21MAB101T - CALCULUS AND LINEAR ALGEBRA

Dr. M.SURESH
Assistant Professor
Department of Mathematics
SRM Institute of Science and Technology
Kattankulathur

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Example-1

Find the eigenvalues and eigenvectors for the matrix

$$A = \begin{pmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{pmatrix}$$

Solution. The characteristic equation is $|A - \lambda I| = 0$ where

$$A = \begin{pmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{pmatrix}$$

The general form of characteristic equation is

$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$

where

S_1 = Sum of the main diagonal entries $\Rightarrow\ S_1 = 8$

S_2= Sum of the minors of the main diagonal entries $\Rightarrow\ S_2 = 4$

S_3 = Determinant of the matrix A $\Rightarrow\ S_3 = -48$

$\lambda^3 - 8\lambda^2 + 4\lambda - 48 = 0$

$\implies \quad \lambda = -2, 4, 6$

The eigenvalues are $\lambda = -2, 4, 6$

The eigenvectors are given by $(A - \lambda I)\ X = 0$

$$\begin{pmatrix} 5-\lambda & 0 & 1 \\ 0 & -2-\lambda & 0 \\ 1 & 0 & 5-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

For $\lambda = -2$ we have

$$\begin{pmatrix} 7 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 7 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$$7x_1 + 0x_2 + x_3 = 0$$

$$0x_1 + 0x_2 + 0x_3 = 0$$

$$x_1 + 0x_2 + 7x_3 = 0$$

Solving first and third equations we have

$$\frac{x_1}{0-0} = \frac{x_2}{1-49} = \frac{x_3}{0-0}$$

$$\frac{x_1}{0} = \frac{x_2}{-48} = \frac{x_3}{0}$$

$$\frac{x_1}{0} = \frac{x_2}{-1} = \frac{x_3}{0}$$

The eigenvector corresponding to the eigenvalue $\lambda = -2$ is

$$X_1 = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}.$$

For $\lambda = 4$ we have

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & -6 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$$x_1 + 0x_2 + x_3 = 0$$

$$0x_1 - 6x_2 + 0x_3 = 0$$

$$x_1 + 0x_2 + x_3 = 0$$

Solving second and third equations we have

$$\frac{x_1}{-6-0} = \frac{x_2}{0-0} = \frac{x_3}{0-(-6)}$$
$$\frac{x_1}{-6} = \frac{x_2}{0} = \frac{x_3}{6}$$
$$\frac{x_1}{-1} = \frac{x_2}{0} = \frac{x_3}{1}$$

The eigenvector corresponding to the eigenvalue $\lambda = 4$ is

$$X_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}.$$

For $\lambda = 6$ we have

$$\begin{pmatrix} -1 & 0 & 1 \\ 0 & -8 & 0 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$$
\begin{aligned}
-x_1 + 0x_2 + x_3 &= 0 \\
0x_1 - 8x_2 + 0x_3 &= 0 \\
x_1 + 0x_2 - x_3 &= 0
\end{aligned}
$$

Solving second and third equations we have

$$
\begin{aligned}
\frac{x_1}{8-0} = \frac{x_2}{0-0} &= \frac{x_3}{8-0} \\
\frac{x_1}{8} = \frac{x_2}{0} &= \frac{x_3}{8} \\
\frac{x_1}{1} = \frac{x_2}{0} &= \frac{x_3}{1}
\end{aligned}
$$

The eigenvector corresponding to the eigenvalue $\lambda = 6$ is

$$
X_3 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.
$$

Example 7

Find the eigenvalues and eigenvectors for the matrix

$$\begin{pmatrix} 3 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 3 \end{pmatrix}$$

Solution. The characteristic equation is $|A - \lambda I| = 0$ where

$$A = \begin{pmatrix} 3 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 3 \end{pmatrix}$$

The general form of characteristic equation is

$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$

where

S_1 = Sum of the main diagonal entries $\Rightarrow$ $S_1 = 8$

S_2= Sum of the minors of the main diagonal entries $\Rightarrow$ $S_2 = 19$

S_3 = Determinant of the matrix A $\Rightarrow$ $S_3 = 12$

$\lambda^3 - 8\lambda^2 + 19\lambda - 12 = 0$

$\implies \quad \lambda = 1, 3, 4$

The eigenvalues are $\lambda = 1, 3, 4$

The eigenvectors are given by $(A - \lambda I)\ X = 0$

$$\begin{pmatrix} 3-\lambda & -1 & 0 \\ -1 & 2-\lambda & -1 \\ 0 & -1 & 3-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

For $\lambda = 1$ we have

$$\begin{pmatrix} 2 & -1 & 0 \\ -1 & 1 & -1 \\ 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$$\begin{aligned} 2x_1 - x_2 + 0x_3 &= 0 \\ -x_1 + x_2 - x_3 &= 0 \\ 0x_1 - x_2 + 2x_3 &= 0 \end{aligned}$$

Solving second and third equations we have

$$\begin{aligned} \frac{x_1}{2-1} = \frac{x_2}{0-(-2)} &= \frac{x_3}{1-0} \\ \frac{x_1}{1} = \frac{x_2}{2} &= \frac{x_3}{1} \end{aligned}$$

The eigenvector corresponding to the eigenvalue $\lambda = 1$ is

$$X_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}.$$

For $\lambda = 3$ we have

$$\begin{pmatrix} 0 & -1 & 0 \\ -1 & -1 & -1 \\ 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$$\begin{aligned} 0x_1 - x_2 + 0x_3 &= 0 \\ -x_1 - x_2 - x_3 &= 0 \\ 0x_1 - x_2 + 0x_3 &= 0 \end{aligned}$$

Solving first two equations we have

$$\frac{x_1}{1+0} = \frac{x_2}{0+0} = \frac{x_3}{0-1}$$

$$\frac{x_1}{1} = \frac{x_2}{0} = \frac{x_3}{-1}$$

The eigenvector corresponding to the eigenvalue $\lambda = 3$ is

$$X_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}.$$

For $\lambda = 4$ we have

$$\begin{pmatrix} -1 & -1 & 0 \\ -1 & -2 & -1 \\ 0 & -1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$$-x_1 - x_2 + 0x_3 = 0$$

$$-x_1 - 2x_2 - x_3 = 0$$

$$0x_1 - x_2 - x_3 = 0$$

Solving first two equations we have

$$\frac{x_1}{1+0} = \frac{x_2}{0-1} = \frac{x_3}{2-1}$$
$$\frac{x_1}{1} = \frac{x_2}{-1} = \frac{x_3}{1}$$

The eigenvector corresponding to the eigenvalue $\lambda = 4$ is

$$X_3 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}.$$

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