21MAB101T - CALCULUS AND LINEAR ALGEBRS

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Characteristic Equation

Let A be any square matrix of order $n \times n$ and I be a unit matrix of same order. Then $|A - \lambda I|$ is called characteristic polynomial of matrix.

Then the equation $|A - \lambda I| = 0$ is called characteristic equation of matrix A.

Eigen value

The roots of the characteristic equation $|A - \lambda I| = 0$ are called Eigen values of matrix. It is denoted by λ

It is also called as Characteristic roots or latent roots.

Here we consider the problems with Eigen values as a real numbers

Eigen vector

Corresponding to each value of λ,there is a non-zero vector $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \neq 0$ such that $(A - \lambda I)X = 0$

Here X is called Eigen vector or characteristic vector or latent vector.

Procedure to find Eigen Values

- For a 2nd order square matrix, the characteristic equation will be of the form $\lambda^2 - S_1\lambda + S_2 = 0$

  where S_1 is sum of main diagonal elements and

  S_2 is the determinant of the matrix

- For a 3rd order square matrix, the characteristic equation will be of the form $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$

  where S_1 is sum of main diagonal elements,

  S_2 is sum of minors of main diagonal elements and

  S_3 is the determinant of the matrix

Note

- If the matrix A is of 2nd order, then there will be TWO Eigen Values
- If the matrix A is of 3rd order, then there will be THREE Eigen Values
- Minor of ij^{th} element is the determinant obtained by deleting i^{th} row and j^{th} column

Procedure to find Eigen Vectors

- Let us consider 3rd order square matrix
- WKT $(A - \lambda I)X = 0$ where $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \neq 0$, an non-zero Eigen vector, i.e, not all values are zero
- For each value of λ, we get three equations. By using cross-multiplication method, we can get the solution of the equations.
- That solution is called Eigen vector corresponding to that Eigen value
- Similarly, for all the possible Eigen values, we have to get the corresponding Eigen vectors

Method of Cross-Multiplication

Let the equations be

$$\begin{aligned} a_1x + b_1y + c_1z = 0 \\ a_2x + b_2y + c_2z = 0 \\ a_3x + b_3y + c_3z = 0 \end{aligned}$$

By considering first two equations we get

$$\frac{x}{b_1c_2 - b_2c_1} = \frac{y}{a_2c_1 - a_1c_2} = \frac{z}{a_1b_2 - a_2b_1}$$

Ratio of that solution is Eigen vector

Note : For cross-multiplication, we should consider non-identical equations

Example :

Let the equations be $x + y - z = 0$ and $x - 2z = 0$
Solving, by cross multiplication, we get

$$\frac{x}{-2-0} = \frac{y}{-1-(-2)} = \frac{z}{0-1}$$
$$\frac{x}{-2} = \frac{y}{1} = \frac{z}{-1}$$
$$\frac{x}{-2} = \frac{y}{1} = \frac{z}{-1}$$

Eigen vector $X = \begin{pmatrix} -2 \\ 1 \\ -1 \end{pmatrix}$ OR $\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$

Synthetic division method

Example 1:
Let us consider an equation $x^3 - 6x^2 + 11x - 6 = 0 \qquad ---> (1)$

By trial and error method, we can find one solution for the above equation, by putting $x = 0, 1, -1, 2, -2....$

Here $x = 1$ is one such solution

To find the remaining two solutions, we have to find the quadratic factor by synthetic division method

Consider the co-efficients and constants

$$
\begin{array}{c|cccc}
1 & 1 & -6 & 11 & -6 \\
 & 0 & 1 & -5 & 6 \\
\hline
 & 1 & -5 & 6 & \multicolumn{1}{|c}{0} \\
\cline{5-5}
\end{array}
$$

From eqn (1) we get $x^3 - 6x^2 + 11x - 6 = (x-1)(x^2 - 5x + 6) = 0$

On factorizing we get $x^2 - 5x + 6 = (x-2)(x-3) = 0$

The remaining two roots are $x = 2, x = 3$

Example 2:

Let us consider an equation $x^3 - 7x^2 + 16x - 12 = 0 \qquad ---> (1)$

Here $x = 2$ is one such solution

To find the remaining two solutions, we have to find the quadratic factor by synthetic division method

Consider the co-efficients and constants

$$\begin{array}{r|rrrr} 2 & 1 & -7 & 16 & -12 \\ & 0 & 2 & -10 & 12 \\ \hline & 1 & -5 & 6 & \boxed{0} \end{array}$$

Eqn (1) $\implies (x-2)(x^2 - 5x + 6) = 0$

On factorizing, we get $x^2 - 5x + 6 = (x-2)(x-3) = 0$

The remaining two roots are $x = 2, x = 3$

Example 3:

Let us consider an equation $x^3 - 12x^2 + 36x - 32 = 0$ $\qquad -- > (1)$

Here $x = 2$ is one such solution

To find the remaining two solutions, we have to find the quadratic factor by synthetic division method

Consider the co-efficients and constants

$$\begin{array}{c|rrrr} 2 & 1 & -12 & 36 & -32 \\ & 0 & 2 & -20 & 32 \\ \hline & 1 & -10 & 16 & \boxed{0} \end{array}$$

Eqn (1) $\implies (x-2)(x^2 - 10x + 16) = 0$

On factorizing we get $x^2 - 10x + 16 = (x-2)(x-8) = 0$

The remaining two roots are $x = 2, x = 8$

Example 4:

Let us consider an equation $x^3 - 4x^2 - x + 4 = 0 \qquad ---> (1)$

Here $x = 1$ is one such solution

Consider the co-efficients and constants

$$\begin{array}{r|rrrr} 1 & 1 & -4 & -1 & 4 \\ & 0 & 1 & -3 & -4 \\ \hline & 1 & -3 & -4 & \boxed{0} \end{array}$$

Eqn (1) $\implies (x-1)(x^2 - 3x - 4) = 0$

On factorizing we get $x^2 - 3x - 4 = (x-4)(x+1) = 0$

The remaining two roots are $x = -1, x = 4$

Non-Symmetric Matrices with Non Repeated Eigenvalues

Example 1

Find the eigenvalues and eigenvectors for the matrix

$$A = \begin{pmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{pmatrix}$$

Solution. The characteristic equation is $|A - \lambda I| = 0$ where

$$A = \begin{pmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{pmatrix}$$

The general form of characteristic equation is

$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$

where

S1 = Sum of the main diagonal entries

S2 = Sum of the minors of the main diagonal entries

S3 = Determinant of the matrix A

S1 = 1 + 2 + 3 = 6

$$S_2 = \begin{vmatrix} 2 & 1 \\ 2 & 3 \end{vmatrix} + \begin{vmatrix} 1 & -1 \\ 2 & 3 \end{vmatrix} + \begin{vmatrix} 1 & 0 \\ 1 & 2 \end{vmatrix}$$

$$S_2 = (6 - 2) + (3 - [-2]) + (2 - 0)$$

$$S_2 = (6 - 2) + (3 + 2) + (2 - 0)$$

$$S_2 = 4 + 5 + 2$$

$$S_2 = 11$$

$S_3 = |A|$

$$S_3 = \begin{vmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{vmatrix}$$

$$S_3 = 1 * \begin{vmatrix} 2 & 1 \\ 2 & 3 \end{vmatrix} - 0 * \begin{vmatrix} 1 & -1 \\ 2 & 3 \end{vmatrix} + (-1) \begin{vmatrix} 1 & 0 \\ 1 & 2 \end{vmatrix}$$

$S_3 = (6 - 2) - 0 - (2 - 4)$

$S_3 = 4 - 0 - (-2)$

$S_3 = 4 + 2$

$S_3 = 6$

The general form of characteristic equation is

$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$

$\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$

$(\lambda-1)(\lambda-2)(\lambda-3)=0$

The eigenvalues are $\lambda = 1, 2, 3.$

Now to find the eigenvector

The eigenvectors are given by $(A-\lambda I)\ X=0$

$$\begin{pmatrix} 1-\lambda & 0 & -1 \\ 1 & 2-\lambda & 1 \\ 2 & 2 & 3-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

For $\lambda = 1$ we have

$$\begin{pmatrix} 0 & 0 & -1 \\ 1 & 1 & 1 \\ 2 & 2 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$$0x_1 + 0x_2 - x_3 = 0$$

$$x_1 + x_2 + x_3 = 0$$

$$2x_1 + 2x_2 + 2x_3 = 0$$

Solving first two equations by rule of cross-multiplication we have

$$\frac{x_1}{0-(-1)} = \frac{x_2}{-1-0} = \frac{x_3}{0-0}$$
$$\frac{x_1}{1} = \frac{x_2}{-1} = \frac{x_3}{0}$$

The eigenvector corresponding to the eigenvalue $\lambda = 1$ is

$$X_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}.$$

For $\lambda = 2$ we have

$$\begin{pmatrix} -1 & 0 & -1 \\ 1 & 0 & 1 \\ 2 & 2 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$
$$-x_1 + 0x_2 - x_3 = 0$$
$$x_1 + 0x_2 + x_3 = 0$$
$$2x_1 + 2x_2 + x_3 = 0$$

Solving second and third equations by rule of cross-multiplication we have

$$\frac{x_1}{0-2} = \frac{x_2}{2-1} = \frac{x_3}{2-0}$$
$$\frac{x_1}{-2} = \frac{x_2}{1} = \frac{x_3}{2}$$
$$\frac{x_1}{2} = \frac{x_2}{-1} = \frac{x_3}{-2}$$

The eigenvector corresponding to the eigenvalue $\lambda = 2$ is

$$X_2 = \begin{pmatrix} 2 \\ -1 \\ -2 \end{pmatrix}.$$

For $\lambda = 3$ we have

$$\begin{pmatrix} -2 & 0 & -1 \\ 1 & -1 & 1 \\ 2 & 2 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$$
\begin{aligned}
-2x_1 + 0x_2 - x_3 &= 0 \\
x_1 - x_2 + x_3 &= 0 \\
2x_1 + 2x_2 + 0x_3 &= 0
\end{aligned}
$$

Solving second and third equations by rule of cross-multiplication we have

$$
\begin{aligned}
\frac{x_1}{0-2} = \frac{-x_2}{0-2} &= \frac{x_3}{2+2} \\
\frac{x_1}{-2} = \frac{x_2}{2} &= \frac{x_3}{4} \\
\frac{x_1}{1} = \frac{x_2}{-1} &= \frac{x_3}{-2}
\end{aligned}
$$

The eigenvector corresponding to the eigenvalue $\lambda = 3$ is

$$
X_3 = \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix}.
$$

Example 2

Find the eigenvalues and eigenvectors for the matrix

$$A = \begin{pmatrix} 1 & 1 & -2 \\ -1 & 2 & 1 \\ 0 & 1 & -1 \end{pmatrix}$$

Solution. The characteristic equation is $|A - \lambda I| = 0$ where

$$A = \begin{pmatrix} 1 & 1 & -2 \\ -1 & 2 & 1 \\ 0 & 1 & -1 \end{pmatrix}$$

$S_1 =$ Sum of the main diagonal entries $\Rightarrow\ S_1 = 2$

$S_2 =$ Sum of the minors of the main diagonal entries $\Rightarrow\ S_2 = -1$

$S_3 =$ Determinant of the matrix A $\Rightarrow\ S_3 = -2$

$\lambda^3 - 2\lambda^2 - \lambda + 2 = 0$

$\implies \quad \lambda = -1, 1, 2 \Rightarrow$ The eigenvalues are $\lambda = -1, 1, 2$.

The eigenvectors are given by $(A - \lambda I)\ X = 0$

$$\begin{pmatrix} 1-\lambda & 1 & -2 \\ -1 & 2-\lambda & 1 \\ 0 & 1 & -1-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

For $\lambda = -1$ we have

$$\begin{pmatrix} 2 & 1 & -2 \\ -1 & 3 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$$\begin{aligned} 2x_1 + 1x_2 - 2x_3 &= 0 \\ -1x_1 + 3x_2 + x_3 &= 0 \\ 0x_1 + 1x_2 + 0x_3 &= 0 \end{aligned}$$

Solving first two equations by rule of cross-multiplication we have

$$\frac{x_1}{7} = \frac{x_2}{0} = \frac{x_3}{7}$$
$$\frac{x_1}{1} = \frac{x_2}{0} = \frac{x_3}{1}$$

The eigenvector corresponding to the eigenvalue $\lambda = -1$ is

$$X_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

For $\lambda = 1$ we have

$$\begin{pmatrix} 0 & 1 & -2 \\ -1 & 1 & 1 \\ 0 & 1 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$
$$0x_1 + 1x_2 - 2x_3 = 0$$
$$-1x_1 + 1x_2 + x_3 = 0$$
$$0x_1 + 1x_2 - 2x_3 = 0$$

Solving these equations we have

$$\frac{x_1}{3} = \frac{x_2}{2} = \frac{x_3}{1}$$

The eigenvector corresponding to the eigenvalue $\lambda = 1$ is

$$X_2 = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}.$$

For $\lambda = 2$ we have

$$\begin{pmatrix} -1 & 1 & -2 \\ -1 & 0 & 1 \\ 0 & 1 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$$\begin{aligned} -1x_1 + 1x_2 - 2x_3 &= 0 \\ -1x_1 + 0x_2 + x_3 &= 0 \\ 0x_1 + x_2 - 3x_3 &= 0 \end{aligned}$$

Solving these equations we have

$$\frac{x_1}{1} = \frac{x_2}{3} = \frac{x_3}{1}$$

The eigenvector corresponding to the eigenvalue $\lambda = 2$ is

$$X_3 = \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}.$$

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