21MAB101T - CALCULUS AND LINEAR ALGEBRA

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Properties of Eigenvalues and Eigenvectors

Property 1

The sum of the eigenvalues of a matrix A is equal to the sum of the principal diagonal elements of A, (the trace of A)and the product of the eigenvalues is the determinant of A.

Property 2

A square matrix and its transpose have the same eigenvalues.

Property 3

An eigenvector cannot correspond to two different eigenvalues.

Property 4

The eigenvalues of a triangular matrix are the diagonal element of that matrix.

Property 5

If $\lambda_1, \lambda_2, \cdots, \lambda_n$ are the eigenvalues of the square matrix A with corresponding eigenvectors $X_1, X_2, \cdots, X_n$, then

$$k\lambda_1, k\lambda_2, \cdots, k\lambda_n$$

are the eigenvalues of kA with corresponding eigenvectors $X_1, X_2, \cdots, X_n$.

Property 6

If $\lambda_1, \lambda_2, \cdots \lambda_n$ are the eigenvalues of the square matrix A with the corresponding eigenvectors $X_1, X_2, \cdots, X_n$, then

$$\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \cdots, \frac{1}{\lambda_n}$$

are the eigenvalues of A^{-1} with the corresponding eigenvectors $X_1, X_2, \cdots, X_n$ provided $\lambda_i \neq 0$ for all i.

Property 7

If $\lambda_1, \lambda_2, \cdots, \lambda_n$ are the eigenvalues of the square matrix A with corresponding eigenvectors $X_1, X_2, \cdots, X_n$, then

$$\lambda_1^r, \lambda_2^r, \cdots, \lambda_n^r$$

are the eigenvalues of A^r with corresponding eigenvectors $X_1, X_2, \cdots, X_n$.

Property 8

The eigenvalues of a real symmetric matrix are real.

Property 9

The eigenvector corresponding to distinct eigenvalues of a real symmetric matrix are orthogonal.

Examples on Properties

Example 1

Find the sum and product of the eigenvalues of
$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{pmatrix}$

Solution.

Sum of the eigenvalues = trace of $A = 1 + 2 + 3 = 6$

Product of the eigenvalues

$$\begin{aligned} &= \text{determinant of } A = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{vmatrix} \\ &= 1[6-4] - (1)[3-2] + 1[2-2] = 2 - 1 + 0 = 1 \end{aligned}$$

Example 2

Find the sum and product of the eigenvalues of
$A = \begin{pmatrix} 1 & 2 & 5 \\ 2 & 2 & 2 \\ 1 & 2 & 7 \end{pmatrix}$

Solution.

Sum of the eigenvalues = trace of $A = 1 + 2 + 7 = 10$
Product of the eigenvalues

$$\begin{aligned} &= \text{determinant of } A = \begin{vmatrix} 1 & 2 & 5 \\ 2 & 2 & 2 \\ 1 & 2 & 7 \end{vmatrix} \\ &= 1[14-4] - (2)[14-2] + 5[4-2] = 10 - 24 + 10 = -4 \end{aligned}$$

Example 3

Two eigenvalues of a matrix $A = \begin{pmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{pmatrix}$ are equal to unity each. Find the third one.

Solution. Let the third eigenvalue be λ_3.
Then the sum of the eigenvalue =trace of $A = 2 + 3 + 2 = 7$

$$\begin{aligned} 1 + 1 + \lambda_3 &= \text{trace of } A \\ &= 7 \\ \lambda_3 &= 5 \end{aligned}$$

Example 4

The product of two eigenvalues of $A = \begin{pmatrix} -2 & 2 & 3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{pmatrix}$ is 3.

Find the third one.

Solution . Let the third eigenvalue be λ_3.
Product of the eigenvalues

$$\begin{aligned} &= \text{determinant of } A \\ &= \begin{vmatrix} -2 & 2 & 3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{vmatrix} \\ &= (-2)[0-12] - (2)[0-6] + 3[-4+1] \\ &= 24 + 12 - 9 = 27 \end{aligned}$$

Now $\lambda_1\lambda_2\lambda_3 = 3\lambda_3 = 27 \implies \lambda_3 = 9$.
Hence, the third eigenvalue is $\lambda_3 = 9$.

Example 5

Two eigenvalues of a matrix $A = \begin{pmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{pmatrix}$ are 3 and 6. Find the eigenvalues of A^{-1}.

Solution. Let the third eigenvalue be λ_3.
Then the sum of the eigenvalue = trace of $A = 3 + 5 + 3 = 11$

$$\begin{aligned} 3 + 6 + \lambda_3 &= \text{trace of } A = 11 \\ \lambda_3 &= 2 \end{aligned}$$

Hence the eigenvalues of A are $2, 3, 6$ andso the eigenvalues of A^{-1} are

$$\frac{1}{2}, \frac{1}{3}, \frac{1}{6}$$

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Example 6

Find the eigenvalues of A^3 if the matrix A is

$$A = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 3 & -1 \\ 0 & 0 & 4 \end{pmatrix}.$$

Solution. The given matrix is an upper triangular matrix.

Then the eigenvalues are nothing but the diagonal elements.

Hence, $1, 3, 4$ are the eigenvalues of A,and the eigenvalues of A^3

are $1^3, 3^3, 4^3 = 1, 27, 64$.

Example 7

One of the eigenvalues of $\begin{pmatrix} 7 & 4 & -4 \\ 4 & -8 & -1 \\ 4 & -1 & -8 \end{pmatrix}$ is -9. Find the other two.

Solution. Let the other two eigenvalues be λ_1 and λ_2.
Then the sum of the eigenvalues = trace of the matrix A

$$\begin{aligned} \lambda_1 + \lambda_2 - 9 &= 7 - 8 - 8 \\ \lambda_1 + \lambda_2 &= 0 \\ \lambda_1 &= -\lambda_2 \end{aligned}$$

Product of the eigenvalues

$$\begin{aligned} &= \text{determinant of } A = \begin{vmatrix} 7 & 4 & -4 \\ 4 & -8 & -1 \\ 4 & -1 & -8 \end{vmatrix} \\ &= 7[64-1]-(4)[-32+4]+(-4)[-4+32] \\ &= 7(63)-4(-28)-4(28) = 7\times 63 \end{aligned}$$

$$\begin{aligned} \lambda_1(-\lambda_1)(-9) &= |A| = 7\times 63 \\ \Rightarrow \lambda_1^2 &= 7\times 7 \\ \lambda_1 &= 7 \text{ and } \lambda_2 = -7 \end{aligned}$$

Example 8

For a given matrix A of order 3, $|A| = 32$, and two eigenvalues are 8 and 2. Find the sum of the eigenvalues.

Solution. Let the third eigenvalue be λ_3.

$$\begin{aligned}
\text{The product of the eigenvalue} &= |A| = 32 \\
(8)(2)\lambda_3 &= 32 \\
\lambda_3 &= 2
\end{aligned}$$

Then the sum of the eigenvalue $= 8 + 2 + 2 = 12$.

Example 9

Given $A = \begin{pmatrix} -1 & 0 & 0 \\ 2 & -3 & 0 \\ 1 & 4 & 2 \end{pmatrix}$ is -9. Find the eigenvalue of A^2.

Solution.

Since the given matrix is triangular matrix, the eigenvalues are nothing but the diagonal elements of A.

The eigenvalues are $= -1, -3, 2$

The eigenvalues of A^2 are $= 1, 9, 4$

Practice Problem

Problem-1

1. Find the sum and product of the eigenvalues

$$\begin{pmatrix} 4 & 2 & -2 \\ 5 & 3 & 2 \\ -2 & 4 & 1 \end{pmatrix}$$

Problem-2

Find the sum and product of the eigenvalues

$$\begin{pmatrix} 5 & 4 & 3 \\ 1 & 2 & 6 \\ 2 & -4 & 2 \end{pmatrix}$$

Problem-3

Find the eigenvalues of both A^3 and A^{-1} if

$$A = \begin{pmatrix} 5 & 4 & 1 \\ 0 & 4 & 2 \\ 0 & 0 & 3 \end{pmatrix}$$

Problem-4

Find the eigenvalues of A^{-1} if

$$A = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 3 & 2 \\ 0 & 0 & 4 \end{pmatrix}$$

Problem-5

Find the eigenvalues of A^{-1}, given that $A = \begin{pmatrix} 2 & 0 & -1 \\ 0 & 2 & -2 \\ 1 & -1 & 2 \end{pmatrix}$

Problem-6

Two eigenvalues of $A = \begin{pmatrix} 2 & 0 & -1 \\ 0 & 2 & -2 \\ 1 & -1 & 2 \end{pmatrix}$ are equal and they are double the third.Find the eigenvalues of $\dfrac{A^{-1}}{2}$ and A^2.

Answers

1. Sum $= 8$, Product $= -90$.
2. Sum $= 9$, Product $= 156$.
3. Eigenvalues of A^3 are $125, 64, 27$ and
   Eigenvalues of A^{-1} are $\frac{1}{5}, \frac{1}{4}, \frac{1}{3}$
4. Eigenvalues of A^{-1} are $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}$
5. Eigenvalues of A^{-1} is $\frac{1}{2}, \frac{1}{2}, \frac{1}{2}$
6. Eigenvalues of $\frac{A^{-1}}{2}$ is $\frac{1}{2}, \frac{1}{4}, \frac{1}{4}$ and eigenvalues of A^2 is $1, 4, 4$