) $*i \parallel *j$ because one is both a left and right option from the other. (That is, if $i > j$, the first player moves $*i - *j$ to $*j - *j = 0$ and wins.) This directly implies that there are no dominated nor reversible options.

7.9: (page 138) We find $\text{mex}\{0, 3, 4, 8, 2\} = 1 = \text{mex}\{0, 6, 4\}$, so the game is $*$. From $G - *$, if the first player moves to $G - 0$ or $0 - *$, the second can reply to $0 - 0$. If, on the other hand, the first player moves to $*j - *$ for $j > 1$, the second player replies to $* - * = 0$.

7.11: (page 138) By definition, the nim-sum addition in the kth binary place (or column) is independent from the addition in other binary places. Furthermore, since addition is commutative and associative, so is the columnwise nim-sum, for it is just the parity of the sum.

For $a \oplus a$, in each binary place we have $0 + 0 = 0$ or $1 + 1 = 0$.

If $a \oplus b = c$ then

$$a \oplus b \oplus c = (a \oplus b) \oplus c = c \oplus c = 0.$$

If $0 = a \oplus b \oplus c$ then

$$c = 0 \oplus c = (a \oplus b \oplus c) \oplus c = a \oplus b(\oplus c \oplus c) = a \oplus b.$$

7.14: (page 140)

(a) $3 \oplus 5 \oplus 7 = 1$; all heaps are odd so remove 1 from any heap.

(b) $2 \oplus 3 \oplus 5 \oplus 7 = 3$ whose leftmost bit is second from the right. In binary, 2, 3, and 7 have a leftmost bit there, so there is a winning move on each of these heaps. Move $2 \to 1$ or $3 \to 0$ or $7 \to 4$.

(c) $2 \oplus 4 \oplus 8 \oplus 32 \oplus = 46$; the only winning move is to reduce the heap of 32 to $32 \oplus 46 = 14$.

(d) $2 \oplus 4 \oplus 10 \oplus 12 = 0$ so there is no winning move.

7.17: (page 140) Suppose a player moves to 0 (according to NIM strategy). Whenever the opponent increases the size of a heap (drawing down the bag), one of the moves dictated by the NIM strategy is simply to pocket those same counters. So there exists a move back to 0, winning by induction. (Eventually the bag runs dry, and the game reverts to NIM.)

7.18: (page 141) As in Exercise 7.17, the player who sees a position with nim-sum 0 can win by always removing counters and, eventually, his opponent will only be able to make reducing moves.

The game is technically loopy since both players on their first move take one counter and then on their second moves add one counter and so on. However, the first part of the exercise shows that the player who is in a winning position never has to add counters to win.

7.20: (page 142) The strategy is exactly the same except if the number of coins is odd, the leftmost coin on square i is equivalent to a nim-heap of size i.

7.23: (page 142) The nim-value of G is the mex of the nim-values of its options. Since there are only n options, at least one of the values $\{0, 1, 2, \ldots, n\}$ does not occur as the nim-value of an option. Hence, $\mathcal{G}(G) \leq n$.

7.27: (page 143)

$$\mathcal{G}(n) = \begin{cases} 0 & \text{if } n \equiv 0 \pmod 5 \\ 1 & \text{if } n \equiv 1 \pmod 5 \\ 0 & \text{if } n \equiv 2 \pmod 5 \\ 1 & \text{if } n \equiv 3 \pmod 5 \\ 2 & \text{if } n \equiv 4 \pmod 5 \end{cases}$$

7.30: (page 145)

$1231451671\ldots$	purely sapp regular, $p = 3$, $s = 2$, $S = \{1, 2\}$;
$1123123123\ldots$	periodic, $p = 3$, pre-period 1;
$1122334455\ldots$	purely arithmetic periodic, $p = 2$, $s = 1$;
$0112325272\ldots$	sapp regular, $p = 2$, $s = 2$, $S = \{1\}$, pre-period 01;
$0120120120\ldots$	purely periodic, $p = 3$;
$1112233445\ldots$	arithmetic periodic, $p = 2$, $s = 1$, pre-period 1.

7.32: (page 147) The sequence is $00011122203314\dot{0}020131\dot{2}$.

7.35: (page 148) Once you have gone as far as 001122031021021021 you can be confident the sequence is $0011220\dot{3}10\dot{2}$. We have, $l = 8$, $p = 3$, and $a = 7$, so it suffices to inspect $\mathcal{G}(n)$ for $n \in \{8, \ldots, 17\}$. In particular, confirm $\mathcal{G}(n) = \mathcal{G}(n+3)$ for the 7 values $8 \leq n \leq 14$.

7.37: (page 149) The first 20 values ($n = 0$ through $n = 19$) are

$$00110213223344554657.$$

7.42: (page 151) Suppose that the sequence (7.1) matches term-by-term for the two values $n = n_0$ and $n = n_0'$ for $n_0' > n_0$. Define $p = n_0' - n_0$ and fix $n_0' \leq n \leq n_0' + 2a$. Then, adding equations from (7.1) form two telescoping sums, and we have

$$\begin{aligned} G(n_0+1) - G(n_0) &= G(n_0'+1) - G(n_0'); \\ G(n_0+2) - G(n_0+1) &= G(n_0'+2) - G(n_0'+1); \\ &\vdots \\ G(n) - G(n-1) &= G(n+p) - G(n+p-1). \end{aligned}$$

Adding these equations yields $G(n+p) - G(n_0') = G(n) - G(n_0)$ and so $G(n+p) - G(n) = G(n_0') - G(n_0) = s$.

Or, use induction:

$$\begin{aligned}
\mathcal{G}(n+p) &= \mathcal{G}(n+p-1) + (\mathcal{G}(n+p) - \mathcal{G}(n+p-1)) \\
&= s + \mathcal{G}(n-1) + (\mathcal{G}(n) - \mathcal{G}(n-1)) \text{ by induction and (7.1)} \\
&= s + \mathcal{G}(n).
\end{aligned}$$

In the base case, when $n = n_0$, $\mathcal{G}(n_0 + p) = \mathcal{G}(n_0') = s + \mathcal{G}(n_0)$ by definition of s.

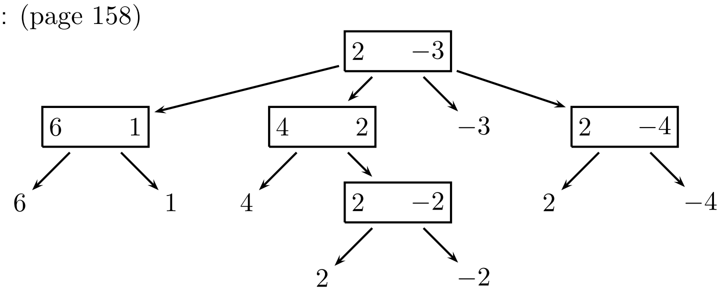
7.44: (page 152) The sequence has period 12 and saltus 4: $\dot{0}0110213223\dot{3}(+4)$. Since $a = 4$, $p = 12$, and $l = 0$, it suffices to compute values through $l+2a+p = 20$ and confirm that $\mathcal{G}(n) = \mathcal{G}(n-p)$ for $12 \le n \le 20$.

7.45: (page 153) It suffices to compute through $n = l+2a+p = 8+2\cdot 10+3 = 31$.

7.47: (page 155) The next values are 4, 2, 6, 4, 1.

Chapter 8 Solutions

Prep 8.1: (page 158)



8.4: (page 161) The bottom game has **LS** $= 2$ and **RS** $= -2$. The middle left game has **LS** $= 4$ and **RS** $= 2$. The middle right game has **LS** $= -3$ and **RS** $= -5$. The top game has **LS** $= 3$ and **RS** $= -3$.

8.8: (page 166) The values for a single heap are periodic, with period 4, specifically, heap size $n \equiv 0 \pmod 4$ has value 0, heap size $n \equiv 1 \pmod 4$ has value 1, heap size $n \equiv 2 \pmod 4$ has value $\{1 \mid 0\}$, and heap size $n \equiv 3 \pmod 4$ has value $\{1, \{1 \mid 0\} \mid 0\}$. The mean value of $\{1 \mid 0\}$ is $\frac{1}{2}$ as is the mean value of $\{1, \{1 \mid 0\} \mid 0\}$. Therefore, the disjunctive sum of 5 heaps of size 10, 20 of size 11, 31 of size 31 has approximate value $6(\frac{1}{2}) + 20(\frac{1}{2}) + 132(\frac{1}{2}) = 78$.

Note: For this game, $\{1 \mid 0\} + \{1 \mid 0\} = 1$ and $4 \cdot \{1, \{1 \mid 0\} \mid 0\} = 2 + X$, where X is a *tiny*, i.e., $X = \{0 \mid \{0 \mid Y\}\}$.

8.10: (page 172) If none of X_t, Y_t, and $(X+Y)_t$ is a number then

$$\begin{aligned} X_t + Y_t &= \left\{ X_t + {Y^L}_t - t, {X^L}_t + Y_t - t \,\middle|\, X_t + {Y^R}_t + t, {X^R}_t + Y_t + t \right\} \\ &= \left\{ (X+Y^L)_t - t, X^L + Y_t - t \,\middle|\, (X+Y^R)_t - t, (X^R + Y_t - t \right\} \\ &= (X+Y)_t. \end{aligned}$$

8.15: (page 175) P has temperature 1 and Q has temperature 2. So both players are advised to move on Q first. However, when Left moves first she can do better by moving on P, for Right must immediately respond locally.

In this example, Thermostrat gives the same advice as Hotstrat. In particular, the compound thermograph of P and Q is maximally left at temperature 0, and at that temperature the Q's thermograph is widest, the difference between left and right stops being 4.

8.16: (page 175) The goal of a strategy is to do well against *any* opponent, and not just an opponent following the same strategy. If your opponent makes a low-temperature move, that should not oblige you to make poor moves for the remainder of the game.

8.19: (page 176)

$$\begin{aligned} &\tfrac{1}{2} \cdot 1* = \{1 \mid *\}, \\ &\tfrac{1}{4} \cdot 1* = \{1 \mid \{0 \mid -1*\}\}, \\ &\tfrac{1}{8} \cdot 1* = \{1 \mid \{0 \mid \{-1 \mid -2*\}\}\}, \ldots \end{aligned}$$

8.21: (page 177) In each step of the induction, the birthdays of both game sums $A+B+C$ and $A \cdot U + B \cdot U + C \cdot U$ (perhaps $-t$) always decrease.

8.23: (page 179)

$$* \int \frac{11}{4} + * \int \frac{5}{4} - * \int \frac{9}{2} = * \int -1/2 = \{0 \mid -1/2*\}$$

Chapter 9 Solutions

Prep 9.1: (page 184) See Section 5.2.

9.4: (page 187) Since $*m = -*m$, $\uparrow *n > *m$ *if and only if* $\uparrow *(n \oplus m) > 0$. By the lemma, this is equivalent to $n \oplus m \neq 1$ or equivalently $n \oplus 1 \neq m$.

9.7: (page 187) By induction, the left options $(a-j, a+i+j)$ and the right options $(a+i+j, a-j)$ have value $*(a-j)$ for $0 < j \leq a$ so that

$$(a, a+i) = \{0, *1, *2, \ldots, *(a-1) \mid 0, *1, *2, \ldots, *(a-1)\} = *a$$

9.8: (page 188) The game is positive, for Left moving first moves to $(1, a+1, 1)$ and moves second to a single heap.

$$\begin{aligned}(a, a+i, 1) &= \{(2a+i, 1), (1, 2a+i-1, 1), \ldots, \\ &\quad\ (a-1, a+i+1, 1))(a, a+i+1)\} \\ &= \{*, 0, \uparrow*(2 \oplus 1), \ldots, \uparrow*(a-1 \oplus 1) \mid *a\} \\ &= \{0 \mid *a\}\text{, since the other left options all reverse out through } 0 \\ &= \uparrow*(a \oplus 1).\end{aligned}$$

9.9: (page 188) As in the last exercise, the game is positive, and so most of Left's options reverse out:

$$\begin{aligned}(a+i, a, 1) &= \{(a-1, a+i+1, 1), \ldots, (2, a+i+a-2, 1), 0, * \mid *(a+1)\} \\ &= \{\uparrow*(a-1 \oplus 1), \ldots, \uparrow*(2 \oplus 1), 0, * \mid *(a+1)\} \\ &= \{0 \mid *(a+1)\} \\ &= \uparrow*((a+1) \oplus 1)).\end{aligned}$$

9.11: (page 188) In base $\uparrow = \{0 \mid *\}$, $\uparrow^1 = \uparrow$ and

$$\uparrow^2 = \{0 \mid * - \uparrow\} = \{0 \mid \downarrow*\}.$$

9.14: (page 190) Let $H = .i_1 i_2 \ldots$ and suppose without loss of generality that the first non-zero digit i_n is positive. Let m be the sum of the absolute values of all the digits in positions to the right of i_n. Then, $H \geq G^n - m \cdot G^{m+1} > 0$.

9.18: (page 191) From Theorem 9.15,

$$.\underbrace{11\ldots1}_{n+1}* = \left\{ .\underbrace{11\ldots1}_{n+1},\ .\underbrace{11\ldots1}_{n}* \;\middle|\; *+*,\ .\underbrace{11\ldots1}_{n+1} \right\}$$

Right's move to $*+* = 0$ dominates, while Left's move to

$$.\underbrace{11\ldots1}_{n+1}$$

reverses through $*$ to 0.

9.24: (page 194)

Reflexive: $G - G = 0$ is infinitesimal.

Symmetric: $G - H$ is infinitesimal if and only if $H - G$ is.

Transitive: If $G - H$ and $H - J$ are infinitesimal, so is $(G-H)+(H-J) = G-J$. (The sum of two infinitesimals is infinitesimal for if $-x < G_1 < x$ and $-x < G_2 < x$ then $-2x < G_1 + G_2 < 2x$.)

9.35: (page 199) To see that $\mathrm{AW}(\boldsymbol{+}_2) = 0$, confirm $\boldsymbol{+}_2 < \uparrow☆$. Also, $\mathrm{AW}(\{0 \mid \boldsymbol{+}_2\} = 1$ for $☆ < \{0 \mid \boldsymbol{+}_2\} < \Uparrow☆$.

9.36: (page 199) Since $.p = p \cdot \uparrow$ then $\mathrm{AW}(.p) = p$. $.01$ is infinitesimal with respect to $.1$, and so must have atomic weight 0. So,

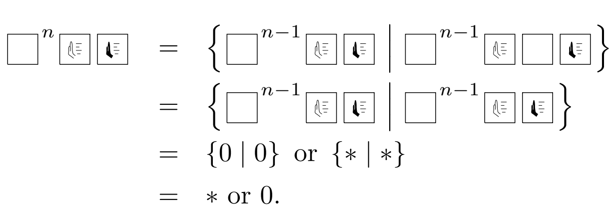
$$\mathrm{AW}(.pq) = \mathrm{AW}(.p + .0q) = \mathrm{AW}(.p) + q\,\mathrm{AW}(.01) = p.$$

Alternatively, since $.01 = \{0 \mid \downarrow *\}$ then $\mathrm{AW}(.01) = \{-2 \mid 1\} = 0$. (Note that since $.01 > 0$, with or without invoking the exception, the atomic weight calculus of Theorem 9.39 gives 0.)

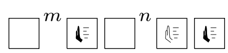
9.40: (page 201) "$\{n-1 \mid n+2\}$" is a number, and we are in the exceptional case. Two integers fit, those being n and $n+1$. When $n \geq 1$, $g > ☆$, and so $\mathrm{AW}(g) = n+1$. If, however, $n = 0$ or $n = -1$, $g \parallel ☆$, and so $\mathrm{AW}(g) = 0$. When $n < -1$, $\mathrm{AW}(g) = n$.

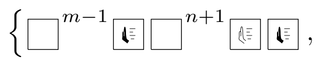
9.41: (page 201) The largest integer in "$\{0-2 \mid 0+2\}$" is 1, so it suffices to confirm that Left wins moving second on $g + ☆$. Left can reply to most Right moves leaving a game of the form $*p + *p$. However, if Right moves to $g + *p$ for $p > m$, Left replies to $g + *(n+1)$. From there, Right has no choice but to let Left leave a pair of matching $*$s.

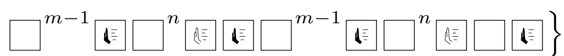
9.42: (page 202) Either player's moves  one square left, and the game ends when it falls off the board. Hence, the game is a variant of SHE LOVES ME SHE LOVES ME NOT. For those fond of formality, by induction,



9.43: (page 202)

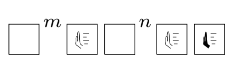


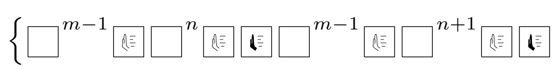
$=$ 

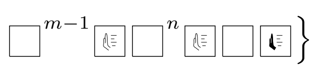


$$
\begin{aligned}
&= \{m\cdot{\uparrow} + (n+m)\cdot *,\ m\cdot{\uparrow} + (n+m-1)\cdot * \mid m\cdot{\uparrow} + (n+m-1)\cdot *\}\\
&= \{m\cdot{\uparrow} + *,\ m\cdot{\uparrow} \mid m\cdot{\uparrow} + (n+m-1)\cdot *\}\\
&= \{0 \mid m\cdot{\uparrow} + (n+m-1)\cdot *\}\ \text{reversible options}\\
&= (m+1)\cdot{\uparrow} + (n+m)\cdot *.
\end{aligned}
$$

9.44: (page 203)



$=$ 



$$
\begin{aligned}
&= \{m\cdot{\downarrow} + (n+m-1)\cdot * \mid m\cdot{\downarrow} + (n+m)\cdot *,\ m\cdot{\downarrow} + (n+m-1)\cdot *\}\\
&= (m+1)\cdot{\downarrow} + (n+m)\cdot *,\ \text{as in the last exercise.}
\end{aligned}
$$

9.45: (page 203) If either end is gray, or if the ends have opposite colors, the first player wins and the game is incomparable with zero. If the ends are both black (respectively, white) the game is positive (respectively, negative).

9.49: (page 205) The game lasts two moves and the second player wins.

Appendix A Solutions

A.2: (page 220) The i^{th} positive odd integer is $2i-1$. Implicit in the statement is that $n \geq 0$, for these are the only values of n for which, "the first n ..." makes any sense.

$$
\begin{aligned}
\sum_{1\le i\le n}(2i-1) &= (2n-1) + \sum_{1\le i\le n-1}(2i-1)\\
&= (2n-1) + (n-1)^2,\ \text{(by induction)}\\
&= (2n-1) + (n^2-2n+1) = n^2.
\end{aligned}
$$

For the base case, when $n = 0$,

$$\sum_{1 \le i \le 0} (2i - 1) = 0,$$

for any empty sum is 0.

A.4: (page 221) Let $a = n \pmod 1)1$.

(a) If $a = 0$, all legal moves leave a pile that is not equivalent to 0, and so lose by induction. Since all moves lose, the second player wins. If, on the other hand, $a \neq 0$, $n - a \equiv 0$, and so removing a coins wins by induction.

(b) (No base case is required, since the first sentence, "If $a = 0$, all legal moves leave a pile that is not equivalent to 0" handles $n = 0$ properly. "All legal moves..." is vacuously satisfied.)

Appendix B Solutions

B.1: (page 234)

```
> G=2||1|*
2||1|*
> G+G
3|2*
> G+G+G+G
5*
> Mean(G)
5/4
> Temperature(G)
3/4
> G>1
true
> G<>1*
true
>
```

B.2: (page 234)

```
> LeftOptions(^^^)
[0]
> RightOptions(^^^)
[^^*]
```

B.3: (page 234) H is infinitesimal and is bounded between $2 \cdot ⧿_2 < H < 2 \cdot ⧾_2$:

```
> H = C(DomineeringRectangle(4,4))
+-(0,2|0,2+Tiny(2)|2|0,Miny(2))
> H < 1/128
true
> H < Tiny(2)
false
> H < Tiny(2)+Tiny(2)
true
```

B.4: (page 234)

```
> Heat(*6,1)
```

B.5: (page 234) You can enter something like

```
> a=5; b=3; c=2; a|b+Tiny(c) == a+Tiny(c) | b+Tiny(c)
false
> a=5; b=2; c=2; a|b+Tiny(c) == a+Tiny(c) | b+Tiny(c)
true
```

In the worksheet, you can edit the line and hit enter again, so you need not type the expression over and over. Once you learn to create functions you can type:

```
> f(a_,b_,c_) := a|b+Tiny(c) == a+Tiny(c) | b+Tiny(c)
proc(a,b,c)  (0 values cached)
> f(5,3,2)
false
> f(5,2,2)
true
```

B.6: (page 235)

```
> H = DomineeringRectangle(4,4)
> SensibleLeftLines(H)
```

(Of course, Right's lines are symmetric.)

B.7: (page 235)

```
> G = Clobber("lrlr","rll.","lrl.")
> SensibleLeftLines(G)
> SensibleRightLines(G)
> AtomicWeight(G)
```

B.8: (page 235)

```
C(Clobber("lllllll","lrlllll","lllllll")) AtomicWeight($)
```

Appendix D

Rulesets

This appendix contains a glossary of the rules to many of the games appearing in the book. If a game only made a brief appearance in a problem or example, then it may not appear here. In most sample games, Left moves first, but the choice was arbitrary as either player can start from any position. Moves in the sample games were chosen to illustrate the rules and do not necessarily represent good play! Unless otherwise noted, the winner is determined by *normal play;* that is, the player who makes the last legal play wins.

We use the following standing conventions throughout, without explicit comment:

- Black or blue pieces, vertices, etc., belong to Left; white or red pieces, vertices, etc., belong to Right.
- In games involving moving pieces on a one-dimensional strip, Left typically moves from left to right while Right moves from right to left.
- If we only specify Left's moves, then Right's moves are similar allowing for appropriate reversals of color and/or direction.

Many games are played on a "board" which amounts to an underlying graph — often a path or a grid. Such games often have variants where the underlying graph is changed, but we do not usually mention such variants explicitly.

ALL-SMALL *ruleset*

The ALL-SMALL variant of a *ruleset* is typically obtained by declaring that play in a component ends when either player has no legal move in *ruleset*.

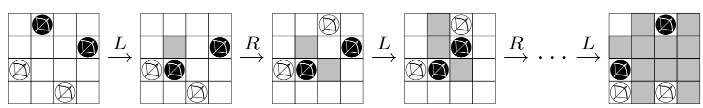
ALLBUT(S)

See SUBTRACTION$(L \mid R)$.

AMAZONS

Position: A rectangular board with pieces called amazons (either black or white) and destroyed squares. The standard starting position is a 10×10 board with white amazons in the fourth and seventh squares of the bottom row, and on the left and right edges of the fourth row, with black amazons positioned symmetrically in the upper half of the board.

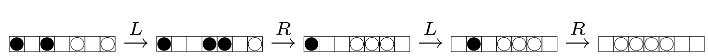
Moves: A player's amazon moves like a chess queen — any number of squares in a horizontal, vertical or diagonal line. Once it gets to its final position, the amazon throws an arrow that also moves like a chess queen. The arrow destroys the board square it lands in. Both the amazon and the arrow it throws cannot cross or enter a destroyed square or a square occupied by an amazon of either color.



BOXCARS

Position: A finite strip of squares, each square is empty or occupied by a black or white *boxcar*. Contiguous strings of boxcars must all be of the same color.

Moves: A boxcar occupies one square, a *train* is any maximal string of contiguous pieces (including just a single boxcar) all of the same color. Left chooses a black train and moves it one square to the right. If this results in a longer string of contiguous boxcars then every boxcar in the string turns black. Right moves leftward. Trains cannot be moved off the end of the board.

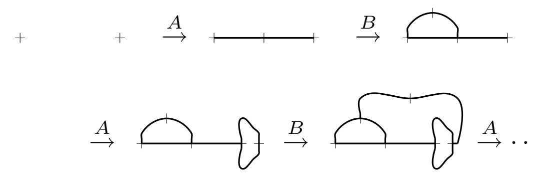


BRUSSELS SPROUTS

Position: A collection of vertices in the plane with four arms emanating from each vertex. Some of these arms may be connected by non-crossing edges. No arm may be connected to itself.

Moves: Draw a curve joining one arm to another that does not cross any other curve or pass through any other vertex. On this curve place another vertex with two new arms, one on each side of the curve.

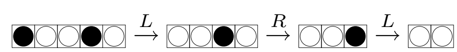
A typical start position has n vertices and no edges.



CHERRIES

Position: A row of red, blue, or green cherries.

Moves: Left removes a blue or green cherry from either end of the row; Right removes a red or green cherry similarly.

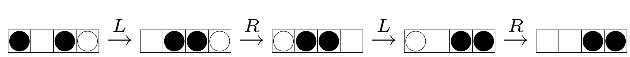


Variants: RED-BLUE CHERRIES has no green-colored cherries.

CLEAR THE POND

Position: A finite strip of squares, each square is empty or occupied by a black or white piece.

Moves: Left moves a black piece to the right to the next empty space or off the board if there is no empty space. Right moves leftward.

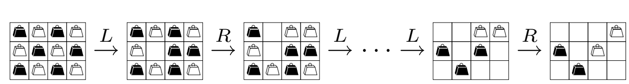


Variants: In the ALL-SMALL variant, the game component is over when only pieces of one color remain. See also CLOBBER THE POND.

CLOBBER

Position: A rectangular board, each square is empty or is occupied by a black or a white stone. Play often begins with a board filled with alternating black and white stones in a checkerboard pattern.

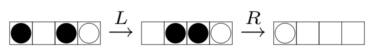
Moves: Left moves a black stone onto an orthogonally adjacent white stone and removes the white stone. Right moves leftward.



CLOBBER THE POND

Position: A finite strip of squares, each square is empty or occupied by a black or white piece.

Moves: Left moves a black piece rightwards to the next empty space or off the board if there is no empty space. All pieces (of either color) that this piece jumps over are removed. Right moves leftward.



Variants: CLEAR THE POND

COL

See SNORT.

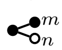
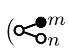
CRAM

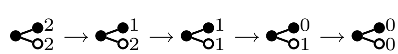
See DOMINEERING.

CUTTHROAT

Position: A graph in which each vertex is colored red or blue.

Moves: Left removes a blue vertex. When a vertex is removed, all the incident edges and all monochromatic connected components are also removed.

Let  (), $m+n \geq 1$, be a star with a blue (red) center, m blue leaves and n red leaves.

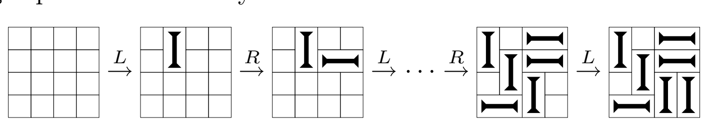


Variants: CUTTHROAT STARS is CUTTHROAT restricted to a collection of star graphs.

DOMINEERING

Position: A subset of the squares of a grid. Play often begins with a complete $n \times n$ square.

Moves: Left places a domino to remove two adjacent vertical squares. Right places horizontally.



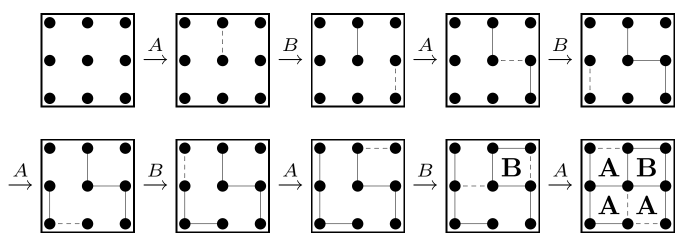
Variants: CRAM is impartial; either player can play a domino vertically or horizontally.

DOTS & BOXES

Position: A grid of dots, with lines drawn connecting some pairs of orthogonally adjacent dots. Any 1×1 completed square box of lines contains a player's initial. Play often begins with an empty rectangular grid of dots.

Moves: A move consists of connecting two adjacent dots by a line. If that move completes one or two boxes, the player initials the box(es) and must move again unless all boxes have been claimed.

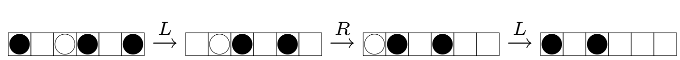
Winner: The game ends when all boxes are claimed, and the player who has claimed the most boxes wins.



DRAG OVER

Position: A finite strip of squares, each square is empty or occupied by a black or white piece.

Moves: A piece of the current player's color is moved one square to the left replacing any piece that may be on that square. All pieces to the right of the piece moved are also moved one square to the left. Pieces can be moved off the end of the strip.



Variants: In ALL-SMALL DRAG OVER the game is over when only pieces of one color remain on the board. See also RUN OVER, SHOVE, and PUSH.

ELEPHANTS & RHINOS

Position: A finite strip of squares, each square is empty or occupied by a black (elephant) or white (rhino) piece.

Moves: Left moves an elephant one square to the right onto an empty square. No beast can budge or land on another. Right moves leftward.

ENDNIM

Position: A row of stacks of boxes.

Moves: A player chooses one end of the row and removes any number of boxes from the stack at that end. Once a stack is empty the next stack is available.

$$5, 3, 4 \xrightarrow{L} 1, 3, 4 \xrightarrow{R} 1, 3, 1 \xrightarrow{L} 1, 3, 1 \xrightarrow{R} 1, 3 \xrightarrow{L} 3 \xrightarrow{R} 0$$

Variants: In PARTIZAN ENDNIM, Left can only remove boxes from the left end and Right from the right end.

EROSION

Position: A heap that contains l left counters and r right counters.

Moves: If $l > r$, Left may legally move by removing exactly r left counters. Similarly, if $r > l$, Right may remove l right counters.

The heap is denoted (l, r).

$$(14, 9) \xrightarrow{L} (5, 9) \xrightarrow{R} (5, 4) \xrightarrow{L} (1, 4) \xrightarrow{R} (1, 3)$$

FORKLIFT

Position: A row of stacks of boxes.

Moves: A forklift can pick up any number of boxes from an end stack. Left takes from the left end and Right from the right end; each puts them on the next stack in the row provided that what remains of the first stack is lower than the height of the second stack (so that the forklift can put the boxes down). The game is finished when there is only one stack.

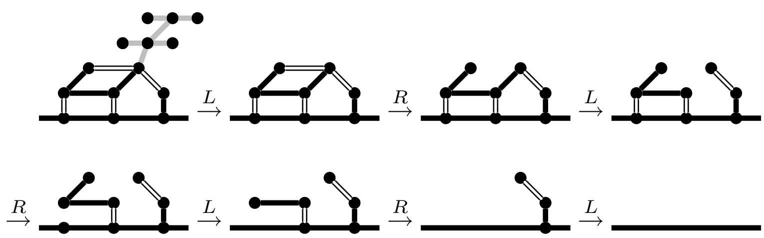
$$2, 1, 3, 2 \xrightarrow{L} 3, 3, 2 \xrightarrow{R} 3, 4, 1 \xrightarrow{L} 1, 6, 1 \xrightarrow{R} 1, 7 \xrightarrow{L} 8$$

Variants: In IMPARTIAL FORKLIFT players can take from either end.

HACKENBUSH

Position: A graph with edges colored blue, red, and green; in this black-and-white text, diagrams will use black, white, and gray, respectively. There is one special vertex called the *ground*, which is shown by a long horizontal line.

Moves: Left cuts a black or gray edge and removes any portion of the graph no longer connected to the ground.



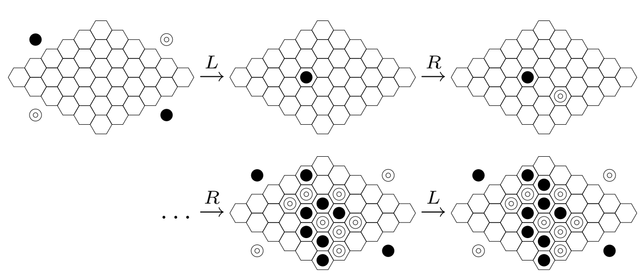
Variants: LR-HACKENBUSH positions have no gray (or green) edges. Green (or impartial) HACKENBUSH positions have only gray (meaning green) edges.

HEX

Position: An $n \times n$ hexagonal tiling possibly containing some black and white stones. Play usually begins with an empty board (but see the note in the variants).

Moves: Black and White alternate placing stones on empty hexagons.

Winner: Black hopes to connect the upper-left side to the lower-right with a path of her color, while Right tries to connect the lower-left to the upper right.

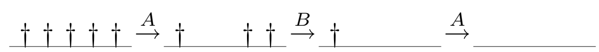


Variants: In practice, particularly on a smaller board, this is a game best played with the *swap rule* to negate the first-player advantage. Alice places a stone (say a black stone) and then Bob decides whether to continue with white, or whether he wants to swap colors and play black. If he swaps, Alice plays again, this time placing a white stone.

KAYLES

Position: A row of pins, possibly containing some gaps.

Moves: Throw a ball that removes either one or two adjacent pins.

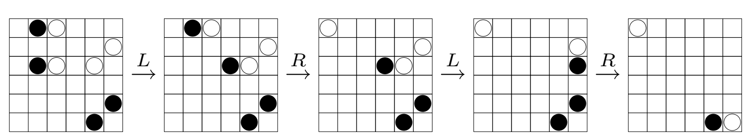


Variants: GENERALIZED KAYLES or OCTAL games where a heap can be reduced in size and then split into two heaps.

KONANE

Position: A rectangular board, each square is empty or is occupied by a black or a white stone. Play often begins with a square or rectangular board filled with alternating black and white stones in a checkerboard pattern with two adjacent stones removed from central squares.

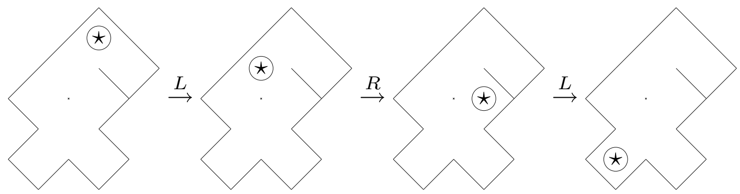
Moves: Left jumps a black stone over an orthogonally adjacent white stone onto an empty square. In a single move, jumps of this type *may* be chained together, but only in a single direction; that is, the jumping stone may not make a 90° turn.



MAZE

Position: A rectangular grid oriented so that the edges are at 45° degree angles to the horizontal, with some highlighted edges and one piece (or multiple pieces) on some square(s).

Moves: Left moves the piece any number of squares in a southwesterly direction, Right moves the piece similarly in a southeasterly direction. The piece may not cross a highlighted edge. If multiple pieces are on the board, they do not interfere with one another making the position a disjunctive sum.



Variants: In MAIZE each player can only move the piece one square. This piece is denoted $\odot$.

NIM

Position: Several heaps of counters.

Moves: A player chooses a heap and removes some counters.

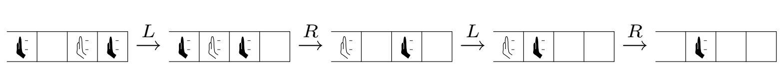
$$9,4,2 \xrightarrow{A} 3,4,2 \xrightarrow{B} 3,4,1 \xrightarrow{A} 3,2,1 \xrightarrow{B} 0,2,1 \xrightarrow{A} 0,0,1 \xrightarrow{B} 0,0,0$$

Variants: In SUBTRACTION games, and ALLBUT subtraction games, the number of counters that a player can remove is restricted in some way; in GREEDY NIM a player must take from the largest heap.

PUSH

Position: A finite strip of squares, each square is empty or occupied by a black or white piece. (The pieces for PUSH look like those for SHOVE, but with fewer lines behind the hand.)

Moves: A piece of the current player's color is pushed one square to the left. No square can be occupied by more than one piece, so any pieces immediately to the left of this piece are also pushed one square. Pieces may be pushed off the end of the strip.

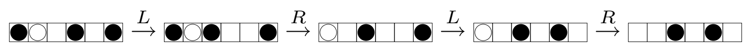


Variants: In ALL-SMALL PUSH the game is over when only pieces of one color remain on the board. See also SHOVE.

RUN OVER

Position: A finite strip of squares, each square is empty or occupied by a black or white piece.

Moves: A piece of the current player's color is moved one square to the left replacing any piece that may be on that square. Pieces may be moved off the end of the strip.

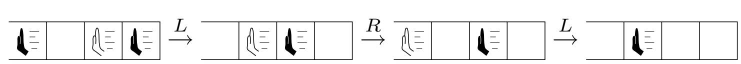


Variants: In ALL-SMALL RUN OVER the game is over when only pieces of one color remain on the board. See also DRAG OVER, SHOVE, and PUSH.

SHOVE

Position: A finite strip of squares, each square is empty or occupied by a black or white piece. (The pieces for SHOVE look like those for PUSH, but with more lines behind the hand.)

Moves: A piece of the current player's color and all pieces to its left are pushed one square to the left. Pieces may be shoved off the end of the strip. Right moves leftward.

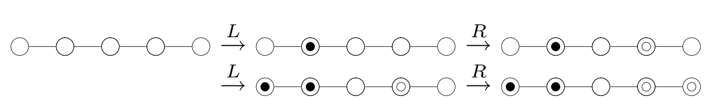


Variants: In ALL-SMALL SHOVE the game is over when only pieces of one color remain on the board. See also PUSH.

SNORT

Position: A graph with each vertex either uncolored or colored white or black.

Moves: Left colors an uncolored vertex black, subject to the proviso that the vertex being colored is not adjacent to a white vertex.



Variants: In COL, Left is not allowed to color a vertex adjacent to a black vertex.

SPLITTLES$(L \mid R)$

Position: A heap of counters.

Moves: Left subtracts some element of L from n and then (optionally) splits the remaining heap into two heaps.

If $L = \{2, 3, 7\}$ and $R = \{2, 4\}$

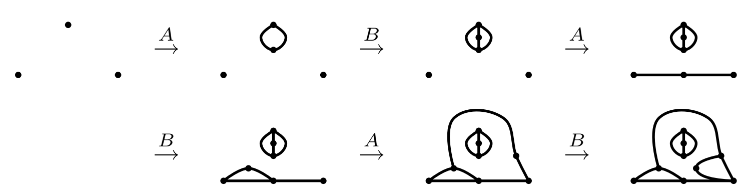
$$20 \xrightarrow{L} \{9, 8\} \xrightarrow{R} \{3, 4, 8\} \xrightarrow{L} \{3, 4, 6\} \xrightarrow{R} \{3, 6\} \xrightarrow{L} \{3, 2\} \xrightarrow{R} \{1, 2\} \xrightarrow{L} \{1\}$$

Variants: Denote the impartial game SPLITTLES$(S \mid S)$ by SPLITTLES(S). See also SUBTRACTION$(L \mid R)$.

SPROUTS

Position: A collection of vertices in the plane connected by non-crossing edges. The maximum degree of a vertex is always 3. Play often begins with n vertices and no edges.

Moves: Draw a curve joining one vertex to another (or itself) that does not cross any other curve or pass through any other vertex. On this curve place another vertex. (Since the maximum-allowed degree is 3, the curve must join two vertices of degree at most 2, or a vertex of degree 0 or 1 to itself.)



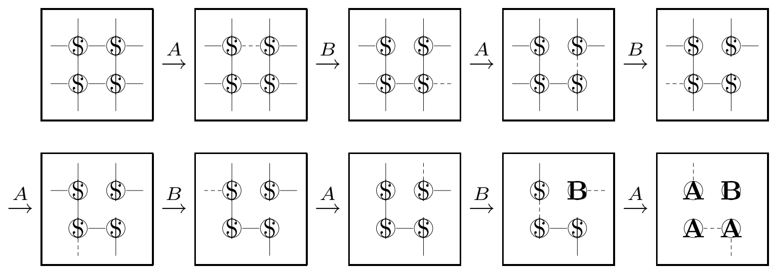
Variants: BRUSSELS SPROUTS.

STRINGS & COINS

Position: A grid of coins, and a set of strings with one or both ends connected to a coin.

Moves: Cut (i.e., entirely remove) a string. If, after the move, one or two coin(s) have no strings, the player pockets the coin(s) and, if moves remain, must move again.

Winner: The player who pocketed the most coins.



SUBTRACTION$(L \mid R)$

Position: A non-negative integer n.

Moves: Left subtracts some element of L from n. Right subtracts an element of R from n. The result of the subtraction must be non-negative.

If $L = \{2, 3\}$ and $R = \{1, 4, 5\}$,

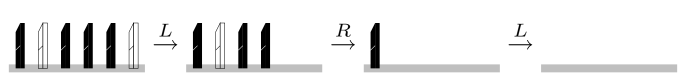
$$17 \xrightarrow{L} 14 \xrightarrow{R} 13 \xrightarrow{L} 10 \xrightarrow{R} 6 \xrightarrow{L} 3 \xrightarrow{R} 2 \xrightarrow{L} 0$$

Variants: With one subtraction set, SUBTRACTION(S) is the impartial game SUBTRACTION$(S \mid S)$. In the literature, SUBTRACTION(S) is the standard *subtraction game*, and SUBTRACTION$(L \mid R)$ is termed a *partizan subtraction game*. The game ALLBUT(S) is the game SUBTRACTION$(\{1, 2, \ldots\} \setminus S)$; S should be a finite set.

TOPPLING DOMINOES

Position: A row of black, white, or gray dominoes.

Moves: Left chooses a black or gray domino and topples it either left or right, every domino in that direction also topples and is removed from the game. Right topples a white or gray domino.

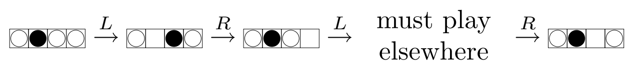


Variants: TOPPLING TOWERS is the two-dimensional version played on a checkerboard. If a tower topples into empty space then the "chain reaction" stops. The game can be extended into any number of dimensions.

WOODPUSH

Position: A finite strip of squares, each square is empty or occupied by a black or white piece.

Moves: A player may either *push* or *retreat*. (Left pushes rightward, Right pushes leftward, and retreats are in the opposite direction.) A retreat involves moving one square backward (or off the end) and is legal only if that square is empty. A push moves a continuous row of pieces ahead (as in PUSH) but the move *must* include a piece of the opposing color. Repetition of a global board positions is not allowed; (i.e., a threat must be played first.)



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