21MAB101T - CALCULUS AND LINEAR ALGEBRA

Unit-II Functions of Several Variables

Dr. M.SURESH
Assistant Professor
Department of Mathematics
SRM Institute of Science and Technology
Kattankulathur

September 20, 2024

Introduction

If $z = f(x, y)$ where $x = g(t)$ and $y = h(t)$.

Substituting the value of x and y in $f(x, y)$ then z can be expressed as a function of t. Then the derivative of z with respect to t is called **the total derivative of** z.

But to find the derivative of z with respect to t without substituting the value of x and y in $f(x, y)$ we have the following theorem.

Theorem If $z = f(x, y)$ where $x = g(t)$ and $y = h(t)$.Then the derivative of u with respect to t is called the total derivative of u and is given by

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\ \frac{dx}{dt} + \frac{\partial z}{\partial y}\ \frac{dy}{dt}$$

Remark. The equation (1) can also be written as

$$dz = \frac{\partial z}{\partial x}\, dx + \frac{\partial z}{\partial y}\, dy \tag{1}$$

Suppose if $u = f(x, y, z)$ where $x = g(t)$, $y = h(t)$ and $z = k(t)$.

Then the derivative of u with respect to t is given by

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\, \frac{dx}{dt} + \frac{\partial u}{\partial y}\, \frac{dy}{dt} + \frac{\partial u}{\partial z}\, \frac{dz}{dt}$$

Example-1

If $z = x^2 + y^2$ where $x = t^3$, $y = t^2$, find $\frac{dz}{dt}$

Solution. Given that

$$z = x^2 + y^2 \qquad x = t^3 \qquad y = t^2$$

$$\frac{\partial z}{\partial x} = 2x \qquad \frac{dx}{dt} = 3t^2 \qquad \frac{dy}{dt} = 2t$$

$$\frac{\partial z}{\partial y} = 2y$$

$$\begin{aligned}
\frac{dz}{dt} &= \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} \\
&= 2x(3t^2) + 2y(2t) \\
&= 2t^3(3t^2) + 2t^2(2t) \\
\frac{dz}{dt} &= 2t^3(2 + 3t^2)
\end{aligned}$$

Example-2

If $z = x^2 - 3xy^2$ where $x = e^t$, $y = e^{-t}$, find $\dfrac{dz}{dt}$

Solution.

$$z = x^2 - 3xy^2 \qquad x = e^t \qquad y = e^{-t}$$

$$\frac{\partial z}{\partial x} = 2x - 3y^2 \qquad \frac{dx}{dt} = e^t \qquad \frac{dy}{dt} = -e^{-t}$$

$$\frac{\partial z}{\partial y} = -6xy$$

$$\begin{aligned}
\frac{dz}{dt} &= \frac{\partial z}{dx}\frac{dx}{\partial t} + \frac{\partial z}{\partial y}\frac{dy}{dt} \\
&= (2x - 3y^2)(e^t) - 6xy(-e^{-t}) \\
&= (2e^t - 3(e^{-t})^2)(e^t) - 6e^te^{-t}(-e^{-t}) \\
&= 2e^{2t} - 3e^{-t}
\end{aligned}$$

Example-3

If $u = x^2 + y^2 + z^2$ where $x = t$, $y = \cos t$, $z = \sin t$ Find $\dfrac{du}{dt}$

Solution.

$$u = x^2 + y^2 + z^2 \qquad x = t \qquad y = \cos t \qquad z = \sin t$$

$$\frac{\partial u}{\partial x} = 2x \qquad \frac{dx}{dt} = 1 \qquad \frac{dy}{dt} = -\sin t \qquad \frac{dz}{dt} = \cos t$$

$$\frac{\partial u}{\partial y} = 2y$$

$$\frac{\partial u}{\partial z} = 2z$$

$$\begin{aligned}
\frac{du}{dt} &= \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} + \frac{\partial u}{\partial z}\frac{dz}{dt} \\
&= 2x(1) + 2y(-\sin t) + 2z(\cos t) \\
&= 2(t)(1) + 2(\cos t)(-\sin t) + 2(\sin t)(\cos t) = 2t
\end{aligned}$$

Example-4

If $u = xy + yz + zx$ where $x = t$, $y = e^t$, $z = t^2$ Find $\dfrac{du}{dt}$

Solution.

$$u = xy + yz + zx \qquad x = t \qquad y = e^t \qquad z = t^2$$

$$\frac{\partial u}{\partial x} = (y + z) \qquad \frac{dx}{dt} = 1 \qquad \frac{dy}{dt} = e^t \qquad \frac{dz}{dt} = 2t$$

$$\frac{\partial u}{\partial y} = (x + z)$$

$$\frac{\partial u}{\partial z} = (y + x)$$

$$\begin{aligned} \frac{du}{dt} &= \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} + \frac{\partial u}{\partial z}\frac{dz}{dt} \\ &= (y+z)(1) + (x+z)(e^t) + (y+x)(2t) \end{aligned}$$

$$\frac{du}{dt} = (t + e^t) + (t + t^2)(e^t) + (e^t + t^2)(2t)$$

$$\frac{du}{dt} = t + 2t^3 + e^t(1 + 3t + t^2)$$

Example 5

If $u = x^2 + y^2 + z^2$ where $x = e^{2t}$, $y = e^{2t}\cos 3t$, $z = e^{2t}\sin 3t$

Find $\dfrac{du}{dt}$

Solution.

$$\begin{array}{llll}
u = x^2 + y^2 + z^2 & x = e^{2t} & y = e^{2t}\cos 3t & z = e^{2t}\sin 3t \\
\dfrac{\partial u}{\partial x} = 2x & \dfrac{dx}{dt} = 2e^{2t} & \dfrac{dy}{dt} = 2e^{2t}\cos 3t & \dfrac{dz}{dt} = 2e^{2t}\sin 3t \\
\dfrac{\partial u}{\partial y} = 2y & & \quad - 3e^{2t}\sin 3t & \quad + 3e^{2t}\cos 3t \\
\dfrac{\partial u}{\partial z} = 2z & & &
\end{array}$$

$$
\begin{aligned}
\frac{du}{dt} &= \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} + \frac{\partial u}{\partial z}\frac{dz}{dt} \\
&= 2x(2e^{2t}) + 2y(2e^{2t}\cos 3t - 3e^{2t}\sin 3t) \\
&\quad +2z(2e^{2t}\sin 3t + 3e^{2t}\cos 3t) \\
&= 2e^{2t}2e^{2t} + 2e^{2t}\cos 3t 2e^{2t}(\cos 3t - 3\sin 3t) \\
&\quad +2e^{2t}\sin 3t 2e^{2t}(\sin 3t - 3\cos 3t) \\
&= 4e^{4t}\left[1 + \cos^2 3t - 3\sin 3t\cos 3t + \sin^2 3t - 3\sin 3t\cos 3t\right] \\
&= 4e^{4t}\left[1 + \cos^2 3t + \sin^2 3t\right] \\
&= 4e^{4t}\left[1 + 1\right] \\
&= 8e^{4t}
\end{aligned}
$$

Exercise

1. If $z = x^2y + 3xy^4$ where $x = e^t$, $y = \sin t$ find $\dfrac{dz}{dt}$
2. If $z = x^2 + y^2$ where $x = t^3$, $y = 1 + t^2$ find $\dfrac{dz}{dt}$
3. If $z = x^2 \sin y$ where $x = 1 + t^2$, $y = 2t$ find $\dfrac{dz}{dt}$
4. If $z = x \tan^{-1}(xy)$ where $x = t^2$, $y = 2t$ find $\dfrac{dz}{dt}$
5. If $u = \log(x + y + z)$ where $x = e^{-t}$, $y = \cos t$, $z = \sin t$ find $\dfrac{du}{dt}$

Answers

1. $e^t \sin t(2 + 3\sin^3 t) + e^t \cos t(e^t + 12\sin^3 t)$
2. $6t^5 + 4t^3 + 4t$
3. $4t(1 + t^2)\sin^3 t + 2(1 + t^2)\cos t$
4. $2t\tan^{-1}(2t^3) + \dfrac{6t^4}{1 + 4t^6}$
5. $\dfrac{\cos t - \sin t - e^{-t}}{e^{-t} + \cos t + \sin t}$