21MAB101T - CALCULUS AND LINEAR ALGEBRA

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Symmetric Matrices With Repeated Eigenvalues

Example-1

Find the eigenvalues and eigenvectors for the matrix

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

Solution. The characteristic equation is $|A - \lambda I| = 0$ where

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

The general form of characteristic equation is

$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$

where

S_1 = Sum of the main diagonal entries $\Rightarrow\ S_1 = 0$

S_2= Sum of the minors of the main diagonal entries $\Rightarrow\ S_2 = -3$

S_3 = Determinant of the matrix A $\Rightarrow\ S_3 = 2$

$\lambda^3 - 3\lambda - 2 = 0$

$\implies \quad \lambda = -1, -1, 2$

The eigenvalues are $\lambda = -1, -1, 2$

The eigenvectors are given by $(A - \lambda I)\ X = 0$,

$$\begin{pmatrix} 0-\lambda & 1 & 1 \\ 1 & 0-\lambda & 1 \\ 1 & 1 & 0-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

Case (i): For $\lambda = 2$

$$\begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$$\begin{aligned} -2x_1 + x_2 + x_3 &= 0 \\ x_1 - 2x_2 + x_3 &= 0 \\ x_1 + x_2 - 2x_3 &= 0 \end{aligned}$$

Solving first two equations we have

$$\begin{aligned} \frac{x_1}{1+2} = \frac{x_2}{1-(-2)} &= \frac{x_3}{4-1} \\ \frac{x_1}{3} = \frac{x_2}{3} &= \frac{x_3}{3} \\ \frac{x_1}{1} = \frac{x_2}{1} &= \frac{x_3}{1} \end{aligned}$$

The eigenvector corresponding to the eigenvalue $\lambda = 2$ is

$$X_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

Case (ii): For $\lambda = -1$

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$$x_1 + x_2 + x_3 = 0$$

$$x_1 + x_2 + x_3 = 0$$

$$x_1 + x_2 + x_3 = 0$$

These three equation represents the same equation,

$$x_1 + x_2 + x_3 = 0.$$

$$\text{Put } x_1 = 0 \text{ we get } x_2 = -x_3$$
$$\frac{x_2}{1} = \frac{x_3}{-1}$$

The eigenvector corresponding to the eigenvalue $\lambda = -1$ is

$$X_2 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}.$$

Case (iii): For $\lambda = -1$

Since the Matrix A is symmetric, all three eigen vectors are mutually orthogonal.

Let $X_3 = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ be orthogonal to X_1 and X_2, then

$$X_1^T X_3 = 0 \quad \& \quad X_2^T X_3 = 0$$

$$X_1^T X_3 = 0$$

$$\begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = 0$$

$$X_2^T X_3 = 0$$

$$\begin{bmatrix} 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = 0$$

Then we get

$$\begin{aligned} a + b + c &= 0 \\ 0a + b - c &= 0 \end{aligned}$$

$$\frac{a}{-1-1} = \frac{b}{0+1} = \frac{c}{1-0}$$

$$\frac{a}{-2} = \frac{b}{1} = \frac{c}{1}$$

The Eigen Vector for $\lambda = 1$ is

$$X_3 = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$$

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Example-2

Find the eigenvalues and eigenvectors for the matrix

$$A = \begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix}$$

Solution. The characteristic equation is $|A - \lambda I| = 0$ where

$$A = \begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix}$$

The characteristic equation is $|A - \lambda I| = 0$

$$\Rightarrow \lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$$

$$
\begin{aligned}
S_1 &= 2+2+2 = 6 \\
S_2 &= \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} \\
&= (4-1)+(4-1)+(4-1) \\
&= 9 \\
S_3 = |A| &= \begin{vmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{vmatrix} \\
&= (2)(4-1) - (-1)(-2+1) + (1)(1-2) \\
&= 6-1-1 = 4
\end{aligned}
$$

$$S_1 = 6, S_2 = 9, S_3 = 4$$

$$\Rightarrow \lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$$

The characteristic equation is $\lambda^3 - 6\lambda^2 + 9\lambda - 4 = 0$

$$\Rightarrow (\lambda - 4)(\lambda - 1)(\lambda - 1) = 0$$

$$\lambda = 4, 1, 1.$$

The eigenvalues are $\lambda = 4, 1, 1.$

The eigenvectors are given by $[A - \lambda I]\ X = 0$

$$\begin{bmatrix} 2-\lambda & -1 & 1 \\ -1 & 2-\lambda & -1 \\ 1 & -1 & 2-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left.\begin{aligned} &(2-\lambda)x_1 - x_2 + x_3 = 0 \\ &-x_1 + (2-\lambda)x_2 - x_3 = 0 \\ &x_1 - x_2 + (2-\lambda)x_3 = 0 \end{aligned}\right\} \tag{1}$$

Case (i): For $\lambda = 4$ in (1), we get

$$-2x_1 - x_2 + x_3 = 0$$
$$-x_1 - 2x_2 - x_3 = 0$$
$$x_1 - x_2 - 2x_3 = 0$$

$$\frac{x_1}{2+1} = \frac{x_2}{1-4} = \frac{x_3}{2+1}$$

$$\frac{x_1}{3} = \frac{x_2}{-3} = \frac{x_3}{3}$$

$$\frac{x_1}{1} = \frac{x_2}{-1} = \frac{x_3}{1}$$

The eigenvector for $\lambda = 4$ is

$$X_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}.$$

Case (ii): For $\lambda = 1$ in (1), we get

$$x_1 - x_2 + x_3 = 0$$

$$-x_1 + x_2 - x_3 = 0$$

$$x_1 - x_2 + x_3 = 0$$

All equations are same, so put $x_1 = 0$.

$$x_2 - x_3 = 0$$

$$x_2 = x_3$$

$$\frac{x_2}{1} = \frac{x_3}{1}$$

The eigenvector for $\lambda = 1$ is

$$X_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}.$$

Case (iii): For $\lambda = 1$

Since the Matrix A is symmetric, all three eigen vectors are mutually orthogonal.

Let $X_3 = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ be orthogonal to X_1 and X_2, then

$$X_1^T X_3 = 0 \quad \& \quad X_2^T X_3 = 0$$

$$X_1^T X_3 = 0$$

$$\begin{bmatrix} 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = 0$$

$$X_2^T X_3 = 0$$

$$\begin{bmatrix} 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = 0$$

Then we get

$$\begin{aligned} a - b + c &= 0 \\ 0a + b + c &= 0 \end{aligned}$$

$$\frac{a}{-1-1} = \frac{b}{0-1} = \frac{c}{1-0}$$

$$\frac{a}{-2} = \frac{b}{-1} = \frac{c}{1}$$

$$\frac{a}{2} = \frac{b}{1} = \frac{c}{-1}$$

The Eigen Vector for $\lambda = 1$ is

$$X_3 = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$$

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Example-3

Find the eigenvalues and eigenvectors for the matrix

$$A = \begin{pmatrix} 2 & 0 & 4 \\ 0 & 6 & 0 \\ 4 & 0 & 2 \end{pmatrix}$$

Solution. The characteristic equation is $|A - \lambda I| = 0$ where

$$A = \begin{pmatrix} 2 & 0 & 4 \\ 0 & 6 & 0 \\ 4 & 0 & 2 \end{pmatrix}$$

The characteristic equation is $|A - \lambda I| = 0$

$$\Rightarrow \lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$$

$$S_1 = 2 + 6 + 2 = 10$$

$$\begin{aligned} S_2 &= \begin{vmatrix} 6 & 0 \\ 0 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 4 \\ 4 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 0 \\ 0 & 6 \end{vmatrix} \\ &= 12 + (4 - 16) + 12 \\ &= 12 \end{aligned}$$

$$\begin{aligned} S_3 = |A| &= \begin{vmatrix} 2 & 0 & 4 \\ 0 & 6 & 0 \\ 4 & 0 & 2 \end{vmatrix} \\ &= (2)(12) + 4(-24) \\ &= -72 \end{aligned}$$

$$S_1 = 10, S_2 = 12, S_3 = -72$$

$$\Rightarrow \lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$$

The characteristic equation is $\lambda^3 - 10\lambda^2 + 12\lambda + 72 = 0$

$$\Rightarrow (\lambda + 2)(\lambda - 6)(\lambda - 6) = 0$$

$$\lambda = -2, 6, 6.$$

The eigenvalues are $\lambda = -2, 6, 6.$

The eigenvectors are given by $[A - \lambda I]\ X = 0$

$$\begin{bmatrix} 2-\lambda & 0 & 4 \\ 0 & 6-\lambda & 0 \\ 4 & 0 & 2-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left.\begin{aligned} (2-\lambda)x_1 + 0x_2 + 4x_3 = 0 \\ 0x_1 + (6-\lambda)x_2 + 0x_3 = 0 \\ 4x_1 + 0x_2 + (2-\lambda)x_3 = 0 \end{aligned}\right\} \tag{1}$$

Case (i): For $\lambda = -2$ in (1), we get

$$4x_1 + 0x_2 + 4x_3 = 0$$

$$0x_1 + 8x_2 + 0x_3 = 0$$

$$4x_1 + 0x_2 + 4x_3 = 0$$

$$\frac{x_1}{0-32} = \frac{x_2}{0-0} = \frac{x_3}{32-0}$$

$$\frac{x_1}{-32} = \frac{x_2}{0} = \frac{x_3}{32}$$

$$\frac{x_1}{1} = \frac{x_2}{0} = \frac{x_3}{-1}$$

The eigenvector for $\lambda = -2$ is

$$X_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}.$$

Case (ii): For $\lambda = 6$ in (1), we get

$$-4x_1 + 0x_2 + 4x_3 = 0$$

$$0x_1 + 0x_2 + 0x_3 = 0$$

$$4x_1 + 0x_2 + 4x_3 = 0$$

first and third equations are same, second equation is 0. so take first equation

$$-4x_1 + 4x_3 = 0$$

The coefficient of x_2 is already 0 we can directly take this as

$$-4x_1 = -4x_3$$

$$\frac{x_1}{1} = \frac{x_3}{1}$$

The eigenvector

$$X_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

Case (iii): For $\lambda = 6$

Since the Matrix A is symmetric, all three eigen vectors are mutually orthogonal.

Let $X_3 = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ be orthogonal to X_1 and X_2, then

$$X_1^T X_3 = 0 \quad \& \quad X_2^T X_3 = 0$$

$$X_1^T X_3 = 0$$

$$\begin{bmatrix} 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = 0$$

$$X_2^T X_3 = 0$$

$$\begin{bmatrix} 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = 0$$

Then we get

$$a + 0b + c = 0$$
$$a + 0b - c = 0$$

$$\frac{a}{0-0} = \frac{b}{1+1} = \frac{c}{0-0}$$

$$\frac{a}{0} = \frac{b}{2} = \frac{c}{0}$$

$$\frac{a}{0} = \frac{b}{1} = \frac{c}{0}$$

The Eigen Vector $X_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

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Find the eigenvalues and eigenvectors of the following matrices

1. $\begin{pmatrix} 4 & 2 & -2 \\ 5 & 3 & 2 \\ -2 & 4 & 1 \end{pmatrix}$

2. $\begin{pmatrix} 2 & -2 & 3 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{pmatrix}$

3. $\begin{pmatrix} -9 & 2 & 6 \\ 5 & 0 & -3 \\ -16 & 4 & 11 \end{pmatrix}$

4. $\begin{pmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{pmatrix}$

5. $\begin{pmatrix} 2 & 1 & 1 \\ 2 & 3 & 2 \\ 3 & 3 & 4 \end{pmatrix}$

6. $\begin{pmatrix} -9 & 4 & 4 \\ -8 & 3 & 4 \\ -16 & 8 & 7 \end{pmatrix}$

7. $\begin{pmatrix} 2 & -2 & 2 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{pmatrix}$

8. $\begin{pmatrix} -2 & 5 & 4 \\ 5 & 7 & 5 \\ 4 & 5 & -2 \end{pmatrix}$

9. $\begin{pmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{pmatrix}$

10. $\begin{pmatrix} 2 & 4 & -6 \\ 4 & 2 & -6 \\ -6 & -6 & -15 \end{pmatrix}$

11. $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{pmatrix}$

12. $\begin{pmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{pmatrix}$

13. $\begin{pmatrix} 2 & 0 & 1 \\ 0 & 3 & 3 \\ 1 & 0 & 2 \end{pmatrix}$

Answers

1. The eigenvalues are $\lambda_1 = -1, \lambda_2 = 2, \lambda_3 = 5$
and the eigenvectors are

$$X_1 = \begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix} \qquad X_2 = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \qquad X_3 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

2. The eigenvalues are $\lambda_1 = -2, \lambda_2 = 1, \lambda_3 = 3$
and the eigenvectors are

$$X_1 = \begin{pmatrix} 11 \\ 1 \\ -14 \end{pmatrix} \qquad X_2 = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \qquad X_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

Answers

3. The eigenvalues are $\lambda_1 = -1, \lambda_2 = 1, \lambda_3 = 2$
and the eigenvectors are

$$X_1 = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} \qquad X_2 = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} \qquad X_3 = \begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix}$$

4. The eigenvalues are $\lambda_1 = 1, \lambda_2 = 1, \lambda_3 = 5$
and the eigenvectors are

$$X_1 = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} \qquad X_2 = \begin{pmatrix} 1 \\ 2 \\ -5 \end{pmatrix} \qquad X_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

Answers

5. The eigenvalues are $\lambda_1 = 1, \lambda_2 = 1, \lambda_3 = 7$
and the eigenvectors are

$$X_1 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \qquad X_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \qquad X_3 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

6. The eigenvalues are $\lambda_1 = -1, \lambda_2 = -1, \lambda_3 = 3$
and the eigenvectors are

$$X_1 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \qquad X_2 = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} \qquad X_3 = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$$

Answers

7. The eigenvalues are $\lambda_1 = -2, \lambda_2 = 2, \lambda_3 = 2$
and the eigenvectors are

$$X_1 = \begin{pmatrix} -4 \\ 1 \\ 7 \end{pmatrix} \qquad X_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

8. The eigenvalues are $\lambda_1 = -3, \lambda_2 = -6, \lambda_3 = 12$
and the eigenvectors are

$$X_1 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \qquad X_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \qquad X_3 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$$

Answers

9. The eigenvalue are $\lambda_1 = -2, \lambda_2 = 4, \lambda_3 = 6$
and the eigenvectors are

$$X_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \qquad X_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \qquad X_3 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

10. The eigenvalues are $\lambda_1 = -2, \lambda_2 = 9, \lambda_3 = 18$
and the eigenvectors are

$$X_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \qquad X_2 = \begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix} \qquad X_3 = \begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix}$$

Answers

11. The eigenvalues are $\lambda_1 = 0, \lambda_2 = 0, \lambda_3 = 14$
and the eigenvectors are

$$X_1 = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} \qquad X_2 = \begin{pmatrix} 3 \\ 6 \\ -5 \end{pmatrix} \qquad X_3 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

12. The eigenvalues are $\lambda_1 = -2, \lambda_2 = 3, \lambda_3 = 6$
and the eigenvectors are

$$X_1 = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} \qquad X_2 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \qquad X_3 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$$

Answers

13. The eigenvalue are $\lambda_1 = 1, \lambda_2 = 3, \lambda_3 = 3$
and the eigenvectors are

$$X_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \qquad X_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \qquad X_3 = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$