21MAB101T - CALCULUS AND LINEAR ALGEBRA

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Consider the expression

$$\begin{aligned}
&a_{11}x_1^2 + a_{12}x_1x_2 + \cdots + a_{1n}x_1x_n \\
&+a_{21}x_2x_1 + a_{22}x_2^2 + \cdots + a_{2n}x_2x_n \\
&+\cdots \\
&+a_{n1}x_nx_1 + a_{n2}x_nx_2 + + \cdots + a_{nn}x_n^2
\end{aligned}$$

which is a homogeneous polynomial of second degree and is known as quadratic form with the variables $x_1, x_2, \cdots x_n$ and is denoted by

$$Q = \sum_{i=1}^{n}\sum_{j=1}^{n} \quad a_{ij}x_ix_j$$

The above expression can be written as

$$Q = X^T A X$$

where

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \qquad X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

The matrix A is called matrix of the quadratic form Q.

Note

The matrix A of the above quadratic form must be a symmetric matrix. we make it symmetric by the substitution

$$c_{ij} = \frac{1}{2}(a_{ij} + a_{ji}) \implies \quad a_{ii} = c_{ii}, \quad a_{ij} = a_{ji}$$

The rank of the $n \times n$ symmetric matrix A is the rank of the quadratic form Q.
If the rank is less than n where n is the number of variables then the quadratic form is called singular otherwise it is nonsingular.

Linear Transform of a Quadratic Form

Let $Q = X^T AX$ be a quadratic form in the n variables $x_1, x_2, \cdots, x_n$.
Consider the transformation $X = CY$, that transforms the variable set $X = (x_1, x_2, \cdots, x_n)^T$ to a new variables set $Y = (y_1, y_2, \cdots, y_n)^T$, where C is a non-singular matrix.
We can easily verify that the transformation $X = CY$ expresses each of the variables $x_1, x_2, \cdots, x_n$ as homogeneous linear expressions in $y_1, y_2, \cdots, y_n$.
Hence $X = CY$ is called a non-singular linear transformation.

By this transformation, $Q = X^T AX$ is transformed to

$$\begin{aligned} Q &= (CY)^T A(CY) \\ &= Y^T(C^T AC)Y \\ &= Y^T DY, \\ &\quad \text{where } D = C^T AC \end{aligned}$$

Also

$$\begin{aligned} D^T &= (C^T AC)^T \\ &= C^T A^T C \\ &= C^T AC \\ &\quad \text{since } A \text{ is symmetric } A = A^T \\ &= D \end{aligned}$$

This shows that D is also a symmetric matrix.

Hence D is the matrix of the quadratic form $Y^T DY$ in the variables $y_1, y_2, \cdots, y_n$.

Thus $Y^T DY$ is the linear transform of the quadratic form $X^T AX$ under the linear transformation $X = CY$,where $D = C^T AC$.

Orthogonal Transformation of a Quadratic forms to the Canonical Form

Consider the linear transformation $X = CY$, where C is chosen such that $D = C^T AC$, a diagonal matrix, then the quadratic form Q gets reduced to

$$\begin{aligned} Q &= Y^T DY \\ &= (y_1, y_2, \cdots, y_n) \times \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix} \times \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} \\ &= \lambda_1 y_1^2 + \lambda_2 y_2^2 + \cdots + \lambda_n y_n^2 \end{aligned}$$

The above expression of Q is called the sum of the squares form of Q or the canonical form of Q.

If C is an orthogonal matrix in the transformation $X = CY$,and if $X = CY$ transforms the quadratic form Q to the canonical form then Q is said to be **the canonical form by an orthogonal transformation**.

We list out the procedure of orthogonal transformation of a quadratic forms to the canonical form

Step 1. Convert the quadratic forms Q to $X^T AX$ where A is real symmetric matrix.

Step 2. Find the eigenvalues of the real symmetric matric A

Step 3. Find the eigenvectors for the corresponding eigenvalues where the eigenvectors are linearly independent and pairwise orthogonal.

Step 4. Normalize each eigenvector by dividing with its magnitude

Step 5. Construct normalized modal matrix N and its transpose N^T.

Step 6. Perform the transform $N^T AN = D$.

Step 7. Finally find $Y^T DY$

Nature of Quadratic Forms

When the quadratic form X^TAX is reduced to the canonical form Y^TDY,it will contain only r terms, if the rank of A is r. The terms in the canonical form may be positive,zero or negative.

Index

The **index** p of the quadratic form is the number of positive terms in the canonical forms.

Signature

The **signature** s of the quadratic form is the excess of the number of positive terms over the number of negative terms in the canonical forms that is $p-(r-p)=2p-r$.

Positive definite

The quadratic form $Q = X^T AX$ in n variables is said to be positive definite, if $r = n$ and $p = n$ or if all the eigenvalues of A are positive.

negative definite

The quadratic form $Q = X^T AX$ in n variables is said to be negative definite, if r=n and p=0 or if all the eigenvalues of A are negative.

Positive Semi-definite

The quadratic form $Q = X^T AX$ in n variables is said to be positive semidefinite, if $r < n$ and $p = r$ or if all the eigenvalues of $A \geq 0$ and at least one eigenvalue is zero.

negative Semi-definite

The quadratic form $Q = X^T AX$ in n variables is said to be negative semidefinite, if $r < n$ and $p = 0$ or if all the eigenvalues of $A \leq 0$ and at least one eigenvalue is zero.

Indefinite

The quadratic form $Q = X^T AX$ in n variables is said to be indefinite in all other cases.

Example 1

Write down the matrix of the quadratic form
$x_1^2 + 2x_2^2 + 3x_3^2 + 4x_1x_2 + 6x_2x_3 + 8x_1x_3$

Solution. The given quadratic form is $Q = X^TAX$ where

$$A = \begin{pmatrix} \text{coeff. of} x_1^2 & \frac{1}{2}\text{coeff. of} x_1x_2 & \frac{1}{2}\text{coeff. of} x_1x_3 \\ \frac{1}{2}\text{coeff. of} x_1x_2 & \text{coeff. of} x_2^2 & \frac{1}{2}\text{coeff. of} x_2x_3 \\ \frac{1}{2}\text{coeff. of} x_1x_3 & \frac{1}{2}\text{coeff. of} x_2x_3 & \text{coeff. of} x_3^2 \end{pmatrix}$$

$$X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \qquad X^T = \begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix}$$

Hence the matrix of the quadratic form is

$$A = \begin{pmatrix} 1 & 2 & 4 \\ 2 & 2 & 3 \\ 4 & 3 & 3 \end{pmatrix}$$

Example 2.

Reduce the quadratic form is $8x_1^2 + 7x_2^2 + 3x_3^2 - 12x_1x_2 - 8x_2x_3 + 4x_1x_3$ to canonical form through an orthogonal transformation and hence show that it is positive definite .

Solution. The given quadratic form is $Q = X^T AX$ where

$$A = \begin{pmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{pmatrix} \quad X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \quad X^T = \begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix}$$

The characteristic equation is $|A - \lambda I| = 0$

$$\lambda^3 - 18\lambda^2 + 45\lambda = 0$$

$$\lambda(\lambda - 3)(\lambda - 15) = 0$$

$$\implies \quad \lambda = 0, 3, 15$$

The eigenvalues are $\lambda = 0, 3, 15$.

The eigenvectors are given by $(A - \lambda I)\ X = 0$

$$\begin{pmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

For $\lambda = 0$ we have

$$\begin{pmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$$8x_1 - 6x_2 + 2x_3 = 0$$

$$-6x_1 + 7x_2 - 4x_3 = 0$$

$$2x_1 - 4x_2 + 3x_3 = 0$$

Solving first two equations we have

$$\frac{x_1}{24-14} = \frac{x_2}{-12-(-32)} = \frac{x_3}{56-36}$$
$$\frac{x_1}{10} = \frac{x_2}{20} = \frac{x_3}{20}$$
$$\frac{x_1}{1} = \frac{x_2}{2} = \frac{x_3}{2}$$

The eigenvector corresponding to the eigenvalue $\lambda = 0$ is

$$X_1 = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}.$$

For $\lambda = 3$ we have

$$\begin{pmatrix} 5 & -6 & 2 \\ -6 & 4 & -4 \\ 2 & -4 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$$5x_1 - 6x_2 + 2x_3 = 0$$

$$-6x_1 + 4x_2 - 4x_3 = 0$$

$$2x_1 - 4x_2 + 0x_3 = 0$$

first two equations we have

$$\frac{x_1}{24-8} = \frac{x_2}{-12-(-20)} = \frac{x_3}{20-36}$$

$$\frac{x_1}{16} = \frac{x_2}{8} = \frac{x_3}{-16}$$

$$\frac{x_1}{2} = \frac{x_2}{1} = \frac{x_3}{-2}$$

The eigenvector corresponding to the eigenvalue $\lambda = 3$ is

$$X_2 = \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}.$$

For $\lambda = 15$ we have

$$\begin{pmatrix} -7 & -6 & 2 \\ -6 & -8 & -4 \\ 2 & -4 & -12 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$$-7x_1 - 6x_2 + 2x_3 = 0$$

$$-6x_1 - 8x_2 - 4x_3 = 0$$

$$2x_1 - 4x_2 - 12x_3 = 0$$

Solving first two equations we have

$$\frac{x_1}{24+16} = \frac{x_2}{-12-28} = \frac{x_3}{56-36}$$
$$\frac{x_1}{40} = \frac{x_2}{-40} = \frac{x_3}{20}$$
$$\frac{x_1}{2} = \frac{x_2}{-2} = \frac{x_3}{1}$$

The eigenvector corresponding to the eigenvalue $\lambda = 15$ is

$$X_3 = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}.$$

Hence the eigenvectors are

$$X_1 = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \qquad X_2 = \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix} \qquad X_3 = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$$

and the normalized eigenvectors are

$$X_1^N = \frac{1}{3} \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \quad X_2^N = \frac{1}{3} \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix} \quad X_3^N = \frac{1}{3} \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$$

The normalized modal matrix is

$$N = \frac{1}{3} \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{pmatrix} \qquad N^T = \frac{1}{3} \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{pmatrix}$$

$$
\begin{aligned}
N^TAN &= \frac{1}{3}\begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{pmatrix} \times \begin{pmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{pmatrix} \\
&\quad \times \frac{1}{3}\begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{pmatrix} \\
D &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 15 \end{pmatrix}
\end{aligned}
$$

Now consider the orthogonal transformation $X = NY$. Then the given quadratic form is transformed to

$$
\begin{aligned}
Q &= X^T A X \\
&= (NY)^T A (NY) \\
&= Y^T (N^T A N) Y \\
&= \begin{pmatrix} y_1 & y_2 & y_3 \end{pmatrix} \times \begin{pmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 15 \end{pmatrix} \times \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \\
Q &= 3y_2^2 + 15y_3^2
\end{aligned}
$$

The nature of the quadratic form are

Rank $(r) = 2$

Index $(p) = 2$

Signature $(s) = 2p - r = 2 * 2 - 2 = 2$

Since, all the eigen values are **positive and one eigen value is zero**

Hence, the quadratic form is **positive semi-definite**.

Example 3

Reduce the quadratic form
$x_1^2 + 5x_2^2 + x_3^2 + 2x_1x_2 + 2x_2x_3 + 6x_1x_3$ to canonical form through an orthogonal transformation.

Solution. The given quadratic form is $Q = X^TAX$ where

$$A = \begin{pmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{pmatrix} \quad X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \quad X^T = \begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix}$$

The characteristic equation is $|A - \lambda I| = 0$

$$\begin{aligned} \lambda^3 - 7\lambda^2 + 36 &= 0 \\ (2+\lambda)(3-\lambda)(6-\lambda) &= 0 \\ \implies \quad \lambda &= -2, 3, 6 \end{aligned}$$

The eigenvalues are $\lambda = -2, 3, 6$.

The eigenvectors are given by $(A - \lambda I)\ X = 0$

$$\begin{pmatrix} 1-\lambda & 1 & 3 \\ 1 & 5-\lambda & 1 \\ 3 & 1 & 1-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

For $\lambda = -2$ we have

$$\begin{pmatrix} 3 & 1 & 3 \\ 1 & 7 & 1 \\ 3 & 1 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$$\begin{aligned} 3x_1 + x_2 + 3x_3 &= 0 \\ x_1 + 7x_2 + x_3 &= 0 \\ 3x_1 + x_2 + 3x_3 &= 0 \end{aligned}$$

Solving first two equations we have

$$\frac{x_1}{1-21} = \frac{-x_2}{3-3} = \frac{x_3}{21-1}$$
$$\frac{x_1}{-20} = \frac{x_2}{0} = \frac{x_3}{20}$$
$$\frac{x_1}{-1} = \frac{x_2}{0} = \frac{x_3}{1}$$

The eigenvector corresponding to the eigenvalue $\lambda = -2$ is

$$X_1 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}.$$

For $\lambda = 3$ we have

$$\begin{pmatrix} -2 & 1 & 3 \\ 1 & 2 & 1 \\ 3 & 1 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$$\begin{aligned} -2x_1 + x_2 + 3x_3 &= 0 \\ x_1 + 2x_2 + x_3 &= 0 \\ 3x_1 + x_2 - 2x_3 &= 0 \end{aligned}$$

Solving first two equations we have

$$\begin{aligned} \frac{x_1}{1-6} = \frac{x_2}{3-(-2)} &= \frac{x_3}{-4-1} \\ \frac{x_1}{-5} = \frac{x_2}{5} &= \frac{x_3}{-5} \\ \frac{x_1}{-1} = \frac{x_2}{1} &= \frac{x_3}{-1} \end{aligned}$$

The eigenvector corresponding to the eigenvalue $\lambda = 3$ is

$$X_2 = \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}.$$

For $\lambda = 6$ we have

$$\begin{pmatrix} -5 & 1 & 3 \\ 1 & -1 & 1 \\ 3 & 1 & -5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$$-5x_1 + x_2 + 3x_3 = 0$$

$$x_1 - x_2 + x_3 = 0$$

$$3x_1 + x_2 - 5x_3 = 0$$

Solving first two equations we have

$$\frac{x_1}{1+3} = \frac{-x_2}{-5-3} = \frac{x_3}{5-1}$$
$$\frac{x_1}{4} = \frac{x_2}{8} = \frac{x_3}{4}$$
$$\frac{x_1}{1} = \frac{x_2}{2} = \frac{x_3}{1}$$

The eigenvector corresponding to the eigenvalue $\lambda = 6$ is

$$X_3 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}.$$

Hence the eigenvectors are

$$X_1 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \qquad X_2 = \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix} \qquad X_3 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$$

and the normalized eigenvectors are

$$X_1^N = \begin{pmatrix} \frac{-1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix} \qquad X_2^N = \begin{pmatrix} \frac{-1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{-1}{\sqrt{3}} \end{pmatrix} \qquad X_3^N = \begin{pmatrix} \frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{pmatrix}$$

The normalized modal matrix is

$$N = \begin{pmatrix} \frac{-1}{\sqrt{2}} & \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \end{pmatrix} \qquad N^T = \begin{pmatrix} \frac{-1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{pmatrix}$$

$$
\begin{aligned}
N^TAN &= \begin{pmatrix} \frac{-1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{pmatrix} \times \begin{pmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{pmatrix} \\
&\quad \times \begin{pmatrix} \frac{-1}{\sqrt{2}} & \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \end{pmatrix} \\
&= \begin{pmatrix} -2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{pmatrix}
\end{aligned}
$$

Now consider the orthogonal transformation $X = NY$. Then the given quadratic form is transformed to

$$
\begin{aligned}
Q &= X^T A X \\
&= (NY)^T A (NY) \\
&= Y^T (N^T A N) Y \\
&= \begin{pmatrix} y_1 & y_2 & y_3 \end{pmatrix} \times \begin{pmatrix} -2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{pmatrix} \times \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \\
&= -2y_1^2 + 3y_2^2 + 6y_3^2
\end{aligned}
$$

The nature of the quadratic form are

Rank $(r) = 3$

Index $(p) = 2$

Signature $(s) = 2p - r = 2 * 2 - 3 = 1$

Since, $r = n = 3$ and $p = 2 < n = 3$.

Hence the quadratic form is In-definite.

Example 4

Reduce the quadratic form is $3x_1^2 + 5x_2^2 + 3x_3^2 - 2x_1x_2 + 2x_2x_3 - 2x_1x_3$ to canonical form through an orthogonal transformation .

Solution. The given quadratic form is $Q = X^TAX$ where

$$A = \begin{pmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{pmatrix} \quad X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \quad X^T = \begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix}$$

The characteristic equation is $|A - \lambda I| = 0$

$$\lambda^3 - 11\lambda^2 + 36\lambda - 36 = 0$$
$$(2-\lambda)(3-\lambda)(6-\lambda) = 0$$
$$\implies \quad \lambda = 2, 3, 6$$

The eigenvalues are $\lambda = 2, 3, 6$.

The eigenvectors are given by $(A - \lambda I)\ X = 0$

$$\begin{pmatrix} 3-\lambda & -1 & 1 \\ -1 & 5-\lambda & -1 \\ 1 & -1 & 3-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

For $\lambda = 2$ we have

$$\begin{pmatrix} 1 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$$x_1 - x_2 + x_3 = 0$$

$$-x_1 + 3x_2 - x_3 = 0$$

$$x_1 - x_2 + x_3 = 0$$

Solving first two equations we have

$$\frac{x_1}{1-3} = \frac{x_2}{-1-(-1)} = \frac{x_3}{3-1}$$
$$\frac{x_1}{-2} = \frac{x_2}{0} = \frac{x_3}{2}$$
$$\frac{x_1}{-1} = \frac{x_2}{0} = \frac{x_3}{1}$$

The eigenvector corresponding to the eigenvalue $\lambda = 2$ is

$$X_1 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}.$$

For $\lambda = 3$ we have

$$\begin{pmatrix} 0 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$$0x_1 - x_2 + x_3 = 0$$

$$-x_1 + 2x_2 - x_3 = 0$$

$$x_1 - x_2 + 0x_3 = 0$$

Solving first two equations we have

$$\frac{x_1}{1-2} = \frac{x_2}{-1-0} = \frac{x_3}{0-1}$$

$$\frac{x_1}{-1} = \frac{x_2}{-1} = \frac{x_3}{-1}$$

$$\frac{x_1}{1} = \frac{x_2}{1} = \frac{x_3}{1}$$

The eigenvector corresponding to the eigenvalue $\lambda = 3$ is

$$X_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

For $\lambda = 6$ we have

$$\begin{pmatrix} -3 & -1 & 1 \\ -1 & -1 & -1 \\ 1 & -1 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$$-3x_1 - x_2 + x_3 = 0$$

$$-x_1 - x_2 - x_3 = 0$$

$$x_1 - x_2 - 3x_3 = 0$$

Solving first two equations we have

$$\frac{x_1}{1+1} = \frac{-x_2}{3+1} = \frac{x_3}{3-1}$$
$$\frac{x_1}{2} = \frac{x_2}{-4} = \frac{x_3}{2}$$
$$\frac{x_1}{1} = \frac{x_2}{-2} = \frac{x_3}{1}$$

The eigenvector corresponding to the eigenvalue $\lambda = 6$ is

$$X_3 = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}.$$

Hence the eigenvectors are

$$X_1 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \qquad X_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \qquad X_3 = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

and the normalized eigenvectors are

$$X_1^N = \begin{pmatrix} \frac{-1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix} \qquad X_2^N = \begin{pmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{pmatrix} \qquad X_3^N = \begin{pmatrix} \frac{1}{\sqrt{6}} \\ \frac{-2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{pmatrix}$$

The normalized modal matrix is

$$N = \begin{pmatrix} \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{3}} & \frac{-2}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \end{pmatrix} \qquad N^T = \begin{pmatrix} \frac{-1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{-2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{pmatrix}$$

$$
\begin{aligned}
N^T A N &= \begin{pmatrix} \frac{-1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{-2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{pmatrix} \times \begin{pmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{pmatrix} \\
&\quad \times \begin{pmatrix} \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{3}} & \frac{-2}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \end{pmatrix} \\
&= \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{pmatrix}
\end{aligned}
$$

Now consider the orthogonal transformation $X = NY$. Then the given quadratic form is transformed to

$$\begin{aligned}
Q &= X^T A X \\
&= (NY)^T A (NY) \\
&= Y^T (N^T A N) Y \\
&= \begin{pmatrix} y_1 & y_2 & y_3 \end{pmatrix} \times \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{pmatrix} \times \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \\
&= 2y_1^2 + 3y_2^2 + 6y_3^2
\end{aligned}$$

Example 5

Reduce the quadratic form $2x_1x_2 + 2x_2x_3 + 2x_1x_3$ to canonical form through an orthogonal transformation and discuss the nature.

Solution. The given quadratic form is $Q = X^TAX$ where

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \quad X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \quad X^T = \begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix}$$

The characteristic equation is $|A - \lambda I| = 0$

$$\begin{aligned} \lambda^3 - 3\lambda - 2 &= 0 \\ (\lambda+1)(\lambda+1)(\lambda-2) &= 0 \\ \implies \quad \lambda &= 2, -1, -1 \end{aligned}$$

The eigenvalues are $\lambda = 2, -1, -1$.

The eigenvectors are given by $(A - \lambda I)\ X = 0$

$$\begin{pmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

For $\lambda = 2$ we have

$$\begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$$-2x_1 + x_2 + x_3 = 0$$

$$x_1 - 2x_2 + x_3 = 0$$

$$x_1 + x_2 - 2x_3 = 0$$

Solving first two equations we have

$$\frac{x_1}{1+2} = \frac{-x_2}{-2-1} = \frac{x_3}{4-1}$$
$$\frac{x_1}{3} = \frac{x_2}{3} = \frac{x_3}{3}$$

The eigenvector corresponding to the eigenvalue $\lambda = 2$ is

$$X_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

For $\lambda = -1$ we have

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$$x_1 + x_2 + x_3 = 0$$

$$x_1 + x_2 + x_3 = 0$$

$$x_1 + x_2 + x_3 = 0$$

All the three are the same equation. Hence
set $x_3 = 0$ then $x_1 + x_2 = 0$, and set $x_1 = 1 \Longrightarrow x_2 = -1$
The eigenvector corresponding to the eigenvalue $\lambda = -1$ is

$$X_2 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}.$$

For $\lambda = -1$, let $X_3 = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$, such that $X_1^T \cdot X_3 = 0$ and $X_2^T \cdot X_3 = 0$

$$\begin{aligned} \begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} &= 0 \\ a + b + c &= 0 \\ \begin{pmatrix} 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} &= 0 \\ a - b + 0c &= 0 \end{aligned}$$

$$\begin{aligned} a+b+c &= 0 \\ a-b+0c &= 0 \\ \frac{a}{0+1} &= \frac{-b}{0-1} = \frac{c}{-1-1} \\ \frac{a}{1} &= \frac{b}{1} = \frac{c}{-2} \end{aligned}$$

The another eigenvector corresponding to the eigenvalue $\lambda = -1$ is

$$X_3 = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}.$$

Hence the eigenvectors are

$$X_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \qquad X_2 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \qquad X_3 = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$$

and the normalized eigenvectors are

$$X_1^N = \begin{pmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{pmatrix} \quad X_2^N = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 0 \end{pmatrix} \quad X_3^N = \begin{pmatrix} \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \\ -\frac{2}{\sqrt{6}} \end{pmatrix}$$

The normalized modal matrix is

$$N = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & -\frac{2}{\sqrt{6}} \end{pmatrix} \quad N^T = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} \end{pmatrix}$$

$$\begin{aligned} N^TAN &= \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & -\frac{2}{\sqrt{6}} \end{pmatrix} \times \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \\ &\quad \times \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} \end{pmatrix} \\ N^TAN &= \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \end{aligned}$$

Now consider the orthogonal transformation $X = NY$. Then the given quadratic form is transformed to

$$\begin{aligned} Q &= X^T A X \\ &= (NY)^T A (NY) \\ &= Y^T (N^T A N) Y \\ &= \begin{pmatrix} y_1 & y_2 & y_3 \end{pmatrix} \times \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \times \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \\ &= 2y_1^2 - y_2^2 - y_3^2 \end{aligned}$$

The nature of the quadratic form are

Rank $= 3$

Index $= 1$

Signature $= 1 - (3 - 1) = 1 - 2 = -1$

Hence, the nature of the quadratic form is indefinite.

Example 6

Reduce the quadratic form
$3x_1^2 + 2x_1x_2 + 2x_1x_3 - 2x_2x_3 + 3x_2^2 + 3x_3^2$ to canonical form.
Discuss its nature.

Solution. The given quadratic form is X^TAX, where its matrix is
$A = \begin{pmatrix} 3 & 1 & 1 \\ 1 & 3 & -1 \\ 1 & -1 & 3 \end{pmatrix}$ and $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$
The characteristic equation is $|A - \lambda I| = 0$

$$\begin{aligned} \lambda^3 - 9\lambda^2 + 24\lambda - 16 &= 0 \\ (\lambda - 1)(\lambda - 4)(\lambda - 4) &= 0 \\ \implies \quad \lambda &= 1, 4, 4 \end{aligned}$$

The eigenvalues are $\lambda = 1, 4, 4$.

The eigenvectors are given by $(A - \lambda I)\ X = 0$

$$\begin{pmatrix} 3-\lambda & 1 & 1 \\ 1 & 3-\lambda & -1 \\ 1 & -1 & 3-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

For $\lambda = 1$ we have

$$\begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$$2x_1 + x_2 + x_3 = 0$$

$$x_1 + 2x_2 - x_3 = 0$$

$$x_1 - x_2 + 2x_3 = 0$$

Solving first two equations we have

$$\frac{x_1}{-1-2} = \frac{-x_2}{-2-1} = \frac{x_3}{4-1}$$
$$\frac{x_1}{-3} = \frac{x_2}{3} = \frac{x_3}{3}$$
$$\frac{x_1}{-1} = \frac{x_2}{1} = \frac{x_3}{1}$$

The eigenvector corresponding to the eigenvalue $\lambda = 1$ is

$$X_1 = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}.$$

For $\lambda = 4$ we have

$$\begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & -1 \\ 1 & -1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$$-x_1 + x_2 + x_3 = 0$$

$$x_1 - x_2 - x_3 = 0$$

$$x_1 - x_2 - x_3 = 0$$

These three equation represents the same equation,

$$x_1 - x_2 - x_3 = 0$$

$$\text{put } \mathrm{x}_1 = 0 \text{ we get } \mathrm{x}_2 = -\mathrm{x}_3$$

$$\frac{x_2}{1} = \frac{x_3}{-1}$$

The eigenvector corresponding to the eigenvalue $\lambda = 4$ is

$$X_2 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}.$$

For $\lambda = 4$, let $X_3 = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$, such that $X_1^T \cdot X_3 = 0$ and $X_2^T \cdot X_3 = 0$.

Since the given matrix is symmetric

$$\begin{pmatrix} -1 & 1 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0$$

or

$$-a + b + c = 0$$

$$\begin{pmatrix} 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0$$

or

$$0a + b - c = 0$$

solving these two we get

$$\frac{a}{-1-1} = \frac{-b}{1} = \frac{c}{-1}$$
$$\frac{a}{2} = \frac{b}{1} = \frac{c}{1}$$

The another eigenvector corresponding to the eigenvalue $\lambda = 4$ is

$$X_3 = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}.$$

Hence the eigenvectors are

$$X_1 = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \qquad X_2 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \qquad X_3 = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$$

and the normalized eigenvectors are

$$X_1^N = \frac{1}{\sqrt{3}} \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \quad X_2^N = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \quad X_3^N = \frac{1}{\sqrt{6}} \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$$

The normalized modal matrix is

$$N = \begin{pmatrix} \frac{-1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{pmatrix} \qquad N^T = \begin{pmatrix} \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{pmatrix}$$

$$\begin{aligned} N^T A N &= \begin{pmatrix} \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{pmatrix} \times \begin{pmatrix} 3 & 1 & 1 \\ 1 & 3 & -1 \\ 1 & -1 & 3 \end{pmatrix} \\ &\times \begin{pmatrix} \frac{-1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{pmatrix} \\ D &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix} \end{aligned}$$

Now consider the orthogonal transformation $X = NY$. Then the given quadratic form is transformed to

$$
\begin{aligned}
Q &= X^T A X \\
&= (NY)^T A (NY) \\
&= Y^T (N^T A N) Y \\
&= \begin{pmatrix} y_1 & y_2 & y_3 \end{pmatrix} \times \begin{pmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix} \times \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \\
&= y_1^2 + 4y_2^2 + +4y_3^2
\end{aligned}
$$

Thus the required canonical form is $y_1^2 + 4y_3^2 + +4y_3^2$

Nature of the quadratic form:

Rank = 3,

index = 3,

signature = 3-0=3

The nature of the quadratic form is **positive definite**.

Example 7

Reduce the quadratic form to the canonical form by an orthogonal transformation
$x_1^2 + x_2^2 + x_3^2 + 4x_1x_2 + 4x_2x_3 + 4x_1x_3$. Discuss its nature.

Solution. The given quadratic form is $Q = X^T AX$ where

$$A = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix} \quad X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \quad X^T = \begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix}$$

The characteristic equation is $|A - \lambda I| = 0$

$$\lambda^3 - 3\lambda^2 - 9\lambda - 5 = 0$$
$$(\lambda + 2)(\lambda + 1)(\lambda - 5) = 0$$
$$\implies \lambda = -1, -1, 5$$

The eigenvalues are $\lambda = -1, -1, 5$.

The eigenvectors are given by $(A - \lambda I)\ X = 0$

$$\begin{pmatrix} 1-\lambda & 2 & 2 \\ 2 & 1-\lambda & 2 \\ 2 & 2 & 1-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

For $\lambda = 5$ we have

$$\begin{pmatrix} -4 & 2 & 2 \\ 2 & -4 & 2 \\ 2 & 2 & -4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$$-4x_1 + 2x_2 + 2x_3 = 0$$

$$2x_1 - 43x_2 + 2x_3 = 0$$

$$2x_1 + 2x_2 - 4x_3 = 0$$

Solving first two equations we have

$$\frac{x_1}{12} = \frac{x_2}{12} = \frac{x_3}{12}$$
$$\frac{x_1}{1} = \frac{x_2}{1} = \frac{x_3}{1}$$

The eigenvector corresponding to the eigenvalue $\lambda = 5$ is

$$X_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

For $\lambda = -1$ we have

$$\begin{pmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$$x_1 + x_2 + x_3 = 0$$

$$x_1 + x_2 + x_3 = 0$$

$$x_1 + x_2 + x_3 = 0$$

These three equation represents the same equation,

$$x_1 + x_2 + x_3 = 0$$

$$\text{put } \mathrm{x}_1 = 0 \text{ we get } \mathrm{x}_2 = -\mathrm{x}_3$$

$$\frac{x_2}{1} = \frac{x_3}{-1}$$

The eigenvector corresponding to the eigenvalue $\lambda = -1$ is

$$X_2 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}.$$

For $\lambda = -1$, let $X_3 = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$, such that $X_1^T \cdot X_3 = 0$ and $X_2^T \cdot X_3 = 0$.
Since the given matrix is symmetric

$$\begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0$$

or

$$a + b + c = 0$$

$$\begin{pmatrix} 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0$$

or

$$0a + b - c = 0$$

solving these two we get

$$\frac{a}{-2} = \frac{-b}{-1} = \frac{c}{1}$$
$$\frac{a}{-2} = \frac{b}{1} = \frac{c}{1}$$

The eigenvector corresponding to the eigenvalue $\lambda = -1$ is

$$X_3 = \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}.$$

Hence the eigenvectors are

$$X_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \qquad X_2 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \qquad X_3 = \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$$

Clearly $X_1^T X_2 = X_2^T X_3 = X_3^T X_1 = 0$
and the normalized eigenvectors are

$$X_1^N = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad X_2^N = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \quad X_3^N = \frac{1}{\sqrt{6}} \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$$

The normalized modal matrix is

$$N = \begin{pmatrix} \frac{1}{\sqrt{3}} & 0 & \frac{-2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{pmatrix} \qquad N^T = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{-2}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{pmatrix}$$

$$
\begin{aligned}
N^T A N &= \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{-2}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{pmatrix} \times \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix} \\
&\quad \times \begin{pmatrix} \frac{1}{\sqrt{3}} & 0 & \frac{-2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{pmatrix} \\
D &= \begin{pmatrix} 5 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}
\end{aligned}
$$

Now consider the orthogonal transformation $X = NY$. Then the given quadratic form is transformed to

$$\begin{aligned} Q &= X^T A X \\ &= (NY)^T A (NY) \\ &= Y^T (N^T A N) Y \\ &= \begin{pmatrix} y_1 & y_2 & y_3 \end{pmatrix} \times \begin{pmatrix} 5 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \times \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \\ Q &= 5y_1^2 - y_2^2 - y_3^2 \end{aligned}$$

Thus the required canonical form is $5y_1^2 - y_2^2 - y_3^2$

Nature of the quadratic form:

Rank = 3,

Index = 1,

Signature = 1-2=-1

Hence the nature of the quadratic form is **Indefinite.**

Example 8

Reduce the quadratic form to the canonical form by an orthogonal transformation $x_1^2 + 3x_2^2 + 3x_3^2 - 2x_2x_3$.

Solution. The given quadratic form is $Q = X^T AX$ where

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & -1 & 3 \end{pmatrix} \quad X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \quad X^T = \begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix}$$

The characteristic equation is $|A - \lambda I| = 0$

$$(1-\lambda)(2-\lambda)(4-\lambda) = 0$$
$$\implies \lambda = 1, 2, 4$$

The eigenvalues are $\lambda = 1, 2, 4$.

The eigenvectors are given by $(A - \lambda I)\ X = 0$

$$\begin{pmatrix} 1-\lambda & 0 & 0 \\ 0 & 3-\lambda & -1 \\ 0 & -1 & 3-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

For $\lambda = 1$ we have

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$$0x_1 + 0x_2 + 0x_3 = 0$$

$$0x_1 + 2x_2 - x_3 = 0$$

$$0x_1 - x_2 + 2x_3 = 0$$

Solving second and third equations we have

$$\frac{x_1}{4-1} = \frac{-x_2}{0-0} = \frac{x_3}{0-0}$$
$$\frac{x_1}{3} = \frac{x_2}{0} = \frac{x_3}{0}$$
$$\frac{x_1}{1} = \frac{x_2}{0} = \frac{x_3}{0}$$

The eigenvector corresponding to the eigenvalue $\lambda = 1$ is

$$X_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

For $\lambda = 2$ we have

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$$\begin{aligned} -x_1 + 0x_2 + 0x_3 &= 0 \\ 0x_1 + x_2 - x_3 &= 0 \\ 0x_1 - x_2 + x_3 &= 0 \end{aligned}$$

Solving first two equations we have

$$\begin{aligned} \frac{x_1}{0-0} = \frac{x_2}{0-1} &= \frac{x_3}{-1-0} \\ \frac{x_1}{0} = \frac{x_2}{-1} &= \frac{x_3}{-1} \\ \frac{x_1}{0} = \frac{x_2}{1} &= \frac{x_3}{1} \end{aligned}$$

The eigenvector corresponding to the eigenvalue $\lambda = 2$ is

$$X_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}.$$

For $\lambda = 2$ we have

$$\begin{pmatrix} -3 & 0 & 0 \\ 0 & -1 & -1 \\ 0 & -1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$$-3x_1 + 0x_2 + 0x_3 = 0$$

$$0x_1 - x_2 - x_3 = 0$$

$$0x_1 - x_2 - x_3 = 0$$

Solving first two equations we have

$$\frac{x_1}{0-0} = \frac{-x_2}{3-0} = \frac{x_3}{3-0}$$
$$\frac{x_1}{0} = \frac{x_2}{-1} = \frac{x_3}{1}$$
$$\frac{x_1}{0} = \frac{x_2}{1} = \frac{x_3}{-1}$$

The eigenvector corresponding to the eigenvalue $\lambda = 4$ is

$$X_3 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}.$$

Hence the eigenvectors are

$$X_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad X_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \qquad X_3 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

Clearly $X_1^T X_2 = X_2^T X_3 = X_3^T X_1 = 0$
and the normalized eigenvectors are

$$X_1^N = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad X_2^N = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \quad X_3^N = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

The normalized modal matrix is

$$N = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \qquad N^T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$$

$$
\begin{aligned}
N^T A N &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \times \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & -1 & 3 \end{pmatrix} \\
&\times \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \\
D &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix}
\end{aligned}
$$

Now consider the orthogonal transformation $X = NY$. Then the given quadratic form is transformed to

$$\begin{aligned}
Q &= X^T A X \\
&= (NY)^T A (NY) \\
&= Y^T (N^T A N) Y \\
&= \begin{pmatrix} y_1 & y_2 & y_3 \end{pmatrix} \times \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix} \times \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \\
Q &= y_1^2 + 2y_2^2 + 4y_3^2
\end{aligned}$$

Example 9

Reduce the quadratic form is $x_1^2 + 2x_2^2 + x_3^2 - 2x_1x_2 + 2x_2x_3$ to canonical form through an orthogonal transformation and hence show that it is positive definite.

Solution. The given quadratic form is $Q = X^T AX$ where

$$A = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix} \quad X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \quad X^T = \begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix}$$

The characteristic equation is $|A - \lambda I| = 0$

$$\lambda(\lambda - 1)(\lambda - 3) = 0$$
$$\implies \lambda = 0, 1, 3$$

The eigenvalues are $\lambda = 0, 1, 3$.

The eigenvectors are given by $(A - \lambda I)\ X = 0$

$$\begin{pmatrix} 1-\lambda & -1 & 0 \\ -1 & 2-\lambda & 1 \\ 0 & 1 & 1-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

For $\lambda = 0$ we have

$$\begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$$x_1 - x_2 + 0x_3 = 0$$

$$-x_1 + 2x_2 + x_3 = 0$$

$$0x_1 + x_2 + x_3 = 0$$

Solving first two equations we have

$$\frac{x_1}{-2-0} = \frac{-x_2}{1-0} = \frac{x_3}{2-1}$$
$$\frac{x_1}{-1} = \frac{x_2}{-1} = \frac{x_3}{1}$$
$$\frac{x_1}{1} = \frac{x_2}{1} = \frac{x_3}{-1}$$

The eigenvector corresponding to the eigenvalue $\lambda = 0$ is

$$X_1 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}.$$

For $\lambda = 1$ we have

$$\begin{pmatrix} 0 & -1 & 0 \\ -1 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$$0x_1 - x_2 + 0x_3 = 0$$

$$-x_1 + x_2 + x_3 = 0$$

$$0x_1 + x_2 + 0x_3 = 0$$

first two equations we have

$$\frac{x_1}{-1-0} = \frac{x_2}{0-0} = \frac{x_3}{0-1}$$

$$\frac{x_1}{-1} = \frac{x_2}{0} = \frac{x_3}{-1}$$

$$\frac{x_1}{1} = \frac{x_2}{0} = \frac{x_3}{1}$$

The eigenvector corresponding to the eigenvalue $\lambda = 3$ is

$$X_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

For $\lambda = 3$ we have

$$\begin{pmatrix} -2 & -1 & 0 \\ -1 & -1 & 1 \\ 0 & 1 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$$-2x_1 - x_2 + 0x_3 = 0$$
$$-60x_1 - x_2 + x_3 = 0$$
$$0x_1 + x_2 - 2x_3 = 0$$

Solving first two equations we have

$$\frac{x_1}{-1-0} = \frac{-x_2}{-2-0} = \frac{x_3}{2-1}$$
$$\frac{x_1}{-1} = \frac{x_2}{2} = \frac{x_3}{1}$$

The eigenvector corresponding to the eigenvalue $\lambda = 15$ is

$$X_3 = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}.$$

Hence the eigenvectors are

$$X_1 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \qquad X_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \qquad X_3 = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$$

and the normalized eigenvectors are

$$X_1^N = \begin{pmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \end{pmatrix} \quad X_2^N = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix} \quad X_3^N = \begin{pmatrix} -\frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{pmatrix}$$

The normalized modal matrix is

$$N = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \\ -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{pmatrix} \quad N^T = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{pmatrix}$$

$$N^TAN = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{pmatrix} \times \begin{pmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{pmatrix}$$

$$\times \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \\ -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{pmatrix}$$

$$D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

Now consider the orthogonal transformation $X = NY$. Then the given quadratic form is transformed to

$$
\begin{aligned}
Q &= X^T A X \\
&= (NY)^T A (NY) \\
&= Y^T (N^T A N) Y \\
&= \begin{pmatrix} y_1 & y_2 & y_3 \end{pmatrix} \times \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix} \times \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \\
Q &= y_2^2 + 3y_3^2
\end{aligned}
$$

The nature of the quadratic form are

Rank $= 2$

Index $= 2$

Signature $= 2 - (2 - 2) = 2$

Hence the quadratic form is **positive semi-definite**.

Example 10

Reduce the quadratic form $10x_1^2 + 2x_2^2 + 5x_3^2 - 4x_1x_2 + 6x_2x_3 - 10x_3x_1$ to the canonical form orthogonal reduction and hence find the rank , index and signature, it's nature.

Solution. The given quadratic form is $Q = X^TAX$ where

$$A = \begin{pmatrix} 10 & -2 & -5 \\ -2 & 2 & 3 \\ -5 & 3 & 5 \end{pmatrix} \quad X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \quad X^T = \begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix}$$

The characteristic equation is $|A - \lambda I| = 0$

$$\lambda^3 - 17\lambda^2 + 42\lambda = 0$$

$$\lambda(\lambda - 3)(\lambda - 14) = 0$$

$$\implies \quad \lambda = 0, 3, 14$$

The eigenvalues are $\lambda = 0, 3, 14$.

The eigenvectors are given by $(A - \lambda I)\ X = 0$

$$\begin{pmatrix} 10-\lambda & -2 & -5 \\ -2 & 2-\lambda & 3 \\ -5 & 3 & 5-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

For $\lambda = 0$ we have

$$\begin{pmatrix} 10 & -2 & -5 \\ -2 & 2 & 3 \\ -5 & 3 & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$$10x_1 - 2x_2 - 5x_3 = 0$$

$$-2x_1 + 2x_2 + 3x_3 = 0$$

$$-5x_1 + 3x_2 + 5x_3 = 0$$

Solving first two equations we have

$$\begin{aligned}\frac{x_1}{-6+10} = \frac{-x_2}{30-10} = \frac{x_3}{20-4} \\ \frac{x_1}{4} = \frac{x_2}{-20} = \frac{x_3}{16} \\ \frac{x_1}{1} = \frac{x_2}{-5} = \frac{x_3}{4}\end{aligned}$$

The eigenvector corresponding to the eigenvalue $\lambda = 0$ is

$$X_1 = \begin{pmatrix} 1 \\ -5 \\ 4 \end{pmatrix}.$$

For $\lambda = 3$ we have

$$\begin{pmatrix} 7 & -2 & -5 \\ -2 & - & 3 \\ -5 & 3 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$$7x_1 - 2x_2 - 5x_3 = 0$$

$$-2x_1 - x_2 + 3x_3 = 0$$

$$-5x_1 + 3x_2 + 2x_3 = 0$$

first two equations we have

$$\frac{x_1}{-6-5} = \frac{x_2}{10-21} = \frac{x_3}{-7-4}$$

$$\frac{x_1}{-11} = \frac{x_2}{-11} = \frac{x_3}{-11}$$

$$\frac{x_1}{1} = \frac{x_2}{1} = \frac{x_3}{1}$$

The eigenvector corresponding to the eigenvalue $\lambda = 3$ is

$$X_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

For $\lambda = 14$ we have

$$\begin{pmatrix} -4 & -2 & -5 \\ -2 & -12 & 3 \\ -5 & 3 & -9 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$$-4x_1 - 2x_2 - 5x_3 = 0$$

$$-2x_1 - 12x_2 + 3x_3 = 0$$

$$-5x_1 + 3x_2 - 9x_3 = 0$$

Solving first two equations we have

$$\frac{x_1}{-6-60} = \frac{-x_2}{-12-10} = \frac{x_3}{48-4}$$
$$\frac{x_1}{-66} = \frac{x_2}{22} = \frac{x_3}{44}$$
$$\frac{x_1}{3} = \frac{x_2}{-1} = \frac{x_3}{-2}$$

The eigenvector corresponding to the eigenvalue $\lambda = 14$ is

$$X_3 = \begin{pmatrix} 3 \\ -1 \\ -2 \end{pmatrix}.$$

Hence the eigenvectors are

$$X_1 = \begin{pmatrix} 1 \\ -5 \\ 4 \end{pmatrix} \qquad X_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \qquad X_3 = \begin{pmatrix} 3 \\ -1 \\ -2 \end{pmatrix}$$

and the normalized eigenvectors are

$$X_1^N = \begin{pmatrix} \frac{1}{\sqrt{42}} \\ -\frac{5}{\sqrt{42}} \\ \frac{4}{\sqrt{24}} \end{pmatrix} \quad X_2^N = \begin{pmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{pmatrix} \quad X_3^N = \begin{pmatrix} \frac{3}{\sqrt{14}} \\ -\frac{1}{\sqrt{14}} \\ -\frac{2}{\sqrt{14}} \end{pmatrix}$$

The normalized modal matrix is

$$N = \begin{pmatrix} \frac{1}{\sqrt{42}} & \frac{1}{\sqrt{3}} & \frac{3}{\sqrt{14}} \\ -\frac{5}{\sqrt{42}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{14}} \\ \frac{4}{\sqrt{42}} & \frac{1}{\sqrt{3}} & -\frac{2}{\sqrt{14}} \end{pmatrix} \quad N^T = \begin{pmatrix} \frac{1}{\sqrt{42}} & -\frac{5}{\sqrt{42}} & \frac{4}{\sqrt{42}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{3}{\sqrt{14}} & -\frac{1}{\sqrt{14}} & -\frac{2}{\sqrt{14}} \end{pmatrix}$$

$$N^TAN = \begin{pmatrix} \frac{1}{\sqrt{42}} & -\frac{5}{\sqrt{42}} & \frac{4}{\sqrt{42}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{3}{\sqrt{14}} & -\frac{1}{\sqrt{14}} & -\frac{2}{\sqrt{14}} \end{pmatrix} \times \begin{pmatrix} 10 & -2 & -5 \\ -2 & 2 & 3 \\ -5 & 3 & 5 \end{pmatrix}$$

$$\times \begin{pmatrix} \frac{1}{\sqrt{42}} & \frac{1}{\sqrt{3}} & \frac{3}{\sqrt{14}} \\ -\frac{5}{\sqrt{42}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{14}} \\ \frac{4}{\sqrt{42}} & \frac{1}{\sqrt{3}} & -\frac{2}{\sqrt{14}} \end{pmatrix}$$

$$D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 14 \end{pmatrix}$$

Now consider the orthogonal transformation $X = NY$. Then the given quadratic form is transformed to

$$
\begin{aligned}
Q &= X^T A X \\
&= (NY)^T A (NY) \\
&= Y^T (N^T A N) Y \\
&= \begin{pmatrix} y_1 & y_2 & y_3 \end{pmatrix} \times \begin{pmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 14 \end{pmatrix} \times \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}
\end{aligned}
$$

The nature of the quadratic form are

Rank $= 2$

Index $= 2$

Signature $= 2 - (2 - 2) = 2$

Hence the quadratic form is positive semi-definite.

Example 11

Reduce the quadratic form to the canonical form by an orthogonal transformation $2x_1^2 + 5x_2^2 + 3x_3^2 + 4x_1x_2$.

Solution. The given quadratic form is $Q = X^T AX$ where

$$A = \begin{pmatrix} 2 & 2 & 0 \\ 2 & 5 & 0 \\ 0 & 0 & 3 \end{pmatrix} \quad X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \quad X^T = \begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix}$$

The characteristic equation is $|A - \lambda I| = 0$

$$\begin{aligned} \lambda^3 - 10\lambda^2 + 27\lambda - 18 &= 0 \\ (1-\lambda)(3-\lambda)(6-\lambda) &= 0 \\ \implies \quad \lambda &= 1, 3, 6 \end{aligned}$$

The eigenvalues are $\lambda = 1, 2, 6$.

The eigenvectors are given by $(A - \lambda I)\ X = 0$

$$\begin{pmatrix} 2-\lambda & 2 & 0 \\ 2 & 5-\lambda & 0 \\ 0 & 0 & 3-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

For $\lambda = 1$ we have

$$\begin{pmatrix} 1 & 2 & 0 \\ 2 & 4 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$$x_1 + 2x_2 + 0x_3 = 0$$

$$2x_1 + 4x_2 + 0x_3 = 0$$

$$0x_1 + 0x_2 + 2x_3 = 0$$

Solving second and third equations we have

$$\frac{x_1}{8-0} = \frac{-x_2}{4-0} = \frac{x_3}{0-0}$$
$$\frac{x_1}{8} = \frac{x_2}{-4} = \frac{x_3}{0}$$
$$\frac{x_1}{2} = \frac{x_2}{-1} = \frac{x_3}{0}$$

The eigenvector corresponding to the eigenvalue $\lambda = 1$ is

$$X_1 = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}.$$

For $\lambda = 3$ we have

$$\begin{pmatrix} -1 & 2 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$$-x_1 + 2x_2 + 0x_3 = 0$$

$$2x_1 + 2x_2 + 0x_3 = 0$$

$$0x_1 + 0x_2 + 0x_3 = 0$$

Solving first two equations we have

$$\frac{x_1}{0-0} = \frac{x_2}{0-0} = \frac{x_3}{-2-2}$$

$$\frac{x_1}{0} = \frac{x_2}{-1} = \frac{x_3}{-4}$$

$$\frac{x_1}{0} = \frac{x_2}{0} = \frac{x_3}{-1}$$

The eigenvector corresponding to the eigenvalue $\lambda = 2$ is

$$X_2 = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}.$$

For $\lambda = 6$ we have

$$\begin{pmatrix} -4 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$$\begin{aligned} -4x_1 + 2x_2 + 0x_3 &= 0 \\ 2x_1 - x_2 + 0x_3 &= 0 \\ 0x_1 + 0x_2 - 3x_3 &= 0 \end{aligned}$$

Solving second and third equations we have

$$\frac{x_1}{3-0} = \frac{-x_2}{-6-0} = \frac{x_3}{0-0}$$
$$\frac{x_1}{3} = \frac{x_2}{6} = \frac{x_3}{0}$$
$$\frac{x_1}{1} = \frac{x_2}{2} = \frac{x_3}{0}$$

The eigenvector corresponding to the eigenvalue $\lambda = 4$ is

$$X_3 = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}.$$

Hence the eigenvectors are

$$X_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad X_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \qquad X_3 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

Clearly $X_1^T X_2 = X_2^T X_3 = X_3^T X_1 = 0$
and the normalized eigenvectors are

$$X_1^N = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad X_2^N = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \quad X_3^N = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

The normalized modal matrix is

$$N = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \qquad N^T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$$

$$
\begin{aligned}
N^TAN &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \times \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & -1 & 3 \end{pmatrix} \\
&\times \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \\
D &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix}
\end{aligned}
$$

Now consider the orthogonal transformation $X = NY$. Then the given quadratic form is transformed to

$$
\begin{aligned}
Q &= X^T A X \\
&= (NY)^T A (NY) \\
&= Y^T (N^T A N) Y \\
&= \begin{pmatrix} y_1 & y_2 & y_3 \end{pmatrix} \times \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix} \times \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \\
Q &= y_1^2 + 2y_2^2 + 4y_3^2
\end{aligned}
$$

Exercise

1. Reduce the quadratic form to canonical form by means of an orthogonal transformation. Discuss the nature of the quadratic form $x_1^2 + 2x_2^2 + x_3^2 - 2x_1x_2 + 2x_2x_3$
2. Reduce the quadratic form to canonical form. Discuss the nature of the quadratic form $2x_1^2 + 2x_2^2 + 2x_3^2 + 2x_1x_2$
3. Reduce the quadratic forms to canonical form. Determine nature of the quadratic form $2x_1^2 + 5x_2^2 + 3x_3^2 + 4x_1x_2$
4. Reduce the quadratic forms to canonical form. Discuss the nature of the quadratic form $6x_1^2 + 3x_2^2 + 3x_3^2 - 2x_1x_2 - 4x_2x_3 + 4x_1x_3$
5. Reduce the quadratic form $2x_1^2 + x_2^2 + x_3^2 + 2x_1x_2 - 2x_1x_3 - 4x_2x_3$ to canonical form by an orthogonal transformation. Also find the rank , index, signature and the nature of the quadratic form.

Answers

1. $y_2^2 + y_3^2$; Rank $= 2$; Index$= 2$; Positive semi-definite.
2. $y_1^2 + y_2^2 + y_3^2$; Rank $= 2$; Index$= 2$; Positive semi-definite.
3. $y_1^2 + 3y_2^2 + 6y_3^2$; Rank $= 3$; Index$= 3$; Signature $= 3$.
4. $4y_1^2 + y_2^2 + y_3^2$
5. $-y_1^2 + y_2^2 + 4y_3^2$; indefinite; Rank $= 3$, index $= 2$, signature $= 1$