21MAB101T - CALCULUS AND LINEAR ALGEBRA

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Theorem

Every square matrix satisfies its own characteristic equation.

Application of the Cayley-Hamilton theorem

1. The inverse of any nonsingular square matrix can be calculated.

2. The higher positive integral power of the matrix A can be found out.

Examples-1

Verify Cayley-Hamilton theorem for the matrix

$$\begin{pmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{pmatrix}$$

Solution. The characteristic equation is $|A - \lambda I| = 0$

$$\text{Let } A = \begin{pmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{pmatrix}$$

The general form of characteristic equation is

$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$

where

S_1 = Sum of the main diagonal entries $\Rightarrow\ S_1 = 4$

S_2= Sum of the minors of the main diagonal entries
$\Rightarrow S_2 = -20$

S_3 = Determinant of the matrix A $\Rightarrow\ S_3 = 35$

The characteristic equation is $\lambda^3 - 4\lambda^2 - 20\lambda - 35 = 0$

By Cayley-Hamilton theorem

$A^3 - 4A^2 - 20A - 35I = 0$

$$A^2 = \begin{pmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{pmatrix} \times \begin{pmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 20 & 23 & 23 \\ 15 & 22 & 37 \\ 10 & 9 & 14 \end{pmatrix}$$

$$A^3 = \begin{pmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{pmatrix} \times \begin{pmatrix} 20 & 23 & 23 \\ 15 & 22 & 37 \\ 10 & 9 & 14 \end{pmatrix}$$

$$= \begin{pmatrix} 135 & 152 & 232 \\ 140 & 163 & 208 \\ 60 & 76 & 111 \end{pmatrix}$$

$A^3 - 4A^2 - 20A - 35I$

$$= \begin{pmatrix} 135 & 152 & 232 \\ 140 & 163 & 208 \\ 60 & 76 & 111 \end{pmatrix} - 4\begin{pmatrix} 20 & 23 & 23 \\ 15 & 22 & 37 \\ 10 & 9 & 14 \end{pmatrix} - 20\begin{pmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{pmatrix}$$

$$- 35\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 135 & 152 & 232 \\ 140 & 163 & 208 \\ 60 & 76 & 111 \end{pmatrix} - \begin{pmatrix} 80 & 92 & 92 \\ 60 & 88 & 148 \\ 40 & 36 & 56 \end{pmatrix} - \begin{pmatrix} 20 & 60 & 140 \\ 80 & 40 & 60 \\ 20 & 40 & 20 \end{pmatrix}$$

$$- \begin{pmatrix} 35 & 0 & 0 \\ 0 & 35 & 0 \\ 0 & 0 & 35 \end{pmatrix}$$

$$A^3 - 4A^2 - 20A - 35I = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Hence, Cayley-Hamilton theorem is verified,

$A^3 - 4A^2 - 20A - 35I = 0$

Example 2

Find characteristic equation of the matrix

$$A = \begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix}.$$

Hence find A^{-1} and A^4.

Solution. Given that

$$A = \begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix}$$

The characteristic equation is $|A - \lambda I| = 0$

The general form of characteristic equation is

$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$
where

S_1 = Sum of the main diagonal entries $\Rightarrow$ $S_1 = 6$

S_2= Sum of the minors of the main diagonal entries $\Rightarrow$ $S_2 = 9$

S_3 = Determinant of the matrix A $\Rightarrow$ $S_3 = 4$

The characteristic equation is $\lambda^3 - 6\lambda^2 + 9\lambda - 4 = 0$

By Cayley-Hamilton theorem

$$\begin{aligned}
A^3 - 6A^2 + 9A - 4I &= 0 \qquad (1)\\
\text{Multiply (1) by } A^{-1} & \\
A^2 - 6A + 9I - 4A^{-1} &= 0\\
A^{-1} &= \frac{1}{4}(A^2 - 6A + 9I) \qquad (2)\\
\text{Multiply (1) by } A & \\
A^4 - 6A^3 + 9A^2 - 4A &= 0\\
A^4 &= 6A^3 - 9A^2 + 4A\\
&= 6(6A^2 - 9A + 4I) - 9A^2 + 4A\\
&= 27A^2 - 50A + 24I \qquad (3)
\end{aligned}$$

$$A^2 = \begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix} \times \begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{pmatrix}$$

From (2) we have

$$\begin{aligned} 4A^{-1} &= A^2 - 6A + 9I \\ &= \begin{pmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{pmatrix} - 6\begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix} + 9\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{pmatrix} \end{aligned}$$

$$A^{-1} = \frac{1}{4}\begin{pmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{pmatrix}$$

From (3) we have

$$\begin{aligned} A^4 &= 27A^2 - 50A + 24I \\ &= 27\begin{pmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{pmatrix} - 50\begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix} \\ &\quad +24\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ A^4 &= \begin{pmatrix} 86 & -85 & 85 \\ -85 & 86 & -85 \\ 85 & -85 & 86 \end{pmatrix} \end{aligned}$$

Example 3

Using Cayley- Hamilton theorem, find A^{-1} when

$$A = \begin{pmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix}.$$

.

Solution. Given that

$$A = \begin{pmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix}$$

The characteristic equation is $|A - \lambda I| = 0$

The general form of characteristic equation is

$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$

where

S_1 = Sum of the main diagonal entries $\Rightarrow$ $S_1 = 3$

S_2= Sum of the minors of the main diagonal entries $\Rightarrow$ $S_2 = -1$

S_3 = Determinant of the matrix A $\Rightarrow$ $S_3 = 9$

The characteristic equation is $\lambda^3 - 3\lambda^2 - \lambda - 9 = 0$

By Cayley-Hamilton theorem

$$\begin{aligned} A^3 - 3A^2 - A - 9I &= 0 \qquad (4) \\ \text{Multiply by } A^{-1} & \\ A^2 - 3A - I - 9A^{-1} &= 0 \\ A^{-1} &= \frac{1}{9}(A^2 - 3A - I) \qquad (5) \end{aligned}$$

$$A^2 = \begin{pmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix} \times \begin{pmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 4 & -3 & 6 \\ 2 & 2 & 4 \\ 0 & -2 & 5 \end{pmatrix}$$

From (2) we have

$$\begin{aligned}
9A^{-1} &= A^2 - 3A - I \\
&= \begin{pmatrix} 4 & -3 & 6 \\ 2 & 2 & 4 \\ 0 & -2 & 5 \end{pmatrix} - 3\begin{pmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 0 & -3 & -3 \\ -3 & -2 & 7 \\ -3 & 1 & 1 \end{pmatrix} \\
A^{-1} &= \frac{1}{9}\begin{pmatrix} 0 & -3 & -3 \\ -3 & -2 & 7 \\ -3 & 1 & 1 \end{pmatrix}
\end{aligned}$$

Example. 4

Find A^{-1} if $A = \begin{pmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{pmatrix}$ using Cayley-Hamilton theorem.

Solution.

$$\text{Let } A = \begin{pmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{pmatrix}$$

The characteristic equation is $|A - \lambda I| = 0$

The general form of characteristic equation is

$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$

where

S_1 = Sum of the main diagonal entries $\Rightarrow$ $S_1 = 2$

S_2= Sum of the minors of the main diagonal entries $\Rightarrow$ $S_2 = -5$

S_3 = Determinant of the matrix A $\Rightarrow$ $S_3 = 6$

The characteristic equation is $\lambda^3 - 2\lambda^2 - 6\lambda - 6 = 0$

By Cayley-Hamilton theorem

$$A^3 - 2A^2 - 5A + 6I = 0$$

To find A^{-1}, pre-multiply (1) by A^{-1} we get

$$
\begin{aligned}
A^2 - 2A - 5I + 6A^{-1} &= 0\\
6A^{-1} &= -A^2 + 2A + 5I\\
A^{-1} &= \frac{1}{6}[-A^2 + 2A + 5I]
\end{aligned}
$$

$$
\begin{aligned}
A^2 = A \times A \quad &= \begin{pmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{pmatrix}\\
&= \begin{pmatrix} 6 & 1 & 1 \\ 7 & 0 & 11 \\ 3 & -1 & 8 \end{pmatrix}
\end{aligned}
$$

$$
\begin{aligned}
-A^2 + 2A + 5I &= \begin{pmatrix} -6 & -1 & -1 \\ -7 & 0 & -11 \\ -3 & 1 & -8 \end{pmatrix} + 2\begin{pmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{pmatrix} \\
&\quad +5\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & -3 & 7 \\ -1 & 9 & -13 \\ 1 & 3 & -5 \end{pmatrix}
\end{aligned}
$$

$$
A^{-1} = \frac{1}{6}[-A^2 + 2A + 5I] = \frac{1}{6}\begin{pmatrix} 1 & -3 & 7 \\ -1 & 9 & -13 \\ 1 & 3 & -5 \end{pmatrix}
$$

Example-5

Use Cayley-Hamilton theorem to find the value of the matrix given by $A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I$ if the matrix
$A = \begin{pmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{pmatrix}$

Solution.

$$\text{Let } A = \begin{pmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{pmatrix}$$

The characteristic equation is $|A - \lambda I| = 0$

The general form of characteristic equation is

$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$
where

S_1 = Sum of the main diagonal entries $\Rightarrow$ $S_1 = 5$

S_2= Sum of the minors of the main diagonal entries $\Rightarrow$ $S_2 = 7$

S_3 = Determinant of the matrix A $\Rightarrow$ $S_3 = 3$

The characteristic equation is $\lambda^3 - 5\lambda^2 + 7\lambda - 3 = 0$

$A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I$
By Cayley-Hamilton theorem

$$A^3 - 5A^2 + 7A - 3I = 0 \tag{6}$$

Given that

$$\begin{aligned}
&= (A^8 - 5A^7 + 7A^6 - 3A^5) + A^4 - 5A^3 + 8A^2 - 2A + I \quad (7)\\
&= A^5(A^3 - 5A^2 + 7A - 3I) + A(A^3 - 5A^2 + 8A - 2I) + I\\
&= A^5(0) + A[A^3 - 5A^2 + 7A - 3I + A + I] + I\\
&= 0 + A[0 + A + I] + I\\
&= A^2 + A + I \quad (8)
\end{aligned}$$

Now

$$A^2 = \begin{pmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{pmatrix}\begin{pmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{pmatrix}$$

$$A^2 + A + I = \begin{pmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{pmatrix} + \begin{pmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 8 & 5 & 5 \\ 0 & 3 & 0 \\ 5 & 5 & 8 \end{pmatrix}$$

$$A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I = \begin{pmatrix} 8 & 5 & 5 \\ 0 & 3 & 0 \\ 5 & 5 & 8 \end{pmatrix}$$

Example 6

Using Cayley-Hamilton theorem to find
$A^4 - 4A^3 - 5A^2 + A + 2I$ when $A = \begin{pmatrix} 3 & 2 \\ 4 & 5 \end{pmatrix}$.

Solution.

Let $A = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}$

The characteristic equation is $|A - \lambda I| = 0$

$$\begin{vmatrix} 1-\lambda & 2 \\ 4 & 3-\lambda \end{vmatrix} = 0$$
$$(1-\lambda)(3-\lambda) - 8 = 0$$
$$\lambda^2 - 4\lambda - 5 = 0$$

By Cayley-Hamilton theorem

$$A^2 - 4A + 5I = 0$$

Now consider

$$\begin{aligned} A^4 - 4A^3 - 5A^2 + A + 2I &= A^2(A^2 - 4A + 5I) + A + 2I \\ &= A^2(0) + A + 2I \\ &= A + 2I \\ &= \begin{pmatrix} 3 & 2 \\ 4 & 5 \end{pmatrix} + \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \\ &= \begin{pmatrix} 5 & 2 \\ 4 & 7 \end{pmatrix} \end{aligned}$$

Exercise

Verify Cayley - Hamilton Theorem and find its inverse of the following matrices.

1. $\begin{pmatrix} 7 & 2 & -2 \\ -6 & -1 & 2 \\ 6 & 2 & -1 \end{pmatrix}$

2. $\begin{pmatrix} 8 & -6 & 2 \\ -6 & -1 & 2 \\ 2 & -4 & 3 \end{pmatrix}$

3. $\begin{pmatrix} 1 & 0 & -2 \\ 2 & 2 & 4 \\ 0 & 0 & 2 \end{pmatrix}$

4. $\begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}$

5. $\begin{pmatrix} 4 & 3 & 1 \\ 2 & 1 & -2 \\ 1 & 2 & 1 \end{pmatrix}$

6. $\begin{pmatrix} 1 & 2 & -2 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{pmatrix}$

7. If $A = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix}$ Prove that $A^3 - 3A^2 - 9A - 5I = 0$
Hence find A^4 and A^{-1}.

8. Find A^n using Cayley-Hamilton theorem, taking $\begin{pmatrix} 7 & 2 \\ 3 & 6 \end{pmatrix}$
also find A^3.

9. Calculate A^4 for the matrix $A = \begin{pmatrix} 2 & 1 & 2 \\ 0 & 2 & 3 \\ 0 & 0 & 5 \end{pmatrix}$.

10. Using Cayley-Hamilton theorem, compute A^3 for
$A = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}$.

11. Verify Cayley-Hamilton theorem for the matrix
(i) $A = \begin{pmatrix} 3 & -1 \\ -1 & 5 \end{pmatrix}$
(ii) $A = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}$.

12. Given that $A = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \\ 3 & -1 & 1 \end{pmatrix}$, Express $A^6 - 5A^5 + 8A^4 - 2A^3 - 9A^2 + 35A + 6I$ as a linear polynomial in A, using Cayley Hamilton theorem.

13. Obtain the matrix $A^6 - 25A^2 + 122A$ where $A = \begin{pmatrix} 0 & 0 & 2 \\ 2 & 1 & 0 \\ -1 & -1 & 3 \end{pmatrix}$.

Answers

1. $A^{-1} = \frac{1}{3}\begin{pmatrix} -3 & -2 & 2 \\ 6 & 5 & -2 \\ -6 & -2 & 5 \end{pmatrix}$

2. $A^{-1} = \frac{1}{45}\begin{pmatrix} 10 & 6 & -2 \\ 6 & 11 & 4 \\ -2 & 4 & 15 \end{pmatrix}$

3. $A^{-1} = \begin{pmatrix} 1 & 0 & 1 \\ -1 & \frac{1}{2} & -2 \\ 0 & 0 & \frac{1}{2} \end{pmatrix}$

4. $A^{-1} = \frac{1}{5}\begin{pmatrix} -3 & 2 \\ 4 & -1 \end{pmatrix}$

5. $A^{-1} = \frac{1}{11}\begin{pmatrix} 5 & -1 & -7 \\ -4 & 3 & 10 \\ 3 & -5 & -2 \end{pmatrix}$

6. $A^{-1} = \frac{1}{4}\begin{pmatrix} 4 & 4 & -4 \\ -2 & -1 & 3 \\ -2 & 1 & 1 \end{pmatrix}$

7. $A^4 = \begin{pmatrix} 229 & 228 & 228 \\ 228 & 229 & 228 \\ 228 & 228 & 229 \end{pmatrix}$ $\quad A^{-1} = \frac{1}{5}\begin{pmatrix} -3 & 2 & 2 \\ 2 & 1 & 0 \\ -1 & -1 & 3 \end{pmatrix}$

8. $A^4 = \begin{pmatrix} 463 & 266 \\ 399 & 336 \end{pmatrix}$

9. $A^4 = \begin{pmatrix} 16 & 32 & 567 \\ 0 & 16 & 609 \\ 0 & 0 & 625 \end{pmatrix}$

10. $A^4 = \begin{pmatrix} 41 & 84 \\ 42 & 83 \end{pmatrix}$