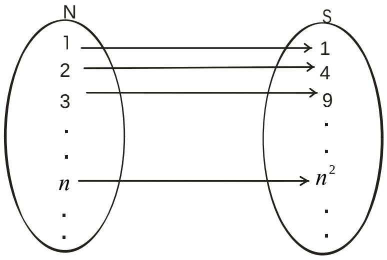
Sequence and Series

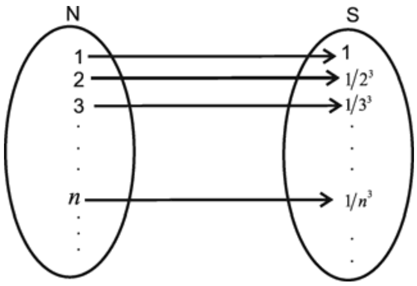
Sequence

A function f:N $\rightarrow$ S, where S is any nonempty set is called a *Sequence* i.e., for each n$\in$ N, $\exists$ a unique element f(n) $\in$ S. The sequence is written as f(1), f(2), f(3), ...f(n)...., and is denoted by {f(n)}, or <f(n)>, or (f(n)). If f(n) = a_n , the sequence is written as $a_1, a_2 .....a_n$ and denoted by , $\{a_n\}\, or < a_n > or\left(a_n\right)$. Here f(n) or a_n are the n^{th} terms of the Sequence.

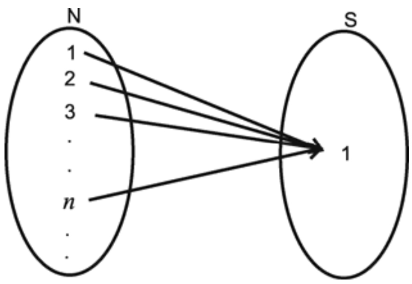
Ex. 1. 1 , 4 , 9 , 16 ,......... n^2 ,.....(or) $< n^2 >$



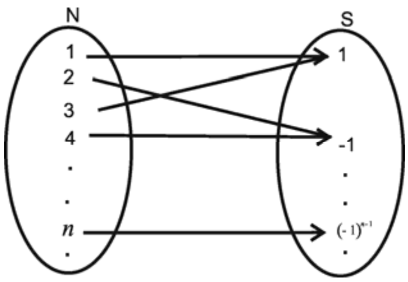
Ex. 2. $\frac{1}{1^3}, \frac{1}{2^3}, \frac{1}{3^3}, ...\frac{1}{n^3}....(or)\left(\frac{1}{n^3}\right)$



Ex. 3. 1, 1, 1......1..... or <1>



Ex 4: 1 , –1, 1, –1, ... or $\left\langle (-1)^{n-1} \right\rangle$



Note :
1. If S $\subseteq$ R then the sequence is called a *real sequence.*
2. The range of a sequence is almost a countable set.

Kinds of Sequences

1. **Finite Sequence:** A sequence $< a_n >$ in which $a_n = 0 \ \forall n > m \in N$ is said to be a finite Sequence. i.e., A finite Sequence has a finite number of terms.
2. **Infinite Sequence:** A sequence, which is not finite, is an infinite sequence.

Bounds of a Sequence and Bounded Sequence

1. If $\exists$ a number 'M' $\ni a_n \leq$ M, $\forall n \in$ N, the Sequence $< a_n >$ is said to be bounded above or bounded on the right.

   Ex. $1, \frac{1}{2}, \frac{1}{3}$, ,....... here $a_n \leq 1 \ \forall n \in$ N

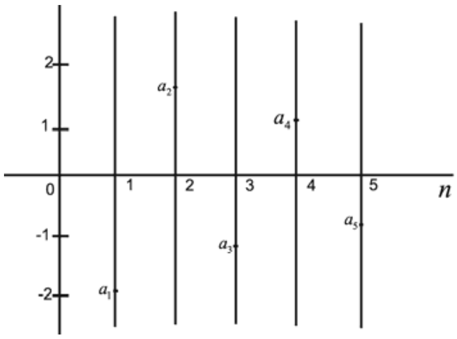
2. If $\exists$ a number 'm' $\ni a_n \geq m, \forall n \in$ N, the sequence $< a_n >$ is said to be bounded below or bounded on the left.

   Ex. 1 , 2 , 3 ,.....here $a_n \geq 1 \ \forall n \in$ N

3. A sequence which is bounded above and below is said to be bounded.

   Ex. Let $a_n = (-1)^n \left(1 + \frac{1}{n}\right)$

n	1	2	3	4	......
a_n	-2	3/2	-4/3	5/4	......



From the above figure (see also table) it can be seen that m = –2 and M = $\frac{3}{2}$.

$\therefore$ The sequence is bounded.

Limits of a Sequence

A Sequence $< a_n >$ is said to tend to limit 'l' when, given any + ve number '$\in$', however small, we can always find an integer 'm' such that $|a_n - l| < \in, \forall n \geq m$, and we write $\underset{n\to\infty}{Lt}\ a_n = l$ or $\langle a_n \to l \rangle$

Ex. If $a_n = \frac{n^2+1}{2n^2+3}$ then $< a_n > \to \frac{1}{2}$.

Convergent, Divergent and Oscillatory Sequences

1. **Convergent Sequence:** A sequence which tends to a finite limit, say 'l' is called a *Convergent Sequence.* We say that the sequence converges to 'l'
2. **Divergent Sequence:** A sequence which tends to $\pm\infty$ is said to be *Divergent* (or is said to diverge).
3. **Oscillatory Sequence:** A sequence which neither converges nor diverges ,is called an *Oscillatory Sequence.*

***Ex.* 1.** Consider the sequence $2, \frac{3}{2}, \frac{4}{3}, \frac{5}{4}, ...$ here $a_n = 1 + \frac{1}{n}$

The sequence $< a_n >$ is convergent and has the limit 1

$a_n - 1 = 1 + \frac{1}{n} - 1 = \frac{1}{n}$ and $\frac{1}{n} < \in$ whenever $n > \frac{1}{\in}$

Suppose we choose $\in = .001$, we have $\frac{1}{n} < .001$ when n > 1000.

***Ex.* 2.** If $a_n = 3 + (-1)^n \frac{1}{'n'} < a_n >$ converges to 3.

Ex. 3. If $a_n = n^2 + (-1)^n .n, < a_n >$ diverges.

Ex. 4. If $a_n = \frac{1}{n} + 2(-1)^n , < a_n >$ oscillates between -2 and 2.

Infinite Series

If $< u_n >$ is a sequence, then the expression $u_1 + u_2 + u_3 + ... + u_n + ...$ is called an infinite series. It is denoted by $\sum_{n=1}^{\infty} u_n$ or simply Σu_n

The sum of the first *n* terms of the series is denoted by s_n

i.e., $s_n = u_1 + u_2 + u_3 + ... + u_n ; s_1, s_2, s_3, ...s_n$ are called *partial sums.*

Convergent, Divergent and Oscillatory Series

Let Σu_n be an infinite series. As $n \to \infty,$ there are three possibilities.

(a) **Convergent series:** As $n \to \infty, s_n \to$ a finite limit, say 's' in which case the series is said to be convergent and 's' is called its sum to infinity.

Thus $\underset{n \to \infty}{Lt}\ s_n = s$ (or) simply $Lt s_n = s$

This is also written as $u_1 + u_2 + u_3 + ... + u_n + ...to\infty = s.$ (or) $\sum_{n=1}^{\infty} u_n = s$ (or) simply $\Sigma u_n = s.$

(b) **Divergent series:** If $s_n \to \infty$ or $-\infty$, the series said to be divergent.

(c) **Oscillatory Series:** If s_n does not tend to a unique limit either finite or infinite it is said to be an *Oscillatory Series.*

Note: Divergent or Oscillatory series are sometimes called non convergent series.

Geometric Series

The series, $1 + x + x^2 + ...x^{n-1} + ...$ is

(i) Convergent when $|x| < 1$, and its sum is $\frac{1}{1-x}$

(ii) Divergent when $x \geq 1$.

(iii) Oscillates finitely when x = -1 and oscillates infinitely when *x* < -1.

Proof: The given series is a geometric series with common ratio '*x*'

$\therefore\ s_n = \frac{1-x^n}{1-x}$ when $x \neq 1$ [By actual division – verify]

(i) When $|x| < 1$:

$$\underset{n\to\infty}{Lt}\ s_n = \underset{n\to\infty}{Lt}\left(\frac{1}{1-x}\right) - \underset{n\to\infty}{Lt}\left(\frac{x^n}{1-x}\right) = \frac{1}{1-x} \qquad \left[\text{since } x^n \to 0 \text{ as } n \to \infty\right]$$

$\therefore$ The series converges to $\dfrac{1}{1-x}$

(ii) When $x \geq 1 : s_n = \dfrac{x^n - 1}{x-1}$ and $s_n \to \infty$ as $n \to \infty$

$\therefore$ The series is divergent.

(iii) When $x = -1$: when n is even, $s_n \to 0$ and when n is odd, $s_n \to 1$

$\therefore$ The series oscillates finitely.

(iv) When $x < -1, s_n \to \infty$ or $-\infty$ according as n is odd or even.

$\therefore$ The series oscillates infinitely.

Some Elementary Properties of Infinite Series

1. The convergence or divergence of an infinites series is unaltered by an addition or deletion of a finite number of terms from it.

2. If some or all the terms of a convergent series of positive terms change their signs, the series will still be convergent.

3. Let Σu_n converge to 's'

Let 'k' be a non – zero fixed number. Then $\Sigma k u_n$ converges to ks.

Also, if Σu_n diverges or oscillates, so does $\Sigma k u_n$

4. Let Σu_n converge to 'l' and Σv_n converge to 'm'. Then

(i) $\Sigma(u_n + v_n)$ converges to ($l + m$) and **(ii)** $\Sigma(u_n + v_n)$ converges to ($l - m$)

Series of Positive Terms

Consider the series in which all terms beginning from a particular term are +ve.

Let the first term from which all terms are +ve be u_1.

Let Σu_n be such a convergent series of +ve terms. Then, we observe that the convergence is unaltered by any rearrangement of the terms of the series.

Theorem

If Σu_n is convergent, then $\underset{n\to\infty}{Lt}\ u_n = 0$.

Proof : $s_n = u_1 + u_2 + \ldots\ldots + u_n$

$s_{n-1} = u_1 + u_2 + \ldots\ldots + u_{n-1,}$, so that, $u_n = s_n - s_{n-1}$

Suppose $\Sigma u_n = l$ then $\underset{n\to\infty}{Lt}\, s_n = l$ and $\underset{n\to\infty}{Lt}\, s_{n-1} = l$

$\therefore \quad \underset{n\to\infty}{Lt}\, u_n = \underset{n\to\infty}{Lt}\left(s_n - s_{n-1}\right) \;;\; \underset{n\to\infty}{Lt}\, s_n - \underset{n\to\infty}{Lt}\, s_{n-1} = l - l = 0$

Note: The converse of the above theorem need not be always true. This can be observed from the following examples.

(i) Consider the series, $1+\frac{1}{2}+\frac{1}{3}+.......+\frac{1}{n}+....$; $u_n = \frac{1}{n},\ \underset{n\to\infty}{Lt}\, u_n = 0$

But from p-series test (1.3.1) it is clear that $\Sigma\frac{1}{n}$ is divergent.

(ii) Consider the series, $\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+.....+\frac{1}{n^2}+......$

$u_n = \frac{1}{n^2},\ \underset{n\to\infty}{Lt}\, u_n = 0,$ by p series test, clearly $\Sigma\frac{1}{n^2}$ converges,

Note : If $\underset{n\to\infty}{Lt}\, u_n \neq 0$ the series is divergent;

Ex. $u_n = \frac{2^n-1}{2^n}$, here $\underset{n\to\infty}{Lt}\, u_n = 1 \quad \therefore \quad \Sigma u_n$ is divergent.

Tests for the Convergence of an Infinite Series

In order to study the nature of any given infinite series of +ve terms regarding convergence or otherwise, a few tests are given below.

P-Series Test

The infinite series, $\sum\limits_{n=1}^{\infty}\frac{1}{n^p} = \frac{1}{1^p}+\frac{1}{2^p}+\frac{1}{3^p}+......,$ is

(i) Convergent when $p > 1$, and **(ii)** Divergent when $p \leq 1$. (JNTU 2002, 2003)

Proof :

***Case* (i)** Let $p > 1$; $p > 1, 3^p > 2^p$; $\Rightarrow \frac{1}{3^p} < \frac{1}{2^p}$

$\therefore \quad \frac{1}{2^p}+\frac{1}{3^p} < \frac{1}{2^p}+\frac{1}{2^p} = \frac{2}{2^p}$

Similarly, $\frac{1}{4^p}+\frac{1}{5^p}+\frac{1}{6^p}+\frac{1}{7^p} < \frac{1}{4^p}+\frac{1}{4^p}+\frac{1}{4^p}+\frac{1}{4^p} = \frac{4}{4^p}$

$\frac{1}{8^p}+\frac{1}{9^p}+....+\frac{1}{16^p} < \frac{8}{8^p}$, and so on.

Adding we get

$$\Sigma\frac{1}{n^p} < 1+\frac{2}{2^p}+\frac{4}{4^p}+\frac{8}{8^p}+....$$

i.e., $$\Sigma\frac{1}{n^p} < 1+\frac{1}{2^{(p-1)}}+\frac{1}{2^{2(p-1)}}+\frac{1}{2^{3(p-1)}}+......$$

The RHS of the above inequality is an infinite geometric series with common ratio $\frac{1}{2^{p-1}} < 1\left(\text{since } p > 1\right)$ The sum of this geometric series is finite.

Hence $\sum\limits_{n=1}^{\infty}\frac{1}{n^p}$ is also finite.

$\therefore$ The given series is convergent.

***Case* (ii)** Let p =1; $\Sigma\frac{1}{n^p} = 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+......$

We have, $\frac{1}{3}+\frac{1}{4} > \frac{1}{4}+\frac{1}{4} = \frac{1}{2}$

$$\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8} > \frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8} = \frac{1}{2}$$

$$\frac{1}{9}+\frac{1}{10}+.......\frac{1}{16} > \frac{1}{16}+\frac{1}{16}+.....\frac{1}{16} = \frac{1}{2} \text{ and so on}$$

$$\therefore \quad \Sigma\frac{1}{n^p} = 1+\left(\frac{1}{2}+\frac{1}{3}\right)+\left(\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}\right)+.....$$

$$\geq 1+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+.....$$

The sum of RHS series is ∞

$$\left(\text{since } s_n = 1+\frac{n-1}{2} = \frac{n+1}{2} \; and \; \underset{n\to\infty}{Lt} s_n = \infty\right)$$

$\therefore$ The sum of the given series is also ∞; $\therefore$ $\sum\limits_{n=1}^{\infty}\frac{1}{n^p}$ (p = 1) diverges.

Case (iii) Let p<1, $\Sigma\frac{1}{n^p} = 1+\frac{1}{2^p}+\frac{1}{3^p}+.....$

Since $p < 1, \frac{1}{2^p} > \frac{1}{2}, \frac{1}{3^p} > \frac{1}{3}......$ and so on

$$\therefore \quad \Sigma\frac{1}{n^p} > 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+.......$$

From the Case (ii), it follows that the series on the RHS of above inequality is divergent.

$\therefore$ $\quad \Sigma \frac{1}{n^p}$ is divergent , when P < 1

Note: *This theorem is often helpful in discussing the nature of a given infinite series.*

Comparison Tests

1. Let Σu_n and Σv_n be two series of +ve terms and let Σv_n be convergent.

Then Σu_n converges,

(a) If $u_n \le v_n, \forall \ n \in N$

(b) or $\frac{u_n}{v_n} \le k \forall n \in N$ where k is > 0 and finite.

(c) or $\frac{u_n}{v_n} \to$ a finite limit > 0

Proof : (a) Let $\Sigma v_n = l$ (finite)

Then, $u_1 + u_2 + ... + u_n + ... \le v_1 + v_2 + ... v_n + ... \le l > 0$

Since l is finite it follows that Σu_n is convergent

(c) $\frac{u_n}{v_n} \le k \Rightarrow u_n \le kv_n, \forall n \in N$, since Σv_n is convergent and k (>0) is finite,

Σkv_n is convergent $\therefore$ Σu_n is convergent.

(d) Since $\underset{n\to\infty}{Lt} \frac{u_n}{v_n}$ is finite, we can find a +ve constant $k, \ni \frac{u_n}{v_n} < k \forall n \in N$

$\therefore$ from (2) , it follows that Σu_n is convergent

2. Let Σu_n and Σv_n be two series of +ve terms and let Σv_n be divergent. Then Σu_n diverges,

* 1. If $u_n \ge v_n, \forall n \in N$

or * 2. If $\frac{u_n}{v_n} \ge k, \forall n \in N$ where k is finite and $\ne 0$

or * 3. If $\underset{n\to\infty}{Lt} \frac{u_n}{v_n}$ is finite and non-zero.

Proof:

1. Let M be a +ve integer however large it may be. Science Σv_n is divergent, a number m can be found such that

$$v_1 + v_2 + ... + v_n > M, \forall n > m$$

$\therefore$ $\quad u_1 + u_2 + ... + u_n > M, \forall n > m \left(u_n \ge v_n\right)$

$\therefore$ $\quad \Sigma u_n$ is divergent

2. $u_1 \geq kv_n \forall n$

Σv_n is divergent $\Rightarrow$ Σkv_n is divergent

$\therefore$ Σu_n is divergent

3. Since $\underset{n\to\infty}{Lt}\dfrac{u_n}{v_n}$ is finite, a + ve constant k can be found such that $\dfrac{u_n}{v_n} > k, \forall n$ (probably except for a finite number of terms)

$\therefore$ From (2), it follows that Σu_n is divergent.

Note :

(a) In (1) and (2), it is sufficient that the conditions with * hold $\forall n > m \in N$

Alternate form of comparison tests : The above two types of comparison tests 2.8.(1) and 2.8.(2) can be clubbed together and stated as follows :

If Σu_n and Σv_n are two series of + ve terms such that $\underset{n\to\infty}{Lt}\dfrac{u_n}{v_n} = k$, where k is non- zero and finite, then Σu_n and Σv_n both converge or both diverge.

(b)
1. The above form of comparison tests is mostly used in solving problems.
2. In order to apply the test in problems, we require a certain series Σv_n whose nature is already known i.e., we must know whether Σv_n is convergent are divergent. For this reason, we call Σv_n as an 'auxiliary series'.
3. In problems, the geometric series (1.2.2.) and the p-series (1.3.1) can be conveniently used as 'auxiliary series'.

Solved Examples

EXAMPLE 1

Test the convergence of the following series:

(a) $\dfrac{3}{1}+\dfrac{4}{8}+\dfrac{5}{27}+\dfrac{6}{64}+.....$ **(b)** $\dfrac{4}{1}+\dfrac{5}{4}+\dfrac{6}{9}+\dfrac{7}{16}+.....$ **(c)** $\sum_{n=1}^{\infty}\left[\left(n^4+1\right)^{1/4}-n\right]$

SOLUTION

(a) **Step 1:** To find "u_n" the n^{th} term of the given series. The numerators 3, 4, 5, 6......of the terms, are in AP.

n^{th} term $t_n = 3+\left(n-1\right).1 = n+2$

Denominators are $1^3, 2^3, 3^3, 4^3.....n^{th}$ term = n^3 ; $\therefore$ $u_n = \dfrac{n+2}{n^3}$

Step 2: To choose the auxiliary series Σv_n. In u_n, the highest degree of n in the numerator is 1 and that of denominator is 3.

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$\therefore$ we take, $\quad v_n = \dfrac{1}{n^{3-1}} = \dfrac{1}{n^2}$

Step 3: $\underset{n\to\infty}{Lt}\dfrac{u_n}{v_n} = \underset{n\to\infty}{Lt}\dfrac{n+2}{n^3}\times n^2 = \underset{n\to\infty}{Lt}\dfrac{n+2}{n} = \underset{n\to\infty}{Lt}\left(1+\dfrac{2}{n}\right) = 1,$ which is non- zero and finite.

Step 4: ***Conclusion:*** $\underset{n\to\infty}{Lt}\dfrac{u_n}{v_n} = 1$

$\therefore$ Σu_n and Σv_n both converge or diverge (by comparison test). But $\Sigma v_n = \Sigma\dfrac{1}{n^2}$ is convergent by p-series test (p = 2 > 1); $\therefore$ Σu_n is convergent.

(b) $\dfrac{4}{1}+\dfrac{5}{4}+\dfrac{6}{9}+\dfrac{7}{16}+.....$

Step 1: 4 , 5, 6, 7, ...in AP , $t_n = 4+(n-1)1 = n+3 \qquad \therefore\ u_n = \dfrac{n+3}{n^2}$

Step 2: Let $\Sigma v_n = \dfrac{1}{n}$ be the auxiliary series

Step 3: $\underset{n\to\infty}{Lt}\dfrac{u_n}{v_n} = \underset{n\to\infty}{Lt}\left(\dfrac{n+3}{n^2}\right)\times n = \underset{n\to\infty}{Lt}\left(1+\dfrac{3}{n}\right) = 1$, which is non-zero and finite.

Step 4: $\therefore$ By comparison test, both Σu_n and Σv_n converge are diverge together.

But $\Sigma v_n = \Sigma\dfrac{1}{n}$ is divergent, by p-series test (p = 1); $\therefore$ Σu_n is divergent.

(c) $\sum\limits_{n=1}^{\infty}\left[\left(n^4+1\right)^{1/4} - n\right] = \left\{n^4\left(1+\dfrac{1}{n^4}\right)\right\}^{\frac{1}{4}} - n = n\left[\left(1+\dfrac{1}{n^4}\right)^{\frac{1}{4}} - 1\right]$

$$= n\left[1+\frac{1}{4n^4}+\frac{\frac{1}{4}\left(\frac{1}{4}-1\right)}{2!}.\frac{1}{n^8}+.....-1\right] = n\left[\frac{1}{4n^4}-\frac{3}{32n^8}+.....\right]$$

$$= \frac{1}{4n^3}-\frac{3}{32n^7}+.... = \frac{1}{n^3}\left[\frac{1}{4}-\frac{3}{32n^4}+.....\right]$$

Here it will be convenient if we take $v_n = \dfrac{1}{n^3}$

$$\underset{n\to\infty}{Lt}\frac{u_n}{v_n}=\underset{n\to\infty}{Lt}\left(\frac{1}{4}-\frac{1}{32n^4}+.....\right)=\frac{1}{4},$$ which is non-zero and finite

$\therefore$ By comparison test, Σu_n and Σv_n both converge or both diverge. But by p-series test $\Sigma v_n=\frac{1}{n^3}$ is convergent. (p = 3 > 1); $\therefore$ Σu_n is convergent.

EXAMPLE 2

If $u_n=\dfrac{\sqrt[3]{3n^2+1}}{\sqrt[4]{2n^3+3n+5}}$ show that Σu_n is divergent

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SOLUTION

As n increases, u_n approximates to

$$\frac{\sqrt[3]{3n^2}}{\sqrt[4]{2n^3}}=\frac{3^{1/3}}{2^{1/4}}\times\frac{n^{2/3}}{n^{3/4}}=\frac{3^{1/3}}{2^{1/4}}\cdot\frac{1}{n^{1/12}}$$

$\therefore$ If we take $v_n=\dfrac{1}{n^{1/12}}$, $\underset{n\to\infty}{Lt}\dfrac{u_n}{v_n}=\dfrac{3^{1/3}}{2^{1/4}}$ which is finite.

[(or) *Hint:* Take $v_n=\dfrac{1}{n^{l_1-l_2}}$, where l_1 and l_2 are indices of 'n' of the largest terms in denominator and nominator respectively of u_n. Here $v_n=\dfrac{1}{n^{\frac{3}{4}-\frac{2}{3}}}=\dfrac{1}{n^{\frac{1}{12}}}$]

By comparison test, Σv_n and Σu_n converge or diverge together. But $\Sigma v_n=\Sigma\dfrac{1}{n^{1/12}}$ is divergent by p – series test (since $p=\dfrac{1}{12}<1$)

$\therefore$ Σu_n is divergent.

EXAMPLE 3

Test for the convergence of the series. $\sqrt{\frac{1}{2}}+\sqrt{\frac{2}{3}}+\sqrt{\frac{3}{4}}+\sqrt{\frac{4}{5}}+......$

SOLUTION

Here, $u_n=\sqrt{\dfrac{n}{n+1}}$; Take $v_n=\dfrac{1}{n^{\frac{1}{2}-\frac{1}{2}}}=\dfrac{1}{n^0}=1$, $\underset{n\to\infty}{Lt}\dfrac{u_n}{v_n}=\underset{n\to\infty}{Lt}\sqrt{\dfrac{1}{1+\dfrac{1}{n}}}=1$ (finite)

Σv_n is divergent by p – series test. ($p = 0 < 1$)

$\therefore$ By comparison test, Σu_n is divergent, (Students are advised to follow the procedure given in ex. 1.2.9(a) and (b) to find "u_n" of the given series.)

EXAMPLE 4

Show that $1+\frac{1}{\lfloor 1}+\frac{1}{\lfloor 2}+.......+\frac{1}{\lfloor n}+.....$ is convergent.

SOLUTION

$$u_n = \frac{1}{\lfloor n} \quad \text{(neglecting } 1^{st} \text{ term)}$$

$$= \frac{1}{1.2.3......n} < \frac{1}{1.2.2.2.....\overline{n-1} times} = \frac{1}{(2^{n-1})}$$

$\therefore$ $$\Sigma u_n < 1+\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+......$$

which is an infinite geometric series with common ratio $\frac{1}{2}<1$

$\therefore$ $\Sigma\frac{1}{2^{n-1}}$ is convergent. (1.2.3(a)). Hence Σu_n is convergent.

EXAMPLE 5

Test for the convergence of the series, $\frac{1}{1.2.3}+\frac{1}{2.3.4}+\frac{1}{3.4.5}+.......$

SOLUTION

$$u_n = \frac{1}{n(n+1)(n+2)}; \qquad \text{Take } v_n = \frac{1}{n^3} \qquad \underset{n\to\infty}{Lt}\frac{u_n}{v_n} = \underset{n\to\infty}{Lt}\frac{n^3}{n^3\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)} = 1 \text{ (finite)}$$

$\therefore$ By comparison test, Σu_n, and Σv_n converge or diverge together. But by p-series test, $\Sigma v_n = \Sigma\frac{1}{n^3}$ is convergent ($p = 3 > 1$); $\therefore$ Σu_n is convergent .

EXAMPLE 6

If $u_n = \sqrt{n^4+1}-\sqrt{n^4-1}$, show that Σu_n is convergent. [JNTU, 2005]

SOLUTION

$$u_n = n^2\left(1+\frac{1}{n^4}\right)^{\frac{1}{2}} - n^2\left(1-\frac{1}{n^4}\right)^{\frac{1}{2}}$$

$$= n^2\left[\left(1+\frac{1}{2n^4}-\frac{1}{8n^8}+\frac{1}{16n^{12}}-....\right)-\left(1-\frac{1}{2n^4}-\frac{1}{8n^8}-\frac{1}{16n^{12}}-.....\right)\right]$$

$$= n^2\left[\frac{1}{n^4}+\frac{1}{8n^{12}}+....\right]=\frac{1}{n^2}\left[1+\frac{1}{8n^{10}}+....\right]$$

Take $v_n = \frac{1}{n^2}$, hence $\underset{n\to\infty}{Lt}\frac{u_n}{v_n}=1$

$\therefore$ By comparison test, Σu_n and Σv_n converge or diverge together. But $\Sigma v_n = \frac{1}{n^2}$ is convergent by p –series test (p = 2 > 1) $\therefore$ Σu_n is convergent.

EXAMPLE 7

Test the series $\frac{1}{1+x}+\frac{1}{2+x}+\frac{1}{3+x}+.....$ for convergence.

SOLUTION

$$u_n = \frac{1}{n+x}; \quad \text{take} \quad v_n = \frac{1}{n}, \quad \text{then} \quad \frac{u_n}{v_n}=\frac{n}{n+x}=\frac{1}{1+\frac{x}{n}}$$

$$\underset{n\to\infty}{Lt}\left(\frac{1}{1+\frac{x}{n}}\right)=1; \Sigma v_n = \Sigma\frac{1}{n}$$ is divergent by p-series test (p =1)

$\therefore$ By comparison test, Σu_n is divergent.

EXAMPLE 8

Show that $\sum_{n=1}^{\infty}\sin\left(\frac{1}{n}\right)$ is divergent.

SOLUTION

$$u_n = \sin\left(\frac{1}{n}\right); \quad \text{take} \quad v_n = \frac{1}{n}$$

$$\underset{n\to\infty}{Lt}\frac{u_n}{v_n}=\underset{n\to\infty}{Lt}\frac{\sin\left(\frac{1}{n}\right)}{\left(\frac{1}{n}\right)}=\underset{t\to 0}{Lt}\frac{\sin t}{t} \text{ (where } t = 1/n) = 1$$

$\therefore$ $\Sigma u_n, \Sigma v_n$ both converge or diverge . But $\Sigma v_n = \Sigma\frac{1}{n}$ is divergent

(p -series test, $p=1$); $\therefore$ Σu_n is divergent.

EXAMPLE 9

Test the series $\Sigma \sin^{-1}\left(\frac{1}{n}\right)$ for convergence.

SOLUTION

$$u_n = \sin^{-1}\frac{1}{n}; \qquad \text{Take} \qquad v_n = \frac{1}{n}$$

$$\underset{n\to\infty}{Lt}\frac{u_n}{v_n} = \underset{n\to\infty}{Lt}\frac{\sin^{-1\left(\frac{1}{n}\right)}}{\left(\frac{1}{n}\right)}; = \underset{\theta\to 0}{Lt}\left(\frac{\theta}{\sin\theta}\right) = 1\left(Taking\,\sin^{-1}\frac{1}{n} = \theta\right)$$

But Σv_n is divergent. Hence Σu_n is divergent.

EXAMPLE 10

Show that the series $1 + \frac{1}{2^2} + \frac{2^2}{3^3} + \frac{3^3}{4^3} + ...$ is divergent.

SOLUTION

Neglecting the first term, the series is $\frac{1}{2^2} + \frac{2^2}{3^3} + \frac{3^3}{4^4} + ...$. Therefore

$$u_n = \frac{n^n}{(n+1)^{n+1}} = \frac{n^n}{(n+1)(n+1)}n = \frac{n^n}{n\left(1+\frac{1}{n}\right).n^n\left(1+\frac{1}{n}\right)^n} = \frac{1}{n\left(1+\frac{1}{n}\right)\left(1+\frac{1}{n}\right)^n};$$

Take $v_n = \frac{1}{n}$

$$\therefore \qquad \underset{n\to\infty}{Lt}\frac{u_n}{v_n} = \underset{n\to\infty}{Lt}\frac{1}{\left(1+\frac{1}{n}\right)\left(1+\frac{1}{n}\right)^n} = \underset{n\to\infty}{Lt}\frac{1}{\left(1+\frac{1}{n}\right)^n.1} = \frac{1}{e}$$

which is finite and $\Sigma v_n = \Sigma\frac{1}{n}$ is divergent by p –series test (p = 1)

$\therefore \quad \Sigma u_n$ is divergent.

EXAMPLE 11

Show that the series $\frac{1}{1.2.3} + \frac{3}{2.3.4} + \frac{5}{3.4.5} + ...\infty$ is convergent. (JNTU 2000)

SOLUTION

$$\frac{1}{1.2.3} + \frac{3}{2.3.4} + \frac{5}{3.4.5} + ...\infty$$

n^{th} term = $u_n = \dfrac{2n-1}{n(n+1)(n+2)} = \dfrac{1}{n^2}\cdot\dfrac{\left(2-\dfrac{1}{n}\right)}{\left(1+\dfrac{1}{n}\right)\left(1+\dfrac{2}{n}\right)}$

Take $v_n = \dfrac{1}{n^2}$

$$\underset{n\to\infty}{Lt}\frac{u_n}{v_n} = \underset{n\to\infty}{Lt}\frac{1}{n^2}\frac{\left(2-\dfrac{1}{n}\right)}{\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)} \div \left(\frac{1}{n^2}\right)$$

$$\underset{n\to\infty}{Lt}\frac{u_n}{v_n} = \frac{2-0}{(1+0)(1+0)} = 2 \text{ which is finite and non-zero}$$

$\therefore$ By comparison test $\sum u_n$ and $\sum v_n$ converge or diverge together

But $\sum v_n = \sum \dfrac{1}{n^2}$ is convergent. $\therefore$ $\sum u_n$ is also convergent.

EXAMPLE 12

Test whether the series $\displaystyle\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}+\sqrt{n+1}}$ is convergent (JNTU 1997, 1999, 2003)

SOLUTION

The given series is $\displaystyle\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}+\sqrt{n+1}}$

$$u_n = \frac{1}{\sqrt{n}+\sqrt{n+1}}$$

$$= \frac{\sqrt{n+1}-\sqrt{n}}{\left(\sqrt{n}+\sqrt{n+1}\right)\left(\sqrt{n+1}-\sqrt{n}\right)} = \sqrt{n+1}-\sqrt{n}$$

$$u_n = \sqrt{n}\left\{\left(1+\frac{1}{n}\right)^{1/2} - 1\right\} = \sqrt{n}\left\{\left(1+\frac{1}{2n}-\frac{1}{8n^2}+.....\right)-1\right\}$$

$$u_{n=}\sqrt{n}\left\{\frac{1}{2n}-\frac{1}{8n^2}+...\right\} = \frac{1}{\sqrt{n}}\left\{\frac{1}{2}-\frac{1}{8n}+....\right\}$$

Take $\quad v_n = \dfrac{1}{\sqrt{n}}$

$$\underset{n\to\infty}{Lt}\frac{u_n}{v_n} = \underset{n\to\infty}{Lt}\frac{1}{\sqrt{n}}\left\{\frac{1}{2}-\frac{2}{8n}+......\right\}\div\left(\frac{1}{\sqrt{n}}\right)=\frac{1}{2}$$

which is finite and non-zero .

Using comparison test $\sum u_n$ and $\sum v_n$ converge or diverge together.

But $\quad \sum v_n = \sum \dfrac{1}{\sqrt{n}}$ is divergent $\left(\text{since } p = 1/2\right)$

$\therefore \quad \sum u_n$ is also divergent.

EXAMPLE 13

Test for convergence $\displaystyle\sum_{n=1}^{\infty}\left[\sqrt[3]{n^3+1}-n\right]$ [JNTU 1996, 2003, 2003]

n^{th} term $\quad u_n = n\left[\left(1+\dfrac{1}{n^3}\right)^{1/3}-1\right] = n\left[1+\dfrac{1}{3n^3}+\dfrac{1/3\left(1/3-1\right)}{1.2}.\dfrac{1}{n^6}+.....-1\right]$

$$= \frac{1}{3n^2}-\frac{1}{9n^5}+......=\frac{1}{n^2}\left(\frac{1}{3}-\frac{1}{9n^3}+......\right) \text{ ; Let } v_n = \frac{1}{n^2}$$

Then $\quad \underset{n\to\infty}{Lt} u_n/v_n = \underset{n\to\infty}{Lt}\left(1/3 - 1/9n^3+....\right) = 1/3 \neq 0$

$\therefore$ By comparison test, $\sum u_n$ and $\sum v_n$ both converge or diverge.

But $\sum v_n$ is convergent by p -series test $\left(\text{since } p = 2 > 1\right)$ $\therefore$ $\sum u_n$ is convergent.

EXAMPLE 14

Show that the series, $\dfrac{2}{1^p}+\dfrac{3}{2^p}+\dfrac{4}{3^p}+.......$ is convergent for p > 2 and divergent for $p \leq 2$

SOLUTION

n^{th} term of the given series = $u_n = \dfrac{n+1}{n^p} = \dfrac{n\left(1+1/n\right)}{n^p} = \dfrac{\left(1+1/n\right)}{n^{p-1}}$

Let us take $v_n = \dfrac{1}{n^{p-1}}$; $\underset{n\to\infty}{Lt} u_n/v_n = 1 \neq 0$;

$\therefore\ \sum u_n$ and $\sum v_n$ both converge or diverge by comparison test.

But $\sum v_n = \sum 1/n^{p-1}$ converges when p -1>1 ; i.e., p >2 and diverges when $p-1 \le 1$ i.e $p \le 2$; Hence the result.

EXAMPLE 15

Test for convergence $\sum_{n=1}^{\infty}\left(\frac{2^n+3}{3^n+1}\right)^{1/2}$ (JNTU 2003)

SOLUTION

$$u_n = \left[\frac{2^n\left(1+3/2^n\right)}{3^n\left(1+1/3^n\right)}\right]^{1/2};\quad \text{Take}\quad v_n = \sqrt{\frac{2^n}{3^n}};\qquad \frac{u_n}{v_n} = \left(\frac{1+3/2^n}{1+1/3^n}\right)^{1/2}$$

$\underset{n\to\infty}{Lt}\frac{u_n}{v_n} = 1 \ne 0$; $\therefore$ By comparison test, $\sum u_n$ and $\sum v_n$ behave the same way.

But $\sum v_n = \sum_{n=1}^{\infty}\left(\frac{2}{3}\right)^{n/2} = \sqrt{\frac{2}{3}} + \frac{2}{3} + \left(\frac{2}{3}\right)^{3/2} + \ldots..,$ which is a geometric series with common ratio $\sqrt{2/3}$ (<1) $\therefore\ \sum v_n$ is convergent. Hence $\sum u_n$ is convergent.

EXAMPLE 16

Test for convergence of the series, $\frac{1}{4.7.10} + \frac{4}{7.10.13} + \frac{9}{10.13.16} + \ldots\ldots$ (JNTU 2003)

SOLUTION

4, 7, 10,..............is an A . P; $\quad t_n = 4 + (n-1)3 = 3n+1$

7, 10, 13,............is an A . P; $\quad t_n = 7 + (n-1)3 = 3n+4$

and 10 , 13 , 16 ,.............is an A. P; $\quad t_n = 10 + (n-1)3 = 3n+7$

$$\therefore\quad u_n = \frac{n^2}{(3n+1)(3n+4)(3n+7)} = \frac{n^2}{3n\left(1+1/3n\right).3n\left(1+4/3n\right).3n\left(1+7/3n\right)}$$

$$= \frac{1}{27n\left(1+1/3n\right)\left(1+4/3n\right)\left(1+7/3n\right)};$$

Taking $v_n = \frac{1}{n}$, we get

$\underset{n\to\infty}{Lt} \frac{u_n}{v_n} = \frac{1}{27} \neq 0$; $\therefore$ By comparison test, both $\sum u_n$ and $\sum v_n$ behave in the same manner. But by p –series test, $\sum v_n$ is divergent, since p = 1. $\therefore$ $\sum u_n$ is divergent .

EXAMPLE 17

Test for convergence $\sum \frac{\sqrt{2n^2 - 5n + 1}}{4n^3 - 7n^2 + 2}$ (JNTU 2003)

SOLUTION

n^{th} term of the given series = $u_n = \frac{\sqrt{2n^2 - 5n + 1}}{4n^3 - 7n^2 + 2}$

Let $v_n = \frac{1}{n^2}$

$$\underset{n\to\infty}{Lt} \frac{u_n}{v_n} = \underset{n\to\infty}{Lt} . \left[\frac{n\sqrt{2 - 5/n + 1/n^2}}{n^3 \left(4 - 7/n + 2/n^3\right)} \times \frac{n^2}{1} \right] = \underset{n\to\infty}{Lt} \left[\frac{\sqrt{2 - 5/n + 1/n^2}}{\left(4 - 7/n + 2/n^3\right)} \right] = \frac{\sqrt{2}}{4} \neq 0$$

$\therefore$ By comparison test, $\sum u_n$ and $\sum v_n$ both converge or diverge.

But $\sum v_n$ is convergent. [p series test – p = 2 > 1] $\therefore$ $\sum u_n$ is convergent .

EXAMPLE 18

Test the series $\sum u_n$, whose n^{th} term is $\frac{1}{\left(4n^2 - i\right)}$

SOLUTION

$$u_n = \frac{1}{\left(4n^2 - i\right)}; \qquad \text{Let } v_n = \frac{1}{n^2}, \qquad \underset{n\to\infty}{Lt} \frac{u_n}{v_n} = \underset{n\to\infty}{Lt} \left[\frac{n^2}{n^2 \left(4 - i/n^2\right)} \right] = \frac{1}{4} \neq 0$$

$\therefore$ $\sum u_n$ and $\sum v_n$ both converge or diverge by comparison test. But $\sum v_n$ is convergent by p –series test $\left(p = 2 > 1\right)$; $\therefore$ $\sum u_n$ is convergent.

***Note*:** Test the series $\sum_{n=1}^{\infty} \frac{1}{4n^2 - 1}$

EXAMPLE 19

If $u_n = \left(\frac{1}{n}\right).\sin\left(\frac{1}{n}\right)$, show that $\sum u_n$ is convergent .

SOLUTION

Let $v_n = \frac{1}{n^2}$, so that $\sum v_n$ is convergent by p –series test .

$$\underset{n\to\infty}{Lt}\left(\frac{u_n}{v_n}\right) = \underset{n\to\infty}{Lt}\frac{\sin\left(1/n\right)}{\left(1/n\right)} = \underset{t\to 0}{Lt}\left(\frac{\sin t}{t}\right)$$

where $t = 1/n$, Thus $\underset{n\to\infty}{Lt}\left(\frac{u_n}{v_n}\right) = 1 \neq 0$

$\therefore$ By comparison test, $\sum u_n$ is convergent.

EXAMPLE 20

Test for convergence $\sum \frac{1}{\sqrt{n}}\tan(1/n)$

SOLUTION

Take $v_n = 1/n^{3/2}$; $\underset{n\to\infty}{Lt}\left[u_n/v_n\right] = 1 \neq 0$ (as in above example)

Hence by comparison test, $\sum u_n$ converges as $\sum v_n$ converges.

EXAMPLE 21

Show that $\sum_{n=1}^{\infty}\sin^2\left(\frac{1}{n}\right)$ is convergent.

SOLUTION

Let $u_n = \sin^2\left(\frac{1}{n}\right)$; Take $v_n = \frac{1}{n^2}$, $\underset{n\to\infty}{Lt}\left(\frac{u_n}{v_n}\right) = \underset{n\to\infty}{Lt}\left[\frac{\sin\left(1/n\right)}{1/n}\right]^2 = \underset{t\to 0}{Lt}\left(\frac{\sin t}{t}\right)^2$

where $t = 1/n$; $\underset{n\to\infty}{Lt}\left(u_n/v_n\right) = 1^2 = 1 \neq 0$

$\therefore$ By comparison test, $\sum u_n$ and $\sum v_n$ behave the same way.

But $\sum v_n$ is convergent by p- series test, since p = 2 > 1; $\therefore$ $\sum u_n$ is convergent.

EXAMPLE 22

Show that $\sum_{n=2}^{\infty} 1\Big/\log\left(n^n\right)$ is divergent.

SOLUTION

$$u_n = 1\Big/n\log n \ ; \ \log 2 < 1 \Rightarrow 2\log 2 < 2 \Rightarrow 1\Big/2\log 2 > 1\Big/2 \ ;$$

Similarly $\quad 1\Big/3\log 3 > 1\Big/3, \ldots\ldots 1\Big/n\log n > 1\Big/n, n \in N$

$\therefore \quad \Sigma\, 1\Big/n\log n > \Sigma\, 1\Big/n$; But $\Sigma\, 1\Big/n$ is divergent by p-series test.

By comparison test, given series is divergent. [If $\sum v_n$ is divergent and $u_n \geq v_n \forall n$ then $\sum u_n$ is divergent.]

(***Note :*** This problem can also be done using Cauchy's integral Test.

EXAMPLE 23

Test the convergence of the series $\sum_{n=1}^{\infty}(c+n)^{-r}(d+n)^{-s}$, where c, d, r, s are all +ve.

SOLUTION

The n^{th} term of the series $= u_n = \dfrac{1}{(c+n)^r(d+n)^s}$.

Let $\quad v_n = \dfrac{1}{n^{r+s}}$ Then $\dfrac{u_n}{v_n} = \dfrac{n^{r+s}}{n^r\left(1+\dfrac{c}{n}\right)^r .n^s\left(1+\dfrac{d}{n}\right)^s} = \dfrac{1}{\left(1+\dfrac{c}{n}\right)^r\left(1+\dfrac{d}{n}\right)^s}$

$\underset{n\to\infty}{Lt}\dfrac{u_n}{v_n} = 1 \neq 0$, $\therefore \sum u_n$ and $\sum v_n$ both converge are diverge, by comparison test.

But by *p*-series test, $\sum v_n$ converges if $(r + s) > 1$ and diverges if $(r + s) \leq 1$

$\therefore \sum u_n$ converges if $(r + s) > 1$ and diverges if $(r + s) \leq 1$.

EXAMPLE 24

Show that $\sum_{1}^{\infty} n^{-\left(1+1/n\right)}$ is divergent.

SOLUTION

$$u_n = n^{-\left(1+1/n\right)} = \frac{1}{n.n^{1/n}} \qquad \text{Take} \qquad v_n = \frac{1}{n} \ ; \ \underset{n\to\infty}{Lt}\frac{u_n}{v_n} = \underset{n\to\infty}{Lt}\frac{1}{n^{1/n}} = 1 \neq 0$$

For let $\underset{n\to\infty}{Lt}\dfrac{1}{n^{1/n}}=y$ say; log y = $\underset{n\to\infty}{Lt}-\dfrac{1}{n}.\log n$ = $-\underset{n\to\infty}{Lt}\dfrac{1/n}{1}=0$

$\therefore$ $y=e^0=1$ ($\left(\dfrac{\infty}{\infty}\right)$ using L Hospitals rule)

By comparison test both $\sum u_n$ and $\sum v_n$ converge or diverge. But *p*-series test, $\sum v_n$ diverges (since p =1); Hence $\sum u_n$ diverges.

EXAMPLE 25

Test for convergence the series $\displaystyle\sum_{n=1}^{\infty}\frac{(n+a)^r}{(n+b)^p\,(n+c)^q}$, *a, b, c , p, q, r,* being +ve.

SOLUTION

$$u_n=\frac{(n+a)^r}{(n+b)^p(n+c)^q}=\frac{n^r\left(1+a/n\right)^r}{n^p\left(1+b/n\right)^p n^q\left(1+c/n\right)^q}=\frac{1}{n^{p+q-r}}.\frac{\left(1+a/n\right)^r}{\left(1+b/n\right)^p\left(1+c/n\right)^q};$$

Take $v_n=\dfrac{1}{n^{p+q-r}}$; $\underset{n\to\infty}{Lt}\dfrac{u_n}{v_n}=1\neq 0$;

Applying comparison tests both $\sum u_n$ and $\sum v_n$ converge or diverge.

But by *p*-series test, $\sum v_n$ converges if $(p+q-r)>1$and diverges if $(p+q-r)\leq 1$.

Hence $\sum u_n$ converges if $(p+q-r)>1$ and diverges if $(p+q-r)\leq 1$.

EXAMPLE 26

Test the convergence of the following series whose n^{th} terms are:

(a) $\dfrac{(3n+4)}{(2n+1)(2n+3)(2n+5)}$; (b) $\tan\dfrac{1}{n}$; (c) $\left(\dfrac{1}{n^2}\right)\left(\dfrac{n+1}{n+3}\right)^n$

(d) $\dfrac{1}{(3^n+5^n)}$; (e) $\dfrac{1}{n.3^n}$

SOLUTION

(a) *Hint :* Take $v_n=\dfrac{1}{n^2};\sum v_n$ is convergent; $\underset{n\to\infty}{Lt}\left(\dfrac{u_n}{v_n}\right)=\dfrac{3}{8}\neq 0$ (Verify)

Apply comparison test:

$\sum u_n$ is convergent [the student is advised to work out this problem fully]

(b) Proceed as in Example 8; $\sum u_n$ is convergent.

(c) *Hint :* Take $v_n = \dfrac{1}{n^2}$; $\underset{n\to\infty}{Lt}\left(\dfrac{u_n}{v_n}\right) = \underset{n\to\infty}{Lt}\dfrac{\left(1+\frac{1}{n}\right)^n}{\left(1+\frac{3}{n}\right)^n} = \dfrac{e}{e^3} = \dfrac{1}{e^2} \neq 0$

$v_n = \dfrac{1}{n^2}$ is convergent (work out completely for yourself)

(d) $u_n = \dfrac{1}{3^n + 5^n} = \dfrac{1}{5^n}.\dfrac{1}{\left[1+\left(\dfrac{3}{5}\right)^n\right]}$; Take $v_n = \dfrac{1}{5^n}$; $\underset{n\to\infty}{Lt}\left(\dfrac{u_n}{v_n}\right) = 1 \neq 0$

$\sum u_n$ and $\sum v_n$ behave the same way. But $\sum v_n$ is convergent since it is a geometric series with common ratio $\dfrac{1}{5} < 1$

$\therefore$ $\sum u_n$ is convergent by comparison test .

(e) $\dfrac{1}{n.3^n} \leq \dfrac{1}{3^n}, \forall n \in N$, since $n.3^n \geq 3^n$;

$\therefore$ $\sum \dfrac{1}{n.3^n} \leq \sum \dfrac{1}{3^n}$ …..(1)

The series on the R.H .S of (1) is convergent since it is geometric series with $r = \dfrac{1}{3} < 1$.

$\therefore$ By comparison test $\sum \dfrac{1}{n.3^n}$ is convergent.

EXAMPLE 27

Test the convergence of the following series.

(a) $1 + \dfrac{1+2}{1^2+2^2} + \dfrac{1+2+3}{1^2+2^2+3^2} + \dfrac{1+2+3+4}{1^2+2^2+3^2+4^2}$ + ...

(b) $1 + \dfrac{1^2+2^2}{1^3+2^3} + \dfrac{1^2+2^2+3^2}{1^3+2^3+3^3} + \dfrac{1^2+2^2+3^2+4^2}{1^3+2^3+3^3+4^3}$ + ...

SOLUTION

(a) $$u_n = \frac{1+2+3+....+n}{1^2+2^2+3^2+.....n^2} = \frac{n\frac{(n+1)}{2}}{n(n+1)\frac{(2n+1)}{6}} = \frac{3}{(2n+1)}$$

Take $v_n = \frac{1}{n}$; $\underset{n\to\infty}{Lt}\frac{u_n}{v_n} = \underset{n\to\infty}{Lt}\left(\frac{3n}{2n+1}\right) = \frac{3}{2} \neq 0$

$\sum u_n$ and $\sum v_n$ behave alike by comparison test.

But $\sum v_n$ is diverges by p-series test. Hence $\sum u_n$ is divergent.

(b) $$u_n = \frac{1^2+2^2+....+n^2}{1^3+2^3+.....n^3} = \frac{n(n+1)\frac{(2n+1)}{6}}{n^2\frac{(n+1)^2}{4}} = \frac{2(2n+1)}{3n(n+1)}$$

Hint : Take $v_n = \frac{1}{n}$ and proceed as in (a) and show that $\sum u_n$ is divergent.

Exercise 1.1

1. Test for convergence the infinite series whose n^{th} term is:

(a) $\dfrac{1}{n-\sqrt{n}}$ **[Ans :** divergent**]**

(b) $\dfrac{\sqrt{n+1}-\sqrt{n}}{n}$ **[Ans :** convergent**]**

(c) $\sqrt{n^2+1}-n$ **[Ans :** divergent**]**

(d) $\dfrac{\sqrt{n}}{n^2-1}$ **[Ans :** convergent**]**

(e) $\sqrt{n^3+1}-\sqrt{n^3}$ **[Ans :** divergent**]**

(f) $\dfrac{1}{\sqrt{n(n+1)}}$ **[Ans :** divergent**]**

(g) $\dfrac{\sqrt{n}}{n^2+1}$ **[Ans :** convergent**]**

(h) $\dfrac{2n^3+5}{4n^5+1}$ **[Ans :** convergent**]**

2. Determine whether the following series are convergent or divergent.

(a) $\frac{1}{1+3^{-1}}+\frac{2}{1+3^{-2}}+\frac{3}{1+3^{-3}}+$ .... **[Ans :** divergent**]**

(b) $\frac{12}{1^3}+\frac{22}{2^3}+\frac{32}{3^3}+........+\frac{2+10n}{n^3}+.....$ .... **[Ans :** convergent**]**

(c) $\frac{1}{\sqrt{1}+\sqrt{2}}+\frac{1}{\sqrt{2}+\sqrt{3}}+\frac{1}{\sqrt{3}+\sqrt{4}}+......$ .... **[Ans :** divergent**]**

(d) $\frac{2}{3^2}+\frac{3}{4^2}+\frac{4}{5^2}+......$ .... **[Ans :** divergent**]**

(e) $\frac{1}{1^2}+\frac{1}{2^3}+\frac{1}{3^4}+$ .... **[Ans :** convergent**]**

(f) $\sum_{n=1}^{\infty}\frac{\sqrt[3]{n^2+1}}{\sqrt[4]{4n^2+2n+3}}$ .... **[Ans :** divergent**]**

(g) $\sum_{1}^{\infty}\left(8^{1/n}-1\right)$ .... **[Ans :** divergent**]**

(h) $\sum_{1}^{\infty}\frac{3n^3+8}{5n^5+9}$ .... **[Ans :** convergent**]**

(i) $\frac{1}{1.3}+\frac{2}{3.5}+\frac{3}{5.7}+$ .... **[Ans :** divergent**]**

D' Alembert's Ratio Test

Let (i) $\sum u_n$ be a series of +ve terms and (ii) $\underset{n\to\infty}{Lt}\frac{u_{n+1}}{u_n}=k\left(\geq 0\right)$

Then the series $\sum u_n$ is (i) convergent if $k < 1$ and (ii) divergent if $k > 1$.

Proof :

***Case* (i)** $\underset{n\to\infty}{Lt}\frac{u_{n+1}}{u_n}=k\left(<1\right)$

From the definition of a limit, it follows that

$\exists m>0$ and $l\left(0<l<1\right)\ni\frac{u_{n+1}}{u_n}<l\forall n\geq m$

i.e., $\quad \frac{u_{m+1}}{u_m} < l \;,\; \frac{u_{m+2}}{u_{m+1}} < l \;, ...$

$$\therefore \quad u_m + u_{m+1} + u_{m+2} + ...\; = u_m \left[1 + \frac{u_{m+1}}{u_m} + \frac{u_{m+2}}{u_m} + ... \right]$$

$$u_m \left[1 + \frac{u_{m+1}}{u_m} + \frac{u_{m+2}}{u_{m+1}} \cdot \frac{u_{m+1}}{u_m} + ... \right]$$

$$< u_m \left(1 + l + l^2 + ... \right) = u_m \cdot \frac{1}{1-l} \left(l < 1 \right)$$

But $u_m \cdot \frac{1}{1-l}$ is a finite quantity $\therefore \sum_{n=m}^{\infty} u_n$ is convergent

By adding a finite number of terms $u_1 + u_2 + ... + u_{m-1}$, the convergence of the series is unaltered. $\sum_{n=m}^{\infty} u_n$ is convergent.

***Case* (ii)** $\underset{n \to \infty}{Lt} \frac{u_{n+1}}{u_n} = k > 1$

There may be some finite number of terms in the beginning which do not satisfy the condition $\frac{u_{n+1}}{u_n} \geq 1$. In such a case we can find a number *'m'* $\ni \frac{u_{n+1}}{u_n} \geq 1, \forall n \geq m$

Omitting the first *'m'* terms, if we write the series as $u_1 + u_2 + u_3 +$ ...., we have

$$\frac{u_2}{u_1} \geq 1, \frac{u_3}{u_2} \geq 1, \frac{u_4}{u_3} \geq 1 \quad \text{.......... and so on}$$

$$\therefore \quad u_1 + u_2 + ... + u_n = u_1 \left(1 + \frac{u_2}{u_1} + \frac{u_3}{u_2} \cdot \frac{u_2}{u_1} + ... \right) \quad \text{(to } n \text{ terms)}$$

$\geq \; u_1$ (1 + 1 + 1.1 +.......to n terms)

$= \; nu_1$

$$\underset{n \to \infty}{Lt} \sum_{n=1}^{n} u_n \geq \underset{n \to \infty}{Lt} n.u_1 \text{ which } \to \infty \; ; \; \therefore \; \sum u_n \text{ is divergent .}$$

***Note*: 1** The ratio test fails when k = 1. As an example, consider the series, $\sum_{n=1}^{\infty} \frac{1}{n^p}$

Here $$\underset{n\to\infty}{Lt} \frac{u_{n+1}}{u_n} = \underset{n\to\infty}{Lt} \left(\frac{n}{n+1}\right)^p = \underset{n\to\infty}{Lt} \left(\frac{1}{1+\frac{1}{n}}\right)^p = 1$$

i.e., k = 1 for all values of p,

But the series is convergent if p > 1 and divergent if $p \le 1$, which shows that when k = 1, the series may converge or diverge and hence the test fails .

***Note:* 2** *Ratio test can also be stated as follows:*

If $\sum u_n$ is series of +ve terms and if $\underset{n\to\infty}{Lt} \frac{u_n}{u_{n+1}} = k$, then $\sum u_n$ is convergent

If k > 1 and divergent if k < 1 (the test fails when k = 1).

Solved Examples

Test for convergence of Series

EXAMPLE 28

(a) $\frac{x}{1.2} + \frac{x^2}{2.3} + \frac{x^3}{3.4} +$ .... (JNTU 2003)

SOLUTION

$$u_n = \frac{x^n}{n(n+1)};\; u_{n+1} = \frac{x^{n+1}}{(n+1)(n+2)};\quad \frac{u_{n+1}}{u_n} = \frac{x^{n+1}}{(n+1)(n+2)}\cdot\frac{n(n+1)}{x^n} = \frac{1}{\left(1+\frac{2}{n}\right)}x\,.$$

Therefore $\underset{n\to\infty}{Lt} \frac{u_{n+1}}{u_n} = x$

$\therefore$ By ratio test $\sum u_n$ is convergent When $|x|$ < 1 and divergent when $|x|$ > 1;

When x = 1, $u_n = \frac{1}{n^2(1+1/n)}$; Take $v_n = \frac{1}{n^2}$; $\underset{n\to\infty}{Lt} \frac{u_n}{v_n} = 1$

$\therefore$ By comparison test $\sum u_n$ is convergent.

Hence $\sum u_n$ is convergent when $|x| \le 1$ and divergent when $|x| > 1$.

(b) $1+3x+5x^2+7x^3+.......$

SOLUTION

$$u_n=(2n-1)x^{n-1};\quad u_{n+1}=(2n+1)x^n;\quad \underset{n\to\infty}{Lt}\frac{u_{n+1}}{u_n}=\underset{n\to\infty}{Lt}\left(\frac{2n+1}{2n-1}\right)x=x$$

$\therefore$ By ratio test $\sum u_n$ is convergent when $|x|<1$ and divergent when $|x|>1$

When $x=1: u_n=2n-1;\ \underset{n\to\infty}{Lt}\ u_n=\infty;\ \therefore\ \sum u_n$ is divergent.

Hence $\sum u_n$ is convergent when $|x|<1$ and divergent when $|x|\geq 1$

(c) $\displaystyle\sum_{n=1}^{\infty}\frac{x^n}{n^2+1}$ ....

SOLUTION

$$u_n=\frac{x^n}{n^2+1};\qquad u_{n+1}=\frac{x^{n+1}}{(n+1)^2+1}.$$

Hence $$\frac{u_{n+1}}{u_n}=\left(\frac{n^2+1}{n^2+2n+2}\right)x,\quad \underset{n\to\infty}{Lt}\frac{u_{n+1}}{u_n}=\underset{n\to\infty}{Lt}\left[\frac{n^2\left(1+\frac{1}{n^2}\right)}{n^2\left(1+\frac{2}{n}+\frac{2}{n^2}\right)}\right](x)=x$$

$\therefore$ By ratio test, $\sum u_n$ is convergent when $|x|<1$ and divergent when $|x|>1$ When $x=1: u_n=\dfrac{1}{n^2+1}$; Take $v_n=\dfrac{1}{n^2}$

$\therefore$ By comparison test, $\sum u_n$ is convergent when $|x|\leq 1$ and divergent when $|x|>1$

EXAMPLE 29

Test the series $\displaystyle\sum_{n\to\infty}^{\infty}\left(\frac{n^2-1}{n^2+1}\right)x^n, x>0$ for convergence.

SOLUTION

$$u_n=\left(\frac{n^2-1}{n^2+1}\right)x^n;u_{n+1}=\left[\frac{(n+1)^2-1}{(n+1)^2+1}\right]x^{n+1}$$

$$\underset{n\to\infty}{Lt}\frac{u_{n+1}}{u_n}=\underset{n\to\infty}{Lt}\left[\left(\frac{n^2+2n}{n^2+2n+2}\right)\frac{\left(n^2+1\right)}{\left(n^2-1\right)}\right].x$$

$$=\underset{n\to\infty}{Lt}\left[\frac{n^4\left(1+2/n\right)\left(1+1/n^2\right)}{n^4\left(1+2/n+2/n^2\right)\left(1-1/n^2\right)}\right]=x$$

$\therefore$ By ratio test, $\sum u_n$ is convergent when $x<1$ and divergent when $x>1$ when $x = 1$,

$$u_n=\frac{n^2-1}{n^2+1}\text{ Take }v_n=\frac{1}{n^0}$$

Applying p-series and comparison test, it can be seen that $\sum u_n$ is divergent when $x = 1$.

$\therefore$ $\sum u_n$ is convergent when $x < 1$ and divergent $x \geq 1$

EXAMPLE 30

Show that the series $1+\frac{2^p}{\lfloor 2}+\frac{3^p}{\lfloor 3}+\frac{4^p}{\lfloor 4}+.....$, is convergent for all values of p.

SOLUTION

$$u_n=\frac{n^p}{\lfloor n};\ u_{n+1}=\frac{\left(n+1\right)^p}{\lfloor n+1}$$

$$\underset{n\to\infty}{Lt}\frac{u_{n+1}}{u_n}=\underset{n\to\infty}{Lt}\left[\frac{\left(n+1\right)^p}{\lfloor n+1}\times\frac{\lfloor n}{n^p}\right]=\underset{n\to\infty}{Lt}\left\{\frac{1}{\left(n+1\right)}\left(\frac{n+1}{n}\right)^p\right\}$$

$$=\underset{n\to\infty}{Lt}\frac{1}{\left(n+1\right)}\times\underset{n\to\infty}{Lt}\left(1+\frac{1}{n}\right)^p=0<1;$$

$\sum u_n$ is convergent for all ‘ p ‘ .

EXAMPLE 31

Test the convergence of the following series

$$\frac{1}{1^p}+\frac{1}{3^p}+\frac{1}{5^p}+\frac{1}{7^p}+\ ....$$

SOLUTION

$$u_n=\frac{1}{\left(2n-1\right)^p};\qquad u_{n+1}=\frac{1}{\left(2n+1\right)^p}$$

$$\frac{u_{n+1}}{u_n} = \frac{(2n-1)^p}{(2n+1)^p} = \frac{2^p.n^p\,(1-1/2n)^p}{2^p\,n^p\,(1+1/2n)^p}\,; \qquad \underset{n\to\infty}{Lt}\,\frac{u_{n+1}}{u_n} = 1$$

$\therefore$ Ratio test fails.

Take $v_n = \frac{1}{n^p}$; $\frac{u_n}{v_n} = \frac{n^p}{(2n-1)^p} = \frac{1}{2^p\left(1-\frac{1}{2n}\right)^p}$; $\underset{n\to\infty}{Lt}\,\frac{u_n}{v_n} = \frac{1}{2^p}$,

which is non – zero and finite

$\therefore$ By comparison test, $\sum u_n$ and $\sum v_n$ both converge or both diverge.

But by p – series test, $\sum v_n = \sum \frac{1}{n^p}$ converges when $p > 1$ and diverges when $p \le 1$

$\therefore$ $\sum u_n$ is convergent if $p > 1$ and divergent if $p \le 1$.

EXAMPLE 32

Test the convergence of the series $\sum_{n=1}^{\infty} \frac{(n+1)x^n}{n^3}; x > 0$

SOLUTION

$$u_n = \frac{(n+1)\,x^n}{n^3}\,; u_{n+1}\,\frac{(n+2)\,x^{n+1}}{(n+1)^3}$$

$$\frac{u_{n+1}}{u_n} = \frac{n+2}{(n+1)^3}.x^{n+1}.\frac{n^3}{(n+1)\,x^n} = \left(\frac{n+2}{n+1}\right)\left(\frac{n}{n+1}\right)^3.x$$

$$\underset{n\to\infty}{Lt}\,\frac{u_{n+1}}{u_n} = \underset{n\to\infty}{Lt}\left(\frac{1+\frac{2}{n}}{1+\frac{1}{n}}\right)\frac{1}{\left(1+\frac{1}{n}\right)^3}.x = x$$

$\therefore$ By ratio test, $\sum u_n$ converges when $x < 1$ and diverges when $x > 1$.

When $x = 1$, $u_n = \frac{n+1}{n^3}$

Take $v_n = \frac{1}{n^2}$; By comparison test $\sum u_n$ is convergent (give proof)

$\therefore$ $\sum u_n$ is convergent if $x \le 1$ and divergent if $x > 1$.

EXAMPLE 33

Test the convergence of the series (JNTU 2002)

$$\text{(i)} \sum_{n-1}^{\infty}\left(\frac{n^2}{2^n}+\frac{1}{n^2}\right) \quad \text{(ii)}\ 1+\frac{2.5.8}{1.5.9}+\frac{2.5.8.11}{1.5.9.13}+\ \ldots \quad \text{(iii)}\ \frac{1}{3}+\frac{1.2}{3.5}+\frac{1.2.3}{3.5.7}+$$

SOLUTION

(i) $\displaystyle\sum_{n-1}^{\infty}\left(\frac{n^2}{2^n}+\frac{1}{n^2}\right) = \sum_{n=1}^{\infty}\frac{n^2}{2^n}+\sum_{n=1}^{\infty}\frac{1}{n^2}$ Let $u_n=\dfrac{n^2}{2^n};v_n=\dfrac{1}{n^2}$

$$u_{n+1}=\frac{(n+1)^2}{2^{n+1}};\frac{u_{n+1}}{u_n}=\frac{(n+1)^2}{2^{n+1}}.\frac{2^n}{n^2} \qquad \underset{n\to\infty}{Lt}\frac{u_{n+1}}{u_n}=\underset{n\to\infty}{Lt}\frac{1}{2}.\left(1+\frac{1}{n}\right)^2=\frac{1}{2}<1$$

$\therefore$ By ratio test $\sum u_n$ is convergent. By p –series test, $\sum v_n$ is convergent.

$\therefore$ The given series $\left(\sum u_n+\sum v_n\right)$ is convergent.

(ii) Neglecting the first term, the series can be taken as, $\dfrac{2.5.8}{1.5.9}+\dfrac{2.5.8.11}{1.5.9.13}+$

Here, 1st term has 3 fractions ,2nd term has 4 fractions and so on .

$\therefore$ n^{th} term contains (n + 2) fractions

2. 5. 8.......are in A. P.

$\therefore$ $(n+2)^{th}$ term = 2 + (n + 1) 3 = 3n + 5 ;

$\therefore$ 1. 5. 9,.......are in A. P.

$\therefore$ $(n+2)^{th}$ term = 1 + (n + 1) 4 = 4n + 5

$$\therefore \quad u_n=\frac{2.5.8.....(3n+5)}{1.5.9.....(4n+5)}$$

$$u_{n+1}=\frac{2.5.8.....(3n+5)(3n+8)}{1.5.9.....(4n+5)(4n+9)}$$

$$\frac{u_{n+1}}{u_n}=\frac{(3n+8)}{(4n+9)}\ ; \qquad \underset{n\to\infty}{Lt}\frac{u_{n+1}}{u_n}=\underset{n\to\infty}{Lt}\frac{n\left(3+\frac{8}{n}\right)}{n\left(4+\frac{9}{n}\right)}=\frac{3}{4}<1$$

$\therefore$ By ratio test, $\sum u_n$ is convergent.

(iii) 1, 2, 3, ... are in A. P n^{th} term = n ; 3. 5. 7..........are in A.P. n^{th} term = $2n + 1$

$$\therefore \quad u_n = \left[\frac{1.2.3.....n}{3.5.7.....(2n+1)}\right]$$

$$u_{n+1} = \left[\frac{1.2.3.....n(n+1)}{3.5.7.....(2n+1)(2n+3)}\right]$$

$$\frac{u_{n+1}}{u_n} = \left(\frac{n+1}{2n+3}\right)$$

$$\underset{n\to\infty}{Lt}\frac{u_{n+1}}{u_n} = \underset{n\to\infty}{Lt}\frac{n.\left(1+\frac{1}{n}\right)}{n\left(2+\frac{3}{n}\right)} = \frac{1}{2} < 1$$

$\therefore$ By ratio test, $\sum u_n$ is convergent.

EXAMPLE 34

Test for convergence $\sum_{n=1}^{\infty}\frac{1.3.5....(2n-1)}{2.4.6....2n}.x^{n-1}\,(x>0)$ (JNTU 2001)

SOLUTION

The given series of +ve terms has $u_n = \frac{1.3.5....(2n-1)}{2.4.6....2n}.x^{n-1}$

and $u_{n+1} = \frac{1.3.5....(2n+1)}{2.4.6....(2n+2)}x^n$

$$\underset{n\to\infty}{Lt}\frac{u_{n+1}}{u_n} = \underset{n\to\infty}{Lt}\left(\frac{2n+1}{2n+2}\right)x = \underset{n\to\infty}{Lt}\frac{2n\left(1+\frac{1}{2n}\right)}{2n\left(1+\frac{2}{2n}\right)}.x = x$$

$\therefore$ By ratio test, $\sum u_n$ is converges when $x < 1$ and diverges when $x > 1$ when $x = 1$, the test fails.

Then $u_n = \frac{1.3.5....(2n-1)}{2.4.6.....2n} < 1$ and $\underset{n\to\infty}{Lt}u_n \neq 0$

$\therefore$ $\sum u_n$ is divergent. Hence $\sum u_n$ is convergent when $x < 1$, and divergent when $x \geq 1$

EXAMPLE 35

Test for the convergence of $1+\frac{2}{5}x+\frac{6}{9}x^2+........+\left(\frac{2^n-2}{2^n+1}\right)x^{n-1}+.....(x>0)$

(JNTU 2003)

SOLUTION

Omitting 1st term, $u_n=\left(\frac{2^n-2}{2^n+1}\right)x^{n-1},(n\geq 2)$ and 'u_n' are all +ve.

$$u_{n+1}=\frac{\left(2^{n+1}-2\right)}{\left(2^{n+1}+1\right)}x^n\ ;\ \underset{n\to\infty}{Lt}\left(\frac{u_{n+1}}{u_n}\right)=\underset{n\to\infty}{Lt}.\left(\frac{2^{n+1}-2}{2^{n+1}+1}\right)\times\left(\frac{2^n+1}{2^n-2}\right).x$$

$$=\underset{n\to\infty}{Lt}\left[\frac{2^{n+1}\left(1-\frac{1}{2^n}\right)}{2^{n+1}\left(1+\frac{1}{2^{n+1}}\right)}.\frac{2^n\left(1+\frac{1}{2^n}\right)}{2^n\left(1-\frac{2}{2^n}\right)}.x\right]=x\ ;$$

Hence, by ratio test, $\sum u_n$ converges if $x<1$ and diverges if $x>1$.

When $x=1$, the test fails. Then $u_n=\frac{2^n-2}{2^n+1}$; $\underset{n\to\infty}{Lt}\ u_n=1\neq 0$; $\therefore\ \sum u_n$ diverges

Hence $\sum u_n$ is convergent when $x<1$ and divergent $x>1$

EXAMPLE 36

Using ratio test show that the series $\sum_{n=0}^{\infty}\frac{(3-4i)^n}{n!}$ converges (JNTU 2000)

SOLUTION

$$u_n=\frac{(3-4i)^n}{n!}\ ;\quad u_{n+1}=\frac{(3-4i)^{n+1}}{(n+1)!}\ ;\ \underset{n\to\infty}{Lt}\left(\frac{u_{n+1}}{u_n}\right)=\underset{n\to\infty}{Lt}\left(\frac{3-4i}{n+1}\right)=0<1$$

Hence, by ratio test, $\sum u_n$ converges.

EXAMPLE 37

Discuss the nature of the series, $\frac{2}{3.4}x+\frac{3}{4.5}x^2+\frac{4}{5.6}x^3+........\infty\,(x>0)$ (JNTU 2003)

SOLUTION

Since $x>0$, the series is of +ve terms ;

$$u_n = \frac{(n+1)}{(n+2)(n+3)} x^n > u_{n+1} = \frac{(n+2)}{(n+3)(n+4)} x^{n+1}$$

$$\underset{n\to\infty}{Lt} \frac{u_{n+1}}{u_n} = \left[\frac{(n+2)^2 .x}{(n+1)(n+4)}\right] = \underset{n\to\infty}{Lt}\left[\frac{n^2\left(1+\frac{2}{n}\right)^2 .x}{n^2\left(1+\frac{5}{n}+\frac{4}{n^2}\right)}\right] = x;$$

Therefore by ratio test, $\sum u_n$ converges if $x < 1$ and diverges if $x > 1$

When $x = 1$, the test fails; Then $u_n = \frac{(n+1)}{(n+2)(n+3)}$;

Taking $v_n = \frac{1}{n}$; $\underset{n\to\infty}{Lt} = \frac{u_n}{v_n} = 1 \neq 0$

$\therefore$ By comparison test $\sum u_n$ and $\sum v_n$ behave same way. But $\sum v_n$ is divergent by p-series test. ($p = 1$);

$\therefore$ $\sum u_n$ is diverges when $x = 1$

$\therefore$ $\sum u_n$ is convergent when $x < 1$ and divergent when $x \geq 1$

EXAMPLE 38

Discuss the nature of the series $\sum \frac{3.6.9.....3n.5^n}{4.7.10.....(3n+1)(3n+2)}$ (JNTU 2003)

SOLUTION

Here, $u_n = \frac{3.6.9.....3n}{4.7.10.....(3n+1)} \frac{5^n}{(3n+2)}$;

$$u_{n+1} = \frac{3.6.9.....3n(3n+3)5^{n+1}}{4.7.10.....(3n+1)(3n+4)(3n+5)};$$

$$\underset{n\to\infty}{Lt} \frac{u_{n+1}}{u_n} = \underset{n\to\infty}{Lt} \frac{(3n+2)(3n+3).5}{(3n+4)(3n+5)}$$

$$= \underset{n\to\infty}{Lt}\left[\frac{5.9n^2\left(1+\frac{2}{3n}\right)\left(1+\frac{3}{3n}\right)}{9n^2\left(1+\frac{4}{3n}\right)\left(1+\frac{5}{3n}\right)}\right] = 5 > 1$$

$\therefore$ By ratio test, $\sum u_n$ is divergent.

EXAMPLE 39

Test for convergence the series $\sum_{n=1}^{\infty} n^{1-n}$

SOLUTION

$$u_n = n^{1-n};\ u_{n+1} = (n+1)^{-n};$$

$$\frac{u_{n+1}}{u_n} = \frac{(n+1)^{-n}}{n^{1-n}} = \frac{n^n}{n(n+1)^n} = \frac{1}{n}\left(\frac{n}{n+1}\right)^n$$

$$\underset{n\to\infty}{Lt}\frac{u_{n+1}}{u_n} = \underset{n\to\infty}{Lt}\frac{1}{n}.\left(\frac{1}{1+1/n}\right)^n = 0.\frac{1}{e} = 0 < 1$$

$\therefore$ By ratio test $\sum u_n$, is convergent

EXAMPLE 40

Test the series $\sum_{n=1}^{\infty}\frac{2n^3}{\lfloor n}$, for convergence.

SOLUTION

$$u_n = \frac{2n^3}{\lfloor n};\ u_{n+1} = \frac{2(n+1)^3}{\lfloor n+1}$$

$$\frac{u_{n+1}}{u_n} = \frac{2(n+1)^3}{\lfloor n+1} \times \frac{\lfloor n}{2n^3} = \frac{(n+1)^2}{n^3} = \frac{\left(1+1/n\right)^2}{n};$$

$$\underset{n\to\infty}{Lt}\frac{u_{n+1}}{u_n} = 0 < 1;$$

$\therefore$ By ratio test, $\sum u_n$ is convergent.

EXAMPLE 41

Test convergence of the series $\sum \frac{2^n n!}{n^n}$

SOLUTION

$$u_n = \frac{2^n n!}{n^n};\ u_{n+1} = \frac{2^{n+1}(n+1)!}{(n+1)^{n+1}};$$

$$\frac{u_{n+1}}{u_n} = \frac{2^{n+1}(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{2^n n!} = 2\left(\frac{n}{n+1}\right)^n$$

$$\underset{n\to\infty}{Lt} \frac{u_{n+1}}{u_n} = 2 \underset{n\to\infty}{Lt} \frac{1}{\left(1+ \frac{1}{n}\right)^n} = \frac{2}{e} < 1 \quad \text{(since } 2 < e < 3\text{)}$$

$\therefore$ By ratio test, $\sum u_n$ is convergent.

EXAMPLE 42

Test the convergence of the series $\sum u_n$ where u_n is

(a) $\dfrac{n^2+1}{3^n+1}$ (b) $\dfrac{x^{n-1}}{(2n+1)^a}, (a > 0)$ (c) $\left(\dfrac{1.2.3....n}{4.7.10.....3n+3}\right)^2$

(d) $\dfrac{\sqrt{1+2^n}}{\sqrt{1+3^n}}$ (e) $\left(\dfrac{3n^3+7n^2}{5n^9+11}\right)x^n$

SOLUTION

(a) $$\underset{n\to\infty}{Lt}\left(\frac{u_{n+1}}{u_n}\right) = \underset{n\to\infty}{Lt}\left[\frac{(n+1)^2+1}{3^{n+1}+1} \times \frac{3^n+1}{n^2+1}\right]$$

$$= \underset{n\to\infty}{Lt}\left[\frac{n^2\left(1+ \frac{2}{n} + \frac{2}{n^2}\right)}{n^2\left(1+ \frac{1}{n^2}\right)} \cdot \frac{3^n\left(1+ \frac{1}{3^n}\right)}{3^{n+1}\left(1+ \frac{1}{3^{n+1}}\right)}\right]$$

$$= \frac{1}{3} < 1$$

$\therefore$ By ratio test, $\sum u_n$ is convergent.

(b) $$\underset{n\to\infty}{Lt}\left(\frac{u_{n+1}}{u_n}\right) = \underset{n\to\infty}{Lt}\left[\frac{x^n}{(2n+3)^a} \times \frac{(2n+1)^a}{x^{n-1}}\right]$$

$$= \underset{n\to\infty}{Lt}\left[\frac{2^a n^a\left(1+ \frac{1}{2n}\right)^a}{2^a n^a\left(1+ \frac{3}{2n}\right)^a} .x\right] = x$$

By ratio test, $\sum u_n$ convergence if $x < 1$ and diverges if $x > 1$.

When x = 1, the test fails; Then, $u_n = \dfrac{1}{(2n+1)^a}$; Taking $v_n = \dfrac{1}{n^a}$ we have,

$$\underset{n\to\infty}{Lt}\left(\frac{u_n}{v_n}\right) = \underset{n\to\infty}{Lt}\left(\frac{n}{2n+1}\right)^a = \underset{n\to\infty}{Lt}\frac{1}{\left(2+\frac{1}{n}\right)^a} = \frac{1}{2^a} \neq 0 \text{ and finite } (\text{since } a>0).$$

$\therefore$ By comparison test, $\sum u_n$ and $\sum v_n$ have same property

But p –series test, we have

(i) $\sum v_n$ convergent when $a > 1$

and (ii) divergent when $a \leq 1$

$\therefore$ To sum up, (i) $x < 1$, $\sum u_n$ is convergent $\forall a$.

(ii) $x > 1$, $\sum u_n$ is divergent $\forall a$.

(iii) $x = 1$, $a > 1$, $\sum u_n$ is convergent, and

(iv) $x = 1$, $a \leq 1$, $\sum u_n$ is divergent.

(c) $$\underset{n\to\infty}{Lt}\frac{u_{n+1}}{u_n} = \underset{n\to\infty}{Lt}\left[\frac{1.2.3....n(n+1)}{4.7.10....(3n+3)(3n+6)}\times\frac{4.7.10....(3n+3)}{1.2.3....n}\right]^2$$

$$= \underset{n\to\infty}{Lt}\left[\frac{(n+1)}{3(n+2)}\right]^2 = \frac{1}{9} < 1 \; ;$$

$\therefore$ By ratio test, $\sum u_n$ is convergent

(d) $$\underset{n\to\infty}{Lt}\frac{u_{n+1}}{u_n} = \underset{n\to\infty}{Lt}\left[\frac{(1+2^{n+1})}{(1+3^{n+1})}\times\frac{(1+3^n)}{(1+2^n)}\right]^{1/2}$$

$$= \underset{n\to\infty}{Lt}\left[\frac{2^{n+1}\left(1+\frac{1}{2^{n+1}}\right)}{3^{n+1}\left(1+\frac{1}{3^{n+1}}\right)}\times\frac{3^n\left(1+\frac{1}{3^n}\right)}{2^n\left(1+\frac{1}{2^n}\right)}\right]^{1/2} = \left(\frac{2}{3}\right)^{1/2} < 1$$

$\therefore$ By ratio test, $\sum u_n$ is convergent.

(e) $$\underset{n\to\infty}{Lt}\frac{u_{n+1}}{u_n}=\underset{n\to\infty}{Lt}\left[\frac{3(n+1)^3+7(n+1)^2}{5(n+1)^9+11}\times\frac{5n^9+11}{3n^3+7}\times x\right]$$

$$=\underset{n\to\infty}{Lt}\left[\frac{3n^3\left(1+\frac{1}{n}\right)^3+7n^2\left(1+\frac{1}{n}\right)^2}{5n^9\left(1+\frac{1}{n}\right)^9+11}\times\frac{5n^9\left(1+\frac{11}{5n^9}\right)}{3n^3\left(1+\frac{7}{3n^3}\right)}\times x\right]$$

$$=\underset{n\to\infty}{Lt}\left[\frac{3n^3\left\{\left(1+\frac{1}{n}\right)^3+\frac{7}{3n}\left(1+\frac{1}{n}\right)^2\right\}}{5n^9\left\{\left(1+\frac{1}{n}\right)^9+\frac{11}{5n^9}\right\}}\times\frac{5n^9\left(1+\frac{11}{5n^9}\right)}{3n^3\left(1+\frac{7}{3n^3}\right)}\times x\right]=x$$

$\therefore$ By ratio test, $\sum u_n$ converges when $x < 1$ and diverges when $x > 1$.

When $x = 1$, the test fails,

Then $$u_n=\frac{3n^3\left(1+\frac{7}{3n}\right)}{5n^9\left(1+\frac{11}{5n^9}\right)}=\frac{3}{5n^6}\frac{\left(1+\frac{7}{3n}\right)}{\left(1+\frac{11}{5n^9}\right)}$$

Taking $v_n=\frac{1}{n^6}$, we observe that $\underset{n\to\infty}{Lt}\frac{u_n}{v_n}=\frac{3}{5}\neq 0$

$\therefore$ By comparison test and p series test, we conclude that $\sum u_n$ is convergent.

$\therefore$ $\sum u_n$ is convergent when $x\leq 1$ and divergent when $x > 1$.

Exercise – 1.2

1. Test the convergency or divergency of the series whose general term is :

(a) $\frac{x^n}{n}$ .... **[Ans :** $|x|<1cgt, |x\geq|1dgt$ **]**

(b) nx^{n-1} .... **[Ans :** $|x|<1cgt, |x\geq|1dgt$ **]**

(c) $\left(\frac{2^n-2}{2^n+1}\right)x^{n-1}$ .... **[Ans :** $|x|<1cgt, |x\geq|1dgt$ **]**

(d) $\left(\frac{n^2+1}{n^2-1}\right)x^n$ .... **[Ans :** $|x|<1cgt, |x\geq|1dgt$ **]**

(e) $\frac{\lfloor n}{n^n}$ .... **[Ans: cgt.]**

(f) $\dfrac{4^n.\lfloor n}{n^n}$ .... **[Ans: dgt.]**

(g) $\dfrac{\left(n^3+1\right)^n}{\left(3^n+1\right)}$ .... **[Ans: cgt.]**

2. Determine whether the following series are convergent or divergent :

(a) $\dfrac{x}{1.2}+\dfrac{x^2}{3.4}+\dfrac{x^3}{5.6}+$ .... **[Ans :** $|x|\le 1cgt, |x|>1dgt$ **]**

(b) $1+\dfrac{x}{2^2}+\dfrac{x^2}{3^2}+\dfrac{x^3}{4^2}+$............. **[Ans :** $|x|\le 1cgt, |x|>1dgt$ **]**

(c) $\dfrac{1}{1.2.3}+\dfrac{x}{4.5.6}+\dfrac{x^2}{7.8.9}+$ ...... **[Ans :** $|x|\le 1cgt, |x|>1dgt$ **]**

(d) $1+\dfrac{x}{2}+\dfrac{x^2}{5}+\dfrac{x^3}{10}+.....\dfrac{x^n}{n^2+1}+....$ **[Ans :** $|x|\le 1cgt, |x|>1dgt$ **]**

(e) $\dfrac{1.2}{x}+\dfrac{2.3}{x^2}+\dfrac{3.4}{x^3}+$ .... **[Ans :** $|x|>1cgt, |x|\le 1dgt$ **]**

Raabe's Test

Let $\sum u_n$ be series of +ve terms and let $\underset{n\to\infty}{Lt}\left\{n\left(\dfrac{u_n}{u_{n+1}}-1\right)\right\}=k$

Then

(i) If $k > 1$, $\sum u_n$ is convergent. **(ii)** If $k < 1$, $\sum u_n$ is divergent. (The test fails if $k = 1$)

Proof : Consider the series $\sum v_n=\sum\dfrac{1}{n^p}$

$$n\left[\frac{v_n}{v_{n+1}}-1\right]=n\left[\left(\frac{n+1}{n}\right)^p-1\right]=n\left[\left(1+\frac{1}{n}\right)^p-1\right]$$

$$=n\left[\left(1+\frac{p}{n}+\frac{p(p-1)}{\lfloor 2}.\frac{1}{n^2}+....\right)-1\right]$$

$$= p+\frac{p(p-1)}{\lfloor 2}.\frac{1}{n}+ \text{ ....} \qquad \underset{n\to\infty}{Lt}\ n\left\{\frac{v_n}{v_{n+1}}-1\right\}=p$$

***Case* (i)** In this case, $\underset{n\to\infty}{Lt}\ n\left\{\frac{u_n}{u_{n+1}}-1\right\}=k>1$

We choose a number 'p' ∋ $k > p > 1$; Comparing the series $\sum u_n$ with $\sum v_n$ which is convergent , we get that $\sum u_n$ will converge if after some fixed number of terms

$$\frac{u_n}{u_{n+1}}>\frac{v_n}{v_{n+1}}=\left(\frac{n+1}{n}\right)^p$$

i.e. If, $n\left(\frac{u_n}{u_{n+1}}-1\right)>p+\frac{p(p-1)}{\angle 2}\cdot\frac{1}{n}+$ ....from (1)

i.e., If $\underset{n\to\infty}{Lt}\ n\left(\frac{u_n}{u_{n+1}}-1\right)>p$

i.e., If $k > p$, which is true . Hence $\sum u_n$ is convergent .The second case also can be proved similarly.

Solved Examples

EXAMPLE 43

Test for convergence the series

$$x+\frac{1}{2}\cdot\frac{x^3}{3}+\frac{1.3}{2.4}\cdot\frac{x^5}{5}+\frac{1.3.5}{2.4.6}\cdot\frac{x^7}{7}+.....$$ (JNTU 2006, 2008)

SOLUTION

Neglecting the first tem ,the series can be taken as ,

$$\frac{1}{2}\cdot\frac{x^3}{3}+\frac{1.3}{2.4}\cdot\frac{x^5}{5}+\frac{1.3.5}{2.4.6}\cdot\frac{x^7}{7}+.....$$

1.3.5....are in A.P. n^{th} term = $1+(n-1)2=2n-1$

2.4.6...are in A.p. n^{th} term = $2+(n-1)2=2n$

3.5.7.....are in A.P n^{th} term = $3+(n-1)2=2n+1$

∴ u_n (n^{th} term of the series) $=\frac{1.3.5....(2n-1)}{2.4.6....(2n)}\cdot\frac{x^{2n+1}}{2n+1}$

$$u_{n+1} = \frac{1.3.5....(2n-1)(2n+1)}{2.4.6....(2n)(2n+2)} \cdot \frac{x^{2n+3}}{2n+3}$$

$$\frac{u_{n+1}}{u_n} = \frac{1.3.5....(2n+1)}{2.4.6....(2n+2)} \cdot \frac{x^{2n+3}}{(2n+3)} \cdot \frac{2.4.6....2n}{1.3.5....(2n-1)} \cdot \frac{(2n+1)}{x^{2n+1}}$$

$$= \frac{(2n+1)^2 x^2}{(2n+2)(2n+3)}$$

$$\therefore \quad \underset{n\to\infty}{Lt} \frac{u_{n+1}}{u_n} = \underset{n\to\infty}{Lt} \frac{4n^2\left(1+\frac{1}{2n}\right)^2}{4n^2\left(1+\frac{2}{2n}\right)\left(1+\frac{3}{2n}\right)} x^2 = x^2$$

$\therefore$ By ratio test, $\sum u_n$ converges if $|x|<1$ and diverges if $|x|>1$

If $|x|=1$ the test fails.

Then $\quad x^2 = 1$ and $\dfrac{u_n}{u_{n+1}} = \dfrac{(2n+2)(2n+3)}{(2n+1)^2}$

$$\frac{u_n}{u_{n+1}} - 1 = \frac{(2n+2)(2n+3)}{(2n+1)^2} - 1 = \frac{6n+5}{(2n+1)^2}$$

$$\underset{n\to\infty}{Lt}\left\{n\left(\frac{u_n}{u_{n+1}} - 1\right)\right\} = \underset{n\to\infty}{Lt}\left(\frac{6n^2+5n}{4n^2+4n+1}\right)$$

$$= \underset{n\to\infty}{Lt} \frac{n^2\left(6+\frac{5}{n}\right)}{n^2\left(4+\frac{4}{n}+\frac{1}{n^2}\right)} = \frac{3}{2} > 1$$

By Raabe's test, $\sum u_n$ converges. Hence the given series is convergent when $|x| \le 1$ an divergent when $|x|>1$.

EXAMPLE 44

Test for the convergence of the series (JNTU 2007)

$$1 + \frac{3}{7}x + \frac{3.6}{7.10}x^2 + \frac{3.6.9}{7.10.13}x^3 + ...; x > 0$$

SOLUTION

Neglecting the first term,

$$u_n = \frac{3.6.9....3n}{7.10.13....3n+4}.x^n$$

$$u_{n+1} = \frac{3.6.9....3n(3n+3)}{7.10.13....(3n+4)(3n+7)}.x^{n+1}$$

$$\frac{u_{n+1}}{u_n} = \frac{3n+3}{3n+7}.x \quad ; \quad \underset{n\to\infty}{Lt}\frac{u_{n+1}}{u_n} = x$$

$\therefore$ By ratio test, $\sum u_n$ is convergent when x < 1 and divergent when x > 1.

When x = 1 The ratio test fails. Then

$$\frac{u_n}{u_{n+1}} = \frac{3n+7}{3n+3}; \frac{u_n}{u_{n+1}} - 1 = \frac{4}{3n+3}$$

$$\underset{n\to\infty}{Lt}\left\{n\left(\frac{u_n}{u_{n+1}} - 1\right)\right\} = \underset{n\to\infty}{Lt}\left(\frac{4n}{3n+3}\right) = \frac{4}{3} > 1$$

$\therefore$ By Raabe's test, $\sum u_n$ is convergent .Hence the given series converges if $x \le 1$ and diverges if x > 1.

EXAMPLE 45

Examine the convergence of the series $\sum_{n=1}^{\infty}\frac{1^2.5^2.9^2....(4n-3)^2}{4^2.8^2.12^2....(4n)^2}$

SOLUTION

$$u_n = \frac{1^2.5^2.9^2....(4n-3)^2}{4^2.8^2.12^2....(4n)^2}; \qquad u_{n+1} = \frac{1^2.5^2.9^2....(4n-3)^2(4n+1)^2}{4^2.8^2.12^2....(4n)^2(4n+4)^2}$$

$$\underset{n\to\infty}{Lt}\frac{u_{n+1}}{u_n} = \underset{n\to\infty}{Lt}\frac{(4n+1)^2}{(4n+4)^2} = 1 \quad \text{(verify)}$$

$\therefore$ The ratio test fails. Hence by Raabe's test, $\sum u_n$ is convergent. (give proof)

EXAMPLE 46

Find the nature of the series $\sum \frac{(\lfloor n)^2}{\lfloor 2n} x^n, (x>0)$ (JNTU 2003)

SOLUTION

$$u_n = \frac{(\lfloor n)^2}{\lfloor 2n}.x^n \; ; \; u_{n+1} = \frac{(\lfloor n+1)^2}{\lfloor 2n+2}.x^{n+1}$$

$$\frac{u_{n+1}}{u_n} = \frac{(n+1)^2}{(2n+1)(2n+2)} x \; ;$$

$$\underset{n\to\infty}{Lt} \frac{u_{n+1}}{u_n} = \underset{n\to\infty}{Lt} \frac{n^2\left(1+\frac{1}{n}\right)^2}{4n^2\left(1+\frac{1}{2n}\right)\left(1+\frac{2}{2n}\right)}.x = \frac{x}{4}$$

$\therefore$ By ratio test, $\sum u_n$ converges when $\frac{x}{4} < 1$, i. e ; $x < 4$; and diverges when $x > 4$;

When $x = 4$, the test fails.

$$\frac{u_n}{u_{n+1}} = \frac{(2n+1)(2n+2)}{4(n+1)^2}$$

$$\frac{u_n}{u_{n+1}} - 1 = \frac{-2n-2}{4(n+1)^2} = \frac{-1}{2(n+1)} \; ; \quad \underset{n\to\infty}{Lt}\left[n\left(\frac{u_n}{u_{n+1}} - 1\right)\right] = \frac{-1}{2} < 1$$

$\therefore$ By ratio test, $\sum u_n$ is divergent

Hence $\sum u_n$ is convergent when $x < 4$ and divergent when $x \geq 4$

EXAMPLE 47

Test for convergence of the series $\sum \frac{4.7....(3n+1)}{1.2.3....n} x^n$ (JNTU 1996)

SOLUTION

$$u_n = \frac{4.7....(3n+1)}{1.2.3....n} x^n \; ; \; u_{n+1} = \frac{4.7....(3n+1)(3n+4)}{1.2.3....n(n+1).} x^{n+1}$$

$$\underset{n\to\infty}{Lt} \frac{u_{n+1}}{u_n} = \underset{n\to\infty}{Lt}\left[\frac{(3n+4)}{(n+1)}.x\right] = 3x$$

$\therefore$ By ratio test $\sum u_n$ converges if $3x<1$ i.e., $x<\frac{1}{3}$ and diverges if $x>\frac{1}{3}$;

If $x=\frac{1}{3}$, the test fails

When $\quad x=\frac{1}{3},\ n\left[\frac{u_n}{u_{n+1}}-1\right]=n\left[\frac{(n+1)3}{3n+4}-1\right] = n\left[\frac{-1}{3n+4}\right]=-\frac{1}{\left(3+\frac{4}{n}\right)}$

$$\underset{n\to\infty}{Lt}\ n\left[\frac{u_n}{u_{n+1}}-1\right]=-\frac{1}{3}<1$$

$\therefore$ By Raabe's test, $\sum u_n$ is divergent.

$\therefore$ $\sum u_n$ is convergent when $x<\frac{1}{3}$ and divergent when $x\geq\frac{1}{3}$

EXAMPLE 48

Test for convergence $2+\frac{3x}{2}+\frac{4x^2}{3}+\frac{5x^3}{4}+..........(x>0)$ (JNTU 2003)

SOLUTION

The n^{th} term $\quad u_n=\frac{(n+1)}{n}x^{n-1};\ u_{n+1}=\frac{(n+2)}{(n+1)}x^n; \quad \frac{u_{n+1}}{u_n}=\frac{n(n+2)}{(n+1)^2}.x$

$$\underset{n\to\infty}{Lt}\frac{u_{n+1}}{u_n}=\underset{n\to\infty}{Lt}\frac{n^2\left(1+\frac{2}{n}\right)}{n^2\left(1+\frac{1}{n}\right)^2}.x=x$$

$\therefore$ By ratio test, $\sum u_n$ is convergent if $x<1$ and divergent if $x>1$

If $x=1$, the test fails.

Then $\quad \underset{n\to\infty}{Lt}\ n\left[\frac{u_n}{u_{n+1}}-1\right]=\underset{n\to\infty}{Lt}\ n\left[\frac{(n+1)^2}{n(n+2)}-1\right] = \underset{n\to\infty}{Lt}\ n\left[\frac{1}{n(n+2)}\right]=0<1$

$\therefore$ By Raabe's test $\sum u_n$ is divergent

$\therefore$ $\sum u_n$ is convergent when $x<1$ and divergent when $x\geq 1$

EXAMPLE 49

Find the nature of the series $\frac{3}{4}+\frac{3.6}{4.7}+\frac{3.6.9}{4.7.10}+......\infty$ (JNTU 2003)

SOLUTION

$$u_n = \frac{3.6.9.....3n}{4.7.10.....(3n+1)}\ ;\ u_{n+1} = \frac{3.6.9.....3n(3n+3)}{4.7.10.....(3n+1)(3n+4)}$$

$$\frac{u_{n+1}}{u_n} = \frac{3n+3}{3n+4};\ \underset{n\to\infty}{Lt}\frac{u_{n+1}}{u_n} = \underset{n\to\infty}{Lt}\frac{3n\left(1+\frac{3}{3n}\right)}{3n\left(1+\frac{4}{3n}\right)} = 1$$

Ratio test fails.

$$\therefore \qquad \underset{n\to\infty}{Lt}\left[n\left\{\frac{u_n}{u_{n+1}}-1\right\}\right] = \underset{n\to\infty}{Lt}\left[n\left(\frac{3n+4}{3n+3}-1\right)\right]$$

$$= \underset{n\to\infty}{Lt}\frac{n}{3(n+1)} = \underset{n\to\infty}{Lt}\frac{n}{3n\left(1+\frac{1}{n}\right)} = \frac{1}{3} < 1$$

$\therefore$ By Raabe's test $\sum u_n$ is divergent.

EXAMPLE 50

If $p, q > 0$ and the series

$$1+\frac{1}{2}\frac{p}{q}+\frac{1.3.p(p+1)}{2.4.q(q+1)}+\frac{1.3.5}{2.4.6}\frac{p(p+1)(p+2)}{q(q+1)(q+2)}+....$$

is convergent , find the relation to be satisfied by p and q.

SOLUTION

$$u_n = \frac{1.3.5.....(2n-1)}{2.4.6.....2n}\frac{p(p+1).....(p+n-1)}{q(q+1).....(q+n-1)} \text{ [neglecting 1}^{\text{st}}\text{ term]}$$

$$u_{n+1} = \frac{1.3.5.....(2n-1)(2n+1)}{2.4.6.....2n(2n+2)}\frac{p(p+1).....(p+n-1)(p+n)}{q(q+1).....(q+n-1)(q+n)}$$

$$\frac{u_{n+1}}{u_n} = \frac{(2n+1)}{(2n+2)}\frac{(p+n)}{(q+n)};$$

$$\underset{n\to\infty}{Lt}\frac{u_{n+1}}{u_n} = \underset{n\to\infty}{Lt}\left[\frac{2n\left(1+\frac{1}{2n}\right)}{2n\left(1+\frac{1}{2n}\right)}.\frac{n\left(1+\frac{p}{n}\right)}{n\left(1+\frac{q}{n}\right)}\right] = 1$$

$\therefore$ ratio test fails.

Let us apply Raabe's test

$$\underset{n\to\infty}{Lt}\left[n\left(\frac{u_n}{u_{n+1}}-1\right)\right]=\underset{n\to\infty}{Lt}\left[n\left\{\frac{(q+n)(2n+2)}{(p+n)(2n+1)}-1\right\}\right]$$

$$\underset{n\to\infty}{Lt}\left[n\left\{\frac{2q(n+1)-p(2n+1)+n}{n^2\left(1+p/n\right)\left(2+1/n\right)}\right\}\right]$$

$$\underset{n\to\infty}{Lt}\left[\frac{2q\left(1+1/n\right)-p\left(2+1/n\right)+1}{2}\right]=\frac{2q-2p+1}{2}$$

Since $\sum u_n$ is convergent, by Raabe's test, $\dfrac{2q-2p+1}{2}>1$

$\Rightarrow q-p>1/2$, is the required relation.

Exercise 1.3

1. Test whether the series $\sum_{1}^{\infty}u_n$ is convergent or divergent where

$$u_n=\frac{2^2.4^2.6^2.....(2n-2)^2}{3.4.5.....(2n-1)(2n)}.x^{2n}$$ **[Ans :** $|x|\le 1 cgt, |x|>1 dgt$ **]**

2. Test for the convergence the series

$$\sum_{1}^{\infty}\frac{4.7.10.....(3n+1)}{\lfloor n}x^n$$ **[Ans :** $|x|<\frac{1}{3}cgt, |x|\ge\frac{1}{3}dgt$ **]**

3. Test for the convergence the series :

(i) $\dfrac{2^2.4^2}{3^2.3^2}+\dfrac{2^2.4^2.5^2.7^2}{3^2.3^2.6^2.6^2}+\dfrac{2^2.4^2.5^2.7^2.8^2.10^2}{3^2.3^2.6^2.6^2.9^2.9^2}+...$ **[Ans** : divergent**]**

(ii) $\dfrac{3.4}{1.2}x+\dfrac{4.5}{2.3}x^2+\dfrac{5.6}{3.4}x^3+.....(x>0)$ **[Ans :** cgt if $x\le 1$ dgt if , $x>1$**]**

(iii) $\sum\dfrac{1.3.5.....(2n-1)}{2.4.6.....2n}.\dfrac{x^n}{(2n+2)}(x>0)$ **[Ans :** cgt if $x\le 1$ dgt if , $x>1$**]**

(iv) $1+\frac{(\lfloor 1)^2}{\lfloor 2}x+\frac{(\lfloor 2)^2 x^2}{\lfloor 4}+\frac{(\lfloor 3)^2 x^3}{\lfloor 6}+......(x>0)$

[Ans : cgt if $x<4$ and dgt if , $x \geq 4$ **]**

1.3.5 Cauchy's Root Test

Let $\sum u_n$ be a series of +ve terms and let $\underset{n\to\infty}{Lt}\, u_n^{1/n}=l$. Then $\sum u_n$ is convergent when l < 1 and divergent when l > 1

Proof **:** (i) $\underset{n\to\infty}{Lt}\, u_n^{1/n}=l<1 \Rightarrow \exists a$ +ve number $'\lambda'(l<\lambda<1) \ni u_n^{1/n}<\lambda, \forall n>m$

(or) $u_n<\lambda^n, \forall n>m$

Since $\lambda<1, \sum \lambda^n$ is a geometric series with common ratio < 1 and therefore convergent.

Hence $\sum u_n$ is convergent.

(ii) $\underset{n\to\infty}{Lt}\, u_n^{1/n}=l>1$

$\therefore$ By the definition of a limit we can find a number $r \ni u_n^{1/n}>1 \forall n>r$

i.e., $u_n > \forall n>r$

i.e., after the 1st 'r' terms , each term is > 1.

$\underset{n\to\infty}{Lt} \sum u_n=\infty \quad \therefore \quad \sum u_n$ is divergent.

Note : When $\underset{n\to\infty}{Lt}\left(u_n^{1/n}\right)=1,$ the root test can't decide the nature of $\sum u_n$. The fact of this statement can be observed by the following two examples.

1. Consider the series $\sum 1/n^3 :- \underset{n\to\infty}{Lt} u_n^{1/n}=\underset{n\to\infty}{Lt}\left(\frac{1}{n^3}\right)^{1/n}=\underset{n\to\infty}{Lt}\left(\frac{1}{n^{1/n}}\right)^3=1$

2. Consider the series $\sum 1/n$, in which $\underset{n\to\infty}{Lt} u_n^{1/n}=\underset{n\to\infty}{Lt}\frac{1}{n^{1/n}}=1$

In both the examples given above**,** $\underset{n\to\infty}{Lt} u_n^{1/n}=1$. But series (1) is convergent (p-series test)

And series (2) is divergent. Hence when the *limit*=1, the test fails.

Solved Examples

EXAMPLE 51

Test for convergence the infinite series whose n^{th} terms are:

(i) $\frac{1}{n^{2n}}$ **(ii)** $\frac{1}{(\log n)^n}$ **(iii)** $\frac{1}{\left[1+\frac{1}{n}\right]^{n^2}}$ (JNTU 1996, 1998, 2001)

SOLUTION

(i) $u_n = \frac{1}{n^{2n}}, u_n^{1/n} = \frac{1}{n^2}$; $\underset{n\to\infty}{Lt}\, u_n^{1/n} = \underset{n\to\infty}{Lt}\, \frac{1}{n^2} = 0 < 1;$

By root test $\sum u_n$ is convergent.

(ii) $u_n = \frac{1}{(\log n)^n}; u_n^{1/n} = \frac{1}{\log n}$; $\underset{n\to\infty}{Lt}\, u_n^{1/n} = \underset{n\to\infty}{Lt}\, \frac{1}{\log n} = 0 < 1;$

$\therefore$ By root test , $\sum u_n$ is convergent.

(iii) $u_n = \frac{1}{\left(1+\frac{1}{n}\right)^{n^2}}; u_n^{1/n} = \frac{1}{\left(1+\frac{1}{n}\right)^n}$ $\underset{n\to\infty}{Lt} u_n^{1/n} = \underset{n\to\infty}{Lt} \frac{1}{\left(1+\frac{1}{n}\right)^n} = \frac{1}{e} < 1;$

$\therefore$ By root test $\sum u_n$ is convergent.

EXAMPLE 52

Find whether the following series are convergent or divergent.

(i) $\sum_{n=1}^{\infty} \frac{1}{3^n - 1}$ (ii) $1 + \frac{1}{2^2} + \frac{1}{3^3} + \frac{1}{4^4} + \ldots..$ (iii) $\sum_{n=1}^{\infty} \frac{\left[(n+1)x\right]^n}{n^{n+1}}$

SOLUTION

(i) $u_n^{1/n} = \left(\frac{1}{3^n - 1}\right)^{1/n} = \left(\frac{1}{3^n\left(1 - \frac{1}{3^n}\right)}\right)^{1/n}$

$$\underset{n\to\infty}{Lt}\, u_n^{1/n} = \underset{n\to\infty}{Lt}\left(\frac{1}{3^n\left(1-\frac{1}{3^n}\right)}\right)^{1/n} = \frac{1}{3} < 1;$$ By root test, $\sum u_n$ is convergent.

(ii) $u_n = \dfrac{1}{n^n}$; $\underset{n\to\infty}{Lt}\, u_n^{1/n} = \underset{n\to\infty}{Lt}\left(\dfrac{1}{n^n}\right)^{1/n} = 0 < 1$; By root test, $\sum u_n$ is convergent.

(iii) $u_n = \dfrac{\left[(n+1)x\right]^n}{n^{n+1}}$

$$\underset{n\to\infty}{Lt}\, u_n^{1/n} = \underset{n\to\infty}{Lt}\left[\frac{\{(n+1)x\}^n}{n^{n+1}}\right]^{1/n}$$

$$\underset{n\to\infty}{Lt}\left[\left\{\frac{(n+1)x}{n}\right\}^n . \frac{1}{n}\right]^{1/n} = \underset{n\to\infty}{Lt}\left(\frac{n+1}{n}\right)x.\frac{1}{n^{1/n}}$$

$$\underset{n\to\infty}{Lt}\left(1+\frac{1}{n}\right)x.\frac{1}{n^{1/n}} = \underset{n\to\infty}{Lt}\, x.\frac{1}{n^{1/n}} = x \qquad \left(\text{since } \underset{n\to\infty}{Lt}\, x.\frac{1}{n^{1/n}} = 1\right)$$

$\therefore$ $\sum u_n$ is convergent if $|x| < 1$ and divergent if $|x| > 1$ and when $|x| = 1$ the test fails.

Then $u_n = \dfrac{(n+1)^n}{n^{n+1}}$; Take $v_n = \dfrac{1}{n}$

$$\frac{u_n}{v_n} = \frac{(n+1)^n}{n^{n+1}}.n = \frac{(n+1)^n}{n^n} = \left(1+\frac{1}{n}\right)^n; \quad \underset{n\to\infty}{Lt}\frac{u_n}{v_n} = e > 1$$

$\therefore$ By comparison test, $\sum u_n$ is divergent.

($\sum v_n$ diverges by p –series test)

Hence $\sum u_n$ is convergent if $|x| < 1$ and divergent $|x| \geq 1$

EXAMPLE 53

If $u_n = \dfrac{n^{n^2}}{(n+1)^{n^2}}$, show that $\sum u_n$ is convergent.

$$\underset{n\to\infty}{Lt}\ u_n^{1/n} = \underset{n\to\infty}{Lt}\left[\frac{n^{n^2}}{(n+1)^{n^2}}\right]^{1/n}\ ; = \underset{n\to\infty}{Lt} = \frac{n^n}{(n+1)^n} = \underset{n\to\infty}{Lt}\left(\frac{n}{n+1}\right)^n$$

$$= \underset{n\to\infty}{Lt}\left(\frac{1}{1+\frac{1}{n}}\right)^n = \frac{1}{e} < 1;\ \therefore\ \sum u_n \text{ converges by root test .}$$

EXAMPLE 54

Establish the convergence of the series $\frac{1}{3}+\left(\frac{2}{5}\right)^2+\left(\frac{3}{7}\right)^3+\ ....$

SOLUTION

$$u_n = \left(\frac{n}{2n+1}\right)^n \text{ ....(verify);} \qquad \underset{n\to\infty}{Lt}\ u_n^{1/n} = \underset{n\to\infty}{Lt}\left(\frac{n}{2n+1}\right) = \frac{1}{2} < 1$$

By root test, $\sum u_n$ is convergent.

EXAMPLE 55

Test for the convergence of $\sum_{n=1}^{\infty}\sqrt{\frac{n}{n+1}}.x^n$

SOLUTION

$$u_n = \left(\frac{1}{1+\frac{1}{n}}\right)^{\frac{1}{2}}.x^n;\ \underset{n\to\infty}{Lt}\ u_n^{1/n} = \underset{n\to\infty}{Lt}\left(\frac{1}{1+\frac{1}{n}}\right)^{\frac{1}{2}}.x = x$$

$\therefore$ By root test, $\sum u_n$ is convergent if $|x|<1$ and divergent if $|x|>1$.

When $|x|=1$: $u_n = \sqrt{\frac{n}{n+1}}$, taking $v_n = \frac{1}{n^0}$ and applying comparison test , it can be

seen that is divergent

$\sum u_n$ is convergent if $|x|<1$ and divergent if $|x|\geq 1$.

EXAMPLE 56

Show that $\sum_{n=1}^{\infty}\left(n^{1/n}-1\right)^n$ converges.

SOLUTION

$$u_n = \left(n^{1/n} - 1\right)^n$$

$$\underset{n\to\infty}{Lt}\, u_n^{1/n} = \underset{n\to\infty}{Lt}\left(n^{1/n} - 1\right) = 1 - 1 = 0 < 1 \left(\text{since } \underset{n\to\infty}{Lt}\, n^{1/n} = 1\right);$$

$\therefore\quad \sum u_n$ is convergent by root test.

EXAMPLE 57

Examine the convergence of the series whose n^{th} term is $\left(\dfrac{n+2}{n+3}\right)^n .x^n$

SOLUTION

$$u_n = \left(\frac{n+2}{n+3}\right)^n .x^n ;\ \underset{n\to\infty}{Lt}\, u_n^{1/n} = \underset{n\to\infty}{Lt}\left(\frac{n+2}{n+3}\right)x = x$$

$\therefore$ By root test, $\sum u_n$ converges when $|x| < 1$ and diverges when $|x| > 1$.

When $|x| = 1$: $\quad u_n = \left(\dfrac{n+2}{n+3}\right)^n ;\ \underset{n\to\infty}{Lt}\, u_n = \underset{n\to\infty}{Lt}\dfrac{\left(1+\frac{2}{n}\right)^n}{\left(1+\frac{3}{n}\right)^n}$

$$= \frac{e^2}{e^3} = \frac{1}{e} \neq 0 \quad \text{and the terms are all +ve .}$$

$\therefore\ \sum u_n$ is divergent . Hence $\sum u_n$ is convergent if $|x| < 1$ and divergent if $|x| \geq 1$.

EXAMPLE 58

Show that the series,

$$\left[\frac{2^2}{1^2} - \frac{2}{1}\right]^{-1} + \left[\frac{3^3}{2^3} - \frac{3}{2}\right]^{-2} + \left[\frac{4^4}{3^4} - \frac{4}{3}\right]^{-3} + \ldots\ldots \text{ is convergent} \qquad \text{(JNTU 2002)}$$

$$u_n = \left[\frac{(n+1)^{n+1}}{n^{n+1}} - \frac{n+1}{n}\right]^{-n} ; = \left(\frac{n+1}{n}\right)^{-n}\left[\left(\frac{n+1}{n}\right)^n - 1\right]^{-n}$$

$$\left(1+\frac{1}{n}\right)^{-n}\left[\left(1+\frac{1}{n}\right)^n - 1\right]^{-n} ;\ u_n^{1/n} = \left(1+\frac{1}{n}\right)^{-1}\left[\left(1+\frac{1}{n}\right)^n - 1\right]^{-1}$$

$$=\frac{1}{\left(1+\frac{1}{n}\right)}\frac{1}{\left\{\left(1+\frac{1}{n}\right)^{n}-1\right\}}$$

$$\therefore \quad \underset{n\to\infty}{Lt}\ u_n^{1/n}=\frac{1}{1}.\frac{1}{e-1}=\frac{1}{e-1}<1$$

$\therefore$ By root test, $\sum u_n$ is convergent.

EXAMPLE 59

Test $\sum_{m=1}^{\infty} u_m$ for convergence when $u_m=\dfrac{e^{-m}}{\left(1+2/m\right)^{-m^2}}$

SOLUTION

$$\underset{m\to\infty}{Lt}\left(u_m^{1/m}\right)=\underset{m\to\infty}{Lt}\left[\frac{\left(1+2/m\right)^{m^2}}{e^m}\right]^{1/m}\ ;\ \underset{m\to\infty}{Lt}\ \frac{1}{e}\left(1+\frac{2}{m}\right)^{m}=\frac{e^2}{e}=e>1$$

Hence Cauchy's root tells us that $\sum u_m$ is divergent.

EXAMPLE 60

Test the convergence of the series $\sum \dfrac{n}{e^{n^2}}$.

SOLUTION

$$\underset{n\to\infty}{Lt}\ u_n^{1/n}=\underset{n\to\infty}{Lt}\ \frac{n^{1/n}}{e^n}=0<1 \qquad \therefore \text{ By root test, } \sum u_n \text{ is convergent.}$$

EXAMPLE 61

Test the convergence of the series, $\dfrac{2}{1^2}x+\dfrac{3^2}{2^3}x^2+\ldots\dfrac{(n+1)^n.x^n}{n^{n+1}}+\ldots\ldots,x>0$

SOLUTION

$$\underset{n\to\infty}{Lt}\ u_n^{1/n}=\underset{n\to\infty}{Lt}\left[\frac{(n+1)^n.x^n}{n^{n+1}}\right]^{1/n}=\underset{n\to\infty}{Lt}\left[\left(\frac{n+1}{n}\right).\frac{1}{n^{1/n}}.x\right]$$

$$= \underset{n\to\infty}{Lt}\left[\left(1+\frac{1}{n}\right).\frac{1}{n^{1/n}}.x\right] = 1.1.x = x\left[\text{since } \underset{n\to\infty}{Lt}\, n^{1/n} = 1\right]$$

$\therefore$ By root test, $\sum u_n$ converges if $x<1$ and diverges when $x>1$.

When $x=1$, the test fails.

Then $\quad u_n = \left(1+\frac{1}{n}\right)^n.\frac{1}{n}$; Take $v_n = \frac{1}{n}$

$$\underset{n\to\infty}{Lt}\frac{u_n}{v_n} = \underset{n\to\infty}{Lt}\left(1+\frac{1}{n}\right)^n = e \neq 0$$

$\therefore$ By comparison test and p-series test, $\sum u_n$ is divergent.

Hence $\sum u_n$ is convergent when $x<1$ and divergent when $x\geq 1$.

Exercise 1.4

1. Test for convergence the infinite series whose n^{th} terms are:

(a) $\frac{1}{2^n-1}$ .... **[Ans :** convergent**]**

(b) $\frac{1}{(\log)^{2n}}.(n\neq 1)$ .... **[Ans :** convergent**]**

(c) $\left(\frac{3n+1}{4n+3}.x\right)^n$ .... **[Ans :** $|x|<\frac{4}{3}cgt, |x|\geq\frac{4}{3}dgt$ **]**

(d) $\frac{x^n}{\lfloor n}$ .... **[Ans :** cgt for all $x\geq 0$ **]**

(e) $\frac{\lfloor n}{n^n}$ .... **[Ans :** convergent**]**

(f) $\frac{3^n.\angle n}{n^3}$ .... **[Ans :** convergent**]**

(g) $\frac{(2n^2-1)^n}{(2n)^{2n}}$ .... **[Ans :** convergent**]**

(h) $\left(n^{1/n}-1\right)^{2n}$ .... **[Ans :** convergent**]**

The Mean Value Theorem

First let's recall one way the derivative reflects the shape of the graph of a function: since the derivative gives the slope of a tangent line to the curve, we know that when the derivative is positive, the function is increasing, and when the derivative is negative, the function is decreasing. When the derivative is zero, it's hard to really say whether the function is increasing or decreasing, but we can at least say that the graph is "flat" near the point where the derivative vanishes.

We can put this to use in the following, relatively unexciting situation: suppose we have a function f which is continuously differentiable on the closed interval $[a, b]$, and suppose that $f(a) = f(b)$. If $f'(c) > 0$ for some point $c \in [a, b)$, then f is increasing at c; in order for us to have $f(a) = f(b)$, f must decrease at some point in the interval (a, b), so we have $f'(d) < 0$ for some $d \in (c, b)$. Then the intermediate value theorem says that f' vanishes somewhere in the interval (c, d). Similarly, if the derivative is negative anywhere in (a, b), we can find a point in the interval (a, b) where the derivative vanishes. So one way or the other, the derivative vanishes somewhere in the interval (a, b). The following theorem formalizes and generalizes[1] what we've demonstrated:

Theorem 1 (Rolle's)**.** If f is a continuous function on the closed interval $[a, b]$ which is differentiable on the interval (a, b) and $f(a) = f(b)$, then the derivative f' vanishes at some point in the interval (a, b).

Note. The fact that f is continuous on a *closed* interval is important. Consider the function

$$f(x) = \begin{cases} x^{-1} - 1, & x \neq 0 \\ 0, & x = 0 \end{cases}.$$

This function is differentiable on the open interval $(0, 1)$ and $f(0) = f(1)$, but the derivative is never zero. Using a closed interval forces f to behave more nicely.

Example. Let $f(x) = x^2 - 4x + 7$. Then f is continuous on the interval $[0, 4]$ and differentiable on $(0, 4)$, and $f(0) = f(4)$. Rolle's theorem guarantees a point $c \in (0, 4)$ so that $f'(c) = 0$, and we can find this c. We have

$$f'(x) = 2x - 4,$$

so if $f'(c) = 0$, then $2c - 4 = 0$. So $f'(2) = 0$.

[1] The generalization is that the theorem only requires the function to be differentiable; our discussion required the function to be *continuously* differentiable.

Rolle's theorem has a nice conclusion, but there are a lot of functions for which it doesn't apply — it requires a function to assume the same value at each end of the interval in question. We can remedy this, though. Suppose f is continuous on $[a, b]$ and differentiable on (a, b). Then we can define a function g which traces out the secant line passing through $(a, f(a))$ and $(b, f(b))$:

$$g(x) := \frac{f(b) - f(a)}{b - a}(x - a) + f(a).$$

Now consider the function h given by $h(x) := f(x) - g(x)$. Then h is continuous on $[a, b]$, differentiable on (a, b), and $h(a) = h(b) = 0$. So Rolle's theorem says there's a point $c \in (a, b)$ so that $h'(c) = 0$. That is,

$$0 = h'(c) = f'(c) - g'(c) = f'(c) - \frac{f(b) - f(a)}{b - a} \qquad \Rightarrow \qquad f'(c) = \frac{f(b) - f(a)}{b - a}.$$

The geometric interpretation is this: given a differentiable function f on the interval $[a, b]$, we can find a tangent line to f between a and b which is parallel to the secant line passing through $(a, f(a))$ and $(b, f(b))$. This is called the Mean Value Theorem.

Theorem 2 (Mean Value)**.** If f is a continuous function on the closed interval $[a, b]$ which is differentiable on the interval (a, b), then there's a point $c \in (a, b)$ so that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Example. Let $f(x) = x^3 - 4x$. Show that there is precisely one $c \in (-2, 1)$ which satisfies the conclusion of the mean value theorem on $[-2, 1]$.

(*Solution*) The mean value theorem says that there is some $c \in (-2, 1)$ so that

$$f'(c) = \frac{f(1) - f(-2)}{1 - (-2)} = \frac{-3}{3} = -1,$$

so we're looking for a $c \in (-2, 1)$ so that

$$-1 = f'(c) = 3c^2 - 4.$$

That is, $3c^2 = 3$, so $c = \pm 1$. We see that c must be -1, since $1 \notin (-2, 1)$.

Example. Use the mean value theorem to show that

$$\sqrt{y} - \sqrt{x} < \frac{y - x}{2\sqrt{x}}$$

whenever $0 < x < y$. (For what it's worth, I don't like this example, but it's of a type that I've seen asked on midterms before.)

(*Solution*) There's an easy way to do this: observe that $(\sqrt{x}-\sqrt{y})^2 > 0$ whenever $0 < x < y$, so

$$\begin{gathered}0 < x - 2\sqrt{x}\sqrt{y} + y \\ 2\sqrt{x}\sqrt{y} - 2x < y - x \\ \sqrt{y} - \sqrt{x} < \frac{y-x}{2\sqrt{x}}.\end{gathered}$$

But we're asked to use the mean value theorem, so let

$$f(x) = \frac{y-x}{2\sqrt{x}} - \sqrt{y} + \sqrt{x}.$$

We want to show that $f(x) > 0$ for all $0 < x < y$. Notice that $f(y) = 0$, and suppose $f(x) \leq 0$ for some $0 < x < y$. Then the mean value theorem says there's a point $c \in (x, y)$ so that

$$f'(c) = \frac{f(y) - f(x)}{x - y} \geq \frac{f(y)}{x-y} = 0.$$

But

$$f'(c) = \frac{2\sqrt{c}(-1) - (y-c)(c^{-1/2})}{2c} + \frac{1}{2\sqrt{c}} = \frac{-\sqrt{c} - c^{-1/2}y}{2c} + \frac{\sqrt{c}}{2c} = \frac{-y}{2\sqrt{c}}.$$

Since $y > 0$, $f'(c) < 0$ for all $x < c < y$, so there's no x with $f(x) \geq 0$.

Extreme Values

Next, we would like to identify the extreme values of continuous functions. The following, important fact will be needed:

Theorem 3 (Extreme Value)**.** If f is a continuous function on $[a, b]$, then there are values m and M so that

$$m \leq f(x) \leq M,$$

for all $x \in [a, b]$.

This theorem guarantees the existence of extreme values; our goal now is to find them. As with the mean value theorem, the fact that our interval is closed is important. The function $f(x) = x^{-1}$ is continuous on the interval $(0, 1)$, but does not assume a maximum value on this interval.

From our discussion at the beginning of the previous section, we see that when $f'(x) > 0$, f is increasing and when $f'(x) < 0$, f is decreasing; in either case, $f(x)$ is *not* an extreme value, because we can move to the left or right and find a larger or greater value. So when f is differentiable, the only interior points of our interval where a function might assume an extreme value are those points where the derivative vanishes. The function could also assume an extreme value at a point where the derivative doesn't exist; we name the points where either of these things happens.

Definition. A number c in the domain of a function f is called a **critical point** of f if either $f'(c) = 0$ or $f'(c)$ does not exist.

Of course, just because c is a critical point doesn't mean that $f(c)$ is an extreme value. Our graph could have multiple "peaks", one higher than another, or it could also be the case that the derivative vanishes without the function taking a local maximum (think of the graph of $f(x) = x^3$ at $x = 0$). So the critical values (the value a function takes at a critical point) become *candidates* for extreme values. We also must consider the endpoints of our closed interval, since these could lead to extreme values without having a vanishing or non-existent derivative.

Theorem 4. Suppose f is continuous on the closed domain $[a, b]$, and let $c \in [a, b]$ be such that $f(c)$ is the minimum or maximum value of f on $[a, b]$. Then c is either a critical point of f or one of the endpoints a or b.

Example. Find the maximum value of $f(x) = x^{1/2}(1-x)^3$ on the closed interval $[0, 1]$.

(*Solution*) We have

$$f'(x) = \frac{1}{2}x^{-1/2}(1-x)^3 + 3x^{1/2}(1-x)^2(-1),$$

by the product and chain rules. Notice that $f'(x)$ is defined on $(0, 1]$, but not defined at 0. Now if $c \in (0, 1)$ is a critical point, then $f'(c) = 0$, so

$$\frac{1}{2}x^{-1/2}(1-x)^3 = 3x^{1/2}(1-x)^2 \qquad \Rightarrow \qquad 1/6(1-c) = c \qquad \Rightarrow \qquad c = \frac{1}{7}.$$

Now we must compare the value of f at c to the value of f at the endpoints:

$$f(0) = 0, \qquad f\left(\frac{1}{7}\right) = \sqrt{\frac{1}{7}}\left(\frac{6}{7}\right)^3, \qquad f(1) = 0.$$

Whatever the value of $f(c)$, it is positive, and is thus the maximum value of f on $[0, 1]$.

In some cases we can actually determine whether a critical point will give a local maximum or a local minimum in a somewhat easier way, using the **first derivative test for critical points**. For this, let c be a critical point of f. Then if $f'(x)$ is positive just to the left of c and negative just to the right of c, $f(c)$ is a local maximum. On the other hand, if $f'(x)$ is negative just to the left of c and positive just to the right of c, then $f(c)$ is a local minimum. If $f'(x)$ does not change signs at $x = c$, then f has neither a local maximum nor a local minimum at $x = c$.

Example. Consider the function $f(x) = \frac{2x+1}{x^2+1}$. Find the critical points of f and the intervals on which f is increasing or decreasing. Use the first derivative test to determine whether the critical point(s) give local minima or maxima.

(*Solution*) First we use the quotient rule to find that

$$f'(x) = \frac{(x^2+1)(2) - (2x+1)(2x)}{(x^2+1)^2} = \frac{-2x^2 - 2x + 2}{(x^2+1)^2} = \frac{-2(x^2+x-1)}{(x^2+1)^2}.$$

Since $(x^2+1)^2 > 0$ for all x, $f'(x)$ exists for all x. So if c is a critical point of f, then $f'(c) = 0$. But this implies that

$$c^2 + c - 1 = 0 \qquad \Rightarrow \qquad c_1 = \frac{-1-\sqrt{5}}{2} \text{ and } c_2 = \frac{-1+\sqrt{5}}{2}.$$

So f has two critical points, one positive, one negative. We can find the intervals on which f is increasing or decreasing by checking the sign of the derivative in the intervals with endpoints at the critical points. In particular, $-2 < c_1$ and $f'(-2) < 0$, so f must have negative derivative at all points less than c_1. That is, f is decreasing on $(-\infty, c_1)$. Similarly, $2 > c_2$ and $f'(2) < 0$, so f must have negative derivative to the right of c_2. So f is decreasing on the interval (c_2, ∞). Finally, $c_1 < 0 < c_2$ and $f'(0) > 0$, so f is increasing on (c_1, c_2). Since f' is negative just to the left of c_1 and positive just to the right of c_1, $f(c_1)$ is a local min. On the other hand, f' is positive just to the left of c_2 and negative just to the right of c_2, so $f(c_2)$ is a local max for f.

Applied Optimization Problems

We'll conclude by applying our discussion of extreme value problems to two realistic scenarios.

Example. A company that makes sprocket sets for bicycles sells these sprocket sets to a bicycle manufacturer in bulk for \$20 per unit. The total cost of producing x units is given by

$$C(x) = 50,000 + 8x + 0.0003x^2,$$

measured in dollars. In a given week, the manufacturer is willing to purchase up to 30,000 sprocket sets at \$20 per unit. How many sprocket sets should the production company make each week in order to maximize their profit?

(*Solution*) We assume that the production company will not make more than 30,000 sprocket sets per week, wanting every set to be bought by the manufacturer. Then the revenue generated by x sprocket sets is

$$R(x) = 20x,$$

where $0 \le x \le 30,000$. So the profit from producing x sprocket sets is

$$P(x) = R(x) - C(x) = 20x - (50,000 + 8x + 0.0003x^2) = 12x - 50,000 - 0.0003x^2;$$

we want to maximize this function on the interval $0 \le x \le 30,000$. First, we find any critical points:

$$P'(x) = 12 - 0.0006x \qquad \Rightarrow \qquad 0 = 12 - 0.0006c \qquad \Rightarrow \qquad c = 20,000.$$

So the maximum profit will occur when the company produces 0, 20,000, or 30,000 units. We check these production levels:

Production level (units):	0	20,000	30,000
Profit (dollars):	-50,000	70,000	40,000

So the company will maximize its profit by producing 20,000 sprocket sets per week.

Example. A marine energy company plans to place a tidal turbine in the ocean, 5km south of a straight shore that runs east-west. The turbine must be connected via a power line to a transformer that is located on the shore, 12km east of the point on the shore that is nearest to the turbine. The cost of installing this power line is $\$30,000$ for each kilometer of line that is in the water and $\$20,000$ for each kilometer along the shore. Assuming the line will follow a straight line to the shore (which is not necessarily perpendicular to the shore) and then follow the shore to the transformer, what is the minimum cost of installing the power line?

(*Solution*) Let x represent the distance between the point where the power line hits the shore and the point on the shore which is closest to the turbine. Then the part of the power line that is in the water is the hypotenuse of a right triangle with legs of length x and 5km, so the length of the line which is in the water is given by $\sqrt{25+x^2}$. The length of the part of the line which runs along the shore is given by $12-x$, so the total cost of the line is given by

$$c = (30)\sqrt{25+x^2} + (20)(12-x),$$

measured in thousands of dollars. Notice that c is defined on the closed interval $[0,12]$. We see that

$$\frac{dc}{dx} = \frac{30(2x)}{2\sqrt{25+x^2}} - 20,$$

so c has a critical point when $\frac{30x}{\sqrt{25+x^2}} = 20$. This is easily solved to give $x = \sqrt{25/1.25} = 2\sqrt{5} \approx 4.47$km. Finally, we notice that

$$c(0) = 390, \qquad c(2\sqrt{5}) = 50\sqrt{5} + 240 \approx 351.803, \qquad \text{and} \qquad c(12) = 390.$$

So the most cost-effective plan for our power line is to reach the shore approximately 4.47km east of the point on the shore which is nearest the turbine, and the installation of the power line will cost approximately $351,803.

CHAPTER 2:

Limits and Continuity

- The conventional approach to calculus is founded on limits.
- In this chapter, we will develop the concept of a limit by example.
- Properties of limits will be established along the way.
- We will use limits to analyze asymptotic behaviors of functions and their graphs.
- Limits will be formally defined near the end of the chapter.
- Continuity of a function (at a point and on an interval) will be defined using limits.

SECTION 2.1: AN INTRODUCTION TO LIMITS

LEARNING OBJECTIVES

- Understand the concept of (and notation for) a limit of a rational function at a point in its domain, and understand that "limits are local."
- Evaluate such limits.
- Distinguish between one-sided (left-hand and right-hand) limits and two-sided limits – and what it means for such limits to exist.
- Use numerical / tabular methods to guess at limit values.
- Distinguish between limit values and function values at a point.
- Understand the use of neighborhoods and punctured neighborhoods in the evaluation of one-sided and two-sided limits.
- Evaluate some limits involving piecewise-defined functions.

PART A: THE LIMIT OF A FUNCTION AT A POINT

Our study of calculus begins with an understanding of the expression $\lim_{x \to a} f(x)$, where a is a real number (in short, $a \in \mathbb{R}$) and f is a function. This is read as:

"the limit of $f(x)$ as x approaches a."

- **WARNING 1:** $\to$ means "approaches." Avoid using this symbol outside the context of limits.

- $\lim_{x \to a}$ is called a limit operator. Here, it is applied to the function f.

$\lim_{x \to a} f(x)$ is the real number that $f(x)$ approaches as x approaches a, **if such a number exists**. If $f(x)$ does, indeed, approach a real number, we denote that number by L (for limit value). We say the limit **exists**, and we write:

$$\lim_{x \to a} f(x) = L, \quad \text{or} \quad f(x) \to L \text{ as } x \to a.$$

These statements will be **rigorously defined** in Section 2.7.

When we **evaluate** $\lim_{x \to a} f(x)$, we do one of the following:

- We find the limit value L (in simplified form).

  We write: $\lim_{x \to a} f(x) = L$.

- We say the limit is ∞ (infinity) or $-\infty$ (negative infinity).

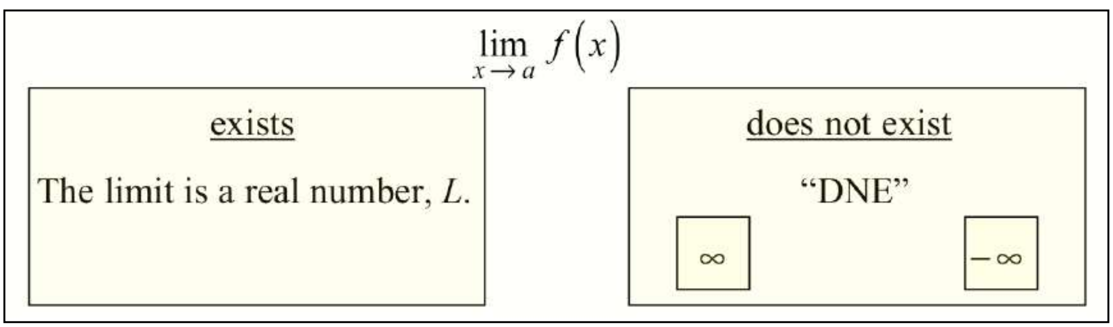
  We write: $\lim_{x \to a} f(x) = \infty$, or $\lim_{x \to a} f(x) = -\infty$.

- We say the limit **does not exist ("DNE")** in some other way.

  We write: $\lim_{x \to a} f(x)$ DNE.

  (The "DNE" notation is used by Swokowski but few other authors.)

If we say the limit is ∞ or $-\infty$, the limit is still **nonexistent**. Think of ∞ and $-\infty$ as "special cases of DNE" that we do write when appropriate; they indicate **why** the limit does not exist.



$\lim_{x \to a} f(x)$ is called a limit at a point, because $x = a$ corresponds to a **point** on the real number line. Sometimes, this is related to a point on the graph of f.

Example 1 (Evaluating the Limit of a Polynomial Function at a Point)

Let $f(x) = 3x^2 + x - 1$. Evaluate $\lim_{x \to 1} f(x)$.

§ Solution

f is a **polynomial** function with implied domain $\text{Dom}(f) = \mathbb{R}$.

We **substitute** ("plug in") $x = 1$ and evaluate $f(1)$.

WARNING 2: Sometimes, the **limit value** $\lim_{x \to a} f(x)$ does not equal the **function value** $f(a)$. (See Part C.)

$\lim_{x \to 1} f(x) = \lim_{x \to 1} \left(3x^2 + x - 1\right)$

WARNING 3: Use **grouping symbols** when taking the limit of an expression consisting of **more than one term**.

$= 3(1)^2 + (1) - 1$

WARNING 4: Do not omit the limit operator $\lim_{x \to 1}$ until this substitution phase.

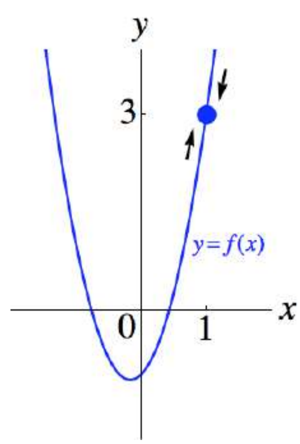
WARNING 5: When performing **substitutions**, be prepared to use **grouping symbols**. Omit them only if you are sure they are unnecessary.

$= 3$

We can write: $\lim_{x \to 1} f(x) = 3$, or $f(x) \to 3$ as $x \to 1$.

• Be prepared to work with function and variable names other than f and x. For example, if $g(t) = 3t^2 + t - 1$, then $\lim_{t \to 1} g(t) = 3$, also.

The graph of $y = f(x)$ is below.



Imagine that the arrows in the figure represent two lovers running towards each other along the parabola. What is the y-coordinate of the point they are approaching as they approach $x = 1$?
It is 3, the limit value.

TIP 1: Remember that ***y*-coordinates** of points along the graph correspond to **function values**. §

Example 2 (Evaluating the Limit of a Rational Function at a Point)

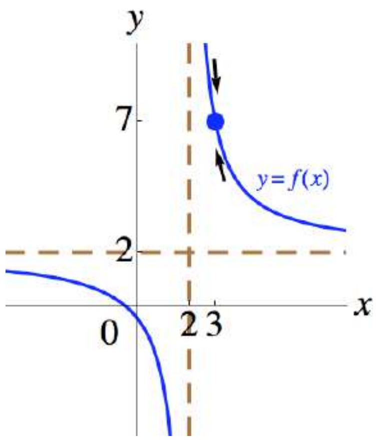
Let $f(x)=\dfrac{2x+1}{x-2}$. Evaluate $\lim\limits_{x\to 3} f(x)$.

§ Solution

f is a **rational** function with implied domain $\mathrm{Dom}(f)=\{x\in\mathbb{R}\mid x\neq 2\}$. We observe that 3 is in the **domain** of f $\left(\text{in short, } 3\in\mathrm{Dom}(f)\right)$, so we **substitute** ("plug in") $x=3$ and evaluate $f(3)$.

$$\begin{aligned}\lim_{x\to 3} f(x) &= \lim_{x\to 3}\frac{2x+1}{x-2}\\ &= \frac{2(3)+1}{(3)-2}\\ &= 7\end{aligned}$$

The graph of $y=f(x)$ is below.



Note: As is often the case, you might not know how to draw the graph until later.

• **Asymptotes.** The dashed lines are asymptotes, which are lines that a graph approaches

- in a "long-run" sense
  (see the horizontal asymptote, or "HA," at $y=2$), or

- in an "explosive" sense
  (see the vertical asymptote, or "VA," at $x=2$).

"HA"s and "VA"s will be defined using limits in Sections 2.3 and 2.4, respectively.

• **“Limits are Local.”** What if the lover on the left is running along the left branch of the graph? In fact, we ignore the left branch, because of the following key principle of limits.

<u>“Limits [at a Point] are Local”</u>

When analyzing $\lim_{x \to a} f(x)$, we only consider the behavior of f in the **“immediate vicinity”** of $x = a$.

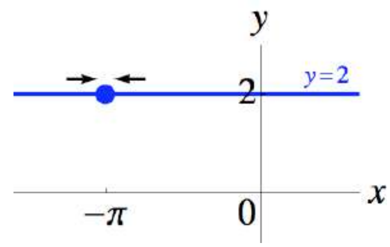
In fact, we may exclude consideration of $x = a$ itself, as we will see in Part C.

In the graph, we only care what happens **“immediately around”** $x = 3$. Section 2.7 will feature a rigorous approach. §

Example 3 (Evaluating the Limit of a Constant Function at a Point)

$$\lim_{x \to -\pi} 2 = 2.$$

(Observe that substituting $x = -\pi$ technically works here, since there is no “*x*” in “2,” anyway.)



• **A constant approaches itself.** We can write $2 \to 2$ (“2 approaches 2”) as $x \to -\pi$. When we think of a sequence of numbers approaching 2, we may think of distinct numbers such as 2.1, 2.01, 2.001, …. However, the **constant sequence** 2, 2, 2, … is also said to approach 2. §

All **constant** functions are also **polynomial** functions, and all **polynomial** functions are also **rational** functions. The following theorem applies to all three Examples thus far.

<u>Basic Limit Theorem for Rational Functions</u>

If f is a rational function, and $a \in \text{Dom}(f)$,

then $\lim_{x \to a} f(x) = f(a)$.

• To evaluate the limit, substitute (“plug in”) $x = a$, and evaluate $f(a)$.

We will justify this theorem in Section 2.2.

PART B: ONE- AND TWO-SIDED LIMITS; EXISTENCE OF LIMITS

$\lim\limits_{x \to a}$ is a **two-sided** limit operator in $\lim\limits_{x \to a} f(x)$, because we must consider the behavior of f as x approaches a from **both** the left **and** the right.

$\lim\limits_{x \to a^-}$ is a **one-sided** left-hand limit operator. $\lim\limits_{x \to a^-} f(x)$ is read as:

"the limit of $f(x)$ as x approaches a **from the left**."

$\lim\limits_{x \to a^+}$ is a **one-sided** right-hand limit operator. $\lim\limits_{x \to a^+} f(x)$ is read as:

"the limit of $f(x)$ as x approaches a **from the right**."

Example 4 (Using a Numerical / Tabular Approach to Guess a Left-Hand Limit Value)

Guess the value of $\lim\limits_{x \to 3^-} (x+3)$ using a **table** of function values.

§ Solution

Let $f(x) = x + 3$. $\lim\limits_{x \to 3^-} f(x)$ is the real number, if any, that $f(x)$ approaches as x approaches 3 from **lesser (or lower) numbers**. That is, we approach $x = 3$ from the **left** along the real number line.

We select an **increasing** sequence of real numbers (x values) approaching 3 such that all the numbers are **close to (but less than) 3**. We evaluate the function at those numbers, and we **guess** the limit value, if any, the function values are approaching. For example:

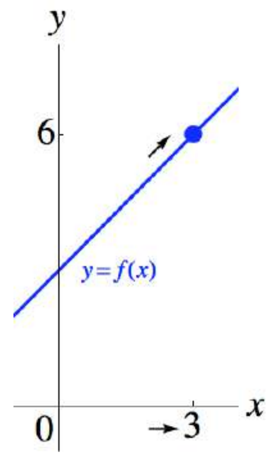
x	2.9	2.99	2.999	$\to 3^-$
$f(x) = x + 3$	5.9	5.99	5.999	$\to 6$ (?)

We guess: $\lim\limits_{x \to 3^-} (x+3) = 6$.

WARNING 6: Do not confuse superscripts with signs of numbers. Be careful about associating the "$-$" superscript with negative numbers. Here, we consider **positive** numbers that are close to 3.

- If we were taking a limit as ***x* approached 0**, then we would associate the "$-$" superscript with **negative** numbers and the "+" superscript with **positive** numbers.

The graph of $y = f(x)$ is below. We only consider the behavior of f **"immediately" to the left of** $x = 3$.



WARNING 7: The numerical / tabular approach is **unreliable**, and it is typically **unacceptable** as a method for evaluating limits on exams. (See Part D, Example 11 to witness a failure of this method.) However, it may help us guess at limit values, and it strengthens our understanding of limits. §

Example 5 (Using a Numerical / Tabular Approach to Guess a Right-Hand Limit Value)

Guess the value of $\lim_{x \to 3^+} (x + 3)$ using a **table** of function values.

§ Solution

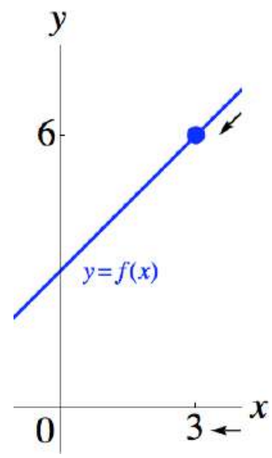
Let $f(x) = x + 3$. $\lim_{x \to 3^+} f(x)$ is the real number, if any, that $f(x)$ approaches as x approaches 3 from **greater (or higher) numbers**. That is, we approach $x = 3$ from the **right** along the real number line.

We select a **decreasing** sequence of real numbers (x values) approaching 3 such that all the numbers are **close to (but greater than) 3**. We evaluate the function at those numbers, and we **guess** the limit value, if any, the function values are approaching. For example:

x	$3^+ \leftarrow$	3.001	3.01	3.1
$f(x) = x + 3$	6 (?) $\leftarrow$	6.001	6.01	6.1

We guess: $\lim_{x \to 3^+} (x + 3) = 6$.

The graph of $y = f(x)$ is below. We only consider the behavior of f **"immediately" to the right of** $x = 3$.



§

<u>Existence of a Two-Sided Limit at a Point</u>

$$\lim_{x \to a} f(x) = L \Leftrightarrow \left[\lim_{x \to a^-} f(x) = L, \text{ and } \lim_{x \to a^+} f(x) = L \right], \quad (a, L \in \mathbb{R}).$$

• A two-sided limit **exists** $\Leftrightarrow$ the corresponding left-hand and right-hand limits **exist**, and they are **equal**.

• If either one-sided limit **does not exist (DNE)**, or if the two one-sided limits are **unequal**, then the two-sided limit **does not exist (DNE)**.

Our guesses, $\lim_{x \to 3^-} (x+3) = 6$ and $\lim_{x \to 3^+} (x+3) = 6$, imply $\lim_{x \to 3} (x+3) = 6$.

In fact, all three limits can be evaluated by **substituting** $x = 3$ into $(x+3)$:

$$\lim_{x \to 3^-} (x+3) = 3+3 = 6; \ \lim_{x \to 3^+} (x+3) = 3+3 = 6; \ \lim_{x \to 3} (x+3) = 3+3 = 6.$$

This procedure is generalized in the following theorem.

<u>Extended Limit Theorem for Rational Functions</u>

If f is a rational function, and $a \in \text{Dom}(f)$,

then $\lim_{x \to a^-} f(x) = f(a)$, $\lim_{x \to a^+} f(x) = f(a)$, and $\lim_{x \to a} f(x) = f(a)$.

• To evaluate each limit, substitute ("plug in") $x = a$, and evaluate $f(a)$.

WARNING 8: Substitution might not work if f is not a rational function.

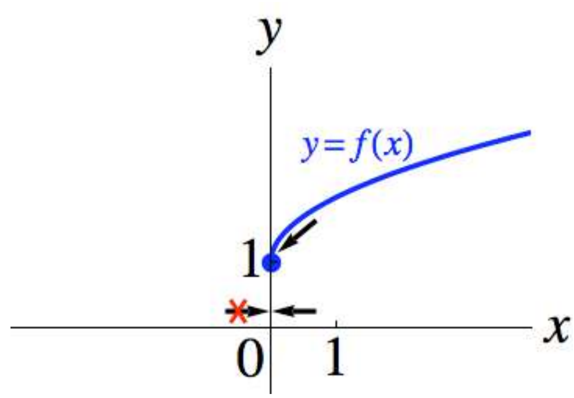
Example 6 (Pitfalls of Substituting into a Function that is Not Rational)

Let $f(x) = \sqrt{x} + 1$. Evaluate $\lim_{x \to 0^+} f(x)$, $\lim_{x \to 0^-} f(x)$, and $\lim_{x \to 0} f(x)$.

§ Solution

Observe that $\text{Dom}(f) = \{x \in \mathbb{R} \mid x \geq 0\} = [0, \infty)$, because $\sqrt{x}$ is **real** when $x \geq 0$, but it is **not real** when $x < 0$.

This is important, because *x* is only allowed to approach 0 (or whatever *a* is) **through** $\text{Dom}(f)$. Here, *x* is allowed to approach 0 from the right but **not** from the left.



Right-Hand Limit: $\lim_{x \to 0^+} f(x) = 1$.

Substituting $x = 0$ works: $\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \left(\sqrt{x} + 1\right) = \sqrt{0} + 1 = 1$.

Left-Hand Limit: $\lim_{x \to 0^-} f(x)$ does not exist (DNE).

Substituting $x = 0$ **does not work** here.

Two-Sided Limit: $\lim_{x \to 0} f(x)$ does not exist (DNE).

This is because the corresponding left-hand limit does not exist (DNE).

Observe that f is **not** a rational function, so the aforementioned theorem does **not** apply, even though $0 \in \text{Dom}(f)$. f is, however, an **algebraic** function, and we will discuss algebraic functions in Section 2.2. §

PART C: IGNORING THE FUNCTION AT *a*

Example 7 (Ignoring the Function at 'a' When Evaluating a Limit; Modifying Examples 4 and 5)

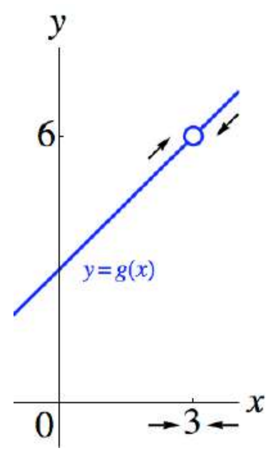
Let $g(x) = x + 3,\ (x \neq 3)$.

(We are deleting 3 from the domain of the function in Examples 4 and 5; this changes the function.)

Evaluate $\lim\limits_{x \to 3^-} g(x)$, $\lim\limits_{x \to 3^+} g(x)$, and $\lim\limits_{x \to 3} g(x)$.

§ Solution

Since $3 \notin \text{Dom}(g)$, we must delete the point $(3, 6)$ from the graph of $y = x + 3$ to obtain the graph of g below.



We say that g has a removable discontinuity at $x = 3$ (see Section 2.8), and the graph of g has a hole at the point $(3, 6)$.

Observe that, as x approaches 3 from the left **and** from the right, $g(x)$ **approaches** 6, even though $g(x)$ never equals 6.

$g(3)$ is undefined, yet the following statements are true:

$$\lim_{x \to 3^-} g(x) = 6,$$
$$\lim_{x \to 3^+} g(x) = 6, \text{ and}$$
$$\lim_{x \to 3} g(x) = 6.$$

There literally **does not have to be a point** at $x = 3$ (in general, $x = a$) for these limits to exist! Observe that substituting $x = 3$ into $(x + 3)$ works. §

Example 8 (Ignoring the Function at 'a' When Evaluating a Limit; Modifying Example 7)

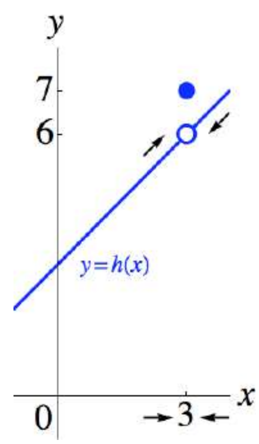
Let the function *h* be defined **piecewise** as follows: $h(x) = \begin{cases} x+3, & x \neq 3 \\ 7, & x = 3 \end{cases}$.

(A piecewise-defined function applies different evaluation rules to different subsets of (groups of numbers in) its domain. This type of function can lead to interesting limit problems.)

Evaluate $\lim_{x \to 3} h(x)$.

§ Solution

h is identical to the function *g* from Example 7, except that $3 \in \text{Dom}(h)$, and $h(3) = 7$. As a result, we must add the point $(3, 7)$ to the graph of *g* to obtain the graph of *h* below.



As with *g*, *h* also has a **removable discontinuity** at $x = 3$, and its graph also has a **hole** at the point $(3, 6)$.

Observe that, as *x* approaches 3 from the left **and** from the right, $h(x)$ also **approaches** 6.

$\lim_{x \to 3} h(x) = 6$ once again, even though $h(3) = 7$.

WARNING 2 repeat (applied to f): Sometimes, the **limit value** $\lim_{x \to a} f(x)$ does not equal the **function value** $f(a)$. §

As in Example 7, observe that substituting $x = 3$ into $(x+3)$ works. §

The existence (or value) of $\lim_{x \to a} f(x)$ **need not** depend on the existence (or value) of $f(a)$.

• Sometimes, it **does help** to know what $f(a)$ is when evaluating $\lim_{x \to a} f(x)$.
In Section 2.8, we will say that f is <u>continuous</u> at $a \Leftrightarrow \lim_{x \to a} f(x) = f(a)$, provided that $\lim_{x \to a} f(x)$ and $f(a)$ exist. We appreciate **continuity**, because we can then simply **substitute** $x = a$ to evaluate a limit, which was what we did when we applied the **Basic Limit Theorem for Rational Functions** in Part A.

• In Examples 7 and 8, we dealt with functions that were **not** continuous at $x = 3$, yet **substituting** $x = 3$ into $(x + 3)$ allowed us to evaluate the one- and two-sided limits at $a = 3$. We will develop theorems that cover these Examples. We first need the following definitions.

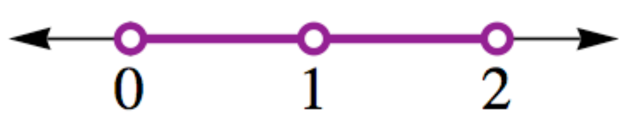
A <u>neighborhood of *a*</u> is an **open interval** along the real number line that is **symmetric** about *a*.

For example, the interval $(0, 2)$ is a **neighborhood** of 1. Since 1 is the **midpoint** of $(0, 2)$, the neighborhood is **symmetric** about 1.

A <u>punctured (or deleted) neighborhood of *a*</u> is constructed by taking a neighborhood of *a* and **deleting** *a* itself.

For example, the set $(0,2)\backslash\{1\}$, which can be written as $(0,1)\cup(1,2)$, is a **punctured neighborhood** of 1. It is a set of numbers that are **"immediately around"** 1 on the real number line.

• The notation $(0,2)\backslash\{1\}$ indicates that we can construct it by taking the **neighborhood** $(0, 2)$ and **deleting** 1.



"Puncture Theorem" for Limits of Locally Rational Functions

Let r be a rational function, and let $a \in \text{Dom}(r)$.
Let $f(x) = r(x)$ on a punctured neighborhood of $x = a$.
Then, $\lim\limits_{x \to a} f(x) = \lim\limits_{x \to a} r(x) = r(a)$.

• To evaluate the limits, substitute ("plug in") $x = a$ into $r(x)$, and evaluate $r(a)$.

• That is, if a function rule is given by a **rational** expression $r(x)$ **locally (immediately) around** $x = a$, where $a \in \text{Dom}(r)$, then **evaluate** the rational expression **at** a to obtain the **limit** of the function at a.

Refer to Examples 7 and 8. Let $r(x) = x + 3$. Observe that r is a rational function, and $3 \in \text{Dom}(r)$. Both the g and h functions were defined by $x + 3$ **locally (immediately) around** $x = 3$. More precisely, they were defined by $x + 3$ on some **punctured neighborhood** of $x = 3$, say $(2.9, 3.1) \setminus \{3\}$. Therefore,

$$\lim_{x \to 3} g(x) = \lim_{x \to 3} r(x) = r(3) = 3 + 3 = 6, \text{ and}$$

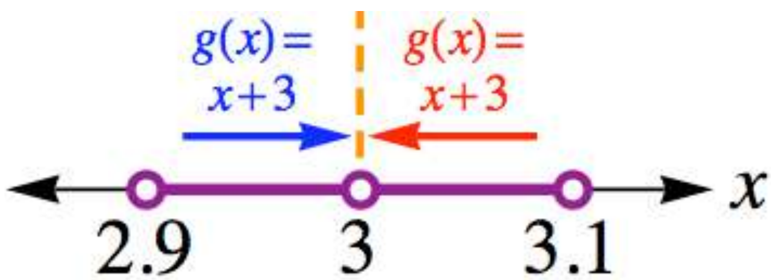
$$\lim_{x \to 3} h(x) = \lim_{x \to 3} r(x) = r(3) = 3 + 3 = 6.$$

It is easier to write:

$$\lim_{x \to 3} g(x) = \lim_{x \to 3} (x + 3) = 3 + 3 = 6, \text{ and}$$

$$\lim_{x \to 3} h(x) = \lim_{x \to 3} (x + 3) = 3 + 3 = 6.$$

The figure below refers to g, but it also applies to h.
The dashed line segment at $x = 3$ reiterates the **puncture** there.



Why does the theorem only require that a function be **locally** rational about *a*? Consider the following Example.

Example 9 (Limits are Local)

Let $f(t)=\begin{cases} t+2, & t<0 \\ \sqrt{t}, & t\geq 0 \end{cases}$. Evaluate $\lim\limits_{t\to -1} f(t)$.

§ Solution

Observe that $f(t)=t+2$ is the **only** rule that is relevant as *t* approaches -1 **locally** from the left **and** from the right. We only consider values of *t* that are **"immediately around"** $a=-1$. **"Limits are Local!"**

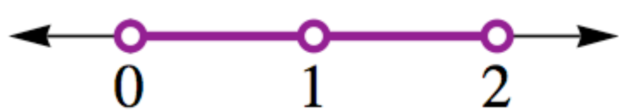
It is **irrelevant** that the rule $f(t)=\sqrt{t}$ is different, or that it is not rational. §

The following definitions will prove helpful in our study of **one-sided limits**.

A <u>left-neighborhood</u> of a is an **open interval** of the form (c,a), where $c<a$.
A <u>right-neighborhood</u> of a is an **open interval** of the form (a,c), where $c>a$.
A **punctured neighborhood** of a consists of **both** a left-neighborhood of a **and** a right-neighborhood of a.

For example, the interval $(0,1)$ is a **left-neighborhood** of 1. It is a set of numbers that are **"immediately to the left"** of 1 on the real number line.

The interval $(1,2)$ is a **right-neighborhood** of 1. It is a set of numbers that are **"immediately to the right"** of 1 on the real number line.



We now modify the "Puncture Theorem" for **one-sided limits**.

• Basically, when evaluating a **left-hand limit** such as $\lim\limits_{x\to a^-} f(x)$, we use the function rule that governs the *x*-values **"immediately to the left"** of *a* on the real number line.

• Likewise, when evaluating a **right-hand limit** such as $\lim\limits_{x\to a^+} f(x)$, we use the rule that governs the *x*-values **"immediately to the right"** of *a*.

Variation of the "Puncture Theorem" for Left-Hand Limits

Let r be a rational function, and let $a \in \text{Dom}(r)$.
Let $f(x) = r(x)$ on a left-neighborhood of $x = a$.
Then, $\lim_{x \to a^-} f(x) = \lim_{x \to a^-} r(x) = r(a)$.

Variation of the "Puncture Theorem" for Right-Hand Limits

Let r be a rational function, and let $a \in \text{Dom}(r)$.
Let $f(x) = r(x)$ on a right-neighborhood of $x = a$.
Then, $\lim_{x \to a^+} f(x) = \lim_{x \to a^+} r(x) = r(a)$.

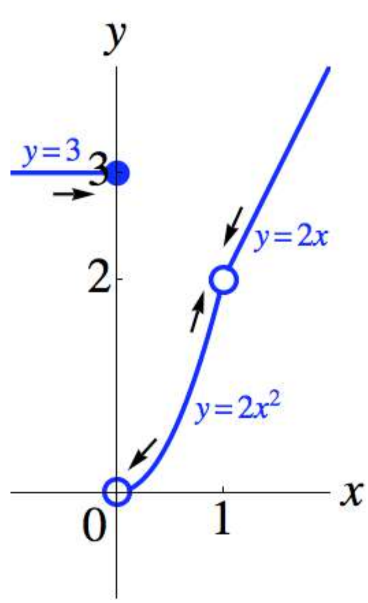
Example 10 (Evaluating One-Sided and Two-Sided Limits of a Piecewise-Defined Function)

Let $f(x) = \begin{cases} 3, & \text{if } x \le 0 \\ 2x^2, & \text{if } 0 < x < 1 \\ 2x, & \text{if } x > 1 \end{cases}$.

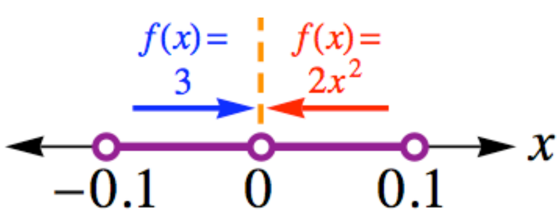
Evaluate the one-sided and two-sided limits of f at 1 and at 0.

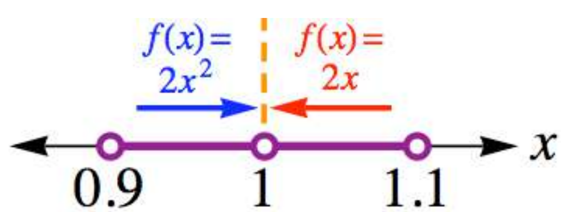
§ Solution

The graph of $y = f(x)$ is below. It helps, but it is **not** required to evaluate limits. Instead, we can evaluate limits of **relevant** function rules.



$\lim\limits_{x \to 1^-} f(x) = \lim\limits_{x \to 1^-} 2x^2$ $= 2(1)^2$ $= 2$	The left-hand limit as $x \to 1^-$: We use the rule $f(x) = 2x^2$, because it applies to a **left-neighborhood** of 1, say $(0.9, 1)$.
$\lim\limits_{x \to 1^+} f(x) = \lim\limits_{x \to 1^+} 2x$ $= 2(1)$ $= 2$	The right-hand limit as $x \to 1^+$: We use the rule $f(x) = 2x$, because it applies to a **right-neighborhood** of 1, say $(1, 1.1)$.
$\lim\limits_{x \to 1} f(x) = 2$	The two-sided limit as $x \to 1$: The left-hand and right-hand limits at 1 **exist**, and they are **equal**, so the two-sided limit exists and equals their common value.
$\lim\limits_{x \to 0^-} f(x) = \lim\limits_{x \to 0^-} 3$ $= 3$	The left-hand limit as $x \to 0^-$: We use the rule $f(x) = 3$, because it applies to a **left-neighborhood** of 0, say $(-0.1, 0)$.
$\lim\limits_{x \to 0^+} f(x) = \lim\limits_{x \to 0^+} 2x^2$ $= 2(0)^2$ $= 0$	The right-hand limit as $x \to 0^+$: We use the rule $f(x) = 2x^2$, because it applies to a **right-neighborhood** of 0, say $(0, 0.1)$.
$\lim\limits_{x \to 0} f(x)$ does not exist (DNE)	The two-sided limit as $x \to 0$: The left-hand and right-hand limits at 0 **exist**, but they are **unequal**, so the two-sided limit does not exist (DNE).





§

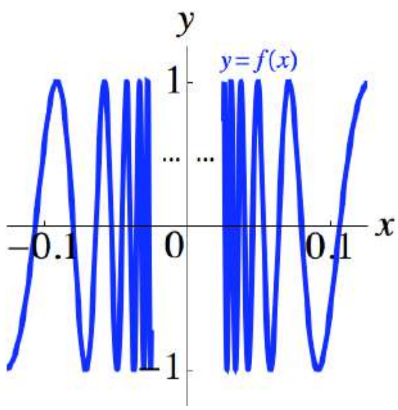
PART D: NONEXISTENT LIMITS

Example 11 (Nonexistent Limits)

Let $f(x)=\sin\left(\frac{1}{x}\right)$. Evaluate $\lim_{x\to 0^+} f(x)$, $\lim_{x\to 0^-} f(x)$, and $\lim_{x\to 0} f(x)$.

§ Solution

The graph of $y=f(x)$ is below. Ask your instructor if s/he might have you even attempt to draw this. In a sense, the classic sine wave is being turned "inside out" relative to the *y*-axis.



As *x* approaches 0 from the right (or from the left), the function values **oscillate** between -1 and 1. They do **not** approach a **single real number**. Therefore,

$\lim_{x\to 0^+} f(x)$ does not exist (DNE),

$\lim_{x\to 0^-} f(x)$ does not exist (DNE), and

$\lim_{x\to 0} f(x)$ does not exist (DNE).

Note 1: The *y*-axis is **not a vertical asymptote (VA)** here, because the graph and the function values are **not "exploding" without bound** around the *y*-axis.

Note 2: Here is an example of how the **numerical / tabular approach** introduced in Part B **might lead us astray**:

x	$0^+ \leftarrow$	$\frac{1}{3\pi}$	$\frac{1}{2\pi}$	$\frac{1}{\pi}$
$f(x)=\sin\left(\frac{1}{x}\right)$	0 (?) $\leftarrow$ NO!	0	0	0

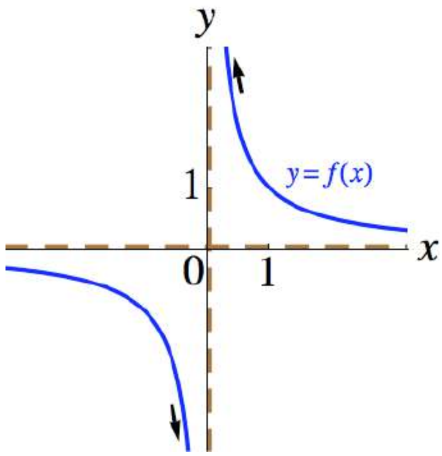
§

Example 12 (Infinite and/or Nonexistent Limits)

Let $f(x) = \frac{1}{x}$. Evaluate $\lim_{x \to 0^+} f(x)$, $\lim_{x \to 0^-} f(x)$, and $\lim_{x \to 0} f(x)$.

§ Solution

The graph of $y = f(x)$ is below. We will discuss this graph in later sections.



As x approaches 0 from the **right**, the function values **increase without bound**.

Therefore, $\lim_{x \to 0^+} f(x) = \infty$.

As x approaches 0 from the **left**, the function values **decrease without bound**.

Therefore, $\lim_{x \to 0^-} f(x) = -\infty$.

∞ and $-\infty$ are **mismatched**.

Therefore, $\lim_{x \to 0} f(x)$ does not exist (DNE).

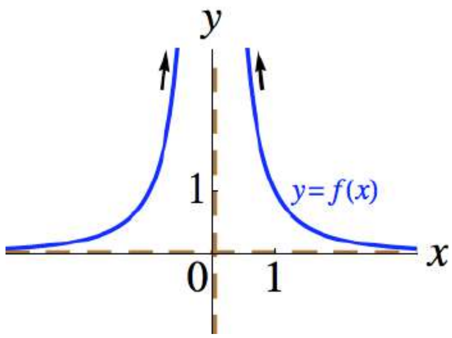
In fact, all three limits **do not exist**. For example, $\lim_{x \to 0^+} f(x)$, **does not exist**, because the function values **do not approach a single real number** as x approaches 0 from the right. The expressions ∞ and $-\infty$ indicate **why** the one-sided limits do not exist, and we write ∞ and $-\infty$ where appropriate. §

Example 13 (Infinite and Nonexistent Limits)

Let $f(x) = \frac{1}{x^2}$. Evaluate $\lim_{x \to 0^+} f(x)$, $\lim_{x \to 0^-} f(x)$, and $\lim_{x \to 0} f(x)$.

§ Solution

The graph of $y = f(x)$ is below. Observe that f is an even function.



$\lim_{x \to 0^+} f(x) = \infty$,

$\lim_{x \to 0^-} f(x) = \infty$, and

$\lim_{x \to 0} f(x) = \infty$. §

Example 14 (A Nonexistent Limit)

Let $f(x) = \dfrac{|x|}{x}$. Evaluate $\lim\limits_{x \to 0^+} f(x)$, $\lim\limits_{x \to 0^-} f(x)$, and $\lim\limits_{x \to 0} f(x)$.

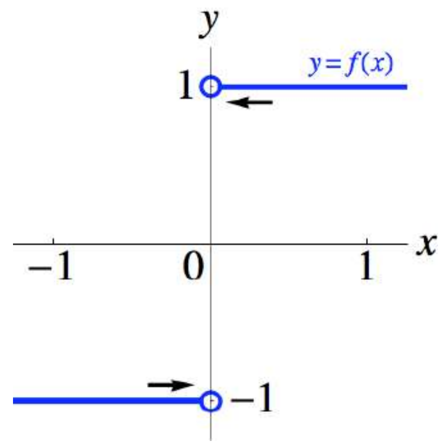
§ Solution

Note: f is **not** a rational function, but it is an **algebraic function**, since $f(x) = \dfrac{|x|}{x} = \dfrac{\sqrt{x^2}}{x}$.

Remember that: $|x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases}$.

Then, $f(x) = \dfrac{|x|}{x} = \begin{cases} \dfrac{x}{x} = 1, & \text{if } x > 0 \\ \dfrac{-x}{x} = -1, & \text{if } x < 0 \end{cases}$, and $f(0)$ is undefined.

The graph of $y = f(x)$ is below.



$\lim\limits_{x \to 0^+} f(x) = 1$,

$\lim\limits_{x \to 0^-} f(x) = -1$, and

$\lim\limits_{x \to 0} f(x)$ does not exist (DNE),

due to the fact that the right-hand and left-hand limits are **unequal**. §

FOOTNOTES

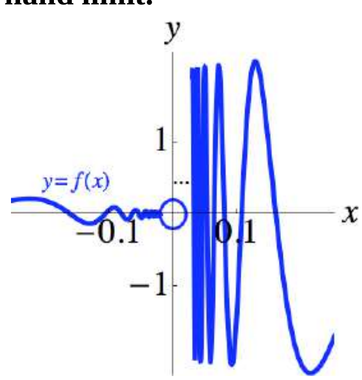
1. **Limits do not require continuity.** In Section 2.8, we will discuss continuity, a property of functions that helps our lovers run along the graph of a function without having to jump or hop. In Exercises 1-3, we could imagine the lovers running towards each other (one from the left, one from the right) while staying on the graph of f and without having to jump or hop, provided they were placed on appropriate parts of the graph. Sometimes, the "run" requires jumping or hopping. Let $f(x)=\begin{cases} 0, & \text{if } x \text{ is a rational number } (x\in\mathbb{Q}) \\ x, & \text{if } x \text{ is an irrational number } (x\notin\mathbb{Q};\ \text{really, } x\in\mathbb{R}\setminus\mathbb{Q}) \end{cases}$.
   It turns out that $\lim\limits_{x\to 0} f(x)=0$.

2. **Misconceptions about limits.**
   See "Why Is the Limit Concept So Difficult for Students?" by Sally Jacobs in the Fall 2002 edition (vol.24, No.1) of *The AMATYC Review*, pp.25-34.
   • Students can be misled by the use of the word "limit" in real-world contexts. For example, a speed limit is a bound that is not supposed to be exceeded; there is no such restriction on limits in calculus.
   • Limit values can sometimes be attained. For example, if a function f is continuous at $x=a$ (see Examples 1-3), then the function value takes on the limit value at $x=a$.
   • Limit values do not have to be attained. See Examples 7 and 8.

   Observations:
   • The dynamic view of limits, which involves ideas of motion and "approaching" (for example, our lovers), may be more accessible to students than the static view preferred by many textbook authors. The static view is exemplified by the formal definitions of limits we will see in Section 2.7. The dynamic view greatly assists students in transitioning to the static view and the formal definitions.
   • Leading mathematicians in 18^{th}- and 19^{th}-century Europe heatedly debated ideas of limits.

3. **Multivariable calculus.** When we go to higher dimensions, there may be more than two possible approaches (not just left-hand and right-hand) when analyzing limits at a point! Neighborhoods can take the form of disks or balls.

4. **An example where a left-hand limit exists but not the right-hand limit.**
   Let $f(x)=\dfrac{x+|x|(1+x)}{x}\sin\left(\dfrac{1}{x}\right)=\begin{cases} -x\sin\left(\dfrac{1}{x}\right), & \text{if } x<0 \\ (2+x)\sin\left(\dfrac{1}{x}\right), & \text{if } x>0 \end{cases}$.



   Then, $\lim\limits_{x\to 0^-} f(x)=0$, which can be proven by the Squeeze (Sandwich) Theorem in Section 2.6. However, $\lim\limits_{x\to 0^+} f(x)$ does not exist (DNE).