Linearised NSE.

NSE

$$\frac{\partial \bar{u}}{\partial t} + (\bar{u}\cdot\nabla)\bar{u} = -\nabla p + \frac{1}{Re}\nabla^2 \bar{u} \quad \text{— (1)}$$

$$\nabla\cdot\bar{u} = 0 \quad \text{— (2)}$$

$$\bar{u}(x,y,z,t) = \bar{U}(y) + u'(x,y,z,t) \quad \text{— (3)}$$

$$p(x,y,z,t) = P(y) + p'(x,y,z,t) \quad \text{— (4)}$$

Using (3) & (4) in (1)

* For this derivation I am using $\nu = Re^{-1}$

$$\frac{\partial \bar{u}}{\partial t} = \frac{\partial \bar{u}'}{\partial t} \quad \because \frac{\partial \bar{U}}{\partial t} = 0$$

$$(u\cdot\nabla)u = (u'+U)\cdot\nabla(u'+U)$$

$$= u'\cdot\nabla u' + U\cdot\nabla u' + u'\cdot\nabla U + U\cdot\nabla U$$

$$\nabla p = \nabla p' + \nabla P$$

$$\nu\nabla^2\bar{u} = \nu\nabla^2 u' + \nu\nabla^2\bar{U}$$

$$\therefore \frac{\partial \bar{u}'}{\partial t} + (\bar{u}'\cdot\nabla)\bar{u}' + \bar{U}\cdot\nabla\bar{u}' + \bar{u}'\cdot\nabla\bar{U} + U\cdot\nabla\bar{U} = -\nabla p' - \nabla P + \nu\nabla^2 u' + \nu\nabla^2 U \quad \text{— (5)}$$

Eqⁿ for the steady flow

$$(\bar{U}\cdot\nabla)\bar{U} = -\nabla P + \nu\nabla^2\bar{U} \quad \text{—⑥}$$

Eqⁿ ⑤ − eqⁿ ⑥ and dropping $O(u'^2)$ terms.

$$\frac{\partial \bar{u}'}{\partial t} + \bar{u}'\cdot\nabla\bar{U} + \bar{U}\cdot\nabla\bar{u}' = -\nabla p' + \nu\nabla^2 u' \quad \text{—⑦}$$

Using ③ in ②

$$\nabla\cdot(u' + U) = 0$$

$$\Rightarrow \nabla\cdot u' + \nabla\cdot\bar{U} = 0 \quad \text{—⑧}$$

for steady flow $\nabla\cdot\bar{U} = 0$ —⑨

⑧ − ⑨,

$$\nabla\cdot\bar{u}' = 0 \quad \text{—⑩}$$

Orr-Sommerfeld eqn.

For parallel flow

$$\bar{U} = U(y)\,\hat{e}_x \qquad \text{--- (11)}$$

Using (11) in (7)

$$\frac{\partial \bar{u}'}{\partial t} + \underbrace{\bar{u}'\cdot\nabla\bar{U}} + \bar{U}\cdot\nabla\bar{u}' = -\nabla p' + \nu\nabla^2 u'$$

$$\underbrace{\Big(\underbrace{u'\frac{\partial}{\partial x}}_{0} + v'\frac{\partial}{\partial y} + \underbrace{w'\frac{\partial}{\partial z}}_{0}\Big)\,U(y)\,\hat{e}_x}$$

$$\underbrace{v'\frac{dU}{dy}\,\hat{e}_x}$$

$$v'\,U^{(1)}\,\hat{e}_x$$

here $U^{(1)} = \dfrac{dU}{dy}$

$$(U\cdot\nabla)u' = \Big(U\frac{\partial}{\partial x} + \underbrace{V\frac{\partial}{\partial y}}_{0} + \underbrace{W\frac{\partial}{\partial z}}_{0}\Big)\bar{u}' = U\frac{\partial \bar{u}'}{\partial x}$$

$$\therefore \left[\frac{\partial \bar{u}'}{\partial t} + v' U^{(1)} \hat{e}_x + U \frac{\partial}{\partial x} \bar{u}' = -\nabla p + \nu \nabla^2 u' \right] \cdot \hat{e}_y$$

$$\underset{(12)}{}$$

$$\Rightarrow \nabla^2 \left[\frac{\partial v'}{\partial t} + U \frac{\partial}{\partial x} v' = -\frac{\partial p'}{\partial y} + \nu \nabla^2 v' \right]$$

$$\Rightarrow \frac{\partial}{\partial t} \nabla^2 v' + \nabla^2 U \frac{\partial v'}{\partial x} + \frac{\partial}{\partial y} \nabla^2 p' = \nu \nabla^4 v' \qquad (13)$$

$\nabla \cdot$ eqn (12)

$$\frac{\partial}{\partial t} \underbrace{\nabla \cdot \bar{u}'}_{=0} + \nabla \cdot \left(v' U^{(1)} \hat{e}_x \right) + \nabla \cdot \left(U \frac{\partial \bar{u}'}{\partial x} \right)$$

$$= -\nabla^2 p' + \nu \nabla^2 \underbrace{(\nabla \cdot \bar{u}')}_{=0}$$

$$\therefore \nabla^2 p' = - \underbrace{\nabla \cdot \left(v' U^{(1)} \hat{e}_x \right)}_{\partial_x (v' U^{(1)}) \,=\, U^{(1)} \partial_x v'} - \underbrace{\nabla \cdot \left(U \frac{\partial \bar{u}'}{\partial x} \right)}_{\partial_i (U \partial_x u_i') \,=\, (\partial_i U)(\partial_x u_i') + U \partial_x \underbrace{\partial_i u_i'}_{=0}}$$

$$\nabla^2 p' = -U^{(1)} \partial_x v' - \underbrace{\partial_x U}_{=0} \partial_x u' - \underbrace{\partial_y U \, \partial_x v'}_{= U^{(1)} \partial_x v'} - \underbrace{\partial_z U}_{=0} \partial_x w'$$

$$\therefore \nabla^2 p' = -2U^{(1)} \partial_x v'$$

$$\therefore \frac{\partial}{\partial y} \nabla^2 p' = -2\left(\frac{\partial}{\partial y} U^{(1)} \frac{\partial v'}{\partial x} + U^{(1)} \partial_y \partial_x v' \right)$$

$$\text{let} \quad U^{(2)} = \frac{d^2 U}{dy^2}$$

$$\therefore \frac{\partial}{\partial y} \nabla^2 p' = -2\left(U^{(2)} \frac{\partial v'}{\partial x} + U^{(1)} \partial_y \partial_x v' \right) \quad \text{— (14)}$$

$$\frac{\partial}{\partial t} \nabla^2 v' + \nabla^2 U \frac{\partial v'}{\partial x} + \frac{\partial}{\partial y} \nabla^2 p' = \nu \nabla^4 v' \quad \text{— (13)}$$

$$\nabla^2\left(U \frac{\partial v'}{\partial x}\right) = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) U \frac{\partial v'}{\partial x}$$

$$= \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} \right) U \frac{\partial v'}{\partial x} + \frac{\partial^2}{\partial y^2} U \frac{\partial v'}{\partial x}$$

$$= \underbrace{U\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} \right) \frac{\partial}{\partial x} v'}_{U \frac{\partial}{\partial x}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} \right) v'} + \frac{\partial}{\partial y} \left\{ \frac{\partial}{\partial y} \left(U \frac{\partial v'}{\partial x} \right) \right\}$$

$$\frac{\partial}{\partial y}\left(U \frac{\partial v'}{\partial x} \right) = \left\{ U^{(1)} \frac{\partial v'}{\partial x} + U \partial_x \partial_y v' \right\}$$

$$= U\frac{\partial}{\partial x}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z^2}\right)v' \quad + \frac{\partial}{\partial y}\left\{ U^{(1)}\frac{\partial v'}{\partial x} + U\,\partial_x\partial_y v' \right\}$$

$$\parallel$$

$$U^{(2)}\frac{\partial v'}{\partial x} + U^{(1)}\partial_x\partial_y v'$$

$$+ \; U^{(1)}\partial_x\partial_y v' + U\,\partial_x\partial_y^2 v'$$

$$= U\frac{\partial}{\partial x}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)v' + U^{(2)}\frac{\partial v'}{\partial x} + 2U^{(1)}\partial_x\partial_y v'$$

$$\nabla^2\left(U\frac{\partial v'}{\partial x}\right) = U\frac{\partial}{\partial x}\nabla^2 v' + U^{(2)}\frac{\partial v'}{\partial x} + 2U^{(1)}\partial_x\partial_y v' \quad — (15)$$

Using (14) & (15) in (13)

$$\frac{\partial}{\partial t}\nabla^2 v' + \nabla^2 U\frac{\partial v'}{\partial x} + \frac{\partial}{\partial y}\nabla^2 p' = \nu\nabla^4 v' \quad — (13)$$

$$\frac{\partial}{\partial t}\nabla^2 v' + U\frac{\partial}{\partial x}\nabla^2 v' + U^{(2)}\frac{\partial v'}{\partial x} + 2U^{(1)}\cancel{\partial_x\partial_y v'}$$

$$-2\left(U^{(2)}\frac{\partial v'}{\partial x} + U^{(1)}\cancel{\partial_y\partial_x v'}\right)$$

$$= \nu\nabla^4 v'$$

$$\Rightarrow \left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\nabla^2 v' - U^{(2)}\frac{\partial v'}{\partial x} = \nu \nabla^4 v'$$

$$\therefore \boxed{\left[\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\nabla^2 - U^{(2)}\frac{\partial}{\partial x}\right] v' = \nu \nabla^4 v'} \quad (16)$$

$$v'(x, y, z, t) = \hat{v}(y)\, e^{i(\alpha x + \beta z)}\, e^{-i\omega t} \quad (17)$$

$$\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} = \frac{\partial^2}{\partial y^2} - \alpha^2 - \beta^2$$

$$= D^2 - (\alpha^2 + \beta^2)$$

$$= D^2 - k^2$$

$$\frac{\partial}{\partial t} \equiv -i\omega$$

$$\frac{\partial}{\partial x} = i\alpha$$

Here $k^2 = \alpha^2 + \beta^2$

$$D \equiv \frac{d}{dy} \qquad D^2 = \frac{d^2}{dy^2}$$

$$\Rightarrow \left[(-i\omega + Ui\alpha)(D^2-k^2) - U^{(2)} i\alpha\right] \hat{v}(y) = \nu (D^2-k^2)^2 \hat{v}(y)$$

$$\Rightarrow (D^4+k^4-2D^2k^2)\hat{v} = \frac{1}{\nu}\left\{\left(-i\underbrace{\frac{\omega}{\alpha}}_{c}\alpha + Ui\alpha\right)(D^2-k^2) - U^{(2)} i\alpha\right\}\hat{v}$$

let $\frac{\omega}{\alpha} = c$

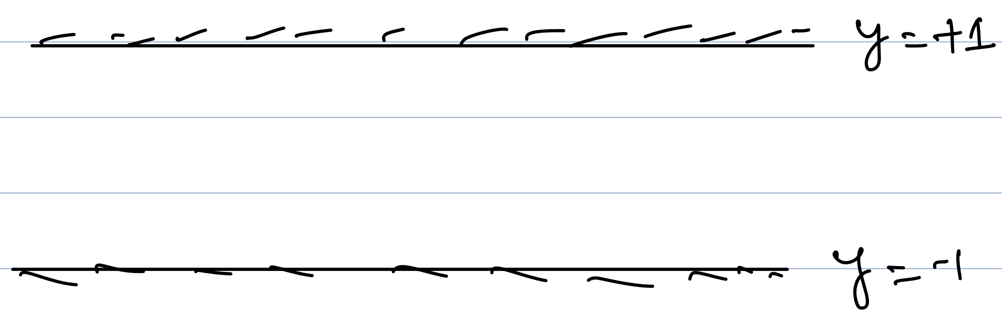
$$\Rightarrow \hat{v}^{(4)} - 2k^2\hat{v}^{(2)} + k^4\hat{v} = \frac{i\alpha}{\nu}\left\{(U-c)(\hat{v}^{(2)} - k^2\hat{v}) - U^{(2)}\hat{v}\right\}$$

$$\Rightarrow \boxed{\hat{v}^{(4)} - 2k^2\hat{v}^{(2)} + k^4\hat{v} = i\alpha Re\left\{(U-c)(\hat{v}^{(2)} - k^2\hat{v}) - U^{(2)}\hat{v}\right\}} \quad (18)$$

Above is the Orr-Sommerfeld eqn.

PPF

Solution of the base flow.



B.C. $\quad U(y=\pm 1)=0$

Base flow eq^n.

$$(\bar{U}\cdot\nabla)\bar{U} = -\nabla P + Re^{-1}\nabla^2\bar{U}$$

For parallel flow $\quad \bar{U} = U(y)\,\hat{e}_x$

$$\therefore (\bar{U}\cdot\nabla)\bar{U} = (\underbrace{U\partial_x}_{0} + \underbrace{V}_{0}\partial_y + \underbrace{W}_{0}\partial_z)\,U(y)\,\hat{e}_x$$

$$= 0$$

$$-\frac{\partial P}{\partial x} + \frac{1}{Re}\nabla^2 U = 0$$

Let $\quad G = -\dfrac{\partial P}{\partial x}$ = Constant pressure gradient.

$$\therefore \frac{d^2U}{dy^2} = -G\,Re$$

$$\Rightarrow \frac{dU}{dy} = -G\,Re\,y + A$$

$$\Rightarrow \quad U(y) = -GRe\frac{y^2}{2} + Ay + B$$

At $y=1$

$$0 = -\frac{GRe}{2} + A + B \quad \text{— (i)}$$

At $y=-1$

$$0 = -\frac{GRe}{2} - A + B \quad \text{— (ii)}$$

(i) − (ii)

$$0 = 0 + 2A + 0 \quad \Rightarrow \quad A = 0$$

(i) + (ii)

$$0 = -GRe + 2B$$

$$\therefore B = \frac{GRe}{2}$$

$$\therefore \quad U(y) = -\frac{GRe}{2}y^2 + \frac{GRe}{2}$$

$$\therefore U(y) = \frac{GRe}{2}(1-y^2)$$

If $G = \frac{2}{Re}$

$$\therefore \boxed{U(y) = 1-y^2}$$

Solution of OS eq^n.

B.C. on perturbation

$$u' = v' = w' = 0 \quad \text{at } y = \pm 1$$

$$\hat{v}^{(4)} - 2k^2 \hat{v}^{(2)} + k^4 \hat{v} = i\alpha Re \{ (U - c)(\hat{v}^{(2)} - k^2 \hat{v}) - U^{(2)} \hat{v} \} \quad (18)$$

We need to solve eq^n (18) using above B.C.

Above B.C. implies.

$$\hat{u}(y) = \hat{v}(y) = \hat{w}(y) = 0 \quad \text{at } y = \pm 1$$

$$\text{or} \quad \hat{v}(y) = 0 \quad \text{At } y = \pm 1$$

Eq^n (18) is 4th order eq^n, therefore we need two more condition to solve OS.

$$\because \quad \nabla \cdot \bar{u}' = 0$$

$$\Rightarrow \quad \frac{\partial u'}{\partial x} + \frac{\partial}{\partial y} v' + \frac{\partial w'}{\partial z} = 0$$

$$\Rightarrow \quad i\alpha \hat{u} + \frac{d\hat{v}}{dy} + i\beta \hat{w} = 0$$

$$\therefore \quad \because \hat{u}(y) = 0 \ \& \ \hat{v}(y) = 0 \quad \text{at } y = \pm 1$$

$$\Rightarrow \quad \frac{d\hat{v}}{dy} = 0 \quad \text{at} \quad y = \pm 1$$

Thus B.C. are.

$$\boxed{\hat{v}(y) = 0 \qquad \frac{d\hat{v}}{dy} = 0 \qquad \text{at} \quad y = \pm 1}$$

Matrix eigenvalue problem.

$$A\hat{v} = \omega B \hat{v}$$

$$\hat{v}^{(4)} - 2k^2\hat{v}^{(2)} + k^4\hat{v} = i\alpha Re \left\{ (U - c)\left(\hat{v}^{(2)} - k^2\hat{v}\right) - U^{(2)}\hat{v} \right\} \quad (18)$$

From eq (18),

Using $c = \frac{\omega}{\alpha}$ and multiplying both sides by $\frac{i}{Re}$

$$\therefore \frac{i}{Re}\left[\hat{v}^{(4)} - 2k^2\hat{v}^{(2)} + k^4\hat{v}\right] = -\alpha \underbrace{\left\{ \left(U - \frac{\omega}{\alpha}\right)\left(\hat{v}^{(2)} - k^2\hat{v}\right) - U^{(2)}\hat{v} \right\}}$$

$$= \alpha U^{(2)}\hat{v} - (\alpha U - \omega)\left(\hat{v}^{(2)} - k^2\hat{v}\right)$$

$$= \alpha U^{(2)}\hat{v} - \left\{ \alpha U\left(\hat{v}^{(2)} - k^2\hat{v}\right) - \omega\left(\hat{v}^{(2)} - k^2\hat{v}\right) \right\}$$

$$= \alpha U^{(2)}\hat{v} + \omega\left(\hat{v}^{(2)} - k^2\hat{v}\right) - \alpha U\left(\hat{v}^{(2)} - k^2\hat{v}\right)$$

$$\Rightarrow \frac{i}{Re}\left[\hat{v}^{(4)} - 2k^2\hat{v}^{(2)} + k^4\hat{v}\right] + \alpha U\left(\hat{v}^{(2)} - k^2\hat{v}\right) - \alpha U^{(2)}\hat{v}$$

$$= \omega\left(\hat{v}^{(2)} - k^2\hat{v}\right)$$

$$\therefore \quad A = \frac{i}{Re}\left(D^4 - 2k^2D^2 + k^4 I\right) + \alpha U\left(D^2 - k^2 I\right) - \alpha U^{(2)} I$$

$$B = D^2 - k^2 I$$

* How to find $\hat{u}$ from $\hat{v}$ in 2D.

From continuity eq^n $\nabla\cdot\bar{u}'=0$

$$\Rightarrow i\alpha\hat{u}+\frac{d\hat{v}}{dy}+i\beta\hat{\omega}=0$$

For 2D $\beta=0$.

$$i\alpha\hat{u}=-\frac{d\hat{v}}{dy}$$

$$\Rightarrow \cancel{i}\alpha\hat{u}=\cancel{+}i\frac{d\hat{v}}{dy}$$

$$\therefore \boxed{\hat{u}=\frac{i}{\alpha}\frac{d\hat{v}}{dy}}$$

* How to obtain v' from $\hat{v}(y)$

$$v'(x,y)=\hat{v}(y)\,e^{i\alpha x}$$

$$v'(x,y)=\hat{v}(y)\left(\cos(\alpha x)+i\sin(\alpha x)\right)$$

$$\Rightarrow v'=\text{Real}\left\{\hat{v}\left(\cos\alpha x+i\sin(\alpha x)\right)\right\}$$

$$=\text{Real}\left\{\left(\hat{v}_R+i\hat{v}_I\right)\left(\cos\alpha x+i\sin\alpha x\right)\right\}$$

$$\boxed{v' = \hat{V}_R \cos \alpha x - \hat{V}_I \sin \alpha x}$$