1.0 LEARNING OBJECTIVES

After reading this unit, you should be able to:

1. In computer arithmetic we learn how data is manipulating by using arithmetic operation
2. Data manipulation is learning to produce result of given problem
3. Learn about the error occur during arithmetic problems
4. Detail study of number representation in memory

1.1 INTRODUCTION

This chapter is to emphasis two types of real numbers viz. fixed point representation,floatingpointrepresentation;withinfloatingpoint (non-normalizedand normalized) and their representations in computer memory. Rules to perform arithmetic operations (Addition, Subtraction, Multiplication, and Division) with normalized floating numbers are also considered. At the last the various types of errors with measurement that can be introduced during numerical computation are alsodefined.

1.2 REPRESENTATION OF FLOATING POINT NUMBERS

For easier understanding we assume that computer can store and operate with decimal numbers, although it does whole work with binary number system. Also only finite number of digits can be stored in the memory of the computer. We will assume that a computer has a memory in which each location can store digits with provision for sign (+ or -)

There are two methods for storing of real numbers in the memory of the computer:

1.2.1 Fixed PointRepresentation

1.2.2 Floating PointRepresentation

1.2.1 Fixed Point Representation

Memory location storing the number 412456.2465

+	4	1	2	4	5	6	2	4

Assumed decimal position

Figure 1.1 Fixed point representations in Memory

This representation is called fixed point representation, since the position of decimal point is fixed after 6 positions from left. In such a representation largest positive number we can store 999999.99 and smallest positive number we can store 000000.01. This range is quite inadequate.

Example 1.1: Following are the examples of fixed point representations in the decimal number system:

210000

0.0005432

65754.546

234.00345

Example 1.2: Following are the examples of fixed point representations in the binary number system:

10111

10.11101

111.00011

0.00011

Disadvantages of fixed point representation

Inadequate range: Range of numbers that can be represented is restricted by number of digits or bits used.

1.2.2 Floating Point Representation

Floating point representation overcomes the above mentioned problem and in position to accommodate a much wider range of numbers than fixed point representation.

In this representation a real number consists of two basic parts:

i) Mantissa part
ii) Exponent part

In such a representation it is possible to float a decimal point within number towards left or right side.

For example:

53436.256 = 5343.6256×10^{1}

534.36256×10^{2}

53.436256×10^{3}

5.3436256×10^{4}

$.53436256 \times 10^{5}$

$.054436256 \times 10^{6}$

and so on

= 534362.56×10^{-1}

5343625.6×10^{-2}

$53436256.0 \times 10^{-3}$

$534362560.0 \times 10^{-4}$

And so on

Floating Point Number	Mantissa	Exponent	
5343.6256 x 10^{1}	5343.6256	1	
534.36256 x 10^{2}	534.36256	2	
53.436256 x 10^{3}	53.436256	3	
5.3436256 x 10^{4}	5.3436256	4	
.53436256 x 10^{5}	0.53436256	5	← **Normalized Floating Point Number**
0.054436256 x 10^{6}	0.053436256	6	
…………….	…………	…………	
534362.56 x 10^{-1}	534362.56	-1	
5343625.6 x 10^{-2}	5343625.6	-2	
53436256.0 x 10^{-3}	53436256.0	-3	
534362560.0 x 10^{-4}	534362560.0	-4	
………….	…………	……	

In general floating representation of a number of any base may be written as:

$$N = \pm \text{Mantissa} \times (\text{Base})^{\pm \text{exponent}}$$

Representation of floating point number in computer memory (with four-digit mantissa)

Let us assume we have hypothetical 8-digit computer out of which four digits are used for mantissa and two digits are used for exponent with a provision of sign of mantissa and sign of exponent.

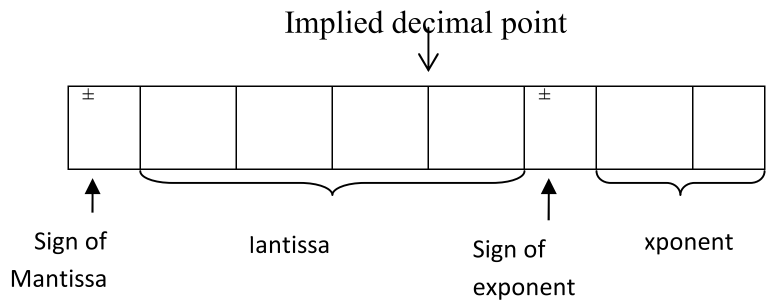

Figure 1.2 Floating point representation in memory (4-digit mantissa)

Normalized Floating Representation

It has been noted that a number may have more than one floating point representations. In order to have unique representation of non-zero numbers a normalized floating point representation is used.

A floating point representation in decimal number system is normalized floating point iff mantissa is less than 1 and greater than equal to .1 or 1/10(base of decimal number system).

i.e.

$$.1 \le |\text{mantissa}| < 1$$

A floating point representation in binary number system is normalized floating point iff mantissa is less than 1 and greater than equal to .5 or 1/2(base of binary number system).

i.e.

$$.5 \le |\text{mantissa}| < 1$$

In general, a floating point representation is called normalized floating point representation iff mantissa lies in the range:

$$1/\text{base} \le |\text{mantissa}| < 1$$

Representation of normalized floating point number in computer memory with four-digit mantissa:

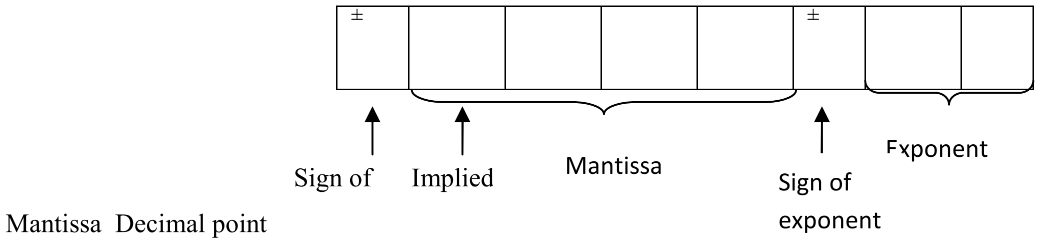

Figure 1.3 Normalized floating point representation in memory (4-digit mantissa)

Note: In computer, storage of floating point numbers is taken place in normalized form.

Disadvantages of floating point representation

- All the eight digits cannot be stored, since two digits are required by exponent.

Some specific rules are to be followed when arithmetic operations are performed with such numbers.

1.3 ARITHMETIC OPERATIONS WITH NORMALIZED FLOATING POINT

1.3.1 Addition

For adding two normalized floating point numbers following rules are to be followed:

a) Their exponents are to be made same if they are not same.
b) Add their mantissa to get the mantissa of resultant.
c) Result is written in normalized floating point number.
d) Check the overflow condition.

Example 1.3

Add .4567E05 to .3456E05

Sol. Here exponents are equal, we have to add only mantissa and exponent remains unchanged.

Addend .3456E05

Augend .4567E05

Sum . 8023E05

Example1.4

Add .3456E05 and .5433E07

Sol. Here exponents are not equal, therefore firstly make exponents same such that mantissa of number with smaller exponent sifted towards R.H.S. equal to the number of digits smaller exponent less than with larger exponent i.e. 7-5=2.

.3456E05 → .0034E07 Addend

.5456E07 Augend

.5490E07 Sum

Example 1.5

Add .3456E03 and .5433E07

Sol. Here exponents are not equal, therefore firstly make exponents same such that mantissa of number with smaller exponent sifted towards R.H.S. equal to the number of digits smaller exponent less than with larger exponent i.e. 7-3=4.

.3456E03 → .0000E07 Addend

.5456E07 Augend

.5456E07 Sum

Example 1.6

Add .4567E05 to .7456E05

Sol. Here exponents are equal, we have to add only mantissa and exponent remains unchanged.

.3456E05 Addend

.7567E05 Augend

1.1023E05->.1102E06 Sum (Last digit of mantissa is chopped)

Example 1.7

Add .4567E99 to .7456E99

Sol. Here exponents are equal, we have to add only mantissa and exponent remains unchanged.

.3456E05 Addend

.7567E05 Augend

1.1023E05->.1102E100 Sum (Last digit of mantissa is chopped)

OVERFLOW

As per exponent part can not store more than two digits, the number is larger than the largest number that can be stored in a memory location. This condition is called overflow condition and computer will intimate an error condition.

1.3.2 Subtraction

Rules to subtract a number from other are as follows:

a. Their exponents are to be made same if they are not same.
b. Subtract mantissa of one number from other to get the mantissa of resultant.
c. Result is written in normalized floating point number.
d. Check the underflow condition

Example 1.8

Subtract .3456E05 from .4567E05

Sol. Here exponents are equal, we have to subtract mantissa and exponents remain unchanged.

.4567E05 Minuend

.3456E05 Subtrahend

.18114E05 Difference

Example 1.9

Subtract .3456E05 from .5433E07

Sol. Here exponents are not equal, therefore firstly make exponents same such that mantissa of number with smaller exponent sifted towards R.H.S. equal to the number of digits smaller exponent less than with larger exponent i.e. 7-5=2.

.5456E07

.3456E05 → .0034E07

.5422E07

Example 1.10

Subtract .3456E03 from .5433E07

Sol. Here exponents are not equal, therefore firstly make exponents same such that mantissa of number with smaller exponent sifted towards R.H.S. equal to the number of digits smaller exponent less than with larger exponent i.e. 7-3=4.

.5433E07

.3456E03 → .0000E07

.5433E07

Example 1.11

Subtract .5345E05 from .5444E05

Sol. Here exponents are equal, we have to subtract only mantissa and exponent remains unchanged.

.5433E05

.5345E05

.0088E05->.8800E03

Example 1.12

Subtract .5345E-99 from .5433E-99

Sol. Here exponents are equal, we have to subtract only mantissa and exponent remains unchanged.

.5433E-99

.5345E-99

.0088E-99 -> .8800E-101 (UNDERFLOW)

As per exponent part can not store more than two digits, the number is smaller than the smallest number that can be stored in a memory location. This condition is called underflow condition and computer will intimate an error condition.

1.3.3 Multiplication

If two normalized floating point numbers are to be multiplied following rules are followed:

a) Exponents are added to give exponent of the product.
b) Mantissas of two numbers are multiplied to give mantissa of the product.
c) Result is written in normalized form.
d) Check for overflow/underflow condition.

$(m1 \times 10^{e1}) \times (m2 \times 10^{e2}) = (m1 \times m2) \times 10^{(e1+e2)}$

Example 1.13 Find the product of following normalized floating point representation with 4 digit mantissa.

.4454E23 and .3456E-45

Sol.

Product of mantissa

$.4454 \times .3456 = .1539\underline{302}$

Discarded

Sum of exponents

23-45 = -18

Resultant Product is 0.1539E-18

Example 1.14 Find the product of following normalized floating point representation with 4-digit mantissa.

.4454E23 and .1456E-45

Sol.

Product of mantissa

.4454 x .1456 = .0648502

Sum of exponent

23-45 = -18

Product is .0648502E-18 -> $.6485\underline{02}$E-19

Resultant product is 0.6485E-19

Example 1.15 Find the product of following normalized floating point representation with 4-digit mantissa.

.4454E50 and .3456E51

Sol.

Product of mantissa

.4454 x .3456 = .1539<u>302</u>

Sum of exponent

50+51 = 101

Product is .1539E101 (OVERFLOW)

As per exponent part cannot store more than two digits, the number is larger than the largest number that can be stored in a memory location. This condition is called overflow condition and computer will intimate an *error condition*.

Example 1.16 Find the product of following normalized floating point representation with 4 digit mantissa.

.4454E-50 and .3456E-51

Sol.

Product of mantissa

.4454 x .3456 = .1539<u>302</u>

Discarded

Sum of exponent

-50-51 = -101

Product is .1539E-101 (UNDERFLOW)

As per exponent part can not store more than two digits, the number is smaller than the smallest number that can be stored in a memory location. This condition is called underflow condition and computer will intimate an *error condition.*

1.3.4 Division

If two normalized floating point numbers are to be divided following rules are to be followed:

a. Exponent of second number is subtracted from first number to obtain of the result.
b. Mantissas of first number is divided by second number to obtain mantissa of the result
c. Result is written in normalized form.
d. Check for overflow/underflow condition.

$(m1 \text{ x } 10^{e1}) \div (m2 \text{ x} 10^{e2}) = (m1 \div m2)\text{x}10^{(e1- e2)}$

Example 1.17 Division of .8888E-05 by .2000 E -03

Sol. .8888E-05 ÷ .2000 E -03 = (.8888 ÷ .2222) E-2

= 4.4440E-2

= .4444E-1

1.4 ERRORS IN NUMBER REPRESENTATION

A computer has finite word length and so only a fixed number of digits are stored and used during computation. This would mean that even in storing an exact decimal number in its converted form in the computer memory, an error is introduced. This error is machine dependent. After the computation is over, the result in the machine form is again converted to decimal form understandable to the users and some more error may be introduced at this stage.

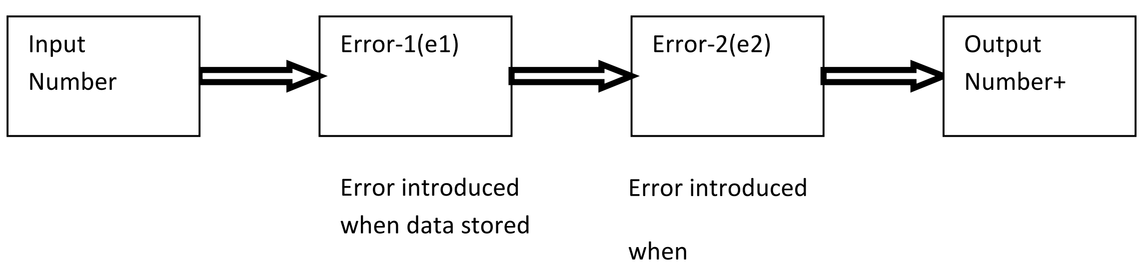

Figure 1.4 Effect of the errors on the result

1.4.1 Measurement of errors

a) Error = True value – Approximate value= E_{true} - E_{cal}

b) Absolute error = | Error| = $\left| E_{true} - E_{cal} \right|$

c) Relative error = $\dfrac{\left| E_{true} - E_{cal} \right|}{\left| E_{true} \right|}$

d) Percentage error = $\dfrac{\left| E_{true} - E_{cal} \right|}{\left| E_{true} \right|} * 100$

Note: For numbers close to 1, absolute error and relative error are nearly equal.

For numbers not close to 1 there can be great difference.

Example: If X = 100500 Xcal = 100000

Absolute error = *100500 - 100000 =500*

Relative error (R_x)= $\dfrac{500}{10000}$ $=0.005$

Example: If X = 1.0000 Xcal = 0.9898

Absolute error = *1.0000 – 0.9898 = 0.0102*

Relative error = $\dfrac{0.0102}{1} = 0.0102$

e) Inherent error

Error arises due to finite representation of numbers.

For example

1/3 = 0.333333 ……..

$\sqrt{2}$ = 1.414………

22/7 = 3.141592653589793…………..

It is noticed that every arithmetic operation performed during computation, gives rise to some error, which once generated may decay or grow in subsequent calculations. In some cases, error may grow so large as to make the computed result totally redundant and we call such a procedure numerically unstable. In some case it can be avoided by changing the calculation procedure, which avoids subtractions of nearly equal numbers or division by a small number or discarded remaining digits of mantissa.

Example Compute midpoint of the numbers

A = 4.568 B= 6.762

Using the four-digit arithmetic.

Solution: Method I

$$C = \frac{A+B}{2} = \frac{4.568+6.762}{2} = .5660x\ 10$$

Method II

$$C= A+ \frac{B-A}{2} =4.568 + \frac{6.762-4.568}{2} = .5665 \text{ x } 10$$

f) Transaction Error

Transaction error arises due to representation of finite number of terms of an infinite series.

For example, finite representation of series Sin x, Log x, e^x etc.

$$\text{Sin } x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} \ldots\ldots\ldots\ldots\ldots\ldots$$

Apart from above type of errors, we face following two types of errors during computation, we come across with large number of significant digits and it will be necessary to cut number up to desired significant digits. Meanwhile two types of errors are introduced.

- Round-off Error
- Chopping-off Error

g) Round-off

Round-off a number to n significant digits, discard all digits to the right of the n^{th} digit, and if this discarded number is:

-less than half a unit in the n^{th} place, leave the n^{th} digit unaltered.

- greater than half a unit in the n^{th} place, increase the n^{th} place digit by unity.

- exactly half a unit in the n^{th} place, increase the n^{th} digit by unity if it is *odd*, otherwise leave it is unchanged.

The number thus round-off said to be correct to n significant digits.

h) Chopping-off

In case of chopping-off a number to n significant digits, discard all digits to the right of the n^{th} digit, and leave the n^{th} digit unaltered.

Note: Chopping-off introduced more error than round-off error.

Example: The numbers given below are rounded-off to five significant digits:

2.45678 to 2.4568

1.45334 to 1.4533

2.45657 to 2.4566

2.45656 to 2.4565

Example: The numbers given below are chopped-off to five significant digits:

2.45678 to 2.4567

1.45334 to 1.4533

2.45657 to 2.4565

2.45656 to 2.4565

1.5 CHECK YOURPROGRESS

1. Convert 1234543into normalizes number.
2. Add .567E03 and .234E04.
3. Subtract .136E04 from 2.34E04.
4. Multiply .12E04 and .24E05.
5. Divide .23456E06 by .23456E02.
6. Calculate the relative error *If X = 200500* *Xcal = 200000*

1.6 SUMMARY

Let's take a quick review of all that we have learned about Computer Arithmetic.

1. We start from learning of representation of numbers in memory. We learn how to represent fixed point numbers and floating point numbers. We learn about disadvantages of fixed and floating point number system.
2. We learn how a number is converted into normalized number. Further we learn about the mathematic operations (Addition, Subtraction, Multiplications, Division) on normalized number.
3. We learn in this chapter about the error. We learn how an error is accrued when a number is converted into normalized number.

1.7 KEYWORDS

Fixed Point Representation: - in this type of rage of numbers that can be represented is restricted by number of digits or bits used.

Floating Point Representation: - it overcomes the problem associate with fixed point representation. It consiststwo-part mantissa and Exponent.

Normalized number: - Number whose mantissa part ranging between 1 and .1 are called normalized number.

Error: -Difference Between actual value (experimental value) to calculated value is called error.

Over flow: - when exponent is greater than two digits is called over flow.

1.8 SELF ASSESSMENTTEST

1. Discuss the errors, if any, introduced by floating point representation of decimal representation of decimal numbers in computers.
2. Describe the various components of computation errors introduced by the computer.
3. Assuming computer can handle 4 digit mantissa, calculate the absolute and relative errors in the following operations were p=0.02455 and q=0.001756.

   (a) p+q (b) p-q (c) px q (d) $p \div q$
4. Assuming the computer can handle only 4 digits in the mantissa, write an algorithm to add, subtract, multiply, and divide two numbers using normalized floating point arithmetic

1.9 ANSWER TO CHECK YOURPROGRESS

1. .1234543E07
2. .2907E04

3. .22040E05
4. .0288E09 (.288E08)
5. 1E04 (.1E05)
6. .0025

1.10 REFERENCES/SUGGESTEDREADINGS

1. Computer Oriented Numerical Methods, V. Rajaraman, PHI.
2. Introductory Methods of Numerical Analysis, S.S. Sastry, PHI.
3. Numerical Methods for Scientific and Engineering Computation, M.K.Jain, S.R.Lyenger, R.K.Jain, Wiley Eastern Limited.
4. Numerical Methods with Programs in C, T. Veerarajan, T. Ramachandarn., Tata McGraw Hill.

SUBJECT: Computer Oriented Numerical Methods	
COURSE CODE: BCA-PC(L)-122	AUTHOR: Prof. Kuldip Bansal
LESSON NO. 2	
ITERATIVE METHODS	
REVISED/UPDATED SLM BY AMIT	

STRUCTURE

2.0 OBJECTIVE

The objective of this lesson is to develop Iterative methods for finding the roots of the algebraic and the transcendental equations.

2.1 INTRODUCTION

To find roots of an equation $f(x) = 0$, up to a desired accuracy, iterative methods are very useful.

2.2 BISECTION METHOD

This method is due to Bolzano and is based on the repeated application of the intermediate value property. Let the function $f(x)$ be continuous between $a$and b. For definiteness, let $f(a)$ be negative &$f(b)$ be positive. Then, the first approximation to the root is

$$x_1 = ½ (a+b)$$

If $f(x_1) = 0$, then x_1 is a root of $f(x) = 0$. Otherwise, the root lies between a and x_1 or x_1 and b according as $f(x_1)$ is positive or negative. Then, we bisect the interval as before and continue the process until the root is found to desired accuracy. If $f(x_1)$ is +ve, so that the root lies between $a$and x_1. Then the second approximation to the root is $x_2 = ½(a+x_1)$. If $f(x_2)$ is –ve, the root lies between x_1 and x_2. Then the third approximation to the root is $x_3 = \frac{1}{2}(x_1+x_2)$ and so on.Graphically it can be shown as

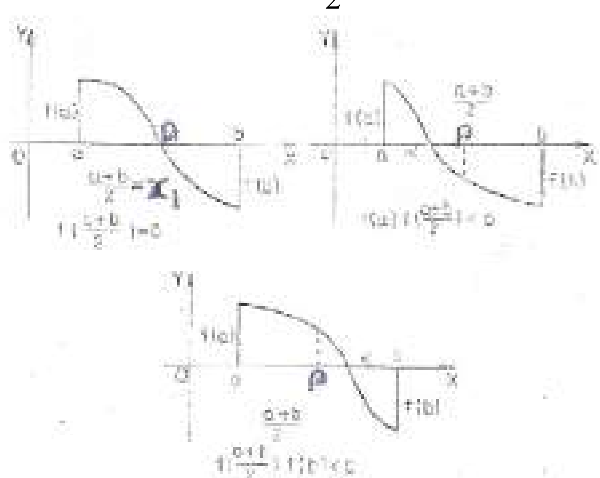

Fig 2.1

At each step the interval is determined in which the root lies. Middle value is next approximation. The process is carried out till the result upto desired accuracy is obtained.

2.3 RATE OF CONVERGENCE

Let $x_0, x_1, x_2, \ldots\ldots$ be the values of a root (α) of an equation at the 0^{th}, 1^{st}, 2^{nd} …….. iterations while its actual value is 3.5567. The values of this root calculated by three different methods, are as given below:

Root	*1st Method*	*2nd Method*	*3rd Method*
x_0	5	5	5
x_1	5.6	3.8527	3.8327
x_2	6.4	3.5693	3.56834
x_3	8.3	3.55798	3.55743
x_4	9.7	3.55687	3.55672
x_5	10.6	3.55676	
x_6	11.9	3.55671	

The values in the 1^{st} method do not converge towards the root 3.5567. In the 2^{nd} and 3^{rd} methods, the values converge to the root after 6^{th} and 4^{th} iterations respectively. Clearly 3^{rd} method converges faster than the 2^{nd} method. This fastness of convergence in any method is represented by its *rate of convergence*.

If e be the error then $e_i = \alpha - x_i = x_{i+1} - x_i$.

If e_{i+1}/e_i is almost constant, convergence is said to be *linear, i.e., slow*.

If e_{i+1}/e_i^p is nearly constant, convergence is said to be of order p, *i.e., faster.*

Since in case of Bisection method, the new interval containing the root, is exactly half the length of the previous one, the interval width is reduced by a factor of ½ at each step. At the end of the n^{th} step, the new interval will therefore, be of length $(b-a)/2^n$. If on repeating this process n times, the latest interval is as small as given ε, then $(b-a)/2^n \leq \varepsilon$

or $\quad n \geq [\log(b\text{-}a) - \log \varepsilon]/\log 2$

This gives the number of iterations required for achieving an accuracy ε. For example, the minimum number of iterations required for converging to a root in the interval (0, 1) for a given ε are as under:

ε:	10^{-2}	10^{-3}	10^{-4}
n:	7	10	14

As the error decreases with each step by a factor of ½, $(i.e.\ e_{x+1}/e_x = ½)$, the convergence in the bisection method is 'linear'.

Example: Find a root of the equation $x^3 - 4x - 9 = 0$, using the bisection method correct to three decimal places.

Solution: Let $\quad f(x) = x^3 - 4x - 9$

Since $f(2)$ is –ve and $f(3)$ is +ve, a root lies between 2 and 3.

∴ First approximation to the root is

$x_1 = ½(2+3) = 2.5$

Thus $f(x_1) = (2.5)^3 - 4(2.5) - 9 = -3.375$, *i.e., – ve.*

∴ The root lies between x_1 and 3. Thus the second approximation to the root is

$x_2 = ½\ (x_1+3) = 2.75.$

Then $\quad f(x_2) = (2.75)^3 - 4(2.75) - 9 = 0.7969$, *i.e., + ve.*

Therefore, the root lies between x_1 and x_2. Thus the third approximation to the root is

$x_3 = ½(x_1+x_2) = 2.625$

Then $f(x_3) = (2.625)^3 - 4(2.625) - 9 = -1.4121$, *i.e, - ve*

The root lies between x_2 and x_3. Thus the fourth approximation to the root is $x_4 = \frac{1}{2}(x_2 + x_3) = 2.6875$.

Repeating this process, the successive approximations are

$x_5 = 2.71875, \ \ x_6 = 2.70313, \ \ x_7 = 2.71094$

$x_8 = 2.70703, \ \ x_9 = 2.70508, \ \ x_{10} = 2.70605$

$x_{11} = 2.70654,$

Hence the root is 2.706.

Example: Find real positive root of the following equation by bisection method:

$$x^3 - 7x + 5 = 0$$

Solution:

Let $f(x) = x^3 - 7x + 5,$

$f(0) = 5, \qquad f(1) = -1$

$\therefore$ Root lies between 0 and 1,

Values of $a, b, \frac{a+b}{2}$ and the signs $\pm$ of functional values are shown as follows:

a	b	$\frac{a+b}{2}$	$f(a)$	$f(b)$	$f(\frac{a+b}{2})$
0	1	0.5	+	-	+
0.5	1	0.75	+	-	+
0.75	1	0.875	+	-	-
0.75	0.875	0.8125	+	-	-

0.75	0.8125	0.7812	+	-	+
0.7812	0.8125	0.7968	+	-	-
0.7812	0.7968	0.7890	+	-	-
0.7812	0.7890	0.7851	+	-	-
0.7812	0.7851	0.7831	+	-	-
0.7812	0.7831	0.7822	+	-	+
2.7822	0.7831	0.7826	+	-	+

Root lies between 0.7826 and 0.7831

$\therefore$ Root is 0.783.

2.4 FALSE POSITION OR REGULA FALSI METHOD

Let $f(x) = 0$ be the equation to be solved and the graph of $y = f(x)$ be drawn. If the line joining the two points $A \leftrightarrow [x_{i-1}, f(x_{i-1})]$ and $B \leftrightarrow [x_i, f(x_i)]$ meets the x-axis at $(x_{i+1}, 0)$, x_{i+1} is the approximate value of the root of the equation $f(x) = 0$.

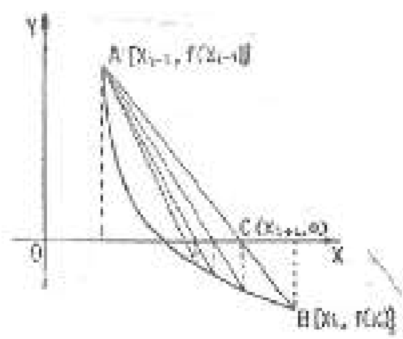

Fig 2.2

The equation of the line joining the points A and B is

$$y - f(x_{i-1}) = \frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}} \times (x - x_{i-1})$$

Putting $y = 0$

$$-f(x_{i-1}) = \frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}} \times (x - x_{i-1})$$

or $\quad x - x_{i-1} = -\dfrac{x_i - x_{i-1}}{f(x_i) - f(x_{i-1})} \times f(x_{i-1})$

$$\therefore \quad x = x_{i-1} - \frac{(x_i - x_{i-1}) f(x_{i-1})}{f(x_i) - f(x_{i-1})}$$

Therefore, the iterative formula is

$$x_{i+1} = x_{i-1} - \frac{(x_i - x_{i-1}) f(x_{i-1})}{f(x_i) - f(x_{i-1})}$$

or $\quad x_{i+1} = \dfrac{x_{i-1} f(x_i) - x_i f(x_{i-1})}{f(x_i) - f(x_{i-1})}$

$$= \begin{vmatrix} x_{i-1} & x_i \\ f(x_{i-1}) & f(x_i) \end{vmatrix} \div [f(x_i) - f(x_{i-1})]$$

2.5 ORDER OF CONVERGENCE OF FALSE POSITION OR REGULA FALSI METHOD

The iterative formula of Regula Falsi Method is

$$x_{i+1} = \frac{x_{i-1} f(x_i) - x_i f(x_{i-1})}{f(x_i) - f(x_{i-1})} \qquad (1)$$

Let α be the root of the equation $f(x) = 0$ and e_{i-1}, e_i, e_{i+1} be the errors when x_{i-1}, x_i, x_{i+1} are the approximate values of the root α.

$\therefore \quad e_{i-1} = x_{i-1} - \alpha \quad or \quad x_{i-1} = e_{i-1} + \alpha$

Similarly, $\quad x_i = e_i + \alpha, \qquad x_{i+1} = e_{i+1} + \alpha$

Substituting the values of x_{i-1}, x_i, x_{i+1} *in (1)*

$$e_{i+1} + \alpha = \frac{(e_{i-1}+\alpha)f(e_i+\alpha)-(e_i+\alpha)f(e_{i-1}+\alpha)}{f(e_i+\alpha)-f(e_{i-1}+\alpha)}$$

$$= \frac{[(e_{i-1}f(e_i+\alpha)-e_i f(e_{i-1}+\alpha)]+\alpha[f(e_i+\alpha)-f(e_{i-1}+\alpha)]}{f(e_i+\alpha)-f(e_{i-1}+\alpha)}$$

$$= \frac{e_{i-1}f(e_i+\alpha)-e_i f(e_{i-1}+\alpha)}{f(e_i+\alpha)-f(e_{i-1}+\alpha)}+\alpha$$

$$\therefore \quad e_{i+1} = \frac{e_{i-1}f(\alpha+e_i)-e_i f(\alpha+e_{i-1})}{f(\alpha+e_i)-f(\alpha+e_{i-1})} \qquad (2)$$

Now expanding $f(\alpha+e_i)$ and $f(\alpha+e_{i-1})$ by Taylor's theorem.

Numerator = $e_{i-1} f(\alpha+e_i) - e_i f(\alpha+e_{i-1})$

$$= e_{i-1}[f(\alpha) + e_i f'(\alpha) + \frac{e_i^2}{2!} f''(\alpha) + ...]$$

$$-e_i[f(\alpha) + e_{i-1} f'(\alpha) + \frac{e_{i-1}^2}{2!} f''(\alpha) + ...]$$

$$= [e_{i-1} - e_i] f(\alpha) + \frac{e_{i-1}e_i^2 - e_i e_{i-1}^2}{2!} f''(\alpha) + ...$$

$$\simeq \frac{e_{i-1}e_i(e_i - e_{i-1})}{2!} f''(\alpha)$$

[$\because f(\alpha) = 0$, α being the root of $f(x) = 0$ Terms containing e_i^3, e^3_{i-1} andhigher degree terms are neglected.]

Denominator = $f(\alpha + e_i) - f(\alpha+e_{i-1})$

$$= [f(\alpha) + e_i f'(\alpha) + \frac{e_i^2}{2!} f''(\alpha) + ...]$$

$$- [f(\alpha) + e_{i-1} f'(\alpha) + \frac{e_{i-1}^2}{2!} f''(\alpha) + ...]$$

$$= (e_i - e_{i-1})\, f'(\alpha) + \frac{e_i^{\,2} - e_{i-1}^2}{2!} f''(\alpha) + \ldots\ldots..$$

$\simeq (e_i - e_{i-1})\, f'(\alpha)$ [Terms containing e_i^2, e^2_{i-1} and higher degree terms are neglected.]

$\therefore$ (2) becomes

$$e_{i+1} = \frac{\frac{1}{2!} e_{i-1} e_i (e_i - e_{i-1})\, f''(\alpha)}{(e_i - e_{i-1}) f'(\alpha)}$$

or $$e_{i+1} = \frac{e_{i-1} e_i f''(\alpha)}{2 f'(\alpha)} = e_{i-1} e_i k, \tag{3}$$

where $k = \dfrac{f''(\alpha)}{2 f'(\alpha)}$

If p is the order of convergence

$$e_i \le e^p_{i-1}\, k' \quad \text{or} \quad \text{taking } e_i = e^p_{i-1}\, k' \tag{4}$$

for all $i \ge n$, k' is a constant.

Eliminating e_{i-1} from (3) and (4)

$$e_{i+1} = e_i \left(\frac{e_i}{k'} \right)^{1/p} k = e_i^{\,1+\frac{1}{p}} \frac{k}{k'^{1/p}} \tag{5}$$

Also $$e_{i+1} = e_i^p k' \tag{6}$$

Equating the values of e_{i+1} from (5) and (6)

$$e_i^{\,1+\frac{1}{p}} \frac{k}{k'^{1/p}} = e_i^p k' \tag{7}$$

Choosing k and k' such that $k = k'.\ k'^{1/p} = k'^{1+1/p}$,(7) becomes

$$e_i^{1+1/p} = e_i^{\,p}$$

$\therefore \quad 1 + \dfrac{1}{p} = p \quad or \quad p^2 - p - 1 = 0$

$\therefore \quad p = \dfrac{1 \pm \sqrt{5}}{2}$

Taking +ve sign, $p = \dfrac{\sqrt{5}+1}{2} = \dfrac{3.236}{2} = 1.618$

$\therefore$ Order of convergence of Regular Falsi Method is 1.618

Example: Solve $x^3 - 9x + 1 = 0$ for the root lying between 2 and 4 by the method of false position.

Solution: Let $f(x) = x^3 - 9x + 1 = 0$

$\therefore \quad f(2) = -9, \quad f(4) = 29$

In the iterative formula

$$x_{i+1} = x_{i-1} - \frac{(x_i - x_{i+1}) f(x_{i-1})}{f(x_i) - f(x_{i-1})}$$

Putting $i = 1$, $\quad x_0 = 2, \quad x_1 = 4$

$f(x_0) = -9 \quad f(x_1) = 29$

$$x_2 = x_0 - \frac{(x_1 - x_0) f(x_0)}{f(x_1) - f(x_0)}$$

$$= 2 - \frac{(4-2)(-9)}{29-(-9)} = 2 + \frac{18}{38} = 2.47$$

For second approximation x_3,

$i = 2$, $\quad x_1 = 2.47, \quad x_2 = 4$

$f(x_1) = -6.063 \quad f(x_2) = 29$

$$x_3 = 2.47 + \frac{1.53 \times 6.063}{35.063} = 2.73$$

For third approximation x_4,

$$i = 3, \qquad \begin{array}{ll} x_2 = 2.73, & x_3 = 4 \\ f(x_2) = -3.2 & f(x_3) = 29 \end{array}$$

$$x_4 = 2.73 + \frac{(1.27)(3.2)}{32.2} = 2.85$$

$\therefore f(2.85) = -2.07$

Putting $i = 4$, the fourth approximation is

$$x_5 = 2.85 + \frac{(1.15)(2.07)}{31.07} = 2.92,$$

$$f(2.92) = -0.37$$

and for $i = 5$

$$x_6 = 2.92 + \frac{(1.08)(3.7)}{29.37} = 2.93$$

$$f(2.93) = 0.21$$

similarly, for

$i = 6$

$$x_7 = 2.93 + \frac{(1.07)(0.21)}{29.21} = 2.937$$

$\therefore$ Root of $f(x) = 0$ is 2.94, correct to two significant figures.

2.6 NEWTON-RAPHSON METHOD

Let $f(x) = 0$ be the equation whose solution is required. If x_i be a point near the root, $f(x)$ may be written as

$$f(x) = (x_i + \overline{x - x_i}).$$

Expanding it by Taylor's series

$$f(x) = f(x_i) + (x-x_i) f'(x_i) + \frac{(x-x_i)^2}{2!} f''(x_i) + ... = 0$$

As a first approximation, $(x-x_i)^2$ and higher degree terms are neglected.

$$\therefore \quad f(x_i) + (x-x_i) f'(x_i) = 0$$

or $\quad x-x_i = -\dfrac{f(x_i)}{f'(x_i)}$

or $\quad x = x_i - \dfrac{f(x_i)}{f'(x_i)}$

Iterative algorithm of Newton-Raphson method is

$$x_{i+1} \quad = \quad x_i - \frac{f(x_i)}{f'(x_i)} \text{ when } f'(x_i) \neq 0$$

Geometrical interpretation of this formula may be given as follows:

Let the graph of $y = f(x)$ be drawn and P_i be any point (x_i, y_i) on it.

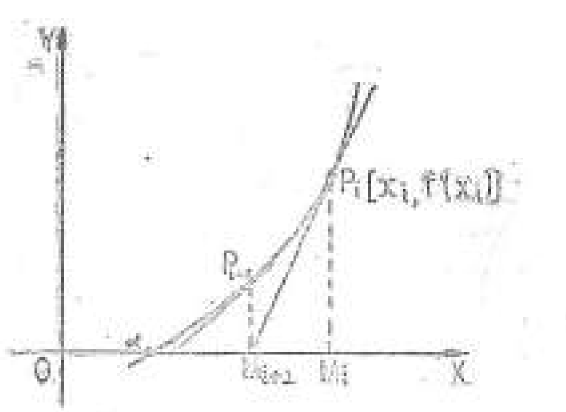

Fig 2.3

Equation of the tangent at p_i is

$$y - f(x_i) = f'(x_i).\ (x - x_i)$$

Putting $y = 0$, *i.e.,* tangent at P_i meets the x -axis at M_{i+1} whose abscissa is given by

$$x - x_i = -\frac{f(x_i)}{f'(x_i)}$$

or $$x = x_i - \frac{f(x_i)}{f'(x_i)}$$

which is nearer to the root α.

$\therefore$ Iterative algorithm is

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}.$$

Thus in this method, we have replaced the part of the curve between the point P_i and x – axis by a tangent to the curve at P_i and so on.

Example: Find the real root of the equation $xe^x - 2 = 0$ correct to two decimal places, using Newton –Raphson method.

Solution : Given $f(x) = xe^x - 2$, we have

$$f'(x) = xe^x + e^x \text{ and } f''(x) = xe^x + 2e^x$$

Therefore, we obtain

$$f(0) = -2 \text{ and } f(1) = e - 2 = 0.71828$$

Hence, the required root lies in the interval(0,1)and is nearer to1. Also $f'(x)$ and $f''(x)$ do not vanish in (0,1); $f(x)$ and $f''(x)$ will have the same sign at $x = 1$. Therefore, we take the first approximation $x_0 = 1$, and using Newton-Raphson method, we get

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = \frac{e+2}{2e} = 0.867879$$

and $\quad f(x_1) = 0.06716$

The second approximation is

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 0.867879 - \frac{0.06716}{4.44902} = 0.85278$$

and

$$f(x_2) = 7.655 \text{ x } 10^{-4}$$

Thus, the required root is 0.853

Example: Find a real root of the equation $x^3 - x - 1 = 0$ using Newton Raphson method, correct to four decimal places.

Solution: Let $f(x) = x^3 - x - 1$, then we note that $f(1) = -1, f(2) = 5$. Therefore, the root lies in the interval (1, 2). We also note that

$$f'(x) = 3x^2 - 1, f''(x) = 6x$$

and

$$f(1) = -1, \qquad f''(1) = 6, \qquad f(2) = 5, \qquad f''(2) = 12$$

Since $f(2)$ and $f''(2)$ are of the same sign, we choose $x_0 = 2$ as the first approximation to the root. The second approximation is computed using Newton-Raphson method as

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 2 - \frac{5}{11} = 1.54545 \text{ and } \qquad f(x_1) = 1.14573$$

The successive approximations are

$$x_2 = 1.54545 - \frac{1.14573}{6.16525} = 1.35961, \quad f(x_2) = 0.15369$$

$$x_3 = 1.35961 - \frac{0.15369}{4.54562} = 1.32579, \qquad f(x_3) = 4.60959 \text{ x } 10^{-3}$$

$$x_4 = 1.32579 - \frac{4.60959 \times 10^{-3}}{4.27316} = 1.32471, \quad f(x_4) = -3.39345 \text{ x } 10^{-5}$$

$$x_5 = 1.32471 + \frac{3.39345 \times 10^{-5}}{4.26457} = 1.324718, \quad f(x_5) = 1.823 \text{ x } 10^{-7}$$

Hence, the required root is 1.3247.

2.7 CONVERGENCE OF NEWTON-REPHSON METHOD

To examine the convergence of Newton-Raphson formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \qquad (1)$$

We compare it with the general iteration formula $x_{n+1} = \phi(x_n)$, and thus obtain

$$\phi(x_n) = x_n - \frac{f(x_n)}{f'(x_n)}$$

or, we write it as

$$\phi(x) = x - \frac{f(x)}{f'(x)}$$

We also know that the iteration method converges if $|\phi'(x)|<1$. Therefore, Newton-Raphson formula (1) converges, provided

$$|f(x)\, f''(x)| < |f'(x)|^2 \qquad (2)$$

in the interval considered. Newton-Raphson formula therefore converges, provided the initial approximation x_0 is chosen sufficiently close to the root and $f(x)$, $f'(x)$ and $f''(x)$ are continuous and bounded in any small interval containing the root.

Definition:

Let

$$x_n = \alpha + \varepsilon_n, \qquad x_{n+1} = \alpha + \varepsilon_{n+1}$$

where α is a root of $f(x) = 0$. If we can prove that $\varepsilon_{n+1} = K\,\varepsilon^p$, where K is a constant and ε_n is the error involved at the n^{th} step, while finding the root by an iterative method, then the rate of convergence of the method is p.

We can now establish that Newton-Raphosn method converges quadratically. Let

$$x_n = \alpha + \varepsilon_n, \qquad x_{n+1} = \alpha + \varepsilon_{n+1},$$

where α is a root of $f(x) = 0$ and ε_n is the error involved at the n^{th} step, while finding the root by Newton-Raphson formula (1). Then, Eq. (1) gives

$$\alpha + \varepsilon_{n+1} = \alpha + \varepsilon_n - \frac{f(\alpha+\varepsilon_n)}{f'(\alpha+\varepsilon_n)},$$

i.e.,

$$\varepsilon_{n+1} = \varepsilon_n - \frac{f(\alpha+\varepsilon_n)}{f'(\alpha+\varepsilon_n)} = \frac{\varepsilon_n f'(\alpha+\varepsilon_n) - f(\alpha+\varepsilon_n)}{f'(\alpha+\varepsilon_n)}$$

Using Taylor's expansion, we get

$$\varepsilon_{n+1} = \frac{1}{f'(\alpha) + \varepsilon_n f''(\alpha) + \ldots\ldots} \left\{ \varepsilon_n \left[f'(\alpha) + \varepsilon_n f''(\alpha) + \ldots \right] - \left[f(\alpha) + \varepsilon_n f'(\alpha) + \frac{\varepsilon_n^2}{2} f''(\alpha) + \ldots\ldots \right] \right\}$$

Since α is a root, $f(\alpha) = 0$. Therefore, the above expression simplifies to

$$\varepsilon_{n+1} = \frac{\varepsilon_n^2}{2} f''(\alpha) \frac{1}{f'(\alpha) + \varepsilon_n f''(\alpha)}$$

$$= \frac{\varepsilon_n^2}{2}\frac{f''(\alpha)}{f'(\alpha)}\left[1+\varepsilon_n\frac{f''(\alpha)}{f'(\alpha)}\right]^{-1}$$

$$= \frac{\varepsilon_n^2}{2}\frac{f''(\alpha)}{f'(\alpha)}\left[1-\varepsilon_n\frac{f''(\alpha)}{f'(\alpha)}\right]$$

or

$$\varepsilon_{n+1} = \frac{\varepsilon_n^2}{2}\frac{f''(\alpha)}{f'(\alpha)} + o(\varepsilon_n^3)$$

On neglecting terms of order (ε_n^3) and higher power, we obtain

$$\varepsilon_{n+1} = K(\varepsilon_n^2), \text{ where} \tag{3}$$

$$K = \frac{f''(\alpha)}{2f'(\alpha)} \tag{4}$$

It shows that Newton-Raphson method has second order convergence or converges quadratically.

Example: Set up Newton's scheme of iteration for finding the square root of a positive number N.

Solution: The square root of N can be carried out as a root of the equation $x^2 - N = 0$. Let $f(x) = x^2 - N$. By Newton's method, we have

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

In this problem, $f(x) = x^2 - N$, $f'(x) = 2x$. Therefore,

$$x_{n+1} = x_n - \frac{x_n^2 - N}{2x_n} = \frac{1}{2}\left(x_n + \frac{N}{x_n}\right) \tag{5}$$

Example: Evaluate $\sqrt{12}$, by Newton's formula.

Solution : Since $\sqrt{9} = 3$, $\sqrt{16} = 4$, we take $x_0 = (3 + 4)/2 = 3.5$. Using equation (5), we have

$$x_1 = \frac{1}{2}\left(x_0 + \frac{N}{x_0}\right) = \frac{1}{2}\left(3.5 + \frac{12}{3.5}\right) = 3.4643$$

$$x_2 = \frac{1}{2}\left(3.4643 + \frac{12}{3.4643}\right) = 3.4641$$

$$x_3 = \frac{1}{2}\left(3.4641 + \frac{12}{3.4641}\right) = 3.4641$$

Hence, $\sqrt{12} = 3.4641$upto four decimal places.

2.8 BAIRSTOW'S METHOD

Lin-Bairstow's method is often useful in finding quadratic factors of polynomial and finding the complete roots of a polynomial equation with real coefficients. Let the polynomial equation is given by

$$f(x) = A_3x^3 + A_2x^2 + A_1x + A_0 = 0 \qquad (1)$$

Let $x^2 + Rx + S$ be a quadratic factor of $f(x)$ and also let $x^2 + rx + s$ be an approximate factor. Usually, first approximations to r and s can be obtained from the last three terms of the given polynomial. Thus,

$$r = \frac{A_1}{A_2} \qquad \text{and} \qquad s = \frac{A_0}{A_2} \qquad (2)$$

Dividing $f(x)$ by $x^2 + rx + s$, let

$$f(x) = (x^2 + rx + s)\,(B_2x + B_1) + Cx + D$$

$$= B_2x^3 + (B_2r + B_1)x^2 + (C + B_1r + sB_2)\,x + (B_1s + D), \quad (3)$$

where the constants B_1, B_2, C and D have to be determined. Equating the coefficients of the like powers of x in equations (1) and (3), we get

$$\left.\begin{aligned} B_2 &= A_3 \\ B_1 &= A_2 - rB_2 \\ C &= A_1 - rB_1 - sB_2 \\ D &= A_0 - sB_1 \end{aligned}\right\} \qquad (4)$$

From (4), it is clear that the coefficients B_1, B_2, C and D are functions of r and s. Since $x^2 + Rx + S$ is a factor of the given polynomial, So

$$C(R, S) = 0 \qquad \text{and} \qquad D(R, S) = 0 \qquad (5)$$

Taking

$$R = r + \Delta r \qquad \text{and} \qquad S = s + \Delta s \qquad (6)$$

Equations (5) can be expanded by Taylor's series and we obtain

$$\left.\begin{aligned} C(R, S) &= C(r, s) + \Delta r \,.\, \frac{\partial C}{\partial r} + \Delta s . \frac{\partial C}{\partial s} = 0 \\ D(R, S) &= D(r, s) + \Delta r \,.\, \frac{\partial D}{\partial r} + \Delta s . \frac{\partial D}{\partial s} = 0 \end{aligned}\right\} \qquad (7)$$

where the derivatives are to be computed at r and s. We solve equation (7) for Δr and Δs. Using these values in (6) will give the next approximation to R and S, respectively. This process can be repeated until successive values of R and S show no significant change.

Example: Find the quadratic factor of the polynomial

$$f(x) = x^3 - 2x^2 + x - 2$$

Solution:

Here, we have $A_3 = 1$, $A_2 = -2$, $A_1 = 1$ and $A_0 = -2$

So, $\quad r = -\dfrac{1}{2} \quad$ and $\quad s = 1$

Using Equations (4), we have

$B_2 = 1$; $B_1 = -\ 2 -\ r$

$C = 1 - r\,(-\,2 - r\,) - s = 1 + 2r + r^2 - s$,

and $D = -\ 2 - s\,(-\,2 - r\,) = -\,2 + 2s + rs$.

Also $[C(r, s)]_{(-\frac{1}{2},1)} = 1 - 1 + \dfrac{1}{4} - 1 = -\dfrac{3}{4}$;

$[D(r,s)]_{(-\frac{1}{2},1)} = -2 + 2 - \dfrac{1}{2} = -\dfrac{1}{2}$;

$$\left(\frac{\partial C}{\partial r}\right)_{\left(-\frac{1}{2},1\right)} = 2 + 2r = 1; \quad \left(\frac{\partial C}{\partial s}\right)_{\left(-\frac{1}{2},1\right)} = -1;$$

$$\left(\frac{\partial D}{\partial r}\right)_{\left(-\frac{1}{2},1\right)} = s = 1;$$

and $$\left(\frac{\partial D}{\partial s}\right)_{\left(-\frac{1}{2},1\right)} = 2 + r = \frac{3}{2}$$

Following equations (7), we get

$\Delta r - \Delta s = \frac{3}{4}$ and $\Delta r + {}^{3}/_{2}\Delta s = \frac{1}{2}$

Hence $\Delta r = \dfrac{13}{20}$ and $\Delta s = -\dfrac{1}{10}$

Thus, we have

$$R = -\frac{1}{2} + \frac{13}{20} = \frac{3}{20} = 0.15$$

and $$S = 1 - \frac{1}{10} = \frac{9}{10} = 0.9$$

Therefore, the quadratic factor is $x^2 + 0.15x + 0.9$, which is now taken as the approximate quadratic factor to get the second approximation. So that, for the second approximation $r = 0.15$ and $s = 0.9$, then

$$C = 1 + 2.15\,(0.15) - 0.9 = 0.4225,$$

$$D = -2 + 2.15\,(0.9) = 0.065,$$

$$\frac{\partial C}{\partial r} = 2 + 2(0.15) = 2.30,$$

$$\frac{\partial C}{\partial S} = -1;\ \frac{\partial D}{\partial r} = 0.9 \text{ and } \frac{\partial D}{\partial s} = 2 + r = 2.15.$$

Hence, equations (7) give us

$$2.3\,\Delta r - \Delta s = -0.4225$$

and $\quad 0.9\Delta r + 2.15\,\Delta s = -\ 0.065$

Solving these equations, we get

$$\Delta r = -0.1665312$$

and $\quad \Delta s = 0.0394783.$

Therefore, the second approximations are obtained as

$$R = 0.15 - 0.16665312 = -0.0165312$$

and $\quad S = 0.9 + 0.0394783 = 0.9394783$

Thus, the second approximation to the quadratic factor is $x^2 - 0.0165312\,x + 0.9394783$. This process is repeated until no significant difference in the values of R and S is there, in two consecutive steps

2.9 CHECK YOUR PROGRESS

1. Find a root of the following equations by using bisection method.

   (i) $x^3 - 2x - 5 = 0$ (ii) $x - \cos x = 0$

2. Using Regula Falsi method, compute the real root of the following equations.

   (i) $xe^x - 2 = 0$ (ii) $x^3 - 4x - 9 = 0$

3. Using Newton Raphson method, evaluate

   (i) $\sqrt{32}$ (ii) 1/3 (iii) 1/15

4. Using Bairstow's method, obtain the quadratic factor of the polynomial given by

   $f(x) = x^3 - 2x^2 + x - 2$

2.10 SUMMARY

1. In this chapter we learn about how to find root of an equation. We learn to use different method to solve equations or to find root of equations.
2. We learn graphical representation of deferent methods.
3. We learn to find rate of convergence of different method (Bisection method, Newton raphson method, regulafalsi method)

2.11 KEYWORDS

1. **Bisection Method:-**x3=(x1+x2)/2, Where x1 is root of equation where function is negative and x2 is root where function value is positive.

2. **Newton Raphson Method:-** $x = x_i - \frac{f(x_i)}{f'(x_i)}$

3. **Regula Falsi Method:-** $x = x_{i-1} - \frac{(x_i - x_{i-1}) f(x_{i-1})}{f(x_i) - f(x_{i-1})}$

4. **Bairstow's Method: -** Lin-Bairstow's method is often useful in finding quadratic factors of polynomial and finding the complete roots of a polynomial equation with real coefficients.

5. **Rate of Convergence: -**It is defined how fast a method provide the accurate result of function or equations.

2.11 SELF ASSESSMENT TEST

1. Find root of X^2-20 using bisection method.

2. Find square root of 3 using newton raphson method.

3. Solve X^2-12 using regulafasi method.

2.12 ANSWER TO CHECK YOUR PROGRESS

(1) (i) 2.687 (ii) 0.937

(2) (i) 0.853 (ii) 2.7065

(3) (i) 5.6569 (ii) 3.4482 (iii) 0.258

(4) $x^2 + 1$

2.13 REFERENCES/ SUGGESTEDREADINGS

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SOLUTION OF SIMULTANEOUS LINEAR EQUATIONS AND ORDINARY DIFFERENTIAL EQUATIONS	
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STRUCTURE

3.14 Self Assessment Questions

3.15 References/ SuggestedReadings

3.0 OBJECTIVE

The objective of this lesson is to describe the numerical methods for finding the solution simultaneous linear equations and ordinary differential equations.

3.1 INTRODUCTION

The general form of a system of m linear equations in n unknowns x_1, x_2 , x_3 , …, x_n can be represented in matrix form as under:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{11} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix} \qquad (1)$$

Using matrix notation, the above system can be written in compact form as

$$[A]\,(X) = (B) \qquad (2)$$

The solution of the system of equations (2) gives $n$unknown values x_1, x_2, …., x_n, which satisfy the system simultaneously. If $m>n$, we may not be able to find a solution, in principle, which satisfy all the equations. If $m<n$, the system usually will have an infinite number of solutions. However, in this lesson, we shall restrict to the case $m = n$. In this case, if $|A| \neq 0$, then the system will have a unique solution, while, if $|A| = 0$, then there exists no solution.

Various numerical methods are available for finding the solution of the system of equations (2), and they are classified as *direct* and *iterative* methods. In direct methods, we get the solution of the system after performing all the steps involved in the procedure. The direct method, we consider is *Gaussian elimination.*Under iterative methods, the initial approximate solution is assumed to be known and is improved towards the exact solution in an iterative way. We consider Gauss-Seidel iterative method.

3.2 GAUSSIAN ELIMINATION METHOD

In the Gaussian elimination method, the solution to the system of equations (2) is obtained in two steps. In the first step, the given system of equations is reduced to an equivalent upper triangular form using elementary transformations. In the second step, the upper triangular system is solved using back substitution procedure by which we obtain the solution in the order x_n, x_{n-1}, x_{n-2}, …, x_2, x_1.

This method is explained by considering a system of n equations in n unknowns in the form as follows

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \cdots a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots a_{2n}x_n = b_2 \\ \vdots \qquad\quad \vdots \qquad\qquad\quad \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots a_{nn}x_n = b_n \end{array}\right\} \tag{3}$$

Step 1: We divide the first equation by a_{11} and then subtract this equation multiplied by a_{21}, a_{31},.....,a_{n1} from 2^{nd}, 3^{rd},....., n^{th} equation. Then the system (3) reduces to the following form:

$$\left.\begin{array}{l} x_1 + a'_{12}x_2 + \ldots + a'_{1n}x_n = b'_1 \\ \qquad a'_{22}x_2 + \cdots + a'_{2n}x_n = b'_2 \\ \qquad \vdots \qquad\quad \vdots \qquad\quad \vdots \\ \qquad a'_{n2}x_2 + \cdots + a'_{nn}x_n = b'_n \end{array}\right\} \tag{4}$$

Here, we observe that the last $(n-1)$ equations are independent of x_1, that is, x_1 is eliminated from the last $(n-1)$ equations.

This procedure is repeated with the second equation of (4), that is, we divide the second equation by a'_{22} and then x_2 is eliminated from 3^{rd}, 4^{th}, …, n^{th} equations of (4). The same procedure is repeated again and again till the given system assumes the following upper triangular form:

$$
\left.\begin{array}{r}
c_{11}x_1 + c_{12}x_2 + \dots + c_{1n}x_n = d_1 \\
c_{22}x_2 + \cdots + c_{2n}x_n = d_2 \\
\vdots \qquad \vdots \\
C_{nn}x_n = d_n
\end{array}\right\} \qquad (5)
$$

Step II: Now, the values of the unknowns are determined by back substitution procedure, in which we obtain x_n from the last equation of (5) and then substituting this value of $x_n$in the preceding equation, we get the value of x_{n-1}. Continuing this way, we can find the values of all other unknowns in the order x_n, x_{n-1}, ….., x_2,x_1.

Example: Solve the following system of equations using Gaussian elimination method

$$
\begin{aligned}
2x + 3y - z &= 5 \\
4x + 4y - 3z &= 3 \\
-2x + 3y - z &= 1
\end{aligned}
$$

Solution: The given system of equations is solved in two stages.

Step 1: We divide the first equation by 2 and then subtract the resulting equation (multiplied by 4 and –2) from the second and third equations, respectively. Thus, we eliminate x from the 2nd and 3rd equations. The resulting new system is given by

$$
\left.\begin{array}{r}
x + \frac{3}{2}y - \frac{z}{2} = \frac{5}{2} \\
-2y - z = -7 \\
6y - 2z = 6
\end{array}\right\} \qquad (1)
$$

Now, we divide the second equation of (1) by –2 and eliminate y from the last equation and the modified system is given by

$$
\left.\begin{array}{r}
x + \frac{3}{2}y - \frac{z}{2} = \frac{5}{2} \\
y + \frac{z}{2} = \frac{7}{2} \\
-5z = -15
\end{array}\right\} \qquad (2)
$$

Step II : From the last equation of (2), we get

$$z = 3 \qquad (3)$$

using this value of z, the second equation of (2) gives

$$y = \frac{7}{2} - \frac{3}{2} = 2 \qquad (4)$$

Using these values of y and z in the first equation of (2), we get

$$x = \frac{5}{2} + \frac{3}{2} - 3 = 1$$

Hence, the solution is given by $x = 1$, $y = 2$, $z = 3$

3.3 PARTIAL AND FULL PIVOTING (*ILL* – CONDITIONS)

The Gaussian elimination method fails if any one of the pivot elements becomes zero. In such a situation, we rewrite the equations in a different order to avoid zero pivots. Changing the order equations is called *pivoting.*

Now we introduce the concept of partial pivoting. In this technique, if the pivot a_{ii} happens to be zero, then the i^{th} column elements are searched for the numerically largest element. Let the j^{th} row (j >i) contains this element, then we interchange the i^{th} equation with the j^{th} equation and proceed for elimination. This process is continued whenever pivots become zero during elimination. For example, let us examine the solution of the following simple system:

$$10^{-6}x_1 + x_2 = 1$$
$$x_1 + x_2 = 2$$

Using Gaussian elimination method with and without partial pivoting, assuming that the solution is required accurate to only four decimal places. The solution by Gaussian elimination method gives $x_1 = 0$, $x_2 = 1$. If we use partial pivoting, the system takes the form

$$x_1 + x_2 = 2$$
$$10^{-6}x_1 + x_2 = 1$$

Using Gaussian elimination method, the solution is found to be $x_1 = 1$, $x_2 = 1$, which is a meaningful and accurate result.

In full pivoting which is also known as *complete* pivoting, we interchange rows as well as columns, such that the largest element in the matrix of the system becomes the pivot element. In this process, the position of the unknown variables also get changed. Full pivoting, in fact, is more complicated than the partial pivoting. Partial pivoting is preferred for hand computation.

Example: Solve the system of equations

$$x + \ y + z = 7$$

$$3x + 3y + 4z = 24$$

$$2x + y + 3z = 16$$

by Gaussion elimination method with partial pivoting.

Solution: In matrix notation, the given system can be written as

$$\begin{bmatrix} 1 & 1 & 1 \\ 3 & 3 & 4 \\ 2 & 1 & 3 \end{bmatrix} \begin{pmatrix} \mathrm{x} \\ y \\ z \end{pmatrix} = \begin{pmatrix} 7 \\ 24 \\ 16 \end{pmatrix} \qquad (1)$$

To start with, we observe that the pivot element $a_{11} = 1$ ($\neq 0$). However, looking at the first column, it shows that the numerically largest element is 3 which is in the second row. Hence, we interchange the first row with the second row and then proceed for elimination. Thus, equation (1) takes the form

$$\begin{bmatrix} 3 & 3 & 4 \\ 1 & 1 & 1 \\ 2 & 1 & 3 \end{bmatrix} \begin{pmatrix} \mathrm{x} \\ y \\ z \end{pmatrix} = \begin{pmatrix} 24 \\ 7 \\ 16 \end{pmatrix} \qquad (2)$$

after partial pivoting.

Step I: Dividing the first row of the system (2) by 3 and then subtracting the resulting row, multiplied by 1 and 2 from the second and third rows of the system (2), we get

$$\begin{bmatrix} 1 & 1 & \frac{4}{3} \\ & & -\frac{1}{3} \\ & -1 & \frac{1}{3} \end{bmatrix} \begin{pmatrix} \mathrm{x} \\ y \\ z \end{pmatrix} = \begin{pmatrix} 8 \\ -1 \\ 0 \end{pmatrix} \qquad (3)$$

The second row in equation (3) cannot be used as the pivot row, as $a_{22} = 0$. Interchanging the second and third rows, we obtain

$$\begin{bmatrix} 1 & 1 & \frac{4}{3} \\ & -1 & \frac{1}{3} \\ & & -\frac{1}{3} \end{bmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 8 \\ 0 \\ -1 \end{pmatrix} \qquad (4)$$

which is in the upper triangular form. From the last row of (4), we get

$$z = 3 \qquad (5)$$

The second row of (4) with this value of z gives

$$-y + 1 = 0 \text{ or } y = 1 \qquad (6)$$

Using these values of y and z, the first row of (4) gives

$$x + 1 + 4 = 8 \quad \text{or } x = 3 \qquad (7)$$

Thus, the required solution is

$$x = 3, \qquad y = 1, \qquad z = 3$$

Example: Solve by Gaussian elimination method with partial pivoting, the following system of equations:

$$0x_1 + 4x_2 + 2x_3 + 8x_4 = 24$$

$$4x_1 + 10x_2 + 5x_3 + 4x_4 = 32$$

$$4x_1 + 5x_2 + 6.5x_3 + 2x_4 = 26$$

$$9x_1 + 4x_2 + 4x_3 + 0x_4 = 21$$

Solution: The given system can be written as

$$\begin{bmatrix} 0 & 4 & 2 & 8 \\ 4 & 10 & 5 & 4 \\ 4 & 5 & 6.5 & 2 \\ 9 & 4 & 4 & 0 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 24 \\ 32 \\ 26 \\ 21 \end{pmatrix} \qquad (1)$$

To start with, we note that the pivot row, that is, the first row has a zero pivot element ($a_{11} = 0$). This row should be interchanged with any row following it, which on becoming a pivot row should not have a zero pivot element. While interchanging rows, it is better to interchange with a row having largest pivotal element. Thus, we interchange the first and fourth rows, which is called partial pivoting and we get

$$\begin{bmatrix} 9 & 4 & 4 & 0 \\ 4 & 10 & 5 & 4 \\ 4 & 5 & 6.5 & 2 \\ 0 & 4 & 2 & 8 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 21 \\ 32 \\ 26 \\ 24 \end{pmatrix} \qquad (2)$$

We note that, in partial pivoting, the unknown vector remains unaltered, while the right-hand side vector gets changed.

Now, carry out Gaussian elimination process in two steps.

Step I: In this case, divide the first row of the system (2) by 9 and then subtracting this resulting row multiplied by 4 from the second and third rows of equation (2), we get

$$\begin{bmatrix} 1 & \frac{4}{9} & \frac{4}{9} & 0 \\ 0 & 8.222 & 3.222 & 4 \\ 0 & 3.222 & 4.722 & 2 \\ 0 & 4 & 2 & 8 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 2.333 \\ 22.667 \\ 16.667 \\ 24 \end{pmatrix} \qquad (3)$$

Now we divide the second pivot row by 8.222 and subtract the resultant row multiplied by 3.222 and 4 from the third and fourth rows of equation (3), we obtain

$$\begin{bmatrix} 1 & \frac{4}{9} & \frac{4}{9} & 0 \\ 0 & 1 & 0.392 & 0.486 \\ 0 & 0 & 3.459 & 0.432 \\ 0 & 0 & 0.432 & 6.054 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 2.333 \\ 2.757 \\ 7.784 \\ 12.973 \end{pmatrix} \qquad (4)$$

Finally, we divide the third pivot row by 3.459 and subtract the resultant row multiplied by 0.432 from fourth row of equation (4), thereby getting the triangular form

$$\begin{bmatrix} 1 & \frac{4}{9} & \frac{4}{9} & 0 \\ 0 & 1 & 0.392 & 0.486 \\ 0 & 0 & 1 & 0.125 \\ 0 & 0 & 0 & 5.999 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 2.333 \\ 2.757 \\ 2.250 \\ 11.999 \end{pmatrix} \qquad (5)$$

Step II: From the last row of equation (5), we get x_4= 2.00. Using this value of x_4 into the third row of equation (5), we obtain

$$x_3 + 0.25 = 2.25 \qquad \text{or } x_3 = 2.00 \qquad (6)$$

Similarly, we get

$$x_2 = 1.00, \qquad x_1 = 1.00$$

Thus, the solution of the given system is given by

$$x_1 = 1.0, \qquad x_2 = 1.0, \qquad x_3 = 2.0, \qquad x_4 = 2.0$$

3.4 GAUSS-SEIDEL ITERATION METHOD

It is *a* well-known iterative method for solving a system of linear equations.

$$a_{11}x_1 + a_{12}x_2 + \ldots\ldots + a_{1n}x_n = b_1,$$

$$a_{21}x_1 + a_{22}x_2 + \ldots\ldots + a_{2n}x_n = b_2$$

....

$$a_{n1}x_1 + a_{n2}x_2 + \ldots\ldots + a_{nn}x_n = b_n$$

In Gauss-Seidel method, the corresponding elements of $x_i^{(r+1)}$ replaces those of $x_i^{(r)}$ as soon as they become available. Hence, it is called the method of successive displacements. In this method $(r+1)^{th}$ approximation or iteration is computed from

$$
\left.\begin{aligned}
x_1^{(r+1)} &= \frac{b_1}{a_{11}} - \frac{a_{12}}{a_{11}} x_2^{(r)} - \cdots - \frac{a_{1n}}{a_{11}} x_n^{(r)} \\
x_2^{(r+1)} &= \frac{b_2}{a_{22}} - \frac{a_{21}}{a_{22}} x_1^{(r+1)} - \cdots - \frac{a_{2n}}{a_{22}} x_n^{(r)} \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
x_n^{(r+1)} &= \frac{b_n}{a_{nn}} - \frac{a_{n1}}{a_{nn}} x_1^{(r+1)} - \cdots - \frac{a_{n,(n-1)}}{a_{nn}} x_{n-1}^{(r+1)}
\end{aligned}\right\} \qquad (1)
$$

Thus, the general procedure can be written in the following compact form

$$
x_i^{(r+1)} = \frac{b_i}{a_{ii}} - \sum_{j=1}^{i-1} \frac{a_{ij}}{a_{ii}} x_j^{(r+1)} - \sum_{j=i+1}^{n} \frac{a_{ij}}{a_{ii}} x_j^{(r)} \qquad (2)
$$

for all $i = 1,2,\ldots\ldots, n$ and $r = 1,2,\ldots$

To describe system (1) in the first equation, we substitute the r-th approximation into the right-hand side and denote the result by $x_1^{(r+1)}$. In the second equation, we substitute $(x_1^{(r+1)}, x_3^{(r)}, \ldots, x_n^{(r)})$ and denote the result $x_2^{(r+1)}$. In the third equation, we substitute $(x_1^{(r+1)}, x_2^{(r+1)}, x_4^{(r+1)}, \ldots, x_n^{(r)})$ and denote the result by $x_3^{(r+1)}$, and so on. This process is continued till we arrive at the desired result.

Example: Find the solution of the following system of equations.

$$
\begin{aligned}
x_1 - 0.25x_2 - 0.25x_3 &= 0.5 \\
-0.25x_1 + x_2 - 0.25x_4 &= 0.5 \\
-0.25x_1 + x_3 - 0.25x_4 &= 0.25 \\
-0.25x_2 - 0.25x_3 + x_4 &= 0.25
\end{aligned}
$$

using Gauss-Seidel method and perform the first four iterations.

Solution: The given system of equations can be rewritten as

$$
x_1 = 0.5 + 0.25x_2 + 0.25x_3
$$

$x_2 = 0.5 + 0.25x_1 + 0.25x_4$ (1)

$x_3 = 0.25 + 0.25x_1 + 0.25x_4$

$x_4 = 0.25 + 0.25x_2 + 0.25x_3$

Taking $x_2 = x_3 = x_4 = 0$ on the right- hand side of the first equation of system (1), we get $x_1^{(1)} = 0.5$ Taking $x_3 = x_4 = 0$ and the current value of x_1, we get

$$x_2^{(1)} = 0.5 + (0.25)(0.5) + 0 = 0.625$$

form the second equation of system (1). Further, we take $x_4 = 0$ and the current value of x_1, we obtain

$$x_3^{(1)} = 0.25 + (0.25)(0.5) + 0 = 0.375$$

from the third equation of system (1). Now, using the current values of x_2 and x_3, the fourth equation of system (1) gives

$$x_4^{(1)} = 0.25 + (0.25)(0.625) + (0.25)(0.375) = 0.5$$

The Gauss-Seidel iterations for the given set of equations can be written as

$$x_1^{(r+1)} = 0.5 + 0.25x_2^{(r)} + 0.25x_3^{(r)}$$

$$x_2^{(r+1)} = 0.5 + 0.25x_1^{(r+1)} + 0.25x_4^{(r)}$$

$$x_3^{(r+1)} = 0.25 + 0.25x_1^{(r+1)} + 0.25x_4^{(r)}$$

$$x_4^{(r+1)} = 0.25 + 0.25x_2^{(r+1)} + 0.25x_3^{(r+1)}$$

Now, by Gauss-Seidel procedure, the second and subsequent approximations can be obtained and the sequence of the first four approximation are given as below:

Iteration	Variables			
Number (*r*)	x_1	x_2	x_3	x_4
1	0.5	0.625	0.375	0.5
2	0.75	0.8125	0.5625	0.5938
3	0.8438	0.8594	0.60941	0.6172
4	0.8672	0.8711	0.6211	0.6231

3.5 SOLUTION OF A DIFFERENTIAL EQUATION

The solution of an ordinary differential equation means finding an explicit expression for *y* in terms of a finite number of elementary functions of *x*.

Let us consider the first order differential equation

$$dy/dx = f(x,y), \text{ given } \quad y(x_0) = y_0 , \qquad \text{---(1)}$$

to study the various numerical methods of solving such equations. In most of these methods, we replace the differential equation by a difference equation and then solve it. These methods yield solutions either as a power series in *x* from which the values of *y* can be found by direct substitution, or a set of values of *x* and *y*. The method of Taylor series belong to the former class of solutions. In this method, *y* in (1) is approximated by a truncated series, each terms of which is a function of *x*. The information about the curve at one point is utilized and the solution is not iterated. As such, these are referred to as *single-step methods*. The methods of Euler, Runga-Kutta, Milne, Adams-Bashforth etc. belong to the latter class of solutions. In these methods, the next point on the curve is evaluated in short steps ahead, by performing iterations till sufficient accuracy is achieved. As such, these methods are called *step-by-step methods*.

Euler and Runga-Kutta methods are used for computing y over a limited range of x-values whereas Milne and Adams methods may be applied for finding y over a wider range of x-values. Therefore Milne and Adams methods require starting values which are found by Taylor series or Runga-Kutta methods.

Initial and boundary conditions:

An ordinary differential equations of the nth order is of the form

$$F(x, y, dy/dx, d^2y/dx^2, \ldots, d^ny / dx^n) = 0 \qquad (2)$$

Its general solution contains n arbitrary constants and is of the form

$$\phi(x,y, \quad c_1, c_2, \ldots\ldots, c_n) = 0 \qquad (3)$$

To obtain its particular solution, n conditions must be given so that the constants $c_1, c_2, \ldots\ldots, c_n$ can be determined. If these conditions are prescribed at one point only (say x_0) then the differential equation together with the conditions constitute an initial value problem of the n^{th} order. If the conditions are prescribed at two or more points, then the problem is termed as boundary value problem.

In this lesson, we shall first describe methods for solving initial value problems and then explain finite difference method for solving boundary value problems.

3.6 TAYLOR'S SERIES METHOD

Consider the first order equation

$$\frac{dy}{dx} = f(x, y) \tag{1}$$

Differentiating (1) we have

$$\frac{d^2 y}{dx^2} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx},$$

i.e., $$y^n = f_x + f_y f' \tag{2}$$

Differentiating this successively, we can get y''', y^{iv} etc. Putting$x = x_0$ and $y = 0$, the values of $(y')_0$, $(y'')_0$, $(y''')_0$ can be obtained. Hence the Taylor's series

$$y = y_0 + (x - x_0)(y')_0 + \frac{(x - x_0)^2}{2!}(y'')_0 + \frac{(x - x_0)^3}{3!}(y''')_0 + \ldots. \tag{3}$$

gives the values of y for every value of x for which (3) converges.

On finding the value y_1 for $x = x_1$ from (3), y', y'' etc. can be evaluated at $x = x_1$ by means of (1), (2) etc. Then y can be expanded about $x = x_1$. In this way, the solution can be extended beyond the range of convergence of series (3).

Remarks: This is a single step method and works well so long as the successive derivatives can be calculated easily. It is useful for finding starting values for the application of powerful methods like Runga-Kutta, Milne and Adams-Bashforth.

Example: Find by Taylor's series method, the values of y at $x = 0.1$ and $x = 0.2$ to five places of decimals from $dy/dx = x^2y - 1$, $y(0) = 1$.

Solution:

Here $(y_0) = 1$.

$\therefore$ Differentiating successively and substituting, we get

$y' = x^2y - 1,$ $\quad (y')_0 = -1$

$y'' = 2xy + x^2y',$ $\quad (y'')_0 = 0$

$y''' = 2y + 4xy' + x^2y'',$ $\quad (y''')_0 = 2$

$y^{iv} = 6y' + 6xy'' + x^2y''',$ $\quad (y^{iv})_0 = -6$, etc.

Putting these values in the Taylor's series, we have

$$y = 1 + x\,(-1) + \frac{x^2}{2!}(0) + \frac{x^3}{3!}(2) + \frac{x^4}{4!}(-6) + ...$$

$$= 1 - x + \frac{x^3}{3} - \frac{x^4}{4} + ...$$

Hence $y\,(0.1) = 0.90033$ and $y(0.2) = 0.80227$.

Example: Apply Taylor's method to obtain approximate value of y at $x = 0.2$ for the differential equation $dy/dx = 2y + 3e^x$, $y(0) = 0$. Compare the numerical solution obtained with the exact solution.

Solution:

We have $y' = 2y + 3e^x$; $\quad y'(0) = 2y(0) + 3e^0 = 3$.

Differentiating successively and substituting $x = 0$, $y = 0$ we get

$y'' = 2y' + 3e^x$, $\quad y''(0) = 2y'(0) + 3 = 9$

$y''' = 2y'' + 3e^x$, $\quad y'''(0) = 2y''(0) + 3 = 21$ $\quad y^{iv} = 2y''' + 3e^x$,

$y^{iv}(0) = 2y'''(0) + 3 = 45$ etc.

Putting these values in the Taylor's series, we have

$$y(x) = y(0) + xy'(0) + \frac{x^2}{2!}y''(0) + \frac{x^3}{3!}y'''(0) + \frac{x^4}{4!}y^{iv}(0) + \ldots\ldots.$$

$$= 0 + 3x + \frac{9}{2}x^2 + \frac{21}{6}x^3 + \frac{45}{24}x^4 + \ldots\ldots.$$

$$= 3x + \frac{9}{2}x^2 + \frac{7}{2}x^3 + \frac{15}{8}x^4 + \ldots\ldots.$$

Hence $y(0.2) = 3(0.2) + 4.5(0.2)^2 + 3.5(0.2)^3 + 1.875(0.4)^4 + \ldots..$

$$= 0.8110 \qquad (1)$$

Now $\frac{dy}{dx} - 2y = 3e^x$ is a Leibniz's linear in x. Its I.F. being e^{-2x} , the solution is

$$ye^{-2x} = \int 3e^x . e^{-2x} dx + c = -3e^{-x} + c$$

or $\quad y = -3e^x + ce^{2x}$

Since $y = 0$ when $x = 0$, this implies c =3. Thus the exact solution is

$$y = 3(e^{2x} - e^x)$$

when $x = 0.2$, $y = 3(e^{0.4} - e^{0.2}) = 0.8112 \qquad (2)$

Comparing (1) and (2) it is clear that (1) approximates to the exact value upto 3 decimal places.

3.7 EULER'S METHOD

Consider the equation

$$\frac{dy}{dx} = f(x, y), \qquad y(x_0) = y_o. \qquad (1)$$

The curve of solution through $P(x_0, y_0)$ for this differential equation is shown in the following figure. Further, we want to find the ordinate of any other point Q on this curve

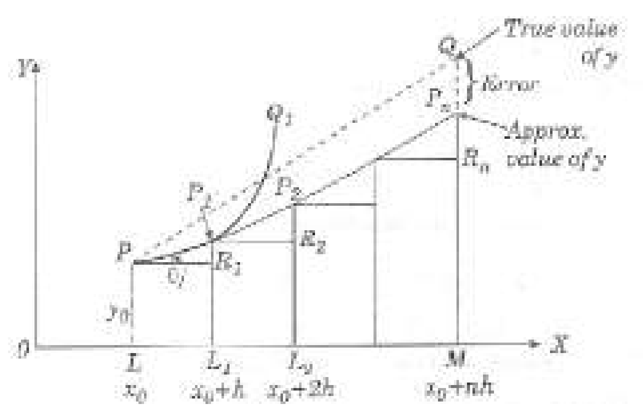

Fig 7.1

Let us divide LM into n sub-intervals each of width h at $L_1, L_2, \ldots\ldots$ so that h is quite small.

In the interval LL_1, we approximate the curve by the tangent at P. If the ordinate through L_1 meets this tangent in $P_1(x_0+h, y_1)$, then

$$y_1 = L_1P_1 = LP + R_1P_1 = y_0 + PR_1 \tan\theta$$

$$= y_0 + h\left(\frac{dy}{dx}\right)_P = y_0 + h\, f(x_0, y_0)$$

Let P_1Q_1 be the curve of solution of (1) through P_1 and let its tangent at P_1 meet the ordinate through L_2 in $P_2(x_0+2h, y_2)$. Then

$$y_2=y_1+h(fx_0+h,y_1) \qquad (2)$$

Repeating this process n times, we finally reach on an approximation MP_n of MQ given by

$$y_n = y_{n-1} + hf(x_0+\overline{n-1}\,h,\ y_{n-1})$$

or $\quad y_{n+1} = y_n + hf(x_n,y_n)$, where $x_n = x_0 + nh$.

This is Euler's method of finding an approximate solution of (1).

Example: Using Euler's method, find an approximate value of y corresponding to $x = 1$, given that $dy/dx = x + y$ and $y = 1$ when $x = 0$.

Solution:

We take $n = 10$ and $h = 0.1$ which is sufficiently small. The various calculations are arranged as follows:

x	y	$x + y = dy/dx$	$old\ y + 0.1\ (dy/dx) = new\ y$
0.0	1.00	1.00	1.00+0.1(1.00)=1.10
0.1	1.10	1.20	1.10+0.1(1.20)=1.22
0.2	1.22	1.42	1.22+0.1(1.42)=1.36
0.3	1.36	1.66	1.36+0.1(1.66)=1.53
0.4	1.53	1.93	1.53+0.1(1.93)=1.72
0.5	1.72	2.22	1.72+0.1(2.22)=1.94
0.6	1.94	2.54	1.94+0.1(2.54)=2.19
0.7	2.19	2.89	2.19+0.1(2.89)=2.48
0.8	2.48	3.29	2.48+0.1(3.29)=2.81
0.9	2.81	3.71	2.81+0.1(3.71)=3.18
1.0	3.18		

Thus the required approximate value of $y = 3.18$

Example: Given $\frac{dy}{dx} = \frac{y-x}{y+x}$ with initial condition $y = 1$ at $x = 0$; find y for $x = 0.1$ by Euler's method.

Solution:

We divide the interval (0, 0.1) into five steps, *i.e.,* we take $n = 5$ and $h = 0.02$. The various calculations are arranged as follows:

x	*y*	*dy/dx*	*old y* + 0.02 *(dy/dx) = new y*
0.00	1.0000	1.0000	1.0000+0.02(1.0000)=1.0200
0.02	1.0200	0.9615	1.0200+0.02(0.9615)=1.0392
0.04	1.0392	0.926	1.0392+0.02(0.926)=1.0577
0.06	1.0577	0.893	1.0577+0.02(0.893)=1.0756
0.08	1.0756	0.862	1.0756+0.02(0.862)=1.0928
0.10	1.0928		

Hence the required approximate value of $y = 1.0928$.

3.8 RUNGE-KUTTA METHOD

The Taylor's series method of solving differential equations numerically involved in finding the higher order derivatives. However there is a class of methods known as RungeKutta methods which do not require the calculations of higher order derivatives and give greater accuracy. The Runge-Kutta formulae possess the advantage of requiring only the function values at some selected points.

3.8.1 First order R-K method:

From Euler's method, we have

$$y_1 = y_0 + hf(x_0, y_0) = y_0 + hy_0' \qquad [\because y' = f(x,y)]$$

Expanding by Taylor's series

$$y_1 = y(x_0+h) = y_0 + hy_0' + \frac{h^2}{2} y_0'' + \ldots\ldots.$$

It follows that the Euler's method agrees with the Taylor's series solution upto the term in *h*.

Hence, Euler's method is the Runge-Kutta method of the first order.

3.8.2 Second order R-K method:

The modified Euler's method gives

$$y_1 = y + \frac{h}{2}[f(x_0, y_0) + f(x_0+h, y_1)] \qquad (1)$$

Substituting $y_1 = y_0 + +hf(x_0, y_0)$ on the right hand side of (1), we obtain

$$y_1 = y_0 + \frac{h}{2}[f_0 + f(x_0+h,\ y_0 + hf_0)], \qquad (2)$$

where $f_0 = f(x_0, y_0)$.

Expanding L.H.S. by Taylor's series, we get

$$y_1 = y(x_0+h) = y_0 + hy_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0''' + \ldots.. \qquad (3)$$

Expanding $f(x_0+h,\ y_0+hf_0)$ by Taylor's series for a function of two variables, (2) gives

$$y_1 = y_0 + \frac{h}{2}\left[f_0 + \left\{ f(x_0, y_0) + h\left(\frac{\partial f}{\partial x}\right)_0 + hf_0\left(\frac{\partial f}{\partial y}\right)_0 + \mathrm{O}(\mathrm{h}^2) \right\} \right]$$

$$= y_0 + \frac{1}{2}\left[hf_0 + hf_0 + h^2\left\{ \left(\frac{\partial f}{\partial x}\right)_0 + \left(\frac{\partial f}{\partial y}\right)_0 f_0 \right\} + \mathrm{O}(\mathrm{h}^3) \right]$$

$$= y_0 + hf_0 + \frac{h^2}{2} f_0' + O(h^3) \qquad \left[\because \frac{df(x,y)}{dx} = \frac{\partial f}{\partial x} + f\frac{\partial f}{\partial y}\right]$$

$$y_1 = y_0 + hy_0' + \frac{h^2}{2!} y_0'' + O(h^3) \qquad (4)$$

Comparing (3) and (4), it follows that this method agrees with the Taylor's series solution upto the term in h^2.

The Runge-Kutta method of the second order is same as modified Euler's method.

$\therefore$ The second order Runge-Kutta formula is

$y_1 = y_0 + \frac{1}{2}(k_1 + k_2)$, where

$k_1 = hf(x_0, y_0)$

$k_2 = hf(x_0 + h, y_0 + k_1)$.

Similarly, the third order Runge-Kutta formula is

$y_1 = y_0 + \frac{1}{6}(k_1 + 4k_2 + k_3)$, where

$k_1 = hf(x_0, y_0)$

$k_2 = hf(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1)$

and $k_3 = hf(x_0 + h, y_0 + k')$, where

$k' = hf(x_0 + h, y_0 + k_1)$.

3.8.3 Fourth order R-K method :

This method is most commonly used and **working rule** for finding the increment k of y corresponding to an increment h of x by Runge-Kutta method for

$$\frac{dy}{dx} = f(x, y), \qquad y(x_0) = y_0$$

is as follows:

Calculate successively

$k_1 = hf(x_0, y_0)$

$k_2 = hf(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1)$,

$k_3 = hf(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_2)$,

$k_4 = hf(x_0 + h, y_0 + k_3)$

Finally compute $k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

and $y_1 = y_0 + k$.

Example: Given $\frac{dy}{dx} = 1 + y^2$, where $y = 0$ when $x = 0$, find $y(0.2)$, y(0.4) and $y(0.6)$ using Runge-Kutta method.

Solution:

we take $h = 0.2$ with $x_0 = y_0 = 0$, we obtain

$k_1 = 0.2$,

$k_2 = 0.2\ (1.01) = 0.202$,

$k_3 = 0.2\ (1+0.010201) = 0.20204$,

$k_4 = 0.2\ (1+0.040820) = 0.20816$,

and $y(0.2) = 0 + \frac{1}{6}(k_1+2k_2+2k_3+k_4)$

$= 0.2027$, which is correct to four decimal places.

To compute y (0.4), we take $x_0 = 0.2$ $y_0 = 0.2027$ and $h = 0.2$. With these values, we get

$k_1 = 0.2\,[1+(0.2027)^2] = 0.2082,$

$k_2 = 0.2\,[1+(0.3068)^2] = 0.2188,$

$k_3 = 0.2\,[1+(0.3121)^2] = 0.2195,$

$k_4 = 0.2\,[1+(0.4222)^2] = 0.2356,$

and $y(0.4) = 0.2027+0.2201$

$= 0.4228$, correct to four decimal places.

Finally, taking $x_0 = 0.4$, $y_0 = 0.4228$ and $h = 0.2$, and proceeding as above, we obtain $y(0.6) = 0.6841$.

3.9 PREDICTOR-CORRECTOR METHODS

In the methods described so far, to solve a differential equation over a single interval, say from $x = x_n$ to $x = x_{n+1}$, we require information only at the beginning of the interval, *i.e.,* at $x = x_n$. *Predictor-corrector methods* are methods which require function values at x_n, x_{n-1}, x_{n-2},...... for the computation of the function value at x_{n+1}. A predictor formula is used to predict the value of y at x_{n+1} and then a corrector formula is used to improve the value of y_{n+1}.

Here we describe two such methods:

1. Milne's method
2. Adams-Bashforth method

3.9.1Milne's method:

Given $dy/dx = f(x,y)$ and $y = y_0$, $x = x_0$; to find an approximate value of y for $x = x_0 + nh$ by Milne's method, we proceed as follows:

The value $y_0 = y(x_0)$ being given, we compute

$y_1 = y(x_0+h)$, $y_2 = (x_0+2h)$, $y_3 = y\,(x_0+3h)$,

by Taylor's series method.

Next we calculate

$$f_0 = f(x_0, y_0), f_1 = f(x_0 + h, y_1), f_2 = f(x_0 + 2h, y_2), f_3 = f(x_0 + 3h, y_3)$$

Then to find $y_4 = y(x_0 + 4h)$, we substitute Newton's forward interpolation formula

$$f(x, y) = f_0 + n\Delta f_0 + \frac{n(n-1)}{2}\Delta^2 f_0 + \frac{n(n-1)(n-2)}{6}\Delta^3 f_0 + \ldots\ldots.$$

in the relation

$$y_4 = y_0 + \int_{x_0}^{x_0+4h} f(x, y)dx.$$

Therefore

$$y_4 = y_0 + \int_{x_0}^{x_0+4h}\left(f_0 + n\Delta f_0 + \frac{n(n-1)}{2}\Delta^2 f_0 + \ldots\ldots\ldots.\right)dx$$

[put $x = x_0 + nh$, $dx = hdn$]

$$= y_0 + h\int_0^4\left(f_0 + n\Delta f_0 + \frac{n(n-1)}{2}\Delta^2 f_0 + \ldots\ldots\ldots.\right)dn$$

$$y_4 = y_0 + h\left(4f_0 + 8\Delta f_0 + \frac{20}{3}\Delta^2 f_0 + \frac{8}{3}\Delta^3 f_0 + \ldots\ldots\ldots.\right)$$

Neglecting fourth and higher order differences and expressing Δf_0, $\Delta^2 f_0$ and $\Delta^3 f_0$ in terms of the function values, we get

$$y_4 = y_0 + \frac{4h}{3}(2f_1 - f_2 + 2f_3)$$

which is called a 'predictor'.

Having found y_4, we obtain a first approximation to

$f_4 = f(x_0 + 4h, y_4)$.

Then a better value of y_4 is found by Simpson's rule as

$$y_4 = y_2 + \frac{h}{3}(f_2 + 4f_3 + f_4)$$

which is called a 'corrector'.

Then an improved value of f_4 is computed and again the corrector is applied to find a still better value of y_4. We repeat this step until y_4 remains unchanged.

Once y_4 and f_4 are obtained to desired degree of accuracy, $y_5 = y(x_0 + 5h)$ is found from the predictor as

$$y_5 = y_1 + \frac{4h}{3}(2f_2 - 4f_3 + 2f_4)$$

and $f_5 = f(x_0 + 5h, y_5)$ is calculated. Then a better approximation to the value of y_5 is obtained from the corrector as

$$y_5 = y_3 + \frac{h}{3}(f_3 + 4f_4 + f_5).$$

We repeat this step till y_5 becomes stationary and we, then proceed to calculate y_6 as before.

This is Milne's Predictor – Corrector method.

Example: Using Runge-Kutta method of order 4, find y for x = 0.1, 0.2, 0.3 given that $dy/dx = xy + y^2$, $y(0) = 1$. Find the solution at $x = 0.4$ using Milne's method.

Solution:

We have $\quad f(x,y) = xy + y^2$.

To find $y(0.1)$, here $x_0 = 0$, $y_0 = 1$, $h = 0.1$.

$\therefore \quad k_1 = h\, f(x_0,y_0) = (0.1)\, f(0,1) = 0.1000$

$$k_2 = h\, f(x_0 + \frac{1}{2}h\,, y_{0+}\frac{1}{2}k_1) = (0.1)\, f(0.05, 1.05) \qquad = 0.1155$$

$$k_3 = h\, f(x_0 + \frac{1}{2}h\,, y_{0+}\frac{1}{2}k_2) = (0.1)\, f(0.05, 1.0577) = 0.1172$$

$$k_4 = h\, f(x_0 + h\,, y_{0+}k_3) = (0.1)\, f(0.1, 1.1172) = 0.1360$$

$$k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= \frac{1}{6}(0.1 + 0.231 + 0.2343 + 0.1360) = 0.1169$$

Thus $y(0.1) = y_1 = y_0 + k = 1.1169$,

To find y (0.2), here $x_1 = 0.1$, $y_1 = 1.1169$, $h = 0.1$.

$$k_1 = hf(x_1, y_1) = (0.1)\, f(0.1, 1.1169) = 0.1359$$

$$k_2 = h\, f(x_1 + \frac{1}{2}h\,, y_{1+}\frac{1}{2}k_1) = (0.1)\, f(0.15, 1.1848) = 0.1581$$

$$k_3 = h\, f(x_1 + \frac{1}{2}h\,, y_1 + \frac{1}{2}k_2) = (0.1)\, f(0.15, 1.1959) = 0.1609$$

$$k_4 = h\, f(x_1 + h, y_1 + k_3) = (0.1)\, f(0.2, 1.2778) = 0.1888$$

$$k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 0.1605$$

Thus $y(0.2) = y_2 = y_1 + k = 1.2773$.

To find $y(0.3)$, here $x_2 = 0.2$, $y_2 = 1.2773$, $h = 0.1$.

$$k_1 = hf(x_2, y_2) = (0.1)\, f(0.2, 1.2773) = 0.1887$$

$$k_2 = h\, f(x_2 + \frac{1}{2}h\,, y_{2+}\frac{1}{2}k_1) = (0.1)\, f(0.25, 1.3716) = 0.2224$$

$$k_3 = h\,f(x_2+\frac{1}{2}h\,,\ y_2+\frac{1}{2}k_2) = (0.1)\,f(0.25,1.3885) = 0.2275$$

$$k_4 = h\,f(x_2+h,\ y_2+ k_3) = (0.1)\,f(0.3,1.5048) = 0.2716$$

$$k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 0.2267$$

Thus $y(0.3) = y_3 = y_2 + k = 1.504$.

Now the starting values for the Milne's method are

$x_0= 0.0$	$y_0= 1.0000$	$f_0= 1.0000$
$x_1= 0.1$	$y_1= 1.1169$	$f_1= 1.3591$
$x_2= 0.2$	$y_2= 1.2773$	$f_2= 1.8869$
$x_3= 0.3$	$y_3= 1.5049$	$f_3= 2.7132$

Using the predictor,

$$y_4 = y_0 + \frac{4h}{3}(2f_1 - f_2 + 2f_3).$$

$x_4 = 0.4$ $\qquad$ $y_4 = 1.8344$ $\qquad$ $f_4 = 4.0988$

and the corrector,

$$y_4 = y_2 + \frac{h}{3}(f_2 + 4f_3 + f_4)$$

$$y_4 = 1.2773 + \frac{0.1}{3}[1.8869 + 4(2.7132) + 4.098]$$

$$= 1.8386, \qquad f_4 = 4.1159$$

Again using the corrector,

$$y_4 = 1.2773 + \frac{0.1}{3}[1.8869 + 4(2.7132) + 4.1159])$$

$= 1.8391,$ $\qquad\qquad\qquad f_4 = 4.1182 \qquad\qquad (1)$

Again using the corrector,

$$y_4 = 1.2773 + \frac{0.1}{3}[1.8869 + 4(2.7132) + 4.1182]$$

$= 1.8392$ which is same as(1)

Hence $y\ (0.4) = 1.8392$

3.9.2 Adams –Bashforth method:

Given $\frac{dy}{dx} = f(x, y)$ and $y_0 = y(x_0)$, we compute

$$y_{-1} = y(x_0 - h),\ \ y_{-2} = y(x_0 - 2h),\ y_{-3} = y(x_0 - 3h),$$

by Taylor's series or Euler's method or Rugne –Kutta method. Next we calculate

$$f_{-1} = f(x_0 - h, y_{-1}), \qquad f_{-2} = f(x_0 - 2h,\ y_{-2}), \qquad f_{-3} = f(x_0 - 3h, y_{-3}).$$

Then to find y_1, we substitute Newton's backward interpolation formula

$$f(x, y) = f_0 + n\nabla f_o + \frac{n(n+1)}{2}\nabla^2 f_0 + \frac{n(n+1)(n+2)}{6}\nabla^3 f_0 + \ldots\ldots$$

in $\qquad y_1 = y_0 + \int_{x_0}^{x_0+h} f(x, y)dx. \qquad (1)$

$$\therefore \qquad y_1 = y_0 + \int_{x_0}^{x_1}\left(f_0 + n\nabla f_0 + \frac{n(n+1)}{2}\nabla^2 f_0 + \ldots\ldots\ldots\right)dx.$$

[put $x = x_0 + nh,\ dx = hdn$]

$$= y_0 + h\int_0^1\left(f_0 + n\nabla f_0 + \frac{n(n+1)}{2}\nabla^2 f_0 + \ldots\ldots\ldots\right)dn$$

$$= y_0 + h\left(f_0 + \frac{1}{2}\nabla f_0 + \frac{5}{12}\nabla^2 f_0 + \frac{3}{8}\nabla^2 f_0 + \ldots\ldots\ldots\right)$$

Neglecting fourth and higher order differences and expressing $\nabla f_0, \nabla^2 f_0$ and $\nabla^3 f_0$ in terms of function values, we get

$$y_1 = y_0 + \frac{h}{24}(55 f_0 - 59 f_{-1} + 37 f_{-2} - 9 f_{-3}) \qquad (2)$$

This is *called Adams-Bashforth predictor formula.* Having found y_1, we find $f_1 = f(x_0+h_1, y_1)$. Then to find a better value of y_1, we derive a corrector formula by substituting Newton's backward formula at f_1, *i.e.,*

$$f(x,y) = f_1 + n\nabla f_1 + \frac{n(n+1)}{2}\nabla^2 f_1 + \frac{n(n+1)(n+2)}{6}\nabla^3 f_1 + \ldots\ldots \qquad \text{in (1).}$$

$$\therefore \qquad y_1 = y_0 + \int_{x_0}^{x_1}\left(f_1 + n\nabla f_1 + \frac{n(n+1)}{2}\nabla^2 f_1 + \ldots\ldots\ldots\right)dx.$$

[put $x = x_1 + nh,\ dx = h\, dn$]

$$= y_0 + h\int_{-1}^{0}\left(f_1 + n\nabla f_1 + \frac{n(n+1)}{2}\nabla^2 f_1 + \ldots\ldots\ldots\right)dn$$

$$or \qquad y_1 = y_0 + h\left(f_1 - \frac{1}{2}\nabla f_1 - \frac{1}{12}\nabla^2 f_1 - \frac{1}{24}\nabla^2 f_1 - \ldots\ldots\ldots\right)$$

Neglecting fourth and higher order differences andexpressing $\nabla f_1, \nabla^2 f_1$ and $\nabla^3 f_1$ in terms of function values, we obtain

$$y_1 = y_0 + \frac{h}{24}(9 f_1 + 19 f_0 - 5 f_{-1} + f_{-2}) \qquad (3)$$

which is called *Adams-Moulton corrector formula.*

Then an improved value of f_1 is calculated and again the corrector (3) is applied to find a still better value y_1. This step is repeated till y_1 remains unchanged and then we proceed to calculate y_2 as above.

Example: Given $\frac{dy}{dx} = x^2(1+y)$ and $y(1) = 1$, $y(1.1) = 1.233$, $y(1.2) = 1.548$, $y(1.3) = 1.979$, evaluate $y(1.4)$ by Adams-Bashforth method.

Solution:

Here $f(x, y) = x^2(1+y)$.

Starting values of the Adams-Bashforth method with $h = 0.1$ are

$x = 1.0$, $y_{-3} = 1.000$, $f_{-3} = (1.0)^2\ (1+1.000) = 2.000$

$x = 1.1$, $y_{-2} = 1.233$, $f_{-2} = 2.702$

$x = 1.2$, $y_{-1} = 1.548$, $f_{-2} = 3.669$

$x = 1.3$, $y_0 = 1.979$, $f_0 = 5.035$

Using the predictor,

$$y_1 = y_0 + \frac{h}{24}(55 f_0 - 59 f_{-1} + 37 f_{-2} - 9 f_{-3})$$

$x = 1.4$, $y_1 = 2.573$, $f_1 = 7.004$

Using the corrector,

$$y_1 = y_0 + \frac{h}{24}(9 f_1 + 19 f_0 - 5 f_{-1} + f_{-2})$$

$$y_1 = 1.979 + \frac{0.1}{24}(9 \times 7.004 + 19 \times 5.035 - 5 \times 3.669 + 2.702)$$

$$= 2.575$$

Hence $y(1.4) = 2.575$.

3.10 CHECK YOUR PROGRESS

1. Solve the equations x + 4y – z = -5; x + y –6 z = -12;

   3x - y – z = 4, by using Gauss elimination method.

2. Solve the following by Gauss – Seidel iteration method:

   (i) 2x + y + 6z = 9; 8x + 3y + 22 = 13; x + 5y + z = 7

   (ii) 5x + 2y + z = 12; x + 4y + 22 = 15; x + 2y + 5z = 20

3. Using Taylor's series method, evaluate y(0.1) if y(x) satisfies

   $$\frac{dy}{dx} = xy + 1, \quad y(0) = 1$$

4. Given

   $$\frac{dy}{dx} = \frac{y - x}{y + x}, \quad \text{with } y(0) = 1,$$

   find y for x = 0.1

5. Apply Runge – Kutta method to find an approximate value of y for x = 0.2 in steps of 0.1, if

   $$\frac{dy}{dx} = x + y^2, \quad \text{where } y(0) = 1.$$

6. Apply Milne's method to find a solution of the differential equation

   $$\frac{dy}{dx} = x - y^2, \qquad \text{in the range } 0 < x < 1 \text{ for } y = 0 \text{ at } x = 0.$$

3.11 SUMMARY

Linear Equations: -In mathematics, a linear equation is an equation that may be put in the form where are the variables, and are the coefficients, which are often real numbers. The coefficients may be considered as parameters of the equation, and may be arbitrary expressions, provided they do not contain any of the variables.

Ordinary Differential Equations:- A most general form of an ordinary differential equation (**ode**) is given by

$$f(x, y, y', \ldots, y^{(m)}) = 0$$

where **x** is the independent variable and **y** is a function of **x**. $y', y'' \ldots y^{(m)}$ are respectively, first, second and m^{th} derivatives of **y** with respect to **x**.

Some definitions:

- the highest derivative in the **ode** is called the **order** of the **ode**.
- the degree of the highest order derivative in the **ode** is called the **degree** of the **ode**.
- the differential equation is linear if no product of the dependent variable **y(x)** with itself or with any one of its derivatives occur in the equation otherwise is non-linear.

3.12 KEYWORDS

1. **Gaussian Elimination Method:-** The process of row reduction makes use of elementary row operations, and can be divided into two parts. The first part (sometimes called forward elimination) reduces a given system to *row echelon form*, from which one can tell whether there are no solutions, a unique solution, or infinitely many solutions. The second part (sometimes called back substitution) continues to use row operations until the solution is found; in other words, it puts the matrix into *reduced* row echelon form.

2. **Gauss – Seidel Iteration Method:- Gauss–Seidel method**. In numerical linear algebra, the **Gauss–Seidel method**, also known as the Liebmann **method** or the **method** of successive displacement, is an **iterative method** used to solve a linear system of equations.
3. **Euler's Method:-** $y_{n+1} = y_n + hf(x_n, y_n)$, where $x_n = x_0 + nh$.
4. **Runge – KuttaMethod:-**RungeKutta methods which do not require the calculations of higher order derivatives and give greater accuracy. The Runge-Kutta formulae possess the advantage of requiring only the function values at some selected points.
5. **Taylor's Series Method:-**

$$y = y_0 + (x-x_0)(y')_0 + \frac{(x-x_0)^2}{2!}(y'')_0 + \frac{(x-x_0)^3}{3!}(y''')_0 + ...$$

3.13 SELF ASSESSMENT TEST

1. Solve the following system of equations using Gaussian elimination method

   $1x + 2y + 1z = 3$

   $3x + 2y + 1z = 3$

   $1x - 2y - 5z = 1$

2. Find the solution of the following system of equations.

   $3x - y + z = 1$

   $3x + 6y + 2z = 0$

   $3x + 3y + 7z = 4$

   using Gauss-Seidel method and perform the first four iterations.

3. Find y(0.5) for y'=-2x-y, y(0)=-1, with step length 0.1,Euler's method.

4. Find y(0.2) for y'=-y, y(0)=1, with step length 0.1

5. Find y(0.3) for $y'=(x.y^2+y)$, y(0)=1, with step length 0.1, Using R-K fourth order method.

6. Find y(0.4) for $y'=y-x^3$, y(0)=1, by Milen's predictor corrector method.

3.14 ANSWER TO CHECK YOUR PROGRESS

(1) $x = 117/71$, $y = -81/71$, $z = 148/71$

(2) (i) $x = 1, y = 1, z = 1$ (ii) $x = 1, y = 2, z = 3$

(3) 1.1053425

(4) 1.0928

(5) 1.2736

(6) 0.4555

3.15 REFERENCES/ SUGGESTED READINGS

1 Computer Oriented Numeical Methods, V. Rajaraman, PHI**.**

2 Introductory Methods of Numerical Analysis, S.S. Sastry, PHI.

3 Numerical Methods for Scientific and Engineering Computation, M.K.Jain, S.R.Lyenger, R.K.Jain, Wiley Eastern Limited.

4 Numerical Methods with Programs in C, T. Veerarajan, T. Ramachandarn., Tata McGraw Hill.

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LESSON NO. 4	
INTERPOLION	
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STRUCTURE

4.0 OBJECTIVE

In this lesson, the objective is to introduce the concepts of finite differences, thereby, to derive the interpolation formulae using the forward and backward difference operators and tables, for the given equi-spaced set of tabular values.

4.1 INTRODUCTION

The problem of approximating a given function f(x) by polynomials P9x) is generally used for the construction of the function f(x), when it is not given in the form and only the values of f(x) are given at a set of distinct points. The deviation of f(x) from P(x), i.e., f(x) – P(x), x ϵ [a, b], is called the error of approximation.

Let y = f(x), $x_0 \leq x \leq x_n$, be a function, then corresponding to every value of x in the range $x_0 \leq x \leq x_n$, there exists one or more values of y. If the function f(x) is single-valued and continuous and that it is known explicitly, then the values of f(x) corresponding to certain given values of x, say $x_0, x_1 \ldots x_n$ can easily be computed and tabulated. Conversely, given a set of tabulated values $(x_0, y_0), (x_1, y_1), (x_2, y_2), \ldots\ldots, (x_n, y_n)$ satisfying the relation y = f(x) where the explicit nature of f(x) is not known, then it is required to find a function, say $\phi(x)$, such that f(x) and $\phi(x)$ agree at the set of tabulated points. Such a process is called interpolation. If $\phi(x)$ is a polynomial, then the process is called polynomial interpolation and $\phi(x)$ is called the interpolating polynomial. Similarly, different types of interpolation arise depending on whether $\phi(x)$ is a finite trigonometric series, series of Bessel functions, etc. But, we shall be concerned with polynomial interpolation only.

The study of interpolation is based on the calculus of finite differences, which deals with the changes that take place in the value of the function (dependent variable) due to finite changes in the independent variable.

In this lesson, we derive two important interpolation formulae by means of forward and backward differences of a function.

4.2 ERRORS IN POLYNOMIAL INTERPOLATION

Let the function y = f(x) be defined by the (n+1) points (x_i, y_i), i = 0, 1, 2,….., n, and is continuous and differentiable (n+1) times. Let y = f(x) be approximated by a polynomial $\phi_n(x)$ of degree not exceeding n such that

$$\phi_n(x_i) = y_i, \quad i = 0, 1,2,\ldots\ldots\ n. \qquad \text{---(1)}$$

Now, if we use $\phi_n(x)$ to obtain approximate values of f(x) at some points other than those defined by (1), what would be the accuracy of this approximation? Since the expression $f(x) - \phi_n(x)$ vanishes for $x = x_0, x_1, \ldots.. x_n$, so we consider

$$f(x) - \phi_n(x) = L\pi_{n+1}(x), \qquad \text{---(2)}$$

where $\pi_{n+1}(x) = (x - x_0)\,(x - x_1).....(x - x_n),$ ---(3)

and L is to be determined such that equation (2) holds for any intermediate value of x, say $x = x'$, $x_0 < x' < x_n$. Clearly, we have

$$L = \frac{f(x') - \phi_n(x')}{\pi_{n+1}(x')}. \qquad \text{---(4)}$$

We construct a function F(x) such that

$$F(x) = f(x) - \phi_n(x) - L\,\pi_{n+1}(x), \qquad \text{---(5)}$$

where L is given by (4) above. From the definition of F(x), it is clear that

$$F(x_0) = F(x_1) = \ldots = F(x_n) = F(x') = 0,$$

that is, F(x) vanishes at (n+2) points in the interval $x_0 \le x \le x_n$. Consequently, by the repeated application of Rolle's theorem $F'(x)$ must vanish (n+1) times, $F''(x)$ must vanish n times, etc., in the interval $x_0 \le x \le x_n$. In particular, $F^{(n+1)}(x)$ must vanish once in the interval. Let this point be given by $x=\xi$, $x_0<\xi< x_n$. Differentiating (5), (n +1) times with respect to x and put x = ξ, we get

$$0 = f^{(n+1)}(\xi) - L.\,(n+1)! \qquad \text{---(6)}$$

Expression (6) implies

$$L = \frac{f^{(n+1)}(\xi)}{(n+1)!} \qquad \text{---(7)}$$

Comparison of (4) and (7) gives us

$$f(x') - \phi_n(x') = \frac{f^{(n+1)}(\xi)}{(n+1)!}\pi_{n+1}(x'). \qquad \text{---(8)}$$

Since x = x′ is any intermediate value of x, so dropping the prime on x′, we obtain

$$f(x) - \phi_n(x) = \frac{\pi_{n+1}(x)}{(n+1)!} f^{(n+1)}(\xi),\ x_0<\xi<x_n, \qquad \text{---(9)}$$

which is the required expression for the error. It is extremely useful in theoretical work in different branches of numerical analysis. In particular, we will use it to determine errors in Newton's interpolation formulae.

4.3 FINITE DIFFERENCES

Suppose that the function y = f(x) be tabulated for equally spaced set of values say $x_i = x_0 + ih$, i = 0, 1,2,….n, and we have a table of values (x_i, y_i), i = 0, 1,2,3…..n. Finding the values of f(x) for some intermediate values of x, or the derivative of f(x) for some x in the range $x_0 \le x \le x_n$, is based on the concept of the differences of a function. The following three types of differences are found useful.

Forward Differences:

If $y_0, y_1, y_2, \ldots, y_n$ denote a set of values of y, then $y_1 - y_0, y_2 - y_1, \ldots, y_n - y_{n-1}$ are called the differences of y. These differences of y denoted by $\Delta y_0, \Delta y_1, \ldots, \Delta y_{n-1}$ respectively are called the first forward differences, and we have

$$\Delta y_0 = y_1 - y_0,\ \Delta y_1 = y_2 - y_1, \ldots, \Delta y_{n-1} = y_n - y_{n-1},$$

or $\quad \Delta y_r = y_{r+1} - y_r,\ r = 0, 1, 2, \ldots\ n-1,$

where Δ is called the forward difference operator. The differences of the first forward differences are called second forward differences and are denoted by $\Delta^2 y_0, \Delta^2 y_1, \ldots$ Similarly, one can define third forward differences, fourth forward differences etc. Thus, we have

$$\Delta^2 y_0 = \Delta y_1 - \Delta y_0 = y_2 - y_1 - (y_1 - y_0) = y_2 - 2y_1 + y_0,$$

$$\Delta^3 y_0 = \Delta^2 y_1 - \Delta^2 y_0 = y_3 - 2y_2 + y_1 - (y_2 - 2y_1 + y_0)$$

$$= y_3 - 3y_2 + 3y_1 - y_0$$

$$\Delta^4 y_0 = \Delta^3 y_1 - \Delta^3 y_0$$

$$= y_4 - 3y_3 + 3y_2 - y_1 - (y_3 - 3y_2 + 3y_1 - y_0)$$

$$= y_4 - 4y_3 + 6y_2 - 4y_1 + y_0$$

And, in general

$$\Delta^p y_r = \Delta^{p-1} y_{r+1} - \Delta^{p-1} y_r = y_n - {}^n c_1\, y_{n-1} + {}^n c_2\, y_{n-2} + \ldots + (-1)^n y_0.$$

It is, therefore, clear that any higher order difference can easily be expressed in terms of the ordinates, since the coefficients occurring on the right side are the binomial coefficients.

These differences are systematically set out as follows in what is called a forward difference table.

Forward Difference Table

x	y	Δ	Δ^2	Δ^3	Δ^4	Δ^5	Δ^6
x_0	y_0						
		Δy_0					
x_1	y_1		$\Delta^2 y_0$				
		Δy_1		$\Delta^3 y_0$			
x_2	y_2		$\Delta^2 y_1$		$\Delta^4 y_0$		
		Δy_2		$\Delta^3 y_1$		$\Delta^5 y_0$	
x_3	y_3		$\Delta^2 y_2$		$\Delta^4 y_1$		$\Delta^6 y_0$
		Δy_3		$\Delta^3 y_2$		$\Delta^5 y_1$	
x_4	y_4		$\Delta^2 y_3$		$\Delta^4 y_2$		
		Δy_4		$\Delta^3 y_3$			
x_5	y_5		$\Delta^2 y_4$				
		Δy_5					
x_6	y_6						

Backward Differences:

The differences $y_1 - y_0$, $y_2 - y_1$, ...,$y_n - y_{n-1}$ when denoted by ∇y_1, ∇y_2, …., ∇y_n respectively, are called the first backward differences. So that $\nabla y_1 = y_1 - y_0$, $\nabla y_2 = y_2 - y_1$,…..,$\nabla y_n = y_n - y_{n-1}$, where ∇ is called the backward difference operator. One can define backward differences of higher orders as

$$\nabla^2 y_2 = \nabla y_2 - \nabla y_1 = y_2 - y_1 - (y_1 - y_0) = y_2 - 2y_1 + y_0,$$

$$\nabla^3 y_3 = \nabla^2 y_3 - \nabla^2 y_2 = y_3 - 3y_2 + 3y_1 - y_0 .$$

These differences are exhibited in the form of a backward difference table as given below

Backward Difference Table

x	y	∇	∇^2	∇^3	∇^4	∇^5	∇^6
x_0	y_0						
x_1	y_1	∇y_1					
x_2	y_2	∇y_2	$\nabla^2 y_2$				
x_3	y_3	∇y_3	$\nabla^2 y_3$	$\nabla^3 y_3$			
x_4	y_4	∇y_4	$\nabla^2 y_4$	$\nabla^3 y_4$	$\nabla^4 y_4$		
x_5	y_5	∇y_5	$\nabla^2 y_5$	$\nabla^3 y_5$	$\nabla^4 y_5$	$\nabla^5 y_5$	
x_6	y_6	∇y_6	$\nabla^2 y_6$	$\nabla^3 y_6$	$\nabla^4 y_6$	$\nabla^5 y_5$	$\nabla^6 y_6$

Central Differences:

The central difference operator δ is defined by the relations

$$y_1 - y_0 = \delta y_{1/2},\ y_2 - y_1 = \delta y_{3/2}, \ldots..,\ y_n - y_{n-1} = \delta y_{n-½}.$$

The higher order central differences are defined as

$$\delta y_{3/2} - \delta y_{½} = \delta^2 y_1,\ \delta y_{5/2} - \delta y_{3/2} = \delta^2 y_2, \ldots\ldots.$$

$\delta^2 y_2 - \delta^2 y_1 = \delta^3 y_{3/2}$, and so on.

These differences are shown in the following table:

Central Difference Table

x	y	δ	δ^2	δ^3	δ^4	δ^5	δ^6
x_0	y_0						
		$\delta y_{1/2}$					
x_1	y_1		$\delta^2 y_1$				

		$\delta y_{3/2}$		$\delta^3 y_{3/2}$			
x_2	y_2		$\delta^2 y_2$		$\delta^4 y_2$		
		$\delta y_{5/2}$		$\delta^3 y_{5/2}$		$\delta^5 y_{5/2}$	
x_3	y_3		$\delta^2 y_3$		$\delta^4 y_3$		$\delta^6 y_3$
		$\delta y_{7/2}$		$\delta^3 y_{7/2}$		$\delta^5 y_{7/2}$	
x_4	y_4		$\delta^2 y_4$		$\delta^4 y_4$		
		$\delta y_{9/2}$		$\delta^3 y_{9/2}$			
x_5	y_5		$\delta^2 y_5$				
		$\delta y_{11/2}$					
x_6	y_6						

We see from this table that the central differences on the same horizontal line have the same suffix. Also, the differences of odd order are know only for half values of the suffix and those of even order for only integral values of the suffix.

Observation: It is noted that in all the three tables, the same numbers occur in the same position and it is only the notation which changes, e.g.,

$$y_1 - y_0 = \Delta y_0 = \nabla y_1 = \delta y_{1/2}$$

Other Difference Operators:

The operators Δ, ∇ and δ have already been defined. Besides these, there are operators E and μ, which are defined as follows :

Shift operator E is defined as the operation of increasing the argument x by h, so that

$$Ef(x) = f(x + h),\ E^2 f(x) = f(x + 2h),\ E^3\ f(x)= f(x+3h)\ \text{ etc.}$$

The inverse operator E^{-1} is defined by

$$E^{-1}\ f(x) = f(x - h)$$

If y_x is the function f(x), then

$$Ey_x = y_{x+h},\ E^{-1}y_x = y_{x-h},\ E^n y_x = y_{x+nh},$$

where n may be any real number.

Averaging operator μ is defined by the relation

$$\mu y_x = \frac{1}{2}\left(y_{x+\frac{h}{2}} + y_{x-\frac{h}{2}}\right)$$

Remark: In the difference calculus E is regarded as the fundamental operator and Δ, ∇, δ, μ can be expressed in terms of E.

Relations between the operators: Show that

(i) $\Delta = E - 1$ (ii) $\nabla = 1 - E^{-1}$

(iii) $\delta = E^{1/2} - E^{-1/2}$ (iv) $\mu = \frac{1}{2}(E^{1/2} + E^{-1/2})$

(v) $\Delta = E\nabla = \nabla E = \delta E^{1/2}$ (vi) $E = e^{hD}$

Proof:

i) $\Delta y_x = y_{x+h} - y_x = Ey_x - y_x = (E - 1)y_x$

This shows that the operators Δ and E are connected by the symbolic relation

$\Delta = E - 1$ or $E = 1 + \Delta$

ii) $\nabla y_x = y_x - y_{x-h} = y_x - E^{-1}y_x$

$= (1 - E^{-1})\, y_x$

$\therefore$ $\nabla = 1 - E^{-1}$

(iii) $\delta y_x = y_{x+\frac{h}{2}} - y_{x-\frac{h}{2}}$

$= E^{1/2}y_x - E^{-1/2}y_x = (E^{1/2} - E^{-1/2})y_x$

$$\delta = E^{1/2} - E^{-1/2}$$

(iv) $\mu y_x = \frac{1}{2}(y_{x+\frac{h}{2}} + y_{x-\frac{h}{2}})$

$= \frac{1}{2}(E^{1/2}y_x + E^{-1/2}y_x) = \frac{1}{2}(E^{1/2} + E^{-1/2})y_x$

$\therefore \quad \mu = \frac{1}{2}(E^{1/2} + E^{-1/2})$

(v) $E\nabla y_x = E(y_x - y_{x-h}) = Ey_x - Ey_{x-h}$

$= y_{x+h} - y_x = \Delta y_x$

$\therefore \quad E\nabla = \Delta$

$\nabla Ey_x = \nabla y_{x+h} = y_{x+h} - y_x = \Delta y_x$

$\therefore \quad \nabla E = \Delta$

$\delta E^{1/2}y_x = \delta y_{x+\frac{h}{2}} = y_{x+\frac{h}{2}+\frac{h}{2}} - y_{x+\frac{h}{2}-\frac{h}{2}}$

$= y_{x+h} - y_x = \Delta y_x$

$\therefore \quad \delta E^{1/2} = \Delta$

Hence $\Delta = E\nabla = \nabla E = \delta E^{1/2}$

(vi) $Ef(x) = f(x + h)$

$= f(x) + hf'(x) + \frac{h^2}{2!} f''(x) + \ldots\ldots\ldots\ldots$

[by Taylor's series]

$= f(x) + hDf(x) + \frac{h^2}{2!}D^2f(x) + \ldots\ldots$

$$= \left(1 + hD + \frac{h^2D^2}{2!} + \frac{h^3D^3}{3!} + \ldots\ldots\ldots\ldots\right) f(x)$$

$$= e^{hD} f(x)$$

$$E = e^{hD}$$

Example: Prove that

$$e^x = \left(\frac{\Delta^2}{E}\right) e^x . \frac{Ee^x}{\Delta^2 e^x},$$

the interval of differencing being h.

Solution:

Since $\left(\frac{\Delta^2}{E}\right) e^x = \Delta^2 . E^{-1} e^x = \Delta^2 e^{x-h}$

$$= \Delta^2 e^x e^{-h} = e^{-h} \Delta^2 e^x$$

$$\therefore \left(\frac{\Delta^2}{E}\right) e^x . \frac{Ee^x}{\Delta^2 e^x} = e^{-h} \Delta^2 e^x . \frac{Ee^x}{\Delta^2 e^x} = e^{-h} E e^x$$

$$= e^{-h} . e^{x+h} = e^x.$$

Example: Prove with the usual notations, that

i. $(E^{1/2} + E^{-1/2}) (1 + \Delta)^{1/2} = 2 + \Delta$

ii. $\Delta = ½ \delta^2 + \delta\sqrt{(1+\delta^2/4)}$

iii. $\Delta^3 y_2 = \nabla^3 y_5$.

Solution:

i. $(E^{1/2} + E^{-1/2}) (1+\Delta)^{1/2}$

$$= (E^{1/2} + E^{-1/2}) E^{1/2}$$

$= E + 1 = 1 + \Delta + 1$

$= 2 + \Delta.$

ii. $\frac{1}{2}\delta^2 + \delta\sqrt{1+\delta^2/4}$

$= \frac{1}{2}(E^{1/2} - E^{-1/2})^2 + (E^{1/2} - E^{-1/2})\sqrt{1+(E^{1/2}-E^{-1/2})^2/4}$

$= \frac{1}{2}(E+E^{-1} - 2) + (E^{1/2} - E^{-1/2})\sqrt{(E+E^{-1}+2)/4}$

$= \frac{1}{2}(E + E^{-1} - 2) + (E^{1/2} - E^{-1/2})(E^{1/2} + E^{-1/2})/2$

$= \frac{1}{2}[(E + E^{-1} - 2) + (E - E^{-1})] = \frac{1}{2}(2E - 2)$

$= E - 1 = \Delta.$

iii. $\Delta^3 y_2 = (E-1)^3 y_2$ $\quad [\because \Delta = E - 1]$

$= (E^3 - 3E^2 + 3E - 1)\, y_2$

$= y_5 - 3y_4 + 3y_3 - y_2$ $\quad$ ---(1)

$\nabla^3 y_5 = (1 - E^{-1})^3 y_5$ $\quad [\because \nabla = 1 - E^{-1}]$

$= (1 - 3E^{-1} + 3E^{-2} - E^{-3})y_5$

$= y_5 - 3y_4 + 3y_3 - y_2$ $\quad$ ---(2)

From (1) and (2),

$\Delta^3 y_2 = \nabla^3 y_5.$

Example: Using the method of separation of symbols, prove that

(i) $u_1x + u_2x^2 + u_3x^3 + \ldots\ldots\ldots\ldots$

$$= \frac{x}{1-x}u_1 + \left(\frac{x}{1-x}\right)^2 \Delta u_1 + \left(\frac{x}{1-x}\right)^3 \Delta^2 u_1 + \ldots\ldots.$$

(ii) $u_0 + \frac{u_1 x}{1!} + \frac{u_2 x^2}{2!} + \frac{u_3 x^3}{3!} + \dots\dots\dots$

$$= e^x\left(u_0 + x\Delta u_0 + \frac{x^2}{2!}\Delta^2 u_0 + \frac{x^3}{3!}\Delta^3 u_0 + \dots\dots\right)$$

Solution:

(i) L.H.S. $= xu_1 + x^2 Eu_1 + x^3E^2u_1 + \dots\dots$ $\quad [\because u_{x+h} = E^h u_x]$

$= x(1 + xE + x^2E^2 + \dots\dots)u_1$

$= x \cdot \frac{1}{1-xE} u_1,$ [taking sum of infinite G.P.]

$= x\left[\frac{1}{1-x(1+\Delta)}\right]u_1$ $\quad [\because E = 1 + \Delta]$

$= x\left(\frac{1}{1-x-x\Delta)}\right)u_1 = \frac{x}{1-x}\left(1-\frac{x\Delta}{1-x}\right)^{-1} u_1$

$= \frac{x}{1-x}\left(1 - \frac{x\Delta}{1-x} + \frac{x^2\Delta^2}{(1-x)^2} + \dots\dots\right)u_1$

$= \frac{x}{1-x}u_1 + \frac{x^2}{(1-x)^2}\Delta u_1 + \frac{x^3}{(1-x)^3}\Delta^2 u_1 + \dots\dots\dots$

= R.H.S.

(ii) L.H.S. $= u_0 + \frac{x}{1!}Eu_0 + \frac{x^2}{2!}E^2u_0 + \frac{x^3}{3!}E^3u_0 + \dots\dots\dots$

$= \left(1 + \frac{xE}{1!} + \frac{x^2E^2}{2!} + \frac{x^3E^3}{3!} + \dots\dots\right)u_0$

$= e^{xE}u_0 = e^{x(1+\Delta)}u_0$

$= e^x . e^{x\Delta}u_0$

$$= e^x\left(1+\frac{x\Delta}{1!}+\frac{x^2\Delta^2}{2!}+\frac{x^3\Delta^3}{3!}+........\right)u_0$$

$$= e^x\left(u_0+\frac{x}{1!}\Delta u_0+\frac{x^2}{2!}\Delta^2 u_0++\frac{x^3}{3!}\Delta^3 u_0+......\right)$$

= R.H.S.

4.4 DETECTION OF ERRORS BY USE OF DIFFERENCE TABLES

Difference tables can be used to check errors in tabular values. Suppose there is an error of +1 unit in a certain tabular value. As higher differences are formed, this error spreads out and is considerably magnified. It affects the difference table as shown below in the table.

Error Difference Table shows the following characteristics:

i. The effect of the error increases with the order of the differences.

ii. The errors in any one column are the binomial coefficients with alternating signs.

Table

y	Δ	Δ^2	Δ^3	Δ^4	Δ^5
0					
	0				
0		0			
	0		0		
0		0		0	
	0		0		1
0		0		1	

	0		1		-5
0		1		-4	
	1		-3		10
1		-2		6	
	-1		3		-10
0		1		-4	
	0		-1		5
0		0		1	
	0		0		-1
0		0		0	
	0		0		
0		0			
	0				
0					

(iii) The algebraic sum of the errors in any difference column is zero, and

(iv) The maximum error occurs opposite the function value containing the error.

These facts can be used to detect errors by difference tables.

Example:

x	y	Δ	Δ^2	Δ^3	Δ^4
1	3010				
		414			
2	3424		-36		
		378		-39	

3	3802		-75		+178
		303		+139	
4	4105		+64		-271
		367		-132	
5	4472		-68		+181
		299		+49	
6	4771		-19		-46
		280		+3	
7	5051		-16		
		264			
8	5315				

The term – 271 in the fourth difference column has fluctuations of 449 and 452 on either side of it. Comparison with the error difference table suggest that there is an error of -45 in the entry for x = 4. The correct value of y is therefore 4105 + 45 = 4150, which shows that the last two digits have been transposed, a very common form of error. The reader is advised to form a new difference table with this correction, and to check that the third differences are now practically constant.

If an error is present in a given data, the differences of some order will become alternating in sign. Hence, higher order differences should be formed till the error is revealed as in the above example. If there are errors in several tabular values, then it is not easy to detect the errors by differencing.

4.5 DIFFERENCES OF A POLYNOMIAL

The nth differences of a polynomial of the nth degree are constant and all higher order differences are zero.

Let the polynomial of the nth degree in x, be

$$f(x) = ax^n + bx^{n-1} + cx^{n-2} + \ldots\ldots\ldots + kx + l$$

$$\therefore \quad \Delta f(x) = f(x + h) - f(x)$$

$$= a[(x + h)^n - x^n] + b[(x + h)^{n-1} - x^{n-1}] + \ldots\ldots + kh$$

$$= anhx^{n-1} + b'x^{n-2} + c'x^{n-3} + \ldots\ldots.. + k'x + l', \qquad \text{---(1)}$$

where $b', c', \ldots\ldots.. l'$ are the new constant co-efficients.

Thus the first difference of a polynomial of the nth degree is a polynomial of degree (n – 1).

Similarly $\Delta^2 f(x) = \Delta[f(x + h) - f(x)]$

$$= \Delta f(x+h) - \Delta f(x)$$

$$= anh[(x+h)^{n-1} - x^{n-1}] + b'[(x+h)^{n-2} - x^{n-2}] + \ldots\ldots + k'h$$

$$= an(n-1)h^2x^{n-2} + b''x^{n-3} + c''x^{n-4} + \ldots\ldots + k'$$

$\therefore$ The second differences represent a polynomial of degree (n – 2). Continuing this process, for the nth differences we get a polynomial of degree zero, i.e.,

$$\Delta^n f(x) = an(n - 1)(n - 2)\ldots\ldots.\ 1h^n$$

$$= an!h^n, \qquad \text{--- (2)}$$

which is a constant. Hence the $(n+1)^{th}$ and higher differences of a polynomial of nth degree will be zero.

Remark: The converse of this theorem is also true, i.e., if the nth differences of a function tabulated at equally spaced intervals are constant, the function is a polynomial of degree n. This fact is important in numerical analysis as it enables us to approximate a function by a polynomial of nth degree, if its nth order differences become nearly constant.

Example: Evaluate

$\Delta^{10}[(1-ax)(1-bx^2)(1-cx^3)(1-dx^4)]$

Solution:

$\Delta^{10}[(1-ax)(1-bx^2)(1-cx^3)(1-dx^4)]$

$= \Delta^{10}[abcdx^{10} + (\ )x^9 + (\ )x^8 + \ldots\ldots+1]$

$= abcd\ \Delta^{10}(x^{10})$ $[\because \Delta^{10}(x^n) = 0 \text{ for } n < 10]$

$= abcd\ (10!)$ [by (2) above]

4.6 NEWTON'S FORWARD INTERPOLATION FORMULA

Let the function $y = f(x)$ take the values $y_0, y_1, y_2, \ldots$ corresponding to the values $x_0, x_0+h, x_0+2h, \ldots$ of x. Suppose it is required to find f(x) for $x = x_0+ph$, where p is any real number.

For any real number p, we have defined E such that

$E^p f(x) = f(x+ph)$

$\therefore \quad y_p = f(x_0+ph) = E^p f(x_0) = (1+\Delta)^p y_0$ $[\because E=1+\Delta]$

$$= \left\{1 + p\Delta + \frac{p(p-1)}{2!}\Delta^2 + \frac{p(p-1)(p-2)}{3!}\Delta^3 + \cdots y_0,\right\}$$

[Using Binomial theorem]

i.e., $y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \ldots\ldots$ ---(1)

If $y = f(x)$ is a polynomial of the nth degree, then $\Delta^{n+1}y_0$ and higher differences will be zero. Hence (1) will become

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \ldots\ldots$$

$$\ldots\ldots + \frac{p(p-1)\ldots\ldots(p-\overline{n-1})}{n!}\Delta^n y_0 \quad \text{---(2)}$$

Remark: This formula is used for interpolating the values of y near the beginning of a set of tabulated values and extrapolating values of y a little backward (i.e. to the left) of y_0.

4.7 NEWTON'S BACKWARDINTERPOLATION FORMULA

Let the function y = f(x) take the values y_0, y_1, y_2,…. corresponding to the values x_0, x_0+h, x_0+2h,…. of x. Suppose it is required to evaluate f(x) for $x = x_n+ph$, where p is any real number. Then we have

$$y_p = f(x_n+ph) = E^p f(x_n) = (1-\nabla)^{-p} y_n \qquad [\because E^{-1} = 1-\Delta]$$

$$= \left\{1 + p\nabla + \frac{p(p+1)}{2!}\nabla^2 + \frac{p(p+1)(p+2)}{3!}\nabla^3 + \cdots\cdots y_n, \right\}$$

[Using Binomial theorem]

i.e., $$y_p = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n + \ldots\ldots \quad \text{---(1)}$$

It is called Newton's backward interpolation formula as it contains y_n and backward differences of y_n.

Remark: This formula is used for interpolating the values of y near the end of a set of tabulated values and also for extrapolating values of y a little ahead (to the right) of y_n.

Example: The table gives the distance in nautical miles of the visible horizon for the given heights in feet above the earth's surface:

x = height:	100	150	200	250	300	350	400

y = distance: 10.63 13.03 15.04 16.81 18.42 19.90 21.27

Find the values of y when (i) x = 218 (ii) x = 410.

Solution

The difference table is as under

x	y	Δ	Δ^2	Δ^3	Δ^4
100	10.63				
		2.40			
150	13.03		-0.39		
		2.01		0.15	
200	15.04		-0.24		-0.07
		1.77		0.08	
250	16.81		-0.16		-0.05
		1.61		0.03	
300	18.42		-0.13		-0.01
		1.48		0.02	
350	19.90		-0.11		
		1.37			
400	21.27				

(i) If we take $x_0 = 200$, then $y_0 = 15.04$, $\Delta y_0 = 1.77$, $\Delta^2 y_0 = -0.16$,

$\Delta^3 = 0.03$ etc.

Since x = 218 and h = 50, $\therefore p = \frac{x - x_0}{h} = \frac{18}{50} = 0.36$

$\therefore$ Using Newton's forward interpolation formula, we get

$$y_{218} = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0$$

$$+ \frac{p(p-1)(p-2)(p-3)}{4!}\Delta^4 y_0 + \ldots..$$

$$f(218) = 15.04 + 0.36(1.77) + \frac{0.36(-0.64)}{2}(-0.16)$$

$$+\frac{0.36(-0.64)(-1.64)}{6}(0.03) + \frac{0.36(-0.64)(-1.64)(-2.64)}{24}(-0.01)$$

$$= 15.04 + 0.637 + 0.018 + 0.002 + 0.0004$$

= 15.697, i.e., 15.7 nautical miles.

(ii) Since x = 410 is near the end of the table, we use Newton's backward interpolation formula.

$\therefore$ Taking $x_n = 400$, $p = \frac{x - x_n}{h} = \frac{10}{50} = 0.2$

Using the line of backward difference

$y_n = 21.27$, $\nabla y_n = 1.37$, $\nabla^2 y_n = -0.11$, $\nabla^3 y_n = 0.02$ etc.

Newton's backward formula gives

$$y_{410} = y_{400} + p\nabla y_{400} + \frac{p(p+1)}{2!}\nabla^2 y_{400} + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_{400} +$$

$$+ \frac{p(p+1)(p+2)(p+3)}{4!}\nabla^4 y_{400}$$

$$= 21.27 + 0.2\,(1.37) + \frac{0.2(1.2)}{2!}(-0.11)$$

$$+ \frac{0.2(1.2)(2.2)}{3!}(0.02) + \frac{0.2(1.2)(2.2)(3.2)}{4!}(-0.01)$$

$$= 21.27 + 0.274 - 0.0132 + 0.0018 - 0.0007$$

= 21.53 nautical miles.

Example: From the following table, estimate the number of students who obtained marks between 40 and 45:

Marks :	30-40	40-50	50-60	60-70	70-80
No. of student:	31	42	51	35	31

Solution:

First we prepare the cumulative frequency table, as follows:

Marks less than (x):	40	50	60	70	80
No. of students (y_x):	31	73	124	159	190

Now the difference table is

x	y_x	Δy_x	$\Delta^2 y_x$	$\Delta^3 y_x$	$\Delta^4 y_x$
40	31				
		42			
50	73		9		
		51		-25	
60	124		-16		37
		35		12	
70	159		-4		
		31			
80	190				

We shall find y_{45}, i.e., number of student with marks less than 45. Taking $x_0 = 40$, $x = 45$, we have

$p = \dfrac{x - x_0}{h} = \dfrac{5}{10} = 0.5$ [$\because$ h = 10]

$\therefore$ Using Newton's forward interpolation formula, we get

$$y_{45} = y_{40} + p\,\Delta y_{40} + \frac{p(p-1)}{2!}\Delta^2 y_{40} + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_{40} + \frac{p(p-1)(p-2)(p-3)}{3!}\Delta^4 y_{40}.$$

$$= 31 + 0.5 \times 42 + \frac{0.5(-0.5)}{2} \times 9 + \frac{0.5(-0.5)(-1.5)}{6} \times (-25) + \frac{0.5(-0.5)(-1.5)(-2.5)}{24} \times 37$$

$= 31+21+1.125-1.5625-1.4453$

$= 47.87$

The number of students with marks less than 45 is 47.87, i.e., 48. But the number of students with marks less than 40 is 31.

Hence the number of students getting marks between 40 and 45=48-31=17.

Example: Find the cubic polynomial which takes the following values :

x:	0	1	2	3
f (x) :	1	2	1	10

Hence or otherwise evaluate f(4).

Solution:

The difference table is

x	f(x)	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
0	1			

		1		
1	2		-2	
		-1		12
2	1		10	
		9		
3	10			

We take $x_0 = 0$ and $p = \frac{x-0}{h} = x$ $\qquad [\because h = 1]$

$\therefore$ Using Newton's forward interpolation formula, we get

$$f(x) = f(0) + \frac{x}{1}\Delta f(0) + \frac{x(x-1)}{1.2}\Delta^2 f(0) + \frac{x(x-1)(x-2)}{1.2.3}\Delta^3 f(0)$$

$$= 1 + x(1) + \frac{x(x-1)}{2}(-2) + \frac{x(x-1)(x-2)}{6}(12)$$

$$= 2x^3 - 7x^2 + 6x + 1,$$

which is the required polynomial.

To compute f(4), we take $x_n = 3$, $x = 4$ so that $p = \frac{x - x_n}{h} = 1$

Using Newton's backward interpolation formula, we get

$$f(4) = f(3) + p\nabla f(3) + \frac{p(p+1)}{1.2}\nabla^2 f(3) + \frac{p(p+1)(p+2)}{1.2.3}\nabla^3 f(3)$$

$$= 10 + 9 + 10 + 12 = 41$$

Which is the same value as that obtained by substituting x = 4 in the cubic polynomial above.

4.8 CHECK YOUR PROGRESS

1. Show that

$$\Delta\left[\frac{1}{f(x)}\right] = \frac{-\Delta f(x)}{f(x)f(x+1)}$$

2. Evaluate

(i) $\Delta^2\left[\frac{1}{x^2+5x+6}\right]$ (ii) $\Delta^n\left[\frac{1}{x}\right]$

3 With usual notations, show that

(i) $(1+\Delta)(1-\nabla)=1$ (ii) $\mu\delta=\frac{1}{2}(\Delta+\nabla)$

(iii) $\nabla^r f_k = \Delta^r f_{k-r}$

4. Evaluate

$$\Delta^4\left[(1-x)(1-2x)(1-3x)(1-4x)\right], \quad h = 1$$

5. Estimate the value of f(22) and f(42)from the following data:

x:	20	25	30	35	40	45
f(x):	354	332	291	260	231	204

6. Find the number of men getting wages between Rs. 10 and 15 from the following data

Wages (Rs.):	0-10	10-20	20-30	30-40
Frequency:	9	30	35	42

7. Construct Newton's forward interpolation polynomial for the following data:

x:	4	6	8	10
y:	1	3	8	16

4.9 SUMMARY

1. We start to learn about the error in polynomial interpolation and detection of that error by using difference table.

2. This chapter tells us how to find value near the top and near the end of table.

3. We learn Newton's forward and backward interpolation formula for detecting the value at any intermediate level.

4.10 KEYWORDS

Polynomial Interpolation:- In numerical analysis, **polynomial interpolation** is the interpolation of a given data set by the polynomial of lowest possible degree that passes through the points of the dataset.

Polynomial interpolation is a method of estimating values between known data points. When graphical data contains a gap, but data is available on either side of the gap or at a few specific points within the gap, an estimate of values within the gap can be made by interpolation.

Newton's Forward Interpolation Formula:-

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \ldots$$

Newton's Backward Interpolation Formula:-

$$y_p = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n + \ldots$$

4.11 SELF ASSESSMENT TEST

1. Find solution using Newton's forward difference formula

x:	1891	1901	1911	1921	1931
f(x):	46	66	81	93	101

X= 1895

2. Find solution using Newton's forward difference formula

x:	0	1	2	3
f(x):	1	0	1	10

X= -1

3. In the table below the value of y are consecutive terms of a series of which the number 21.6 is the 6^{th} term. Find the 1^{st} and the 10^{th} term of the series.

X:	3	4	5	6	7	8	9
Y:	2.7	6.4	12.5	21.6	34.3	51.2	72.9

4. Construct difference table for following data

X:	1	2	4	5	6	7	8
Y:	23	33	45	56	65	77	89

4.12 ANSWER TO CHECK YOUR PROGRESS

(2) (i) -2/(x +2) (x +3) (x +4)

(ii) $(-1)^n \dfrac{n(n-1)(n-2)......2.1}{x(x+1)(x+2)......(x+n)}$

(4) 576

(5) 352; 219

(6) 24

(7) 1.625

4.13 REFERENCES/ SUGGESTEDREADINGS

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STRUCTURE

5.0 OBJECTIVE

The objective of this lesson is to continue with the objective of the previous lesson by deriving some more interpolation formulae such as, central difference interpolation formulae, which are based on central difference table and operators; Interpolation formulae for unequi-spaced set of values and also the inverse interpolation.

5.1 INTRODUCTION

The Newton's forward and backward interpolation formulae are applicable for interpolation near the beginning and end of the tabulated values. Now we shall develop central difference formulae which are best suited for interpolation near the middle of the table, for equi-spaced set of values.

All the interpolation formulae discussed, have the disadvantage of requiring the values of the independent variable to be equally spaced, i.e., equi-spaced set of values. It is therefore desirable to have interpolation formulae with unequally spaced values of the argument and therefore, we discuss one such formula, Lagrange's interpolation formula. Also, the inverse interpolation is discussed, in which we can find the value of the argument for a given value of the function, from the set of tabulated values.

5.2 CENTRAL DIFFERENCE INTERPOLATION FORMULA

If x takes the values $x_0 - 2h$, $x_0 - h$, x_0, $x_0 + h$, $x_0 + 2h$ and the corresponding values of $y = f(x)$ are y_{-2}, y_{-1}, y_0, y_1, y_2, then we can write the difference table in the two notations as follows :

x	y	1st diff.	2nd diff.	3rd diff.	4th diff.
$x_0 - 2h$	y_{-2}				
		Δy_{-2} $(=\delta y_{-3/2})$			
$x_0 - h$	y_{-1}		$\Delta^2 y_{-2}(=\delta^2 y_{-1})$		
		$\Delta y_{-1}(=\delta y_{-1/2})$		$\Delta^3 y_{-2}(=\delta^3 y_{-1/2})$	
x_0	y_0		$\Delta^2 y_{-1}(=\delta^2 y_0)$		$\Delta^4 y_{-2}(=\delta^4 y_0)$
		$\Delta y_0(=\delta y_{1/2})$		$\Delta^3 y_{-1}(=\delta^3 y_{1/2})$	
$x_0 + h$	y_1		$\Delta^2 y_0(=\delta^2 y_1)$		
		$\Delta y_1(=\delta y_{3/2})$			
$x_0 + 2h$	y_2				

5.2.1 Gauss's Forward Interpolation Formula:

The Newton's forward interpolation formula is

$$y_p = y_0 + p\,\Delta y_0 + \frac{p(p-1)}{1.2}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{1.2.3}\Delta^3 y_0 + \qquad \text{---(1)}$$

We have $\Delta^2 y_0 - \Delta^2 y_{-1} = \Delta^3 y_{-1}$,

i.e., $\Delta^2 y_0 = \Delta^2 y_{-1} + \Delta^3 y_{-1}.$ ---(2)

Similarly, $\Delta^3 y_0 = \Delta^3 y_{-1} + \Delta^4 y_{-1}$, ---(3)

$\Delta^4 y_0 = \Delta^4 y_{-1} + \Delta^5 y_{-1}$ etc. ---(4)

Also, $\Delta^3 y_{-1} - \Delta^3 y_{-2} = \Delta^4 y_{-2}$,

i.e. $\Delta^3 y_{-1} = \Delta^3 y_{-2} + \Delta^4 y_{-2}$.

Similarly, $\Delta^4 y_{-1} = \Delta^4 y_{-2} + \Delta^5 y_{-2}$ etc. ---(5)

Substituting for $\Delta^2 y_0, \Delta^3 y_0, \Delta^4 y_0$ …… from (2), (3), (4), (5) in (1), we get

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{1.2}(\Delta^2 y_{-1} + \Delta^3 y_{-1})$$

$$+\ \frac{p(p-1)(p-2)}{1.2.3}(\Delta^3 y_{-1}+\Delta^4 y_{-1})$$

$$+\ \frac{p(p-1)(p-2)(p-3)}{1.2.3.4}\ (\Delta^4 y_{-1}+\Delta^5 y_{-1}) + \ldots..$$

Hence

$$y_p = y_o + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{3!}\Delta^3 y_{-1}$$

$$+\ \frac{(p+1)p(p-1)(p-2)}{4!}\Delta^4 y_{-2} + \ldots.., \qquad \text{---(6)}$$

which is known as Gauss's forward interpolation formula.

In the central difference notations, this formula will be

$$y_p = y_0 + p\delta y_{1/2} + \frac{p(p-1)}{2!}\delta^2 y_0 + \frac{(p+1)p(p-1)}{3!}\delta^3 y_{1/2}$$

$$+\ \frac{(p+1)p(p-1)(p-2)}{4!}\delta^4 y_0 + \ldots.. \qquad \text{---(7)}$$

Remark : This formula is used to interpolate the values of y for p ($0<p<1$) measured forwardly from the origin.

5.2.2 Gauss's Backward Interpolation Formula:

The Newton's forward interpolation formula is

$$y_p = y_0 + p\ \Delta y_0 + \frac{p(p-1)}{1.2}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{1.2.3}\Delta^3 y_0 + \qquad \text{---(1)}$$

We have $\Delta y_0 - \Delta y_{-1} = \Delta^2\ y_{-1}$,

i.e., $\Delta y_0 = \Delta y_{-1} + \Delta^2 y_{-1}$. ---(2)

Similarly, $\Delta^2 y_0 = \Delta^2 y_{-1} + \Delta^3 y_{-1}$, ---(3)

$\Delta^3 y_0 = \Delta^3 y_{-1} + \Delta^4 y_{-1}$ etc. ---(4)

Also, $\Delta^3 y_{-1} - \Delta^3 y_{-2} = \Delta^4 y_{-2}$,

i.e. $\Delta^3 y_{-1} = \Delta^3 y_{-2} + \Delta^4 y_{-2}$ ---(5)

Similarly, $\Delta^4 y_{-1} = \Delta^4 y_{-2} + \Delta^5 y_{-2}$ etc. ---(6)

Substituting for Δy_0, $\Delta^2 y_0$, $\Delta^3 y_0$ …… from (2), (3), (4), (5), (6) in (1), we get

$$y_p = y_0 + p(\Delta y_{-1} + \Delta^2 y_{-1}) + \frac{p(p-1)}{1.2}(\Delta^2 y_{-1} + \Delta^3 y_{-1})$$

$$+ \frac{p(p-1)(p-2)}{1.2.3}(\Delta^3 y_{-1} + \Delta^4 y_{-1})$$

$$+ \frac{p(p-1)(p-2)(p-3)}{1.2.3.4}(\Delta^4 y_{-1} + \Delta^5 y_{-1}) + \ldots..$$

$$= y_0 + p\Delta y_{-1} + \frac{(p+1)p}{1.2}\Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{1.2.3}\Delta^3 y_{-1}$$

$$+ \frac{(p+1)p(p-1)(p-2)}{1.2.3.4}\Delta^4 y_{-1}$$

$$+ \frac{(p-1)p(p-2)(p-3)}{1.2.3.4}\Delta^5 y_{-1} + \ldots..$$

$$= y_0 + p\Delta y_{-1} + \frac{(p+1)p}{1.2}\Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{1.2.3}(\Delta^3 y_{-2} + \Delta^4 y_{-2})$$

$$+ \frac{(p+1)p(p-1)(p-2)}{1.2.3.4}(\Delta^4 y_{-2} + \Delta^5 y_{-2}) + \ldots..$$

[Using (5) and (6)]

Hence

$$y_p = y_0 + p\Delta y_{-1} + \frac{(p+1)p}{2!}\Delta^2 y_{-2} + \frac{(p+1)p(p-1)}{3!}\Delta^3 y_{-2}$$

$$+ \frac{(p+2)(p+1)p(p-1)}{4!}\Delta^4 y_{-2} + \ldots\ldots., \qquad \text{---(7)}$$

which is called Gauss's backward interpolation formula.

In the central difference notations, this formula will be

$$y_p = y_0 + p\delta y_{-1/2} + \frac{(p+1)p}{2!}\delta^2 y_0 + \frac{(p+1)p(p-1)}{3!}\delta^3 y_{-1/2}$$

$$+ \frac{(p+2)(p+1)p(p-1)}{4!}\delta^4 y_0 + \ldots\ldots. \qquad \text{---(8)}$$

Remark: It is used to interpolate the values of y for a negative value of p lying between –1 and 0.

Gauss's forward and backward formulae are themselves not of much practical use. However, these serve as intermediate steps for obtaining the following two important formulae:

5.2.3Sterling's Formula:

Gauss's forward interpolation formula is

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{3!}\Delta^3 y_{-1}$$

$$+ \frac{(p+1)p(p-1)(p-2)}{4!}\Delta^4 y_{-2} + \ldots\ldots \qquad \text{---(1)}$$

Gauss's backward interpolation formula is

$$y_p = y_0 + p\Delta y_{-1} + \frac{(p+1)p}{2!}\Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{3!}\Delta^3 y_{-2}$$

$$+ \frac{(p+2)(p+1)p(p-1)}{4!}\Delta^4 y_{-2} + \ldots\ldots \qquad \text{---(2)}$$

Taking the mean of (1) and (2), we obtain

$$y_p = y_0 + p\left(\frac{\Delta y_0 + \Delta y_{-1}}{2}\right) + \frac{p^2}{2!}\Delta^2 y_{-1} + \frac{p(p^2-1)}{3!}\left(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right)$$

$$+ \frac{p^2(p^2-1)}{4!}\Delta^4 y_{-2} + \ldots\ldots \qquad \text{---(3)}$$

which is called Stirling's formula

In the central difference notations, (3) takes the form

$$y_p = y_0 + p\mu\delta y_0 + \frac{p^2}{2!}\delta^2 y_0 + \frac{p(p^2-1^2)}{3!}\mu\delta^3 y_0$$

$$+ \frac{p^2(p^2-1^2)}{4!}\delta^4 y_0 + \ldots\ldots \quad , \qquad \text{---(4)}$$

since $\frac{1}{2}(\Delta y_0 + \Delta y_{-1}) = \frac{1}{2}(\delta y_{1/2} + \delta y_{-1/2}) = \mu\delta y_0$,

$\frac{1}{2}(\Delta^3 y_{-1} + \Delta^3 y_{-2}) = \frac{1}{2}(\delta^3 y_{1/2} + \delta^3 y_{-1/2}) = \mu\delta^3 y_0$ etc.

Remark: This formula involves means of the odd differences just above and below the central line and even differences on this line as shown below :

$$\ldots\ldots y_0 \ldots\ldots \begin{pmatrix}\Delta y_{-1}\\ \Delta y_0\end{pmatrix} \ldots\ldots \Delta^2 y_{-1} \ldots\ldots \begin{pmatrix}\Delta^3 y_{-2}\\ \Delta^3 y_{-1}\end{pmatrix} \ldots\ldots \Delta^4 y_{-2} \ldots\ldots \begin{pmatrix}\Delta^5 y_{-3}\\ \Delta^5 y_{-2}\end{pmatrix} \ldots \Delta^6 y_{-3} \ldots\ldots \text{ central line}$$

5.2.4 Bessel's Formula:

Guass's forward interpolation formula is

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{3!}\Delta^3 y_{-1}$$

$$+ \frac{(p+1)p(p-1)(p-2)}{4!}\Delta^4 y_{-2} + \ldots\ldots \qquad \text{---(1)}$$

We have $\Delta^2 y_0 - \Delta^2 y_{-1} = \Delta^3 y_{-1}$,

i.e. $\Delta^2 y_{-1} = \Delta^2 y_0 - \Delta^3 y_{-1}$ ---(2)

Similarly, $\Delta^4 y_{-2} = \Delta^4 y_{-1} - \Delta^5 y_{-2}$, etc. ---(3)

Now (1) can be written as

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\left(\frac{1}{2}\Delta^2 y_{-1} + \frac{1}{2}\Delta^2 y_{-1}\right) + \frac{p(p^2-1)}{3!}\Delta^3 y_{-1}$$

$$+ \frac{p(p^2-1)(p-2)}{4!}\left(\frac{1}{2}\Delta^4 y_{-2} + \frac{1}{2}\Delta^4 y_{-2}\right) + \ldots\ldots$$

$$= y_0 + p\Delta y_0 + \frac{1}{2}\frac{p(p-1)}{2!}\Delta^2 y_{-1} + \frac{1}{2}\frac{p(p-1)}{2!}(\Delta^2 y_0 - \Delta^3 y_{-1})$$

$$+ \frac{p(p^2-1)}{3!}\Delta^3 y_{-1} + \frac{1}{2}\frac{p(p^2-1)(p-2)}{4!}\Delta^4 y_{-2}$$

$$+ \frac{1}{2}\frac{p(p^2-1)(p-2)}{4!}(\Delta^4 y_{-1} - \Delta^5 y_{-2}) + \ldots..$$

[using (2), (3) etc.]

$$= y_0 + p\Delta y_0 + \frac{p(p-1)}{2!} \cdot \frac{\Delta^2 y_{-1} + \Delta^2 y_0}{2} + \frac{p(p-1)}{2!}\left(\frac{p+1}{3} - \frac{1}{2}\right)\Delta^3 y_{-1}$$

$$+ \frac{p(p^2-1)(p-2)}{4!} \cdot \frac{\Delta^4 y_{-2} + \Delta^4 y_{-1}}{2} + \ldots\ldots$$

Hence

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!} \cdot \frac{\Delta^2 y_{-1} + \Delta^2 y_0}{2} + \frac{(p-\frac{1}{2})p(p-1)}{3!}\Delta^3 y_{-1}$$

$$+ \frac{(p+1)p(p-1)(p-2)}{4!} \cdot \frac{\Delta^4 y_{-2} + \Delta^4 y_{-1}}{2} + \ \ldots.., \qquad \text{---(4)}$$

which is known as the Bessel's formula.

In the central difference notations, (4) becomes

$$y_p = y_0 + p\delta y_{1/2} + \frac{p(p-1)}{2!}\mu\delta^2 y_{1/2} + \frac{(p-\frac{1}{2})p(p-1)}{3!}\delta^3 y_{1/2}$$

$$+\frac{(p+1)p(p-1)(p-2)}{4!}\mu\delta^4 y_{1/2} + \ldots, \qquad \text{---(5)}$$

since $\frac{1}{2}(\Delta^2 y_{-1} + \Delta^2 y_0) = \mu\delta^2 y_{1/2}$, $\frac{1}{2}(\Delta^4 y_{-2} + \Delta^4 y_{-1}) = \mu\delta^4 y_{1/2}$, etc.

Remark: This is a very useful formula for practical purposes. It involves odd differences below the central line and means of even differences of and below his line.

Choice of an interpolation formula :

The right choice of an interpolation formula, depends on the position of the interpolated value in the given data.

The following rules will be found useful :

1. To find a tabulated value near the beginning of the table, use Newton's forward formula.
2. To find a value near the end of the table, use Newton's backward formula.
3. To find an interpolated value near the centre of the table, use either Stirling's or Bessel's formula.]

If interpolation is required for p lying between $-\frac{1}{4}$ and $\frac{1}{4}$, use Stirling's formula. If interpolation is desired for p laying between $\frac{1}{4}$ and $\frac{3}{4}$, use Bessel's formula

Example: Employ Stirling's formula to compute $y_{12.2}$ from the following table ($y_x = 1 + \log_{10} \sin x$) :

x°:	10	11	12	13	14
$10^5 u_x$:	23,967	28,060	31,788	35,209	38,368

Solution:

Taking the origin at x = 12°, h = 1 and p = x – 12 , we have the following central table :

p	y_x	Δy_x	$\Delta^2 y_x$	$\Delta^3 y_x$	$\Delta^4 y_x$
-2	0.23967				
		0.04093			
-1	0.28060		-0.00365		
		0.03728		.00058	
0	0.31788		-0.00307		-0.00013
		0.03421		-0.00045	
1	0.35209		-0.00062		
		0.03159			
2	0.38368				

At x = 12.2, p = 0.2. (As p lies between $-\frac{1}{4}$ and $\frac{1}{4}$, the use of Stirling's formula will be quite suitable.)

Stirling's formula is

$$y_p = y_0 + \frac{p}{1} \cdot \frac{\Delta y_{-1} + \Delta y_0}{2} + \frac{p^2}{2!}\Delta^2 y_{-1} + \frac{p(p^2-1)}{3!}\frac{\Delta^3 y_{-2} + \Delta^3 y_{-1}}{2}$$

$$+ \frac{p^2(p^2-1)}{4!}\Delta^4 y_{-2} + \ldots\ldots$$

When p = 0.2, we have

$$y_{0.2} = 0.31788 + 0.2\left(\frac{0.03728 + 0.03421}{2}\right) + \frac{(0.2)^2}{2}(-0.00307)$$

$$+ \frac{(0.2)[(0.2)^2 - 1]}{6}\left(\frac{0.00058 - 0.00045}{2}\right) + \frac{(0.2)^2[(0.2)^2 - 1]}{24}\ (-\ 0.00013)$$

$$= 0.31788 + 0.00715 - 0.00006 - 0.000002 + 0.0000002$$

$$= 0.32497.$$

Example: Apply Bessel's formula to obtain y_{25}, given $y_{20} = 2854$, $y_{24} = 3162$, $y_{28} = 3544$, $y_{32} = 3992$.

Solution:

Taking the origin at $x_0 = 24$, h = 4, we have $p = \frac{1}{4}$ (x-24).

∴ The central difference table is

p	y	Δy	$\Delta^2 y$	$\Delta^3 y$
-1	2854			
		308		
0	<u>3162</u>		<u>74</u>	
		<u>382</u>		<u>-8</u>
1	3544		<u>66</u>	
		448		
2	3992			

At x = 25, p =(25-24)/4 = ¼ . (As p lies between ¼ and ¾, the use of Bessel's formula will yield accurate result.)

Bessel's formula is

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!} \cdot \frac{\Delta^2 y_{-1} + \Delta^2 y_0}{2} + \frac{(p-\frac{1}{2})p(p-1)}{3!}\Delta^3 y_{-1} + \ldots\ldots \quad \text{---(1)}$$

When p = 0.25, we have

$$y_p = 3162 + 0.25 \times 382 + \frac{0.25(-0.75)}{2}\left(\frac{74+66}{2}\right)$$

$$+\frac{(-0.25)(0.25)(-0.75)}{6}(-8)$$

$$= 3162 + 95.5 - 6.5625 - 0.0625 = 3250.875 \text{ approx.}$$

5.3 INTERPOLATION WITH UNEQUAL INTERVALS

The various interpolation formulae derived so far being applicable only to equally spaced values of the argument. It is, therefore, desirable to develop interpolation formulae for unequally spaced values of x. Lagrange's interpolation formula is one such formula and is as follows:

5.3.1 Lagrange's Interpolation Formula:

If y = f(x) takes the value $y_0, y_1, \ldots\ldots y_n$ corresponding to $x = x_0, x_1, \ldots\ldots, x_n$ then

$$f(x) = \frac{(x-x_1)(x-x_2).....(x-x_n)}{(x_0-x_1)(x_0-x_2)....(x_0-x_n)} y_0 + \frac{(x-x_0)(x-x_2).....(x-x_n)}{(x_1-x_0)(x_1-x_2)....(x_1-x_n)} y_1$$

$$+......+\frac{(x-x_0)(x-x_1).....(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)....(x_n-x_{n-1})} y_n \qquad ---(1)$$

This is known as Lagrange's interpolation formula for unequal intervals.

Proof: Let y = f(x) be a function which takes the values (x_0, y_0) (x_1, y_1),.... (x_n, y_n). Since there are n +1 pairs of values of x and y, we can represent f(x) by a polynomial in x of degree n. Let this polynomial be of the form

$$y = f(x) = a_0(x-x_1)(x-x_2)\ldots\ldots(x-x_n) + a_1(x-x_0)(x-x_2)\ldots\ldots(x-x_n)$$
$$+ a_2(x-x_0)(x-x_1)(x-x_3)\ldots\ldots(x-x_n)+\ldots\ldots$$
$$+ a_n(x-x_0)(x-x_1)\ldots\ldots(x-x_{n-1}) \qquad --- (2)$$

Putting $x= x_0$, $y = y_0$, in (2), we get

$$y_0 = a_0(x_0-x_1)(x_0-x_2)\ldots\ldots\ldots..(x_0-x_n)$$

$$\Rightarrow \quad a_0= y_0/[(x_0-x_1)(x_0-x_2)\ldots\ldots\ldots..(x_0-x_n)]$$

Similarly putting $x = x_1$, $y = y_1$ in (2), we have

$$a_1 = y_1/[(x_1-x_0)(x_1-x_2)\ldots\ldots\ldots..(x_1-x_n)]$$

Proceeding the same way, we find $a_2, a_3\ldots\ldots a_n$. Substituting the values of $a_0, a_1, \ldots.., a_n$ in (2), we get (1).

Remark: Lagrange's formula can be applied whether the values $x_i$are equally spaced or not. It is easy to remember but quite cumbersome to apply.

Example: Given the values

x:	5	7	11	13	17
f(x):	150	392	1452	2366	5202

Evaluate f(9), using Lagrange's formula

Solution:

Here $x_0 = 5, x_1 = 7, x_2 = 11, x_3 = 13, x_4 = 17$

$y_0 = 150, y_1 = 392, y_2 = 1452, y_3 = 2366, y_4 = 5202$

Putting x = 9 and substituting the above values in Lagrange's formula, we get

$$f(9) = \frac{(9-7)(9-11)(9-13)(9-17)}{(5-7)(5-11)(5-13)(5-17)}\times 150+\frac{(9-5)(9-11)(9-13)(9-17)}{(7-5)(7-11)(7-13)(7-17)}\times 392$$

$$+\frac{(9-5)(9-7)(9-13)(9-17)}{(11-5)(11-7)(11-13)(11-17)}\times 1452+\frac{(9-5)(9-7)(9-11)(9-17)}{(13-5)(13-7)(13-11)(13-17)}\times 2366$$

$$+\frac{(9-5)(9-7)(9-11)(9-13)}{(17-5)(17-7)(17-11)(17-13)}\times 5202$$

$$= -\frac{50}{3}+\frac{3136}{15}+\frac{3872}{3}-\frac{2366}{3}+\frac{578}{5}=810$$

5.4 INVERSE INTERPOLATION

So far, we have been finding the value of y corresponding to a certain value of x from a given set of values of x and y. On the other hand, the process of finding the value of x for a value of y is called the inverse interpolation. When the values of x are unequally spaced, Lagrange's method is used and when the values of x are equally spaced, the Iterative method should be used.

5.4.1 Lagrange's Method:

This is similar to Lagrange's interpolation formula, the only difference being that x is assumed to be expressible as a polynomial in y.

Lagrange's formula is merely a relation between two variables either of which may be taken as the independent variable. Therefore, on inter-changing x and y in the Lagrange's formula, we obtain.

$$x = \frac{(y-y_1)(y-y_2).......(y-y_n)}{(y_0-y_1)(y_0-y_{21}).......(y_0-y_n)}x_0 + \frac{(y-y_0)(y-y_2).......(y-y_n)}{(y_1-y_0)(y_1-y_2).......(y_1-y_n)}x_1$$

$$+..........+\frac{(y-y_0)(y-y_1).......(y-y_{n-1})}{(y_n-y_0)(y_n-y_1).......(y_n-y_{n-1})}x_n,$$

which is used for inverse interpolation.

Example: The following table gives the values of x and y :

x:	1.2	2.1	2.8	4.1	4.9	6.2
y:	4.2	6.8	9.8	13.4	15.5	19.6

Find the value of x corresponding to y = 12, using Lagrange's technique.

Solution:

Here $x_0 = 1.2, x_1 = 2.1, x_2 = 2.8, x_3 = 4.1$ $x_4 = 4.9, x_5 = 6.2$

and $y_0 = 4.2, y_1 = 6.8, y_2 = 9.8, y_3 = 13.4, y_4 = 15.5, y_5 = 19.6$

Taking y = 12, the above formula gives

$$x = \frac{(12-6.8)(12-9.8)(12-13.4)(12-15.5)(12-19.6)}{(4.2-6.8)(4.2-9.8)(4.2-13.4)(4.2-15.5)(4.2-19.6)}\times 1.2$$

$$+\frac{(12-4.2)(12-9.8)(12-13.4)(12-15.5)(12-19.6)}{(6.8-4.2)(6.8-9.8)(6.8-13.4)(6.8-15.5)(6.8-19.6)}\times 2.1$$

$$+\frac{(12-4.2)(12-6.8)(12-13.4)(12-15.5)(12-19.6)}{(9.8-4.2)(9.8-6.8)(9.8-13.4)(9.8-15.5)(9.8-19.6)}\times 2.8$$

$$+\frac{(12-4.2)(12-6.8)(12-9.8)(12-15.5)(12-19.6)}{(13.4-4.2)(13.4-6.8)(13.4-9.8)(13.4-15.5)(13.4-19.6)}\times 4.1$$

$$+\frac{(12-4.2)(12-6.8)(12-9.8)(12-13.4)(12-19.6)}{(15.5-4.2)(15.5-6.8)(15.5-9.8)(15.5-13.4)(15.5-19.6)}\times 4.9$$

$$+\frac{(12-4.2)(12-6.8)(12-9.8)(12-13.4)(12-15.5)}{(19.6-4.2)(19.6-6.8)(19.6-9.8)(19.6-13.4)(19.6-15.5)}\times 6.2$$

$= 0.022-0.234+1.252+3.419-0.964+0.055$

$= 3.55.$

5.4.2 Iterative Method or Method of Successive Approximations:

Newton's forward interpolation formula is

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \ldots\ldots$$

From this, we obtain

$$p = \frac{1}{\Delta y_0}\left[y_p - y_0 - \frac{p(p-1)}{2!}\Delta^2 y_0 - \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 - \ldots\ldots\right] \qquad \text{---(1)}$$

Neglecting the second and higher order differences, we obtain the first approximation to p as

$$p_1 = (y_p - y_0)/\Delta y_0 \qquad \text{---(2)}$$

To find the second approximation, retaining the term with second differences in (1) and replacing p by p_1, we get

$$p_2 = \frac{1}{\Delta y_0}[y_p - y_0 - \frac{p_1(p_1-1)}{2!}\Delta^2 y_0] \qquad \text{---(3)}$$

To find the third approximation, retaining the term with third differences in (1) and replacing every p by p_2, we have

$$p_3 = \frac{1}{\Delta y_0}[y_p - y_0 - \frac{p_2(p_2-1)}{2!}\Delta^2 y_0 - \frac{p_2(p_2-1)(p_2-2)}{3!}\Delta^3 y_0],$$

and so on. This process is continued till two successive approximations of p agree with each other, to the desired accuracy.

Remark: This technique can equally well be applied by starting with any other interpolation formula.

Example: The following values of y = f(x) are given

x:	10	15	20
y:	1754	2648	3564

Find the value of x for y = 3000 by iterative method.

Solution:

Taking $x_0 = 10$ and h = 5, the difference table is

x	y	Δy	$\Delta^2 y$
10	1754		
		894	
15	2648		22
		916	
20	3564		

Here $y_p = 3000$, $y_0 = 1754$, $\Delta y_o = 894$ and $\Delta^2 y_o = 22$.

$\therefore$ The successive approximations to p are

$$p_1 = \frac{1}{894}(3000 - 1754) = 1.39$$

$$p_2 = \frac{1}{894}\left[3000 - 1754 - \frac{1.39(1.39-1)}{2}\times 22\right] = 1.387,$$

$$p_3 = \frac{1}{894}\left[3000 - 1754 - \frac{1.387(1.387-1)}{2}\times 22\right],$$

$$= 1.3871$$

We, therefore, take p = 1.387 correct to three decimal places.

Hence the value of x (corresponding to y =3000) = x_0 +ph

= 10 + 1.387 x 5 =16.935.

Example: Using inverse interpolation, find the real root of the equation $x^3 + x - 3 = 0$, witch is close to 1.2.

Solution:

The difference table is

x	v	y(=x^3 + x-3)	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
1	-0.2	-1				
			0.431			
1.1	-0.1	-0.569		0.066		
			0.497		0.006	
1.2	0.0	-0.072		0.072		-0.00013
			0.569		-0.006	
1.3	0.1	0.497		0.078		
			0.647			
1.4	0.2	1.144				

Clearly the root of the given equation lies between 1.2 and 1.3.

Assuming the origin at x =1.2 and using Stirling's formula.

$$y = y_o + x\,\frac{\Delta y_o + \Delta y_{-1}}{2} + \frac{x^2}{2}\Delta^2 y_{-1} + \frac{x(x^2-1)}{6}\frac{\left(\Delta^3 y_{-1} + \Delta^3 y_{-2}\right)}{2},$$

we get

$$0 = -0.072 + x\frac{0.569+0.467}{2} + \frac{x^2}{2}(0.072)$$

$$+\frac{x(x^2-1)}{6}.\frac{0.006+0.006}{2} \qquad [\because y = 0]$$

or $\quad 0 = -0.072 + 0.532x+0.036x^2 + 0.001x^3$

This equation can be written as

$$x = \frac{0.072}{0.532} - \frac{0.036}{0.532}x^2 - \frac{0.001}{0.532}x^3 \qquad \text{.....(i)}$$

$\therefore$ First approximation $x^{(1)} = \frac{0.072}{0.532} = 0.1353$

Putting $x = x^{(1)}$ on R.H.S. of (i), we get second approximation $x^{(2)}$ as

$x^{(2)} = 0.1353-0.067(0.1353)^2-1.8797(0.1353)^3$

$= 0.134$

Hence the desired root = 1.2 +0.1 x 0.134 = 1.2134

5.5 CHECK YOUR PROGRESS

1. Use Stirling's formula to evaluate f(1.22) from the following

x:	1.0	1.1	1.2	1.3	1.4
f(x):	0.841	0.891	0.932	0.963	0.985

2. Use Bessel's formula to obtain y_{25}, given

$y_{20} = 24 \quad y_{24} = 32 \quad y_{28} = 35 \quad y_{32} = 40$

3. Use Lagrange's interpolation formula to find the value of y when x = 10, if the following values of x and y are given:

x:	5	6	9	11
f(x):	12	13	11	16

4. From the following data:

x:	1.8	2.0	2.2	2.4	2.6
f(x):	2.9	3.6	4.4	5.5	6.7

find x when y = 5 using the Iterative method.

5. The equation $x^3 - 15x + 4 = 0$ has a root close to 0.3, obtain this root upto 4 decimal places using inverse interpolation.

5.6 SUMMARY

1. In this chapter we learn to find the value which lies near the center of table.
2. We learn about the center difference interpolation method. For this we use gauss's forward and gauss's backward interpolation formulas.
3. We learn to solve problem having unequal interval. For this purpose we use **Lagrange's Interpolation Formula.**

5.7 KEYWORDS

Gauss's Forward Interpolation Formula: -

$$y_p = y_o + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{3!}\Delta^3 y_{-1}$$

$$+ \frac{(p+1)p(p-1)(p-2)}{4!}\Delta^4 y_{-2} + \ldots..,$$

Gauss's Backward Interpolation Formula:

$$y_p = y_0 + p\Delta y_{-1} + \frac{(p+1)p}{2!}\Delta^2 y_{-2} + \frac{(p+1)p(p-1)}{3!}\Delta^3 y_{-2}$$

$$+ \frac{(p+2)(p+1)p(p-1)}{4!}\Delta^4 y_{-2} + \ldots\ldots.,$$

Stirling'sformula:-

$$y_p = y_0 + p\left(\frac{\Delta y_0 + \Delta y_{-1}}{2}\right) + \frac{p^2}{2!}\Delta^2 y_{-1} + \frac{p(p^2-1)}{3!}\left(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right)$$

$$+ \frac{p^2(p^2-1)}{4!}\Delta^4 y_{-2} + \ldots\ldots$$

Bessel's formula:-

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!} \cdot \frac{\Delta^2 y_{-1} + \Delta^2 y_0}{2} + \frac{(p-\frac{1}{2})p(p-1)}{3!} \Delta^3 y_{-1}$$

$$+ \frac{(p+1)p(p-1)(p-2)}{4!} \cdot \frac{\Delta^4 y_{-2} + \Delta^4 y_{-1}}{2} + \quad .....,$$

Lagrange's Interpolation Formula:-

$$f(x) = \frac{(x-x_1)(x-x_2).....(x-x_n)}{(x_0-x_1)(x_0-x_2)....(x_0-x_n)} y_0 + \frac{(x-x_0)(x-x_2).....(x-x_n)}{(x_1-x_0)(x_1-x_2)....(x_1-x_n)} y_1$$

$$+......+ \frac{(x-x_0)(x-x_1).....(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)....(x_n-x_{n-1})} y_n$$

This is known as Lagrange's interpolation formula for unequal intervals.

5.8 SELF ASSESSMENT TEST

1. Use Stirling's formula to evaluate f(22.2) from the following

x:	2.0	2.1	2.2	2.3	2.4
f(x):	0.23967	0.28062	0.31788	0.35209	0.38368

2. Using gauss forword interpolation formula, find f(337.5)

X	310	320	330	340	350	360
Y	2.4914	2.5052	2.5185	2.5315	2.5441	2.5563

3. Using gauss forword interpolation formula, find f(30)

X	21	25	29	33	37
Y	18.4708	17.8144	17.1070	16.3432	15.5154

4. Using gauss backward interpolation formula, find f(1966)

X	1931	1941	1951	1961	1971	1981
Y	12	15	20	27	39	52

5. Using gauss backward interpolation formula, find f(1974)

X	1939	1949	1959	1969	1979	1989
Y	12	15	20	27	39	52

6. Using Bessel's interpolation formula, find f(25)

X	20	24	28	32
Y	2854	3162	3544	3992

7. Using Bessel's interpolation formula, find f(25)

X	20	24	28	32
Y	24	32	35	40

5.9 ANSWER TO CHECK YOUR PROGRESS

(1) 0.934

(2) 32.945

(3) 14.63

(4) 2.3

(5) 0.2679

5.10 REFERENCES/ SUGGESTEDREADINGS

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NUMERICAL DIFFERENTIATION AND INTEGRATION	
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STRUCTURE

6.0 OBJECTIVE

Objective of this lesson is to develop numerical techniques for finding the differentiation and integration of the functions, which are given in the tabulated forms, using the interpolation formulae. In the end, the errors in numerical integration formulae are also obtained.

6.1 INTRODUCTION

If a function y = f(x) be defined at a set of n+1 distinct points x_0, x_1,…..,x_{n-1},x_n lying in some interval [a, b] such that a = $x_0<x_1<x_2<…..<x_n$ = b. From the given tabulated data (set of values of x and y), we require to find differentiation of different orders at tabular or non tabular points. Also, we require to find the values of the definite integral $\int_a^b f(x)dx$, where f(x) is either given explicitly or defined by a tabulated data.

6.2 NUMERICAL DIFFERENTIATION

It is the process of calculating the value of the derivative of a function at some assigned value of x from the given set of values (x_i, y_i). To compute dy/dx, we first replace the exact relation y= f(x) by the best interpolating polynomial y = ϕ(x) and then differentiate the latter as many times as we desire. The choice of the interpolation formula to be used, will depend on the assigned value of x at which dy/dx is desired.

If the values of x are equi-spaced and dy/dx is required near the beginning of the table, we employ Newton's forward formula. If it is required near the end of the table, we use Newton's backward formula. For values near the middle of the table, dy/dx is calculated by means of Stirling's or Bessel's formula.

If the values of x are not equi-spaced, we use Newton's divided difference formula to represent the function.

Hence corresponding to each of the interpolation formulae, we can derive a formula for finding the derivative.

Consider the function y = f(x) which is tabulated for the values x_i $(= x_0 + ih)$, i= 0, 1,2, ...n.

6.2.1 Derivatives using Forward Difference Formula:

Newton's forward interpolation formula is

$$y = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \ldots\ldots \qquad \text{---(1)}$$

Differentiating both sides w.r.t. p, we have

$$\frac{dy}{dp} = \Delta y_0 + \frac{2p-1}{2!}\Delta^2 y_0 + \frac{3p^2-6p+2}{3!}\Delta^3 y_0 + \ldots\ldots$$

Since p = $\frac{(x-x_0)}{h}$, therefore $\frac{dp}{dx} = \frac{1}{h}$.

Now

$$\frac{dy}{dx} = \frac{dy}{dp}.\frac{dp}{dx} = \frac{1}{h}\left[\Delta y_o + \frac{2p-1}{2!}\Delta^2 y_0 + \frac{3p^2-6p+2}{3!}\Delta^3 y_0 + \frac{4p^3-18p^2+22p-6}{4!}\Delta^4 y_0 + \ldots\right] \qquad \text{---(2)}$$

At x = x_0, p = 0. Hence putting p = 0,

$$\left(\frac{dy}{dx}\right)_{x_0} = \frac{1}{h}\left[\Delta y_o - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \ldots\ldots\ldots\right] \qquad \text{---(3)}$$

Again differentiating (2) w.r.t. x, we get

$$\frac{d^2y}{dx^2} = \frac{d}{dp}\left(\frac{dy}{dp}\right)\frac{dp}{dx}$$

$$= \frac{1}{h}\left[\frac{2}{2!}\Delta^2 y_0 + \frac{6p\ \ -6}{3!}\Delta^3 y_0 + \frac{12p^2 - 36p + 22}{4!}\Delta^4 y_0 + ...\right]\frac{1}{h}$$

Putting p = 0, we obtain

$$\left(\frac{d^2y}{dx^2}\right)_{x_0} = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 + ...\right] \qquad \text{---(4)}$$

6.2.2 Derivatives using Backward Differences Formula:

Newton's backward interpolation formula is

$$y = y_n + p\nabla y_n + \frac{P(p+1)}{2!}\nabla^2 y_n + \frac{P(P+1)(p+2)}{3!}\nabla^3 y_n + ...$$

Differentiating both sides w.r.t. p, we get

$$\frac{dy}{dx} = \nabla y_n + \frac{2p+1}{2!}\nabla^2 y_n + \frac{3p^2 + 6p + 2}{3!}\nabla^3 y_n + ...$$

Since $p = \frac{x - x_n}{h}$, therefore $\frac{dp}{dx} = \frac{1}{h}$

Now $\frac{dy}{dx} = \frac{dy}{dp}.\frac{dp}{dx}$

$$= \frac{1}{h}\left[\nabla y_n + \frac{2p+1}{2!}\nabla^2 y_n + \frac{3p^3 + 6p + 2}{3!}\nabla^2 y_n + ...\right] \qquad \text{---(5)}$$

At x = x_n, p = 0. Hence putting p = 0, we get

$$\left(\frac{dy}{dx}\right)_{x_n} = \frac{1}{h}\left[\nabla y_n + \frac{1}{2}\nabla^2 y_n + \frac{1}{3}\nabla^3 y_n + \frac{1}{4}\nabla^4 y_n..........\right] \qquad \text{---(6)}$$

Again differentiating (5) w.r.t. x, we have

$$\frac{d^2y}{dx^2} = \frac{d}{dp}\left(\frac{dy}{dx}\right)\frac{dp}{dx}$$

$$= \frac{1}{h}\left[\nabla^2 y_n + \frac{6P+6}{3!}\nabla^3 y_n + \frac{6p^2+18p+11}{12}\nabla^4 y_n + ...\right]$$

Putting p = 0, we obtain

$$\left(\frac{d^2y}{dx^2}\right)_{x_n} = \frac{1}{h^2}\left[\nabla^2 y_n + \nabla^3 y_n + \frac{11}{12}\nabla^4 y_n ....\right] \qquad \text{---(7)}$$

6.2.3 Derivatives using Central Difference Formulae:

Stirling's formula is

$$y_P = y_0 + \frac{p}{1!}\left(\frac{\Delta y_0 + \Delta y_{-1}}{2}\right) + \frac{p^2}{2!}\Delta^2 y_{-1}$$

$$+ \frac{p(p^2-1^2)}{3!}\left(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right) + \frac{p^2(p^2-1^2)}{4!}\Delta^4 y_{-2} + ... \qquad \text{---(8)}$$

Differentiating both sides w.r.t. p, we get

$$\frac{dy}{dx} = \left(\frac{\Delta y_0 + \Delta y_{-1}}{2}\right) + \frac{2p}{2!}\Delta^2 y_{-1} + \frac{3p^2-1}{3!}\left(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right) + \frac{4p^3-2p}{4!}\Delta^4 y_{-2} + ...$$

Since p = $\frac{x-x_0}{h}$, $\quad\therefore\quad \frac{dp}{dx} = \frac{1}{h}$.

Now

$$\frac{dy}{dx} = \frac{dy}{dp}.\frac{dp}{dx}$$

$$= \frac{1}{h}\left[\left(\frac{\Delta y_0 + \Delta y_{-1}}{2}\right) + p\Delta^2 y_{-1} + \frac{3p^2-1}{6}\left(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right) + \frac{2p^3-p}{12}\Delta^4 y_{-2} + ...\right]$$

At x = x_0, p = 0. Hence putting p = 0, we get

$$\left(\frac{dy}{dx}\right)_{x_0} = \frac{1}{h}\left[\frac{\Delta y_0 + \Delta y_{-1}}{2} - \frac{1}{6}\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2} + \frac{1}{30}\frac{\Delta^5 y_{-2} + \Delta^5 y_{-3}}{2} + ...\right] \quad \text{--(9)}$$

similarly $$\left(\frac{d^2 y}{dx^2}\right)_{x_0} = \frac{1}{h^2}\left[\Delta^2 y_{-1} - \frac{1}{12}\Delta^4 y_{-2} + \frac{1}{90}\Delta^6 y_{-3} .....\right] \qquad \text{---(10)}$$

Similarly, we can use any other interpolation formula for computing the derivatives.

Example : Given that

x:	1.0	1.1	1.2	1.3	1.4	1.5	1.6
y:	7.989	8.403	8.781	9.129	9.451	9.750	10.031,

Find $\frac{dy}{dx}$ and $\frac{d^2 y}{dx^2}$ at (a) $x = 1.1$ (b) x = 1.6

Solution:

The difference table is

x	y	Δ	Δ^2	Δ^3	Δ^4	Δ^5	Δ^6
1.0	7.989						
		0.414					
1.1	8.403		-0.036				
		0.378		0.006			
1.2	8.781		-0.030		-0.002		
		0.348		0.004		0.002	
1.3	9.129		-0.026		0.000		-0.003
		0.322		0.004		-0.001	
1.4	9.451		-0.023		-0.001		

		0.299		0.005
1.5	9.750		-0.018	
		0.281		
1.6	10.031			

a) For x = 1.1, we take

$$\left(\frac{dy}{dx}\right)_{x_0} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \frac{1}{5}\Delta^5 y_0 - \frac{1}{6}\Delta^6 y_0 + ...\right] \text{--(i)}$$

and $$\left(\frac{d^2y}{dx^2}\right)_{x_0} = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 - \frac{5}{6}\Delta^5 y_0 + \frac{137}{180}\Delta^6 y_0 - ....\right] \quad \text{---(ii)}$$

Here h = 0.1, x_0 = 1.1, Δy_0 = 0.378, $\Delta^2 y_0$ = -0.03 etc.

Substituting these values in (i) and (ii), we get

$$\left(\frac{dy}{dx}\right)_{1.1} = \frac{1}{0.1}\left[0.378 - \frac{1}{2}(-0.03) + \frac{1}{3}(0.004) - \frac{1}{4}(0) + \frac{1}{5}(-0.001) - \frac{1}{6}(-0.003)\right]$$

= 3.946

$$\left(\frac{d^2y}{dx^2}\right)_{1.1} = \frac{1}{(0.1)^2}\left[-0.03 - (0.004) + \frac{11}{12}(0) - \frac{5}{6}(-0.001) + \frac{137}{180}(-0.003)\right]$$

= -3.545

b) For x = 1.6, we use the above difference table and the backward difference operator ∇ instead of Δ, and we take

$$\left(\frac{dy}{dx}\right)_{x_n} = \frac{1}{h}\left[\nabla y_n + \frac{1}{2}\nabla^2 y_n + \frac{1}{3}\nabla^3 y_n + \frac{1}{4}\nabla^4 y_n + \frac{1}{5}\nabla^5 y_n + \frac{1}{6}\nabla^6 y_n + ...\right] \text{---(iii)}$$

and

$$\left(\frac{d^2y}{dx^2}\right)_{x_n} = \frac{1}{h^2}\left[\nabla^2 y_n + \nabla^3 y_n + \frac{11}{12}\nabla^4 y_n + \frac{5}{6}\nabla^5 y_n + \frac{137}{180}\nabla^6 y_n + ...\right] \text{---(iv)}$$

Here h = 0.1, x_n = 1.6, ∇y_n = 0.281, $\nabla^2 y_n$ = -0.018 etc.

Putting these values in (iii) and (iv), we get

$$\left(\frac{dy}{dx}\right)_{1.6} = \frac{1}{0.1}\left[0.281 + \frac{1}{2}(-0.018) + \frac{1}{3}(0.005) + \frac{1}{4}(-0.001) + \frac{1}{5}(-0.001) + \frac{1}{6}(-0.003)\right]$$

= 2.727

$$\left(\frac{d^2y}{dx^2}\right)_{1.6} = \frac{1}{(0.1)^2}\left[-0.018 + 0.005 + \frac{11}{12}(-0.001) + \frac{5}{6}(-0.001) + \frac{137}{180}(-0.003)\right]$$

= -1.703

Example: From the following table find $\frac{dx}{dt}$ & $\frac{d^2x}{dt^2}$ at t = 0.3

T:	0	0.1	0.2	0.3	0.4	0.5	0.6
X	30.13	31.62	32.87	33.64	33.95	33.81	33.24

Solution :

The difference table is:

t	x	Δ	Δ^2	Δ^3	Δ^4	Δ^5	Δ^6
0	30.13						
		1.49					
0.1	31.62		-0.24				
		1.25		-0.24			
0.2	32.87		-0.48		0.26		

		0.77		0.02		-0.27	
0.3	33.64		-0.46		-0.01		0.29
		0.31		0.01		0.02	
0.4	33.95		-0.45		0.01		
		-0.14		0.02			
0.5	33.81		-0.43				
		-0.57					
0.6	33.24						

As the derivatives are required near the middle of the table, we use Stirling's formulae:

$$\left(\frac{dx}{dt}\right)_{t_0} = \frac{1}{h}\left(\frac{\Delta x_0 + \Delta x_{-1}}{2}\right) - \frac{1}{6}\left(\frac{\Delta^3 x_{-1} + \Delta^3 x_{-2}}{2}\right) + \frac{1}{30}\left(\frac{\Delta^5 x_{-2} + \Delta^5 x_{-3}}{2}\right) + \ldots\ldots. \quad \text{--(i)}$$

$$\left(\frac{d^2 x}{dt^2}\right)_{t_0} = \frac{1}{h^2}\left[\Delta^2 x_{-1} - \frac{1}{12}\Delta^4 x_{-2} + \frac{1}{90}\Delta^6 x_{-3} - \ldots..\right] \quad \text{---(ii)}$$

Here h = 0.1, $t_0 = 0.3$, $\Delta x_0 = 0.31$, $\Delta x_{-1} = 0.77$, $\Delta^2 x_{-1} = -0.46$ etc.

Putting these values in (i) and (ii), we get

$$\left(\frac{dx}{dt}\right)_{0.3} = \frac{1}{0.1}\left[\left(\frac{0.31+0.77}{2}\right) - \frac{1}{6}\left(\frac{0.01+0.02}{2}\right) + \frac{1}{30}\left(\frac{0.02-0.27}{2}\right) - \ldots\ldots.\right]$$

$$= 5.33$$

$$\left(\frac{d^2 x}{dt^2}\right)_{0.3} = \frac{1}{(0.1)^2}\left[-0.46 - \frac{1}{12}(-0.01) + \frac{1}{90}(0.29) - \ldots.\right] = -45.6$$

6.3 MAXIMA AND MINIMA OF A TABULATED FUNCTIO N

Newton's forward interpolation formula is

$$y = y_o + p\Delta y_o + \frac{p(p-1)}{2!}\Delta^2 y_o + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_o + ...$$

Differentiating both sides w.r.t. p, we get

$$\frac{dy}{dp} = \Delta y_o + \frac{2p-1}{2}\Delta^2 y_o + \frac{3p^2-6p+2}{2}\Delta^3 y_o + ... \qquad \text{---(1)}$$

For maxima or minima, dy/dp = 0. Hence equating the right hand side of (1) to zero and retaining only upto third differences, we obtain

$$\Delta y_o + \frac{2p-1}{2!}\Delta^2 y_o + \frac{(3p^2-6p+2)}{6}\Delta^3 y_o = 0,$$

i.e. $(\frac{1}{2}\Delta^3 y_0)p^2 + (\Delta^2 y_0 - \Delta^3 y_0)\,p + (\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + 1/3\,\Delta^3 y_0) = 0$

Substituting the values of Δy_0, $\Delta^2 y_0$, $\Delta^3 y_0$ from the difference table, we solve this quadratic for p. Then the corresponding values of x are given by $x = x_0 + ph$ at which y is maximum or minimum.

Example: From the table given below, find the value of x for which y is minimum. Also find this value of y.

x:	3	4	5	6	7	8
y:	0.205	0.240	0.259	0.262	0.250	0.224

Solution:

The difference table is

x	y	Δ	Δ^2	Δ^3
3	0.205			
		0.035		
4	0.240		-0.016	
		0.019		0.000
5	0.259		-0.016	
		0.003		0.001
6	0.262		-0.015	
		-0.012		0.001
7	0.250		-0.014	
		-0.026		
8	0.224			

Taking $x_0 = 3$, we have $y_0 = 0.205$, $\Delta y_0 = 0.035$, $\Delta^2 y_0 = -0.016$ and $\Delta^3 y_0 = 0$.

$\therefore$ Newton's forward difference formula gives

$$y = 0.205 + p(0.035) + \frac{p(p-1)}{2!}(-0.016) \qquad \text{---(i)}$$

Differentiating it w.r.t. p, we have

$$\frac{dy}{dp} = 0.035 + \frac{2p-1}{2!}(-0.016)$$

For y to be minimum, dy/dp = 0

$\therefore \quad 0.035 - 0.008\,(2p - 1) = 0$

Which gives p = 2.6875

$\therefore \quad x = x_0 + ph = 3 + 2.6875 \times 1 = 5.6875.$

Hence y is minimum when x = 5.6875.

Putting p = 2.6875 in (i), the minimum value of y

$$= 0.205 + 2.6875 \times 0.035 + \tfrac{1}{2}\,(2.6875 \times 1.6875)\,(-0.016)$$

$$= 0.2628.$$

6.4 NUMERICAL INTEGRATION

The process of evaluating a definite integral from a set of tabulated values of the integrand f(x) is called numerical integration. This process when applied to a function of a single variable, is known as quadrature.

The problem of numerical integration, like that of numerical differentiation, is solved by representing f(x) by an interpolation formula and then integrating it between the given limits. In this way, we can derive quadrature formulae for approximate integration of a function defined by a set of numerical values only.

Newton-Cotes quadrature formula:

Let $\quad I = \int_a^b f(x)dx$,

where f(x) takes the values $y_o, y_1, y_2, \ldots\ldots, y_n$ for $x = x_0, x_1, x_2, \ldots., x_n$.

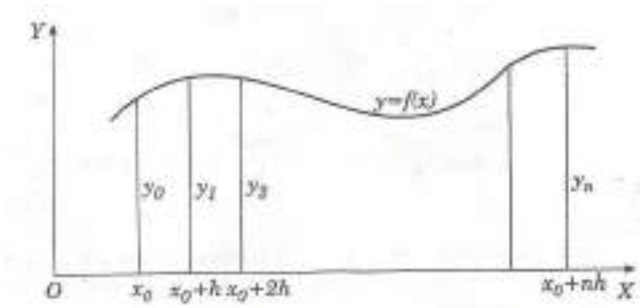

Fig – 5.1

Let us divide the interval (a,b) into n sub- intervals of width h so that $x_0 = a$, $x_1 = x_0+h$, $x_2 = x_0 + 2h$, ……, $x_n = x_0+nh = b$. Then

$$I = \int_{x_0}^{x_0+nh} f(x)dx \qquad \text{[Put } x = x_0 + rh, \ dx = hdr]$$

$$= h\int_0^n f(x_0 + rh)dr$$

$$= h\int_0^n \left[y_o + r\Delta y_0 + \frac{r(r-1)}{2!}\Delta^2 y_0 + \frac{r(r-1)(r-2)}{3!}\Delta^3 y_0 + \dots \right] dr$$

[Approximated by Newton's forward interpolation formula]

Integrating term by term, we obtain

$$\int_{x_0}^{x_0+nh} f(x)dx = nh\left[y_o + \frac{n}{2}\Delta y_0 + \frac{n(2n-3)}{12}\Delta^2 y_0 + \frac{n(n-2)^2}{24}\Delta^3 y_0 \right.$$

$$\left. + \left(\frac{n^4}{5} - \frac{3n^3}{2} + \frac{11n^2}{3} - 3n \right)\frac{\Delta^4 y_0}{24} + \dots\dots \right] \quad \text{---(1)}$$

This is known as Newton-Cotes quadrature formula. From this general formula, we deduce the following important quadrature rules by taking n = 1, 2, 3,…..

6.4.1 Trapezoidal Rule

Putting n = 1 in (1) and taking the curve through (x_0, y_0) and (x_1,y_1) as a straight line, i.e., a polynomial of first order so that differences of order higher than first become zero, we get

$$\int_{x_0}^{x_0+h} f(x)dx = h\left(y_0 + \frac{1}{2}\Delta y_0 \right) = \frac{h}{2}(y_0 + y_1)$$

Similarly

$$\int_{x_0+h}^{x_0+2h} f(x)dx = h\left(y_1 + \frac{1}{2}\Delta y_1 \right) = \frac{h}{2}(y_1 + y_2)$$

………………………………………

$$\int_{x_0+(n-1)h}^{x_0+nh} f(x)dx = \frac{h}{2}\left(y_{n-1} + y_n\right)$$

Adding these n integrals, we obtain

$$\int_{x_0}^{x_0+nh} f(x)dx = \frac{h}{2}\left[\left(y_0 + y_n\right) + 2(y_1 + y_2 + ... + y_{n-1})\right] \qquad \text{---(2)}$$

This is known as the trapezoidal rule.

In this rule, the area of each strip (trapezium) is found separately. Then the area under the curve and the ordinates at x_0 and x_0 +nhis approximately equal to the sum of the areas of the n trapeziums.

6.4.2 Simpson's one-third Rule

Putting n = 2 in (1) above and taking the curve through (x_0, y_0), (x_1, y_1) and (x_2, y_2) as a parabola, i.e., a polynomial of second order so that differences of order higher than second vanish, we get

$$\int_{x_0}^{x_0+2h} f(x)dx = 2h\left(y_0 + \Delta y_0 + \frac{1}{6}\Delta^2 y_0\right) = \frac{h}{3}(y_0 + 4y_1 + y_2)$$

Similarly

$$\int_{x_0+2h}^{x_0+4h} f(x)dx = \frac{h}{3}\left(y_2 + 4y_3 + y_4\right)$$

………………………………

$$\int_{x_0+(n-2)h}^{x_0+nh} f(x)dx = \frac{h}{3}\left(y_{n-2} + 4y_{n-1} + y_n\right), \text{ n being even.}$$

Adding all these integrals, we have when n is even

$$\int_{x_0}^{x_0+nh} f(x)dx = \frac{h}{3}\left[\left(y_0 + y_n\right) + 4(y_1 + y_3 + ... + y_{n-1}) + 2(y_2 + y_4 + ... + y_{n-2})\right]$$

---(3)

This is known as the Simpson's one-third rule or simply Simpson's rule and is most commonly used.

Remark : While applying (3), the given interval must be divided into even number of equal sub-intervals, since we find the area of two strips at a time.

6.4.3 Simpson's three-eighth Rule

Putting n = 3 in (1) above and taking the curve through (x_i, y_i): i = 0, 1, 2, 3, as a polynomial of third order so that differences above the third order vanish, we get

$$\int_{x_0}^{x_0+3h} f(x)dx = 3h\left(y_0 + \frac{3}{2}\Delta y_0 + \frac{3}{2}\Delta^2 y_0 + \frac{1}{8}\Delta^2 y_0 \right)$$

$$= \frac{3h}{8}(y_0 + 3y_1 + 3y_2 + y_3)$$

Similarly,

$$\int_{x_0+3h}^{x_0+5h} f(x)dx = \frac{3h}{8}(y_3 + 3y_4 + 3y_5 + y_6) \text{ and so on.}$$

Adding all such expressions from x_0 to x_0 +nh, where n is a multiple of 3, we obtain

$$\int_{x_0}^{x_0+nh} f(x)dx = \frac{3h}{8}\left[(y_0 + y_n) + 3(y_1 + y_2 + y_4 + y_5 + .. + y_{n-1}) + 2(y_3 + y_6 + .. + y_{n-3})\right], \quad \text{---(4)}$$

which is known as Simpson's three-eighth rule.

In this rule, the number of sub-intervals should be taken as multiple of three.

Example: Evaluate $\int_0^6 \frac{dx}{1+x^2}$ by using (i) Trapezoidal rule, (ii) Simplson's 1/3 rule, (iii) Simpson's 3/8 rule, and compare the results with its actual value.

Solution:

Divide the interval (0,6) into six parts each of width h = 1. The values of f(x) = $\frac{1}{1+x^2}$ are given below

x	0	1	2	3	4	5	6
f(x) y	1 y_0	0.5 y_1	0.2 y_2	0.1 y_3	0.0588 y_4	0.0385 y_5	0.027 y_6

(i) By Trapezoidal rule,

$$\int_0^6 \frac{dx}{1+x^2} = \frac{h}{2}[(y_0 + y_6) + 2(y_1+y_2+y_3+y_4+y_5)]$$

$$= \frac{1}{2}[(1+0.027)+2(0.5+0.2+0.1+0.0588+0.0385)]$$

$$= 1.4108$$

(ii) By Simpson's 1/3 rule,

$$\int_0^6 \frac{dx}{1+x^2} = \frac{h}{3}[(y_0 + y_6) + 4(y_1+y_3+y_5)+2(y_2+y_4)]$$

$$= \frac{1}{3}[(1+0.027) + 4(0.5+0.1+0.0385)+2(0.2+0.0588)]$$

$$= 1.3662.$$

(iii) By Simpson's 3/8 rule,

$$\int_0^6 \frac{dx}{1+x^2} = \frac{3h}{8}[(y_0 + y_6) + 3(y_1+y_2+y_4+y_5)+2y_3)]$$

$$= \frac{3}{8}[(1+0.027) + 3(0.5+0.2+0.0588+0.0385)+2(0.1)]$$

$= 1.3571$

Also $\int_0^6 \frac{dx}{1+x^2} = \left|\tan^{-1} x\right|_0^6 = \tan^{-1} 6 = 1.4056$

This shows that the value of the integral found by Simpson's 1/3 rule is the nearest to the actual value.

Example: The velocity v(km/min) of a moped which starts from rest, is given at fixed intervals of time t (min) as follows :

t:	2	4	6	8	10	12	14	16	18	20
v:	10	18	25	29	32	20	11	5	2	0

Estimate approximately the distance covered in 20 minutes.

Solution:

If s (km) be the distance covered in t (min), then

$$\frac{ds}{dt} = v$$

$$\therefore \qquad \left|s\right|_{t=0}^{20} = \int_0^{20} v\,dt = \frac{h}{3}[X + 4.O + 2.E]. \qquad \text{(By Simpson's rule)}$$

Here h = 2, $v_0 = 0$, $v_1 = 10$, $v_2 = 18$, $v_3 = 25$ etc.

$X = v_0 + v_{10} = 0 + 0 = 0$

$O = v_1 + v_3 + v_5 + v_7 + v_9 = 10 + 25 + 32 + 11+2 = 80$

$E = v_2 + v_4 + v_6 + v_8 = 18 + 29 + 20+ 5 = 72$

Hence the required distance

$$= \left|s\right|_{t=0}^{20} = \frac{2}{3}(0 + 4\times 80 + 2\times 72) = 309.33 \text{ km.}$$

Example:A solid of revolution is formed by rotating about the x-axis, the area between the x-axis, the lines x = 0 and x = 1 and a curve through the points with the following co-ordinates.

x:	0.00	0.25	0.50	0.75	1.00
y:	1.0000	0.9896	0.9589	0.9089	0.8415

Estimate the volume of the solid formed using Simpson's rule.

Solution:

Here h = 0.25 $y_0 = 1$, $y_1 = 0.9896$, $y_2 = 0.9589$ etc.

$\therefore$ Required volume of the solid generated

$$= \int_0^1 \pi y^2 dx = \pi.\frac{h}{3}[(y_0^2 + y_4^2) + 4(y_1^2 + y_3^2) + 2y_2^2]$$

$$= \frac{0.25\pi}{3}[\{1 + (0.8415)^2\} + 4\{(0.9896)^2 + (0.9089)^2\} + 2(0.9589)^2]$$

$$= \frac{0.25 \times 3.1416}{3}[1.7081 + 7.2216 + 1.839]$$

= 0.2618(10.7687) = 2.8192.

Example: Evaluate

$$I = \int_0^1 \frac{1}{1+x} dx$$, correct to three decimal places.

Solution:

We solve this example by both the trapezoidal and Simpson's rules with h = 0.5, 0.25 and 0.125 respectively.

(i) h = 0.5. The values of x and $y = \frac{1}{1+x}$ are tabulated below

x:	0	0.5	1.0
y :	1.0000	0.6667	0.5

(a) Trapezoidal rule gives

$$I = \frac{1}{4}[1.0000 + 2(0.6667) + 0.5]$$

$$= 0.7084$$

(b) Simpson's rule gives

$$I = \frac{1}{6}[1.0000 + 4(0.6667) + 0.5]$$

$$= 0.6945$$

(ii) h = 0.25. The tabulated values of x and $y = \frac{1}{1+x}$ are given below

x:	0	0.25	0.50	0.75	1.00
y:	1.0000	0.8000	0.6667	0.5714	0.5

(a) Trapezoidal rule gives

$$I = \frac{1}{8}[1.0 + 2(0.8000 + 0.6667 + 0.5714) + 0.5]$$

$$= 0.6970$$

(b) Simpson's rule gives

$$I = \frac{1}{12}[1.0 + 4(0.8000 + 0.5714) + 2(0.6667) + 0.5]$$

$$= 0.6932$$

(iii) Finally, we take h = 0.125. The tabulated values of x and y are

x:	0	0.125	0.250	0.375	0.5	0.625	0.750	0.875	1.0
y:	1.0	0.8889	0.8000	0.7273	0.6667	0.6154	0.5714	0.5333	0.5

(a) Trapezoidal rule gives

$$I = \frac{1}{16}[1.0 + 2(0.8889 + 0.8000 + 0.7273 + 0.6667$$

$$+ 0.6154 + 0.5714 + 0.5333) + 0.5]$$

$$= 0.6941$$

(b) Simpson's rule gives

$$I = \frac{1}{24}[1.0 + 4(0.8889 + 0.7273 + 0.6154 + 0.5333)$$

$$+ 2(0.8000 + 0.6667 + 0.5714) + 0.5]$$

$$= 0.6932$$

Hence the value of I may be taken to be equal to 0.693, correct to three decimal places. The exact value of Iis $\log_e 2$, which is equal to 0.693147…. This example demonstrates that, in general, Simpson's rule yields more accurate results than the trapezoidal rule.

6.5 ERRORS IN QUADRATURE FORMULAE

The error in the quadrature formulae is given by

$$E = \int_a^b y\,dx - \int_a^b P(x)\,dx,$$

where P(x) is the polynomial representing the function y = f(x), in the interval [a, b].

1) Error in the Trapezoidal rule

Expanding y = f(x) around $x = x_0$ by Taylor's series, we get

$$y = y_0 + (x - x_0)y_0' + \frac{(x - x_0)^2}{2!} y_0'' + \ldots\ldots \qquad \text{---(1)}$$

$$\therefore \quad \int_{x_0}^{x_0+h} y\,dx = \int_{x_0}^{x_0+h} [y_0 + (x - x_0)y_0' + \frac{(x - x_0)^2}{2!} y_0'' + \ldots\ldots]dx$$

$$= y_0 h + \frac{h^2}{2!} y_0' + \frac{h^3}{3!} y_0'' + \ldots\ldots \qquad \text{---(2)}$$

Also A_1 = area of the first trapezium in the interval $[x_0, x_1]$

$$= \frac{1}{2} h(y_0 + y_1) \qquad \text{---(3)}$$

Putting $x = x_0+h$ and $y = y_1$ in (1), we get

$$y_1 = y_0 + hy_0' + \frac{h^2}{2!} y_0'' + \ldots\ldots$$

Substituting this value of y_1 in (3), we get

$$A_1 = \frac{1}{2} h\left[y_0 + y_0 + hy_0' + \frac{h^2}{2!} y_0'' + \ldots\ldots \right]$$

$$= hy_0 + \frac{h^2}{2} y_0' + \frac{h^3}{2.2!} y_0'' + \ldots\ldots \qquad \text{---(4)}$$

$\therefore$ Error in the interval $[x_0, x_1]$

$$= \int_{x_0}^{x_1} y\,dx - A_1$$

$$= \left(\frac{1}{3!} - \frac{1}{2.2!} \right) h^3 y_0'' + \ldots = -\frac{h^3}{12} y_0'' + \ldots\ldots,$$

i.e., Principal part of the error in $[x_0, x_1] = -\frac{h^3}{12} y_0''$

Similarly principal part of the error in $[x_1, x_2] = -\frac{h^3}{12} y_1''$ and so on.

Hence the total error $E = -\frac{h^3}{12}[y_0'' + y_1'' + \ldots\ldots + y''_{n-1}]$

Assuming that $y''(X)$ is the largest of the n quantities $y_0'' y_1'', \ldots, y''_{n-1}$, we obtain

$$E < -\frac{nh^3}{12} y''(X) = -\frac{(b-a)h^2}{12} y''(X), \quad [\because nh = b-a] \quad \text{--- (5)}$$

Hence the error in the trapezoidal rule is of the order h^2.

2) Error in Simpson's one-third rule

Expanding $y = f(x)$ around $x = x_0$ by Taylor's series, we get (1).

$\therefore$ Over the first double strip, we get

$$\int_{x_0}^{x_2} y dx = \int_{x_0}^{x_0+2h} \left[(y_0 + (x - x_0) y_0' + \frac{(x-x_0)^2}{2!} y_0'' + \ldots\ldots \right] dx$$

$$= 2hy_0 + \frac{4h^2}{2!} y_0' + \frac{8h^3}{3!} y_0'' + \frac{16h^4}{4!} y_0''' + \frac{32h^5}{5!} y_0^{iv} + \ldots\ldots \quad \text{--- (6)}$$

Also A_1 = area over the first double strip by Simpson's 1/3 rule

$$= \frac{1}{3} h(y_0 + 4y_1 + y_2) \quad \text{--- (7)}$$

Putting $x = x_0 + h$ and $y = y_1$ in (1), we get

$$y_1 = y_0 + hy_0' + \frac{h^2}{2\,!} y_0'' + \frac{h^3}{3\,!} y_0''' + \ldots\ldots\ldots\ldots$$

Again putting $x = x_0 + 2h$ and $y = y_2$ in (1), we have

$$y_2 = y_0 + 2hy_0' + \frac{4h^2}{2!} y_0'' + \frac{8h^3}{3!} y_0''' + ...$$

Substituting these values of y_1 and y_2 in (7), we get

$$A_1 = \frac{h}{3}\left[y_0 + 4\left(y_0 + hy_0' + \frac{h^2}{2!} y_0''' + ...\right)\right.$$

$$\left. + \left(y_0 + 2hy_0' + \frac{4h^2}{2!} y_0'' + \frac{8h^3}{3!} y_0''' + ...\right)\right]$$

$$= 2hy_0 + 2h^2 y_0' + \frac{4h^3}{3} y_0'' + \frac{2h^2}{3} y_0''' + \frac{5h^5}{18} y_0^{iv} + ... \qquad \text{---(8)}$$

$\therefore$ Error in the interval $[x_0, x_2] = \int_{x_0}^{x_2} y \ dx - A_1$

$$= \left(\frac{4}{15} - \frac{5}{18}\right) h^5 y_0^{iv}, \qquad [(6)\text{-}(8)],$$

i.e., Principal part of the error in $[x_0, x_2]$

$$= \left(\frac{4}{15} - \frac{5}{18}\right) h^5 y_0^{iv} - \frac{h^5}{90} y_0^{iv}$$

Similarly principal part of the error in $[x_2, x_4] = -\frac{h^5}{90} y_2^{iv}$ and so on.

Hence the total error $E = -\frac{h^5}{90}[y_0^{iv} + y_2^{iv} + ... + y^{iv}{}_{2(n-1)}]$

Assuming the $y^{iv}(X)$ is the largest of $y_0^{iv}, y_2^{iv}, ..., y^{iv}{}_{2n-2}$, we get

$$E < -\frac{nh^5}{90} y_0^{iv}(X) = -\frac{(b-a)h^4}{180} y^{iv}(X), \qquad [\because 2nh = b-a], \text{---(9)}$$

i.e., the error in Simpson's $\frac{1}{3}$ rule is of the order h^4.

(3) Error in Simpson's 3/8 rule. Proceeding as above, here the principal part of the error in the interval $[x_0, x_3]$

$$= -\frac{3h^5}{80} y^{iv} \qquad \text{--- (10)}$$

6.6 CHECK YOUR PROGRESS

Find the first and second derivatives of f(x) at x = 1.5 if

x:	1.5	2.0	2.5	3.0	3.5	4.0
f(x):	3.375	7.00	13.625	24.00	38.875	59.00

1. Find the first and second derivatives of the function tabulated below at x = 1.1,

x:	1.00	1.2	1.4	1.6	1.8	2.0
f(x):	0.0	0.128	0.544	1.296	2.432	4.00

2. Find the first derivative at x = 4 from the following values of x and y,

x:	1	2	4	8	10
f(x):	0	1	5	21	27

3. Calculate the value of $\int_0^{\pi/2} \sin\ x\, dx$ by Simpson's one-third rule using 10 sub-intervals.

4. A curve is drawn to pass through the points given in the following table:

x:	1	1.5	2	2.5	3	3.5	4
y:	2	2.4	2.7	2.8	3	2.6	2.1

Find the area bounded by the curve, x-axis and the lines x = 1, x = 4.

6.7 SUMMAERY

NUMERICAL DIFFERENTIATION:- Methods based on interpolation uses the ponomial approximation obtained by nterpolation to find the derivative of the function, which is known at discrete points in the interval **[a, b]**.Derivatives are find by using Forward difference Formula ,Backward difference formula, central difference formula.

NUMERICAL INTEGRATION: -Numerical Integration is nothing but finding an approximate value to $\mathbf{I} = \int_a^b f(x)dx$.

There are two different strategies to develop numerical integration formulae. One is similar to what we have adopted to numerical differentiation. That is, we approximate a polynomial for the given function and integrate that polynomial within the limits of the integration. This restricts us to integrate a function known at discrete tabular points. If these points are uniformly spaced then the corresponding integration formulae are called as Newton - Cotes formulae for numerical integration. On the other hand if we know the function explicitly but could not integrate in the usual means because of the nature of the function then we can use the concept called quadrature rule to find an approximate value to the integration.

In this method the function is evalualted at some predetermined abscissa (nodal) points and then these values are added after multiplying them with some weights which are again predeternmined, to find an approximate value to the given integral.

6.8 KEYWORDS

1. Derivatives using Forward Difference Formula:

$$\left(\frac{dy}{dx}\right)_{x_0} = \frac{1}{h}\left[\Delta y_o - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \ldots\ldots\ldots\right]$$

$$\left(\frac{d^2 y}{dx^2}\right)_{x_0} = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 + \ldots\ldots\right]$$

2. Derivatives using Backward Differences Formula:-

$$\left(\frac{dy}{dx}\right)_{x_n} = \frac{1}{h}\left[\nabla y_n + \frac{1}{2}\nabla^2 y_n + \frac{1}{3}\nabla^3 y_n + \frac{1}{4}\nabla^4 y_n \ldots\ldots\ldots\ldots\right]$$

$$\left(\frac{d^2 y}{dx^2}\right)_{x_n} = \frac{1}{h^2}\left[\nabla^2 y_n + \nabla^3 y_n + \frac{11}{12}\nabla^4 y_n \ldots\right]$$

3. Derivatives using Central Difference Formulae:

$$\left(\frac{dy}{dx}\right)_{x_0} = \frac{1}{h}\left[\frac{\Delta y_0 + \Delta y_{-1}}{2} - \frac{1}{6}\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2} + \frac{1}{30}\frac{\Delta^5 y_{-2} + \Delta^5 y_{-3}}{2} + \ldots\ldots\ldots\right]$$

$$\left(\frac{d^2 y}{dx^2}\right)_{x_0} = \frac{1}{h^2}\left[\Delta^2 y_{-1} - \frac{1}{12}\Delta^4 y_{-2} + \frac{1}{90}\Delta^6 y_{-3} \ldots\ldots\right]$$

4. Trapezoidal Rule:- $\int_{x_0}^{x_0+nh} f(x)dx = \frac{h}{2}\left[(y_0 + y_n) + 2(y_1 + y_2 + \ldots. + y_{n-1})\right]$

5. Simpson's one-third Rule:-

$$\int_{x_0}^{x_0+nh} f(x)dx = \frac{h}{3}\left[(y_0 + y_n) + 4(y_1 + y_3 + \ldots. + y_{n-1}) + 2(y_2 + y_4 + \ldots\ldots + y_{n-2})\right]$$

5. **Simpson's three-eighth Rule**

$$\int_{x_0}^{x_0+nh} f(x)dx = \frac{3h}{8}\left[(y_0 + y_n) + 3(y_1 + y_2 + y_4 + y_5 + .. + y_{n-1}) + 2(y_3 + y_6 + .. + y_{n-3})\right]$$

6.9 SELF ASSESSMENT TEST

1. From the following table of values of X and Y, obtain first and second Derivatives for X=1.2

x	1.0	1.2	1.4	1.6	1.8	2.0	2.2
y	2.7183	3.3201	4.0552	4.9530	6.0496	7.3891	9.0250

2. From the following table of values of X and Y, obtain first and second Derivatives for X=1.4

x	1.4	1.6	1.8	2.0	2.2
y	4.0552	4.9530	6.0496	7.3891	9.0250

3. From the following table of values of X and Y, obtain first and second Derivatives for X=2.2

x	1.0	1.2	1.4	1.6	1.8	2.0	2.2
y	2.7183	3.3201	4.0552	4.9530	6.0496	7.3891	9.0250

4. From the following table of values of X and Y, obtain first and second Derivatives for X=1.4

x	1.4	1.6	1.8	2.0	2.2
y	4.0552	4.9530	6.0496	7.3891	9.0250

5. From the following table of values of X and Y, obtain first and second Derivatives for X=5

x	2	4	9	10
y	4	56	711	980

6. From the following table, find the area bounded by curve and x axis from X=7.47 to X=7.52 using trapezoidal, simpson's 1/3, simpson's 3/8 rule.

x	7.47	7.48	7.49	7.50	7.51	7.52
f(x)	1.93	1.95	1.98	2.01	2.03	2.06

6.10 ANSWER TO CHECK YOUR PROGRESS

(1) 4.75;9

(2) 0.63;6.6

(3) 2.8326

(4) 0.9985

(5) 7.78

6.11 REFERENCES/ SUGGESTEDREADINGS

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