21MAB101T - CALCULUS AND LINEAR ALGEBRA

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Similar Matrices

Two matrices A and C are said to be similar, if there exists a non-singular matrix B such that

$$B^{-1}AB = C$$

The transformation of a matrix A to another matrix C by $B^{-1}AB = C$ is known as similarity transformation.

Diagonalizable

A square matrix A is said to be diagonalizable if there exists a non-singular matrix B such that

$$B^{-1}AB = D$$

where D is a diagonal matrix.

Let A be a square matrix with distinct eigenvalues. Then the eigenvectors are **linearly independent** .

Let B be matrix whose columns are the eigenvectors of A.Then A can be diagonalized by the similarity transformation
$B^{-1}AB = D$

Where D is a diagonal matrix whose diagonal elements are the eigenvalues of matrix A.

The diagonalization process can also be applied to a matrix whose eigenvalues are not necessarily distinct but the eigenvectors are linearly independent.

Procedure of Similarity Transformation

Step 1. Find the eigenvalues of the given matrix A

Step 2. Find the eigenvectors for the corresponding eigenvalues where the eigenvectors are linearly independent.

Step 3. Construct modal matrix B and its inverse B^{-1}.

Step 4. Finally perform the transform $B^{-1}AB = D$.

Orthogonal Transformation

The orthogonal transformation or orthogonal reduction can be applicable only for real symmetric matrices.

If A is a real symmetric matric then the eigenvectors are linearly independent and pairwise orthogonal.

Normalize each eigenvectors by dividing each element by the square root of sum of the square of its elements and the normalized modal matrix N can be formed by these normalized eigenvectors.

Moreover the resulting matrix is orthogonal. By the property of orthogonal matrices $N^{-1} = N^T$.

Hence the similarity transformation $N^{-1}AN = D$ becomes $N^{T}AN = D$ where D is a diagonal matrix whose diagonal elements are the eigenvalues of A.

Transforming A to D by means of the transform $N^{T}AN = D$ known as the orthogonal transformation or orthogonal reduction.

Procedure of Orthogonal Transformation

Step 1. Find the eigenvalues of the given real symmetric matric A

Step 2. Find the eigenvectors for the corresponding eigenvalues where the eigenvectors are linearly independent and pairwise orthogonal.

Step 3. Normalize each eigenvector by dividing with its magnitude which nothing but the length of the vector.

Step 4. Construct normalized modal matrix N and its transpose N^T.

Step 5. Finally perform the transform $N^T AN = D$.

Solved Problems

Example 1.

Diagonalize the matrix

$$A = \begin{pmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{pmatrix}$$

Solution.

Let

$$A = \begin{pmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{pmatrix}$$

The characteristic equation is $|A - \lambda I| = 0$

The general form of characteristic equation is

$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$
where

S_1 = Sum of the main diagonal entries $\Rightarrow$ $S_1 = 18$

S_2= Sum of the minors of the main diagonal entries $\Rightarrow$ $S_2 = 45$

S_3 = Determinant of the matrix A $\Rightarrow$ $S_3 = 0$

$\lambda^3 - 18\lambda^2 + 45\lambda = 0$

$\implies \lambda = 0, 3, 15$

The eigenvalues are $\lambda = 0, 3, 15$

The eigenvectors are given by $(A - \lambda I)\ X = 0$,

$$\begin{pmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

For $\lambda = 0$ we have

$$\begin{pmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$$8x_1 - 6x_2 + 2x_3 = 0$$

$$-6x_1 + 7x_2 - 4x_3 = 0$$

$$2x_1 - 4x_2 + 3x_3 = 0$$

Solving first two equations we have

$$\frac{x_1}{24-14} = \frac{x_2}{12-(-32)} = \frac{x_3}{56-36}$$
$$\frac{x_1}{10} = \frac{x_2}{20} = \frac{x_3}{20}$$
$$\frac{x_1}{1} = \frac{x_2}{2} = \frac{x_3}{2}$$

The eigenvector corresponding to the eigenvalue $\lambda = 0$ is

$$X_1 = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$$

For $\lambda = 3$ we have

$$\begin{pmatrix} 5 & -6 & 2 \\ -6 & 4 & -4 \\ 2 & -4 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$$5x_1 - 6x_2 + 2x_3 = 0$$

$$-6x_1 + 4x_2 - 4x_3 = 0$$

$$2x_1 - 4x_2 + 0x_3 = 0$$

Solving last two equations we have

$$\frac{x_1}{0-16} = \frac{x_2}{-8-0} = \frac{x_3}{24-8}$$

$$\frac{x_1}{-6} = \frac{x_2}{-8} = \frac{x_3}{16}$$

$$\frac{x_1}{2} = \frac{x_2}{1} = \frac{x_3}{-2}$$

The eigenvector corresponding to the eigenvalue $\lambda = 3$ is

$$X_2 = \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}.$$

For $\lambda = 15$ we have

$$\begin{pmatrix} -7 & -6 & 2 \\ -6 & -8 & -4 \\ 2 & -4 & -12 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$$-7x_1 - 6x_2 + 2x_3 = 0$$

$$-6x_1 - 8x_2 - 4x_3 = 0$$

$$2x_1 - 4x_2 - 12x_3 = 0$$

Solving first two equations we have

$$\frac{x_1}{24+16} = \frac{x_2}{-12-28} = \frac{x_3}{56-36}$$
$$\frac{x_1}{40} = \frac{x_2}{-40} = \frac{x_3}{20}$$
$$\frac{x_1}{2} = \frac{x_2}{-2} = \frac{x_3}{1}$$

The eigenvector corresponding to the eigenvalue $\lambda = 15$ is

$$X_3 = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$$

Hence the eigenvectors are

$$X_1 = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \qquad X_2 = \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix} \qquad X_3 = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$$

and the normalized eigenvectors are

$$X_1^N = \frac{1}{3} \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \quad X_2^N = \frac{1}{3} \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix} \quad X_3^N = \frac{1}{3} \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$$

The normalized modal matrix is

$$N = \frac{1}{3} \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{pmatrix} \qquad N^T = \frac{1}{3} \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{pmatrix}$$

$$
\begin{aligned}
N^T A N &= \frac{1}{3}\begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{pmatrix} \times \begin{pmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{pmatrix} \times \\
& \frac{1}{3}\begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{pmatrix} \\
D &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 15 \end{pmatrix}
\end{aligned}
$$

Example 2

Diagonalize the matrix

$$A = \begin{pmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{pmatrix}$$

by means of orthogonal transformation.

Solution. The characteristic equation is $|A - \lambda I| = 0$

$$\lambda^3 - 11\lambda^2 + 36\lambda - 36 = 0$$
$$(\lambda - 2)(\lambda - 3)(\lambda - 6) \Longrightarrow \quad \lambda = 2, 3, 6$$

The eigenvalues are $\lambda = 2, 3, 6$.

The eigenvectors are given by $(A-\lambda I)\ X=0$

$$\begin{pmatrix} 3-\lambda & -1 & 1 \\ -1 & 5-\lambda & -1 \\ 1 & -1 & 3-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

For $\lambda = 2$ we have

$$\begin{pmatrix} 1 & -1 & 1 \\ -1 & 3 & -1 \\ 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$$\begin{aligned} x_1 - x_2 + x_3 &= 0 \\ -x_1 + 3x_2 - x_3 &= 0 \\ x_1 - x_2 + x_3 &= 0 \end{aligned}$$

Solving first two equations we have

$$\frac{x_1}{1-3} = \frac{x_2}{1-1} = \frac{x_3}{3-1}$$
$$\frac{x_1}{-2} = \frac{x_2}{0} = \frac{x_3}{2}$$
$$\frac{x_1}{-1} = \frac{x_2}{0} = \frac{x_3}{1}$$

The eigenvector corresponding to the eigenvalue $\lambda = 2$ is

$$X_1 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}.$$

For $\lambda = 3$ we have

$$\begin{pmatrix} 0 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$$\begin{aligned} 0x_1 - x_2 + x_3 &= 0 \\ -x_1 + 2x_2 - x_3 &= 0 \\ x_1 - x_2 + 0x_3 &= 0 \end{aligned}$$

Solving first two equations we have

$$\begin{aligned} \frac{x_1}{1-2} = \frac{x_2}{-1-0} &= \frac{x_3}{0-1} \\ \frac{x_1}{-1} = \frac{x_2}{-1} &= \frac{x_3}{-1} \\ \frac{x_1}{1} = \frac{x_2}{1} &= \frac{x_3}{1} \end{aligned}$$

The eigenvector corresponding to the eigenvalue $\lambda = 3$ is

$$X_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

For $\lambda = 6$ we have

$$\begin{pmatrix} -3 & -1 & 1 \\ -1 & -1 & -1 \\ 1 & -1 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$$-3x_1 - x_2 + x_3 = 0$$

$$-x_1 - x_2 - x_3 = 0$$

$$x_1 - x_2 - 3x_3 = 0$$

Solving first two equations we have

$$\begin{aligned}\frac{x_1}{1+1} = \frac{-x_2}{3+1} = \frac{x_3}{3-1}\\ \frac{x_1}{2} = \frac{x_2}{-4} = \frac{x_3}{2}\\ \frac{x_1}{1} = \frac{x_2}{-2} = \frac{x_3}{1}\end{aligned}$$

The eigenvector corresponding to the eigenvalue $\lambda = 6$ is

$$X_3 = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}.$$

Hence the eigenvectors are

$$X_1 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \qquad X_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \qquad X_3 = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

and the normalized eigenvectors are

$$X_1^N = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \quad X_2^N = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad X_3^N = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

The normalized modal matrix is

$$N = \begin{pmatrix} \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{3}} & \frac{-2}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \end{pmatrix} \qquad N^T = \begin{pmatrix} \frac{-1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{-2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{pmatrix}$$

$$
\begin{aligned}
N^TAN &= \begin{pmatrix} \frac{-1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{-2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{pmatrix} \times \begin{pmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{pmatrix} \\
&\quad \times \begin{pmatrix} \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{3}} & \frac{-2}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \end{pmatrix} \\
D &= \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{pmatrix}
\end{aligned}
$$

Example 3

Diagonalize the matrix

$$A = \begin{pmatrix} 7 & -2 & 0 \\ -2 & 6 & -2 \\ 0 & -2 & 5 \end{pmatrix}$$

by means of orthogonal transformation.

Solution. The characteristic equation is $|A - \lambda I| = 0$

$$\begin{aligned} \lambda^3 - 18\lambda^2 + 99\lambda - 162 = 0 \\ (\lambda - 3)(\lambda - 6)(\lambda - 9) = 0 \\ \implies \quad \lambda = 3, 6, 9 \end{aligned}$$

The eigenvalues are $\lambda = 3, 6, 9$.

The eigenvectors are given by $(A - \lambda I)\ X = 0$

$$\begin{pmatrix} 7-\lambda & -2 & 0 \\ -2 & 6-\lambda & -2 \\ 0 & -2 & 5-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

For $\lambda = 3$ we have

$$\begin{pmatrix} 4 & -2 & 0 \\ -2 & 3 & -2 \\ 0 & -2 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$$4x_1 - 2x_2 + 0x_3 = 0$$

$$-2x_1 + 3x_2 - 2x_3 = 0$$

$$0x_1 - 2x_2 + 2x_3 = 0$$

Solving first two equations we have

$$\frac{x_1}{4-0} = \frac{x_2}{8-0} = \frac{x_3}{12-4}$$
$$\frac{x_1}{4} = \frac{x_2}{8} = \frac{x_3}{8}$$
$$\frac{x_1}{1} = \frac{x_2}{2} = \frac{x_3}{2}$$

The eigenvector corresponding to the eigenvalue $\lambda = 3$ is

$$X_1 = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}.$$

For $\lambda = 6$ we have

$$\begin{pmatrix} 1 & -2 & 0 \\ -2 & 0 & -2 \\ 0 & -2 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$$x_1 - 2x_2 + 0x_3 = 0$$

$$-2x_1 + 0x_2 - 2x_3 = 0$$

$$0x_1 - 2x_2 - x_3 = 0$$

Solving first two equations we have

$$\frac{x_1}{4} = \frac{x_2}{2} = \frac{x_3}{-4}$$

$$\frac{x_1}{2} = \frac{x_2}{1} = \frac{x_3}{-2}$$

The eigenvector corresponding to the eigenvalue $\lambda = 6$ is

$$X_2 = \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}.$$

For $\lambda = 9$ we have

$$\begin{pmatrix} -2 & -2 & 0 \\ -2 & -3 & -2 \\ 0 & -2 & -4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$$\begin{aligned} -2x_1 - 2x_2 + 0x_3 &= 0 \\ -2x_1 - 3x_2 - 2x_3 &= 0 \\ 0x_1 - 2x_2 - 4x_3 &= 0 \end{aligned}$$

Solving first two equations we have

$$\frac{x_1}{4} = \frac{-x_2}{4} = \frac{x_3}{2}$$
$$\frac{x_1}{2} = \frac{x_2}{-2} = \frac{x_3}{1}$$

The eigenvector corresponding to the eigenvalue $\lambda = 9$ is

$$X_3 = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}.$$

Hence the eigenvectors are

$$X_1 = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \qquad X_2 = \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix} \qquad X_3 = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$$

and the normalized eigenvectors are

$$X_1^N = \frac{1}{3}\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \quad X_2^N = \frac{1}{3}\begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix} \quad X_3^N = \frac{1}{3}\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$$

The normalized modal matrix is

$$N = \frac{1}{3}\begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{pmatrix}$$

$$N^T = \frac{1}{3}\begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{pmatrix}$$

$$N^T AN = \frac{1}{3}\begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{pmatrix} \times \begin{pmatrix} 7 & -2 & 0 \\ -2 & 6 & -2 \\ 0 & -2 & 5 \end{pmatrix}$$
$$\times \frac{1}{3}\begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{pmatrix}$$
$$D = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 9 \end{pmatrix}$$

Example 4

Diagonalize the matrix

$$A = \begin{pmatrix} 2 & 0 & 4 \\ 0 & 6 & 0 \\ 4 & 0 & 2 \end{pmatrix}$$

by means of orthogonal transformation.

Solution. The characteristic equation is $|A - \lambda I| = 0$

$$\begin{aligned} \lambda^3 - 10\lambda^2 + 12\lambda - 72 &= 0 \\ (\lambda + 2)(\lambda - 6)(\lambda - 6) &= 0 \\ \implies \quad \lambda &= -2, 6, 6 \end{aligned}$$

The eigenvalues are $\lambda = -2, 6, 6$.

The eigenvectors are given by $(A - \lambda I)\ X = 0$

$$\begin{pmatrix} 2-\lambda & 0 & 4 \\ 0 & 6-\lambda & 0 \\ 4 & 0 & 2-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

For $\lambda = -2$ we have

$$\begin{pmatrix} 4 & 0 & 4 \\ 0 & 8 & 0 \\ 4 & 0 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$$4x_1 + 0x_2 + 4x_3 = 0$$
$$0x_1 + 8x_2 + 0x_3 = 0$$
$$4x_1 + 0x_2 + 4x_3 = 0$$

Solving first two equations we have

$$\frac{x_1}{-32} = \frac{x_2}{0} = \frac{x_3}{32}$$
$$\frac{x_1}{-1} = \frac{x_2}{0} = \frac{x_3}{1}$$

The eigenvector corresponding to the eigenvalue $\lambda = -2$ is

$$X_1 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}.$$

For $\lambda = 6$ we have

$$\begin{pmatrix} -4 & 0 & 4 \\ 0 & 0 & 0 \\ 4 & 0 & -4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$$\begin{aligned} -4x_1 + 0x_2 + 4x_3 &= 0 \\ 0x_1 + 0x_2 + 0x_3 &= 0 \\ 4x_1 + 0x_2 - 4x_3 &= 0 \\ \text{put } \mathrm{x}_2 = 0 \text{ we get } 4\mathrm{x}_1 \ &= 4\mathrm{x}_3 \\ \frac{x_1}{1} &= \frac{x_3}{1} \end{aligned}$$

The eigenvector corresponding to the eigenvalue $\lambda = 6$ is

$$X_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

Let

$$X_3 = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \text{ be an eigenvector orthogonal to } \mathrm{X}_1 \text{ and } \mathrm{X}_2$$

Since the given matrix is symmetric

$$X_1^T X_3 = 0$$

$$\begin{pmatrix} -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0$$

$$-a + 0b + c = 0$$

and $X_2^T X_3 = 0$

$$\begin{pmatrix} 1 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0$$

or

$$a + 0b + c = 0$$

solving these two equations we get

$$\frac{a}{0} = \frac{b}{2} = \frac{c}{0}$$

The another eigenvector corresponding to the eigenvalue $\lambda = 6$ is

$$X_3 = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}.$$

Hence the eigenvectors are

$$X_1 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \qquad X_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \qquad X_3 = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}$$

and the normalized eigenvectors are

$$X_1^N = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \quad X_2^N = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \quad X_3^N = \frac{1}{2} \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}$$

The normalized modal matrix is

$$N = \begin{pmatrix} \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{pmatrix} \qquad N^T = \begin{pmatrix} \frac{-1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \end{pmatrix}$$

$$\begin{aligned} N^T AN &= \begin{pmatrix} \frac{-1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \end{pmatrix} \times \begin{pmatrix} 2 & 0 & 4 \\ 0 & 6 & 0 \\ 4 & 0 & 2 \end{pmatrix} \times \begin{pmatrix} \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{pmatrix} \\ D &= \begin{pmatrix} -2 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{pmatrix} \end{aligned}$$

Example 5

Diagonalize the matrix

$$A = \begin{pmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{pmatrix}$$

by means of orthogonal transformation.

Solution. The characteristic equation is $|A - \lambda I| = 0$

$$\begin{aligned} \lambda^3 - 12\lambda^2 + 36\lambda - 32 &= 0 \\ (\lambda - 8)(\lambda - 2)(\lambda - 2) &= 0 \\ \implies \quad \lambda &= 8, 2, 2 \end{aligned}$$

The eigenvalues are $\lambda = 8, 2, 2$.

The eigenvectors are given by $(A - \lambda I)\ X = 0$

$$\begin{pmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

For $\lambda = 8$ we have

$$\begin{pmatrix} -2 & -2 & 2 \\ -2 & -5 & -1 \\ 2 & -1 & -5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$$-2x_1 - 2x_2 + 2x_3 = 0$$

$$-2x_1 - 5x_2 - x_3 = 0$$

$$2x_1 - x_2 - 5x_3 = 0$$

Solving first two equations we have

$$\frac{x_1}{2+10} = \frac{-x_2}{2+4} = \frac{x_3}{10-4}$$
$$\frac{x_1}{12} = \frac{x_2}{-6} = \frac{x_3}{6}$$
$$\frac{x_1}{2} = \frac{x_2}{-1} = \frac{x_3}{1}$$

The eigenvector corresponding to the eigenvalue $\lambda = 8$ is

$$X_1 = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}.$$

For $\lambda = 2$ we have

$$\begin{pmatrix} 4 & -2 & 2 \\ -2 & 1 & -1 \\ 2 & -1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$$4x_1 - 2x_2 + 2x_3 = 0$$

$$-2x_1 + x_2 - x_3 = 0$$

$$2x_1 - x_2 + x_3 = 0$$

These three equation represents are the same equations,

$$x_1 - x_2 + x_3 = 0$$

$$\text{put } \mathrm{x}_1 = 0 \text{ we get } \mathrm{x}_2 = \mathrm{x}_3$$

$$\frac{x_1}{-1} = \frac{x_3}{1}$$

The eigenvector corresponding to the eigenvalue $\lambda = 2$ is

$$X_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}.$$

Let

$$X_3 = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \text{ and } \mathrm{X}_3 \text{ is orthogonal to } \mathrm{X}_1 \text{ and } \mathrm{X}_2$$

Since the given matrix is symmetric $\Rightarrow$ $X_1^T X_3 = 0$

$$\begin{pmatrix} 2 & -1 & 1 \end{pmatrix} \cdot \begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0$$

or

$$2a - b + c = 0$$

$$X_2^T X_3 = 0$$

$$\begin{pmatrix} 0 & 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0$$

or

$$0a + b + c = 0$$

solving these two we get

$$\frac{a}{-1-1} = \frac{b}{0-(-2)} = \frac{c}{2}$$
$$\frac{a}{-2} = \frac{b}{-2} = \frac{c}{2}$$
$$\frac{a}{1} = \frac{b}{1} = \frac{c}{-1}$$

The another eigenvector corresponding to the eigenvalue $\lambda = 2$ is

$$X_3 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}.$$

Hence the eigenvectors are

$$X_1 = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \quad X_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \quad X_3 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$$

and the normalized eigenvectors are

$$X_1^N = \frac{1}{\sqrt{6}} \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \quad X_2^N = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \quad X_3^N = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$$

The normalized modal matrix is

$$N = \begin{pmatrix} \frac{2}{\sqrt{6}} & 0 & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{3}} \end{pmatrix} \quad N^T = \begin{pmatrix} \frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \end{pmatrix}$$

$$\begin{aligned} N^TAN &= \begin{pmatrix} \frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \end{pmatrix} \times \begin{pmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{pmatrix} \\ &\times \begin{pmatrix} \frac{2}{\sqrt{6}} & 0 & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{3}} \end{pmatrix} \\ D &= \begin{pmatrix} 8 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \end{aligned}$$

Exercise

Diagonalize the following matrices by orthogonal transformation

1. $\begin{pmatrix} 2 & 1 & -1 \\ 1 & 1 & -2 \\ -1 & -2 & 1 \end{pmatrix}$

2. $\begin{pmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{pmatrix}$

3. $\begin{pmatrix} 3 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 3 \end{pmatrix}$

4. $\begin{pmatrix} 10 & -2 & -5 \\ -2 & 2 & 3 \\ -5 & 3 & 5 \end{pmatrix}$

Answers

1. $\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix}$

2. $\begin{pmatrix} 5 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -3 \end{pmatrix}$

3. $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{pmatrix}$

4. $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 14 \end{pmatrix}$