3.3 THE ITERATION METHOD

In iteration method the basic idea is to expend the recurrence and express it as a summation of terms dependent only on 'n' and the initial conditions.

Example 6 Consider the recurrence : $T(n)=3T\left(\left\lfloor\frac{n}{4}\right\rfloor\right)+n$

Solution. We iterate it as follows :

$$T(n)=n+3T\left(\left\lfloor\frac{n}{4}\right\rfloor\right)=n+3\left(\left\lfloor\frac{n}{4}\right\rfloor+3T\left(\left\lfloor\frac{n}{16}\right\rfloor\right)\right)$$

$$=n+3\left(\left\lfloor\frac{n}{4}\right\rfloor+3\left(\left\lfloor\frac{n}{16}\right\rfloor+3T\left(\left\lfloor\frac{n}{64}\right\rfloor\right)\right)\right)$$

$$=n+3\left\lfloor\frac{n}{4}\right\rfloor+9\left\lfloor\frac{n}{16}\right\rfloor+27T\left(\left\lfloor\frac{n}{64}\right\rfloor\right)$$

$$\leq n+\frac{3n}{4}+\frac{9n}{16}+\ldots 3^iT\left(\frac{n}{4^i}\right)$$

The series terminates when $\frac{n}{4^i}=1 \quad \Rightarrow \quad n=4^i$ or $i=\log_4 n$

$$T(n)\leq n+\frac{3n}{4}+\frac{9n}{16}+\frac{27n}{64}+\ldots+3^{\log_4 n}T(1)$$

$$\leq n+\frac{3n}{4}+\frac{9n}{16}+\frac{27n}{64}+\ldots+3^{\log_4 n}\,\theta(1)$$

$$\leq n\sum_{i=0}^{\infty}\left(\frac{3}{4}\right)^i+\theta(n^{\log_4 3}) \text{ as } 3^{\log_4 n}=n^{\log_4 3}$$

$$\leq n\frac{1}{1-\frac{3}{4}}+o(n) \text{ as } \log_4 3<1 \text{ i.e., } \theta(n^{\log_4 3})=o(n)$$

$$=4n+o(n)=O(n)$$

Example 7 Solve the recurrence relation by iteration : $T(n)=T(n-1)+n^4$ **(UPTU MCA 2002-03)**

Solution. $T(n)=T(n-1)+n^4$

$$=[T(n-2)+(n-1)^4]+n^4$$

$$=T(n-2)+(n-1)^4+n^4$$

$$=T(n-3)+(n-2)^4+(n-1)^4+n^4$$

....

$$=n^4+(n-1)^4+(n-2)^4+\ldots+2^4+1^4+T(0)$$

$$=\sum_{i=1}^{n} i^4+T(0)$$

$$T(n)=\Theta(n^5)$$

Example 8 *Solve the following recurrence relation :*

$$T(n)=2\,T(n/2)+3n^2$$
$$T(n)=11.$$

Solution. $T(n)=2\,T(n/2)+3n^2$

$$=3n^2+2T(n/2)$$

$$=3n^2+2\left[3\left(\frac{n}{2}\right)^2+2T\left(\frac{n}{4}\right)\right]$$

$$=3n^2+2.3\left(\frac{n}{2}\right)^2+4\left[3.\left(\frac{n}{4}\right)^2+2T\left(\frac{n}{8}\right)\right]$$

$$=3n^2+\frac{3n^2}{2}+4.3\frac{n^2}{4^2}+2^3T\left(\frac{n}{2^3}\right)$$

$$=3n^2+\frac{3n^2}{2}+\frac{3n^2}{4}+2^3T\left(\frac{n}{2^3}\right)$$

$$=3n^2+\frac{3n^2}{2}+\frac{3n^2}{2^2}+2^3T\left(\frac{n}{2^3}\right)$$

$$\therefore \quad T(n)=3n^2+\frac{3n^2}{2^1}+\frac{3n^2}{2^2}+\ldots+2^iT\left(\frac{n}{2^i}\right)$$

The series terminates when

$$\frac{n}{2^i}=1 \Rightarrow n=2^i \text{ or } i=\log_2 n$$

$$\therefore \quad T(n)=3n^2+\frac{3n^2}{2^1}+\frac{3n^2}{2^2}+\ldots+2^{\log_2 n}T(1)$$

$$=3n^2+\frac{3n^2}{2}+\frac{3n^2}{2^2}+\ldots+n^{\log_2 2}T(1)$$

$$=3n^2\left[1+\frac{1}{2}+\frac{1}{2^2}+\ldots\right]+n.11$$

$$\leq 3n^2.\left[\frac{1}{1-1/2}\right]+11n$$

$$\leq 3n^2.\,2+11n$$

$$\leq 6n^2+11n$$

$$\therefore \quad T(n)=O(n^2)$$

Example 9 *Solve the recurrence relation*

$$T(n)=1 \quad \textit{for } n=1$$
$$=2\,T(n-1) \quad \textit{for } n>1$$

Solution. $T(n)=2T(n-1)$

$$=2[2T(n-2)]=4T(n-2)$$
$$=4[2T(n-3)]=8T(n-3)$$
$$=2^3T(n-3)=2^4T(n-4).$$

In general $T(n)=2^iT(n-i)$

Put $i=n-1$ we get $T(n)=2^{n-1}T(1)=2^{n-1}$

Example 10 *Consider the recurrence*

$$T(n)=T(n-1)+1 \text{ and } T(1)=\theta(1). \text{ Solve it}$$

Solution. $T(n)=T(n-1)+1$

$$=(T(n-2)+1)+1=(T(n-3)+1)+1+1$$
$$=T(n-4)+4=T(n-5)+1+4$$
$$=T(n-5)+5=T(n-k)+k$$

where $k=n-1$

i.e., $T(n-k)=T(1)=\theta(1)$

i.e., $T(n)=\theta(1)+(n-1)=1+n-1=n=\theta(n).$

Example 11 $T(n)=T\left(\frac{n}{3}\right)+n^{4/3}$. *Solve this recurrence by iteration method.*

Solution. $T(n)=T\left(\frac{n}{3}\right)+n^{4/3}=n^{4/3}+T\left(\frac{n}{3}\right)$

$$=n^{4/3}+\left(\frac{n}{3}\right)^{4/3}+T\left(\frac{n}{3^2}\right)$$

$\therefore$ $$T(n)=n^{4/3}+\left(\frac{n}{3}\right)^{4/3}+\left(\frac{n}{3^2}\right)^{4/3}+\ldots\left(\frac{n}{3^k}\right)^{4/3}$$

Let $k=\log_3 n$ when $\frac{n}{3^k}=1$ i.e., $3^k=n.$

$\therefore$ $$T(n)=n^{4/3}+n^{4/3}\left(\frac{1}{3^{4/3}}\right)+n^{4/3}\left(\frac{1}{3^{4/3}}\right)^2+\ldots n^{\frac{4}{3}}\left(\frac{1}{3^{4/3}}\right)^k$$

$$=n^{4/3}\sum_{i=0}^{k}\left(\frac{1}{3^{4/3}}\right)^i$$

$$\le n^{4/3}\sum_{i=0}^{\infty}\left(\frac{1}{3^{4/3}}\right)^i$$

$$\le n^{4/3}\frac{1}{1-\frac{1}{3^{4/3}}}=O(n^{4/3}).$$

Example 12 Solve $T(n)=T(n-1)+\frac{1}{n}$.

Solution. By iteration method.

$$T(n)=\frac{1}{n}+T(n-1)=\frac{1}{n}+\frac{1}{n-1}+T(n-2)$$

$$=\frac{1}{n}+\frac{1}{n-1}+\frac{1}{n-2}+T\ (n-3)=\frac{1}{n}+\frac{1}{n-1}+\frac{1}{n-2}+\ldots+T(1)$$

$$=\sum_{i=0}^{n-2}\frac{1}{n-i}+T(1)\le\sum_{i=0}^{\infty}\frac{1}{n-i}+\theta(1)$$

Let $n-i=x$

$-di=dx$

Thus, it can be transformed into an integral.

$$\sum_{i=0}^{n}\frac{1}{n-i}=-\int_{n}^{0}\frac{dx}{x}=\log n.$$

Thus $T(n)=\theta(\log n)+\theta(1)=\theta(\log n)$

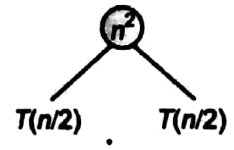
3.3.1 Recursion Tree

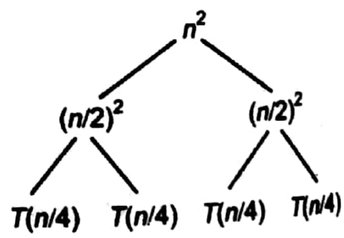
Recursion tree method is a pictorial representation of an iteration method, which is in the form of a tree, where at each level nodes are expanded. In general, we consider second term in recurrence as root. It is useful when divide and conquer algorithm is used.

Example 13 *Consider* $T(n)=2T\left(\frac{n}{2}\right)+n^2$.

We have to obtain the asymptotic bound using recursion tree method. **(UPTU MCA 2003-04)**

Solution. The recursion tree for the above recurrence is



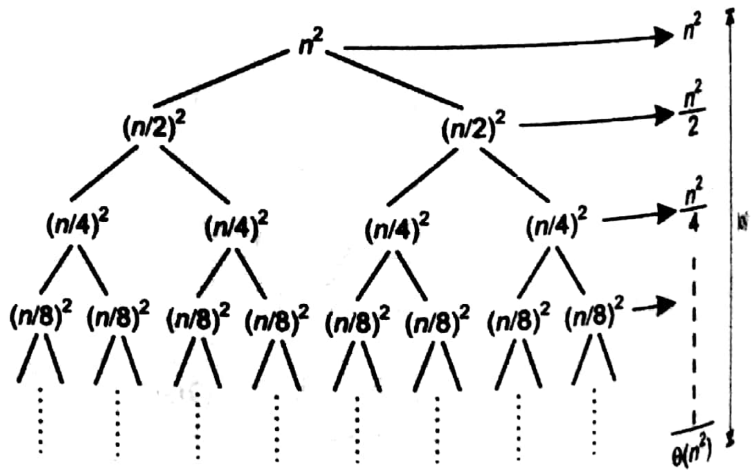


$$T(n)=n^2+\frac{n^2}{2}+\frac{n^2}{4}+\ldots\lg n \text{ times}$$

$$\le n^2\sum_{i=0}^{\infty}\left(\frac{1}{2^i}\right)$$

$$\le n^2\left(\frac{1}{1-1/2}\right)\le 2n^2$$

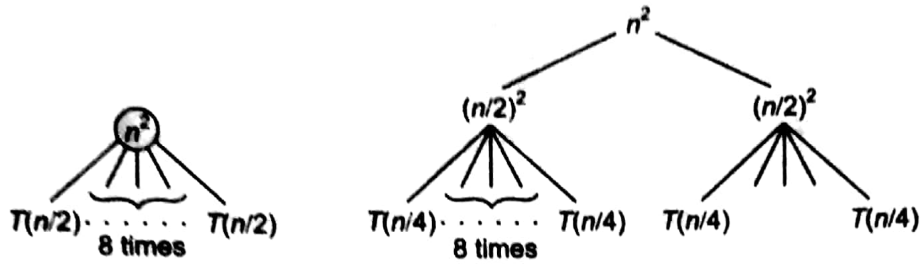
$\therefore\ T(n)=\theta(n^2)$

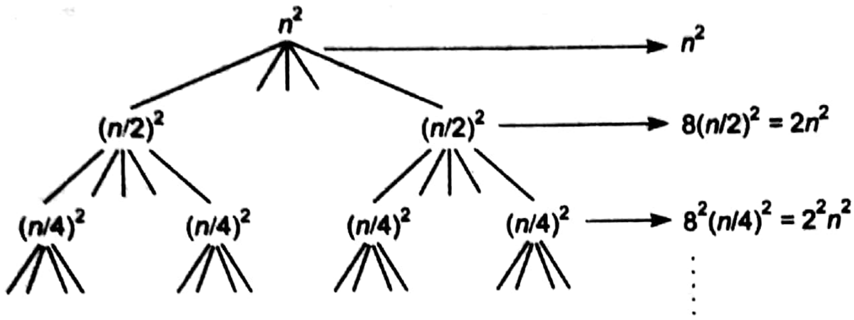


Example 14 *Solve the following recurrence by using the recursion tree method :*

$$T(n)=8T\left(\frac{n}{2}\right)+n^2$$

Solution. The recursion tree for the above recurrence is





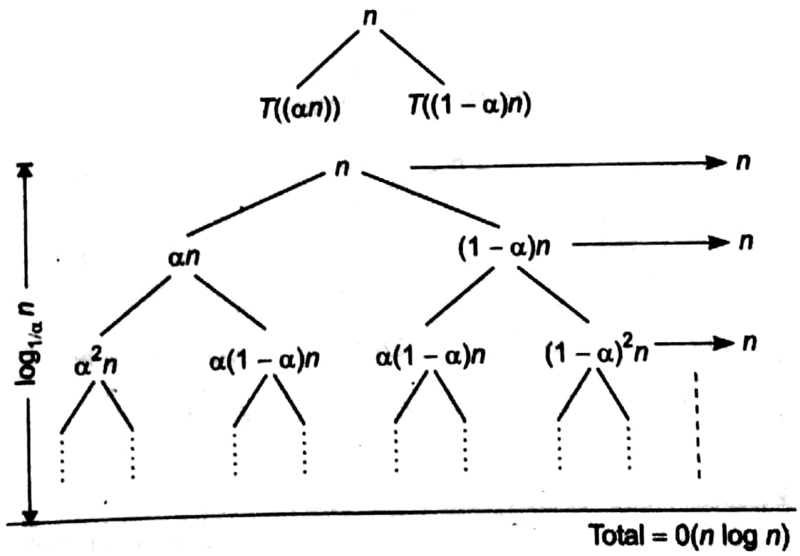
$$\begin{aligned}T(n)&=n^2+2n^2+2^2n^2+\ldots\\&=O(n^2)\end{aligned}$$

Example 15 *Solve the following recurrences :* (UPTU B.Tech. 2003-04)

$$T(n)=T(\alpha . n)+T(1-\alpha)n+n \qquad 0<\alpha<1$$

Solution. $T(n)=T(\alpha\, n)+T(1-\alpha)n+n$

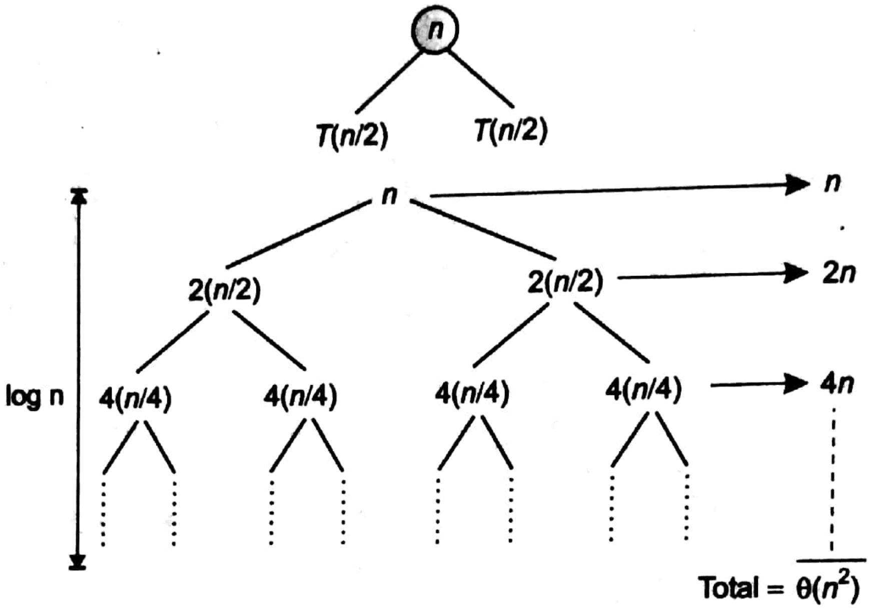
The recursion tree for the above recurrence is



Example 16 *Consider the following recurrence* $T(n)=4T\left(\left\lfloor \frac{n}{2} \right\rfloor\right)+n$

Obtain the asymptotic bound using recursion tree method.

Solution. The recursion tree for the above recurrence



We have $n+2n+4n+\ldots\log_2 n$ times

$$=n(1+2+4+\ldots\log_2 n \text{ times})$$

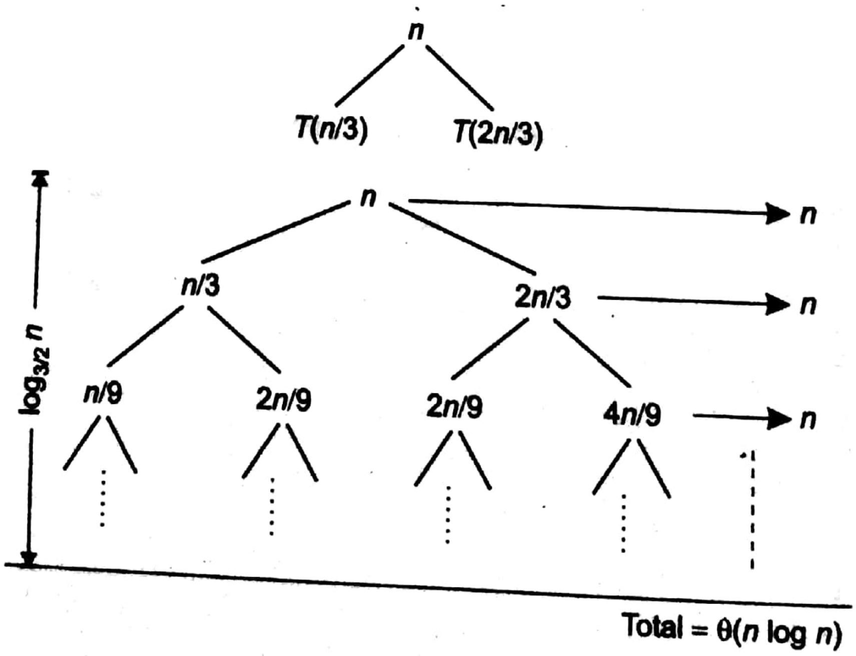
$$=n\frac{(2^{\log_2 n}-1)}{(2-1)}=\frac{n(n-1)}{1}=n^2-n=\theta(n^2)$$

$\therefore \quad T(n)=\theta(n^2)$

Example 17 *Consider the following recurrence* $T(n)=T\left(\frac{n}{3}\right)+T\left(\frac{2n}{3}\right)+n$

Obtain the asymptotic bound using recursion tree method.

Solution. The given recurrence has the following recursion tree.



When we add the values across the levels of the recursion tree, we get a value of n for every level. The longest path from the root to a leaf is

$$n \to \frac{2}{3}n \to \left(\frac{2}{3}\right)^2 n \to \ldots 1$$

Since $\left(\frac{2}{3}\right)^i n = 1$ when $i = \log_{3/2} n$.

Thus the height of the tree is $\log_{3/2} n$.

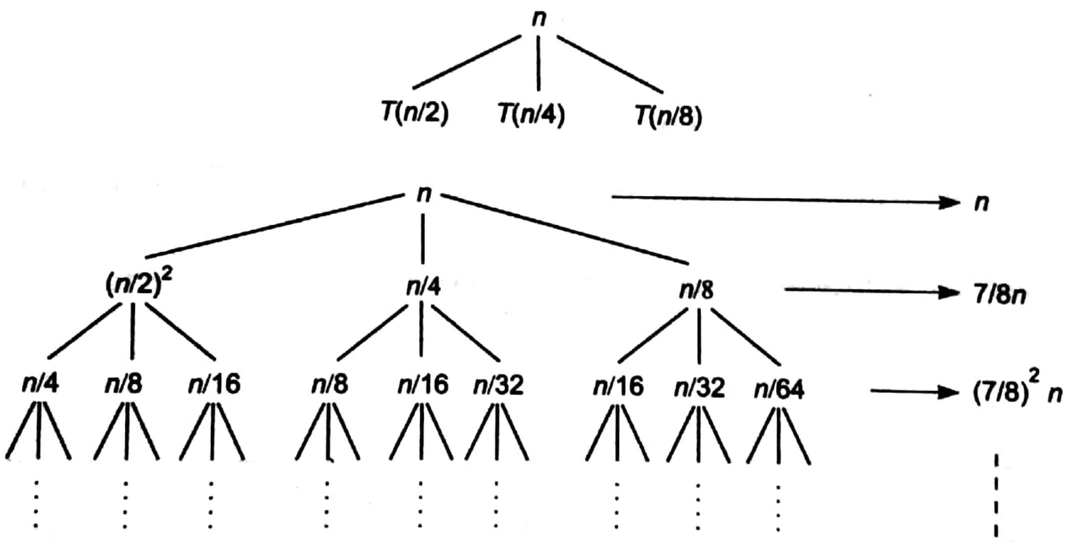
$\therefore \quad T(n) = n + n + n + \ldots + \log_{3/2} n$ times.

$\quad = \theta(n \log n)$

<u>*Example 18*</u> *Solve the following recursion :*

$$T(n) = T\left(\frac{n}{2}\right) + T\left(\frac{n}{4}\right) + T\left(\frac{n}{8}\right) + n \text{ by recursion tree method.}$$

Solution. The given recurrence has the following recursion tree :



$$T(n) \le n + \frac{7}{8}n + \left(\frac{7}{8}\right)^2 n + \ldots$$

$$\le n\left[1 + \frac{7}{8} + \left(\frac{7}{8}\right)^2 + \ldots\right]$$

$$\le n.\frac{1}{1 - \frac{7}{8}}$$

$$\le 8n$$

$$T(n) = \theta(n).$$