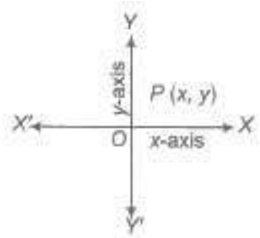
Coordinate Geometry

The branch of Mathematics in which geometrical problem are solved through algebra by using the coordinate system, is known as coordinate geometry.

Rectangular Axis

Let XOX’ and YOY’ be two fixed straight lines, which meet at right angles at O. Then,



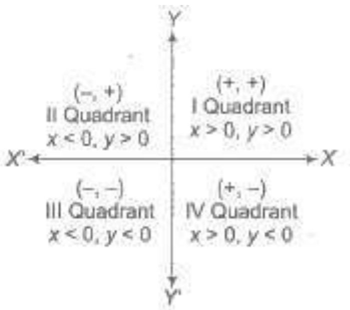
(i) X’OX is called axis of X or the X-axis or abscissa.

(ii) Y’OY is called axis of Yor the Y-axis or ordinate.

(iii) The ordered pair of real numbers (x, y) is called cartesian coordinate .

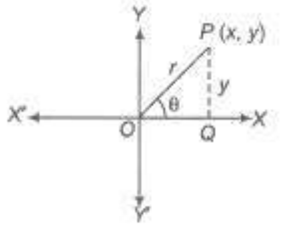
Quadrants

The X and Y-axes divide the coordinate plane into four parts, each part is called a quadrant which is given below.



Polar Coordinates

In ΔOPQ,



$\cos\theta = x / r$ and $\sin\theta = y / r \Rightarrow x = r\cos\theta$ and $y = r\sin\theta$

where, $r = \sqrt{x^2 + y^2}$

The polar coordinate is represented by the symbol $P(r,\theta)$.

Distance Formula

(i) Distance between two points P (x_1, y_1) and Q (x_2, y_2), is

$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

(ii) If points are (r_1, θ_1) arid (r_2, θ_2), then distance between them is

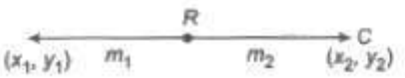
$\sqrt{r^2_1 + r^2_2 - 2r_1r_2\cos(\theta_1 - \theta_2)}$.

(iii) Distance of a point (x_1, y_1) from the origin is $\sqrt{x^2_1 + y^2_1}$.

Section Formula

(i) The coordinate of the point which divides the joint of (x_1, y_1) and (x_2, y_2) in the ratio $m_1 : m_2$ internally, is

$$\left(\frac{m_1x_2 + m_2x_1}{m_1 + m_2}, \frac{m_1y_2 + m_2y_1}{m_1 + m_2}\right)$$



and externally is $\left(\frac{m_1x_2 - m_2x_1}{m_1 - m_2}, \frac{m_1y_2 - m_2y_1}{m_1 - m_2}\right)$.

(ii) X-axis divides the line segment joining (x_1, y_1) and (x_2, y_2) in the ratio $-y_1 : y_2$.

Similarly, Y-axis divides the same line segment in the ratio $-x_1 : x_2$.

(iii) Mid-point of the joint of (x_1, y_1) and (x_2, y_2) is $(x_1 + x_2 / 2 , y_1 + y_2 / 2)$

(iv) Centroid of ΔABC with vertices (x_1, y_1), (x_2, y_2) and (x_3, y_3), is

$(x_1 + x_2 + x_3 / 3 , y_1 + y_2 + y_3 / 3)$.

(v) Circumcentre of ΔABC with vertices $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$, is

$$\left(\frac{x_1 \sin 2A + x_2 \sin 2B + x_3 \sin 2C}{\sin 2A + \sin 2B + \sin 2C}, \frac{y_1 \sin 2A + y_2 \sin 2B + y_3 \sin 2C}{\sin 2A + \sin 2B + \sin 2C}\right).$$

(vi) Incentre of Δ ABC with vertices $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ and whose sides are a, band c, is

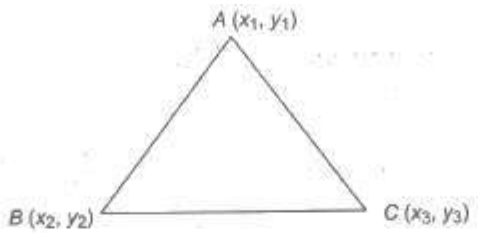
$(ax_1 + bx_2 + cx_3 / a + b + c , ay_1 + by_2 + cy_3 / a + b + c)$.

(vii) Excentre of ΔABC with vertices $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ and whose sides are a, band c, is given by

$$I_1 = \left(\frac{-ax_1 + bx_2 + cx_3}{-a + b + c}, \frac{-ay_1 + by_2 + cy_3}{-a + b + c}\right),$$
$$I_2 = \left(\frac{ax_1 - bx_2 + cx_3}{a - b + c}, \frac{ay_1 - by_2 + cy_3}{a - b + c}\right)$$
$$I_3 = \left(\frac{ax_1 + bx_2 - cx_3}{a + b - c}, \frac{ay_1 + by_2 - cy_3}{a + b - c}\right)$$

Area of Triangle/Quadrilateral

(i) Area of ΔABC with vertices $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$, is



$$\Delta = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \frac{1}{2}\begin{vmatrix} x_1 - x_3 & x_2 - x_3 \\ y_1 - y_3 & y_2 - y_3 \end{vmatrix}$$

These points A, Band C will be collinear, if Δ = O.

(ii) Area of the quadrilateral formed by joining the vertices

$$(x_1, y_1), (x_2, y_2), (x_3, y_3) \text{ and } (x_4, y_4) \text{ is } \frac{1}{2}\begin{vmatrix} x_1 - x_3 & x_2 - x_4 \\ y_1 - y_3 & y_2 - y_4 \end{vmatrix}.$$

(iii) Area of trapezium formed by joining the vertices

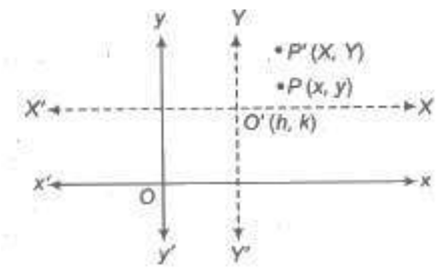
$(x_1, y_1), (x_2, y_2), (x_3, y_3)$ and (x_4, y_4) is

$$\frac{1}{2}[(y_2 + y_1)(x_1 - x_2) + (y_1 + y_3)(x_3 - x_1) + (y_2 + y_3)(x_3 - x_2)]$$

Shifting/Rotation of Origin/Axes

Shifting of Origin

Let the origin is shifted to a point O'(h, k). If P(x, y) are coordinates of a point referred to old axes and P' (X, Y) are the coordinates of the same points referred to new axes, then

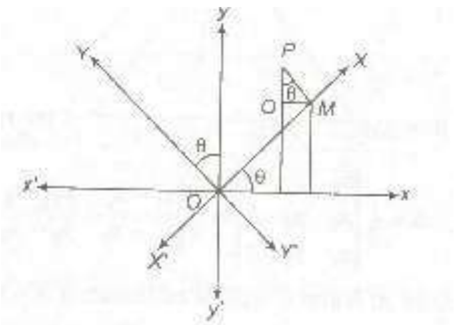


$x = X + h, y = Y + k$

Rotation of Axes

Let (x, y) be the coordinates of any point P referred to the old axes and (X, Y) be its coordinates referred to the new axes (after rotating the old axes by angle θ). Then,

X = x cos θ + y sin θ and Y = y cos θ + x sin θ



Shifting of Origin and Rotation of Axes

If origin is shifted to point (h, k) and system is also rotated by an angle θ in anti-clockwise, then coordinate of new point P' (x', y') is obtained by replacing

x'= h + x cos θ + y sin θ

and y' = k – x sin θ + y cos θ

Locus

The curve described by a point which moves under given condition(s) is called its locus.

Equation of Locus

The equation of the locus of a point which is satisfied by the coordinates of every point.

Algorithm to Find the Locus of a Point

Step I Assume the coordinates of the point say (h,k) whose locus is to be found.

Step II Write the given condition in mathematical form involving h, k.

Step III Eliminate the variable(s), if any.

Step IV Replace h by x and k by y in the result obtained in step III. The equation so obtained is the locus of the point, which moves under some stated condition(s).

Straight Line

Any curve is said to be a straight line, if two points are taken on the curve such that every point on the line segment joining any two points on it lies on the curve.

General equation of a line is ax + by + c = o.

Slope (Gradient) of a Line)

The trigonometric tangent of the angle that a line makes with the positive direction of the X-axis in anti-clockwise sense is called the slope or gradient of the line.

So, slope of a line, $m = \tan \theta$

where, θ is the angle made by the line with positive direction of X-axis.

Important Results on Slope of Line

(i) Slope of a line parallel to X-axis, m = 0.
(ii) Slope of a line parallel to Y-axis, $m = \infty$.
(iii) Slope of a line equally inclined with axes is 1 or -1 as it makes an angle of 45° or 135°, with X-axis.
(iV) Slope of a line passing through (x, y,) and (x_2, y_2) is given by

$m = \tan \theta = y_2 - y_1 / x_2 - x_1$.

Angle between Two Lines

The angle e between two lines having slopes m_1 and m_2 is

$$\tan\theta = \left|\frac{m_2 - m_1}{1 + m_1 m_2}\right|.$$

(i) Two lines are parallel, iff $m_1 = m_2$.
(ii) Two lines are perpendicular to each other, iff $m_1 m_2 = -1$.

Equation of a Straight Line

General equation of a straight line is $Ax + By + C = 0$.

(i) The equation of a line parallel to X-axis at a distance b from it, is given by

$y = b$

(ii) The equation of a line parallel to Y-axis at a distance a from it, is given by

$x = a$

(iii) Equation of X-axis is

$y = 0$

(iv) Equation of Y-axis is

$x = 0$

Different Form of the Equation of a Straight Line

(i) **Slope Intercept Form** The equation of a line with slope m and making an intercept c on Y-axis, is

$y = mx + c$

If the line passes through the origin, then its equation will be

$y= mx$

(ii) **One Point Slope Form** The equation of a line which passes through the point (x_1, y_1) and has the slope of m is given by

$(y - y_1) = m (x - x_1)$

(iii) **Two Points Form** The equation of a line' passing through the points (x_1, y_1) and (x_2, y_2) is given by

$(y - y_1) = (y_2 - y_1 / x_2 - x_1) (x - x_1)$

This equation can also be determined by the determinant method, that is

$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$$

(iv) **The Intercept Form** The equation of a line which cuts off intercept a and b respectively on the X and Y-axes is given by

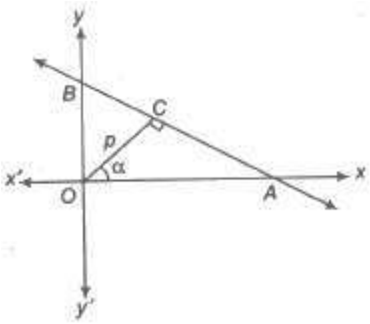
$x / a + y / b = 1$

The general equation $Ax + By + C = 0$ can be converted into the intercept form, as

$x / -(C A) + y / -(C B) = 1$

(v) **The Normal Form** The equation of a straight line upon which the length of the perpendicular from the origin is p and angle made by this perpendicular to the X-axis is α, is given by

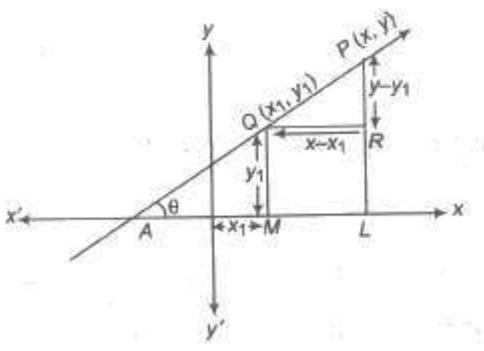
$x \cos \alpha + Y \sin \alpha = p$



(vi) T**he Distance** (Parametric) **Form** The equation of a straight line passing through (x_1, y_1) and making an angle θ with the positive direction of x-axis, is

$x - x_1 / \cos \theta = y - y_1 / \sin \theta = r$

where, r is the distance between two points P(x, y) and $Q(x_1, y_1)$.



Thus, the coordinates of any point on the line at a distance r from the given point (x_1, y_1) are $(x_1 + r\cos\theta, y_1 + r\sin\theta)$. If P is on the right side of (x_1, y_1) then r is positive and if P is on the left side of (x_1, y_1) then r is negative.

Position of Point(s) Relative to a Given Line

Let the equation of the given line be $ax + by + C = 0$ and let the Coordinates of the two given points be $P(x_1, y_1)$ and $Q(x_2, y_2)$.

(i) The two points are on the same side of the straight line $ax + by + c = 0$, if $ax_1 + by_1 + c$ and $ax_2 + by_2 + c$ have the same sign.

(ii) The two points are on the opposite side of the straight line $ax + by + c = 0$, if $ax_1 + by_1 + c$ and $ax_2 + by_2 + c$ have opposite sign.

(iii) A point (x_1, y_1) will lie on the side of the origin relative to a line $ax + by + c = 0$, if $ax_1 + by_1 + c$ and c have the same sign.

(iv) A point (x_1, y_1) will lie on the opposite side of the origin relative to a line $ax + by + c = 0$, if $ax_1 + by_1 + c$ and c have the opposite sign.

(v) Condition of concurrency for three given lines $ax_1 + by_1 + c_1 = 0$, $ax_2 + by_2 + c_2$ and $ax_3 + by_3 + c_3 = 0$ is $a_3(b_1c_2 – b_2c_1) + b_3(c_1a_2 – a_1c_2) + c_3(a_1b_2 – a_2b_1) = 0$

or $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$

(vi) **Point of Intersection of Two Lines** Let equation of lines be $ax_1 + by_1 + c_1 = 0$ and $ax_2 + by_2 + c_2 = 0$, then their point of intersection is

$(b_1c_2 – b_2c_1 / a_1b_2 – a_2b_1, c_1a_2 – c_2a_1 / a_1b_2 – a_2b_1)$.

Line Parallel and Perpendicular to a Given Line

(i) The equation of a line parallel to a given line $ax + by + c = 0$ is $ax + by + \lambda = 0$, where λ is a constant.

(ii) The equation of a line perpendicular to a given line $ax + by + c =$ is $bx – ay + \lambda = 0$, where λ is a constant.

Image of a Point with Respect to a Line

Let the image of a point (x_1, y_1) with respect to $ax + by + c = 0$ be (x_2, y_2), then

$x_2 – x_1 / a = y_2 – y_1 / b = – 2 (ax_1 + by_1 + c) / a^2 + b^2$

(i) The image of the point $P(x_1, y_1)$ with respect to X-axis is $Q(x_1 – y_1)$.

(ii) The image of the point $P(x_1, y_1)$ with respect to Y-axis is $Q(-x_1, y_1)$.

(iii) The image of the point $P(x_1, y_1)$ with respect to mirror $Y = x$ is $Q(y_1, x_1)$.

(iv) The image. of the point $P(x_1, y_1)$ with respect to the line mirror y == x tan θ is

$x = x_1 \cos 2\theta + y_1 \sin 2\theta$
$Y = x_1 \sin 2\theta – y_1 \cos 2\theta$

(v) The image of the point $P(x_1, y_1)$ with respect to the origin is the point $(-x_1, y_1)$.

(vi) The length of perpendicular from a point (x_1, y_1) to a line $ax + by + c = 0$ is

$$\left|\frac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}}\right|$$

Equation of the Bisectors

The equation of the bisectors of the angle between the lines

$a_1x + b_1y + c_1 = 0$

and $a_2x + b_2y + c_2 = 0$

are given by

$a_1x + b_1y + c_1 / \sqrt{a^2_1 + b^2_1} =$ &plusmn $a_2x + b_2y + c_2 / \sqrt{a^2_2 + b^2_2}$

(i) If $a_1 a_2 + b_1 b_2 > 0$, then we take positive sign for obtuse and negative sign for acute.

(ii) If $a_1 a_2 + b_1 b_2 < 0$, then we take negative sign for obtuse and positive sign for acute .

Pair of Lines

General equation of a pair of straight lines is $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$.

Homogeneous Equation of Second Degree

A rational, integral, algebraic equation in two variables x and y is said to be a homogeneous equation of the second degree, if the sum of the indices of x and y in each term is equal to 2.

The general form of homogeneous equation of the second degree x and y is $ax^2 + 2hxy + by^2 = 0$, which passes through the origin.

Important Properties

If $ax^2 + 2hxy + by^2 = 0$ be an equation of pair of straight lines.

(i) Slope of first line, $m_1 = – h + \sqrt{h^2} – ab / b$

and slope of second line, $m_2 = – h – \sqrt{h^2} – ab / b$

$m_1 + m_2 = – 2h / b = –$ Coefficient of xy / Coefficient of y^2

and $m_1 m_2 = a / b =$ Coefficient of x^2 / Coefficient of y^2

Here, m_1 and m_2 are

(a) real and distinct, if $h^2 > ab$.

(b) coincident, if $h^2 = ab$.

(c) imaginary, if $h^2 < ab$.

(ii) Angle between the pair of lines is given by

$\tan \theta = {}^2\sqrt{h^2} – ab / a + b$

(a) If lines are coincident, then $h^2 = ab$
(b) If lines are perpendicular, then $a + b = o$.

(iii) The joint equation of bisector of the angles between the lines represented by the equation $ax^2 + 2hxy + by^2 = 0$ is

$x^2 – b^2 / a – b = xy / h \Rightarrow hx^2 – (a – b)xy – hy^2=0.$

(iv) The necessary and sufficient condition $ax^2 + 2hxy + by^2 + 2gx + 2fy + C = 0$ to represent a pair of straight lines, if $abc + 2fgh – af^2 – bg^2 – ch^2 = 0$

or $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$

(v) The equation of the bisectors of the angles between the lines represented by $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ are given by

$(x – x^1)^2 – (y – y^1)^2 / a – b = (x – x^1) (y – y^1) / h,$

where, (x^1, y^1) is the point of intersection of the lines represented by the given equation.

(vi) The general equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + C = 0$ will represent two parallel lines, if $g^2 – ac > 0$ and $a / h = h / b = g / f$ and the distance between them is $2\sqrt{g^2 – ac / a(a + b)}$ or $2\sqrt{f^2 – bc / b(a + b)}$.

(vii) If the equation of a pair of straight lines is $ax^2 + 2hxy + by^2 + 2gx + 2fy + C = 0$, then the point of intersection is given by

$(hf – bg / ab – h^2, gh – af / ab – h^2)$.

(viii) The equation of the pair of lines through the origin and perpendicular to the pair of lines given by $ax^2 + 2hxy + by^2 = 0$ is $bx^2 – 2hxy + ay^2 = 0$.

(ix) Equation of the straight lines having the origin to the points of intersection of a second degree curve $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ and a straight line $Lx + my + n = 0$ is

$ax^2 + 2hxy + by^2 + 2gx(Lx + my / – n) + 2fy(Lx + my / – n) + c (Lx + my / – n)^2 = 0$.

Important Points to be Remembered

1. A triangle is an isosceles, if any two of its median are equal.

2. In an equilateral triangle, orthocentre, centroid, circumcentre, incentre coincide.

3. The circumcentre of a right angled triangle is the mid-point of the hypotenuse.

4. Orthocentre, centroid, circumcentre of a triangle are collinear, Centroid divides the line joining orthocentre and circumcentre in the ratio 2: 1.

5. If D, E and F are the mid-point of the sides BC, CA and AB of MBC, then the centroid of Δ ABC = centroid of Δ DEF.

6. Orthocentre of the right angled Δ ABC, right angled at A is A

7. Circumcentre of the right angled Δ ABC, right angled at A is B + C / 2.

8. The distance of a point (x^1, y^1) from the $ax + by + c = 0$ is

$d = |ax_1 + by_1 + c / \sqrt{a^2 + b^2}|$

9. Distance between two parallel lines $a_1x + b_1y + c_1 = 0$ and $a_1x + b_1y + c_2 = 0$ is given by

$d = |c_2 – c_1 / \sqrt{a^2 + b^2}|$.

10. The area of the triangle formed by the lines $y =m_1x + c_1$, $y = m_2x + c_2$ and $y = m_3x + c_3$ is .

$$\Delta = \frac{1}{2}\left|\sum \frac{(c_1 - c_2)^2}{m_1 - m_2}\right|.$$

11. Area of the triangle formed by the line ax + by + c = 0 with the coordinate axes is $\Delta = c^2 / 2|ab|$.

12. The foot of the perpendicular(h,k) from (x_1, y_1) to the line ax + by + c = 0 is given by $h - x_1 / a = k - y_1 / b = -(ax_1 + by_1 + c) / a^2 + b^2$.

13. Area of rhombus formed by ax ± by ± c = 0 is $|2c^2 / ab|$.

14. Area of the parallelogram formed by the lines

$a_1x + b_1y + c_1 = 0$, $a_2x + b_2y + c_2 = 0$, $a_1x + b_1y + d_1 = 0$ and $a_2x + b_2y + d_2 = 0$ is

$|(d_1 - c_1)(d_2 - c_2 / a_1b_2 - a_2b_1|$.

15. (a) Foot of the perpendicular from (a, b) on x – y = 0 is

(a + b / 2, a + b / 2).

(b) Foot of the perpendicular from (a,b) on x + y = 0 is

(a – b / 2, a – b / 2).

16. The image of the line $a_1x + b_1y + c_1 = 0$ about the line ax + by + c = 0 is .

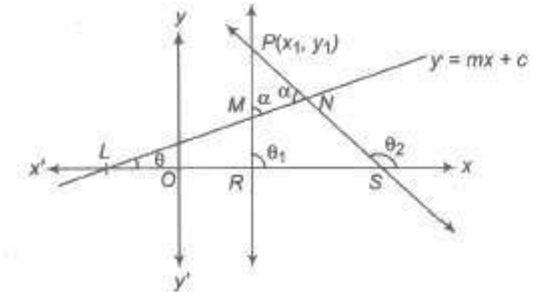
$2(aa_1 + bb_1)(ax + by + c) = (a^2 + b^2)(a_1x + b_1y + c_1)$.

17. Given two vertices (x_1, y_1) and (x_2, y_2) of an equilateral MBC, then its third vertex is given by.

$[x_1 + x_2 \pm \sqrt{3}(y_1 - y_2) / 2, y_1 + y_2 \mp \sqrt{3}(x_1 - x_2) / 2]$

18. The equation of the straight line which passes through a given point (x_1, y_1) and makes an angle α with the given straight line y = mx + c are

$$(y - y_1) = \frac{m \pm \tan\alpha}{1 \mp m\tan\alpha}(x - x_1)$$



19. The equation of the family of lines passing through the intersection of the lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ is

$(a_1x + b_1y + c_1) + l(a_2x + b_2y + c_2) = 0$

where, λ is any real number.

20. Line $ax + by + c = 0$ divides the line joining the points (x_1, y_1) and (x_2, y_2) in the ratio $\lambda : 1$, then $\lambda = -(a_1x + b_1y + c / a_2x + b_2y + c)$.

If λ is positive it divides internally and if λ. is negative, then it divides externally.

21. Area of a polygon of n-sides with vertices $A_1(x_1, y_1)$, $A_2 (x_2, y_2)$,… ,$A_n(x_n, y_n)$

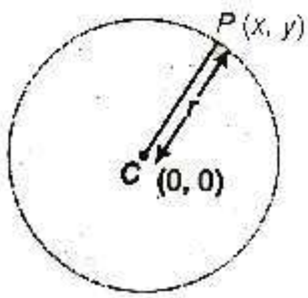
$$= \frac{1}{2}\left[\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} + \begin{vmatrix} x_2 & y_2 \\ x_3 & y_3 \end{vmatrix} + \ldots + \begin{vmatrix} x_n & y_n \\ x_1 & y_1 \end{vmatrix}\right]$$

22. Equation of the pair of lines through (α, β) and perpendicular to the pair of lines $ax^2 + 2hxy + by^2 = 0$ is $b (x - \alpha\}^2 - 2h (x - \alpha)(y - \beta) + a (y - \beta)^2 = 0$.

Maths Class 11 Chapter 11 Conic section part-1 Circles

Circles

Circle is defined as the locus of a point which moves in a plane such that its distance from a fixed point in that plane is constant.

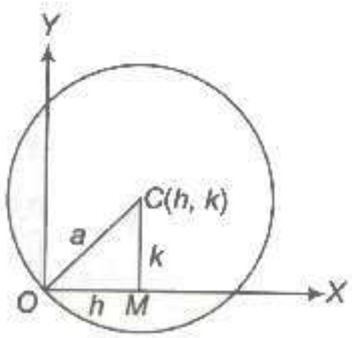


Standard Forms of a Circle

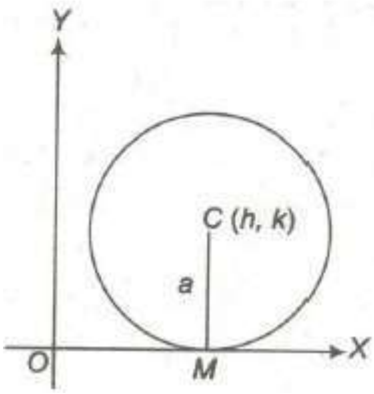
(i) Equation of circle having centre (h, k) and radius $(x — h)^2 + (y — k)^2 = a^2$.

If centre is (0, 0), then equation of circle is $x^2 + y^2 = a^2$.

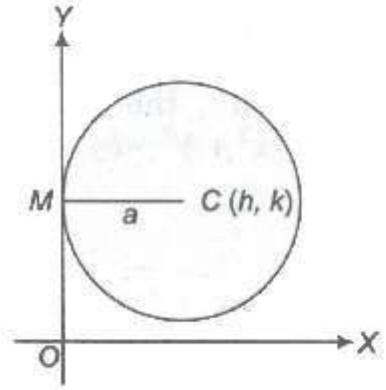
(ii) When the circle passes through the origin, then equation of the circle is $x^2 + y^2 — 2hx — 2ky = 0$.



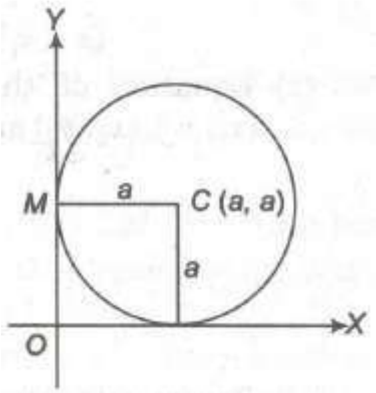
(iii) When the circle touches the X-axis, the equation is $x^2 + y^2 — 2hx — 2ay + h^2 = O$.



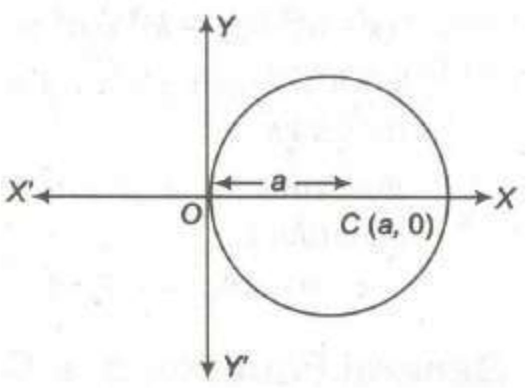
(iv) Equation of the circle, touching the Y-axis is $x^2 + y^2 — 2ax — 2ky + k^2 = 0$.



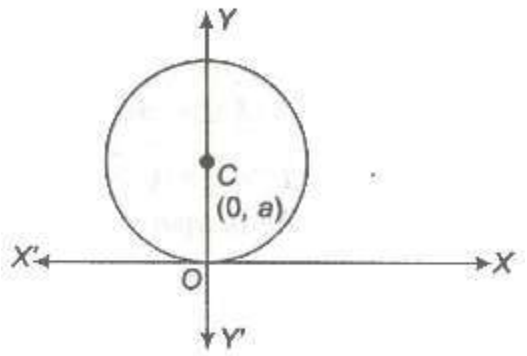
(v) Equation of the circle, touching both axes is $x^2 + y^2 — 2ax — 2ay + a^2 = O$.



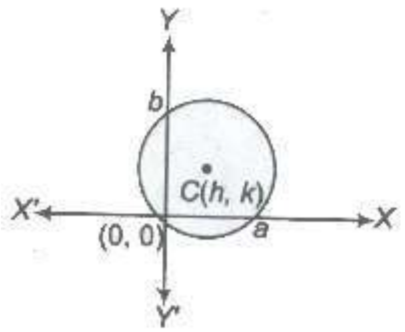
(vi) Equation of the circle passing through the origin and centre lying on the X-axis is $x^2 + y^2 — 2ax = O$.



(vii) Equation of the circle passing through the origin and centre lying on the Y-axis is $x^2 + y^2 – 2ay = 0$.



(viii) Equation of the circle through the origin and cutting intercepts a and b on the coordinate axes is $x^2 + y^2 — by = 0$.



(ix) Equation of the circle, when the coordinates of end points of a diameter are (x_1, y_1) and (x_2, y_2) is

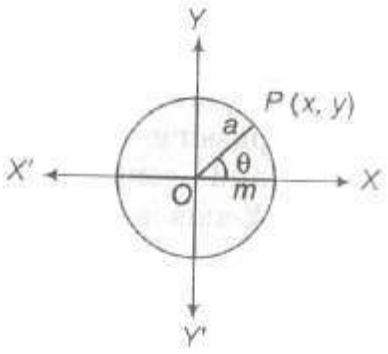
$(x — x_1)(x — x_2) + (y – y_1)(y — y_2) = 0$.

(x) Equation of the circle passes through three given points (x_1, y_1), (x_2, y_2) and (x_3, y_3) is

$$\begin{vmatrix} x^2+y^2 & x & y & 1 \\ x_1^2+y_1^2 & x_1 & y_1 & 1 \\ x_2^2+y_2^2 & x_2 & y_2 & 1 \\ x_3^2+y_3^2 & x_3 & y_3 & 1 \end{vmatrix} = 0.$$

(xi) Parametric equation of a circle

$(x – h)^2 + (y – k)^2 = a^2$ is
$x = h + a \cos\theta$, $y = k + a \sin\theta$,
$0 \le \theta \le 2\pi$
For circle $x^2 + y^2 = a^2$, parametric equation is
$x = a \cos \theta$, $y = a \sin \theta$



General Equation of a Circle

The general equation of a circle is given by $x^2 + y^2 + 2gx + 2fy + c = 0$, where centre of the circle = $(- g, – f)$

Radius of the circle = $\sqrt{g^2 + f^2 – c}$

1. If $g^2 + f^2 – c > 0$, then the radius of the circle is real and hence the circle is also real.
2. If $g^2 + f^2 – c = 0$, then the radius of the circle is 0 and the circle is known as point circle.
3. If $g^2 + f^2 – c < 0$, then the radius of the circle is imaginary. Such a circle is imaginary, which is not possible to draw.

Position of a Point with Respect to a Circle

A point (x_1, y_1) lies outside on or inside a circle

$S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$, according as $S_1 >$, = or < 0
where, $S_1 = x_1^2 + y_1^2 + 2gx_1, + 2fy_1 + c$

Intercepts on the Axes

The length of the intercepts made by the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ with X and Y-axes are

$2\sqrt{g^2 – c}$ and $2\sqrt{g^2 – c}$.

1. If $g^2 > c$, then the roots of the equation $x^2 + 2gx + c = 0$ are real and distinct, so the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ meets the X-axis in two real and distinct points.
2. If $g^2 = c$, then the roots of the equation $x^2 + 2gx + c = 0$ are real and equal, so the circle touches X-axis, then intercept on X-axis is O.
3. If $g^2 < c$, then the roots of the equation $x^2 + 2gx + c = 0$ are imaginary, so the given circle does not meet X-axis in real point. Similarly, the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ cuts the Y-axis in real and distinct points touches or does not meet in real point according to $f^2 >, = \text{or} < c$

Equation of Tangent

A line which touch only one point of a circle.

1. Point Form

1. The equation of the tangent at the point $P(x_1, y_1)$ to a circle $x^2 + y^2\ 2gx + 2fy + c= 0$ is $xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$
2. The equation of the tangent at the point $P(x_1, y_1)$ to a circle $x^2 + y^2$ is $xx_1 + yy_1 = r^2$

2. Slope Form

(i) The equation of the tangent of slope m to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$

are $y + f = m(x + g) \pm \sqrt{(g^2 + f^2 - c)(1 + m^2)}$

(ii) The equation of the tangents of slope m to the circle $(x – a)^2\ (y – b)^2 = r^2$ are $y – b = m(x – a) \pm r\sqrt{(1 + m^2)}$ and the coordinates of the points of contact are

$$\left(a \pm \frac{mr}{\sqrt{1+m^2}}, b \mp \frac{r}{\sqrt{1+m^2}}\right).$$

(iii) The equation of tangents of slope m to the circle $x^2 + y^2 = r^2$ are $y = mx \pm r\sqrt{(1 + m^2)}$ and the coordinates of the point of contact are

$$\left(\pm \frac{rm}{\sqrt{1+m^2}}, \mp \frac{r}{\sqrt{1+m^2}}\right).$$

3. Parametric Form

The equation of the tangent to the circle $(x – a)^2 + (y – b)^2 = r^2$ at the point $(a + r \cos \theta, b + r \sin\theta)$ is $(x – a) \cos \theta + (y – b) \sin \theta = r$.

Equation of Normal

A line which is perpendicular to the tangent.

1. Point Form

1. (i) The equation of normal at the point (x_1, y_1) to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is $y – y_1 = [(y_1 + f)(x – x_1)]/(x_1 + g)$
   $(y_1 + f)x – (x_1 + g)y + (gy_1 – fx_1) = 0$
2. (ii) The equation of normal at the point (x_1, y_1) to the circle $x^2 + y^2 = r^2$ is $x/x_1 = y/y_1$

2. Parametric Form

The equation of normal to the circle $x^2 + y^2 = r^2$ at the point ($r\cos\theta$, $r\sin\theta$) is

$(x/r\cos\theta) = (y/r\sin\theta)$

or $y = x\tan\theta$.

Important Points to be Remembered

(i) The line $y = mx + c$ meets the circle in unique real point or touch the circle

$x_2 + y_2 + r_2$, if $r = |c/\sqrt{1} + m_2$

and the point of contacts are $\left(\frac{\pm mr}{\sqrt{1+m^2}}, \frac{\mp r}{\sqrt{1+m^2}}\right)$.

(ii) The line $lx + my + n = 0$ touches the circle $x_2 + y_2 = r_2$, if $r_2(l_2 + m_2) = n_2$.

(iii) Tangent at the point P (θ) to the circle $x_2 + y_2 = r_2$ is $x\cos\theta + y\sin\theta = r$.

(iv) The point of intersection of the tangent at the points $P(\theta_1)$ and $Q(\theta_2)$ on the circle $x^2 + y^2 = r^2$

$$x = \frac{r\cos\left(\frac{\theta_1+\theta_2}{2}\right)}{\cos\left(\frac{\theta_1-\theta_2}{2}\right)} \text{ and } y = \frac{r\sin\left(\frac{\theta_1+\theta_2}{2}\right)}{\cos\left(\frac{\theta_1-\theta_2}{2}\right)}.$$

(v) Normal at any point on the circle is a straight line which is perpendicular to the tangent to the curve at the point and it passes through the centre of circle.

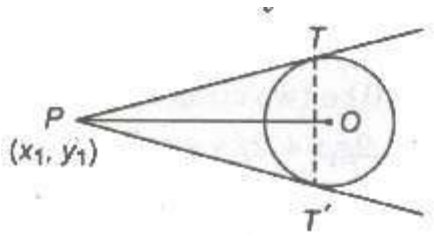
(vi) Power of a point (x_1, y_1) with respect to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is $x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$.

(vii) If P is a point and C is the centre of a circle of radius r, then the maximum and minimum distances of P from the circle are CP + r and CP — r , respectively.

(viii) If a line is perpendicular to the radius of a circle at its end points on the circle, then the line is a tangent to the circle and vice-versa.

Pair of Tangents

(i) The combined equation of the pair of tangents drawn from a point $P(x_1, y_1)$ to the circle $x^2 + y^2 = r^2$ is



$(x^2 + y^2 – r^2)(x_1^2+ y_1^2 – r_1^2) = (xx_1 + yy_1 – r^2)^2$
or $SS_1 = T^2$
where, $S = x^2 + y^2 – r^2$, $S_1 = x_1^2+ y_1^2 – r_1^2$
and $T = xx_1 + yy_1 – r^2$

(ii) The length of the tangents from the point $P(x_1, y_1)$ to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is equal to

$$\sqrt{x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c} = \sqrt{S_1}$$

(iii) Chord of contact TT' of two tangents, drawn from $P(x_1, y_1)$ to the circle $x^2 + y^2 = r^2$ or $T = 0$.

Similarly, for the circle

$x^2 + y^2 + 2gx + 2fy + c = 0$ is

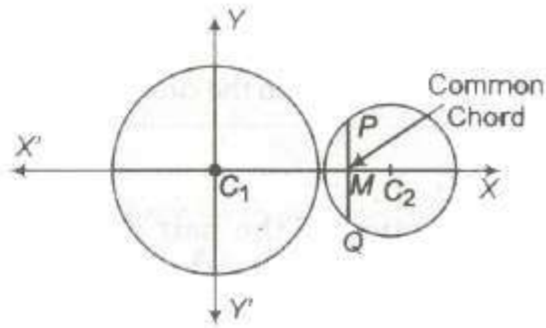
$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$

(iv) Equation of Chord Bisected at a Given Point The equation of chord of the circle $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ bisected at the point (x_1, y_1) is give by $T = S_1$.
i.e., $xx_1 + yy_1 + g (x + x_1) + f (y + y_1) + c$
$= x_1^2 + y_1^2 + 2gx_1 + fy_1 + c$

(v) Director Circle The locus of the point of intersection of two perpendicular tangents to a given circle is called a director circle. For circle $x^2 + y^2 = r^2$, the equation of director circle is $x^2 + y^2 = 2r^2$.

Common Chord

The chord joining the points of intersection of two given circles is called common chord.



(i) If $S_1 = 0$ and $S_1 = 0$ be two circles, such that

$S_1 \equiv x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$
and $S_2 \equiv x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0$
then their common chord is given by $S_1 — S_2 = 0$

(ii) If C_1, C_2 denote the centre of the given circles, then their common chord

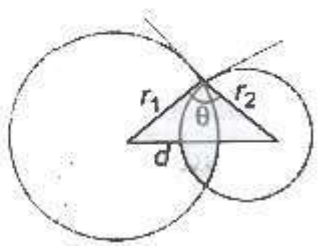
$PQ = 2\ PM = 2\sqrt{(C_1P)^2 – C_1M)^2}$

(iii) If r_1, and r_2 be the radii of ‘two circles, then length of common chord is
$\frac{2r_1r_2}{\sqrt{r_1^2 + r_2^2}}$.

Angle of Intersection of Two Circles

The angle of intersection of two circles is defined as the angle between the tangents to the two circles at their point of intersection is given by

$\cos \theta = (r_1^2 + r_2^2 – d^2)/(2r_1r_2)$



Orthogonal Circles

Two circles are said to be intersect orthogonally, if their angle of intersection is a right angle.

If two circles

$S_1 \equiv x^2 + y^2 + 2g_1x + 2f_1y + C_1 = 0$ and

$S_2 \equiv x^2 + y^2 + 2g_2x + 2f_2y + C_2 = 0$ are orthogonal, then $2g_1g_2 + 2f_1f_2 = c_1 + c_2$

Family of Circles

(i) The equation of a family of circles passing through the intersection of a circle $x^2 + y^2 + 2gx + 2fy + c = 0$ and line

$L = lx + my + n = 0$ is $S + \lambda L = 0$

where, X, is any real number.

(ii) The equation of the family of circles passing through the point $A(x_1, y_1)$ and $B(x_1, y_1)$ is

$$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) + \lambda \begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$$

(iii) The equation of the family of -circles touching the circle

$S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ at point $P(x_1, Y_1)$ is

$xx^2 + y^2 + 2gx + 2fy + c + \lambda, [xx_1 + yy_1 + g(x + x_1) + f(Y+ Y_1) + c] = 0$ or $S + \lambda L = 0$, where $L = 0$ is the equation of the tangent to

$S = 0$ at (x_1, y_1) and $X \in R$

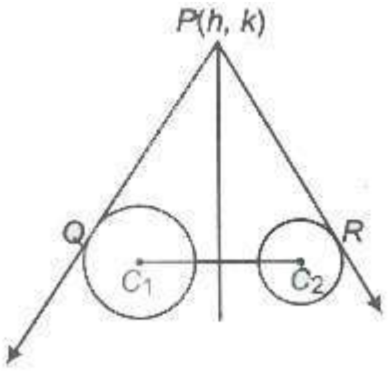
(iv) Any circle passing through the point of intersection of two circles S_1 and S_2 is $S_1 +\lambda(S_1 — S_2) = 0$.

Radical Axis

The radical axis of two circles is the locus of a point which moves in such a way that the length of the tangents drawn from it to the two circles are equal.

A system of circles in which every pair has the same radical axis is called a coaxial system of circles.

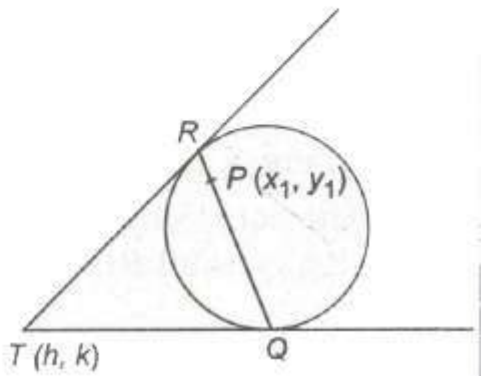
The radical axis of two circles $S_1 = 0$ and $S_2 = 0$ is given by $S_1 — S_2 = 0$.



1. The radical axis of two circles is always perpendicular to the line joining the centres of the circles.
2. The radical axis of three vertices, whose centres are non-collinear taken in pairs of concurrent.
3. The centre of the circle cutting two given circles orthogonally, lies on their radical axis.
4. Radical Centre The point of intersection of radical axis of three circles whose centre are non-collinear, taken in pairs, is called their radical centre.

Pole and Polar

If through a point P (x_1, y_1) (within or outside a circle) there be drawn any straight line to meet the given circle at Q and R, the locus of the point of intersection of tangents at Q and R is called the polar of P and po.:.at P is called the pole of polar.



1. Equation of polar to the circle $x^2 + y^2 = r^2$ is $xx_1 + yy_1 = r^2$.
2. Equation of polar to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is $xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$
3. Conjugate Points Two points A and B are conjugate points with respect to a given circle, if each lies on the polar of the other with respect to the circle.
4. Conjugate Lines If two lines be such that the pole of one lies on the other, then they are called conjugate lines with respect to the given circle.

Coaxial System of Circles

A system of circle is said to be coaxial system of circles, if every pair of the circles in the system has same radical axis.

1. The equation of a system of coaxial circles, when the equation of the radical axis $P \equiv lx + my + n = 0$ and one of the circle of the system $S = x^2 + y^2 + 2gx + 2fy + c = 0$, is $S + \lambda P = 0$.
2. Since, the lines joining the centres of two circles is perpendicular to their radical axis. Therefore, the centres of all circles of a coaxial system lie on a straight line, which is perpendicular to the common radical axis.

Limiting Points

Limiting points of a system of coaxial circles are the centres of the point circles belonging to the family.

Let equation of circle be $x^2 + y^2 + 2gx + c = 0$

∴ Radius of circle = $\sqrt{g^2 — c}$

For limiting point, $r = 0$

∴ $\sqrt{g^2 — c} = 0$ ⇒$g = \pm \sqrt{c}$

Thus, limiting points of the given coaxial system as $(\sqrt{c}, 0)$ and $(—\sqrt{c}, 0)$.

Important Points to be Remembered

(i) Circle touching a line L=O at a point (x_1, y_1) on it is

$(x — x_1)^2 + (y — y_1)^2 + XL = 0$.

(ii) Circumcircle of a A with vertices (x_1, y_1), (x_2, y_2), (x_3, y_3) is

$$\frac{(x-x_1)(x-x_2)+(y-y_1)(y-y_2)}{(x_3-x_1)(x_3-x_2)+(y_3-y_1)(y_3-y_2)} \text{ or } \frac{\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix}}{\begin{vmatrix} x_3 & y_3 & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix}}$$

(iii) A line intersect a given circle at two distinct real points, if the length of the perpendicular from the centre is less than the radius of the circle.

(iv) Length of the intercept cut off from the line $y = mx + c$ by the circle $x^2 + y^2 = a^2$ is

$$2\sqrt{\frac{a^2(1+m^2)-c^2}{1+m^2}}$$

(v) In general, two tangents can be drawn to a circle from a given point in its plane. If m_1 and m_2 are slope of the tangents drawn from the point $P(x_1, y_1)$ to the circle $x^2 + y^2 = a^2$, then

$$m_1 + m_2 = \frac{2x_1y_1}{x_1^2 - a^2} \quad \text{and} \quad m_1 \times m_2 = \frac{y_1^2 - a^2}{x_1^2 - a^2}$$

(vi) Pole of $lx + my + n = 0$ with respect to $x^2 + y^2 = a^2$ is $\left(-\frac{a^2 l}{n}, -\frac{a^2 m}{n}\right)$.

(vii) Let $S_1 = 0$, $S_2 = 0$ be two circles with radii r_1 , r_2, then $S_1/r_1 \pm S_2/r_2 = 0$ will meet at right angle.

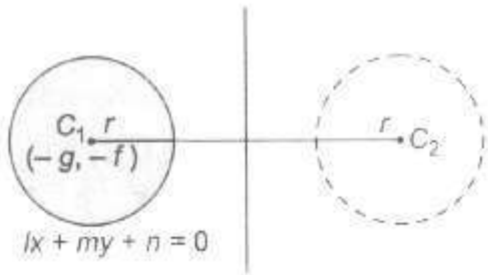
(viii) The angle between the two tangents from (x_1, y_1) to the circle $x^2 + y^2 = a^2$ is $2 \tan^{-1} (a/\sqrt{S_1})$.

(ix) The pair of tangents from (0, 0) to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ are at right angle, if $g^2 + f^2 = 2c$.

(x) If (x_1, y_1) is one end of a diameter of the circle $x^2 + y^2 + 2gx + 2fy + c = 0$, then the other end will be $(-2g - x_1, -2f - y_1)$.

Image of the Circle by the Line Minor

Let the circle be $x^2 + y^2 + 2gx + 2fy + c = 0$



and line minor $lx + my + n = 0$.

Then, the image of the circle is

$(x — X_1)^2 + (y — y_1)^2 = r^2$

where, $r = \sqrt{g^2 + f^2 — c}$

Diameter of a Circle

The locus of the middle points of a system of parallel chords of a circle is called a diameter of the circle.

(i) The equation of the diameter bisecting parallel chords $y = mx + c$ of the circle $x^2 + y^2 = a^2$ is $x + my = 0$.

(ii) The diameter corresponding to a system of parallel chords of a circle always passes through the centre of the circle and is perpendicular to the parallel chords.

Common Tangents of Two Circles

Let the centres and radii of two circles are C_1, C_2 and r_1, r_2, respectively.

1. (i) When one circle contains another circle, no common tangent is possible.
   Condition, $C_1C_2 < r_1 – r_2$
2. (ii) When two circles touch internally, one common tangent is possible.
   Condition , $C_1C_2 = r_1 – r_2$
3. (iii) When two circles intersect, two common tangents are possible.
   Condition, $|r_1 — r_2| < C_1C_2 < |r_1 + r_2|$
4. (iv) When two circles touch externally, three common tangents are possible.
   Condition, $C_1C_2 = r_1 + r_2$
5. (v) When two circles are separately, four common tangents are possible.
   Condition, $C_1C_2 > r_1 + r_2$

Important Points to be Remembered

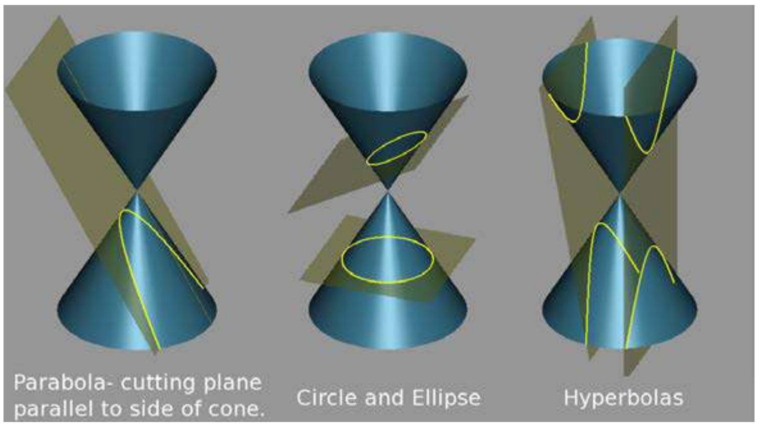
Let AS is a chord of contact of tangents from C to the circle $x^2 + y^2 = r^2$. M is the mid-point of AB.

14. Mathematics for Orbits: Ellipses, Parabolas, Hyperbolas

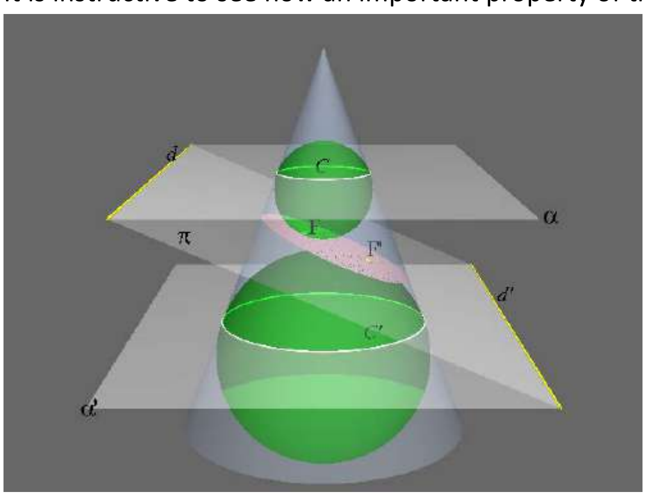
Michael Fowler

Preliminaries: Conic Sections

Ellipses, parabolas and hyperbolas can all be generated by cutting a cone with a plane (see diagrams, from Wikimedia Commons). Taking the cone to be $x^2 + y^2 = z^2$, and substituting the z in that equation from the planar equation $\vec{r} \cdot \vec{p} = p$, where $\vec{p}$ is the vector perpendicular to the plane from the origin to the plane, gives a quadratic equation in x, y. This translates into a quadratic equation in the plane—take the line of intersection of the cutting plane with the x, y plane as the y axis in both, then one is related to the other by a scaling $x' = \lambda x$. To identify the conic, diagonalized the form, and look at the coefficients of x^2, y^2. If they are the same sign, it is an ellipse, opposite, a hyperbola. The parabola is the exceptional case where one is zero, the other equates to a linear term.



It is instructive to see how an important property of the ellipse follows immediately from this construction. The slanting plane in the figure cuts the cone in an ellipse. Two spheres inside the cone, having circles of contact with the cone C, C', are adjusted in size so that they both just touch the plane, at points F, F' respectively.



It is easy to see that such spheres exist, for example start with a tiny sphere inside the cone near the point, and gradually inflate it, keeping it spherical and touching the cone, until it touches the plane. Now consider a point P on the ellipse. Draw two lines: one from P to the point F where the small sphere touches, the other up the cone, aiming for the vertex, but stopping at the point of intersection with the circle C. *Both these lines are tangents to the small sphere, and so have the same length*. (The tangents to a sphere from a point outside it form a cone, they are all of equal length.) Now repeat with F', C'. We find that $PF + PF' = PC + PC'$, the distances to the circles measured along the line through the vertex. So $PF + PF' = CC'$ in the obvious notation— F, F' are therefore evidently the foci of the ellipse.

The Ellipse

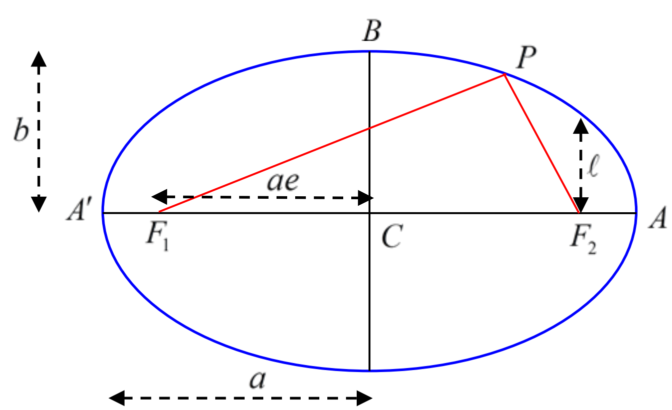
Squashed Circles and Gardeners

The simplest nontrivial planetary orbit is a circle: $x^2 + y^2 = a^2$ is centered at the origin and has radius a. An ellipse is a circle scaled (squashed) in one direction, so an ellipse centered at the origin with semimajor axis a and semiminor axis $b < a$ has equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

in the standard notation, a circle of radius a scaled by a factor b/a in the y direction. (It's usual to orient the larger axis along x.)

A *circle* can also be defined as the set of points which are the same distance a from a given point, and an *ellipse* can be defined as the set of points such that the *sum of the distances from two fixed points is a constant length* (which must obviously be greater than the distance between the two points!). This is sometimes called the *gardener's definition*: to set the outline of an elliptic flower bed in a lawn, a gardener would drive in two stakes, tie a loose string between them, then pull the string tight in all different directions to form the outline.



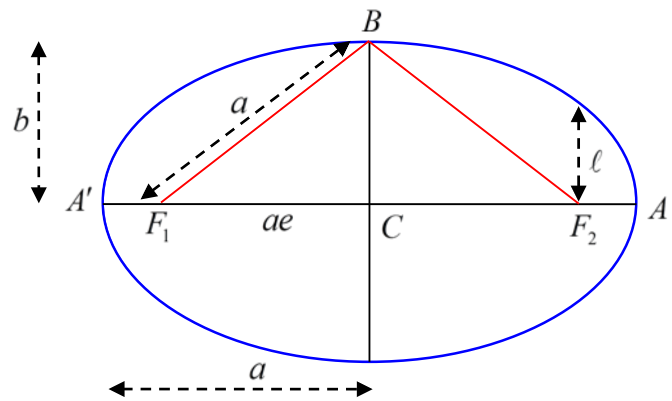
In the diagram, the stakes are at F_1, F_2, the **red lines** are the **string**, P is an arbitrary point on the ellipse.

CA is called the semimajor axis length a, CB the semiminor axis, length b.

F_1, F_2 are called the foci (plural of focus).

Notice first that *the string has to be of length* $2a$, because it must stretch along the major axis from F_1 to A then back to F_2, and for that configuration there's a double length of string along F_2A and a single length from F_1 to F_2. But the length $A'F_1$ is the same as F_2A, so the total length of string is the same as the total length $A'A = 2a$.

Suppose now we put P at B. Since $F_1B = BF_2$, and the string has length $2a$, the length $F_1B = a$.



We get a useful result by applying Pythagoras' theorem to the triangle F_1BC,

$$(F_1C)^2 = a^2 - b^2.$$

(We shall use this shortly.)

Evidently, for a circle, $F_1C = 0$.

Eccentricity

The *eccentricity* e of the ellipse is defined by

$$e = F_1C / a = \sqrt{1-(b/a)^2}, \text{ note } e < 1.$$

Eccentric just means off center, this is *how far the focus is off the center of the ellipse*, as a fraction of the semimajor axis. The eccentricity of a circle is zero. The eccentricity of a long thin ellipse is just below one.

F_1 and F_2 on the diagram are called the *foci* of the ellipse (plural of *focus*) because if a point source of light is placed at F_1, and the ellipse is a mirror, it will reflect—and therefore *focus*—all the light to F_2.

Equivalence of the Two Definitions

We need to verify, of course, that this gardener's definition of the ellipse is equivalent to the squashed circle definition. From the diagram, the total string length

$$2a = F_1P + PF_2 = \sqrt{(x+ae)^2 + y^2} + \sqrt{(x-ae)^2 + y^2}$$

and squaring both sides of

$$2a - \sqrt{(x+ae)^2 + y^2} = \sqrt{(x-ae)^2 + y^2}$$

then rearranging to have the residual square root by itself on the left-hand side, then squaring again,

$$(x+ae)^2 + y^2 = (a+ex)^2,$$

from which, using $e^2 = 1 - (b^2/a^2)$, we find $x^2/a^2 + y^2/b^2 = 1$.

Ellipse in Polar Coordinates

In fact, in analyzing planetary motion, it is more natural to *take the origin of coordinates at the center of the Sun* rather than the center of the elliptical orbit.

It is also more convenient to take (r,θ) coordinates instead of (x,y) coordinates, because the strength of the gravitational force depends only on r. Therefore, the relevant equation describing a planetary orbit is the (r,θ) equation with the origin at one focus, here we follow the standard usage and choose the origin at F_2.

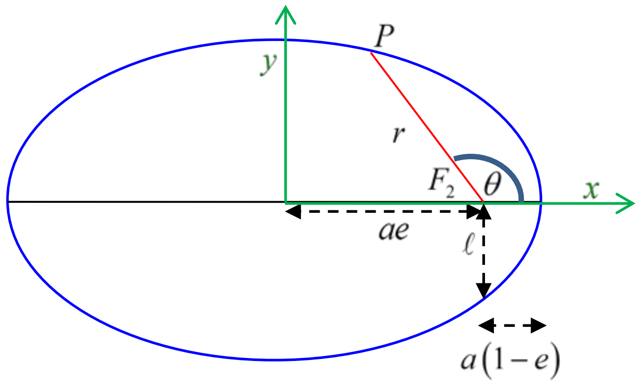
For an ellipse of semi major axis a and eccentricity e the equation is:

$$\frac{a(1-e^2)}{r} = 1 + e\cos\theta.$$

This is also often written

$$\frac{\ell}{r} = 1 + e\cos\theta$$

where ℓ is the *semi-latus rectum*, the perpendicular distance from a focus to the curve (so $\theta = \pi/2$ **)**, see the diagram below: but notice again that *this equation has* F_2 *as its origin*! **(For** $\theta < \pi/2,\ r < \ell$ **.)**



(It's easy to prove $\ell = a(1-e^2)$ using Pythagoras' theorem, $(2a-\ell)^2 = (2ae)^2 + \ell^2$.)

The directrix: writing $r\cos\theta = x$, the equation for the ellipse can also be written as

$$r = a(1-e^2) - ex = e(x_0 - x),$$

where $x_0 = (a/e) - ae$ (the origin $x = 0$ being the focus).

The line $x = x_0$ is called the *directrix*.

For any point on the ellipse, its distance from the focus is e times its distance from the directrix.

Deriving the Polar Equation from the Cartesian Equation

Note first that (following standard practice) coordinates (x, y) and (r, θ) have different origins!

Writing $x = ae + r\cos\theta,\ y = r\sin\theta$ in the Cartesian equation,

$$\frac{(ae + r\cos\theta)^2}{a^2} + \frac{(r\sin\theta)^2}{b^2} = 1,$$

that is, with slight rearrangement,

$$r^2\left(\frac{\cos^2\theta}{a^2} + \frac{\sin^2\theta}{b^2}\right) + r\frac{2e\cos\theta}{a} - (1-e^2) = 0.$$

This is a quadratic equation for r and can be solved in the usual fashion, but looking at the coefficients, it's evidently a little easier to solve the corresponding quadratic for $u = 1/r$.

The solution is:

$$\frac{1}{r} = u = \frac{e\cos\theta}{a\left(1-e^2\right)} \pm \frac{1}{a\left(1-e^2\right)},$$

from which

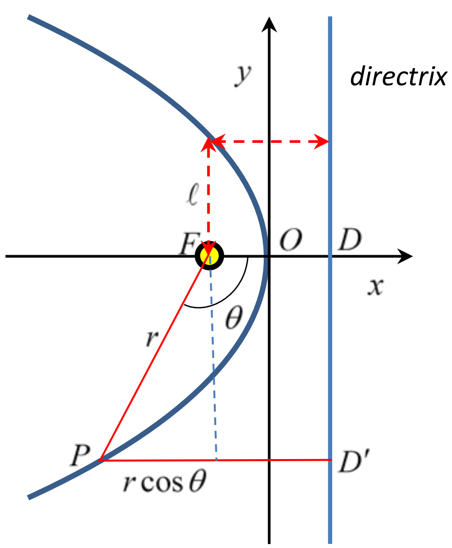
$$\frac{a\left(1-e^2\right)}{r} = \frac{\ell}{r} = 1 + e\cos\theta$$

where we drop the other root because it gives negative r, for example for $\theta = \pi/2$. This establishes the equivalence of the two equations.

The Parabola

The parabola can be defines as the limiting curve of an ellipse as one focus (in the case we're examining, that would be F_1) going to infinity. The eccentricity evidently goes to one, $e \to 1$, since the center of the ellipse has gone to infinity as well. The semi-latus rectum ℓ is still defined as the perpendicular distance from the focus to the curve, the equation is

$$\ell = r\left(1 + \cos\theta\right).$$



Note that this describes a parabola opening to the *left*. Taking $OF = 1$, the equation of this parabola is

$$y^2 = -4x.$$

All parabolas look the same, apart from scaling (maybe just in one direction).

The line perpendicular to the axis and the same distance from the curve along the axis as the focus is, but outside the curve, is the parabola's *directrix*. That is, $FO = OD$.

Each point on the curve is the same distance from the focus as it is from the directrix. This can be deduced from the limit of the ellipse property that the sum of distances to the two foci is constant. Let's call the other focus ∞. Then $FP + P\infty = FO + O\infty = D\infty = D'\infty$. So from the diagram, $FP = PD'$.

Exercises: 1. Prove by finding the slope, etc., that any ray of light emitted by a point lamp at the focus will be reflected by a parabolic mirror to go out parallel to the axis.

2. From the diagram above, show that the equality $FP = PD'$ easily gives the equation for the parabola, both in (r,θ) and in (x,y) coordinates.

The Hyperbola

Cartesian Coordinates

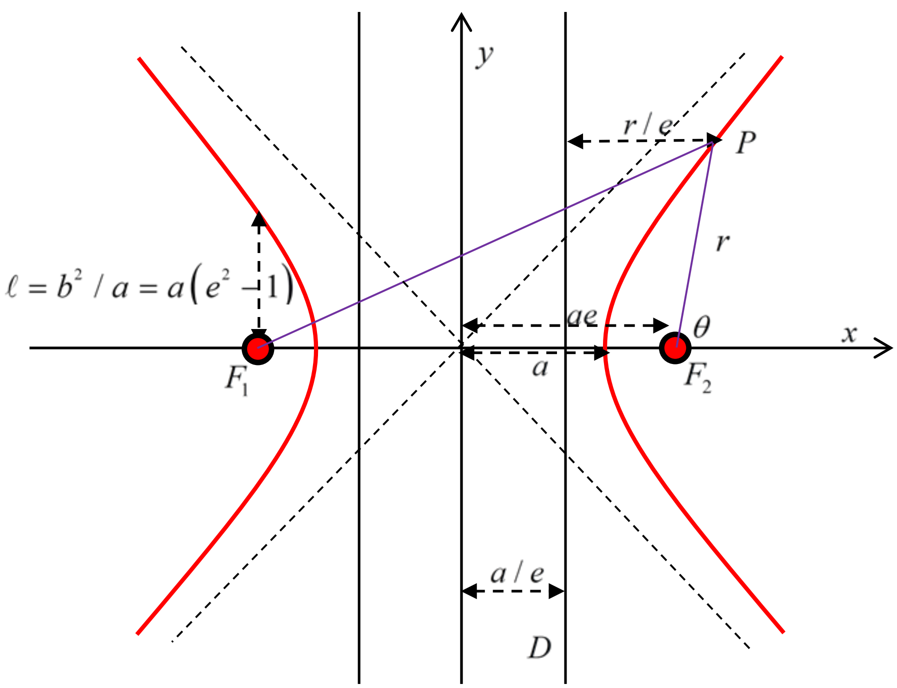
The hyperbola has eccentricity $e > 1$. In Cartesian coordinates, it has equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

and has two branches, both going to infinity approaching asymptotes $x = \pm(a/b)y$. The curve intersects the x axis at $x = \pm a$, the foci are at $x = \pm ae$, for any point on the curve,

$$r_{F_1} - r_{F_2} = \pm 2a$$

the sign being opposite for the two branches.



The semi-latus rectum, as for the earlier conics, is the perpendicular distance from a focus to the curve, and is $\ell = b^2/a = a(e^2 - 1)$. Each focus has an associated directrix, the distance of a point on the curve from the directrix multiplied by the eccentricity gives its distance from the focus.

Polar Coordinates

The (r,θ) equation with respect to a focus can be found by substituting $x = r\cos\theta + ae,\ \ y = r\sin\theta$ in the Cartesian equation and solving the quadratic for $u = 1/r$.

Notice that θ *has a limited range*: the equation for the right-hand curve with respect to its own focus F_2 has

$$\tan\theta_{\text{asymptote}} = \pm b/a \text{, so } \cos\theta_{\text{asymptote}} = \pm 1/e.$$

The equation for this curve is

$$\frac{\ell}{r} = 1 - e\cos\theta$$

in the range

$$\theta_{\text{asymptote}} < \theta < 2\pi - \theta_{\text{asymptote}}.$$

This equation comes up with various signs! The left hand curve, with respect to the left hand focus, would have a positive sign $+e$. With origin at F_1, (on the *left*) the equation of the *right*-hand curve is $\frac{\ell}{r} = e\cos\theta - 1$, finally with the origin at F_2 the left-hand curve is $\frac{\ell}{r} = -1 - e\cos\theta$. These last two describe *repulsive* inverse square scattering (Rutherford).

Note: A Useful Result for Rutherford Scattering

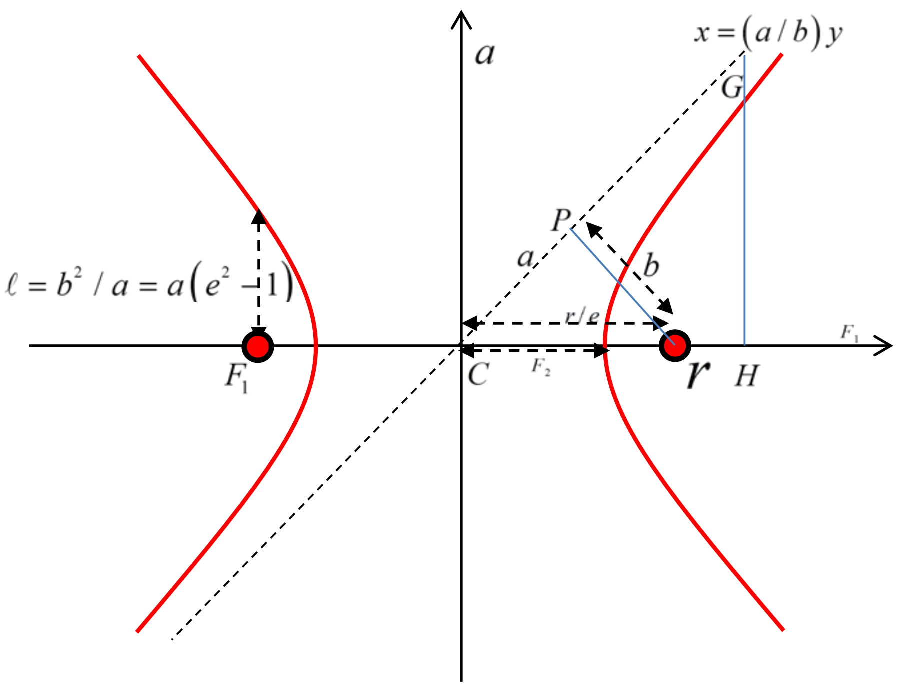
If we define the hyperbola by

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1,$$

then *the perpendicular distance from a focus to an asymptote is just* b.

This equation is the same (including scale) as

$$\ell/r=-1-e\cos\theta,\text{ with }\ell=b^2/a=a\left(e^2-1\right).$$



Proof: The triangle CPF_2 is similar to triangle CHG, so $PF_2/PC=GH/CH=b/a$, and since the square of the hypotenuse CF_2 is $a^2e^2=a^2+b^2$, the distance $F_2P=b$.

I find this a surprising result because in analyzing Rutherford scattering (and other scattering) the impact parameter, the distance of the ingoing particle path from a parallel line through the scattering center, is denoted by b. Surely this can't be a coincidence? But I can't find anywhere that this was the original motivation for the notation.

28

DIFFERENTIAL EQUATIONS

Having studied the concept of differentiation and integration, we are now faced with the question where do they find an application.

In fact these are the tools which help us to determine the exact takeoff speed, angle of launch, amount of thrust to be provided and other related technicalities in space launches.

Not only this but also in some problems in Physics and Bio-Sciences, we come across relations which involve derivatives.

One such relation could be $\frac{ds}{dt} = 4.9\ t^2$ where s is distance and t is time. Therefore, $\frac{ds}{dt}$ represents velocity (rate of change of distance) at time t.

Equations which involve derivatives as their terms are called differential equations. In this lesson, we are going to learn how to find the solutions and applications of such equations.

OBJECTIVES

After studying this lesson, you will be able to :

- define a differential equation, its order and degree;
- determine the order and degree of a differential equation;
- form differential equation from a given situation;
- illustrate the terms "general solution" and "particular solution" of a differential equation through examples;
- solve differential equations of the following types :

  (i) $\frac{dy}{dx} = f(x)$ (ii) $\frac{dy}{dx} = f(x)g(y)$

  (iii) $\frac{dy}{dx} = \frac{f(x)}{g(y)}$ (iv) $\frac{dy}{dx} + P(x)y = Q(x)$ (v) $\frac{d^2y}{dx^2} = f(x)$

- find the particular solution of a given differential equation for given conditions.

EXPECTED BACKGROUND KNOWLEDGE

- Integration of algebraic functions, rational functions and trigonometric functions

28.1 DIFFERENTIAL EQUATIONS

As stated in the introduction, many important problems in Physics, Biology and Social Sciences, when formulated in mathematical terms, lead to equations that involve derivatives. Equations which involve one or more differential coefficients such as $\frac{dy}{dx}, \frac{d^2y}{dx^2}$ (or differentials) etc. and independent and dependent variables are called differential equations.

For example,

(i) $\frac{dy}{dx} = \cos x$ (ii) $\frac{d^2y}{dx^2} + y = 0$ (iii) $x dx + y dy = 0$

(iv) $\left(\frac{d^2y}{dx^2}\right)^2 + x^2\left(\frac{dy}{dx}\right)^3 = 0$ (vi) $y = \frac{dy}{dx} + \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$

28.2 ORDER AND DEGREE OF A DIFFERENTIAL EQUATION

Order : It is the order of the highest derivative occurring in the differential equation.

Degree : It is the degree of the highest order derivative in the differential equation after the equation is free from negative and fractional powers of the derivatives. For example,

	Differential Equation	**Order**	**Degree**
(i)	$\frac{dy}{dx} = \sin x$	One	One
(ii)	$\left(\frac{dy}{dx}\right)^2 + 3y^2 = 5x$	One	Two
(iii)	$\left(\frac{d^2s}{dt^2}\right)^2 + t^2\left(\frac{ds}{dt}\right)^4 = 0$	Two	Two
(iv)	$\frac{d^3v}{dr^3} + \frac{2}{r}\frac{dv}{dr} = 0$	Three	One
(v)	$\left(\frac{d^4y}{dx^4}\right)^2 + x^3\left(\frac{d^3y}{dx^3}\right)^5 = \sin x$	Four	Two

Example 28.1 Find the order and degree of the differential equation :

$$\frac{d^2y}{dx^2} + \left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}} = 0$$

Solution : The given differential equation is

$$\frac{d^2y}{dx^2} + \left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}} = 0 \quad \text{or} \quad \frac{d^2y}{dx^2} = -\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}$$

The term $\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}$ has fractional index. Therefore, we first square both sides to remove fractional index.

Squaring both sides, we have

$$\left(\frac{d^2y}{dx^2}\right)^2 = \left[1 + \left(\frac{dy}{dx}\right)^2\right]^3$$

Hence each of the order of the diferential equation is 2 and the degree of the differential equation is also 2.

Note : Before finding the degree of a differential equation, it should be free from radicals and fractions as far as derivatives are concerned.

28.3 LINEAR AND NON-LINEAR DIFFERENTIAL EQUATIONS

A differential equation in which the dependent variable and all of its derivatives occur only in the first degree and are not multiplied together is called a **linear differential equation**. A differential equation which is not linear is called non-linear differential equation . For example, the differential equations

$$\frac{d^2y}{dx^2} + y = 0 \quad \text{and} \quad \cos^2 x \frac{d^3y}{dx^3} + x^3 \frac{dy}{dx} + y = 0 \text{ are linear.}$$

The differential equation $\left(\frac{dy}{dx}\right)^2 + \frac{y}{x} = \log x$ is non-linear as degree of $\frac{dy}{dx}$ is two.

Further the differential equation $y \frac{d^2y}{dx^2} - 4 = x$ is non-linear because the dependent variable y and its derivative $\frac{d^2y}{dx^2}$ are multiplied together.

28.4 FORMATION OF A DIFFERENTIAL EQUATION

Consider the family of all straight lines passing through the origin (see Fig. 28.1).

This family of lines can be represented by

$$y = mx \qquad .....(1)$$

Differentiating both sides, we get

$$\frac{dy}{dx} = m \qquad .....(2)$$

From (1) and (2), we get

$$y = x\frac{dy}{dx} \qquad .....(3)$$

So $y = mx$ and $y = x\frac{dy}{dx}$ represent the same family.

Clearly equation (3) is a differential equation.

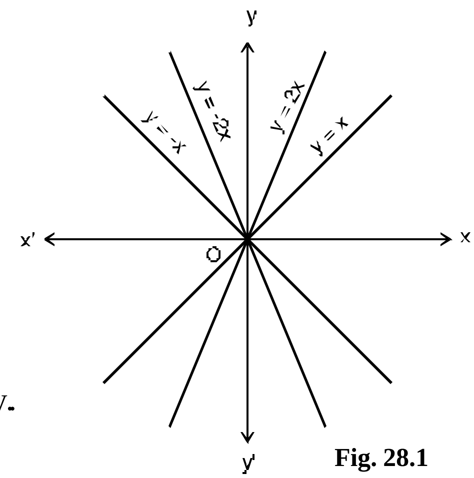


Fig. 28.1

Working Rule : To form the differential equation corresponding to an equation involving two variables, say x and y and some arbitrary constants, say, a, b, c, etc.

(i) Differentiate the equation as many times as the number of arbitrary constants in the equation.

(ii) Eliminate the arbitrary constants from these equations.

Remark

If an equation contains n arbitrary constants then we will obtain a differential equation of n^{th} order.

Example 28.2 Form the differential equation representing the family of curves.

$$y = ax^2 + bx. \qquad ......(1)$$

Differentiating both sides, we get

$$\frac{dy}{dx} = 2ax + b \qquad ......(2)$$

Differentiating again, we get

$$\frac{d^2y}{dx^2} = 2a \qquad ......(3)$$

$$\Rightarrow \qquad a = \frac{1}{2}\frac{d^2y}{dx^2} \qquad ......(4)$$

(The equation (1) contains two arbitrary constants. Therefore, we differentiate this equation two times and eliminate 'a' and 'b').

On putting the value of 'a' in equation (2), we get

$$\frac{dy}{dx} = x\frac{d^2y}{dx^2} + b$$

$$\Rightarrow \qquad b = \frac{dy}{dx} - x\frac{d^2y}{dx^2} \qquad \text{......(5)}$$

Substituting the values of 'a' and 'b' given in (4) and (5) above in equation (1), we get

$$y = x^2\left(\frac{1}{2}\frac{d^2y}{dx^2}\right) + x\left(\frac{dy}{dx} - x\frac{d^2y}{dx^2}\right)$$

or $$y = \frac{x^2}{2}\frac{d^2y}{dx^2} + x\frac{dy}{dx} - x^2\frac{d^2y}{dx^2}$$

or $$y = x\frac{dy}{dx} - \frac{x^2}{2}\frac{d^2y}{dx^2}$$

or $$\frac{x^2}{2}\frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = 0$$

which is the required differential equation.

Example 28.3 Form the differential equation representing the family of curves

$y = a\cos(x + b)$.

Solution : $$y = a\cos(x + b) \qquad \text{.....(1)}$$

Differentiating both sides, we get

$$\frac{dy}{dx} = -a\sin(x + b) \qquad \text{....(2)}$$

Differentiating again, we get

$$\frac{d^2y}{dx^2} = -a\cos(x + b) \qquad \text{.....(3)}$$

From (1) and (3), we get

$$\frac{d^2y}{dx^2} = -y \qquad \text{or} \qquad \frac{d^2y}{dx^2} + y = 0$$

which is the required differential equation.

Example 28.4 Find the differential equation of all circles which pass through the origin and whose centres are on the x-axis.

Solution : As the centre lies on the x-axis, its coordinates will be (a, 0).

Since each circle passes through the origin, its radius is a.

Then the equation of any circle will be

$$(x - a)^2 + y^2 = a^2 \qquad (1)$$

To find the corresponding differential equation, we differentiate equation (1) and get

$$2(x-a)+2y\frac{dy}{dx}=0$$

or $$x-a+y\frac{dy}{dx}=0$$

or $$a=y\frac{dy}{dx}+x$$

Substituting the value of 'a' in equation (1), we get

$$\left(x-y\frac{dy}{dx}-x\right)^2+y^2=\left(y\frac{dy}{dx}+x\right)^2$$

or $$\left(y\frac{dy}{dx}\right)^2+y^2=x^2+\left(y\frac{dy}{dx}\right)^2+2xy\frac{dy}{dx}$$

or $$y^2=x^2+2xy\frac{dy}{dx}$$

which is the required differential equation.

Remark

If an equation contains one arbitrary constant then the corresponding differential equation is of the first order and if an equation contains two arbitrary constants then the corresponding differential equation is of the second order and so on.

Example 28.5 Assuming that a spherical rain drop evaporates at a rate proportional to its surface area, form a differential equation involving the rate of change of the radius of the rain drop.

Solution : Let r(t) denote the radius (in mm) of the rain drop after t minutes. Since the radius is decreasing as t increases, the rate of change of r must be negative.

If V denotes the volume of the rain drop and S its surface area, we have

$$V=\frac{4}{3}\pi r^3 \qquad .....(1)$$

and $$S=4\pi r^2 \qquad .....(2)$$

It is also given that

$$\frac{dV}{dt}\propto S$$

or $$\frac{dV}{dt}=-kS$$

or $$\frac{dV}{dr}\cdot\frac{dr}{dt}=-kS \qquad .....(3)$$

Using (1), (2) and (3) we have

$$4\pi r^2\cdot\frac{dr}{dt}=-4k\pi r^2$$

or $$\frac{dr}{dt} = k$$

which is the required differential equation.

CHECK YOUR PROGRESS 28.1

1. Find the order and degree of the differential equation

$$y = x\frac{dy}{dx} + \frac{1}{\frac{dy}{dx}}$$

2. Write the order and degree of each of the following differential equations.

(a) $\left(\frac{ds}{dt}\right)^4 + 3s\frac{d^2s}{dt^2} = 0$ (b) $y = 2x\frac{dy}{dx} + x\sqrt{1 + \left(\frac{dy}{dx}\right)^2}$

(c) $\sqrt{1-x^2}\,dx + \sqrt{1-y^2}\,dy = 0$ (d) $\left(\frac{d^2s}{dt^2}\right)^2 + 3\left(\frac{ds}{dt}\right)^3 + 4 = 0$

3. State whether the following differential equations are linear or non-linear.

(a) $\left(xy^2 - x\right)dx + \left(y - x^2y\right)dy = 0$ (b) $dx + dy = 0$

(c) $\frac{dy}{dx} = \cos x$ (d) $\frac{dy}{dx} + \sin^2 y = 0$

4. Form the differential equation corresponding to

$(x-a)^2 + (y-b)^2 = r^2$ by eliminating 'a' and 'b'.

5. (a) Form the differential equation corresponding to

$y^2 = m\left(a^2 - x^2\right)$

(b) Form the differential equation corresponding to

$y^2 - 2ay + x^2 = a^2$, where a is an arbitrary constant.

(c) Find the differential equation of the family of curves $y = Ae^{2x} + Be^{-3x}$ where A and B are arbitrary constants.

(d) Find the differential equation of all straight lines passing through the point (3,2).

(e) Find the differential equation of all the circles which pass through origin and whose centres lie on y-axis.

28.5 GENERAL AND PARTICULAR SOLUTIONS

Finding solution of a differential equation is a reverse process. Here we try to find an equation which gives rise to the given differential equation through the process of differentiations and elimination of constants. The equation so found is called the primitive or the solution of the differential equation.

Remarks

(1) If we differentiate the primitive, we get the differential equation and if we integrate the differential equation, we get the primitive.

(2) Solution of a differential equation is one which satisfies the differential equation.

Example 28.6 Show that $y = C_1 \sin x + C_2 \cos x$, where C_1 and C_2 are arbitrary constants, is a solution of the differential equation :

$$\frac{d^2y}{dx^2} + y = 0$$

Solution : We are given that

$$y = C_1 \sin x + C_2 \cos x \qquad .....(1)$$

Differentiating both sides of (1), we get

$$\frac{dy}{dx} = C_1 \cos x - C_2 \sin x \qquad .....(2)$$

Differentiating again, we get

$$\frac{d^2y}{dx^2} = -C_1 \sin x - C_2 \cos x$$

Substituting the values of $\frac{d^2y}{dx^2}$ and y in the given differential equation, we get

$$\frac{d^2y}{dx^2} + y = C_1 \sin x + C_2 \cos x + (-C_1 \sin x - C_2 \cos x)$$

or $$\frac{d^2y}{dx^2} + y = 0$$

In integration, the arbitrary constants play important role. For different values of the constants we get the different solutions of the differential equation.

A solution which contains as many as arbitrary constants as the order of the differential equation is called the **General Solution** or complete primitive.

If we give the particular values to the arbitrary constants in the general solution of differential equation, the resulting solution is called a **Particular Solution**.

Remark

General Solution contains as many arbitrary constants as is the order of the differential equation.

Example 28.7 Show that $y = cx + \frac{a}{c}$ (where c is a constant) is a solution of the differential equation.

$$y = x\frac{dy}{dx} + a\frac{dx}{dy}$$

Solution : We have $y = cx + \frac{a}{c}$ .....(1)

Differentiating (1), we get

$$\frac{dy}{dx} = c \qquad \Rightarrow \qquad \frac{dx}{dy} = \frac{1}{c}$$

On substituting the values of $\frac{dy}{dx}$ and $\frac{dx}{dy}$ in R.H.S of the differential equation, we have

$$x(c) + a\left(\frac{1}{c}\right) = cx + \frac{a}{c} = y$$

$\Rightarrow$ R.H.S. = L.H.S.

Hence $y = cx + \frac{a}{c}$ is a solution of the given differential equation.

Example 28.8 If $y = 3x^2 + C$ is the general solution of the differential equation $\frac{dy}{dx} - 6x = 0$, then find the particular solution when y = 3, x = 2.

Solution : The general solution of the given differential equation is given as

$$y = 3x^2 + C \qquad ....(1)$$

Now on substituting y = 3, x = 2 in the above equation , we get

$$3 = 12 + C \qquad \text{or} \qquad C = -9$$

By substituting the value of C in the general solution (1), we get

$$y = 3x^2 - 9$$

which is the required particular solution.

28.6 TECHNIQUES OF SOLVING A DIFFERENTIAL EQUATION

28.6.1 When Variables are Separable

(i) Differential equation of the type $\frac{dy}{dx} = f(x)$

Consider the differential equation of the type $\frac{dy}{dx} = f(x)$

or $dy = f(x)\,dx$

On integrating both sides, we get

$$\int dy = \int f(x)\,dx$$

$$y = \int f(x)\,dx + c$$

where c is an arbitrary constant. This is the general solution.

Note : It is necessary to write c in the general solution, otherwise it will become a particular solution.

Example 28.9 Solve

$$(x+2)\frac{dy}{dx} = x^2 + 4x - 5$$

Solution : The given differential equation is $(x+2)\frac{dy}{dx} = x^2 + 4x - 5$

or $\quad \frac{dy}{dx} = \frac{x^2 + 4x - 5}{x+2} \quad$ or $\quad \frac{dy}{dx} = \frac{x^2 + 4x + 4 - 4 - 5}{x+2}$

or $\quad \frac{dy}{dx} = \frac{(x+2)^2}{x+2} - \frac{9}{x+2} \quad$ or $\quad \frac{dy}{dx} = x + 2 - \frac{9}{x+2}$

or $$dy = \left(x + 2 - \frac{9}{x+2}\right)dx \qquad .....(1)$$

On integrating both sides of (1), we have

$$\int dy = \int\left(x + 2 - \frac{9}{x+2}\right)dx \quad \text{or} \quad y = \frac{x^2}{2} + 2x - 9\log|x+2| + c,$$

where c is an arbitrary constant, is the required general solution.

Example 28.10 Solve

$$\frac{dy}{dx} = 2x^3 - x$$

given that y = 1 when x = 0

Solution : The given differential equation is $\frac{dy}{dx} = 2x^3 - x$

or $$dy = \left(2x^3 - x\right)dx \qquad ...(1)$$

On integrating both sides of (1), we get

$$\int dy = \int\left(2x^3 - x\right)dx \quad \text{or} \quad y = 2.\frac{x^4}{4} - \frac{x^2}{2} + C$$

or $$y = \frac{x^4}{2} - \frac{x^2}{2} + C \qquad .....(2)$$

where C is an arbitrary constant.

Since y = 1 when x = 0, therefore, if we substitute these values in (2) we will get

$$1 = 0 - 0 + C \qquad \Rightarrow \qquad C = 1$$

Now, on putting the value of C in (2), we get

$$y = \frac{1}{2}\left(x^4 - x^2\right) + 1 \text{ or } \qquad y = \frac{1}{2}x^2\left(x^2 - 1\right) + 1$$

which is the required particular solution.

(ii) Differential equations of the type $\dfrac{dy}{dx} = f(x) \cdot g(y)$

Consider the differential equation of the type

$$\frac{dy}{dx} = f(x) \cdot g(y)$$

or

$$\frac{dy}{g(y)} = f(x)\,dx \qquad .....(1)$$

In equation (1), x's and y's have been separated from one another. Therefore, this equation is also known differential equation with variables separable.

To solve such differential equations, we integrate both sides and add an arbitrary constant on one side.

To illustrate this method, let us take few examples.

Example 28.11 Solve

$$\left(1 + x^2\right)dy = \left(1 + y^2\right)dx$$

Solution : The given differential equation

$$\left(1 + x^2\right)dy = \left(1 + y^2\right)dx$$

can be written as

$$\frac{dy}{1 + y^2} = \frac{dx}{1 + x^2} \quad \text{(Here variables have been seperated)}$$

On integrating both sides of (1), we get

$$\int \frac{dy}{1 + y^2} = \int \frac{dx}{1 + x^2}$$

or

$$\tan^{-1} y = \tan^{-1} x + C$$

where C is an arbitrary constant.

This is the required solution.

Example 28.12 Solve

$$\left(x^2 - yx^2\right)\frac{dy}{dx} + y^2 + xy^2 = 0$$

Solution : The given differential equation

$$\left(x^2 - yx^2\right)\frac{dy}{dx} + y^2 + xy^2 = 0$$

can be written as $$x^2(1-y)\frac{dy}{dx} + y^2(1+x) = 0$$

or $$\frac{(1-y)}{y^2}dy = \frac{-(1+x)}{x^2}dx$$

(Variables separated) ....(1)

If we integrate both sides of (1), we get

$$\int\left(\frac{1}{y^2} - \frac{1}{y}\right)dy = \int\left(-\frac{1}{x^2} - \frac{1}{x}\right)dx$$

where C is an arbitrary constant.

or $$-\frac{1}{y} - \log|y| = \frac{1}{x} - \log|x| + C$$

or $$\log\left|\frac{x}{y}\right| = \frac{1}{x} + \frac{1}{y} + C$$

Which is the required general solution.

Example 28.13 Find the particular solution of

$$\frac{dy}{dx} = \frac{2x}{3y^2 + 1}$$

when y (0) = 3 (i.e. when x = 0, y = 3).

Solution : The given differential equation is

$$\frac{dy}{dx} = \frac{2x}{3y^2+1} \text{ or } \left(3y^2+1\right)dy = 2x\,dx \text{ (Variables separated)} \quad ....(1)$$

If we integrate both sides of (1), we get

$$\int\left(3y^2+1\right)dy = \int 2x\,dx,$$

where C is an arbitrary constant.

$$y^3 + y = x^2 + C \quad ....(2)$$

It is given that, $y(0) = 3$.

$\therefore$ on substituting y = 3 and x = 0 in (2), we get

$$27 + 3 = C$$

$\therefore$ $$C = 30$$

Thus, the required particular solution is

$$y^3 + y = x^2 + 30$$

28.6.2 Homogeneous Differential Equations

Consider the following differential equations :

(i) $y^2 + x^2 \dfrac{dy}{dx} = xy \dfrac{dy}{dx}$ (ii) $\left(x^3 + y^3\right)dx - 3xy^2 dy = 0$

(iii) $\dfrac{dy}{dx} = \dfrac{x^3 + xy^2}{y^2 x}$

In equation (i) above, we see that each term except $\dfrac{dy}{dx}$ is of degree 2

[as degree of y^2 is 2, degree of x^2 is 2 and degree of xy is 1 + 1 =2]

In equation (ii) each term except $\dfrac{dy}{dx}$ is of degree 3.

In equation (iii) each term except $\dfrac{dy}{dx}$ is of degree 3, as it can be rewritten as

$$y^2 x \frac{dy}{dx} = x^3 + x\,y^2$$

Such equations are called **homogeneous equations**.

Remarks

Homogeneous equations do not have constant terms.

For example, differential equation

$$\left(x^2 + 3yx\right)dx - \left(x^3 + x\right)dy = 0$$

is not a homogeneous equation as the degree of the function except $\dfrac{dy}{dx}$ in each term is not the same. [degree of x^2 is 2, that of 3yx is 2, of x^3 is 3, and of x is 1]

28.6.3 Solution of Homogeneous Differential Equation :

To solve such equations, we proceed in the following manner :

(i) write one variable = v. (the other variable).
(i.e. either y = vx or x = vy)

(ii) reduce the equation to separable form

(iii) solve the equation as we had done earlier.

Example 28.14 Solve

$$\left(x^2 + 3xy + y^2\right)dx - x^2 dy = 0$$

Solution : The given differential equation is

$$\left(x^2 + 3xy + y^2\right)dx - x^2 dy = 0$$

Notes

or $$\frac{dy}{dx} = \frac{x^2 + 3xy + y^2}{x^2} \qquad .....(1)$$

It is a homogeneous equation of degree two. (Why?)

Let $y = vx$. Then

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

$\therefore$ From (1), we have

$$v + x\frac{dv}{dx} = \frac{x^2 + 3x.vx + (vx)^2}{x^2} \quad \text{or} \quad v + x\frac{dv}{dx} = x^2\left[\frac{1 + 3v + v^2}{x^2}\right]$$

or $$v + x\frac{dv}{dx} = 1 + 3v + v^2 \quad \text{or} \quad x\frac{dv}{dx} = 1 + 3v + v^2 - v$$

or $$x\frac{dv}{dx} = v^2 + 2v + 1 \quad \text{or} \quad \frac{dv}{v^2 + 2v + 1} = \frac{dx}{x}$$

or $$\frac{dv}{(v+1)^2} = \frac{dx}{x} \qquad .....(2)$$

Further on integrating both sides of (2), we get

$$\frac{-1}{v+1} + C = \log|x|,$$ where C is an arbitrary constant.

On substituting the value of v, we get

$$\frac{x}{y + x} + \log|x| = C$$ which is the required solution.

28.6.4 Differential Equation

Consider the equation

$$\frac{dy}{dx} + Py = Q \qquad .....(1)$$

where P and Q are functions of x. This is linear equation of order one.

To solve equation (1), we first multiply both sides of equation (1) by $e^{\int Pdx}$ (called integrating factor) and get

$$e^{\int Pdx}\frac{dy}{dx} + Pye^{\int Pdx} = Qe^{\int Pdx}$$

or $$\frac{d}{dx}\left(ye^{\int Pdx}\right) = Qe^{\int Pdx} \qquad .....(2)$$

$$\left[\because \frac{d}{dx}\left(ye^{\int Pdx}\right) = e^{\int Pdx}\frac{dy}{dx} + Py.e^{\int Pdx}\right]$$

On integrating, we get

$$ye^{\int P\,dx} = \int Qe^{\int P\,dx}\,dx + C \qquad .....(3)$$

where C is an arbitrary constant,

or $$y = e^{-\int P\,dx}\left[\int Qe^{\int P\,dx}\,dx + C\right]$$

Note : $e^{\int P\,dx}$ is called the integrating factor of the equation and is written as I.F in short.

Remarks

(i) We observe that the left hand side of the linear differential equation (1) has become $\frac{d}{dx}\left(ye^{\int P\,dx}\right)$ after the equation has been multiplied by the factor $e^{\int P\,dx}$.

(ii) The solution of the linear differential equation

$$\frac{dy}{dx} + Py = Q$$

P and Q being functions of x only is given by

$$ye^{\int P\,dx} = \int Q\left(e^{\int P\,dx}\right)dx + C$$

(iii) The coefficient of $\frac{dy}{dx}$, if not unity, must be made unity by dividing the equation by it throughout.

(iv) Some differential equations become linear differential equations if y is treated as the independent variable and x is treated as the dependent variable.

For example, $\frac{dx}{dy} + Px = Q$, where P and Q are functions of y only, is also a linear differential equation of the first order.

In this case $\quad$ I.F. $= e^{\int P\,dy}$

and the solution is given by

$$x\,(\text{I.F.}) = \int Q.(\text{I.F.})\,dy + C$$

Example 28.15 Solve

$$\frac{dy}{dx} + \frac{y}{x} = e^{-x}$$

Solution : Here $P = \frac{1}{x}, Q = e^{-x}$ (Note that both P an Q are functions of x)

I.F. (Integrating Factor) $e^{\int P\,dx} = e^{\int \frac{1}{x}dx} = e^{\log x} = x \qquad (x > 0)$

On multiplying both sides of the equation by I.F., we get

$$x \cdot \frac{dy}{dx} + y = x \cdot e^{-x} \quad \text{or} \quad \frac{d}{dx}(y \cdot x) = xe^{-x}$$

Integrating both sides, we have

$$yx = \int xe^{-x}dx + C$$

where C is an arbitrary constant

or $\quad xy = -xe^{-x} + \int e^{-x}dx + C$

or $\quad xy = -xe^{-x} - e^{-x} + C$

or $\quad xy = -e^{-x}(x+1) + C$

or $\quad y = -\left(\frac{x+1}{x}\right)e^{-x} + \frac{C}{x}$

Note: In the solution x > 0.

Example 28.16 Solve :

$$\sin x \frac{dy}{dx} + y\cos x = 2\sin^2 x \cos x$$

Solution : The given differential equation is

$$\sin x \frac{dy}{dx} + y\cos x = 2\sin^2 x \cos x$$

or $\quad \frac{dy}{dx} + y\cot x = 2\sin x\cos x \qquad .....(1)$

Here $\quad P = \cot x, Q = 2\sin x\cos x$

$$\text{I.F.} = e^{\int P dx} = e^{\int \cot x dx} = e^{\log \sin x} = \sin x$$

On multiplying both sides of equation (1) by I.F., we get (sin x >0)

$$\frac{d}{dx}(y\sin x) = 2\sin^2 x\cos x$$

Further on integrating both sides, we have

$$y\sin x = \int 2\sin^2 x\cos x\, dx + C$$

where C is an arbitrary constant (sin x >0)

or $\quad y\sin x = \frac{2}{3}\sin^3 x + C,\quad$ which is the required solution.

Example 28.17 Solve $\quad \left(1+y^2\right)\frac{dx}{dy} = \tan^{-1} y - x$

Solution : The given differential equation is

$$\left(1+y^2\right)\frac{dx}{dy} = \tan^{-1} y - x$$

or $$\frac{dx}{dy} = \frac{\tan^{-1} y}{1+y^2} - \frac{x}{1+y^2}$$

or $$\frac{dx}{dy} + \frac{x}{1+y^2} = \frac{\tan^{-1} y}{1+y^2} \qquad \text{....(1)}$$

which is of the form $\frac{dx}{dy} + Px = Q$, where P and Q are the functions of y only.

$$\text{I.F.} = e^{\int P dy} = e^{\int \frac{1}{1+y^2} dy} = e^{\tan^{-1} y}$$

Multiplying both sides of equation (1) by I.F., we get

$$\frac{d}{dy}\left(xe^{\tan^{-1} y}\right) = \frac{\tan^{-1} y}{1+y^2}\left(e^{\tan^{-1} y}\right)$$

On integrating both sides, we get

or $$\left(e^{\tan^{-1} y}\right)x = \int e^t . t dt + C,$$

where C is an arbitrary constant and $t = \tan^{-1} y$ and $dt = \frac{1}{1+y^2} dy$

or $$\left(e^{\tan^{-1} y}\right)x = te^t - \int e^t + C$$

or $$\left(e^{\tan^{-1} y}\right)x = te^t - e^t + C$$

or $$\left(e^{\tan^{-1} y}\right)x = \tan^{-1} y \, e^{\tan^{-1} y} - e^{\tan^{-1} y} + C \qquad \text{(on putting } t = \tan^{-1} y\text{)}$$

or $$x = \tan^{-1} y - 1 + Ce^{-\tan^{-1} y}$$

CHECK YOUR PROGRESS 28.2

1. (i) Is $y = \sin x$ a solution of $\frac{d^2 y}{dx^2} + y = 0$?

   (ii) Is $y = x^3$ a solution of $x\frac{dy}{dx} - 4y = 0$?

2. Given below are some solutions of the differential equation $\frac{dy}{dx} = 3x$.

   State which are particular solutions and which are general solutions.

(i) $2y = 3x^2$ (ii) $y = \frac{3}{2}x^2 + 2$

(iii) $2y = 3x^2 + C$ (iv) $y = \frac{3}{2}x^2 + 3$

3. State whether the following differential equations are homogeneous or not ?

(i) $\frac{dy}{dx} = \frac{x^2}{1+y^2}$ (ii) $\left(3xy + y^2\right)dx + \left(x^2 + xy\right)dy = 0$

(iii) $\left(x+2\right)\frac{dy}{dx} = x^2 + 4x - 9$ (iv) $\left(x^3 - yx^2\right)dy + \left(y^3 + x^3\right)dx = 0$

4. (a) Show that y = a sin 2x is a solution of $\frac{d^2y}{dx^2} + 4y = 0$

(b) Verify that $y = x^3 + ax^2 + c$ is a solution of $\frac{d^3y}{dx^3} = 6$

5. The general solution of the differential equation

$\frac{dy}{dx} = \sec^2 x$ is $y = \tan x + C$.

Find the particular solution when

(a) $x = \frac{\pi}{4}, y = 1$ (b) $x = \frac{2\pi}{3}, y = 0$

6. Solve the following differential equations :

(a) $\frac{dy}{dx} = x^5 \tan^{-1}\left(x^3\right)$ (b) $\frac{dy}{dx} = \sin^3 x \cos^2 x + xe^x$

(c) $\left(1+x^2\right)\frac{dy}{dx} = x$ (d) $\frac{dy}{dx} = x^2 + \sin 3x$

7. Find the particular solution of the equation $e^x \frac{dy}{dx} = 4$, given that y = 3 , when x = 0

8. Solve the following differential equations :

(a) $\left(x^2 - yx^2\right)\frac{dy}{dx} + y^2 + xy^2 = 0$ (b) $\frac{dy}{dx} = xy + x + y + 1$

(c) $\sec^2 x \tan y\, dx + \sec^2 y \tan x\, dy = 0$ (d) $\frac{dy}{dx} = e^{x-y} + e^{-y}x^2$

9. Find the particular solution of the differential equation

$\log\left(\frac{dy}{dx}\right) = 3x + 4y$, given that y = 0 when x = 0

10. Solve the following differential equations :

(a) $\left(x^2 + y^2\right)dx - 2xy\,dy = 0$ (b) $x\frac{dy}{dx} + \frac{y^2}{x} = y$

(c) $\dfrac{dy}{dx} = \dfrac{\sqrt{x^2 - y^2} + y}{x}$ (d) $\dfrac{dy}{dx} = \dfrac{y}{x} + \sin\left(\dfrac{y}{x}\right)$

11. Solve : $\dfrac{dy}{dx} + y\sec x = \tan x$

12. Solve the following differential equations :

(a) $\left(1 + x^2\right)\dfrac{dy}{dx} + y = \tan^{-1} x$ (b) $\cos^2 x \dfrac{dy}{dx} + y = \tan x$

(c) $x\log x \dfrac{dy}{dx} + y = 2\log x,\ x > 1$

13. Solve the following differential equations:

(a) $(x + y + 1)\dfrac{dy}{dx} = 1$

[Hint: $\dfrac{dx}{dy} = x + y + 1$ or $\dfrac{dx}{dy} - x = y + 1$ which is of the form $\dfrac{dx}{dy} + Px = Q$]

(b) $\left(x + 2y^2\right)\dfrac{dy}{dx} = y,\ y > 0$ [Hint: $y\dfrac{dx}{dy} = x + 2y^2$ or $\dfrac{dx}{dy} - \dfrac{x}{y} = 2y$]

28.7 DIFFERENTIAL EQUATIONS OF HIGHER ORDER

Till now we were dealing with the differential equations of first order. In this section, simple differential equations of second order and third order will be discussed.

28.7.1 Differential Equations of the Type $\dfrac{d^2y}{dx^2} = f(x)$

Consider the differential equation

$$\frac{d^2y}{dx^2} = f(x)$$

It may be noted that it is a differential equation of second order. So its general solution will contain two arbitrary constants.

Now we have, $$\frac{d^2y}{dx^2} = f(x) \qquad ....(1)$$

or $$\frac{d}{dx}\left(\frac{dy}{dx}\right) = f(x)$$

Integrating both sides of (1) , we get

$$\frac{dy}{dx} = \int f(x)\,dx + C_1,$$ where C_1 is an arbitrary constant

Let $$\int f(x)\,dx = \phi(x)$$

Then $$\frac{dy}{dx} = \phi(x) + C_1 \qquad .....(2)$$

Again on integrating both sides of (2), we get

$$y = \int \phi(x)\,dx + C_1 x + C_2,$$

where C_2 is another arbitrary constant. Therefore in order to find the particular solution we need two conditions. [See Example 28.19]

Example 28.18 Solve $\frac{d^2y}{dx^2} = xe^x$

Solution : The given differential equation is

$$\frac{d^2y}{dx^2} = xe^x \qquad .....(1)$$

Now integrating both sides of (1), we have

$$\frac{dy}{dx} = \int xe^x dx + C_1,$$

where C_1 is an arbitrary constant

or $$\frac{dy}{dx} = xe^x - \int e^x dx + C_1$$

or $$\frac{dy}{dx} = xe^x - e^x + C_1 \qquad .....(2)$$

Again on integrating both sides of (2), we get

$$y = \int \left(xe^x - e^x + C_1 \right) dx + C_2,$$

where C_2 is another arbitrary constant.

or $$y = xe^x - \int e^x dx - e^x + C_1 x + C_2$$

or $$y = xe^x - 2e^x + C_1 x + C_2$$

which is the required general solution.

Example 28.19 Find the particular solution of the differential equation

$$\frac{d^2y}{dx^2} = x^2 + \sin 3x$$

for which $y(0) = 0$ and $\frac{dy}{dx} = 0$ when $x = 0$

Solution : The given differential equation is

$$\frac{d^2y}{dx^2} = x^2 + \sin 3x \qquad .....(1)$$

SUCCESSIVE DIFFERENTIATION AND LEIBNITZ'S THEOREM

1.1 Introduction

Successive Differentiation is the process of differentiating a given function successively n times and the results of such differentiation are called successive derivatives. The higher order differential coefficients are of utmost importance in scientific and engineering applications.

Let $\boldsymbol{f}(\boldsymbol{x})$ be a differentiable function and let its successive derivatives be denoted by $\boldsymbol{f}'(\boldsymbol{x}), \boldsymbol{f}''(\boldsymbol{x}), \ldots, \boldsymbol{f}^{(n)}(\boldsymbol{x})$.

Common notations of higher order Derivatives of $y = f(x)$

1st Derivative: $f'(x)$ **or** y' **or** y_1 **or** $\frac{dy}{dx}$ **or** Dy

2nd Derivative: $f''(x)$ **or** y'' **or** y_2 **or** $\frac{d^2y}{dx^2}$ **or** D^2y

$\vdots$

n^{th} Derivative: $f^{(n)}(x)$ **or** $y^{(n)}$ **or** y_n **or** $\frac{d^ny}{dx^n}$ **or** D^ny

1.2 Calculation of n^th Derivatives

i. **n^{th} Derivative of e^{ax}**

Let y $= e^{ax}$

$$y_1 = ae^{ax}$$
$$y_2 = a^2e^{ax}$$
$$\vdots$$
$$y_n = a^n e^{ax}$$

ii. **n^{th} Derivative of $(ax + b)^m$, m is a +ve integer greater than n**

Let y = $(ax + b)^m$

$$y_1 = m\,a(ax + b)^{m-1}$$
$$y_2 = m(m-1)a^2(ax + b)^{m-2}$$
$$\vdots$$
$$y_n = m(m-1)\ldots(m-n+1)a^n(ax + b)^{m-n}$$

$$= \frac{m!}{(m-n)!} a^n (ax + \mathrm{b})^{m-n}$$

iii. $\boldsymbol{n^{th}}$ **Derivative of** $\boldsymbol{y = \log(ax + b)}$

Let $y = \log(ax + b)$

$$y_1 = \frac{a}{(ax+b)}$$

$$y_2 = \frac{-a^2}{(ax+b)^2}$$

$$y_3 = \frac{2!\, a^3}{(ax+b)^3}$$

$$\vdots$$

$$y_n = (-1)^{n-1} \frac{(n-1)!\, a^n}{(ax+b)^n}$$

iv. $\boldsymbol{n^{th}}$ **Derivative of** $\boldsymbol{y = \sin(ax + b)}$

Let $y = \sin(ax + b)$

$$y_1 = a\cos(ax + b) = a \sin\left(ax + b + \frac{\pi}{2}\right)$$

$$y_2 = a^2 \cos\left(ax + b + \frac{\pi}{2}\right) = a^2 \sin\left(ax + b + \frac{2\pi}{2}\right)$$

$$\vdots$$

$$y_n = a^n \sin\left(ax + b + \frac{n\pi}{2}\right)$$

Similarly if $\boldsymbol{y = \cos(ax + b)}$

$$y_n = a^n \cos\left(ax + b + \frac{n\pi}{2}\right)$$

v. $\boldsymbol{n^{th}}$ **Derivative of** $\boldsymbol{y = e^{ax}\sin(ax + b)}$

Let $y = e^{ax}\sin(bx + c)$

$$y_1 = a\, e^{ax}\sin(bx + c) + e^{ax} b \cos(bx + c)$$

$$= e^{ax}\, [a \sin(bx + c) + b \cos(bx + c)]$$

$$= e^{ax}\, [\, r\, cos\alpha \sin(bx + c) + r\, sin\alpha \cos(bx + c)]$$

Putting $a = r\, cos\alpha,\ b = r\, sin\alpha$

$$= e^{ax}\ r\ \sin(bx + c + \alpha)$$

Similarly $y_2 = e^{ax}\ r^2\ \sin(bx + c + 2\alpha)$

$$\vdots$$

$$y_n = e^{ax}\ r^n\ \sin(bx + c + n\alpha)$$

where $r^2 = a^2 + b^2$ and $\tan\alpha = \frac{b}{a}$

$$\therefore y_n = e^{ax}\ (a^2 + b^2)^{\frac{n}{2}}\ \sin\left(bx + c + n\tan^{-1}\frac{b}{a}\right)$$

Similarly if $\boldsymbol{y = e^{ax}\cos(ax + b)}$

$$y_n = e^{ax}\ r^n\ \cos(bx + c + n\alpha)$$

$$= e^{ax}\ (a^2 + b^2)^{\frac{n}{2}}\ \cos\left(bx + c + n\tan^{-1}\frac{b}{a}\right)$$

Summary of Results

Function	$\boldsymbol{n^{th}}$ **Derivative**
y = e^{ax}	$\boldsymbol{y_n} = a^n e^{ax}$
y = $(ax+b)^m$	$\boldsymbol{y_n} = \begin{cases} \frac{m!}{(m-n)!} a^n (ax+\mathrm{b})^{m-n}, m>0, m>n \\ 0, \quad m>0, \quad m<n, \\ n!\ a^n, \quad m=n \\ \frac{(-1)^n n!\, a^n}{(ax+b)^{n+1}}, \quad m=-1 \end{cases}$
$y = \log(ax+b)$	$y_n = (-1)^{n-1} \frac{(n-1)!\, a^n}{(ax+b)^n}$
$y = \sin(ax+b)$	$y_n = a^n \sin\left(ax+b+\frac{n\pi}{2}\right)$
$y = cos(ax+b)$	$y_n = a^n \cos\left(ax+b+\frac{\mathrm{n}\pi}{2}\right)$
y = $e^{ax}\sin(bx+c)$	$y_n = e^{ax}(a^2+b^2)^{\frac{n}{2}} \sin\left(bx+c+n\tan^{-1}\frac{b}{a}\right)$
y = $e^{ax}\cos(bx+c)$	$y_n = e^{ax}(a^2+b^2)^{\frac{n}{2}} \cos\left(bx+c+n\tan^{-1}\frac{b}{a}\right)$

Example 1 Find the n^{th} derivative of $\frac{1}{1-5x+6x^2}$

Solution: Let $y = \frac{1}{1-5x+6x^2}$

Resolving into partial fractions

$$y = \frac{1}{1-5x+6x^2} = \frac{1}{(1-3x)(1-2x)} = \frac{3}{1-3x} - \frac{2}{1-2x}$$

$$\therefore y_n = \frac{3\,(-3)^n(-1)^n n!}{(1-3x)^{n+1}} - \frac{2(-2)^n(-1)^n n!}{(1-2x)^{n+1}}$$

$$\Rightarrow y_n = (-1)^{n+1}\, n! \left[\left(\frac{3}{1-3x}\right)^{n+1} - \left(\frac{2}{1-2x}\right)^{n+1}\right]$$

Example 2 Find the n^{th} derivative of $\sin 6x \cos 4x$

Solution: Let y = $\sin 6x \cos 4x$

$$= \frac{1}{2}(\sin 10\,x + \cos 2\,x)$$

$$\therefore y_n = \frac{1}{2}\left[10^n \sin\left(10x+\frac{n\pi}{2}\right) + 2^n \cos\left(2x+\frac{n\pi}{2}\right)\right]$$

Example 3 Find n^{th} derivative of $sin^2 x cos^3 x$

Solution: Let y = $sin^2 x cos^3 x$

$$= sin^2x cos^2x\ cosx$$
$$= \frac{1}{4} sin^2 2x\ cosx\ = \frac{1}{8}(1 - cos4x)cosx$$
$$= \frac{1}{8} cosx - \frac{1}{8} cos4x\ cosx$$
$$= \frac{1}{8} cosx - \frac{1}{16}(cos3x + cos5x)$$
$$= \frac{1}{16}(2cosx - cos3x - cos5x)$$
$$\therefore y_n = \frac{1}{16}\left[2cos\left(x + \frac{n\pi}{2}\right) - 3^n cos\left(3x + \frac{n\pi}{2}\right) - 5^n cos\left(5x + \frac{n\pi}{2}\right)\right]$$

Example 4 Find the n^{th} derivative of sin^4x

Solution: Let $y = sin^4x\ = (sin^2x)^2$
$$= \left(\frac{1}{2} 2\ sin^2x\right)^2$$
$$= \frac{1}{4}((1 - \cos 2x)^2$$
$$= \frac{1}{4}\left[1 - 2cos2x + \frac{1}{2}(2cos^2 2x)\right]$$
$$= \frac{1}{4}\left[1 - 2cos2x + \frac{1}{2}(1 + cos4x)\right]$$
$$= \frac{3}{8} - \frac{1}{2} cos2x + \frac{1}{8} cos4x$$
$$\therefore y_n = -\frac{1}{2} 2^n cos\left(2x + \frac{n\pi}{2}\right) + \frac{1}{8} 4^n cos\left(4x + \frac{n\pi}{2}\right)$$

Example 5 Find the n^{th} derivative of $e^{3x}\cos x\ \sin^2 2x$

Solution: Let $y = e^{3x}\cos x\ \sin^2 2x$

Now $\cos x\ \sin^2 2x\ = \frac{1}{2}\ (\cos x - \cos x \cos 4x)$
$$\because \sin^2 2x\ = \frac{1}{2}(1 - \cos 4x)$$
$$= \frac{1}{2}\left(\cos x - \frac{1}{2}(\cos 5x\ + \cos 3x\right)$$
$$\Rightarrow\ y\ = e^{3x}\cos x\ \sin^2 2x = \frac{1}{2} e^{3x}\cos x - \frac{1}{4} e^{3x}\cos 5x - \frac{1}{4} e^{3x}\cos 3x$$
$$\therefore y_n = \frac{1}{2} e^{3x}\ (9+1)^{\frac{n}{2}}\ \cos\left(x + n\tan^{-1}\frac{1}{3}\right) - \frac{1}{4} e^{3x}\ (9+25)^{\frac{n}{2}}\ \cos\left(5x + n\tan^{-1}\frac{5}{3}\right)$$
$$- \frac{1}{4} e^{3x}\ (9+9)^{\frac{n}{2}}\ \cos\left(3x + n\tan^{-1}\frac{3}{3}\right)$$
$$= \frac{1}{2} e^{3x}\ 10^{\frac{n}{2}}\ \cos\left(x + n\tan^{-1}\frac{1}{3}\right) - \frac{1}{4} e^{3x}\ 34^{\frac{n}{2}}\ \cos\left(5x + n\tan^{-1}\frac{5}{3}\right)$$
$$- \frac{1}{4} e^{3x}\ 18^{\frac{n}{2}}\ \cos(3x + n\tan^{-1} 1)$$

Example 6 If $y\ = \sin ax\ + \cos ax$, prove that $y_n\ = a^n[1 + (-1)^n \sin 2ax\ ]^{\frac{1}{2}}$

Solution: $y\ = \sin ax\ + \cos ax$
$$\therefore y_n = a^n\left[\sin\left(ax\ + \frac{n\pi}{2}\right) + \cos\left(ax\ + \frac{n\pi}{2}\right)\right]$$

$$= a^n \left[\left\{\sin\left(ax + \frac{n\pi}{2}\right) + \cos\left(ax + \frac{n\pi}{2}\right)\right\}^2\right]^{\frac{1}{2}}$$

$$= a^n \left[\sin^2\left(ax + \frac{n\pi}{2}\right) + \cos^2\left(ax + \frac{n\pi}{2}\right) + 2\sin\left(ax + \frac{n\pi}{2}\right).\cos\left(ax + \frac{n\pi}{2}\right)\right]^{\frac{1}{2}}$$

$$= a^n[1 + \sin(2ax + n\pi)]^{\frac{1}{2}}$$

$$= a^n[1 + \sin 2ax \cos n\pi + \cos 2ax \sin n\pi\,]^{\frac{1}{2}}$$

$$= a^n[1 + (-1)^n \sin 2ax\,]^{\frac{1}{2}} \quad \because \cos n\,\pi = (-1)^n \text{ and } \sin n\,\pi = 0$$

Example 7 Find the n^{th} derivative of $\tan^{-1}\frac{x}{a}$

Solution: Let $y = \tan^{-1}\frac{x}{a}$

$$\Rightarrow y_1 = \frac{dy}{dx} = \frac{1}{a\left(1+\frac{x^2}{a^2}\right)} = \frac{a}{x^2+a^2} = \frac{a}{x^2-(ai)^2}$$

$$= \frac{a}{(x+ai)(x-ai)} = \frac{a}{2ai}\left(\frac{1}{x-ai} - \frac{1}{x+ai}\right)$$

$$= \frac{1}{2i}\left(\frac{1}{x-ai} - \frac{1}{x+ai}\right)$$

Differentiating above $(n-1)$ times w.r.t. x, we get

$$y_n = \frac{1}{2i}\left[\frac{(-1)^{n-1}(n-1)!}{(x-ai)^n} - \frac{(-1)^{n-1}(n-1)!}{(x+ai)^n}\right]$$

Substituting $x = r\,cos\theta,\ a = r\,sin\theta$ such that $\theta = \tan^{-1}\frac{x}{a}$

$$\Rightarrow y_n = \frac{(-1)^{n-1}(n-1)!}{2i}\left[\frac{1}{r^n(\cos\theta - i\,sin\theta)^n} - \frac{1}{r^n(\cos\theta + isin\theta)^n}\right]$$

$$= \frac{(-1)^{n-1}(n-1)!}{2ir^n}\left[(\cos\theta - i\,sin\theta)^{-n} - (\cos\theta + i\,sin\theta)^{-n}\right]$$

Using De Moivre's theorem, we get

$$y_n = \frac{(-1)^{n-1}(n-1)!}{2ir^n}[\cos n\theta + i\sin n\theta - \cos n\theta + i\sin n\theta]$$

$$= \frac{(-1)^{n-1}(n-1)!}{r^n}\sin n\theta$$

$$= \frac{(-1)^{n-1}(n-1)!}{\left(\frac{a}{\sin\theta}\right)^n}\sin n\theta \qquad \because a = r\,sin\theta$$

$$= \frac{(-1)^{n-1}(n-1)!}{a^n}\sin n\theta\ \sin^n\theta \qquad \text{where } \theta = \tan^{-1}\frac{a}{x}$$

Example 8 Find the n^{th} derivative of $\frac{1}{1+x+x^2}$

Solution: Let $y = \frac{1}{1+x+x^2}$

$$= \frac{1}{(x-w)(x-w^2)} \quad \text{where } w = \frac{-1+i\sqrt{3}}{2} \text{ and } w^2 = \frac{-1-i\sqrt{3}}{2}$$

Resolving into partial fractions

$$y = \frac{1}{w-w^2}\left(\frac{1}{x-w} - \frac{1}{x-w^2}\right)$$

$$= \frac{1}{i\sqrt{3}}\left(\frac{1}{x-w} - \frac{1}{x-w^2}\right) = \frac{-i}{\sqrt{3}}\left(\frac{1}{x-w} - \frac{1}{x-w^2}\right)$$

Differentiating n times w.r.t. , we get

$$y_n = \frac{-i}{\sqrt{3}}\left[\frac{(-1)^n n!}{(x-w)^{n+1}} - \frac{(-1)^n n!}{(x-w^2)^{n+1}}\right]$$

$$= \frac{-i\,(-1)^n n!}{\sqrt{3}}\left[\frac{1}{(x-w)^{n+1}} - \frac{1}{(x-w^2)^{n+1}}\right]$$

$$= \frac{i\,(-1)^{n+1} n!}{\sqrt{3}}\left[\frac{1}{\left(x+\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)^{n+1}} - \frac{1}{\left(x+\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)^{n+1}}\right]$$

$$= \frac{i\,2^{n+1}\,(-1)^{n+1} n!}{\sqrt{3}}\left[\frac{1}{\left(2x+1-i\sqrt{3}\right)^{n+1}} - \frac{1}{\left(2x+1+i\sqrt{3}\right)^{n+1}}\right]$$

Substituting $2x + 1 = r\,cos\theta$, $\sqrt{3} = r\,sin\theta$ such that $\theta = \tan^{-1}\frac{\sqrt{3}}{2x+1}$

$$y_n = \frac{i\,2^{n+1}\,(-1)^{n+1} n!}{\sqrt{3}\ r^{n+1}}\left[(cos\theta - isin\theta)^{-(n+1)} - (cos\theta + isin\theta)^{-(n+1)}\right]$$

Using De Moivre's theorem, we get

$$y_n = \frac{i\,2^{n+1}\,(-1)^{n+1} n!}{\sqrt{3}\left(\frac{\sqrt{3}}{sin\theta}\right)^{n+1}}\ [cos(n+1)\theta + i\,sin(n+1)\theta - cos(n+1)\theta + isin(n+1)\theta]$$

$$\because \sqrt{3} = r\,sin\theta$$

$$= \frac{i\,2^{n+1}\,(-1)^{n+1} n!}{\left(\sqrt{3}\right)^{n+2}}\ 2i\,sin(n+1)\theta\ sin^{n+1}\theta$$

$$= \frac{(-2)^{n+2} n!}{\sqrt{3}^{n+2}}\ sin(n+1)\theta\ sin^{n+1}\theta \quad \text{where } \theta = \tan^{-1}\frac{\sqrt{3}}{2x+1}$$

Example 9 If $y = x + \tan x$, show that $\cos^2 x\,\frac{d^2y}{dx^2} - 2y + 2x = 0$

Solution: $y = x + \tan x$

$$\Rightarrow \frac{dy}{dx} = 1 + sec^2 x$$

$$\frac{d^2y}{dx^2} = 2\sec x\,(\sec x \tan x) = 2\ sec^2 x \tan x$$

$$\therefore\ \cos^2 x\,\frac{d^2y}{dx^2} - 2y + 2x = 2\cos^2 x\,sec^2 x \tan x - 2(x + \tan x) + 2x$$

$$= 2\tan x - 2x - 2\tan x + 2x$$

$$= 0$$

Example 10 If $y = \log\left(x + \sqrt{x^2+1}\right)$, show that $(1 + x^2)\frac{d^2y}{dx^2} + x\frac{dy}{dx} = 0$

Solution: $y = \log\left(x + \sqrt{x^2+1}\right)$

$$\Rightarrow \frac{dy}{dx} = \frac{1+\frac{x}{\sqrt{1+x^2}}}{x+\sqrt{1+x^2}} = \frac{1}{\sqrt{1+x^2}}$$

$$\Rightarrow (\sqrt{1+x^2})\,\frac{\mathrm{dy}}{\mathrm{d}x} = 1$$

Differentiating both sides w.r.t. x , we get

$$(\sqrt{1+x^2})\,\frac{\mathrm{d}^2\mathrm{y}}{\mathrm{d}x^2} + \frac{x}{\sqrt{1+x^2}}\frac{\mathrm{dy}}{\mathrm{d}x} = 0$$

$$\Rightarrow (1+x^2)\,\frac{\mathrm{d}^2\mathrm{y}}{\mathrm{d}x^2} + x\frac{dy}{dx} = 0$$

Exercise 1 A

1. Find the n^{th} derivative of $\frac{x^4}{(x-1)(x-2)}$

   Ans. $(-1)^n n!\left[\frac{16}{(\mathrm{x}-2)^{n+1}} - \frac{1}{(\mathrm{x}-1)^{n+1}}\right]$

2. Find the n^{th} derivative of $\cos x\ \cos 2x \cos 3x$

   Ans.$\frac{1}{4}\left[2^n \cos\left(2x + \frac{n\pi}{2}\right) + 4^n \cos\left(4x + \frac{n\pi}{2}\right) + 6^n \cos\left(6x + \frac{n\pi}{2}\right)\right]$

3. If $x = \sin t$, $y = \sin at$, show that $(1-x^2)\frac{\mathrm{d}^2\mathrm{y}}{\mathrm{d}x^2} - x\frac{dy}{dx} + a^2y = 0$

4. If $p^2 = a^2 cos^2\,\theta + b^2 sin^2\,\theta$, show that $p + \frac{\mathrm{d}^2p}{\mathrm{d}\theta^2} = \frac{a^2b^2}{p^3}$

5. If $y = \frac{x}{x^2+a^2}$, find y_n i.e. the n^{th} derivative of y

   Ans. $\frac{(-1)^n n!}{a^{n+1}} cos(n+1)\theta\ sin^{n+1}\theta$ where $\theta = \tan^{-1}\frac{x}{a}$

6. If $y = e^x sin^2\,x$, find y_n i.e. the n^{th} derivative of y

   Ans. $\frac{1}{2}\ e^x\left[1 - 16\left(2\mathrm{x} + \frac{n\pi}{2}\right)\right]$

7. Find n^{th} differential coefficient of $y = \log[(ax + b)(cx + d)]$

   Ans. $y_n = (-1)^{n-1}(n-1)!\left[\frac{a^n}{(ax+b)^n} + \frac{c^n}{(cx+d)^n}\right]$

8. If y = $x\log\frac{x-1}{x+1}$, show that $y_n = (-1)^{n-1}(n-2)!\left[\frac{x-n}{(x-1)^n} + \frac{x+n}{(x+1)^n}\right]$

9. If $y = \tan^{-1}\frac{\sqrt{1+x^2}-1}{x}$, show that $y_n = \frac{1}{2}(-1)^{n-1}(n-1)!\sin n\,\theta\ sin^n\theta$

1.2 LEIBNITZ'S THEOREM

If u and v are functions of x such that their n^{th}derivatives exist, then the n^{th}derivative of their product is given by

$$(u\,v)_n = u_n v + n_{C_1} u_{n-1} v_1 + n_{C_2} u_{n-2} v_2 + \cdots + n_{C_r} u_{n-r} v_r + \cdots + u v_n$$

where u_r and v_r represent r^{th}derivatives of u and v respectively.

Example11 Find the n^{th} derivative of $x \log x$

Solution: Let $u = \log x$ and $v = x$

Then $u_n = (-1)^{n-1}\frac{(n-1)!}{x^n}$ and $u_{n-1} = (-1)^{n-2}\frac{(n-2)!}{x^{n-1}}$

By Leibnitz's theorem, we have

$$(u\,v)_n = u_n v + n_{C_1} u_{n-1} v_1 + n_{C_2} u_{n-2} v_2 + \cdots + n_{C_r} u_{n-r} v_r + \cdots + u v_n$$

$$\Rightarrow (x \log x)_n = (-1)^{n-1}\frac{(n-1)!}{x^n} x + n(-1)^{n-2}\frac{(n-2)!}{x^{n-1}} + 0$$

$$= (-1)^{n-1}\frac{(n-1)!}{x^{n-1}} + n(-1)^{n-2}\frac{(n-2)!}{x^{n-1}}$$

$$= (-1)^{n-2}\frac{(n-2)!}{x^{n-1}}[-(n-1)+n]$$

$$= (-1)^{n-2}\frac{(n-2)!}{x^{n-1}}$$

Example 12 Find the n^{th} derivative of $x^2 e^{3x} \sin 4x$

Solution: Let $u = e^{3x} \sin 4x$ and $v = x^2$

$$\text{Then } u_n = e^{3x}\, 25^{\frac{n}{2}} \sin\left(4x + n\tan^{-1}\frac{4}{3}\right)$$

$$= e^{3x}\, 5^n \sin\left(4x + n\tan^{-1}\frac{4}{3}\right)$$

By Leibnitz's theorem, we have

$$(u\,v)_n = u_n v + n_{C_1} u_{n-1} v_1 + n_{C_2} u_{n-2} v_2 + \cdots + n_{C_r} u_{n-r} v_r + \cdots + u v_n$$

$$\Rightarrow \left(x^2 e^{3x} \sin 4x\right)_n = x^2 e^{3x}\, 5^n \sin\left(4x + n\tan^{-1}\frac{4}{3}\right) +$$

$$2nx e^{3x}\, 5^{n-1} \sin\left(4x + (n-1)\tan^{-1}\frac{4}{3}\right) +$$

$$n(n-1) e^{3x}\, 5^{n-2} \sin\left(4x + (n-2)\tan^{-1}\frac{4}{3}\right) + 0$$

$$= e^{3x}5^n \left[x^2 \sin\left(4x + n\tan^{-1}\frac{4}{3}\right) + \frac{2nx}{5}\sin\left(4x + (n-1)\tan^{-1}\frac{4}{3}\right) + \frac{n(n-1)}{25}\sin\left(4x + (n-2)\tan^{-1}\frac{4}{3}\right)\right]$$

Example 13 If $y = a\cos(\log x) + b\, sin(\log x)$, show that

$$x^2 y_{n+2} + (2n+1)xy_{n+1} + n(n+1)y_n = 0$$

Solution: Here $y = a\cos(\log x) + b\sin(\log x)$

$$\Rightarrow y_1 = \frac{-a}{x}\sin(\log x) + \frac{b}{x}\cos(logx)$$

$$\Rightarrow xy_1 = -a\sin(logx) + b\, cos(logx)$$

Differentiating both sides w.r.t. x , we get

$$xy_2 + y_1 = -\frac{a}{x}\cos(logx) + \frac{-b}{x}\ sin(logx)$$

$$\Rightarrow x^2y_2 + xy_1 = -\{a\cos(logx) + b\ sin(logx)\}$$

$$= -y$$

$$\Rightarrow x^2y_2 + xy_1 + y = 0$$

Using Leibnitz's theorem, we get

$$(y_{n+2}x^2 + n_{C_1}y_{n+1}2x + n_{C_2}y_n.2) + \left(y_{n+1}x + n_{C_1}y_n.1\right) + y_n = 0$$

$$\Rightarrow y_{n+2}x^2 + y_{n+1}2nx + n(n-1)y_n + y_{n+1}x + ny_n + y_n = 0$$

$$\Rightarrow x^2y_{n+2} + (2n+1)xy_{n+1} + (n^2+1)y_n = 0$$

Example 14 If $y = \log(x + \sqrt{1+x^2})$

Prove that $(1+x^2)y_{n+2} + (2n+1)xy_{n+1} + n^2y_n = 0$

Solution: $y = \log(x + \sqrt{1+x^2}$

$$\Rightarrow y_1 = \frac{1}{x+\sqrt{1+x^2}}\left(1 + \frac{1}{2\sqrt{1+x^2}}2x\right) = \frac{1}{\sqrt{1+x^2}}$$

$$\Rightarrow (1+x^2){y_1}^2 = 1$$

Differentiating both sides w.r.t. x, we get

$$(1+x^2)2y_1y_2 + 2x{y_1}^2 = 0$$

$$\Rightarrow (1+x^2)y_2 + xy_1 = 0$$

Using Leibnitz's theorem

$$\left[y_{n+2}(1+x^2)+n_{C_1}y_{n+1}2x+n_{C_2}y_n.2\right]+\left(y_{n+1}x+n_{C_1}y_n.1\right)=0$$

$$\Rightarrow y_{n+2}(1+x^2)+y_{n+1}2nx+n(n-1)y_n+y_{n+1}x+ny_n=0$$

$$\Rightarrow (1+x^2)y_{n+2}+(2n+1)xy_{n+1}+n^2y_n=0$$

Example 15 If $y=\sin\,(m\,sin^{-1}x)$, show that

$(1-x^2)y_{n+2}=(2n+1)xy_{n+1}+(n^2-m^2)y_n$. Also find $y_n(0)$

Solution: Here $y=\sin(m\,sin^{-1}x)$ ......①

$$\Rightarrow y_1=\frac{m}{\sqrt{1-x^2}}\cos(m\,sin^{-1}x) \quad ......②$$

$$\Rightarrow (1-x^2){y_1}^2=\mathrm{m}^2\cos^2(m\,sin^{-1}x)$$

$$\Rightarrow (1-x^2){y_1}^2=\mathrm{m}^2[1-\sin^2(m\,sin^{-1}x)]$$

$$\Rightarrow (1-x^2){y_1}^2=m^2(1-y^2)......③$$

$$\Rightarrow (1\text{-}x^2){y_1}^2+m^2y^2=m^2$$

Differentiating w.r.t. x, we get

$$(1-x^2)2y_1y_2+{y_1}^2(-2x)+m^22yy_1=0$$

$$\Rightarrow (1-x^2)y_2-xy_1+m^2y=0$$

Using Leibnitz's theorem, we get

$$\left[y_{n+2}(1-x^2)+n_{C_1}y_{n+1}(-2x)+n_{C_2}y_n(-2)\right]-\left(y_{n+1}x+n_{C_1}y_n1\right)+m^2y_n=0$$

$$\Rightarrow y_{n+2}(1-x^2)-y_{n+1}2nx-n(n-1)y_n-(y_{n+1}x+ny_n)+m^2y_n=0$$

$$\Rightarrow (1-x^2)y_{n+2}=(2n+1)xy_{n+1}+(n^2-m^2)y_n......④$$

Putting $x=0$ in ①, ②and ③

$y(0)=\ 0, y_1(0)=m$ and $y_2(0)=0$

Putting $x=0$ in ④

$y_{n+2}(0)=(n^2-m^2)y_n(0)$

Putting $n=1,2,3 \ldots \ldots$ in the above equation, we get

$y_3(0)=(1^2-m^2)y_1(0)$

$=(1^2-m^2)m \qquad \because y_1(0)=m$

$$y_4(0) = (2^2 - m^2)y_2(0)$$

$$= 0 \qquad \because y_2(0) = 0$$

$$y_5(0) = \left(3^2 - m^2\right) y_3(0)$$

$$= m(1^2 - m^2)(3^2 - m^2)$$

$$\vdots$$

$$\Rightarrow \quad y_n(0) = \begin{cases} 0\,, & if\ n\ is\ even \\ m(1^2 - m^2)(3^2 - m^2) \dots [(n-2)^2) - m^2]\,, & if\ n\ is\ odd \end{cases}$$

Example 16 If $y = e^{msin^{-1}x}$, show that
$(1 - x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2 + m^2)y_n = 0$. Also find $y_n(0)$.

Solution: Here $y = e^{msin^{-1}x}$ …①

$$\Rightarrow y_1 = \frac{m}{\sqrt{1-x^2}}\, e^{m\,sin^{-1}x}$$

$$= \frac{my}{\sqrt{1-x^2}} \quad \ldots\ldots②$$

$$\Rightarrow (1 - x^2){y_1}^2 = m^2y^2$$

Differentiating above equation w.r.t. x , we get

$$(1 - x^2)2y_1y_2 + {y_1}^2(-2x) = m^2 2yy_1$$

$$\Rightarrow (1 - x^2)y_2 - xy_1 - m^2y = 0 \qquad \ldots\ldots③$$

Differentiating above equation n times w.r.t. x using Leibnitz's theorem, we get

$$[y_{n+2}(1 - x^2) + n_{C_1}y_{n+1}(-2x) + n_{C_2}y_n(-2)] - \left(y_{n+1}x + n_{C_1}y_n 1\right) - m^2y_n = 0$$

$$\Rightarrow y_{n+2}(1 - x^2) - y_{n+1}2nx - n(n-1)y_n - (y_{n+1}x + ny_n) - m^2y_n = 0$$

$$\Rightarrow (1 - x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2 + m^2)y_n = 0 \ldots\ldots④$$

To find $y_n(0)$: Putting $x = 0$ in ①, ② and ③

$y(0) = 1$, $y_1(0) = m$ and $y_2(0) = m^2$

Also putting $x = 0$ in , we get

$$y_{n+2}(0) = (n^2 + m^2)y_n(0)$$

Putting $n = 1,2,3 \ldots$ in the above equation, we get

$y_3(0) = (1^2 + m^2)y_1(0)$

$=(1^2 + m^2)m \qquad \because y_1(0) = m$

$y_4(0) = (2^2 + m^2)y_2(0)$

$= m^2(2^2 + m^2) \qquad \because y_2(0) = m^2$

$y_5(0) = \left(3^2 + m^2\right)y_3(0)$

$= m(1^2 + m^2)(3^2 + m^2)$

$\vdots$

$$\Rightarrow \quad y_n(0) = \begin{cases} m^2(2^2 + m^2) \ldots [(n-2)^2 + m^2] \;, & if\ n\ is\ even \\ m(1^2 + m^2)(3^2 + m^2) \ldots [(n-2)^2 + m^2] \,, & if\ n\ is\ odd \end{cases}$$

Example 17 If $y = tan^{-1}x$, show that

$(1 - x^2)y_{n+2} + 2(n+1)xy_{n+1} + n(n+1)y_n = 0$. Also find $y_n(0)$

Solution: Here $y = tan^{-1}x$.....①

$\Rightarrow y_1 = \frac{1}{1+x^2}$ ......②

$y_2 = \frac{-2x}{1+x^2}$

$\Rightarrow (1 + x^2)y_2 + 2xy_1 = 0$ ......③

Differentiating equation ③ n times w.r.t. x using Leibnitz's theorem

$$[y_{n+2}(1 + x^2) + n_{C_1}y_{n+1}(2x) + n_{C_2}y_n(2)] + 2\left(y_{n+1}x + n_{C_1}y_n 1\right) = 0$$

$$\Rightarrow y_{n+2}(1 + x^2) + y_{n+1}2nx + n(n-1)y_n + 2(y_{n+1}x + ny_n) = 0$$

$$\Rightarrow (1 + x^2)y_{n+2} + 2(n+1)xy_{n+1} + n(n+1)y_n = 0 \ldots\ldots ④$$

To find $y_n(0)$: Putting $x = 0$ in ①, ②and ③, we get

$y(0) = 0$, $y_1(0) = 1$ and $y_2(0) = 0$

Also putting $x = 0$ in ④, we get

$y_{n+2}(0) = -n(n+1)y_n(0)$

Putting $n = 1,2,3$ ... in the above equation, we get

$y_3(0) = -1(2)y_1(0)$

$= -2 \qquad \because y_1(0) = 1$

$y_4(0) = -2(3)y_2(0)$

$= \ 0 \qquad \because y_2(0) = 0$

$y_5(0) = -3(4)y_3(0)$

$= -3(4)(-2) = 4!$

$y_6(0) = -4(5)y_4(0) = 0$

$y_7(0) = -5(6)y_5(0) = -5(6)4! = -(6!)$

$\vdots$

$\Rightarrow \quad y_{2n+1}(0) = (-1)^n(2n)!$ and $\quad y_{2n}(0) = 0$

Example18 If $y = (sin^{-1}x)^2$, show that $(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - n^2y_n = 0$. Also find $y_n(0)$

Solution: Here $y = (sin^{-1}x)^2$.....①

$\Rightarrow y_1 = 2sin^{-1}x.\frac{1}{\sqrt{1-x^2}}$ ......②

Squaring both the sides, we get

$(1-x^2){y_1}^2 = 4\,(sin^{-1}x)^2$

$\Rightarrow (1-x^2){y_1}^2 = 4\,(y)^2$

Differentiating the above equation w.r.t. x, we get

$(1-x^2)2y_1y_2 + {y_1}^2(-2x) - 4y_1 = 0$

$\Rightarrow (1-x^2)y_2 + y_1(-x) - 2 = 0$ ......③

Differentiating the above equation n times w.r.t. x using Leibnitz's theorem, we get

$[y_{n+2}(1-x^2) + n_{C_1}y_{n+1}(-2x) + n_{C_2}y_n(-2)] - \left(y_{n+1}x + n_{C_1}y_n1\right) = 0$

$\Rightarrow y_{n+2}(1-x^2) - y_{n+1}2nx - n(n-1)y_n - (y_{n+1}x + ny_n) = 0$

$\Rightarrow (1-x^2)y_{n+2} - (2n+1)xy_{n+1} - y_nn^2 = 0$......④

To find $y_n(0)$: Putting $x = 0$ in ①, ②and ③, we get

$y(0) = 0$, $y_1(0) = 0$ and $y_2(0) = 2$

Also putting $x = 0$ in ④, we get

$$y_{n+2}(0) = n^2 y_n(0)$$

Putting $n = 1,2,3 \ldots$ in the above equation, we get

$$y_3(0) = 1^2 y_1(0)$$

$$= 0 \qquad \because y_1(0) = 0$$

$$y_4(0) = 2^2 y_2(0)$$

$$= 2^2 2 \qquad \because y_2(0) = 2$$

$$y_5(0) = 3^2 y_3(0) = 0$$

$$y_6(0) = 4^2 y_4(0) = 4^2 2^2 2$$

$$\vdots$$

$$\Rightarrow \quad y_n(0) = \begin{cases} 0, & if\ n\ is\ odd \\ 2.\,2^2.\,4^2 \ldots\ldots\ldots.(n-2)^2, & if\ n\ is\ even \end{cases}$$

Exercise 1 B

1 .Find y_n , if $y = x^3 \cos x$

Ans.$x^3 cos\left(x + \frac{n\pi}{2}\right) + 3nx^2 cos\left[x + \frac{1}{2}(n-1)\pi\right] + 3n(n-1)xcos\left[x + \frac{1}{2}(n-2)\pi\right] + n(n-1)(n-2)cos\left[x + \frac{1}{2}(n-3)\pi\right]$

2. Find y_n , if $y = x^2 e^x \cos x$

Ans. $2^{\frac{n}{2}} e^x cos\left(x + \frac{n\pi}{4}\right)$

3. If $y^{\frac{1}{m}} + y^{\frac{-1}{m}} = 2x$, prove that $(x^2 - 1)y_{n+2} + (2n+1)xy_{n+1} + (n^2 - m^2)y_n = 0$

4. If $y\sqrt{1+x^2} = \log\left(x + \sqrt{1+x^2}\right)$, prove that

$$(1+x^2)y_{n+2} + (2n+3)xy_{n+1} + (n+1)^2 y_n = 0$$

5. If $y = \left[x + \sqrt{1+x^2}\right]^m$, prove that $(x^2+1)y_{n+2} + (2n+1)xy_{n+1} + (n^2 - m^2)y_n = 0$

6 If $y = (sinh^{-1}x)^2$, show that

$(1+x^2)y_{n+2} + (2n+1)xy_{n+1} + n^2 y_n = 0$. Also find $y_n(0)$.

Ans. $y_{2n+1}(0) = 0$ and $y_{2n}(0) = (-1)^{n-1} 2.\,2^2.\,4^2 \ldots\ldots\ldots.(2n-2)^2$

7. If $y = \cos(m\, sin^{-1}x)$, show that

$(1-x^2)y_{n+2} = (2n+1)xy_{n+1} + (n^2 - m^2)y_n$. Also find $y_n(0)$.

8. If $f(x) = \tan x$, prove that $f^n(0) - n_{C_2} f^{n-2}(0) + n_{C_4} f^{n-4}(0) - \cdots \ = sin\frac{n\pi}{2}$

(Partial Derivatives)

1. Partial derivatives

The concept of partial derivative plays a vital role in differential calculus. The different ways of limit discussed in the previous section, yields different type of partial derivatives of a function.

1.1. Definitions. Consider a real valued function $z = f(x, y)$ defined on $E \subset \mathbb{R}^2$ such that E contains a neighbourhood of $(a, b) \in \mathbb{R}^2$. Let Δa be a change in a. If the limit,

$$\lim_{\Delta a \to 0} \frac{f(a + \Delta a, b) - f(a, b)}{\Delta a}$$

exists, then it is called the *partial derivative of f with respect to x at (a, b)* and is denoted by $\left.\frac{\partial f}{\partial x}\right|_{(a,b)}$ or $f_x(a, b)$ or $z_x(a, b)$. Similarly, let Δb be a change in b. If the limit,

$$\lim_{\Delta b \to 0} \frac{f(a, b + \Delta b) - f(a, b)}{\Delta b}$$

exists, then it is called the *partial derivative of f with respect to y at (a, b)* and is denoted by $\left.\frac{\partial f}{\partial y}\right|_{(a,b)}$ or $f_y(a, b)$ or $z_y(a, b)$.

Notations. If the partial derivatives f_x and f_y exist at each point of E, then they are also the real valued functions on E. Further, we can obtain the partial derivatives of these functions, if they are differentiable. In these cases, we fix up the following notations.

$$f_{xx} = \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right), \qquad f_{yy} = \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right),$$
$$f_{yx} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right); \text{ and } f_{xy} = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right).$$

The notations of derivatives of order greater than two should be clear from the above pattern.

1.2. Remark. As we have seen in the above example, in general, f_{xy} and f_{yx} need not be equal, even if they exist. The following proposition gives a sufficient condition for them to be equal. We accept it without proof. However, we shall be dealing only with the functions f for which these two are equal.

1.3. Proposition. *Consider a real valued function $z = f(x, y)$ defined on $E \subset \mathbb{R}^2$ such that E contains a neighbourhood of $(a, b) \in \mathbb{R}^2$. If f_{xy} and f_{yx} exist and are continuous, then $f_{xy} = f_{yx}$.*

Throughout this chapter our blanket assumption will be that the operation of taking partial derivatives is commutative. That is, for our function f of two variables, $f_{xy} = f_{yx}$. In general, we may assume that the second derivatives of functions exists and are continuous, so that, the Proposition 1.3 ensures our requirement.

1.4. Example. For $u = x^3 - 3xy^2$, prove that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$. Also prove that $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$.

SOLUTION. Here $u = x^3 - 3xy^2$. Hence,

$$\frac{\partial u}{\partial x} = 3x^2 - 3y^2; \quad \frac{\partial u}{\partial y} = -6xy; \quad \frac{\partial^2 u}{\partial x \partial y} = -6y = \frac{\partial^2 u}{\partial y \partial x}.$$

$$\frac{\partial^2 u}{\partial x^2} = 6x; \quad \frac{\partial^2 u}{\partial y^2} = -6x.$$

Hence,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 6x - 6x = 0 \qquad \text{and} \qquad \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}.$$

□

2. Homogeneous functions

Let us observe the following expressions carefully.

(1) $f_1(x, y) = x^2y^4 - x^3y^3 + xy^5$.

(2) $f_2(x, y) = x^4y^4 - x^5y^3 + x^6y^2$.

The combined degree of x and y in each term of the first expression is 6 and that in the second expression is 8. Can we determine whether the combined degree of x and y in each term of the expression $\frac{x}{x^4+y^4}$ is same or not? It seems difficult to determine. Let us develop the following tests.

Test 1: Let us take $t = \frac{y}{x}$. Then

$$x^2y^4 - x^3y^3 + xy^5 = x^6(t^4 - t^3 + t^5) = x^6 f(t)$$

and

$$x^4y^4 - x^5y^3 + x^6y^2 = x^8(t^4 - t^3 + t^2) = x^8 g(t),$$

where f and g are functions of one variable t.

Test 2: Now, let us replace x by tx and y by ty. Then

$$f_1(tx, ty) = (tx)^2(ty)^4 - (tx)^3(ty)^3 + (tx)(ty)^5 = t^6 f_1(x, y)$$

and

$$f_2(tx, ty) = (tx)^4(ty)^4 - (tx)^5(ty)^3 + (tx)^6(ty)^2 = t^8 f_2(x, y).$$

2.1. Definitions. A function $z = f(x, y)$ is said to be a *homogeneous function of degree* r, if $f(tx, ty) = t^r f(x, y)$ for some real number r. Otherwise, f is said to be a *nonhomogeneous function.*

2.2. Example. Let $f : \mathbb{R}^2 \setminus \{(x, y) : y = -x\} \to \mathbb{R}$ defined by $f(x, y) = \frac{x-y}{x+y}$. Then prove that f is a homogeneous function of degree 0 and f_x and f_y exist at each point of the domain.

SOLUTION. Clearly, $f(tx, ty) = f(x, y) = t^0 f(x, y)$. Thus f is a homogeneous function of degree 0. Now for any $(x, y) \in \mathbb{R}^2$ with $x + y \neq 0$, we have,

$$f_x(x, y) = \frac{(x + y)(1) - (x - y)(1)}{(x + y)^2} = \frac{2y}{(x + y)^2}$$

and

$$f_y(x, y) = \frac{(x + y)(-1) - (x - y)(1)}{(x + y)^2} = \frac{-2x}{(x + y)^2}.$$

□

2.3. Example. $f : \mathbb{R}^2 \setminus \{(0, 0)\} \to \mathbb{R}$ defined by $f(x, y) = \frac{\sqrt[5]{x} - \sqrt[5]{y}}{x^3 + y^3}$ is a homogeneous function of degree $-\frac{14}{5}$.

2.4. Theorem (Euler's Theorem)**.** *State and prove Euler's Theorem*
Statement : Let $z = f(x, y)$ be a real valued function defined on $E \subset \mathbb{R}^2$. Suppose that f is a homogeneous function of degree n. If f_x and f_y exist on E, then

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nz. \tag{2.4.1}$$

PROOF. Since $z = f(x, y)$ is a homogeneous function of x, y of degree n, we can write

$$z = f(x, y) = x^n g\left(\frac{y}{x}\right). \tag{2.4.2}$$

Differentiating (2.4.2) partially with respect to x, we get,

$$\frac{\partial z}{\partial x} = nx^{n-1} g\left(\frac{y}{x}\right) + x^n g'\left(\frac{y}{x}\right)\left(-\frac{y}{x^2}\right).$$

Hence,

$$x\frac{\partial z}{\partial x} = nx^n g\left(\frac{y}{x}\right) - x^{n-1} y g'\left(\frac{y}{x}\right). \tag{2.4.3}$$

Similarly, differentiating (2.4.2) partially with respect to y, we get,

$$\frac{\partial z}{\partial y} = x^n g'\left(\frac{y}{x}\right)\frac{1}{x} = x^{n-1} g'\left(\frac{y}{x}\right).$$

Hence,

$$y\frac{\partial z}{\partial y} = yx^{n-1} g'\left(\frac{y}{x}\right). \tag{2.4.4}$$

Adding (2.4.3) and (2.4.4) we get,

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nx^n g\left(\frac{y}{x}\right) = nz.$$

This completes the proof. □

We note that the converse of Euler's Theorem also holds. That is, if a function $z = f(x, y)$ satisfies (2.4.1), on a certain domain, then it must be homogeneous on that domain.

2.5. Remark. Now onwards we shall not mention the domain of the functions under discussion. Also, whenever we use the derivatives of functions under discussion, we assume them to be sufficiently many times differentiable.

2.6. Corollary. *Let $z = f(x, y)$ be a real valued function defined on $E \subset \mathbb{R}^2$. Suppose that f is a homogeneous function of degree n and that all the second order partial derivatives of f exist and are continuous. Then prove that*

$$x^2\frac{\partial^2 z}{\partial x^2} + 2xy\frac{\partial^2 z}{\partial x \partial y} + y^2\frac{\partial^2 z}{\partial y^2} = n(n-1)z.$$

Proof. Since $z = f(x, y)$ is a homogeneous function of x, y of degree n, by Euler's Theorem,

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nz. \tag{2.6.1}$$

Differentiating (2.6.1) partially with respect to x, we have,

$$x\frac{\partial^2 z}{\partial x^2} + \frac{\partial z}{\partial x} + y\frac{\partial^2 z}{\partial x \partial y} = n\frac{\partial z}{\partial x},$$

which, on multiplication by x, gives

$$x^2\frac{\partial^2 z}{\partial x^2} + x\frac{\partial z}{\partial x} + xy\frac{\partial^2 z}{\partial x \partial y} = nx\frac{\partial z}{\partial x}.$$

Hence,

$$x^2\frac{\partial^2 z}{\partial x^2} + xy\frac{\partial^2 z}{\partial x \partial y} = (n-1)x\frac{\partial z}{\partial x}. \tag{2.6.2}$$

Similarly, differentiating (2.6.1) partially with respect to y and then multiplying the result by y, we get,

$$y^2\frac{\partial^2 z}{\partial y^2} + xy\frac{\partial^2 z}{\partial y \partial x} = (n-1)y\frac{\partial z}{\partial y}.$$

Since $\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}$, we get,

$$y^2\frac{\partial^2 z}{\partial y^2} + xy\frac{\partial^2 z}{\partial x \partial y} = (n-1)y\frac{\partial z}{\partial y}. \tag{2.6.3}$$

By adding (2.6.2) and (2.6.3) we have,

$$x^2\frac{\partial^2 z}{\partial x^2} + 2xy\frac{\partial^2 z}{\partial x \partial y} + y^2\frac{\partial^2 z}{\partial y^2} = (n-1)(x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y}) = n(n-1)z.$$

This completes the proof. $\square$

2.7. Corollary. *Let $u = u(x, y)$ be a nonhomogeneous real valued function defined on $E \subset \mathbb{R}^2$ and $z = \varphi(u)$ be homogeneous function of degree n. Then prove that*

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = n\frac{\varphi(u)}{\varphi'(u)},$$

provided $\varphi'(u) \neq 0$ for any $(x, y) \in E$.

PROOF. Since $z = \varphi(u)$ is a homogeneous function of x, y of degree n, by Euler's Theorem we have,

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nz = n\varphi(u)$$

$$\Rightarrow x\left(\varphi'(u)\frac{\partial u}{\partial x}\right) + y\left(\varphi'(u)\frac{\partial u}{\partial y}\right) = n\varphi(u) \Rightarrow x\left(\frac{\partial u}{\partial x}\right) + y\left(\frac{\partial u}{\partial y}\right) = n\frac{\varphi(u)}{\varphi'(u)}.$$

□

2.8. Corollary. *(Only statement)Let $u = u(x, y)$ be a nonhomogeneous real valued function defined on $E \subset \mathbb{R}^2$ and $z = \varphi(u)$ be homogeneous function of degree n. Then prove that*

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = \psi(u)[\psi'(u) - 1],$$

where $\psi(u) = n\frac{\varphi(u)}{\varphi'(u)}$, provided $\varphi'(u) \neq 0$ for any $(x, y) \in E$.

2.9. Example. For the following functions, verify Euler's Theorem and find $x^2\frac{\partial^2 z}{\partial x^2} + 2xy\frac{\partial^2 z}{\partial x \partial y} + y^2\frac{\partial^2 z}{\partial y^2}$.

(1) $z = x^n \log\left(\frac{y}{x}\right)$.

(2) $z = \sin^{-1}(\frac{x}{y}) + \tan^{-1}(\frac{y}{x})$.

SOLUTION. (1) Clearly, z is a homogeneous function of degree n.

$$\frac{\partial z}{\partial x} = nx^{n-1}\log\left(\frac{y}{x}\right) + x^n\frac{x}{y}\left(-\frac{y}{x^2}\right) = nx^{n-1}\log\left(\frac{y}{x}\right) - x^{n-1}$$

$$\Rightarrow x\frac{\partial z}{\partial x} = nx^n\log\left(\frac{y}{x}\right) - x^n.$$

Also,

$$\frac{\partial z}{\partial y} = x^n\left(\frac{x}{y}\right)\left(\frac{1}{x}\right) = \frac{x^n}{y} \Rightarrow y\frac{\partial z}{\partial y} = x^n.$$

Hence,

$$\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nx^n\log\left(\frac{y}{x}\right) = nz.$$

Thus Euler's Theorem is verified.

By the Corollary 2.6,

$$x^2\frac{\partial^2 z}{\partial x^2} + 2xy\frac{\partial^2 z}{\partial x \partial y} + y^2\frac{\partial^2 z}{\partial y^2} = n(n-1)z.$$

(2) Replacing x by tx and y by ty, $f(tx, ty) = \sin^{-1}(\frac{x}{y}) + \tan^{-1}(\frac{y}{x}) = t^0 f(x, y)$. Thus $z = f(x, y)$ is a homogeneous function of degree 0. Now,

$$\frac{\partial z}{\partial x} = \frac{1}{\sqrt{1 - \frac{x^2}{y^2}}}\left(\frac{1}{y}\right) + \frac{1}{1 + \frac{y^2}{x^2}}\left(\frac{-y}{x^2}\right) = \frac{1}{\sqrt{y^2 - x^2}} - \frac{y}{x^2 + y^2}$$

$$\Rightarrow x\frac{\partial z}{\partial x} = \frac{x}{\sqrt{y^2 - x^2}} - \frac{xy}{x^2 + y^2}.$$

Also,

$$\frac{\partial z}{\partial y} = \frac{1}{\sqrt{1-\frac{x^2}{y^2}}}\left(\frac{-x}{y^2}\right) + \frac{1}{1+\frac{y^2}{x^2}}\left(\frac{1}{x}\right) = \frac{-x}{y\sqrt{y^2-x^2}} + \frac{x}{x^2+y^2}$$
$$\Rightarrow y\frac{\partial z}{\partial y} = \frac{-x}{\sqrt{y^2-x^2}} + \frac{xy}{x^2+y^2}.$$

Hence,

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 0.$$

Thus Euler's Theorem is verified.

By the Corollary 2.6,

$$x^2\frac{\partial^2 z}{\partial x^2} + 2xy\frac{\partial^2 z}{\partial x \partial y} + y^2\frac{\partial^2 z}{\partial y^2} = n(n-1)z = 0,$$

as $n = 0$. □

2.10. Example. If $u = \sin^{-1}(\frac{x^2y^2}{x+y})$, then prove the following.

(1) $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 3\tan u$.

(2) $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = 3\tan u(3\sec^2 u - 1)$.

SOLUTION. Here $u = \sin^{-1}(\frac{x^2y^2}{x+y})$ is not a homogeneous function of x, y. Writing the given equation differently, we have $\sin u = \frac{x^2y^2}{x+y}$. Let $z = \varphi(u) = \sin u$. Then $z = \frac{x^2y^2}{x+y}$, which is homogeneous of degree 3. Hence by Corollary 2.7, $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 3\frac{\varphi(u)}{\varphi'(u)} = 3\frac{\sin u}{\cos u} = 3\tan u$, which proves (1). Also, by Corollary 2.8, we have,

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = 3\tan u[3\sec^2 u - 1].$$

□

3. Theorem on total differentials

Throughout this section we consider only those functions of two variables that admit continuous partial derivatives on their domain of definition. That is, if we are discussing about a function $z = f(x, y)$, then f_x, f_y exist and are continuous on the domain of f.

3.1. Theorem. *(Only statement) Let $z = f(x, y)$ be defined on E. Then*

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy.$$

4. Differentiation of composite functions

In this section we shall study the differentiation of composite functions. Let $z = f(x, y)$ be function defined on $E \subset \mathbb{R}^2$. In turn one can have $x = \phi(t)$ and $y = \psi(t)$, $t \in F \subset \mathbb{R}$. This makes f a function of one independent variable t. That is,

$$t \in F \mapsto (\phi(t), \psi(t)) \in E \mapsto f(\phi(t), \psi(t)).$$

The following theorem describes the differentiation of f with respect to t in this situation.

4.1. Theorem. *(Only statement) Let $z = f(x, y)$ be function defined on $E \subset \mathbb{R}^2$ and $x = \phi(t)$, $y = \psi(t)$, $t \in F \subset \mathbb{R}$. Then prove that $\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$.*

To extend Theorem 4.1 for functions of three variables, let $u = f(x, y, z)$ be a function of three variables with $x = x(t)$, $y = y(t)$ and $z = z(t)$. Then

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} + \frac{\partial u}{\partial z}\frac{dz}{dt}.$$

5. Change of variables

Like the composite functions we can also consider the following situation. Let $z = f(x, y)$ be function defined on $E \subset \mathbb{R}^2$ and let there be another domain $F \subset \mathbb{R}^2$ such that for each $(x, y) \in E$, $x = \phi(u, v)$, $y = \psi(u, v)$, $(u, v) \in F \subset \mathbb{R}^2$. This is nothing but the change of variable. In this case, the following theorem describes the partial derivatives of f with respect to u and v.

Now we prove Euler's Theorem for three variables. The homogeneous functions of more than two variables are defined as in Definition 2.1. More explicitly, a function $H = f(x_1, x_2, \ldots, x_n)$ of n variables is called *homogeneous* if there exists $r \in \mathbb{R}$ such that for $f(tx_1, tx_2, \ldots, tx_n) = t^r f(x_1, x_2, \ldots, x_n)$ for all $t \in \mathbb{R}$. In this case, the degree of homogeneity of H is r.

5.1. Theorem (Euler's Theorem for Three variables)**.** *Let $H = f(x, y, z)$ be a real valued homogeneous function of three variables x, y, z of degree n defined on $E \subset \mathbb{R}^3$. If f_x, f_y, f_z exist on E, then prove that*

$$x\frac{\partial H}{\partial x} + y\frac{\partial H}{\partial y} + z\frac{\partial H}{\partial z} = nH. \tag{5.1.1}$$

PROOF. Since $H = f(x, y, z)$ is homogeneous function of degree n,

$$H = x^n\varphi\left(\frac{y}{x}, \frac{z}{x}\right) = x^n\varphi(u, v),$$

where $u = \frac{y}{x}$ and $v = \frac{z}{x}$. Hence,

$$\begin{aligned}
\frac{\partial H}{\partial x} &= nx^{n-1}\varphi(u, v) + x^n\left[\frac{\partial \varphi}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial \varphi}{\partial v}\frac{\partial v}{\partial x}\right] \\
&= nx^{n-1}\varphi(u, v) + x^n\left[-\frac{y}{x^2}\frac{\partial \varphi}{\partial u} - \frac{z}{x^2}\frac{\partial \varphi}{\partial v}\right] \\
&= nx^{n-1}\varphi(u, v) - x^{n-2}y\frac{\partial \varphi}{\partial u} - x^{n-2}z\frac{\partial \varphi}{\partial v} \\
\Rightarrow x\frac{\partial H}{\partial x} &= nx^n\varphi(u, v) - x^{n-1}y\frac{\partial \varphi}{\partial u} - x^{n-1}z\frac{\partial \varphi}{\partial v}. \qquad (5.1.2)
\end{aligned}$$

Now,

$$\begin{aligned}
\frac{\partial H}{\partial y} &= x^n\left[\frac{\partial \varphi}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial \varphi}{\partial v}\frac{\partial v}{\partial y}\right] = x^n\left[\frac{1}{x}\frac{\partial \varphi}{\partial u} + 0\frac{\partial \varphi}{\partial v}\right] = x^{n-1}\frac{\partial \varphi}{\partial u} \\
\Rightarrow y\frac{\partial H}{\partial y} &= x^{n-1}y\frac{\partial \varphi}{\partial u}. \qquad (5.1.3)
\end{aligned}$$

Similarly,

$$z\frac{\partial H}{\partial z} = x^{n-1}z\frac{\partial \varphi}{\partial v}. \tag{5.1.4}$$

Adding (5.1.2), (5.1.3) and (5.1.4) we have,

$$x\frac{\partial H}{\partial x} + y\frac{\partial H}{\partial y} + z\frac{\partial H}{\partial z} = nx^n\varphi(u,v) = nH.$$

This completes the proof. □

As noted in case of the functions of two variables, here also we recall that the converse of Euler's Theorem also holds. That is, if a function $z = f(x,y)$ satisfies (5.1.1), on a certain domain, then it must be homogeneous on that domain.

5.2. Example. Find $\frac{dz}{dt}$ when $z = \sin^{-1}(x-y)$, $x = 3t,\ y = 4t^3$. Also verify by the direct substitution.

SOLUTION.

$$\begin{aligned}
\frac{dz}{dt} &= \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} \\
&= \frac{1}{\sqrt{1-(x-y)^2}}\cdot 3 - \frac{1}{\sqrt{1-(x-y)^2}}\cdot 12t^2 \\
&= \frac{3(1-4t^2)}{\sqrt{1-(x-y)^2}} \\
&= \frac{3(1-4t^2)}{\sqrt{1-(3t-4t^3)^2}} \\
&= \frac{3(1-4t^2)}{\sqrt{(1-3t+4t^3)(1+3t-4t^3)}} \\
&= \frac{3(1-4t^2)}{\sqrt{(1-t^2)(1-4t^2)^2}} = \frac{3}{\sqrt{1-t^2}}.
\end{aligned}$$

On the other hand, verifying directly by putting the values of x and y in z, we have

$$\begin{aligned}
& z = \sin^{-1}(3t-4t^3) \\
\Rightarrow & \frac{dz}{dt} = \frac{(3-12t^2)}{\sqrt{1-(3t-4t^3)^2}} = \frac{3(1-4t^2)}{\sqrt{1-(3t-4t^3)^2}} = \frac{3}{\sqrt{1-t^2}}.
\end{aligned}$$

□

5.3. Example. If $z = f(x,y)$, $x = r\cos\theta$, $y = r\sin\theta$, then prove that

$$\left[\frac{\partial z}{\partial x}\right]^2 + \left[\frac{\partial z}{\partial y}\right]^2 = \left[\frac{\partial z}{\partial r}\right]^2 + \frac{1}{r^2}\left[\frac{\partial z}{\partial \theta}\right]^2.$$

SOLUTION. Here x, y are functions of r, θ. Hence z is a composite function of r, θ. Thus,

$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r} = \cos\theta\frac{\partial z}{\partial x} + \sin\theta\frac{\partial z}{\partial y}$$

$$\Rightarrow \left[\frac{\partial z}{\partial r}\right]^2 = \cos^2\theta\left[\frac{\partial z}{\partial x}\right]^2 + 2\sin\theta\cos\theta\frac{\partial z}{\partial x}\frac{\partial z}{\partial y} + \sin^2\theta\left[\frac{\partial z}{\partial y}\right]^2. \tag{5.3.1}$$

Also,

$$\frac{\partial z}{\partial \theta} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial \theta} = -r\sin\theta\frac{\partial z}{\partial x} + r\cos\theta\frac{\partial z}{\partial y}$$

$$\Rightarrow \left[\frac{\partial z}{\partial \theta}\right]^2 = r^2\sin^2\theta\left[\frac{\partial z}{\partial x}\right]^2 - 2r^2\sin\theta\cos\theta\frac{\partial z}{\partial x}\frac{\partial z}{\partial y} + r^2\cos^2\theta\left[\frac{\partial z}{\partial y}\right]^2$$

$$\Rightarrow \frac{1}{r^2}\left[\frac{\partial z}{\partial \theta}\right]^2 = \sin^2\theta\left[\frac{\partial z}{\partial x}\right]^2 - 2\sin\theta\cos\theta\frac{\partial z}{\partial x}\frac{\partial z}{\partial y} + \cos^2\theta\left[\frac{\partial z}{\partial y}\right]^2. \tag{5.3.2}$$

Adding (5.3.1) and (5.3.2) we get,

$$\left[\frac{\partial z}{\partial r}\right]^2 + \frac{1}{r^2}\left[\frac{\partial z}{\partial \theta}\right]^2 = \left[\frac{\partial z}{\partial x}\right]^2 + \left[\frac{\partial z}{\partial y}\right]^2.$$

□

5.4. Example. If $H = f(2x-3y, 3y-4z, 4z-2x)$, then prove that

$$\frac{1}{2}\frac{\partial H}{\partial x} + \frac{1}{3}\frac{\partial H}{\partial y} + \frac{1}{4}\frac{\partial H}{\partial z} = 0.$$

Solution. Let $u = 2x-3y$, $v = 3y-4z$, $w = 4z-2x$. Then $H = f(u,v,w)$. Hence H is a composite function of x, y, z. Therefore,

$$\frac{\partial H}{\partial x} = \frac{\partial H}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial H}{\partial v}\frac{\partial v}{\partial x} + \frac{\partial H}{\partial w}\frac{\partial w}{\partial x} = 2\frac{\partial H}{\partial u} + 0\frac{\partial H}{\partial v} - 2\frac{\partial H}{\partial w} = 2\frac{\partial H}{\partial u} - 2\frac{\partial H}{\partial w}. \tag{5.4.1}$$

Also,

$$\frac{\partial H}{\partial y} = \frac{\partial H}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial H}{\partial v}\frac{\partial v}{\partial y} + \frac{\partial H}{\partial w}\frac{\partial w}{\partial y} = -3\frac{\partial H}{\partial u} + 3\frac{\partial H}{\partial v} + 0\frac{\partial H}{\partial w} = -3\frac{\partial H}{\partial u} + 3\frac{\partial H}{\partial v}. \tag{5.4.2}$$

Finally,

$$\frac{\partial H}{\partial z} = \frac{\partial H}{\partial u}\frac{\partial u}{\partial z} + \frac{\partial H}{\partial v}\frac{\partial v}{\partial z} + \frac{\partial H}{\partial w}\frac{\partial w}{\partial z} = 0\frac{\partial H}{\partial u} - 4\frac{\partial H}{\partial v} + 4\frac{\partial H}{\partial w} = -4\frac{\partial H}{\partial v} + 4\frac{\partial H}{\partial w}. \tag{5.4.3}$$

Hence,

$$\frac{1}{2}\frac{\partial H}{\partial x} + \frac{1}{3}\frac{\partial H}{\partial y} + \frac{1}{4}\frac{\partial H}{\partial z} = \frac{\partial H}{\partial u} - \frac{\partial H}{\partial w} - \frac{\partial H}{\partial u} + \frac{\partial H}{\partial v} - \frac{\partial H}{\partial v} + \frac{\partial H}{\partial w} = 0.$$

□

5.5. Example. If $z = f(x,y)$ and $u = e^x\cos y$, $v = e^x\sin y$. Then prove that $\frac{\partial f}{\partial x} = u\frac{\partial f}{\partial u} + v\frac{\partial f}{\partial v}$.

Solution. $u = e^x\cos y$, $v = e^x\sin y$. Hence,

$$u^2+v^2 = e^{2x} \Rightarrow e^x = \sqrt{u^2+v^2} \Rightarrow x = \frac{1}{2}\log(u^2+v^2).$$

Also,

$$\frac{v}{u} = \tan y \Rightarrow y = \tan^{-1}(\tfrac{v}{u}).$$

Thus x, y are functions of u, v, and so, z is a composite function of u, v. Now,

$$\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u} = \frac{\partial f}{\partial x}\left[\frac{u}{u^2+v^2}\right] + \frac{\partial f}{\partial y}\left[\frac{-v}{u^2+v^2}\right]$$

$$\Rightarrow u\frac{\partial f}{\partial u} = \left[\frac{u^2}{u^2+v^2}\right]\frac{\partial f}{\partial x} - \left[\frac{uv}{u^2+v^2}\right]\frac{\partial f}{\partial y}. \tag{5.5.1}$$

Similarly,

$$v\frac{\partial f}{\partial v} = \left[\frac{v^2}{u^2+v^2}\right]\frac{\partial f}{\partial x} + \left[\frac{uv}{u^2+v^2}\right]\frac{\partial f}{\partial y}. \tag{5.5.2}$$

Adding (5.5.1) and (5.5.2) we get, $u\frac{\partial f}{\partial u} + v\frac{\partial f}{\partial v} = \frac{\partial f}{\partial x}$. □

6. Differentiation of implicit functions

Many a times we are given an expression $f(x,y) = c$, where $c \in \mathbb{R}$ is a constant. Note here that, x and y are associated by a rule however we may not be able to write y as a function of x. In this case, we say that y is a function of x, implicitly described by $f(x,y) = c$ or y is an implicit function of x. We obtain the method of calculating $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ using the tools of partial derivatives.

6.1. Theorem. *Let a function y of x be implicitly described by $f(x,y) = c$. Then prove that*

(1) $\dfrac{dy}{dx} = -\dfrac{f_x}{f_y}$.

(2) $\dfrac{d^2y}{dx^2} = -\dfrac{f_{xx}(f_y)^2 - 2f_{xy}f_xf_y + f_{yy}(f_x)^2}{(f_y)^3}$.

Proof. We know that f is a function of x and y. Also, y is an implicit function of x. So, f is a composite function of x. Hence, differentiating the equation $f(x,y) = c$ with respect to x, we get,

$$\frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} = 0 \Rightarrow \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}} = -\frac{f_x}{f_y}.$$

This proves (1).

Now we prove (2).

$$\begin{aligned}
\frac{d^2y}{dx^2} &= \frac{d}{dx}\left(\frac{dy}{dx}\right) \\
&= \frac{d}{dx}\left(\frac{-f_x}{f_y}\right) \\
&= -\frac{f_y\frac{d}{dx}(f_x) - f_x\frac{d}{dx}(f_y)}{(f_y)^2} \\
&= -\frac{f_y\left(\frac{\partial}{\partial x}(f_x) + \frac{\partial}{\partial y}(f_x)\frac{dy}{dx}\right) - f_x\left(\frac{\partial}{\partial x}(f_y) + \frac{\partial}{\partial y}(f_y)\frac{dy}{dx}\right)}{(f_y)^2} \\
&= -\frac{f_y\left(f_{xx} + f_{xy}\left(-\frac{f_x}{f_y}\right)\right) - f_x\left(f_{yx} + f_{yy}\left(-\frac{f_x}{f_y}\right)\right)}{(f_y)^2} \\
&= -\frac{f_{xx}(f_y)^2 - f_yf_xf_{xy} - f_xf_yf_{yx} + f_{yy}(f_x)^2}{(f_y)^3}
\end{aligned}$$

$$= -\frac{f_{xx}(f_y)^2 - 2f_xf_yf_{xy} + f_{yy}(f_x)^2}{(f_y)^3}.$$

□

6.2. Example. Find $\frac{dy}{dx}$ when

(1) $x\sin(x-y) - (x+y) = 0$. (2) $x^y = y^x$.

PROOF. (1) Let $f(x,y) = x\sin(x-y)-(x+y)$. Since $f(x,y) = 0$, by the previous theorem, we have,

$$\frac{dy}{dx} = -\frac{f_x}{f_y} = -\frac{x\cos(x-y)+\sin(x-y)-1}{x\cos(x-y)(-1)-1}$$
$$= \frac{x\cos(x-y)+\sin(x-y)-1}{x\cos(x-y)+1}.$$

(2) Let $f(x,y) = x^y - y^x$. Since $f(x,y) = 0$, by the previous theorem, we have,

$$\frac{dy}{dx} = -\frac{f_x}{f_y} = -\frac{yx^{y-1} - y^x\log y}{x^y\log x - xy^{x-1}} = \frac{y^x\log y - yx^{y-1}}{x^y\log x - xy^{x-1}}.$$

□

6.3. Example. If $z = xyf(\frac{y}{x})$ and z is constant, then show that

$$\frac{f'(\frac{y}{x})}{f(\frac{y}{x})} = \frac{x[y + x\frac{dy}{dx}]}{y[y - x\frac{dy}{dx}]}.$$

SOLUTION. Let $F(x,y) = xyf(\frac{y}{x})$. Then $F(x,y) = z$, z is constant. Thus y is an implicit function of x. So,

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\frac{dy}{dx} = 0. \tag{6.3.1}$$

Now differentiating $F(x,y)$ with respect to x, we get,

$$\frac{\partial F}{\partial x} = yf(\tfrac{y}{x}) + xyf'(\tfrac{y}{x})\left(-\frac{y}{x^2}\right) = yf(\tfrac{y}{x}) - \frac{y^2}{x}f'(\tfrac{y}{x}) = \frac{y}{x}\left[xf(\tfrac{y}{x}) - yf'(\tfrac{y}{x})\right].$$

Similarly,

$$\frac{\partial F}{\partial y} = xf(\tfrac{y}{x}) + xyf'(\tfrac{y}{x})(\tfrac{1}{x}) = xf(\tfrac{y}{x}) + yf'(\tfrac{y}{x}).$$

Putting these values in (6.3.1), we have,

$$\frac{y}{x}\left[xf(\tfrac{y}{x}) - yf'(\tfrac{y}{x})\right] + xf(\tfrac{y}{x}) + yf'(\tfrac{y}{x})\frac{dy}{dx} = 0$$
$$\Rightarrow \left[y + x\frac{dy}{dx}\right]f(\tfrac{y}{x}) = \frac{y}{x}\left[y - x\frac{dy}{dx}\right]f'(\tfrac{y}{x})$$
$$\Rightarrow \frac{f'(\frac{y}{x})}{f(\frac{y}{x})} = \frac{x}{y}\left[\frac{y + x\frac{dy}{dx}}{y - x\frac{dy}{dx}}\right].$$

□

6.4. Example. If A, B and C are angles of a ΔABC such that $\sin^2 A + \sin^2 B + \sin^2 C = K$, a constant, then prove that $\frac{dB}{dC} = \frac{\tan C - \tan A}{\tan A - \tan B}$.

SOLUTION. Clearly, $A+B+C=\pi$. So, $A = \pi - (B+C)$. Therefore, $\sin A = \sin(B+C)$. Let $f(B,C) = \sin^2(B+C) + \sin^2 B + \sin^2 C - K$. Hence $f(B,C) = 0$, *i.e.*, B is an implicit function of C. So, $\frac{dB}{dC} = -\frac{f_C}{f_B}$. Also,

$$\begin{aligned}
f_B &= \frac{\partial f}{\partial B} \\
&= 2\sin(B+C)\cos(B+C) + 2\sin B\cos B \\
&= \sin 2(B+C) + \sin 2B \\
&= \sin(2\pi - 2A) + \sin 2B \\
&= -\sin 2A + \sin 2B \\
&= 2\cos(B+A)\sin(B-A) \\
&= 2\cos(\pi - C)\sin(B-A) \\
&= -2\cos C\sin(B-A) \\
&= 2\cos C\sin(A-B).
\end{aligned}$$

Similarly, we get,

$$f_C = 2\cos B\sin(A-C).$$

Hence,

$$\begin{aligned}
\frac{dB}{dC} &= -\frac{\cos B\sin(A-C)}{\cos C\sin(A-B)} \\
&= -\frac{\cos B(\sin A\cos C - \cos A\sin C)}{\cos C(\sin A\cos B - \cos A\sin B))} \\
&= -\frac{\sin A\cos B\cos C - \cos A\cos B\sin C}{\sin A\cos B\cos C - \cos A\sin B\cos C}.
\end{aligned}$$

Dividing by $\cos A\cos B\cos C$, we get,
$\frac{dB}{dC} = -\frac{\tan A - \tan C}{\tan A - \tan B} = \frac{\tan C - \tan A}{\tan A - \tan B}$.

$\square$

♣♣♣♣♣♣♣♣♣

The Jacobian

The Jacobian of a Transformation

In this section, we explore the concept of a "derivative" of a coordinate transformation, which is known as the *Jacobian* of the transformation. However, in this course, it is the *determinant* of the Jacobian that will be used most frequently.

If we let $\mathbf{u} = \langle u, v \rangle$, $\mathbf{p} = \langle p, q \rangle$, and $\mathbf{x} = \langle x, y \rangle$, then $(x, y) = T(u, v)$ is given in vector notation by

$$\mathbf{x} = T(\mathbf{u})$$

This notation allows us to extend the concept of a total derivative to the total derivative of a coordinate transformation.

Definition 5.1: A coordinate transformation $T(\mathbf{u})$ is *differentiable* at a point $\mathbf{p}$ if there exists a matrix $J(\mathbf{p})$ for which

$$\lim_{\mathbf{u}\to\mathbf{p}} \frac{||T(\mathbf{u}) - T(\mathbf{p}) - J(\mathbf{p})(\mathbf{u}-\mathbf{p})||}{||\mathbf{u}-\mathbf{p}||} = 0 \tag{1}$$

When it exists, $J(\mathbf{p})$ is the *total derivative* of $T(\mathbf{u})$ at $\mathbf{p}$.

In non-vector notation, definition 5.1 says that the total derivative at a point (p, q) of a coordinate transformation $T(u, v)$ is a matrix $J(u, v)$ evaluated at (p, q). In a manner analogous to that in section 2-5, it can be shown that this matrix is given by

$$J(u, v) = \begin{bmatrix} x_u & x_v \\ y_u & y_v \end{bmatrix}$$

(see exercise 46). The total derivative is also known as the *Jacobian Matrix* of the transformation $T(u, v)$.

EXAMPLE 1 What is the Jacobian matrix for the polar coordinate transformation?

Solution: Since $x = r\cos(\theta)$ and $y = r\sin(\theta)$, the Jacobian matrix is

$$J(r, \theta) = \begin{bmatrix} x_r & x_\theta \\ y_r & y_\theta \end{bmatrix} = \begin{bmatrix} \cos(\theta) & -r\sin(\theta) \\ \sin(\theta) & r\cos(\theta) \end{bmatrix}$$

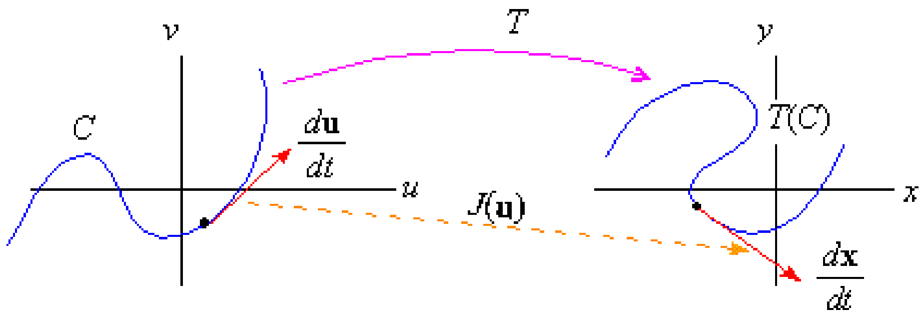
If $\mathbf{u}(t) = \langle u(t), v(t) \rangle$ is a curve in the uv-plane, then $\mathbf{x}(t) = T(u(t), v(t))$ is the image of $\mathbf{u}(t)$ in the xy-plane. Moreover,

$$\frac{d\mathbf{x}}{dt} = \begin{bmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{bmatrix} = \begin{bmatrix} x_u \frac{du}{dt} + x_v \frac{dv}{dt} \\ y_u \frac{du}{dt} + y_v \frac{dv}{dt} \end{bmatrix} = \begin{bmatrix} x_u & x_v \\ y_u & y_v \end{bmatrix} \begin{bmatrix} \frac{du}{dt} \\ \frac{dv}{dt} \end{bmatrix}$$

The last vector is $d\mathbf{u}/dt$. Thus, we have shown that if $\mathbf{x}(t) = T(\mathbf{u}(t))$, then

$$\frac{d\mathbf{x}}{dt} = J(\mathbf{u})\frac{d\mathbf{u}}{dt}$$

That is, the Jacobian maps tangent vectors to curves in the uv-plane to tangent vectors to curves in the xy-plane.



In general, the Jacobian maps any tangent vector to a curve at a given point to a tangent vector to the image of the curve at the image of the point.

EXAMPLE 2 Let $T(u,v) = \left\langle u^2 - v^2, 2uv \right\rangle$

a) Find the velocity of $\mathbf{u}(t) = \left\langle t, t^2 \right\rangle$ when $t = 1$.

b) Find the Jacobian and apply it to the vector in a)

c) Find $\mathbf{x}(t) = T(\mathbf{u}(t))$ in the xy-plane and then find its velocity vector at $t = 1$. Compare to the result in (b).

Solution: a) Since $\mathbf{u}'(t) = \langle 1, 2t \rangle$, the velocity at $t = 1$ is $\mathbf{u}'(1) = \langle 1, 2 \rangle$.

b) Since $x(u,v) = u^2 - v^2$ and $y(u,v) = 2uv$, the Jacobian of $T(u,v)$ is

$$J(u,v) = \begin{bmatrix} x_u & x_v \\ y_u & y_v \end{bmatrix} = \begin{bmatrix} 2u & -2v \\ 2v & 2u \end{bmatrix}$$

Since $\mathbf{u}' = \langle 1, 2t \rangle$, we have

$$\begin{aligned} J(u,v)\,\mathbf{u}' &= \begin{bmatrix} 2u & -2v \\ 2v & 2u \end{bmatrix}\begin{bmatrix} 1 \\ 2t \end{bmatrix} \\ &= \begin{bmatrix} 2u(1) - 2v(2t) \\ 2v(1) + 2u(2t) \end{bmatrix} \\ &= \begin{bmatrix} 2u - 4tv \\ 2v + 4tu \end{bmatrix} \end{aligned}$$

Substituting $\langle u, v\rangle = \left\langle t, t^2\right\rangle$ yields

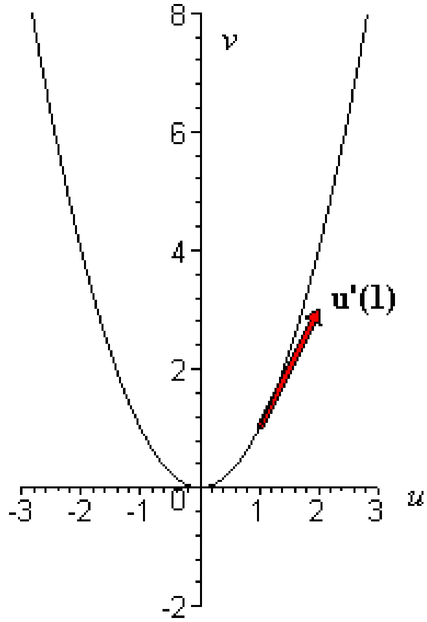
$$\mathbf{x}' = J(u,v)\,\mathbf{u}' = \begin{bmatrix} 2t - 4t\left(t^2\right) \\ 2t^2 + 4t\,(t) \end{bmatrix} = \begin{bmatrix} 2t - 4t^3 \\ 6t^2 \end{bmatrix}$$

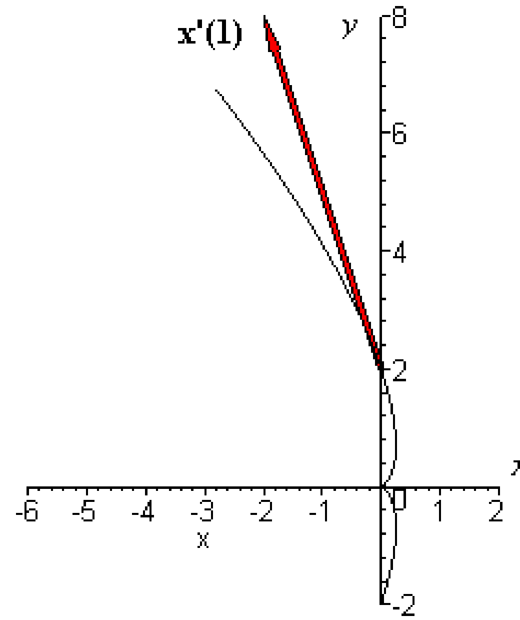
In vector form, $\mathbf{x}'(t) = \left\langle 2t - 4t^3, 6t^2\right\rangle$, so that $\mathbf{x}'(1) = \langle -2, 6\rangle$.

c) Substituting $u = t$, $v = t^2$ into $T(u,v) = \left\langle u^2 - v^2, 2uv\right\rangle$ results in

$$\mathbf{x}(t) = \left(t^2 - t^4, 2t^3\right)$$

which has a velocity of $\mathbf{x}'(t) = \left\langle 2t - 4t^3, 6t^2\right\rangle$. Moreover, $\mathbf{x}'(1) = \langle -2, 6\rangle$.





Check your Reading: At what point in the xy-plane is $\mathbf{x}'(1)$ tangent to the curve?

The Jacobian Determinant

The determinant of the Jacobian matrix of a transformation is given by

$$\det(J) = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$$

However, we often use a notation for $\det(J)$ that is more suggestive of how the determinant is calculated.

$$\frac{\partial(x,y)}{\partial(u,v)} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$$

The remainder of this section explores the Jacobian determinant and some of its more important properties.

EXAMPLE 3 Calculate the Jacobian Determinant of

$$T(u,v) = \left\langle u^2 - v, u^2 + v \right\rangle$$

Solution: If we identify $x = u^2 - v$ and $y = u^2 + v$, then

$$\begin{aligned} \frac{\partial(x,y)}{\partial(u,v)} &= \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u} \\ &= (2u)(1) - (-1)(2u) \\ &= 4u \end{aligned}$$

Before we consider applications of the Jacobian determinant, let's develop some of the its properties. To begin with, if $x(u,v)$ and $y(u,v)$ are differentiable functions, then

$$\begin{aligned} \frac{\partial(y,x)}{\partial(u,v)} &= \frac{\partial y}{\partial u}\frac{\partial x}{\partial v} - \frac{\partial y}{\partial v}\frac{\partial x}{\partial u} \\ &= -\left(\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}\right) \\ &= -\frac{\partial(x,y)}{\partial(u,v)} \end{aligned}$$

from which it follows immediately that

$$\frac{\partial(x,x)}{\partial(u,v)} = \frac{\partial(y,y)}{\partial(u,v)} = 0$$

Similarly, if $f(u,v)$, $g(u,v)$, and $h(u,v)$ are differentiable, then

$$\begin{aligned} \frac{\partial(f+g,h)}{\partial(u,v)} &= \frac{\partial(f+g)}{\partial u}\frac{\partial h}{\partial v} - \frac{\partial(f+g)}{\partial v}\frac{\partial h}{\partial u} \\ &= \frac{\partial f}{\partial u}\frac{\partial h}{\partial v} + \frac{\partial g}{\partial u}\frac{\partial h}{\partial v} - \frac{\partial f}{\partial v}\frac{\partial h}{\partial u} - \frac{\partial g}{\partial v}\frac{\partial h}{\partial u} \\ &= \left(\frac{\partial f}{\partial u}\frac{\partial h}{\partial v} - \frac{\partial f}{\partial v}\frac{\partial h}{\partial u}\right) + \left(\frac{\partial g}{\partial u}\frac{\partial h}{\partial v} - \frac{\partial g}{\partial v}\frac{\partial h}{\partial u}\right) \\ &= \frac{\partial(f,h)}{\partial(u,v)} + \frac{\partial(g,h)}{\partial(u,v)} \end{aligned}$$

The remaining properties in the next theorem can be obtained in similar fashion.

Theorem 5.2: If $f(u,v)$, $g(u,v)$, and $h(u,v)$ are differentiable functions and k is a number, then

$$\begin{array}{ll} \frac{\partial(g,f)}{\partial(u,v)} = -\frac{\partial(f,g)}{\partial(u,v)} & \frac{\partial(f+g,h)}{\partial(u,v)} = \frac{\partial(f,h)}{\partial(u,v)} + \frac{\partial(g,h)}{\partial(u,v)} \\ \frac{\partial(f,f)}{\partial(u,v)} = 0 & \frac{\partial(f-g,h)}{\partial(u,v)} = \frac{\partial(f,h)}{\partial(u,v)} + \frac{\partial(g,h)}{\partial(u,v)} \\ \frac{\partial(kf,g)}{\partial(u,v)} = k\frac{\partial(f,g)}{\partial(u,v)} & \frac{\partial(fg,h)}{\partial(u,v)} = \frac{\partial(f,h)}{\partial(u,v)}\, g + f\, \frac{\partial(g,h)}{\partial(u,v)} \end{array}$$

These and additional properties will be explored in the exercises.

EXAMPLE 4 Verify the property

$$\frac{\partial(fg,h)}{\partial(u,v)} = \frac{\partial(f,h)}{\partial(u,v)}\, g + f\, \frac{\partial(g,h)}{\partial(u,v)}$$

Solution: Direct calculation leads to

$$\begin{aligned} \frac{\partial(fg,h)}{\partial(u,v)} &= \frac{\partial(fg)}{\partial u}\frac{\partial h}{\partial v} - \frac{\partial(fg)}{\partial v}\frac{\partial h}{\partial u} \\ &= \left(\frac{\partial f}{\partial u}\, g + f\, \frac{\partial g}{\partial u}\right)\frac{\partial h}{\partial v} - \left(\frac{\partial f}{\partial v}\, g + f\, \frac{\partial g}{\partial v}\right)\frac{\partial h}{\partial u} \\ &= \left(\frac{\partial f}{\partial u}\, \frac{\partial h}{\partial v} - \frac{\partial f}{\partial v}\frac{\partial h}{\partial u}\right) g + f\left(\frac{\partial g}{\partial u}\, \frac{\partial h}{\partial v} - \frac{\partial g}{\partial v}\frac{\partial h}{\partial u}\right) \\ &= \frac{\partial(f,h)}{\partial(u,v)}\, g + f\, \frac{\partial(g,h)}{\partial(u,v)} \end{aligned}$$

Check Your Reading: If k is constant and $f(u,v)$ is differentiable, then what is

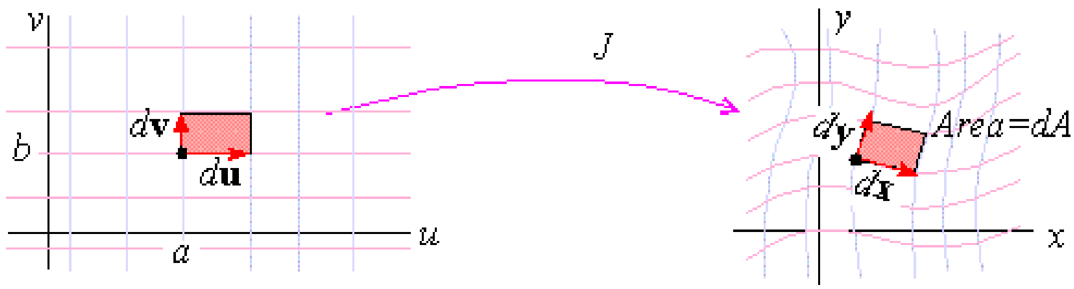
$$\frac{\partial(k,f)}{\partial(u,v)}?$$

The Area Differential

Let $T(u,v)$ be a smooth coordinate transformation with Jacobian $J(u,v)$, and let R be the rectangle spanned by $\mathbf{du} = \langle du, 0\rangle$ and $\mathbf{dv} = \langle 0, dv\rangle$. If du and dv are sufficiently close to 0, then $T(R)$ is approximately the same as the parallelogram spanned by

$$\begin{aligned} d\mathbf{x} &= J(u,v)\, d\mathbf{u} = \langle x_u du, y_u du, 0\rangle \\ d\mathbf{y} &= J(u,v)\, d\mathbf{v} = \langle x_v dv, y_v dv, 0\rangle \end{aligned}$$

If we let dA denote the area of the parallelogram spanned by $d\mathbf{x}$ and $d\mathbf{y}$, then dA approximates the area of $T(R)$ for du and dv sufficiently close to 0.



The cross product of $d\mathbf{x}$ and $d\mathbf{y}$ is given by

$$d\mathbf{x} \times d\mathbf{y} = \left\langle 0, 0, \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} \right\rangle dudv$$

from which it follows that

$$dA = \|d\mathbf{x} \times d\mathbf{y}\| = |x_u y_v - x_v y_u| \, dudv \tag{2}$$

Consequently, the *area differential* dA is given by

$$dA = \left| \frac{\partial(x,y)}{\partial(u,v)} \right| dudv \tag{3}$$

That is, the area of a small region in the uv-plane is scaled by the Jacobian determinant to approximate areas of small images in the xy-plane.

EXAMPLE 5 Find the Jacobian determinant and the area differential of $T(u,v) = \langle u^2 - v^2, 2uv \rangle$ at $(u,v) = (1,1)$, What is the approximate area of the image of the rectangle $[1, 1.4] \times [1, 1.2]$?

Solution: The Jacobian determinant is

$$\begin{aligned} \frac{\partial(x,y)}{\partial(u,v)} &= \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u} \\ &= (2u)(2u) - (-2v)(2v) \\ &= 4u^2 + 4v^2 \end{aligned}$$

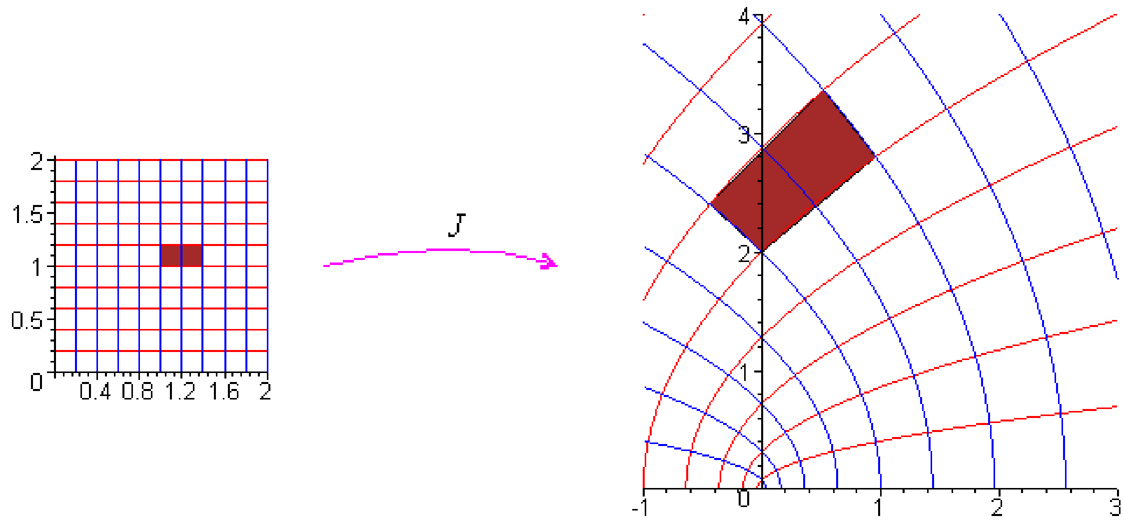
Thus, the area differential is given by

$$dA = \left| \frac{\partial(x,y)}{\partial(u,v)} \right| dudv = \left(4u^2 + 4v^2\right) dudv$$

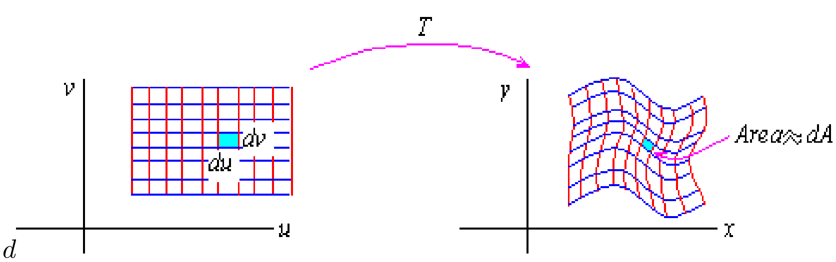
On the rectangle $[1, 1.4] \times [1, 1.2]$, the variable u changes by $du = 0.4$ and v changes by $dv = 0.2$. We evaluate the Jacobian at $(u, v) = (1, 1)$ and obtain the area

$$dA = (4 \cdot 1^2 + 4 \cdot 1^2) \cdot 0.4 \cdot 0.2 = 0.32$$

which is the approximate area in the xy-plane of the image of $[1, 1.4] \times [1, 1.2]$ under $T(u, v)$.



Let's look at another interpretation of the area differential. If the coordinate curves under a transformation $T(u, v)$ are sufficiently close together, then they form a grid of lines that are "practically straight" over short distances. As a result, sufficiently small rectangles in the uv-plane are mapped to small regions in the xy-plane that are practically the same as parallelograms.



11−6−d

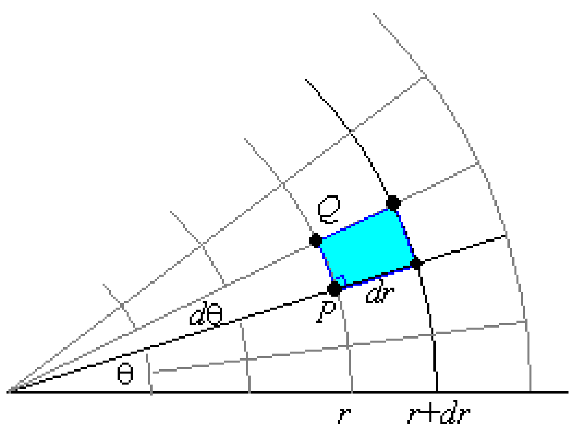
Consequently, the area differential dA approximates the area in the xy-plane of the image of a rectangle in the uv-plane as long as the rectangle in the uv-plane is sufficiently small.

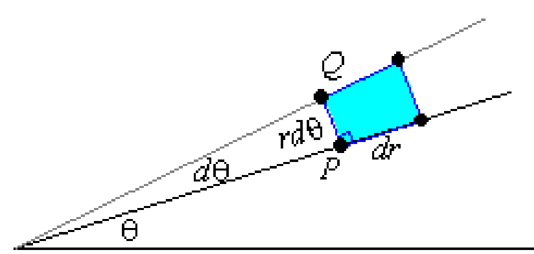
EXAMPLE 6 Find the Jacobian determinant and the area differential for the polar coordinate transformation. Illustrate using the image of a "grid" of rectangles in polar coordinates.

Solution: Since $x = r\cos(\theta)$ and $y = r\sin(\theta)$, the Jacobian determinant is

$$\begin{aligned}\frac{\partial(x,y)}{\partial(r,\theta)} &= \frac{\partial x}{\partial r}\frac{\partial y}{\partial \theta} - \frac{\partial x}{\partial \theta}\frac{\partial y}{\partial r} \\ &= \cos(\theta)\, r\cos(\theta) - -r\sin(\theta)\sin(\theta) \\ &= r\left[\cos^2(\theta) + \sin^2(\theta)\right] \\ &= r\end{aligned}$$

Thus, the area differential is $dA = r dr d\theta$.
Geometrically, "rectangles" in polar coordinates are regions between circular arcs away from the origin and rays through the origin. If the distance changes from r to $r + dr$ for $r > 0$ and some small $dr > 0$, and if the polar angle changes from θ to $\theta + d\theta$ for some small angle $d\theta$, then the region covered is practically the same as a small rectangle with height dr and width ds, which is the distance from θ to $\theta + d\theta$ along a circle of radius r.





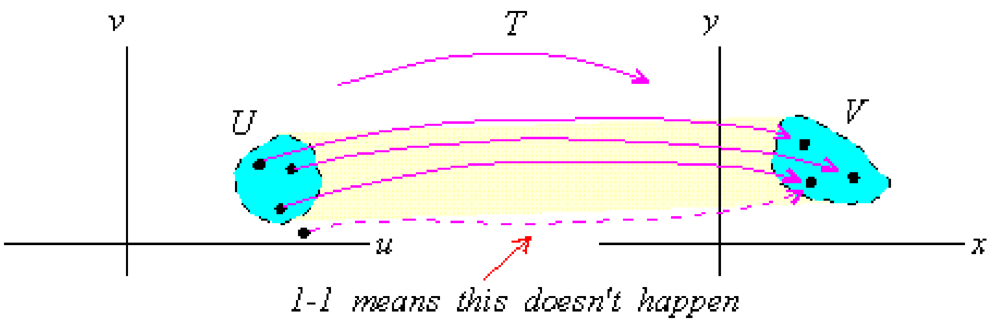
If an arc subtends an angle $d\theta$ of a circle of radius r, then the length of the arc is $ds = r d\theta$. Thus,

$$dA = dr\ ds = r dr d\theta$$

Check your Reading: Do "rectangles" in polar coordinates resemble rectangles if r is arbitrarily close to 0?

The Inverse Function Theorem

Recall taht if a coordinate transformation T maps an open region U in the uv-plane to an open region V in the xy-plane, then T is 1-1 if each point in V is the image of *only one* point in U.



Additionally, if every point in V is the image under $T(u,v)$ of at least one point in U, then $T(u,v)$ is said to map U *onto* V.

If $T(u,v)$ is a 1-1 mapping of a region U in the uv-plane **onto** a region V in the xy-plane, then we define the *inverse transformation* of T from V onto U by

$$T^{-1}(x,y) = (u,v) \quad only \quad if \quad (x,y) = T(u,v)$$

The Jacobian determinant can be used to determine if T has an inverse transformation T^{-1} on at least some small region about a given point.

> **Inverse Function Theorem:** Let $T(u,v)$ be a coordinate transformation on an open region S in the uv-plane and let (p,q) be a point in S. If
>
> $$\left.\frac{\partial(x,y)}{\partial(u,v)}\right|_{(u,v)=(p,q)} \neq 0$$
>
> then there is an open region U containing (p,q) and an open region V containing $(x,y) = T(p,q)$ such that T^{-1} exists and maps V onto U.
>
> *image*

The proof of the inverse function theorem follows from the fact that the Jacobian matrix of $T^{-1}(x,y)$, when it exists, is given by the inverse of the Jacobian of T,

$$J^{-1}(x,y) = \left(\frac{\partial(x,y)}{\partial(u,v)}\right)^{-1} \begin{bmatrix} y_v & -x_v \\ -y_u & x_u \end{bmatrix}$$

which features a Jacobian determinant with a negative power. Thus, J^{-1} exists only if the determinant of $J(u,v)$ is non-zero.

EXAMPLE 7 Where is $T(r,\theta) = \langle r\cos(\theta), r\sin(\theta)\rangle$ invertible?

Solution: The Jacobian determinant for polar coordinates is

$$\frac{\partial(x,y)}{\partial(r,\theta)} = r$$

which is non-zero everywhere except the origin. Thus, at any point (r_0, θ_0) with $r_0 > 0$, there is an open region U in the $r\theta$-plane and an open region V containing $(x,y) = (r_0\cos(\theta_0), r_0\sin(\theta_0))$ such that $T^{-1}(x,y)$ exists and maps V onto U.

We will explore the result in example 7 more fully in the exercises. In particular, we will show that

$$T^{-1}(x,y) = \left\langle \sqrt{x^2+y^2}, 2\tan^{-1}\left(\frac{y}{x+\sqrt{x^2+y^2}}\right)\right\rangle$$

Clearly, T^{-1} is not defined on any open region containing $(0,0)$. Also, if $y=0$ and $x>0$, then

$$2\tan^{-1}\left(\frac{0}{x+\sqrt{x^2+0^2}}\right) = 2\tan^{-1}\left(\frac{0}{x+|x|}\right) = 0$$

But if $y=0$ and $x<0$, then

$$2\tan^{-1}\left(\frac{0}{x+\sqrt{x^2+0^2}}\right) = 2\tan^{-1}\left(\frac{0}{x+|x|}\right) = 2\tan^{-1}\left(\frac{0}{0}\right)$$

That is, a different representation of T^{-1} must be used on any region which intersects the negative real axis.

Exercises

Find the velocity vector in the uv-plane to the given curve. Then find Jacobian matrix and the tangent vector at the corresponding point to the image of the curve in the xy-plane.

1. $T(u,v) = \langle u+v, u-v\rangle$
   $u=t,\ v=t^2$ at $t=1$
2. $T(u,v) = \langle 2u+v, 3u-v\rangle$
   $u=t,\ v=t^2$ at $t=1$
3. $T(u,v) = \langle u^2v, uv^2\rangle$
   $u=t,\ v=3t$ at $t=2$
4. $T(u,v) = \langle u^2-v^2, 2uv\rangle$
   $u=\cos(t),\ v=\sin(t)$ at $t=0$
5. $T(u,v) = \langle u\sec(v), u\tan(v)\rangle$
   $u=t,\ v=\pi$ at $t=1$
6. $T(u,v) = \langle u\cosh(v), u\sinh(v)\rangle$
   $u=t,\ v=t^2$ at $t=1$

Find the Jacobian determinant and area differential of each of the following transformations.

7. $T(u,v) = \langle u+v, u-v\rangle$
8. $T(u,v) = \langle uv, u-v\rangle$
9. $T(u,v) = \left\langle u^2 - v^2, 2uv\right\rangle$
10. $T(u,v) = \left\langle u^3 - 3uv^2, 3u^2v - v^3\right\rangle$
11. $T(u,v) = \langle ue^v, ue^{-v}\rangle$
12. $T(u,v) = \langle e^u\cos(v), e^u\sin(v)\rangle$
13. $T(u,v) = \langle 2u\cos(v), 3u\sin(v)\rangle$
14. $T(u,v) = \left\langle u^2\cos(v), u^2\sin(v)\right\rangle$
15. $T(u,v) = \langle e^u\cos(v), e^{-u}\sin(v)\rangle$
16. $T(u,v) = \langle e^u\cosh(v), e^{-u}\sinh(v)\rangle$
17. $T(u,v) = \langle \sin(u)\sinh(v), \cos(u)\cosh(v)\rangle$
18. $T(u,v) = \langle \sin(uv), \cos(uv)\rangle$

In each of the following, sketch several coordinate curves of the given coordinate system to form a grid of "rectangles" (i.e., make sure the u-curves are close enough to appear straight between the v-curves and vice-versa. Find the area differential and discuss its relationship to the "coordinate curve grid". (19 - 22 are linear transformations and have a constant Jacobian determinant)

19. $T(u,v) = \langle 2u, v\rangle$
20. $T(u,v) = \langle u+1, v\rangle$
21. $T(u,v) = \left\langle \frac{u-v}{\sqrt{2}}, \frac{u+v}{\sqrt{2}}\right\rangle$
22. $T(u,v) = \left\langle \frac{u-\sqrt{3}v}{2}, \frac{\sqrt{3}u+v}{2}\right\rangle$
23. *parabolic coordinates*
 $T(u,v) = \left\langle u^2 - v^2, 2uv\right\rangle$
22. *tangent coordinates*
 $T(u,v) = \left\langle \frac{u}{u^2+v^2}, \frac{v}{u^2+v^2}\right\rangle$
25. *elliptic coordinates*
 $T(u,v) = \langle \cosh(u)\cos(v), \sinh(u)\sin(v)\rangle$
24. *bipolar coordinates*
 $T(u,v) = \left\langle \frac{\sinh(v)}{\cosh(v)-\cos(u)}, \frac{\sin(u)}{\cosh(v)-\cos(u)}\right\rangle$

Some of the exercises below refer to the following formula for the inverse of the Jacobian:

$$J^{-1}(x,y) = \left(\frac{\partial(x,y)}{\partial(u,v)}\right)^{-1}\begin{bmatrix} y_v & -x_v \\ -y_u & x_u \end{bmatrix} \tag{4}$$

27. Find $T^{-1}(x,y)$ for the transformation

$$T(u,v) = \langle u+v, u-v\rangle$$

by letting $x = u+v$, $y = u-v$ and solving for u and v. Then find $J^{-1}(x,y)$ both (a) directly from $T^{-1}(x,y)$ and (b) from the formula (4).

28. Find $T^{-1}(x,y)$ for the transformation

$$T(u,v) = \langle u+4, u-v\rangle$$

Then find $J^{-1}(x,y)$ both (a) directly from $T^{-1}(x,y)$ and (b) from the formula (4).

29. At what points (u,v) does the coordinate transformation

$$T(u,v) = \langle e^u\cos(v), e^u\sin(v)\rangle$$

have an inverse? Can the same inverse be used over the entire uv-plane?

30. At what points (u, v) does the coordinate transformation

$$T(u, v) = \langle u \cosh(v), u \sinh(v) \rangle$$

have an inverse.

31. Show that if $T(u, v) = \langle au + bv, cu + dv \rangle$ where a, b, c, d are constants (i.e., $T(u, v)$ is a linear transformation), then $J(u, v)$ is the matrix of the linear transformation $T(u, v)$.

32. Show that if $T(u, v) = \langle au + bv, cu + dv \rangle$ where a, b, c, d are constants (i.e., $T(u, v)$ is a linear transformation), then

$$\frac{\partial(x, y)}{\partial(u, v)} = ad - bc$$

33. Show that if $f(u, v)$ is differentiable, then

$$\frac{\partial(f, f)}{\partial(u, v)} = 0$$

34. Show that if $f(u, v)$ and $g(u, v)$ are differentiable and if k is constant, then

$$\frac{\partial(kf, g)}{\partial(u, v)} = k \frac{\partial(f, g)}{\partial(u, v)}$$

35. Explain why if $x > 0$, then the inverse of the polar coordinate transformation is

$$T^{-1}(x, y) = \left\langle \sqrt{x^2 + y^2},\ \ \tan^{-1}\left(\frac{y}{x}\right) \right\rangle$$

36. The Jacobian Matrix of $(r, \theta) = T^{-1}(x, y)$ is

$$K(x, y) = \begin{bmatrix} r_x & r_y \\ \theta_x & \theta_y \end{bmatrix}$$

Find $K(x, y)$ for $T^{-1}(x, y)$ in exercise 35, and then use polar coordinates to explain its relationship to

$$J^{-1}(r, \theta) = \frac{1}{r} \begin{bmatrix} r\cos(\theta) & r\sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix}$$

37. Show that if $x < 0$, then the inverse of the polar coordinate transformation is

$$T^{-1}(x, y) = \left\langle \sqrt{x^2 + y^2},\ \ \pi + \tan^{-1}\left(\frac{y}{x}\right) \right\rangle$$

38. Use the following steps to show that if (x, y) is not at the origin or on the negative real axis, then

$$T^{-1}(x, y) = \left\langle \sqrt{x^2 + y^2}, \quad 2\tan^{-1}\left(\frac{y}{x + \sqrt{x^2 + y^2}}\right) \right\rangle$$

is the inverse of the polar coordinate transformation.

a. Verify the identity

$$\tan(\phi) = \frac{\sin(2\phi)}{1+\cos(2\phi)}$$

b. Let $\phi = \theta/2$ in a. Multiply numerator and denominator by r.
c. Simplify to an equation in x, y, and θ.

39. The coordinate transformation of rotation about the origin is given by

$$T(u,v) = \langle \cos(\theta)\,u + \sin(\theta)\,v, -\sin(\theta)\,v + \cos(\theta)\,u \rangle$$

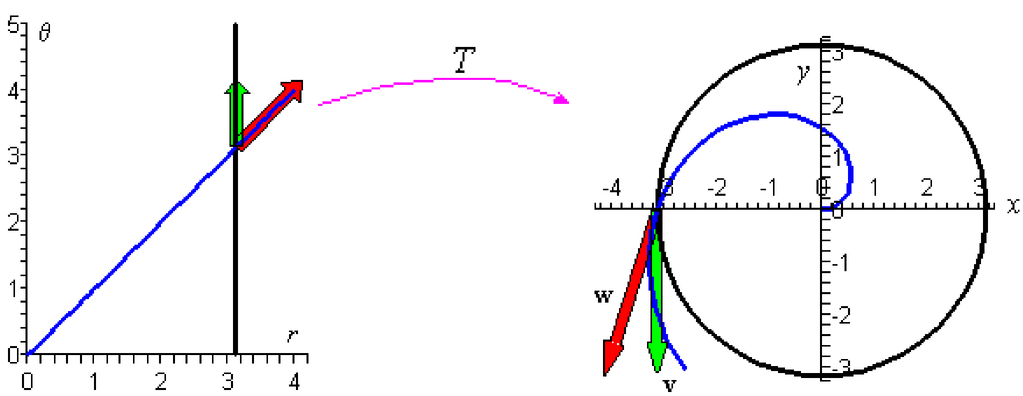
where θ is the angle of rotation. What is the Jacobian determinant and area differential for rotation through an angle θ? Explain the result geometrically.

40. The coordinate transformation of scaling horizontally by $a > 0$ and scaling vertically by $b > 0$ is given by

$$T(u,v) = \langle au, bv \rangle$$

What is its area differential? Explain the result geometrically.

41. A transformation $T(u,v)$ is said to be a *conformal transformation* if its Jacobian matrix preserves angles between tangent vectors. Consider that the vector $\langle 1,0 \rangle$ is parallel to the line $r = \pi$ and that the vector $\langle 1,1 \rangle$ is parallel to the line $r = \theta$. Also, notice that $r = \pi$ and $r = \theta$ intersect at $(r,\theta) = (\pi,\pi)$ at a $45°$ angle.



For $J(r,\theta)$ for polar coordinates, calculate

$$\mathbf{v} = J(\pi,\pi)\begin{bmatrix}1\\0\end{bmatrix} \quad \textit{and} \quad \mathbf{w} = J(\pi,\pi)\begin{bmatrix}1\\1\end{bmatrix}$$

Is the angle between $\mathbf{v}$ and $\mathbf{w}$ a $45°$ angle? Is the polar coordinate transformation conformal?

42. Find the Jacobian and repeat exercise 41 for the transformation

$$T(\rho,\theta) = \langle e^{\rho}\cos(\theta), e^{\rho}\sin(\theta) \rangle$$

43. Write to Learn: Write a short essay in which you calculate the area differential of the transformation $T(\rho,\theta) = \langle e^{\rho}\cos(\theta), e^{\rho}\sin(\theta)\rangle$ both computationally and geometrically.

44. Write to Learn: A coordinate transformation $T(u,v) = \langle f(u,v), g(u,)\rangle$ is said to be *area preserving* if the area of the image of any region R in the uv-plane is the same as the area of R. Write a short essay which uses the area differential to explain why a rotation through an angle θ is area preserving.

45. Proof of a Simplified Inverse Function Theorem: Suppose that the Jacobian determinant of $T(u,v) = \langle f(u,v), g(u,v)\rangle$ is non-zero at a point (p,q) and suppose that $\mathbf{r}(t) = \langle p+mt, q+nt\rangle$, t in $[-\varepsilon,\varepsilon]$, is a line segment in the uv-plane (m and n are numbers). Explain why if ε is sufficiently close to 0, then there is a 1-1 correspondence between the segment $\mathbf{r}(t)$ and its image $T(\mathbf{r}(t))$, t in $[-\varepsilon,\varepsilon]$. (Hint: first show that $x(t) = f(p+mt, q+nt)$ is monotone in t for t in $[-\varepsilon,\varepsilon]$).

46. Write to Learn: Let $T(u,v) = \langle x(u,v), y(u,v)\rangle$ be differentiable at $\mathbf{p} = (p,q)$ and assume that its Jacobian matrix is of the form

$$J = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

By letting $\mathbf{u} = \langle p+h, q\rangle$ in definition 5.1, (so that $\mathbf{u}-\mathbf{p} = [h \quad 0]^t$ in matrix notation), show that

$$\lim_{\mathbf{u}\to\mathbf{p}} \frac{|T(\mathbf{u}) - T(\mathbf{p}) - J(\mathbf{p})(\mathbf{u}-\mathbf{p})|}{||\mathbf{u}-\mathbf{p}||} = 0$$

is transformed into

$$\lim_{h\to 0^+} \frac{||\langle x(p+h,q) - x(p,q), y(p+h,q) - y(p,q)\rangle - \langle ah, ch\rangle||}{h} = 0$$

Use this to show that $a = x_u$ and $c = y_u$. How would you find b and d? Explain your derivations and results in a short essay.

Limits and Continuity

- The conventional approach to calculus is founded on limits.
- In this chapter, we will develop the concept of a limit by example.
- Properties of limits will be established along the way.
- We will use limits to analyze asymptotic behaviors of functions and their graphs.
- Limits will be formally defined near the end of the chapter.
- Continuity of a function (at a point and on an interval) will be defined using limits.

SECTION 2.1: AN INTRODUCTION TO LIMITS

LEARNING OBJECTIVES

• Understand the concept of (and notation for) a limit of a rational function at a point in its domain, and understand that "limits are local."
• Evaluate such limits.
• Distinguish between one-sided (left-hand and right-hand) limits and two-sided limits – and what it means for such limits to exist.
• Use numerical / tabular methods to guess at limit values.
• Distinguish between limit values and function values at a point.
• Understand the use of neighborhoods and punctured neighborhoods in the evaluation of one-sided and two-sided limits.
• Evaluate some limits involving piecewise-defined functions.

PART A: THE LIMIT OF A FUNCTION AT A POINT

Our study of calculus begins with an understanding of the expression $\lim_{x \to a} f(x)$, where a is a real number (in short, $a \in \mathbb{R}$) and f is a function. This is read as:

"the limit of $f(x)$ as x approaches a."

• **WARNING 1:** $\to$ means "approaches." Avoid using this symbol outside the context of limits.

• $\lim_{x \to a}$ is called a limit operator. Here, it is applied to the function f.

$\lim_{x \to a} f(x)$ is the real number that $f(x)$ approaches as x approaches a, **if such a number exists**. If $f(x)$ does, indeed, approach a real number, we denote that number by L (for limit value). We say the limit **exists**, and we write:

$$\lim_{x \to a} f(x) = L, \quad \text{or} \quad f(x) \to L \text{ as } x \to a.$$

These statements will be **rigorously defined** in Section 2.7.

When we **evaluate** $\lim_{x \to a} f(x)$, we do one of the following:

- We find the limit value L (in simplified form).

  We write: $\lim_{x \to a} f(x) = L$.

- We say the limit is ∞ (infinity) or $-\infty$ (negative infinity).

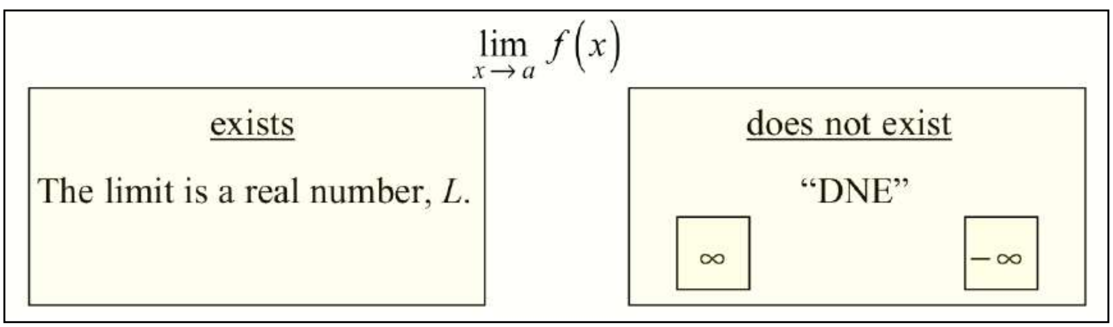
  We write: $\lim_{x \to a} f(x) = \infty$, or $\lim_{x \to a} f(x) = -\infty$.

- We say the limit **does not exist ("DNE")** in some other way.

  We write: $\lim_{x \to a} f(x)$ DNE.

  (The "DNE" notation is used by Swokowski but few other authors.)

If we say the limit is ∞ or $-\infty$, the limit is still **nonexistent**. Think of ∞ and $-\infty$ as "special cases of DNE" that we do write when appropriate; they indicate **why** the limit does not exist.



$\lim_{x \to a} f(x)$ is called a <u>limit at a point</u>, because $x = a$ corresponds to a **point** on the real number line. Sometimes, this is related to a point on the graph of f.

Example 1 (Evaluating the Limit of a Polynomial Function at a Point)

Let $f(x) = 3x^2 + x - 1$. Evaluate $\lim_{x \to 1} f(x)$.

§ Solution

f is a **polynomial** function with implied domain $\text{Dom}(f) = \mathbb{R}$.

We **substitute** ("plug in") $x = 1$ and evaluate $f(1)$.

<u>WARNING 2</u>: Sometimes, the **limit value** $\lim_{x \to a} f(x)$ does not equal the **function value** $f(a)$. (See Part C.)

$$\lim_{x\to 1} f(x) = \lim_{x\to 1}\left(3x^2 + x - 1\right)$$

WARNING 3: Use **grouping symbols** when taking the limit of an expression consisting of **more than one term**.

$$= 3(1)^2 + (1) - 1$$

WARNING 4: Do not omit the limit operator $\lim_{x\to 1}$ until this substitution phase.

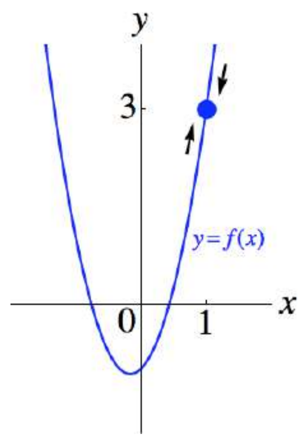
WARNING 5: When performing **substitutions**, be prepared to use **grouping symbols**. Omit them only if you are sure they are unnecessary.

$$= 3$$

We can write: $\lim_{x\to 1} f(x) = 3$, or $f(x) \to 3$ as $x \to 1$.

• Be prepared to work with function and variable names other than f and x. For example, if $g(t) = 3t^2 + t - 1$, then $\lim_{t\to 1} g(t) = 3$, also.

The graph of $y = f(x)$ is below.



Imagine that the arrows in the figure represent two lovers running towards each other along the parabola. What is the y-coordinate of the point they are approaching as they approach $x = 1$? It is 3, the limit value.

TIP 1: Remember that ***y*-coordinates** of points along the graph correspond to **function values**. §

Example 2 (Evaluating the Limit of a Rational Function at a Point)

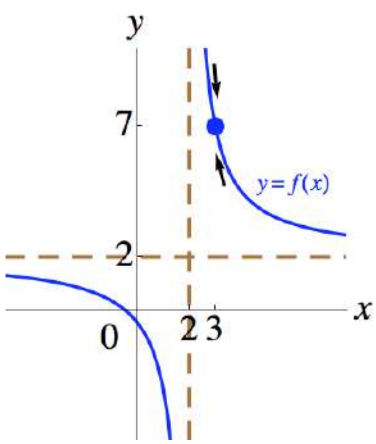
Let $f(x) = \dfrac{2x+1}{x-2}$. Evaluate $\displaystyle\lim_{x \to 3} f(x)$.

§ Solution

f is a **rational** function with implied domain $\mathrm{Dom}(f) = \{x \in \mathbb{R} \mid x \neq 2\}$.
We observe that 3 is in the **domain** of f (in short, $3 \in \mathrm{Dom}(f)$), so we **substitute** ("plug in") $x = 3$ and evaluate $f(3)$.

$$\begin{aligned} \lim_{x \to 3} f(x) &= \lim_{x \to 3} \frac{2x+1}{x-2} \\ &= \frac{2(3)+1}{(3)-2} \\ &= 7 \end{aligned}$$

The graph of $y = f(x)$ is below.



Note: As is often the case, you might not know how to draw the graph until later.

• **Asymptotes.** The dashed lines are asymptotes, which are lines that a graph approaches

- in a "long-run" sense
  (see the horizontal asymptote, or "HA," at $y = 2$), or

- in an "explosive" sense
  (see the vertical asymptote, or "VA," at $x = 2$).

"HA"s and "VA"s will be defined using limits in Sections 2.3 and 2.4, respectively.

• **"Limits are Local."** What if the lover on the left is running along the left branch of the graph? In fact, we ignore the left branch, because of the following key principle of limits.

<u>"Limits [at a Point] are Local"</u>

When analyzing $\lim_{x \to a} f(x)$, we only consider the behavior of f in the **"immediate vicinity"** of $x = a$.

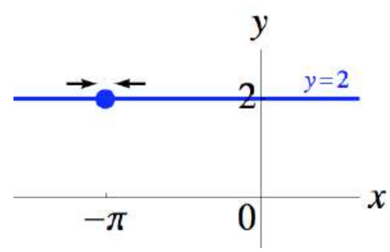
In fact, we may exclude consideration of $x = a$ itself, as we will see in Part C.

In the graph, we only care what happens **"immediately around"** $x = 3$. Section 2.7 will feature a rigorous approach. §

<u>Example 3 (Evaluating the Limit of a Constant Function at a Point)</u>

$$\lim_{x \to -\pi} 2 = 2.$$

(Observe that substituting $x = -\pi$ technically works here, since there is no "*x*" in "2," anyway.)



• **A constant approaches itself.** We can write $2 \to 2$ ("2 approaches 2") as $x \to -\pi$. When we think of a sequence of numbers approaching 2, we may think of distinct numbers such as 2.1, 2.01, 2.001, …. However, the **constant sequence** 2, 2, 2, … is also said to approach 2. §

All **constant** functions are also **polynomial** functions, and all **polynomial** functions are also **rational** functions. The following theorem applies to all three Examples thus far.

<u>Basic Limit Theorem for Rational Functions</u>

If f is a rational function, and $a \in \mathrm{Dom}(f)$,

then $\lim_{x \to a} f(x) = f(a)$.

• To evaluate the limit, substitute ("plug in") $x = a$, and evaluate $f(a)$.

We will justify this theorem in Section 2.2.

PART B: ONE- AND TWO-SIDED LIMITS; EXISTENCE OF LIMITS

$\lim\limits_{x \to a}$ is a **two-sided** limit operator in $\lim\limits_{x \to a} f(x)$, because we must consider the behavior of f as x approaches a from **both** the left **and** the right.

$\lim\limits_{x \to a^-}$ is a **one-sided** left-hand limit operator. $\lim\limits_{x \to a^-} f(x)$ is read as:

"the limit of $f(x)$ as x approaches a **from the left**."

$\lim\limits_{x \to a^+}$ is a **one-sided** right-hand limit operator. $\lim\limits_{x \to a^+} f(x)$ is read as:

"the limit of $f(x)$ as x approaches a **from the right**."

Example 4 (Using a Numerical / Tabular Approach to Guess a Left-Hand Limit Value)

Guess the value of $\lim\limits_{x \to 3^-} (x+3)$ using a **table** of function values.

§ Solution

Let $f(x) = x+3$. $\lim\limits_{x \to 3^-} f(x)$ is the real number, if any, that $f(x)$ approaches as x approaches 3 from **lesser (or lower) numbers**. That is, we approach $x = 3$ from the **left** along the real number line.

We select an **increasing** sequence of real numbers (x values) approaching 3 such that all the numbers are **close to (but less than) 3**. We evaluate the function at those numbers, and we **guess** the limit value, if any, the function values are approaching. For example:

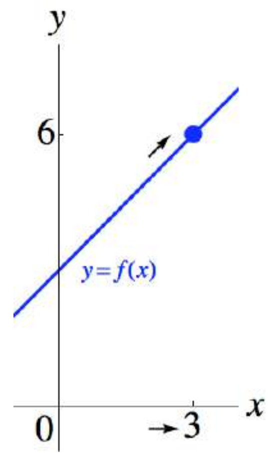
x	2.9	2.99	2.999	$\to 3^-$
$f(x) = x+3$	5.9	5.99	5.999	$\to 6$ (?)

We guess: $\lim\limits_{x \to 3^-} (x+3) = 6$.

WARNING 6: Do not confuse superscripts with signs of numbers. Be careful about associating the "$-$" superscript with negative numbers. Here, we consider **positive** numbers that are close to 3.

- If we were taking a limit as ***x* approached 0**, then we would associate the "$-$" superscript with **negative** numbers and the "+" superscript with **positive** numbers.

The graph of $y = f(x)$ is below. We only consider the behavior of f **"immediately" to the left of** $x = 3$.



WARNING 7: The numerical / tabular approach is **unreliable**, and it is typically **unacceptable** as a method for evaluating limits on exams. (See Part D, Example 11 to witness a failure of this method.) However, it may help us guess at limit values, and it strengthens our understanding of limits. §

Example 5 (Using a Numerical / Tabular Approach to Guess a Right-Hand Limit Value)

Guess the value of $\lim\limits_{x \to 3^+} (x+3)$ using a **table** of function values.

§ Solution

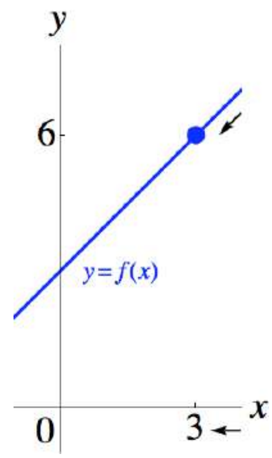
Let $f(x) = x + 3$. $\lim\limits_{x \to 3^+} f(x)$ is the real number, if any, that $f(x)$ approaches as x approaches 3 from **greater (or higher) numbers**. That is, we approach $x = 3$ from the **right** along the real number line.

We select a **decreasing** sequence of real numbers (x values) approaching 3 such that all the numbers are **close to (but greater than) 3**. We evaluate the function at those numbers, and we **guess** the limit value, if any, the function values are approaching. For example:

x	$3^+ \leftarrow$	3.001	3.01	3.1
$f(x) = x + 3$	6 (?) $\leftarrow$	6.001	6.01	6.1

We guess: $\lim\limits_{x \to 3^+} (x+3) = 6$.

The graph of $y = f(x)$ is below. We only consider the behavior of f **"immediately" to the right of** $x = 3$.



§

<u>Existence of a Two-Sided Limit at a Point</u>

$$\lim_{x \to a} f(x) = L \Leftrightarrow \left[\lim_{x \to a^-} f(x) = L, \text{ and } \lim_{x \to a^+} f(x) = L \right], \quad (a, L \in \mathbb{R}).$$

• A two-sided limit **exists** $\Leftrightarrow$ the corresponding left-hand and right-hand limits **exist**, and they are **equal**.

• If either one-sided limit **does not exist (DNE)**, or if the two one-sided limits are **unequal**, then the two-sided limit **does not exist (DNE)**.

Our guesses, $\lim_{x \to 3^-} (x+3) = 6$ and $\lim_{x \to 3^+} (x+3) = 6$, imply $\lim_{x \to 3} (x+3) = 6$.

In fact, all three limits can be evaluated by **substituting** $x = 3$ into $(x+3)$:

$$\lim_{x \to 3^-} (x+3) = 3+3 = 6; \quad \lim_{x \to 3^+} (x+3) = 3+3 = 6; \quad \lim_{x \to 3} (x+3) = 3+3 = 6.$$

This procedure is generalized in the following theorem.

<u>Extended Limit Theorem for Rational Functions</u>

If f is a rational function, and $a \in \text{Dom}(f)$,

then $\lim_{x \to a^-} f(x) = f(a)$, $\lim_{x \to a^+} f(x) = f(a)$, and $\lim_{x \to a} f(x) = f(a)$.

• To evaluate each limit, substitute ("plug in") $x = a$, and evaluate $f(a)$.

WARNING 8: Substitution might not work if f is not a rational function.

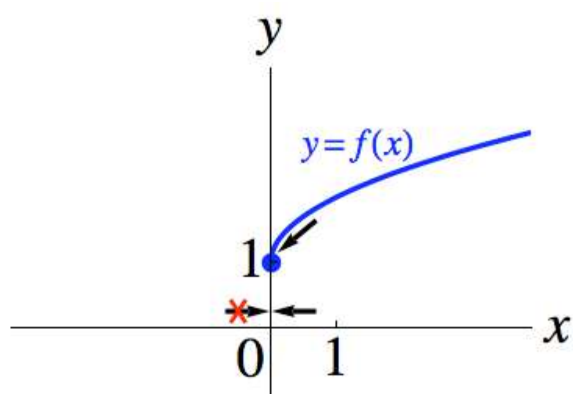
Example 6 (Pitfalls of Substituting into a Function that is Not Rational)

Let $f(x) = \sqrt{x} + 1$. Evaluate $\lim_{x \to 0^+} f(x)$, $\lim_{x \to 0^-} f(x)$, and $\lim_{x \to 0} f(x)$.

§ Solution

Observe that $\text{Dom}(f) = \{x \in \mathbb{R} \mid x \geq 0\} = [0, \infty)$, because $\sqrt{x}$ is **real** when $x \geq 0$, but it is **not real** when $x < 0$.

This is important, because *x* is only allowed to approach 0 (or whatever *a* is) **through** $\text{Dom}(f)$. Here, *x* is allowed to approach 0 from the right but **not** from the left.



Right-Hand Limit: $\lim_{x \to 0^+} f(x) = 1$.

Substituting $x = 0$ works: $\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \left(\sqrt{x} + 1\right) = \sqrt{0} + 1 = 1$.

Left-Hand Limit: $\lim_{x \to 0^-} f(x)$ does not exist (DNE).

Substituting $x = 0$ **does not work** here.

Two-Sided Limit: $\lim_{x \to 0} f(x)$ does not exist (DNE).

This is because the corresponding left-hand limit does not exist (DNE).

Observe that f is **not** a rational function, so the aforementioned theorem does **not** apply, even though $0 \in \text{Dom}(f)$. f is, however, an **algebraic** function, and we will discuss algebraic functions in Section 2.2. §

PART C: IGNORING THE FUNCTION AT *a*

Example 7 (Ignoring the Function at 'a' When Evaluating a Limit; Modifying Examples 4 and 5)

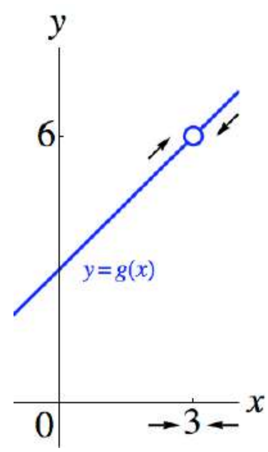
Let $g(x) = x + 3, \ (x \neq 3)$.

(We are deleting 3 from the domain of the function in Examples 4 and 5; this changes the function.)

Evaluate $\lim_{x \to 3^-} g(x)$, $\lim_{x \to 3^+} g(x)$, and $\lim_{x \to 3} g(x)$.

§ Solution

Since $3 \notin \text{Dom}(g)$, we must delete the point $(3, 6)$ from the graph of $y = x + 3$ to obtain the graph of *g* below.



We say that *g* has a removable discontinuity at $x = 3$ (see Section 2.8), and the graph of *g* has a hole at the point $(3, 6)$.

Observe that, as *x* approaches 3 from the left **and** from the right, $g(x)$ **approaches** 6, even though $g(x)$ never equals 6.

$g(3)$ is undefined, yet the following statements are true:

$$\lim_{x \to 3^-} g(x) = 6,$$
$$\lim_{x \to 3^+} g(x) = 6, \text{ and}$$
$$\lim_{x \to 3} g(x) = 6.$$

There literally **does not have to be a point** at $x = 3$ (in general, $x = a$) for these limits to exist! Observe that substituting $x = 3$ into $(x + 3)$ works. §

Example 8 (Ignoring the Function at 'a' When Evaluating a Limit; Modifying Example 7)

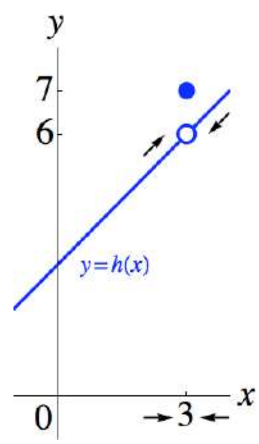
Let the function h be defined **piecewise** as follows: $h(x) = \begin{cases} x+3, & x \neq 3 \\ 7, & x = 3 \end{cases}$.

(A piecewise-defined function applies different evaluation rules to different subsets of (groups of numbers in) its domain. This type of function can lead to interesting limit problems.)

Evaluate $\lim\limits_{x \to 3} h(x)$.

§ Solution

h is identical to the function g from Example 7, except that $3 \in \mathrm{Dom}(h)$, and $h(3) = 7$. As a result, we must add the point $(3, 7)$ to the graph of g to obtain the graph of h below.



As with g, h also has a **removable discontinuity** at $x = 3$, and its graph also has a **hole** at the point $(3, 6)$.

Observe that, as x approaches 3 from the left **and** from the right, $h(x)$ also **approaches** 6.

$\lim\limits_{x \to 3} h(x) = 6$ once again, even though $h(3) = 7$.

WARNING 2 repeat (applied to f): Sometimes, the **limit value** $\lim\limits_{x \to a} f(x)$ does not equal the **function value** $f(a)$. §

As in Example 7, observe that substituting $x = 3$ into $(x+3)$ works. §

The existence (or value) of $\lim_{x \to a} f(x)$ **need not** depend on the existence (or value) of $f(a)$.

• Sometimes, it **does help** to know what $f(a)$ is when evaluating $\lim_{x \to a} f(x)$.
In Section 2.8, we will say that f is <u>continuous</u> at $a \Leftrightarrow \lim_{x \to a} f(x) = f(a)$, provided that $\lim_{x \to a} f(x)$ and $f(a)$ exist. We appreciate **continuity**, because we can then simply **substitute** $x = a$ to evaluate a limit, which was what we did when we applied the **Basic Limit Theorem for Rational Functions** in Part A.

• In Examples 7 and 8, we dealt with functions that were **not** continuous at $x = 3$, yet **substituting** $x = 3$ into $(x+3)$ allowed us to evaluate the one- and two-sided limits at $a = 3$. We will develop theorems that cover these Examples. We first need the following definitions.

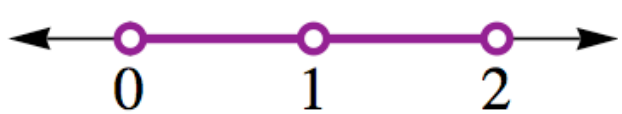
A <u>neighborhood of *a*</u> is an **open interval** along the real number line that is **symmetric** about *a*.

For example, the interval $(0, 2)$ is a **neighborhood** of 1. Since 1 is the **midpoint** of $(0, 2)$, the neighborhood is **symmetric** about 1.

A <u>punctured (or deleted) neighborhood of *a*</u> is constructed by taking a neighborhood of *a* and **deleting** *a* itself.

For example, the set $(0, 2) \setminus \{1\}$, which can be written as $(0, 1) \cup (1, 2)$, is a **punctured neighborhood** of 1. It is a set of numbers that are **"immediately around"** 1 on the real number line.

• The notation $(0, 2) \setminus \{1\}$ indicates that we can construct it by taking the **neighborhood** $(0, 2)$ and **deleting** 1.



"Puncture Theorem" for Limits of Locally Rational Functions

Let r be a rational function, and let $a \in \text{Dom}(r)$.
Let $f(x) = r(x)$ on a punctured neighborhood of $x = a$.
Then, $\lim\limits_{x \to a} f(x) = \lim\limits_{x \to a} r(x) = r(a)$.

• To evaluate the limits, substitute ("plug in") $x = a$ into $r(x)$, and evaluate $r(a)$.

• That is, if a function rule is given by a **rational** expression $r(x)$ **locally (immediately) around** $x = a$, where $a \in \text{Dom}(r)$, then **evaluate** the rational expression **at** a to obtain the **limit** of the function at a.

Refer to Examples 7 and 8. Let $r(x) = x + 3$. Observe that r is a rational function, and $3 \in \text{Dom}(r)$. Both the g and h functions were defined by $x + 3$ **locally (immediately) around** $x = 3$. More precisely, they were defined by $x + 3$ on some **punctured neighborhood** of $x = 3$, say $(2.9, 3.1) \setminus \{3\}$. Therefore,

$$\lim_{x \to 3} g(x) = \lim_{x \to 3} r(x) = r(3) = 3 + 3 = 6, \text{ and}$$

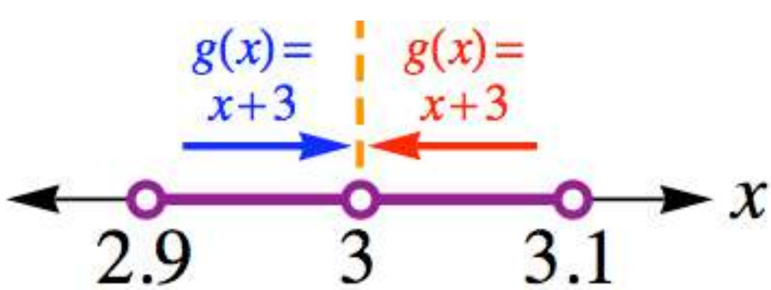
$$\lim_{x \to 3} h(x) = \lim_{x \to 3} r(x) = r(3) = 3 + 3 = 6.$$

It is easier to write:

$$\lim_{x \to 3} g(x) = \lim_{x \to 3} (x + 3) = 3 + 3 = 6, \text{ and}$$

$$\lim_{x \to 3} h(x) = \lim_{x \to 3} (x + 3) = 3 + 3 = 6.$$

The figure below refers to g, but it also applies to h.
The dashed line segment at $x = 3$ reiterates the **puncture** there.



Why does the theorem only require that a function be **locally** rational about *a*? Consider the following Example.

Example 9 (Limits are Local)

Let $f(t)=\begin{cases} t+2, & t<0 \\ \sqrt{t}, & t\geq 0 \end{cases}$. Evaluate $\lim\limits_{t\to -1} f(t)$.

§ Solution

Observe that $f(t)=t+2$ is the **only** rule that is relevant as *t* approaches -1 **locally** from the left **and** from the right. We only consider values of *t* that are **"immediately around"** $a=-1$. **"Limits are Local!"**

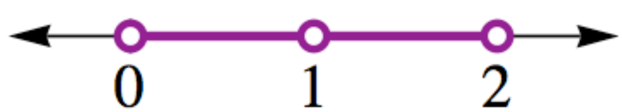
It is **irrelevant** that the rule $f(t)=\sqrt{t}$ is different, or that it is not rational. §

The following definitions will prove helpful in our study of **one-sided limits**.

A left-neighborhood of a is an **open interval** of the form (c,a), where $c<a$.
A right-neighborhood of a is an **open interval** of the form (a,c), where $c>a$.
A **punctured neighborhood** of a consists of **both** a left-neighborhood of a **and** a right-neighborhood of a.

For example, the interval $(0,1)$ is a **left-neighborhood** of 1. It is a set of numbers that are **"immediately to the left"** of 1 on the real number line.

The interval $(1,2)$ is a **right-neighborhood** of 1. It is a set of numbers that are **"immediately to the right"** of 1 on the real number line.



We now modify the "Puncture Theorem" for **one-sided limits**.

• Basically, when evaluating a **left-hand limit** such as $\lim\limits_{x\to a^-} f(x)$, we use the function rule that governs the *x*-values **"immediately to the left"** of *a* on the real number line.

• Likewise, when evaluating a **right-hand limit** such as $\lim\limits_{x\to a^+} f(x)$, we use the rule that governs the *x*-values **"immediately to the right"** of *a*.

Variation of the "Puncture Theorem" for Left-Hand Limits

Let r be a rational function, and let $a \in \mathrm{Dom}(r)$.
Let $f(x) = r(x)$ on a left-neighborhood of $x = a$.
Then, $\lim_{x \to a^-} f(x) = \lim_{x \to a^-} r(x) = r(a)$.

Variation of the "Puncture Theorem" for Right-Hand Limits

Let r be a rational function, and let $a \in \mathrm{Dom}(r)$.
Let $f(x) = r(x)$ on a right-neighborhood of $x = a$.
Then, $\lim_{x \to a^+} f(x) = \lim_{x \to a^+} r(x) = r(a)$.

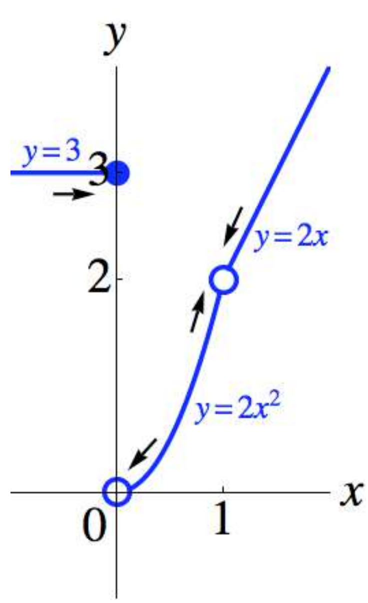
Example 10 (Evaluating One-Sided and Two-Sided Limits of a Piecewise-Defined Function)

Let $f(x) = \begin{cases} 3, & \text{if } x \le 0 \\ 2x^2, & \text{if } 0 < x < 1 \\ 2x, & \text{if } x > 1 \end{cases}$.

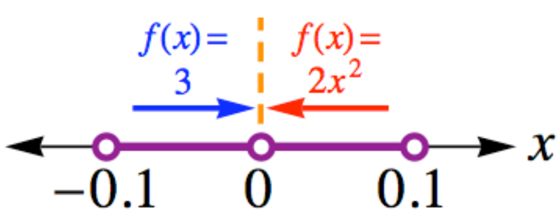
Evaluate the one-sided and two-sided limits of f at 1 and at 0.

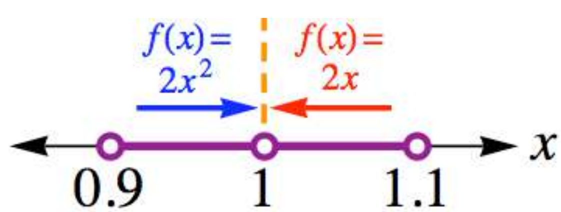
§ Solution

The graph of $y = f(x)$ is below. It helps, but it is **not** required to evaluate limits. Instead, we can evaluate limits of **relevant** function rules.



$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} 2x^2$ $= 2(1)^2$ $= 2$	The left-hand limit as $x \to 1^-$: We use the rule $f(x) = 2x^2$, because it applies to a **left-neighborhood** of 1, say $(0.9, 1)$.
$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} 2x$ $= 2(1)$ $= 2$	The right-hand limit as $x \to 1^+$: We use the rule $f(x) = 2x$, because it applies to a **right-neighborhood** of 1, say $(1, 1.1)$.
$\lim_{x \to 1} f(x) = 2$	The two-sided limit as $x \to 1$: The left-hand and right-hand limits at 1 **exist**, and they are **equal**, so the two-sided limit exists and equals their common value.
$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} 3$ $= 3$	The left-hand limit as $x \to 0^-$: We use the rule $f(x) = 3$, because it applies to a **left-neighborhood** of 0, say $(-0.1, 0)$.
$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} 2x^2$ $= 2(0)^2$ $= 0$	The right-hand limit as $x \to 0^+$: We use the rule $f(x) = 2x^2$, because it applies to a **right-neighborhood** of 0, say $(0, 0.1)$.
$\lim_{x \to 0} f(x)$ does not exist (DNE)	The two-sided limit as $x \to 0$: The left-hand and right-hand limits at 0 **exist**, but they are **unequal**, so the two-sided limit does not exist (DNE).





§

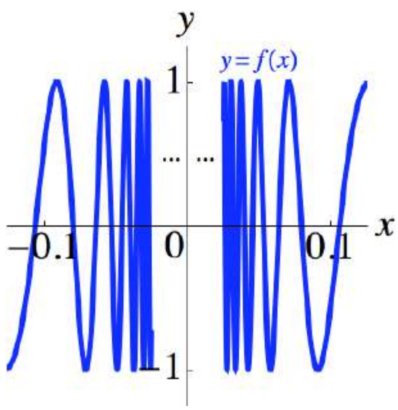
PART D: NONEXISTENT LIMITS

Example 11 (Nonexistent Limits)

Let $f(x) = \sin\left(\frac{1}{x}\right)$. Evaluate $\lim\limits_{x\to 0^+} f(x)$, $\lim\limits_{x\to 0^-} f(x)$, and $\lim\limits_{x\to 0} f(x)$.

§ Solution

The graph of $y = f(x)$ is below. Ask your instructor if s/he might have you even attempt to draw this. In a sense, the classic sine wave is being turned "inside out" relative to the *y*-axis.



As *x* approaches 0 from the right (or from the left), the function values **oscillate** between -1 and 1. They do **not** approach a **single real number**. Therefore,

$$\lim_{x\to 0^+} f(x) \text{ does not exist (DNE),}$$

$$\lim_{x\to 0^-} f(x) \text{ does not exist (DNE), and}$$

$$\lim_{x\to 0} f(x) \text{ does not exist (DNE).}$$

Note 1: The *y*-axis is **not a vertical asymptote (VA)** here, because the graph and the function values are **not "exploding" without bound** around the *y*-axis.

Note 2: Here is an example of how the **numerical / tabular approach** introduced in Part B **might lead us astray**:

x	$0^+ \leftarrow$	$\frac{1}{3\pi}$	$\frac{1}{2\pi}$	$\frac{1}{\pi}$
$f(x) = \sin\left(\frac{1}{x}\right)$	0 (?) $\leftarrow$ NO!	0	0	0

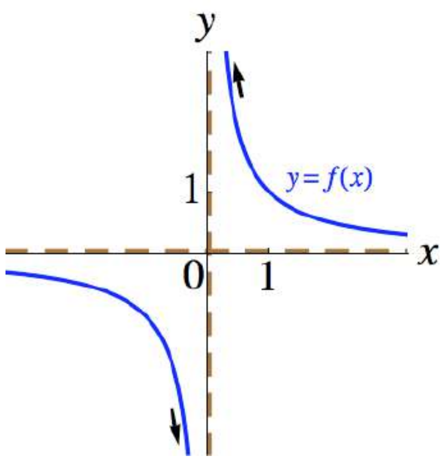
§

Example 12 (Infinite and/or Nonexistent Limits)

Let $f(x)=\frac{1}{x}$. Evaluate $\lim_{x\to 0^+} f(x)$, $\lim_{x\to 0^-} f(x)$, and $\lim_{x\to 0} f(x)$.

§ Solution

The graph of $y=f(x)$ is below. We will discuss this graph in later sections.



As x approaches 0 from the **right**, the function values **increase without bound**.

Therefore, $\lim_{x\to 0^+} f(x)=\infty$.

As x approaches 0 from the **left**, the function values **decrease without bound**.

Therefore, $\lim_{x\to 0^-} f(x)=-\infty$.

∞ and $-\infty$ are **mismatched**.

Therefore, $\lim_{x\to 0} f(x)$ does not exist (DNE).

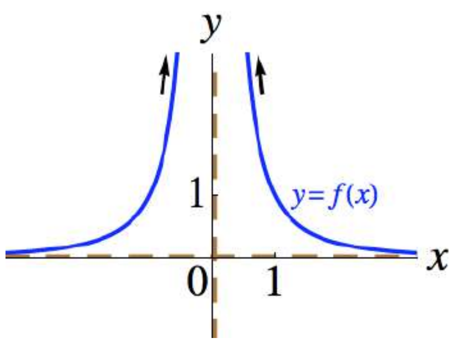
In fact, all three limits **do not exist**. For example, $\lim_{x\to 0^+} f(x)$, **does not exist**, because the function values **do not approach a single real number** as x approaches 0 from the right. The expressions ∞ and $-\infty$ indicate **why** the one-sided limits do not exist, and we write ∞ and $-\infty$ where appropriate. §

Example 13 (Infinite and Nonexistent Limits)

Let $f(x)=\frac{1}{x^2}$. Evaluate $\lim_{x\to 0^+} f(x)$, $\lim_{x\to 0^-} f(x)$, and $\lim_{x\to 0} f(x)$.

§ Solution

The graph of $y=f(x)$ is below. Observe that f is an even function.



$\lim_{x\to 0^+} f(x)=\infty$,

$\lim_{x\to 0^-} f(x)=\infty$, and

$\lim_{x\to 0} f(x)=\infty$. §

Example 14 (A Nonexistent Limit)

Let $f(x)=\dfrac{|x|}{x}$. Evaluate $\lim\limits_{x\to 0^+} f(x)$, $\lim\limits_{x\to 0^-} f(x)$, and $\lim\limits_{x\to 0} f(x)$.

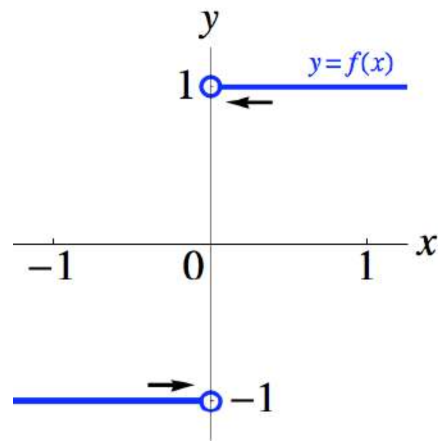
§ Solution

Note: f is **not** a rational function, but it is an **algebraic function**, since $f(x)=\dfrac{|x|}{x}=\dfrac{\sqrt{x^2}}{x}$.

Remember that: $|x|=\begin{cases} x, & \text{if } x\ge 0 \\ -x, & \text{if } x<0 \end{cases}$.

Then, $f(x)=\dfrac{|x|}{x}=\begin{cases} \dfrac{x}{x}=1, & \text{if } x>0 \\ \dfrac{-x}{x}=-1, & \text{if } x<0 \end{cases}$, and $f(0)$ is undefined.

The graph of $y=f(x)$ is below.



$\lim\limits_{x\to 0^+} f(x)=1$,

$\lim\limits_{x\to 0^-} f(x)=-1$, and

$\lim\limits_{x\to 0} f(x)$ does not exist (DNE),

due to the fact that the right-hand and left-hand limits are **unequal**. §

FOOTNOTES

1. **Limits do not require continuity.** In Section 2.8, we will discuss continuity, a property of functions that helps our lovers run along the graph of a function without having to jump or hop. In Exercises 1-3, we could imagine the lovers running towards each other (one from the left, one from the right) while staying on the graph of f and without having to jump or hop, provided they were placed on appropriate parts of the graph. Sometimes, the "run" requires jumping or hopping. Let $f(x) = \begin{cases} 0, & \text{if } x \text{ is a rational number } (x \in \mathbb{Q}) \\ x, & \text{if } x \text{ is an irrational number } (x \notin \mathbb{Q};\ \text{really, } x \in \mathbb{R} \setminus \mathbb{Q}) \end{cases}$.

   It turns out that $\lim_{x \to 0} f(x) = 0$.

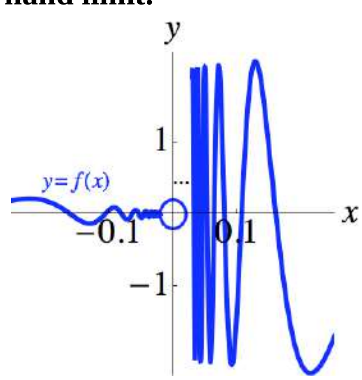
2. **Misconceptions about limits.**
   See "Why Is the Limit Concept So Difficult for Students?" by Sally Jacobs in the Fall 2002 edition (vol.24, No.1) of *The AMATYC Review*, pp.25-34.
   • Students can be misled by the use of the word "limit" in real-world contexts. For example, a speed limit is a bound that is not supposed to be exceeded; there is no such restriction on limits in calculus.
   • Limit values can sometimes be attained. For example, if a function f is continuous at $x = a$ (see Examples 1-3), then the function value takes on the limit value at $x = a$.
   • Limit values do not have to be attained. See Examples 7 and 8.

   Observations:
   • The dynamic view of limits, which involves ideas of motion and "approaching" (for example, our lovers), may be more accessible to students than the static view preferred by many textbook authors. The static view is exemplified by the formal definitions of limits we will see in Section 2.7. The dynamic view greatly assists students in transitioning to the static view and the formal definitions.
   • Leading mathematicians in 18$^{\text{th}}$- and 19$^{\text{th}}$-century Europe heatedly debated ideas of limits.

3. **Multivariable calculus.** When we go to higher dimensions, there may be more than two possible approaches (not just left-hand and right-hand) when analyzing limits at a point! Neighborhoods can take the form of disks or balls.

4. **An example where a left-hand limit exists but not the right-hand limit.**

   Let $f(x) = \dfrac{x + |x|(1+x)}{x} \sin\left(\dfrac{1}{x}\right) = \begin{cases} -x\sin\left(\dfrac{1}{x}\right), & \text{if } x < 0 \\ (2+x)\sin\left(\dfrac{1}{x}\right), & \text{if } x > 0 \end{cases}$.



   Then, $\lim_{x \to 0^-} f(x) = 0$, which can be proven by the Squeeze (Sandwich) Theorem in Section 2.6. However, $\lim_{x \to 0^+} f(x)$ does not exist (DNE).

Notes on Integral Calculus

1 Introduction and highlights

Differential calculus you learned in the past term was about differentiation. You may feel embarrassed to find out that you have already forgotten a number of things that you learned differential calculus. However, if you still remember that differential calculus was about *the rate of change, the slope of a graph, and the tangent of a curve*, you are probably OK.

- The essence of differentiation is *finding the ratio between the difference in the value of $f(x)$ and the increment in x.*

Remember, the derivative or the slope of a function is given by

$$f'(x) = \frac{df}{dx} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}. \tag{1}$$

Integral calculus that we are beginning to learn now is called integral calculus. It will be mostly about *adding an incremental process to arrive at a "total"*. It will cover three major aspects of integral calculus:

1. The meaning of integration.
    - We'll learn that integration and differentiation are inverse operations of each other. They are simply two sides of the same coin (Fundamental Theorem of Caclulus).
2. The techniques for calculating integrals.
3. The applications.

2 Sigma Sum

2.1 Addition re-learned: adding a sequence of numbers

In essence, integration is an advanced form of addition. We all started learning how to add two numbers since as young as we could remember. You might say "Are you kidding? Are you telling me that I have to start my university life by learning addition?".

The answer is positive. You will find out that doing addition is often much harder than calculating an integral. Some may even find sigma sum is the most difficult thing to learn in integral calculus. Although this difficulty is by-passed by using the Fundamental Theorem of Caclulus, you should NEVER forget that *you are actually doing a sigma sum when you are calculating an integral.* This is one secret for correctly formulating the integral in many applied problems with ease!

Now, I use a couple of examples to show that your skills in doing addition still need improvement.

Example 1a: Find the total number of logs in a triangular pile of four layers (see figure).

Solution 1a: Let the total number be S_4, where 'S' stands for 'Sum' and the subscript reminds us that we are calculating the sum for a pile of 4 layers.

$$S_4 = \underbrace{1}_{\text{in layer 1}} + \underbrace{2}_{\text{in layer 2}} + \underbrace{3}_{\text{in layer 3}} + \underbrace{4}_{\text{in layer 4}} = 10.$$

A piece of cake!

Example 1b: Now, find the total number of logs in a triangular pile of 50 layers, i.e. find S_{50}! (Give me the answer in a few seconds without using a calculator).

Solution 1b: Let's start by formulating the problem correctly.

$$S_{50} = \underbrace{1}_{\text{in layer 1}} + \underbrace{2}_{\text{in layer 2}} + \cdots + \underbrace{49}_{\text{in layer 49}} + \underbrace{50}_{\text{in layer 50}} =?$$

where '$\cdots$' had to be used to represent the numbers between 3 and 48 inclusive. This is because there isn't enough space for writing all of them down. Even if there is enough space, it is tedious and unnecessary to write all of them down since the *regularity* of this sequence makes it very clear what are the numbers that are not written down.

Still a piece of cake? Not really if you had not learned Gauss's formula. We'll have to leave it unanswered at the moment.

Example 2: Finally, find the total number of logs in a triangular pile of k layers, i.e. find S_k (k is any positive integer, e.g. $k = 8,888,888$ is one possible choice)!

Solution 2: This is equivalent to calculating the sum of the first k positive integers.

$$S_k = 1 + 2 + \cdots + (k-1) + k.$$

The only thing we can say now is that the answer must be a function of k which is the total number of integers we need to add. Again, we have to leave it unanswered at the moment.

2.2 Regular vs irregular sequences

A **sequence** is a list of numbers written in a definite order. A sequence is *regular* if each term of the sequence is uniquely determined, following a well-defined rule, by its *position/order* in the sequence (often denoted by an integer i). Very often, each term can be generated by an explicit *formula* that is expressed as a function of the position i, e.g. $f(i)$. We can call this formula the *sequence generator* or the *general term*.

For example, the ith term in the sequence of integers is identical to its location in the sequence, thus its sequence generator is $f(i) = i$. Thus, the 9th term is 9 while the 109th term is equal to 109.

Example 3: The sum of the first ten odd numbers is

$$O_{10} = 1 + 3 + 5 + \cdots + 19.$$

Find the sequence generator.

Solution 3: Note that the ith odd number is equal to the ith even number minus 1. The ith even number is simply $2i$. Thus, the ith odd number is $2i-1$, namely $f(i) = 2i-1$. To verify, 5 is the 3rd odd number, i.e. $i = 3$. Thus, $2i - 1 = 2 \times 3 - 1 = 5$ which is exactly the number we expect.

Knowing the sequence generator, we can write down the sum of the first k odd numbers for any positive integer k.

$$O_k = 1 + 3 + 5 + \cdots + (2k - 1).$$

Example 4: Find the sequence generator of the following sum of 100 products of subsequent pairs of integers.

$$P_{100} = \underbrace{1 \cdot 2}_{\text{1st term}} + \underbrace{2 \cdot 3}_{\text{2nd term}} + \underbrace{3 \cdot 4}_{\text{3rd term}} + \cdots + \underbrace{100 \cdot 101}_{\text{100th term}}$$

Solution 4: Since the ith term is equal to the number i multiplied by the subsequent integer which is equal to $i + 1$. Thus, $f(i) = i(i + 1)$.

Knowing the sequence generator, we can write down the sum of k such terms for any positive integer k.

$$P_k = 1 \cdot 2 + 2 \cdot 3 + \cdots + k(k + 1).$$

Example 4: The sequence of the first 8 digits of the irrational number $\pi = 3.1415926\ldots$ is

$$\Pi_8 = 3 + 1 + 4 + 1 + 5 + 9 + 2 + 6.$$

We cannot find the sequence generator since the sequence is irregular, we cannot express the sum of the first k digits of π (for arbitrary k).

2.3 The sigma notation

In order to short-hand the mathematical exression of the sum of a regular sequence, a convenient notation is introduced.

Definition (Σ sum): The sum of the first k terms of a sequence generated by the sequence generator $f(i)$ can be denoted by

$$S_k = f(1) + f(2) + \cdots + f(k) \equiv \sum_{i=1}^{k} f(i)$$

where the symbol Σ (the Greek equivalent of S reads "sigma") means **"take the sum of"**, the general expression for the terms to be added or the sequence generator $f(i)$ is called the **summand**, i is called the **summation index**, 1 and k are, respectively, the starting and the ending indices of the sum.

Thus,

$$\sum_{i=1}^{k} f(i)$$

means *calculate the sum of all the terms generated by the sequence generator $f(i)$ for all integers starting from $i = 1$ and ending at $i = k$.*

Note that the value of the sum is independent of the summation index i, hence i is called a "dummy" variable serving for the sole purpose of running the summation from the starting index to the ending index. Therefore, the sum only depends on the summand and both the starting and the ending indices.

Example 5: Express the sum $S_k = \sum_{i=3}^{k} i^2$ in an expanded form.

Solution 5: The sequence generator is $f(i) = i^2$. Note that the starting index is not 1 but 3!. Thus, the 1st term is $f(3) = 3^2$. The subsequent terms can be determined accordngly. Thus,

$$S_k = \sum_{i=3}^{k} i^2 = f(3) + f(4) + f(5) + \cdots + f(k) = 3^2 + 4^2 + 5^2 + \cdots + k^2.$$

An easy check for a mistake that often occurs. If you still find the "dummy" variable i in an expanded form or in the final evaluation of the sum, your answer must be WRONG.

2.4 Gauss's formula and other formulas for simple sums

Let us return to Examples 1 and 2 about the total number of logs in a triangular pile. Let's start with a pile of 4 layers. Imaging that you could (in a "thought-experiement") put an identical pile with up side down adjacent to the original pile, you obtain a pile that contains twice the number of logs that you want to calculate (see figure).

The advantage of doing this is that, in this double-sized pile, each layer contains an equal number of logs. This number is equal to number on the 1st (top) layer plus the number on the 4th (bottom) layer. In the mean time, the height of the pile remains unchanged (4). Thus, the number in this double-sized pile is $4 \times (4+1) = 20$. The sum S_4 is just half of this number which is 10.

Let's apply this idea to finding the formula in the case of k layers. Note that

$$\begin{aligned} \text{(Original)} \quad S_k &= \quad 1 \quad + \quad 2 \quad + \cdots + \; k-1 \; + \; k. && (2) \\ \text{(Inversed)} \quad S_k &= \quad k \quad + \; k-1 \; + \cdots + \quad 2 \quad + \; 1. && (3) \\ \text{(Adding the two)} \quad 2S_k &= \underbrace{(k+1)+(k+1)+\cdots+(k+1)+(k+1)}_{\text{k terms in total}} = k(k+1). && (4) \end{aligned}$$

Dividing both sides by 2, we obtain Gauss's formula for the sum of the first k positive integers.

$$S_k = \sum_{i=1}^{k} i = 1+2+\cdots+k = \frac{1}{2}k(k+1). \tag{5}$$

This actaully answered the problem in Example 2. The answer to Example 1b is even simpler.

$$S_{50} = \sum_{i=1}^{50} i = \frac{1}{2} \times 50 \times 51 = 1275.$$

The following are two important simple sums that we shall use later. One is the sum of the first k integers squared.

$$S_k = \sum_{i=1}^{k} i^2 = 1^2+2^2+\cdots+k^2 = \frac{1}{6}k(k+1)(2k+1). \tag{6}$$

The other is the sum of the first k integers cubed.

$$S_k = \sum_{i=1}^{k} i^3 = 1^3+2^3+\cdots+k^3 = \left[\frac{1}{2}k(k+1)\right]^2. \tag{7}$$

We shall not illustrate how to derive these formulas. You can find it in numerous calculus text books.

To prove that these formulas work for arbitrarily large integers k, we can use a method called mathematical induction. To save time, we'll just outline the basic ideas here.

The only way we can prove things concerning arbitrarily large numbers is to guarantee that this formula must be correct for $k = N + 1$ if it is correct for $k = N$. This is like trying to arrange a string/sequence of standing dominos. To guarantee that all the dominos in the string/sequence fall one after the other, we need to guarantee that the falling of each domino will necessarily cause the falling of the subsequent one. This is the essence of proving that if the formula is right for $k = N$, it must be right for $k = N + 1$. The last thing you need to do is to knock down the first one and keep your fingers crossed. Knocking down the first domino is equivalent to proving that the formula is correct for $k = 1$ which is very easy to check in all cases (see Keshet's notes for a prove of the sum of the first k integers squared).

2.5 Important rules of sigma sums

Rule 1: Summation involving constant summand.

$$\sum_{i=1}^{k} c = \underbrace{c + c + \cdots + c}_{\text{k terms in total}} = kc.$$

Note that: the total # of terms=ending index - starting index +1.

Rule 2: Constant multiplication: multiplying a sum by a constant is equal to multiplying each term of the sum by the same constant.

$$c\sum_{i=1}^{k} f(i) = \sum_{i=1}^{k} cf(i)$$

Rule 3: Adding two sums with identical starting and ending indices is equal to the sum of sums of the corresponding terms.

$$\sum_{i=1}^{k} f(i) + \sum_{i=1}^{k} g(i) = \sum_{i=1}^{k}[f(i) + g(i)].$$

Rule 4: Break one sum into more than one pieces.

$$\sum_{i=1}^{k} f(i) = \sum_{i=1}^{n} f(i) + \sum_{i=n+1}^{k} f(i), \quad (1 \le n < k).$$

Example 7: Let's go back to solve the sum of the first k odd numbers.

Solution 7:

$$O_k = 1 + 3 + \cdots + (2k-1) = \sum_{1}^{k}(2i-1) = \sum_{1}^{k}(2i) - \sum_{1}^{k} 1 = 2\sum_{1}^{k} i - k,$$

where Rules 1, 2, and 3 are used in the last two steps. Using the known formulas, we obtain

$$O_k = \sum_1^k (2i-1) = k(k+1) - k = k^2.$$

Example 8: Let us now go back to solve Example 4. It is the sum of the first k products of pairs of subsequent integers.

Solution 8:

$$P_k = 1 \cdot 2 + 2 \cdot 3 + \cdots + k \cdot (k+1) = \sum_1^k i(i+1) = \sum_1^k [i^2 + i] = \sum_1^k i^2 + \sum_1^k i,$$

where Rule 3 was used in the last step. Applying the formulas, we learned

$$P_k = \sum_1^k i(i+1) = \frac{1}{6}k(k+1)(2k+1) + \frac{1}{2}k(k+1) = \frac{1}{3}k(k+1)(k+2).$$

This is another simple sum that we can easily remember.

Example 9: Calculate the sum $S = \sum_1^k (i+2)^3$.

Solution 9: This problem can be solved in two different ways. The first is to expand the summand $f(i) = (i+2)^3$ which yield

$$S = \sum_1^k (i+2)^3 = \sum_1^k (i^3 + 6i^2 + 12i + 8) = \sum_1^k i^3 + 6\sum_1^k i^2 + 12\sum_1^k i + 8k.$$

We can solve the resulting three sums separately using the known formulas.

But there is a better way to solve this. This involves substituting the summation index. We find it easier to see how substitution works by expanding the sum.

$$S = \sum_{i=1}^k (i+2)^3 = 3^3 + 4^3 + \cdots + k^3 + (k+1)^3 + (k+2)^3.$$

We see that this is simply a sum of integers cubed. But the sum does not start at 1^3 and end at k^3 like in the formula for the sum of the first k integers cubed.

Thus we re-write the sum with a sigma notation with an new index called l which starts at $l = 3$ and ends at $l = k + 2$ (there is no need to change the symbol for the index, you can keep calling it i if you do not feel any confusion). Thus,

$$S = \sum_{i=1}^k (i+2)^3 = 3^3 + 4^3 + \cdots + k^3 + (k+1)^3 + (k+2)^3 = \sum_{l=3}^{k+2} l^3.$$

We just finished doing a substitution of the summation index. It is equivalent to replacing i by $l = i + 2$. This relation also implies that $i = 1 \;\Rightarrow\; l = 3$ and $i = k \;\Rightarrow\; l = k + 2$. This is actually how you can determine the starting and the ending values of the new index.

Now we can solve this sum using Rule 4 and the known formula.

$$S = \sum_{i=1}^{k}(i+2)^3 = \sum_{l=3}^{k+2} l^3 = \sum_{l=1}^{k+2} l^3 - \sum_{l=1}^{2} l^3 = \left[\frac{1}{2}(k+2)(k+3)\right]^2 - 1^2 - 2^3 = \left[\frac{1}{2}(k+2)(k+3)\right]^2 - 9.$$

2.6 Applications of sigma sum

The area under a curve

We know that the area of a rectangle with length l and width w is $A_{rect} = w \cdot l$.

Starting from this formula we can calculate the area of a triangle and a trapezoid. This is because a triangle and a trapezoid can be transformed into a rectangle (see Figure). Thus, for a triangle of height h and base length b

$$A_{trig} = \frac{1}{2} hb.$$

Similarly, for a trapezoid with base length b, top length t, and height h

$$A_{trap} = \frac{1}{2} h(t + b).$$

Following a very similar idea, the sum of a trapezoid-shaped pile of logs with t logs on top layer, b logs on the bottom layer, and a height of $h = b - t + 1$ layers (see figure) is

$$\sum_{i=t}^{b} i = t + (t + 1) + \cdots + (b - 1) + b = \frac{1}{2} h(t + b) = \frac{1}{2}(b - t + 1)(t + b). \tag{8}$$

Now returning to the problem of calculating the area. Another important formula is for the area of a circle of radius r.

$$A_{circ} = \pi r^2.$$

Now, once we learned sigma and/or integration, we can calculate the area under the curve of any function that is integrable.

Example 10: Calculate the area under the curve $y = x^2$ between 0 and 2 (see figure).

Solution 10: Remember always try to reduce a problem that you do not know how to solve into a problem that you know how to solve.

Let the area be A. Let's divide A into 3 rectangles of equal width $w = 2/3$. Thus,

$$A \approx w \cdot h_1 + w \cdot h_2 + w \cdot h_3 = \begin{cases} w[x_0^2 + x_1^2 + x_2^2] = \frac{1}{3}[0^2 + \left(\frac{1}{3}\right)^2 + \left(\frac{2}{3}\right)^2], & \text{left} - \text{end approx.;} \\ w[x_1^2 + x_2^2 + x_3^2] = \frac{1}{3}[\left(\frac{1}{3}\right)^2 + \left(\frac{2}{3}\right)^2 + \left(\frac{3}{3}\right)^2], & \text{right} - \text{end approx..} \end{cases}$$

By using these rectangles, we introduced large errors in our estimates using the heights based on both the left and the right endpoints of the subintervals. To increase accurary, we need to increase the number of rectangles by making each one thinner. Let us now divide A into n rectangles of equal width $w = 2/n$. Thus, using the height based on the right endpoint of each subinterval, we obtain

$$A \approx S_n = A_1 + A_2 + \cdots + A_n = \sum_{i=1}^{n} w \cdot h_i = w \sum_{i=1}^{n} f(x_i) = \frac{2}{n} \sum_{i=1}^{n} x_i^2.$$

It is very important to keep a clear account of the height of each rectangle. In this case, $h_i = f(x_i) = x_i^2$. Thus, the key is finding the x-coordinate of the right-end of each rectangle. For rectangles of of equal width, $x_i = x_0 + iw = x_0 + i\Delta x/n$, where Δx is the length of the interval x_0 is the left endpoint of the interval. $x_0 = 0$ and $\Delta x = 2$ for this example. Therefore, $x_i = i(2/n)$. Substitute into the above equation

$$A \approx S_n = \frac{2}{n} \sum_{i=1}^{n} x_i^2. = \frac{2}{n} \sum_{i=1}^{n} \left(\frac{2i}{n}\right)^2 = \frac{8}{n^3} \sum_{i=1}^{n} i^2 = \frac{8}{n^3} \frac{1}{6} n(n+1)(2n+1) = \frac{4}{3} \frac{(n+1)(2n+1)}{n^2}.$$

Note that if we divide the area into infinitely many retangles with a width that is infinitely small, the approximate becomes accurate.

$$A = \lim_{n \to \infty} S_n = \lim_{n \to \infty} \frac{4}{3} \frac{(n+1)(2n+1)}{n^2} = \frac{8}{3}.$$

The volume of a solid revolution of a curve

We know the volume of a rectangular block of height h, width w, and length l is

$$V_{rect} = w \cdot l \cdot h.$$

The volume of a cylinder of thichness h and radius r

$$V_{cyl} = A_{cross} \cdot h = \pi r^2 h.$$

Example 11: Calculate the volume of the bowl-shaped solid obtained by rotating the curve $y = x^2$ on $[0, 2]$ about the y-axis.

2.7 The sum of a geometric sequence

3 The Definite Integrals and the Fundamental Theorem

3.1 Riemann sums

Definition 1: Suppose $f(x)$ is finite-valued and piecewise continuous on $[a, b]$. Let $P = \{x_0 = a, x_1, x_2, \ldots, x_n = b\}$ be a partition of $[a, b]$ into n subintervals $I_i = [x_{i-1}, x_i]$ of width $\Delta x_i = x_i - x_{i-1}$, $i = 1, 2, \ldots, n$. (Note in a special case, we can partition it into subintervals of equal width: $\Delta x_i = x_i - x_{i-1} = w = (b-a)/n$ for all i). Let x_i^* be a point in I_i such that $x_{i-1} \leq x_i^* \leq x_i$. (Here are some special ways to choose x_i^*: (i) left endpoint rule $x_i^* = x_{i-1}$, and (ii) the right endpoint rule $x_i^* = x_i$).

The Riemann sums of $f(x)$ on the interval $[a, b]$ are defined by:

$$R_n = \sum_{i=1}^{n} h_i \Delta x_i = \sum_{i=1}^{n} f(x_i^*) \Delta x_i$$

which approximate the area between $f(x)$ and the x-axis by the sum of the areas of n thin rectangles (see figure).

Example 1: Approximate the area under the curve of $y = e^x$ on $[0, 1]$ by 10 rectangles of equal width using the left endpoint rule.

Solution 1: We are actually calculating the Riemann sum R_{10}. The width of each rectangle is $\Delta x_i = w = 1/10 = 0.1$ (for all i). The left endpoint of each subinterval is $x_i^* = x_i - 1 = (i-1)/10$. Thus,

$$R_{10} = \sum_{i=1}^{10} f(x_i^*) \Delta x_i = \sum_{i=1}^{10} e^{(i-1)/10}(0.1) = 0.1 \sum_{j=0}^{9} e^{j/10} = 0.1 \frac{1-e}{1-e^{1/10}} \approx 1.6338.$$

3.2 The definite integral

Definition 1: Suppose $f(x)$ is finite-valued and piecewise continuous on $[a, b]$. Let $P = \{x_0 = a, x_1, x_2, \ldots, x_n = b\}$ be a partition of $[a, b]$ with a length defined by $|p| = Max_{1 \leq i \leq n}\{\Delta x_i\}$ (i.e. the longest of all subintervals). The definite integral of $f(x)$ on $[a, b]$ is

$$\int_a^b f(x)dx = \lim_{|p| \to 0;\ n \to \infty} R_n = \lim_{|p| \to 0;\ n \to \infty} \sum_{i=1}^{n} f(x_i^*) \Delta x_i$$

where the symbol $\int$ means **"to integrate"**, the function $f(x)$ to be integrated is called the **integrand**, x is called the **integration variable** which is a "dummy" variable, a and b are, respectively, the lower (or the starting) limit and the upper (or the ending) limit of the integral. Thus,

$$\int_a^b f(x)dx$$

means *integrate the function $f(x)$ starting from $x = a$ and ending at $x = b$.*

Example 2: Calculate the definite integral of $f(x) = x^2$ on [0.2]. (This is Example 10 of Lecture 1 reformulated in the form of a definite integral).

Solution 2:

$$I = \int_0^2 x^2 dx = \lim_{n\to\infty} \sum_{i=1}^{n} \underbrace{\left(\frac{2i}{n}\right)^2}_{f(x_i^*)} \underbrace{\left(\frac{2}{n}\right)}_{\Delta x_i} = \lim_{n\to\infty} \left(\frac{8}{n^3}\right) \sum_{i=1}^{n} i^2 = \lim_{n\to\infty} \frac{4}{3} \frac{(n+1)(2n+1)}{n^2} = \frac{8}{3}.$$

Important remarks on the relation between an area and a definite integral:

- An area, defined as the physical measure of the size of a 2D domain, is always non-negative.
- The value of a definite integral, sometimes also referred to as an "area", can be both positive and negative.
- This is because: a definite integral = the limit of Riemann sums. But Riemann sums are defined as

$$R_n = \sum_{i=1}^{n} (\text{area of i}^{\text{th}} \text{ rectangle}) = \sum_{i=1}^{n} \underbrace{f(x_i^*)}_{height} \underbrace{\Delta x_i}_{width}.$$

- Note that both the hieght $f(x_i^*)$ and the width Δx_i can be negative implying that R_n can have either signs.

3.3 The fundamental theorem of calculus

3.4 Areas between two curves

4 Applications of Definite Integrals: I

4.1 Displacement, velocity, and acceleration

4.2 Density and total mass

4.3 Rates of change and total change

4.4 The average value of a function

5 Differentials

Definition: The *differential*, dF, of any differentiable function F is an infinitely small increment or change in the value of F.

Remark: dF is measured in the same units as F itself.

Example: If x is the position of a moving body measured in units of m (meters), then its *differential*, dx, is also in units of m. dx is an infinitely small increment/change in the position x.

Example: If t is time measured in units of s (seconds), then its *differential*, dt, is also in units of s. dt is an infinitely small increment/change in time t.

Example: If A is area measured in units of m^2 (square meters), then its *differential*, dA, also is in units of m^2. dA is an infinitely small increment/change in area A.

Example: If V is volume measured in units of m^3 (cubic meters), then its *differential*, dV, also is in units of m^3. dV is an infinitely small increment/change in volume V.

Example: If v is velocity measured in units of m/s (meters per second), then its *differential*, dv, also is in units of m/s. dv is an infinitely small increment/change in velocity v.

Example: If C is the concentration of a biomolecule in our body fluid measured in units of M ($1\ M = 1\ molar = 1\ mole/litre$, where 1 mole is about 6.023×10^{23} molecules), then its *differential*, dC, also is in units of M. dC is an infinitely small increment/change in the concentration C.

Example: If m is the mass of a rocket measured in units of kg (kilograms), then its *differential*, dm, also is in units of kg. dm is an infinitely small increment/change in the mass m.

Example: If $F(h)$ is the culmulative probability of finding a man in Canada whose height is smaller than h ($meters$), then dF is an infinitely small increment in the probability.

Definition: The derivative of a function F with respect to another function x is defined as the quotient between their differentials:

$$\frac{dF}{dx} = \frac{an\ infinitely\ small\ rise\ in\ F}{an\ infinitely\ small\ run\ in\ x}.$$

Example: Velocity as the rate of change in position x with respect to time t can be expressed as

$$v = \frac{infinitely\ small\ change\ in\ position}{infinitely\ small\ time\ interval} = \frac{dx}{dt}.$$

Remark: Many physical laws are correct only when expressed in terms of differentials.

Example: The formula ($distance$) = ($velocity$) × ($time\ interval$) is true either when the velocity is a constant or in terms of differentials, i.e., ($an\ infinitely\ small\ distance$) = ($velocity$) × ($an\ infinitely\ short\ time\ interval$). This is because in an infinitely short time interval, the velocity can be considered a constant. Thus,

$$dx = \frac{dx}{dt}dt = v(t)dt,$$

which is nothing new but $v(t) = dx/dt$.

Example: The formula ($work$) = ($force$) × ($distance$) is true either when the force is a constant or in terms of differentials, i.e., ($an\ infinitely\ small\ work$) = ($force$) × ($an\ infinitely\ small\ distance$). This is because in an infinitely small distance, the force can be considered a constant. Thus,

$$dW = fdx = f\frac{dx}{dt}dt = f(t)v(t)dt,$$

which simply implies: (1) $f = dW/dx$, i.e., force f is nothing but the rate of change in work W with respect to distance x; (2) $dW/dt = f(t)v(t)$, i.e., the rate of change in work W with respect to time t is equal to the product between $f(t)$ and $v(t)$.

Example: The formula ($mass$) = ($density$) × ($volume$) is true either when the density is constant or in terms of differentials, i.e., ($an\ infinitely\ small\ mass$) = ($density$) ×

(*volume of an infinitely small volume*). This is because in an infinitely small piece of volume, the density can be considered a constant. Thus,

$$dm = \rho dV,$$

which implies that $\rho = dm/dV$, i.e., density is nothing but the rate of change in mass with respect to volume.

6 The Chain Rule in Terms of Differentials

When we differentiate a composite function, we need to use the Chain Rule. For example, $f(x) = e^{x^2}$ is a composite function. This is because f is not an exponential function of x but it is an exponential function of $u = x^2$ which is itself a power function of x. Thus,

$$\frac{df}{dx} = \frac{de^{x^2}}{dx} = \frac{de^u}{dx}\frac{du}{du} = (\frac{de^u}{du})(\frac{du}{dx}) = e^u(x^2)' = 2xe^{x^2},$$

where a substitution $u = x^2$ was used to change $f(x)$ into a true exponential function $f(u)$. Therefore, the Chain Rule can be simply interpreted as *the quotient between df and dx is equal to the quotient between df and du multiplied by the quotient between du and dx*. Or simply, divide df/dx by du and then multiply it by du.

However, if we simply regard x^2 as a function different from x, the actual substitution $u = x^2$ becomes unnecessary. Instead, the above derivative can be expressed as

$$\frac{de^{x^2}}{dx} = \frac{de^{x^2}}{dx}\frac{dx^2}{dx^2} = (\frac{de^{x^2}}{dx^2})(\frac{dx^2}{dx}) = e^{x^2}(x^2)' = 2xe^{x^2}.$$

Generally, if $f = f(g(x))$ is a differentiable function of g and g is a differentiable function of x, then

$$\frac{df}{dx} = \frac{df}{dg}\frac{dg}{dx},$$

where df, dg, and dx are, respectively, the differentials of the functions f, g, and x.

Example: Calculate df/dx for $f(x) = \sin(\ln(x^2 + e^x))$.

Solution: f is a composite function of another composite function!

$$\frac{df}{dx} = \frac{d\sin(\ln(x^2+e^x))}{d\ln(x^2+e^x)}\frac{d\ln(x^2+e^x)}{d(x^2+e^x)}\frac{d(x^2+e^x)}{dx} = cos(\ln(x^2+e^x))\frac{1}{(x^2+e^x)}(2x+e^x).$$

7 The Product Rule in Terms of Differentials

The Product Rule says if both $u = u(x)$ and $v = v(x)$ are differentiable functions of x, then

$$\frac{d(uv)}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}.$$

Multiply both sides by the differential dx, we obtain

$$d(uv) = vdu + udv,$$

which is the Product Rule in terms of differentials.

Example: Let $u = x^2$ and $v = e^{\sin(x^2)}$, then

$$d(uv) = vdu + udv = e^{\sin(x^2)}d(x^2) + x^2 d(e^{\sin(x^2)})$$

$$= 2xe^{\sin(x^2)}dx + x^2 e^{\sin(x^2)}\cos(x^2)(2x)dx = 2xe^{\sin(x^2)}[1 + x^2\cos(x^2)]dx.$$

8 Other Properties of Differentials

1. For any differentiable function $F(x)$, $dF = F'(x)dx$ (Recall that $F'(x) = dF/dx$!).
2. For any constant C, $dC = 0$.
3. for any constant C and differentiable function $F(x)$, $d(CF) = CdF = CF'(x)dx$.
4. For any differentiable functions u and v, $d(u \pm v) = du \pm dv$.

9 The Fundamental Theorem in Terms of Differentials

Fundamental Theorem of Calculus: If $F(x)$ is one antiderivative of the function $f(x)$, i.e., $F'(x) = f(x)$, then

$$\int f(x)dx = \int F'(x)dx = \int dF = F(x) + C.$$

Thus, **the integral of the differential of a function F is equal to the function itself plus an arbitrary constant**. This is simply saying that differential and integral are inverse math operations of each other. If we first differentiate a function $F(x)$ and then integrate the derivative $F'(x) = f(x)$, we obtain $F(x)$ itself plus an arbitrary constant. The opposite also is true. If we first integrate a function $f(x)$ and then differentiate the resulting integral $F(x)+C$, we obtain $F'(x) = f(x)$ itself.

Example:

$$\int x^5 dx = \int d(\frac{x^6}{6}) = \frac{x^6}{6} + C.$$

Example:

$$\int e^{-x} dx = \int d(-e^{-x}) = -e^{-x} + C.$$

Example:

$$\int cos(3x)dx = \int d(\frac{sin(3x)}{3}) = \frac{sin(3x)}{3} + C.$$

Example:

$$\int sec^2 x dx = \int dtanx = tanx + C.$$

Example:

$$\int cosh(3x)dx = \int d(\frac{sinh(3x)}{3}) = \frac{sinh(3x)}{3} + C.$$

Example:

$$\int \frac{1}{1+x^2}dx = \int dtan^{-1}x = tan^{-1} + C.$$

Example:

$$\int \frac{1}{1-tanh^2x}dx = \int \frac{1}{sech^2x}dx = \int cosh^2xdx$$

$$= \int \frac{1+cosh(2x)}{2}dx = \frac{1}{2}\int d(x+\frac{sinh(2x)}{2}) = \frac{1}{2}[x+\frac{sinh(2x)}{2}] + C.$$

For more indefinite integrals involving elementary functions, look at the first page of the table of integrals provided at the end of the notes.

10 Integration by Subsitution

Substitution is a necessity when integrating a composite function since we cannot write down the antiderivative of a composite function in a straightforward manner.

Many students find it difficult to figure out the substitution since for different functions the subsitutions are also different. However, there is a general rule in substitution, namely, to change the composite function into a simple, elementary function.

Example: $\int(sin(\sqrt{x})/\sqrt{x})dx$.

Solution: Note that $sin(\sqrt{x})$ is not an elementary sine function but a composite function. The first goal in solving this integral is to change $sin(\sqrt{x})$ into an elementary sine function through substitution. Once you realize this, $u = \sqrt{x}$ is an obvious subsitution. Thus, $du = u'dx = \frac{1}{2\sqrt{x}}dx$, or $dx = 2\sqrt{x}du = 2udu$. Substitute into the integral, we obtain

$$\int \frac{sin(\sqrt{x})}{\sqrt{x}}dx = \int \frac{sin(u)}{u}2udu = 2\int sin(u)du = -2\int dcos(u) = -2cos(u)+C = -2cos(\sqrt{x})+C.$$

Once you become more experienced with subsitutions and differentials, you do not need to do the actual substitution but only symbolically. Note that $x = (\sqrt{x})^2$,

$$\int \frac{sin(\sqrt{x})}{\sqrt{x}}dx = \int \frac{sin(\sqrt{x})}{\sqrt{x}}d(\sqrt{x})^2 = \int \frac{sin(\sqrt{x})}{\sqrt{x}}2\sqrt{x}d\sqrt{x} = 2\int sin(\sqrt{x})d\sqrt{x} = -2cos(\sqrt{x})+C.$$

Thus, as soon as you realize that $\sqrt{x}$ is the substitution, your goal is to change the differential in the integral dx into the differential of $\sqrt{x}$ which is $d\sqrt{x}$.

If you feel that you cannot do it without the actual substitution, that is fine. You can always do the actual substitution. I here simply want to teach you a way that actual subsitution is not a necessity!

Example: $\int x^4 cos(x^5)dx$.

Solution: Note that $cos(x^5)$ is a composite function that becomes a simple cosine function only if the subsitution $u = x^5$ is made. Since $du = u'dx = 5x^4dx$, $x^4dx = \frac{1}{5}du$. Thus,

$$\int x^4 cos(x^5)dx = \frac{1}{5}\int cos(u)du = \frac{sin(u)}{5} + C = \frac{sin(x^5)}{5} + C.$$

Or alternatively,

$$\int x^4 cos(x^5)dx = \frac{1}{5}\int cos(x^5)dx^5 = \frac{1}{5}sin(x^5) + C.$$

Example: $\int x\sqrt{x^2+1}dx$.

Solution: Note that $\sqrt{x^2+1} = (x^2+1)^{1/2}$ is a composite function. We realize that $u = x^2+1$ is a substitution. $du = u'dx = 2xdx$ implies $xdx = \frac{1}{2}du$. Thus,

$$\int x\sqrt{x^2+1}dx = \frac{1}{2}\int \sqrt{u}du = \frac{1}{2}\frac{2}{3}u^{3/2} + C = \frac{(x^2+1)^{3/2}}{3} + C.$$

Or alternatively,

$$\int x\sqrt{x^2+1}dx = \frac{1}{2}\int \sqrt{x^2+1}d(x^2+1) = \frac{1}{2}\frac{2}{3}(x^2+1)^{3/2} + C = \frac{(x^2+1)^{3/2}}{3} + C.$$

In many cases, substitution is required even no obvious composite function is involved.

Example: $\int(\ln x/x)dx$ $(x > 0)$.

Solution: The integrand $\ln x/x$ is not a composite function. Nevertheless, its antiderivative is not obvious to calculate. We need to figure out that $(1/x)dx = d\ln x$, thus by introducing the substitution $u = \ln x$, we obtained a differential of the function $\ln x$ which also appears in the integrand. Therefore,

$$\int(\ln x/x)dx = \int udu = \frac{u^2}{2} + C = \frac{1}{2}(\ln x)^2 + C.$$

It is more natural to consider this substitution is an attempt to change the differential dx into something that is identical to a function that appears in the integrand, namely $d\ln x$. Thus,

$$\int(\ln x/x)dx = \int \ln x d\ln x = \frac{1}{2}(\ln x)^2 + C.$$

Example: $\int \tan x dx$.

Solution: $\tan x$ is not a composite function. Nevertheless, it is not obvious to figure out $\tan x$ is the derivative of what function. However, if we write $\tan x = \sin x/\cos x$, we can regard $1/\cos x$ as a composite function. We see that $u = cosx$ is a candidate for substitution and $du = u'dx = -sin(x)dx$. Thus,

$$\int tanxdx = \int \frac{sinx}{cosx}dx = -\int \frac{du}{u} = -\ln|u| + C = -\ln|cosx| + C.$$

Or alternatively,

$$\int tanxdx = \int \frac{sinx}{cosx}dx = -\int \frac{dcosx}{cosx} = -\ln|cosx| + C.$$

Some substitutions are standard in solving specific types of integrals.

Example: Integrands of the type $(a^2 - x^2)^{\pm 1/2}$.

In this case both $x = asinu$ and $x = acosu$ will be good. $x = a\tanh u$ also works $(1 - tanh^2 u = sech^2 u)$. Let's pick $x = asinu$ in this example. If you ask how can we find out that $x = asinu$

is the substitution, the answer is $a^2 - x^2 = a^2(1 - sin^2u) = a^2cos^2u$. This will help us eliminate the half power in the integrand. Note that with this substitution, $u = sin^{-1}(x/a)$, $sinu = x/a$, and $cosu = \sqrt{1 - x^2/a^2}$. Thus,

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \int \frac{d(asinu)}{\sqrt{a^2 - a^2sin^2u}} = \int \frac{acosudu}{acosu} = \int du = u + C = sin^{-1}\frac{x}{a} + C.$$

Note that $\int dx/\sqrt{1 - x^2} = sin^{-1}x + C$, we can make a simple substitution $x = au$, thus

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \int \frac{d(au)}{\sqrt{a^2 - a^2u^2}} = \int \frac{du}{\sqrt{1 - u^2}} = sin^{-1}u + C = sin^{-1}\frac{x}{a} + C.$$

Similarly,

$$\int \sqrt{a^2 - x^2}dx = \int \sqrt{a^2 - a^2sin^2u}d(asinu) = a^2 \int cos^2udu = a^2 \int \frac{1 + cos(2u)}{2}du$$

$$= \frac{a^2}{2}[u + \frac{sin(2u)}{2}] + C = \frac{a^2}{2}[u + sinucosu] + C = \frac{1}{2}[a^2sin^{-1}(\frac{x}{a}) + x\sqrt{a^2 - x^2}] + C.$$

Example: Integrands of the type $(a^2 + x^2)^{\pm 1/2}$.

$x = asinh(u)$ is a good substitution since $a^2 + x^2 = a^2 + a^2sinh^2(u) = a^2(1 + sinh^2(u)) = a^2cosh^2(u)$, where the hyperbolic identity $1 + sinh^2(u) = cosh^2(u)$ was used. ($x = a\tan u$ is also good since $1 + tan^2u = sec^2u$!). Thus,

$$\int \frac{dx}{\sqrt{a^2 + x^2}} = \int \frac{dasinh(u)}{\sqrt{a^2 + a^2sinh^2(u)}} = \int \frac{acosh(u)du}{acosh(u)} = \int du = u + C = sinh^{-1}(\frac{x}{a}) + C$$

$$= \ln|x + \sqrt{a^2 + x^2}| + C,$$

where $sinh^{-1}(x/a) = \ln|x + \sqrt{a^2 + x^2}| - \ln|a|$ was used. And,

$$\int \sqrt{a^2 + x^2}dx = \int \sqrt{a^2 + a^2sinh^2(u)}dasinh(u) = a^2 \int cosh^2(u)du = \frac{a^2}{2} \int [1 + cosh(2u)]du$$

$$= \frac{a^2}{2}[u + \frac{sinh(2u)}{2}] + C = \frac{1}{2}[a^2 sinh^{-1}(\frac{x}{a}) + a^2 sinh(u)cosh(u)] + C$$

$$= \frac{1}{2}[a^2 \ln|x + \sqrt{a^2 + x^2}| + x\sqrt{a^2 + x^2}] + C,$$

where the following hyperbolic identities were used: $sinh(2u) = 2sinh(u)cosh(u)$, $sinh^{-1}(x/a) = \ln|x + \sqrt{a^2 + x^2}| - \ln|a|$, and $acosh(u) = \sqrt{a^2 + x^2}$.

Example: Integrands of the type $(x^2 - a^2)^{\pm 1/2}$.

$x = acosh(u)$ is prefered since $x^2 - a^2 = a^2 scos^2(u) - a^2 = a^2[cosh^2(u) - 1] = a^2 sinh^2(u)$, where the hyperbolic identity $cosh^2(u) - 1 = sinh^2(u)$ was used. ($x = a \sec u$ is also good since $sec^2 u - 1 = tan^2 u$!). Thus,

$$\int \frac{dx}{\sqrt{x^2 - a^2}} = \int \frac{da\cosh(u)}{\sqrt{a^2\cosh^2(u) - a^2}} = \int \frac{a\sinh(u)du}{a\sinh(u)} = \int du = u + C$$

$$= cosh^{-1}(\frac{x}{a}) + C = \ln|x + \sqrt{x^2 - a^2}| + C.$$

Similarly

$$\int \sqrt{x^2 - a^2}dx = \int \sqrt{a^2\cosh^2(u) - a^2}da\cosh(u) = a^2 \int \sinh^2 u du = \frac{a^2}{2}\int [cosh(2u) - 1]du$$

$$= \frac{1}{2}[x\sqrt{x^2 - a^2} - a^2 \ln|x + \sqrt{x^2 - a^2}|] + C,$$

where $\cosh u = x/a$, $\sinh u = \sqrt{x^2/a^2 - 1}$, and $u = \ln|x + \sqrt{x^2 - a^2}| - \ln a$ were used.

For more details on hyperbolic functions, read the last page of the table of integrals at the end of the notes.

More Exercises on Substitution:

1. Substitution aimed at eliminating a composite function

Example:
(i)

$$\int \frac{4x}{\sqrt[3]{2x^2+3}}dx = \int \frac{d(2x^2+3)}{\sqrt[3]{2x^2+3}} = \int u^{-1/3}du = \frac{3}{2}u^{2/3} + C = \frac{3}{2}(2x^2+3)^{2/3} + C.$$

(ii)

$$\int \frac{x}{\sqrt{2x+3}}dx = \frac{1}{4}\int \frac{2x+3-3}{\sqrt{2x+3}}d(2x+3) = \frac{1}{4}\int \frac{u-3}{u^{1/2}}du = \frac{1}{4}\int [u^{1/2} - 3u^{-1/2}]du$$

$$= \frac{1}{4}[\frac{2}{3}u^{3/2} - 6u^{1/2}] + C = \frac{1}{4}[\frac{2}{3}(2x+3)^{3/2} - 6(2x+3)^{1/2}] + C.$$

Edwards/Penney $5^{th} - ed$ 5.7 Problems.

(1) $\int (3x-5)^{17}dx$ (3) $\int x\sqrt{x^2+9}dx$ (4) $\int \frac{x^2}{\sqrt[3]{2x^3-1}}dx$

(7) $\int x\sin(2x^2)dx$ (9) $\int (1-cosx)^5 sinxdx$ (15) $\int \frac{dx}{\sqrt{7x+5}}dx$

(23) $\int x\sqrt{2-3x^2}dx$ (31) $\int cos^3xsinxdx$ (33) $\int tan^3xsec^2xdx$

(45) $\int \frac{cos(\sqrt{x})}{\sqrt{x}}dx$ (49) $\int_0^{\pi/2} (1+3sin\theta)^{3/2}cos\theta d\theta$ (51) $\int_0^{\pi/2} e^{sinx}cosxdx$

Edwards/Penney $5^{th} - ed$ 9.2 Problems.

(7) $\int \frac{cot(\sqrt{y})csc(\sqrt{y})}{\sqrt{y}}dy$ (11) $\int e^{-cotx}csc^2xdx$ (13) $\int \frac{(\ln t)^10}{t}dt,\ (t>0)$

(21) $\int tan^4(3x)sec^2(3x)dx$ (25) $\int \frac{(1+\sqrt{x})^4}{\sqrt{x}}dx$ (31) $\int x^2\sqrt{x+2}dx.$

2. Substitution to achieve a function in differential that appears in the integrand

Example:
(i)
$$\int \frac{e^x}{1+e^{2x}}dx = \int \frac{de^x}{1+(e^x)^2} = \int \frac{du}{1+u^2} = tan^{-1}(u) + C = tan^{-1}(e^x) + C.$$

(ii)
$$\int \frac{x}{\sqrt{e^{2x^2}-1}}dx = \frac{1}{2}\int \frac{dx^2}{\sqrt{e^{2x^2}(1-e^{-2x^2})}} = \frac{1}{2}\int \frac{du}{\sqrt{e^{2u}(1-e^{-2u})}} = \frac{1}{2}\int \frac{du}{e^u\sqrt{1-(e^{-u})^2}}$$

$$= -\frac{1}{2}\int \frac{de^{-u}}{\sqrt{1-(e^{-u})^2}} = -\frac{1}{2}\int \frac{dy}{\sqrt{1-y^2}} = \frac{1}{2}cos^{-1}(y)+C = \frac{1}{2}cos^{-1}(e^{-u})+C = \frac{1}{2}cos^{-1}(e^{-x^2})+C.$$

Edwards/Penney $5^{th}-ed$ 9.2 Problems (difficult ones!).

(17) $\int \frac{e^{2x}}{1+e^{4x}}dy,\ (u = e^{2x})$ (19) $\int \frac{3x}{\sqrt{1-x^4}}dx,\ (u = x^2)$ (23) $\int \frac{cos\theta}{1+sin^2\theta}d\theta,\ (u = sin\theta)$

(27) $\int \frac{1}{(1+t^2)tan^{-1}t}dt,\ (u = tan^{-1}t)$ (29) $\int \frac{1}{\sqrt{e^{2x}-1}}dx,\ (u = e^{-x})$

3. Special Trigonometric Substitutions

Example:
(i)
$$\int \frac{x^3}{\sqrt{1-x^2}}dx = \int \frac{\sin^3 u d\sin u}{\sqrt{1-\sin^2 u}} = \int \sin^3 u du = -\int (1-\cos^2 u)d\cos u$$

$$= \frac{cos^3 u}{3} - \cos u + C = \frac{(1-x^2)^{3/2}}{3} - \sqrt{1-x^2} + C.$$

Edwards/Penney $5^{th}-ed$ 9.6 Problems (difficult ones!).

(1) $\int \frac{1}{\sqrt{16-x^2}}dx,\ (x = 4\sin u)$ (9) $\int \frac{\sqrt{x^2-1}}{x}dx,\ (x = \cosh u)$ (11) $\int x^3\sqrt{9+4x^2}dx,\ (2x = 3\sinh u)$

(13) $\int \frac{\sqrt{1-4x^2}}{x}dx,\ (2x = \sin u)$ (19) $\int \frac{x^2}{\sqrt{1+x^2}}dx,\ (x = \sinh u)$ (27) $\int \sqrt{9+16x^2}dx,\ (4x = 3\sinh u)$

11 Integration by Parts

Integration by Parts is the integral version of the Product Rule in differentiation. The Product Rule in terms of differentials reads,

$$d(uv) = vdu + udv.$$

Integrating both sides, we obtain

$$\int d(uv) = \int vdu + \int udv.$$

Note that $\int d(uv) = uv + C$, the above equation can be expressed in the following form,

$$\int udv = uv - \int vdu.$$

Generally speaking, we need to use Integration by Parts to solve many integrals that involve the product between two functions. In many cases, Integration by Parts is most efficient in solving integrals of the product between a polynomial and an exponential, a logarithmic, or a trigonometric function. It also applies to the product between exponential and trigonometric functions.

Example: $\int xe^x dx$.

Solution: In order to eliminate the power function x, we note that $(x)' = 1$. Thus,

$$\int xe^x dx = \int xde^x = xe^x - \int e^x dx = xe^x - e^x + C.$$

Example: $\int x^2 cosxdx$.

Solution: In order to eliminate the power function x^2, we note that $(x^2)'' = 2$. Thus, we need to use Integration by Patrs twice.

$$\int x^2 cosx dx = \int x^2 dsinx = x^2 sinx - \int sinx dx^2 = x^2 sinx - \int 2xd(-cosx)$$

$$= x^2 sinx + 2\int xdcosx = x^2 sinx + 2xcosx - 2\int cosxdx = x^2 sinx + 2xcosx - 2sinx + C.$$

Example: $\int x^2 e^{-2x} dx$.

Solution: In order to eliminate the power function x^2, we note that $(x^2)'' = 2$. Thus, we need to use Integration by Patrs twice. However, the number (-2) can prove extremely annoying and easily cause errors. Here is how we use substitution to avoid this problem.

$$\int x^2 e^{-2x} dx = \frac{-1}{2^3}\int (-2x)^2 e^{-2x} d(-2x) = \frac{-1}{8}\int u^2 e^u du = \frac{-1}{8}[u^2 e^u - 2\int ue^u du]$$

$$= \frac{-1}{8}[u^2 e^u - 2(ue^u - e^u)] + C = \frac{-e^u}{8}[u^2 - 2u + 2] + C = \frac{-e^{-2x}}{8}[4x^2 + 4x + 2] + C.$$

When integrating the product between a polynomial and a logarithmic function, the main goal is to eliminate the logarithmic function by differentiating it. This is because $(\ln x)' = 1/x$.

Example: $\int \ln x dx$, $(x > 0)$.

Solution:

$$\int \ln x dx = x\ln x - \int x d\ln x = x\ln x - \int x\frac{dx}{x} = x\ln x - x + C.$$

Example: $\int x(\ln x)^2 dx$, $(x > 0)$.

Solution:

$$\int x(\ln x)^2 dx = \frac{1}{2}\int (\ln x)^2 dx^2 = \frac{1}{2}[x^2(\ln x)^2 - \int x^2(2\ln x)\frac{dx}{x}] = \frac{1}{2}[x^2(\ln x)^2 - \int \ln x dx^2]$$

$$= \frac{1}{2}[x^2(\ln x)^2 - x^2 \ln x + \frac{x^2}{2}] + C = \frac{x^2}{2}[(\ln x)^2 - \ln x + \frac{1}{2}] + C.$$

More Exercises on Integration by Parts:

Example:
(i)

$$\int t \sin t dt = -\int t d \cos t = -[t \cos t - \int \cos t dt] = [\sin t - t \cos t] + C.$$

(ii)

$$\int tan^{-1} x dx = x tan^{-1} x - \int x d tan^{-1} x = x tan^{-1} x - \int \frac{x}{1+x^2} dx$$

$$= x tan^{-1} x - \frac{1}{2} \int \frac{d(1+x^2)}{1+x^2} = x tan^{-1} x - \ln \sqrt{1+x^2} + C.$$

Edwards/Penney $5^{th} - ed$ 9.3 Problems.

(1) $\int x e^{2x} dx$ (5) $\int x \cos(3x) dx$ (7) $\int x^3 \ln x dx, \ (x > 0)$

(11) $\int \sqrt{y} \ln y dy$ (13) $\int (\ln t)^2 dt$ (19) $\int csc^3 \theta d\theta$

(21) $\int x^2 tan^{-1} x dx$ (27) $\int x csc^2 x dx$ (19) $\int csc^3 \theta d\theta$

12 Integration by Partial Fractions

Rational functions are defined as the quotient between two polynomials:

$$R(x) = \frac{P_n(x)}{Q_m(x)}$$

where $P_n(x)$ and $Q_m(x)$ are polynomials of degree n and m respectively. The method of partial fractions is an **algebraic** technique that decomposes $R(x)$ into a sum of terms:

$$R(x) = \frac{P_n(x)}{Q_m(x)} = p(x) + F_1(x) + F_2(x) + \cdots F_k(x),$$

where $p(x)$ is a polynomial and $F_i(x),\ (i = 1, 2, \cdots, k)$ are fractions that can be integrated easily.

The method of partial fractions is an area many students find very difficult to learn. It is related to **algebraic** techniques that many students have not been trained to use. The most typical claim is that there is no fixed formula to use. It is not our goal in this course to cover this topics in great details (Read Edwards/Penney for more details). Here, we only study two simple cases.

Case I: $Q_m(x)$ is a power function, i.e., $Q_m(x) = (x-a)^m$ ($Q_m(x) = x^m$ if $a = 0$!).

Example:

$$\int \frac{2x^2 - x + 3}{x^3} dx.$$

This integral involves the simplest partial fractions:

$$\frac{A + B + C}{D} = \frac{A}{D} + \frac{B}{D} + \frac{C}{D}.$$

Some may feel that it is easier to write the fractions in the following form: $D^{-1}(A+B+C) = D^{-1}A + D^{-1}B + D^{-1}C$. Thus,

$$\int \frac{2x^2 - x + 3}{x^3} dx = \int [\frac{2x^2}{x^3} - \frac{x}{x^3} + \frac{3}{x^3}] dx = \int [\frac{2}{x} - \frac{1}{x^2} + \frac{3}{x^3}] dx = \ln x^2 + \frac{1}{x} - \frac{3}{2x^2} + C.$$

Example:

$$\int \frac{2x^2 - x + 3}{x^2 - 2x + 1} dx = \int \frac{2x^2 - x + 3}{(x-1)^2} dx = \int \frac{2(x-1+1)^2 - (x-1+1) + 3}{(x-1)^2} dx$$

$$= \int \frac{2(x-1)^2 + 4(x-1) + 2 - (x-1) - 1 + 3}{(x-1)^2} dx = \int \frac{2(x-1)^2 + 3(x-1) + 4}{(x-1)^2} dx$$

$$= \int [2 + \frac{3}{x-1} + \frac{4}{(x-1)^2}] d(x-1) = 2(x-1) + 3 \ln |x-1| - \frac{4}{x-1} + C.$$

Case II: $P_n(x) = A$ is a constant and $Q_m(x)$ can be factorized into the form $Q_2(x) = (x-a)(x-b)$, $Q_3(x) = (x-a)(x-b)(x-c)$, or $Q_m(x) = (x-a_1)(x-a_2)\cdots(x-a_m)$.

Example:

$$I = \int \frac{2}{x^2-2x-3}dx = \int \frac{2dx}{(x-3)(x+1)} = 2\int [\frac{A}{x-3} + \frac{B}{x+1}]dx.$$

Since $\frac{A}{x-3} + \frac{B}{x+1} = \frac{1}{(x-3)(x+1)}$ implies that $A(x+1) + B(x-3) = 1$. Setting $x = 3$ in this equation, we obtain $A = 1/4$. Setting $x = -1$, we obtain $B = -1/4$. Thus,

$$I = 2\int [\frac{A}{x-3} + \frac{B}{x+1}]dx = 2\int [\frac{1/4}{x-3} + \frac{-1/4}{x+1}]dx = \frac{1}{2}\int [\frac{dx}{x-3} - \frac{dx}{x+1}] = \frac{1}{2}\ln|\frac{x-3}{x+1}| + C.$$

Example:

$$I = \int \frac{dx}{(x-1)(x-2)(x-3)} = \int [\frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{x-3}]dx.$$

Since $\frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{x-3} = 1$ implies that $A(x-2)(x-3) + B(x-1)(x-3) + C(x-1)(x-2) = 1$. Setting $x = 1$ in this equation, we obtain $A = 1/2$. Setting $x = 2$, we obtain $B = -1$. Setting $x = 3$, we get $C = 1/2$. Thus,

$$I = \int [\frac{1/2}{x-1} + \frac{-1}{x-2} + \frac{1/2}{x-3}]dx = \frac{1}{2}\ln\frac{|(x-1)(x-3)|}{(x-2)^2} + C.$$

More Exercises on Integration by Partial Fractions:

Examples:
(i)

$$\int \frac{dx}{x^2-3x} = \int \frac{dx}{x(x-3)} = \frac{1}{3}\int [\frac{1}{x-3} - \frac{1}{x}]dx = \frac{1}{3}\ln|\frac{x-3}{x}| + C.$$

(ii)

$$\int \frac{dx}{x^2+x-6} = \int \frac{dx}{(x+3)(x-2)} = \frac{1}{5}\int [\frac{1}{x-2} - \frac{1}{x+3}]dx = \frac{1}{5}\ln|\frac{x-2}{x+3}| + C.$$

(iii)

$$\int \frac{x-1}{x+1}dx = \int \frac{x+1-2}{x+1}dx = \int [1 - \frac{2}{x+1}]dx = x - \ln(x+1)^2 + C.$$

Edwards/Penney $5^{th} - ed$ 9.5 Problems.

(Li) $\int \frac{2x^3+3x^2-5}{x^3}dx$ (15) $\int \frac{dx}{x^2-4}$ (Li) $\int \frac{dx}{x^2-2x-8}$

(Li) $\int \frac{x}{x^2+5x+6}dx,\ (Hint: x = x+2-2)$ (Li) $\int \frac{x^2}{1+x^2}dx,\ (Hint: x^2 = 1+x^2-1)$

(Li) $\int \frac{x^2}{x^2+x}dx,\ (Hint:\ x^2 = [(x+1)-1]^2)$ (23) $\int \frac{x^2}{(x+1)^3}dx$

13 Integration by Multiple Techniques

Whenever we encounter a complex integral, often the first thing to do is to use substitution to eliminate the composite function(s).

Example: $I = \int x^{-3}e^{1/x}dx.$

Solution: Note that $e^{1/x}$ is a composite fucntion. The substitution to change it into a simple, elementary function is $u = 1/x$, $du = u'dx = -x^{-2}dx$ or $x^{-2}dx = -du = d(-1/x)$. Thus,

$$I = \int x^{-3}e^{1/x}dx = \int (1/x)e^{1/x}d(-1/x) = -\int ue^u du = -[ue^u - e^u] + C = e^{1/x}[1 - 1/x] + C.$$

Example: $I = \int \cos(\sqrt{x})dx$.

Solution: Note that $cos(\sqrt{x})$ is a composite fucntion. The substitution to change it into a simple, elementary function is $u = \sqrt{x}$, $du = u'dx = \frac{1}{2\sqrt{x}}dx$ or $dx = 2\sqrt{x}du = 2udu$. However, note that $x = (\sqrt{x})^2$,

$$I = \int \cos(\sqrt{x})dx = \int \cos(\sqrt{x})d(\sqrt{x})^2 = \int cos(u)du^2 = 2\int ucos(u)du$$

$$= 2\int udsin(u) = 2[usin(u) + cos(u)] + C = 2[\sqrt{x}sin(\sqrt{x}) + cos(\sqrt{x})] + C.$$

Example: $I = \int sec(x)dx = \int (1/cos(x))dx$.

Solution: This is a difficult integral. Note that $1/cos(x)$ is a composite fucntion. One substitution to change it into a simple, elementary function is $u = cos(x)$. However, $x = cos^{-1}(u)$ and $dx = x'du = \frac{-du}{\sqrt{1-u^2}}$. $1/\sqrt{1-u^2}$ is still a composite function that is difficult to deal with. Here is how we can deal with it. It involves trigonometric substitution followed by partial fractions.

$$I = \int \frac{dx}{cos(x)} = \int \frac{cos(x)dx}{cos^2(x)} = \int \frac{dsin(x)}{1 - sin^2(x)} = \int \frac{du}{1-u^2} = -\int [\frac{1}{u-1} - \frac{1}{u+1}]\frac{du}{2}$$

$$= \ln\sqrt{|\frac{u+1}{u-1}|} + C = \ln|\frac{u+1}{\sqrt{1-u^2}}| + C = \ln|\frac{sin(x)+1}{cos(x)}| + C = \ln|tan(x) + sec(x)| + C,$$

where $u = sin(x)$ and $\sqrt{1-u^2} = cos(x)$ were used!

More Exercises on Integration by Multiple techniques:

1. Substitution followed by Integration by Parts

Example:
(i)

$$\int x^3\sin(x^2)dx = \frac{1}{2}\int x^2\sin(x^2)dx^2 = \frac{1}{2}\int u\sin u du = \frac{-1}{2}\int ud\cos u$$

$$= \frac{1}{2}[\sin u - u\cos u] + C = \frac{1}{2}[\sin(x^2) - x^2\cos(x^2)] + C.$$

(ii)

$$\int xtan^{-1}\sqrt{x}dx = \int (\sqrt{x})^2 tan^{-1}\sqrt{x}d(\sqrt{x})^2 = \int u^2 tan^{-1}udu^2 = \frac{1}{2}\int tan^{-1}udu^4$$

$$= \frac{1}{2}[u^4 tan^{-1}u - \int u^4 dtan^{-1}u] = \frac{1}{2}[u^4 tan^{-1}u - \int \frac{u^4-1+1}{1+u^2}du] = \frac{1}{2}[u^4 tan^{-1}u - \int (u^2-1+\frac{1}{1+u^2})du]$$

$$= \frac{1}{2}[u^4 tan^{-1}u - \frac{u^3}{3} + u - tan^{-1}u] + C = \frac{1}{2}[(x^2-1)tan^{-1}\sqrt{x} - \frac{x^{3/2}}{3} + \sqrt{x}] + C.$$

Edwards/Penney $5^{th}-ed$ 9.3 Problems.

(15) $\int x\sqrt{x+3}dx,\ u=x+3$ (17) $\int x^5\sqrt{x^3+1}dx,\ u=x^3+1$

(23) $\int sec^{-1}\sqrt{x}dx,\ u=\sqrt{x}$ (29) $\int x^3\cos(x^2)dx,\ u=x^2$ (25) $\int tan^{-1}\sqrt{x}dx,\ u=\sqrt{x}$

(31) $\int \frac{\ln x}{x\sqrt{x}}dx,\ u=\sqrt{x}$ (37) $\int e^{-\sqrt{x}}dx,\ u=\sqrt{x}$

2. Substitution followed by Integration by Partial Fractions

Example:
(i)

$$\int \frac{\cos x dx}{\sin^2 x - \sin x - 6} = \int \frac{d\sin x}{\sin^2 x - \sin x - 6} = \int \frac{du}{(u-3)(u+2)} = \frac{1}{5}\ln|\frac{u-3}{u+2}| + C = \frac{1}{5}\ln|\frac{\sin x - 3}{\sin x + 2}| + C$$

Edwards/Penney $5^{th}-ed$ 9.5 Problems (diffifult ones!)

(40) $\int \frac{\sec^2 t}{\tan^3 t + tan^2 t}dt$ (37) $\int \frac{e^{4t}}{(e^{2t}-1)^3}dt$ (39) $\int \frac{1+\ln t}{t(3+2\ln t)^2}dt$

2. Integration by Parts followed by Partial Fractions

Example:
(i)

$$\int (2x+2)\tan^{-1} x dx = \int \tan^{-1} x d(x^2+2x) = (x^2+2x)\tan^{-1} x - \int (x^2+2x) d\tan^{-1} x$$

$$= (x^2+2x)\tan^{-1} x - \int \frac{1+x^2+2x-1}{1+x^2} dx = (x^2+2x)\tan^{-1} x - \int [dx + \frac{d(1+x^2)}{1+x^2} - \frac{dx}{1+x^2}]$$

$$= (x^2+2x)\tan^{-1} x - x - \ln(1+x^2) + \tan^{-1} x + C = (x+1)^2 \tan^{-1} x - x - \ln(1+x^2) + C.$$

Li's Problems (diffifult ones!)

(Li) $\int (2t+1)\ln t dt$

14 Applications of Integrals

The central idea in all sorts of application problems involving integrals is *breaking the whole into pieces and then adding the pieces up to obtain the whole.*

14.1 Area under a curve.

We know the area of a rectangle is:

$$Area = height \times width,$$

(see Fig. 1(a)). This formula can not be applied to the area under the curve in Fig. 1(b) because the top side of the area is NOT a straight, horizontal line. However, if we divide the area into infinitely many rectangles with infinitely thin width dx, the infinitely small area of the rectangle located at x is

$$dA = height \times infintely\ small\ width = f(x)dx.$$

Adding up the area of all the thin rectangles, we obtain the area under the curve

$$Area = A(b) - A(a) = \int\limits_0^{A(b)} dA = \int\limits_a^b f(x)dx,$$

where $A(x) = \int_a^x f(s)ds$ is the area under the curve between a and x.

14.2 Volume of a solid.

We know the volume of a cylinder is

$$Volume = (cross - section\ area) \times length,$$

(see Fig. 2(a)). This formula does not apply to the volume of the solid obtained by rotating a curve around the x-axis (see Fig. 2(b)). This is because the cross-section area is NOT a constant but changes along the length of this “cylinder”. However, if we cut this “cylinder” into infinitely many disks of infinitely small thickness dx, the volume of such a disk located at x is

$$dV = cross - section\ area \times infintely\ small\ thickness = \pi r^2 dx = \pi(f(x))^2 dx.$$

Adding up the volume of all such thin disks, we obtain the volume of the solid

$$Volume = V(b) - V(a) = \int_0^{V(b)} dV = \pi \int_a^b (f(x))^2 dx,$$

where $V(x) = \pi \int_a^x (f(s))^2 ds$ is the volume of the “cylinder” between a and x.

14.3 Arc length.

We know the length of the hypotenuse of a right triangle is

$$hypotenuse = \sqrt{base^2 + height^2},$$

(see Fig. 3(a)). This formula does not apply to the length of the curve in Fig. 3(b) because the curve is NOT a straight line and the geometric shape of the area enclosed in solid lines is NOT

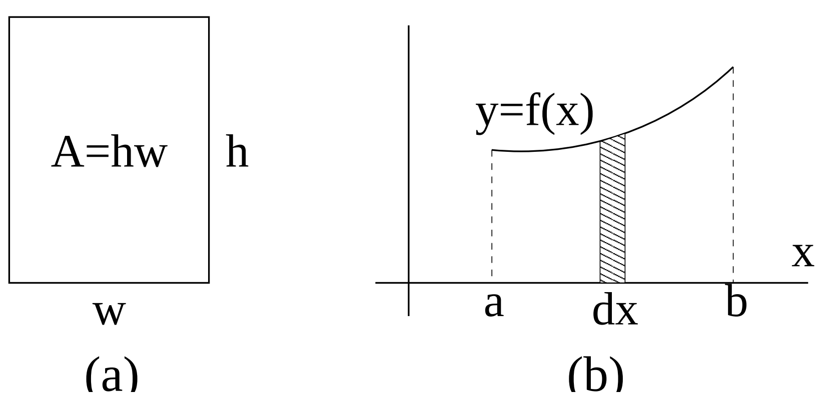


Figure 1: Area under a curve.

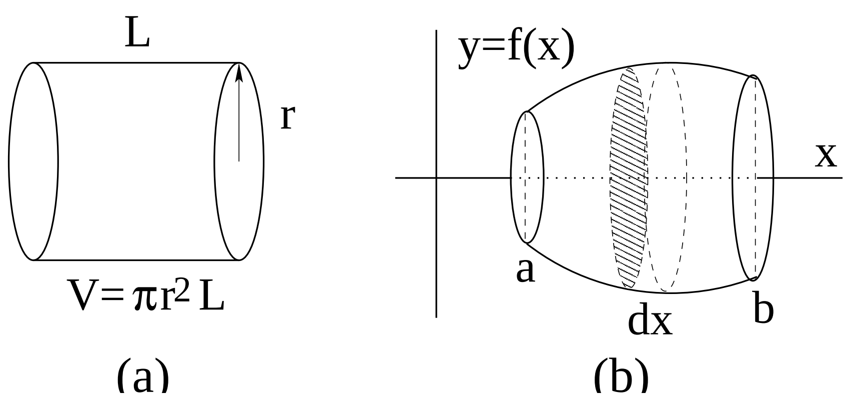


Figure 2: Volume of a solid obtained by rotating a curve.

a right triangle. However, if we "cut" the curve on interval $[a,\ b]$ into infinitely many segments each with a horizontal width dx, each curve segment can be regarded as the hypotenuse of an inifinitely small right triangle with base dx and height dy. Thus, the length of one such curve segments located at x is

$$dl = \sqrt{(infinitely\ small\ base)^2 + (infinitely\ small\ height)^2}$$

$$= \sqrt{dx^2 + dy^2} = \sqrt{[1 + (\frac{dy}{dx})^2]dx^2} = [\sqrt{1 + (y')^2}]dx.$$

The total length of this curve is obtained by adding the lengths of all the segments together.

$$L = \int_0^{l(b)} dl = \int_a^b [\sqrt{1 + (y')^2}]dx,$$

where $l(x) = \int_a^x [\sqrt{1 + [y'(s)]^2}]ds$ is the length of the curve between a and x.

Example: Calculate the length of the curve $y = x^2/2$ on the interval $[-2,\ 2]$.

Solution: Because of the symmetry, we only need to calculate the length on the interval $[0,\ 2]$ and multiply it by two. Thus, noticing that $y' = x$,

$$L = 2\int_{l(0)}^{l(2)} dl = 2\int_0^2 [\sqrt{1 + (y')^2}]dx = 2\int_0^2 [\sqrt{1 + x^2}]dx = 2F(x)|_0^2,$$

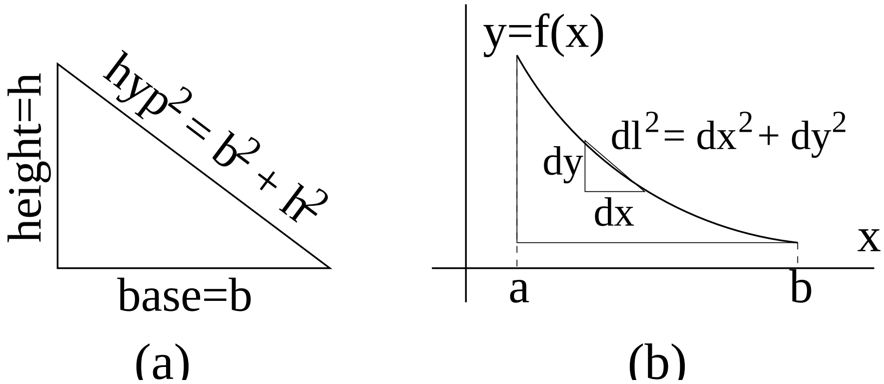


Figure 3: Length of a curve.

where $F(x) = \int[\sqrt{1+x^2}]dx$. Now, this integral can be solved by the standard substitution $x = sinh(u)$. Note that $1 + x^2 = 1 + sinh^2(u) = cosh^2(u)$ and $dx = dsinh(u) = cosh(u)du$, we obtain by subsitution

$$F(x) = \int[\sqrt{1+x^2}]dx = \int cosh(u)cosh(u)du = \int cosh^2(u)du = \frac{1}{2}\int[1 + cosh(2u)]du$$

$$= \frac{1}{2}[u + \frac{1}{2}sinh(2u)] + C = \frac{1}{2}[u + sinh(u)cosh(u)] + C = \frac{1}{2}[\ln|x + \sqrt{1+x^2}| + x\sqrt{1+x^2}] + C,$$

where the inverse substitution $sinh(u) = x$, $cosh(u) = \sqrt{1+x^2}$, and $u = sinh^{-1}(x) = \ln|x + \sqrt{1+x^2}|$ were used in the last step.

Thus,

$$L = 2F(x)|_0^2 = \ln|2 + \sqrt{1+2^2}| + 2\sqrt{1+2^2} = \ln(2+\sqrt{5}) + 2\sqrt{5} \approx 5.916.$$

14.4 From density to mass.

We have learned that the mass contained in a solid is

$$Mass = density \times volume.$$

The formula is no longer valid if the density is NOT a constant but is nonuniformly distributed in the solid. However, if you cut the mass up into infinitely many pieces of inifinitely small volume dV, then the density in this little volume can be considered a constant so that we can apply the above formula. Thus, the infinitely small mass contained in the little volume is

$$dm = density \times (infinitely\ small\ volume) = \rho dV.$$

The total mass is obtained by adding up the masses of all such small pieces.

$$m = \int_0^{m(V(b))} dm = \int_{V(a)}^{V(b)} \rho dV,$$

where $V(x) = \int_0^{V(x)} dV$ is the volume of the portion of the solid between a and x; while $m(V(x)) = \int_0^{m(V(x))} dm$ is the mass contained in the volume $V(x)$.

Example: Calculate the total amount of pollutant in an exhaust pipe filled with polluted liquid which connects a factory to a river. The pipe is 100 m long with a diameter of 1 m. The density of the pollutant in the pipe is $\rho(x) = e^{-x/10}$ (kg/m^3), x is the distance in meters from the factory.

Solution: Since the density only varies as a function of the distance x, we can "cut" the pipe into thin cylindrical disks along the axis of the pipe. Thus, $dV = d(\pi r^2 x) = \pi r^2 dx$. Using the integral form of the formula, we obtain

$$m = \int_{m(0)}^{m(100)} dm = \int_{V(0)}^{V(100)} \rho(x) dV$$

$$= \int_0^{100} \rho(x) \pi r^2 dx = \frac{\pi}{4} \int_0^{100} e^{-x/10} dx = \frac{10\pi}{4}[1 - e^{-10}] \approx 7.85\ (kg).$$

Example: The air density h meters above the earth's surface is $\rho(h) = 1.28e^{-0.000124h}$ (kg/m^3). Find the mass of a cylindrical column of air 4 meters in diameter and 25 kilometers high. (3 points)

Solution: Similar to the previous problem, the density only varies as a function of the altitutde h. Thus, we "cut" this air column into horizontal slices of thin disks with volume $dV = \pi r^2 dh$.

$$m = \int_{m(0)}^{m(25,000)} dm = \int_{V(0)}^{V(25,000)} \rho dV = \int_0^{25,000} \rho(h)\pi r^2 dh = \int_0^{25,000} \rho(h)\pi 2^2 dh$$

$$= \frac{4 \times 1.28\pi}{-0.000124} e^{-0.000124h}|_0^{25,000} = \frac{4 \times 1.28\pi}{0.000124}[1 - e^{-3.1}] = 123,873.71\ (kg).$$

Very often the density is given as mass per unit area (two dimensional density). In this case, $dm = \rho dA$, where dA is an infinitely small area. Similarly, if the density is given as mass per unit length (one dimensional density), then $dm = \rho dx$, where dx is an infinitely small length.

Example: The density of bacteria growing in a circular colony of radius 1 cm is observed to be $\rho(r) = 1 - r^2,\ 0 \leq r \leq 1$ (in units of one million cells per square centimeter), where r is the distance (in cm) from the center of the colony. What is the total number of bacteria in the colony?

Solution: This is a problem involving a two dimensional density. However, the density varies only in the radial direction. Thus, we can divide the circular colony into infinitely many concentric ring areas. We can use the formula ($\#\ of\ cells\ in\ a\ ring$) = $density \times$ ($area\ of\ the\ ring$), i.e., $dm = \rho(r)dA$. The area, dA, of a ring of bacteria colony with radius r and width dr is $dA = \pi(r+dr)^2 - \pi r^2 = 2\pi r dr + \pi dr^2 = 2\pi r dr$, where dr^2 is infinitely smaller than dr, thus is ignored as $dr \to 0$. Another way to calculate the area of the ring is to consider the ring as a rectangular area of length $2\pi r$ (i.e., the circumference) and width dr, $dA = length \times width = 2\pi r dr$. Or simply, $A = \pi r^2$, thus $dA = d(\pi r^2) = \pi dr^2 = 2\pi dr$.

$$m = \int_{m(0)}^{m(1)} dm = \int_{A(0)}^{A(1)} \rho dA = \int_0^1 \rho(r) d(\pi r^2) = \int_0^1 \rho(r) 2\pi r dr$$

$$= \pi \int_0^1 (1 - r^2) dr^2 = -\pi \frac{(1-r^2)^2}{2}|_0^1 = \frac{\pi}{2}\ (millions).$$

Example: The density of a band of protein along a one-dimensional strip of gel in an electrophoresis experiment is given by $\rho(x) = 6(x-1)(2-x)$ for $1 \leq x \leq 2$, where x is the distance along the strip in cm and $\rho(x)$ is the protein density (i.e. protein mass per cm) at distance x. Find the total mass of the protein in the band for $1 \leq x \leq 2$.

Solution: This is a problem involving a one dimensional density. Thus, ($infinitely\ small\ mass$) = $density \times$ ($infinitely\ small\ length$), i.e., $dm = \rho(x)dx$.

$$m = \int_{m(1)}^{m(2)} dm = \int_1^2 \rho(x)dx = -6\int_1^2 (x-1)(x-2)dx = -[2x^3-9x^2+12x]\Big|_1^2 = -4-(-5) = 1.$$

14.5 Work done by a varying force.

Elementary physics tells us that work done by a force is

$$Work = force \times distance.$$

The formula is no longer valid if the force is NOT a constant over the distance covered but changes. However, if we "cut" the distance into infinitely many segments of inifinitely short distances dx, then the force on that little distance can be considered a constant. Thus, the infinitely small work done on a distance dx is

$$dW = force \times (infintely\ small\ distance) = fdx.$$

The total work is obtained by adding up the work done on all such small distances.

$$W = \int_{W(a)}^{W(b)} dW = \int_a^b fdx.$$

Example: Calculate total work done by the earth's gravitational force on a satellite of mass m when it is being launched into the space. This is the lower limit of energy required to launch it into space. Note that the gravitational force experienced by a mass m located at a distance r from the center of the earth is: $f(r) = GM_em/r^2$, where G is the universal gravitational constant and M_e the mass of the earth. Assume that the earth is a sphere with radius R_e and that $r = \infty$ when the satellite is "outside" the earth's gravitational field. (Consider G, M_e, R_e, and m are known constants.)

Solution: We know that work done on an infinitely small distance dr when the satellite is located at a distance r from the center of the earth is: $dW = f(r)dr = (GM_em/r^2)dr$. Thus,

$$W = \int_{W(R_e)}^{W(\infty)} dW = \int_{R_e}^{\infty} f(r)dr = GM_em\int_{R_e}^{\infty} \frac{1}{r^2}dr.$$

Integrals that involve a limit which is ∞ are called *improper integrals*. We shall study improper integrals in more detail later. In many cases, we can just treat ∞ as a normal upper limit of this definite integral.

$$W = GM_e m \int_{R_e}^{\infty} \frac{1}{r^2} dr = GM_e m(-\frac{1}{r})|_{R_e}^{\infty} = \frac{GM_e m}{R_e} = mgR_e,$$

where the gravitaional constant on the surface of the earth $g = GM_e/R_e^2$ was used.

Example: For the time interval $0 \le t \le 10$, write down the definite integral determining the total work done by a force, $f(t)$ ($Newton$), experienced by a particle moving on a straight line with a speed, $v(t) > 0$ (m/s). $f(t)$ and $v(t)$ always point to the same direction and both change with time t (s).

Solution:

$$W = \int_{W(0)}^{W(10)} dW = \int_{x(0)}^{x(10)} f(t)dx = \int_0^{10} f(t)(\frac{dx}{dt})dt = \int_0^{10} f(t)v(t)dt.$$

15 Elementary integrals

All of these follow immediately from the table of derivatives. They should be memorized.

$$\int cf(x)\,dx = c\int f(x)\,dx \qquad \int [\alpha f + \beta g]\,dx = \alpha\int f\,dx + \beta\int g\,dx$$

$$\int c\,dx = cx + C \qquad \int dx = x + C$$

$$\int x^r\,dx = \frac{x^{r+1}}{r+1} + C \quad (r \neq -1) \qquad \int \frac{1}{x}\,dx = \ln|x| + C$$

$$\int e^x\,dx = e^x + C \qquad \int \ln x\,dx = x\ln x - x + C$$

The following are elementary integrals involving trigonometric functions.

$$\int \sin x\,dx = -\cos x + C \qquad \int \cos x\,dx = \sin x + C$$

$$\int \sec^2 x\,dx = \tan x + C \qquad \int \csc^2 x\,dx = -\cot x + C$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a} tan^{-1}\frac{x}{a} + C \qquad \int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a} + C$$

$$\int \tan x\;dx = -\ln|\cos x| + C \qquad \int \cot x\;dx = \ln|\sin x| + C$$

$$\int \sec x\;dx = \ln|\tan x + \sec x| + C \qquad \int \csc x\;dx = \ln|\cot x - \csc x| + C$$

The following are elementary integrals involving hyperbolic functions (for details read the Section on Hyperbolic Functions).

$$\int \sinh x\,dx = \cosh x + C \qquad \int \cosh x\,dx = \sinh x + C$$

$$\int \operatorname{sech}^2 x\,dx = \tanh x + C \qquad \int \operatorname{csch}^2 x\,dx = -\coth x + C$$

$$\int \frac{1}{a^2-x^2}\,dx = \frac{1}{a}\tanh^{-1}\frac{x}{a} + C \qquad \int \frac{1}{\sqrt{a^2+x^2}}\,dx = \sinh^{-1}\frac{x}{a} + C$$

$$\int \tanh x\;dx = \ln(\cosh x) + C \qquad \int \coth x\;dx = \ln|\sinh x| + C$$

$$\int \operatorname{sech} x\;dx = tan^{-1}(\sinh x) + C \qquad \int \operatorname{csch} x\;dx = \ln|\coth x - \operatorname{csch} x| + C$$

16 A selection of more complicated integrals

These begin with the two basic formulas, change of variables and integration by parts.

$\int f(g(x))g'(x)\,dx = \int f(u)\,du$ where $u = g(x)$ (change of variables)

$\int f(g(x))\,dx = \int f(u)\frac{dx}{du}\,du$ where $u = g(x)$ (different form of the same change of variables)

$\int e^{cx}\,dx = \frac{1}{c}e^{cx} + C \quad (c \neq 0)$

$\int a^x\,dx = \frac{1}{\ln a}a^x + C$ (for $a > 0,\ a \neq 1$)

$\int \ln x\,dx = x\ln x - x + C$

$\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\arctan\frac{x}{a} + C,\ a \neq 0$

$\int \frac{1}{x^2-a^2}\,dx = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + C,\ a \neq 0$

$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \arcsin\frac{x}{a} + C,\ a > 0$

$\int \frac{1}{\sqrt{x^2\pm a^2}}\,dx = \ln|x + \sqrt{x^2\pm a^2}| + C$

To compute $\int \frac{1}{x^2+bx+c}\,dx$ we complete the square

$$x^2+bx+c = x^2+bx+\frac{b^2}{4}+c-\frac{b^2}{4} = \left(x+\frac{b}{2}\right)^2 + c - \frac{b^2}{4}$$

If $c - b^2/4 > 0$, set it equal to a^2; if < 0 equal to $-a^2$; and if $= 0$ forget it. In any event you will arrive after the change of variables $u = x + \frac{b}{2}$ at one of the three integrals

$$\int \frac{1}{u^2+a^2}\,du, \quad \int \frac{1}{u^2-a^2}\,du, \quad \int \frac{1}{u^2}\,du$$

$\int \sqrt{x^2\pm a^2}\,dx = \frac{1}{2}\left(x\sqrt{x^2\pm a^2} \pm a^2\ln\left|x+\sqrt{x^2\pm a^2}\right|\right) + C$

$\int x^n e^{cx}\,dx = x^n\frac{e^{cx}}{c} - \frac{n}{c}\int x^{n-1}e^{cx}\,dx$ etc. This is to be used repetaedly until you arrive at the case $n = 0$, which you can do easily.

17 Hyperbolic Functions

Definition: The most frequently used hyperbolic functions are defined by,

$$\sinh x = \frac{e^x - e^{-x}}{2} \qquad \cosh x = \frac{e^x + e^{-x}}{2} \qquad \tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

where $\sinh x$ and $\tanh x$ are odd like $\sin x$ and $\tan x$, while $\cosh x$ is even like $\cos x$ (see Fig. 4 for plots of these functions). The definitions of other hyperbolic functions are identical to the corresponding trig functions.

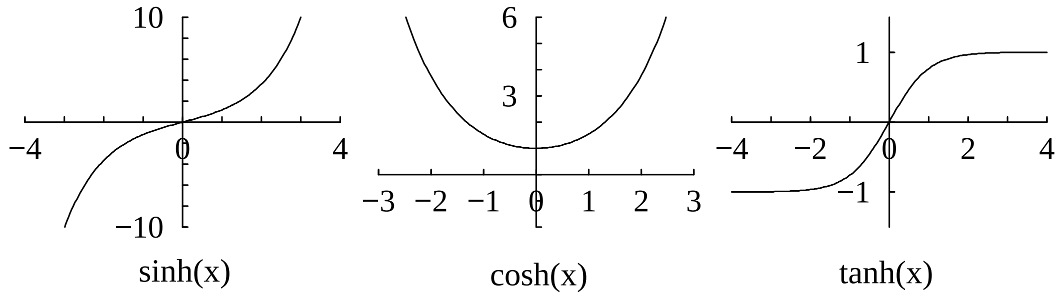


Figure 4: Graphs of $\sinh x$, $\cosh x$, and $\tanh x$.

We can easily derive all the corresponding identities for hyperbolic functions by using the definition. However, it helps us memorize these identities if we know the relationship between $\sinh x$, $\cosh x$ and $\sin x$, $\cos x$ in complex analysis.

Combining $\sinh x = \dfrac{e^x - e^{-x}}{2}$ and $\sin x = \dfrac{e^{ix} - e^{-ix}}{2i}$, we obtain $\sinh(x) = -i\sin(ix)$.

Combining $\cosh x = \dfrac{e^x + e^{-x}}{2}$ and $\cos x = \dfrac{e^{ix} + e^{-ix}}{2}$, we obtain $\cosh(x) = \cos(ix)$.

where the constant $i = \sqrt{-1}$ and $i^2 = -1$. Now, we can write down these identities:

$\cos^2 x + \sin^2 x = 1$	$\cosh^2 x - \sinh^2 x = 1$
$1 + \tan^2 x = \sec^2 x$	$1 - \tanh^2 x = \mathrm{sech}^2 x$
$\cot^2 x + 1 = \csc^2 x$	$\coth^2 x - 1 = \mathrm{csch}^2 x$
$\cos^2 x = (1 + \cos 2x)/2$	$\cosh^2 x = (1 + \cosh 2x)/2$
$\sin^2 x = 1 - \cos^2 x = (1 - \cos 2x)/2$	$\sinh^2 x = \cosh^2 x - 1 = (-1 + \cosh 2x)/2$
$\sin 2x = 2\sin x \cos x$	$\sinh 2x = 2\sinh x \cosh x$
$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$	$\sinh(x \pm y) = \sinh x \cosh y \pm \cosh x \sinh y$
$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$	$\cosh(x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y$

The Advantage of Hyperbolic Functions

The advantage of hyperbolic functions is that their inverse functions can be expressed in terms of elementary functions. We here show how to solve for the inverse hyperbolic functions.

By definition, $u = \sinh x = \dfrac{e^x - e^{-x}}{2}$.

Multiplying both sides by $2e^x$, we obtain

$2ue^x = (e^x)^2 - 1$ which simplifies to $(e^x)^2 - 2u(e^x) - 1 = 0$.

This is a quadratic in e^x, Thus,

$e^x = [2u \pm \sqrt{(2u)^2 + 4}]/2 = u \pm \sqrt{u^2 + 1}$.

The negative sign in $\pm$ does not make sense since $e^x > 0$ for all x.
Thus, $x = \ln e^x = \ln |u + \sqrt{u^2 + 1}|$ which implies

$x = \sinh^{-1} u = \ln |u + \sqrt{u^2 + 1}|$.

Similarly, by definition, $u = \cosh x = \dfrac{e^x + e^{-x}}{2}$.

We can solve this equation for the two branches of the inverse:

$x = \cosh^{-1} u = \ln |u \pm \sqrt{u^2 - 1}| = \pm \ln |u + \sqrt{u^2 - 1}|$.

Similarly, for

$u = \tanh x = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$.

We can solve the equation for the inverse:

$x = \tanh^{-1} u = \dfrac{1}{2} \ln |\dfrac{1 + u}{1 - u}|$.

The Advantage of Trigonometric Functions

Relations between different trigonometric functions are very important since when we differentiate or integrate one trig function we obtain another trig function. When we use trig substitution, it is often a necessity for us to know the definition of other trig functions that is related to the one we use in the substitution.

Example: Calculate the derivative of $y = \sin^{-1} x$.

Solution: $y = \sin^{-1} x$ means $\sin y = x$. Differentiate both sides of $\sin y = x$, we obtain

$$\cos y y' = 1 \;\Rightarrow\; y' = \frac{1}{\cos y}.$$

In order to express y' in terms of x, we need to express $\cos y$ in terms of x. Given that $\sin y = x$, we can obtain $\cos y = \sqrt{\cos^2 y} = \sqrt{1 - \sin^2 y} = \sqrt{1 - x^2}$ by using trig identities. However, it is easier to construct a right triangle with an angle y. The opposite side must be x while the hypotenuse must be 1, thus the adjacent side is $\sqrt{1 - x^2}$. Therefore,

$$\cos y = \frac{\textit{adjacent side}}{\textit{hypotenuse}} = \frac{\sqrt{1 - x^2}}{1} = \sqrt{1 - x^2}.$$

Example: Calculate the integral $\int \sqrt{1 + x^2} dx$.

Solution: This integral requires standard trig substitution $x = \tan u$ or $x = \sinh u$. Let's use $x = \sinh u$. Recall that $1 + \sinh^2 u = \cosh^2 u$ and that $dx = d \sinh u = \cosh u du$,

$$\int \sqrt{1 + x^2} dx = \int \sqrt{1 + \sinh^2 u} d \sinh u = \int \cosh^2 u du$$

$$= \frac{1}{2} \int [1 + \cosh(2u)] du = \frac{1}{2}[u + \frac{1}{2}\sinh(2u)] + C = \frac{1}{2}[u + \sinh u \cosh u] + C.$$

where hyperbolic identities $\cosh^2 u = [1 + \cosh(2u)]/2$ and $\sinh(2u) = 2 \sinh u \cosh u$ were used. However, we need to express the solution in terms of x. Since $x = \sinh u$ was the substitution, we know right away $u = \sinh^{-1} x = \ln |x + \sqrt{1 + x^2}|$ and $\sinh u = x$, but how to express $\cosh u$ in terms of x? We can solve it using hyperbolic identities. $\cosh u = \sqrt{\cosh^2 u} = \sqrt{1 + \sinh^2 u} = \sqrt{1 + x^2}$. Thus,

$$\frac{1}{2}[u + \sinh u \cosh u] = \frac{1}{2}[\ln |x + \sqrt{1 + x^2}| + x\sqrt{1 + x^2}].$$

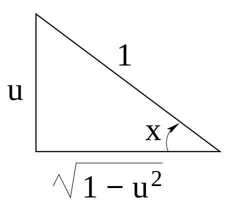
However, the following table shows how can we exploit the similarity between trig and hyperbolic functions to find out the relationship between different hyperbolic functions.

Relations between different trig and between different hyperbolic functions

==

Given one trig func, $\sin x = u = \dfrac{u}{1}$

Draw a right triangle
with an angle x, opp = u, hyp = 1:



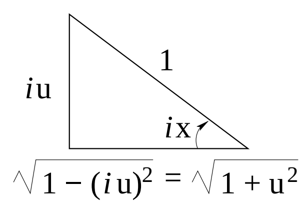
Thus, other trig functions are

$$\cos x = \frac{\text{adj}}{\text{hyp}} = \sqrt{1-u^2}$$

$$\tan x = \frac{\text{opp}}{\text{adj}} = \frac{u}{\sqrt{1-u^2}}$$

Given $\sinh x = -i\sin(ix) = u$, $\sin(ix) = i\text{u} = \dfrac{i\text{u}}{1}$

Draw the corresponding right triangle
with an angle ix, opp = iu, hyp = 1:



By formal analogy,

$$\cosh x = \cos(ix) = \frac{\text{adj}}{\text{hyp}} = \sqrt{1+u^2}$$

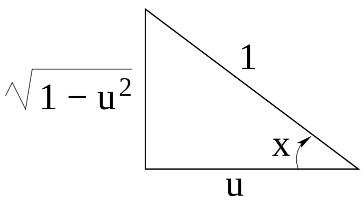
$$\tanh x = -i\tan(ix) = -i\frac{\text{opp}}{\text{adj}} = \frac{u}{\sqrt{1+u^2}}$$

==

==

Given that, $\cos x = u = \dfrac{u}{1}$

Draw a right triangle
with an angle x, adj = u, hyp = 1:



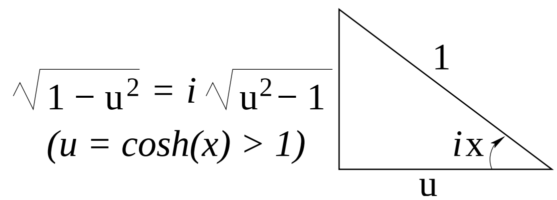
Thus, other trig functions are

$$\sin x = \frac{\text{opp}}{\text{hyp}} = \sqrt{1-u^2}$$

$$\tan x = \frac{\text{opp}}{\text{adj}} = \frac{\sqrt{1-u^2}}{u}$$

Given $\cosh x = \cos(ix) = u = \dfrac{u}{1}$

Draw the corresponding right triangle
with an angle ix, adj = iu, hyp = 1:



By formal analogy,

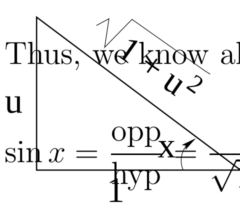
$$\sinh x = -i\sin(ix) = -i\frac{\text{opp}}{\text{hyp}} = \sqrt{u^2-1}$$

$$\tanh x = -i\tan(ix) = -i\frac{\text{opp}}{\text{adj}} = \frac{\sqrt{u^2-1}}{u}$$

==

Given that, $\tan x = u = \dfrac{u}{1}$

Draw a right triangle
with an angle x, opp = u, adj = 1:



Thus, we know all other trig functions

$$\sin x = \frac{\text{opp}}{\text{hyp}} = \frac{u}{\sqrt{1+u^2}}$$

$$\cos x = \frac{\text{adj}}{\text{hyp}} = \frac{1}{\sqrt{1+u^2}}$$

Given $\tanh x = (-i)\tan(ix) = u$, $\tan(ix) = i\text{u} = \frac{i\text{u}}{1}$

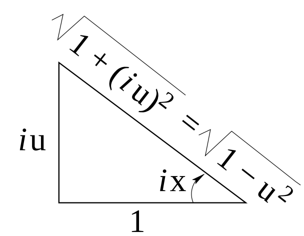
Draw the corresponding right triangle
with an angle ix, opp = iu, adj = 1:

By formal analogy,

$$\sinh x = -i\sin(ix) = -i\frac{\text{opp}}{\text{hyp}} = \frac{u}{\sqrt{1-u^2}}$$

$$\cosh x = \cos(ix) = \frac{\text{adj}}{\text{hyp}} = \frac{1}{\sqrt{1-u^2}}$$

==


$\sqrt{1+(iu)^2} = \sqrt{1-u^2}$
iu
ix
1

Mathematics Notes for Class 12 chapter 10. Vector Algebra

A **vector** has direction and magnitude both but scalar has only magnitude.

Magnitude of a vector a is denoted by |a| or a. It is non-negative scalar.

Equality of Vectors

Two vectors a and b are said to be equal written as a = b, if they have (i) same length (ii) the same or parallel support and (iii) the same sense.

Types of Vectors

(i) **Zero or Null Vector** A vector whose initial and terminal points are coincident is called zero or null vector. It is denoted by 0.

(ii) **Unit Vector** A vector whose magnitude is unity is called a unit vector which is denoted by $\hat{n}$

(iii) **Free Vectors** If the initial point of a vector is not specified, then it is said to be a free vector.

(iv) **Negative of a Vector** A vector having the same magnitude as that of a given vector a and the direction opposite to that of a is called the negative of a and it is denoted by —a.

(v) **Like and Unlike Vectors** Vectors are said to be like when they have the same direction and unlike when they have opposite direction.

(vi) **Collinear or Parallel Vectors** Vectors having the same or parallel supports are called collinear vectors.

(vii) **Coinitial Vectors** Vectors having same initial point are called coinitial vectors.

(viii) **Coterminous Vectors** Vectors having the same terminal point are called coterminous vectors.

(ix) **Localized Vectors** A vector which is drawn parallel to a given vector through a specified point in space is called localized vector.

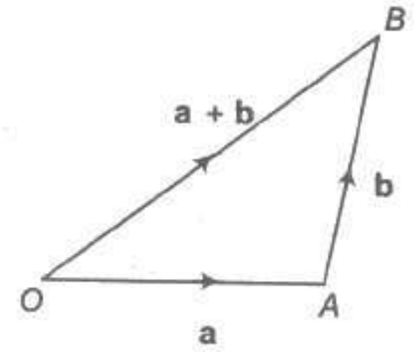
(x) **Coplanar Vectors** A system of vectors is said to be coplanar, if their supports are parallel to the same plane. Otherwise they are called non-coplanar vectors.

(xi) **Reciprocal of a Vector** A vector having the same direction as that of a given vector but magnitude equal to the reciprocal of the given vector is known as the reciprocal of a.

i.e., if $|a| = a$, then $|a^{-1}| = 1 / a$.

Addition of Vectors

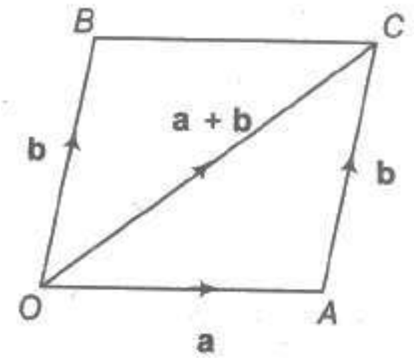
Let **a** and **b** be any two vectors. From the terminal point of a, vector b is drawn. Then, the vector from the initial point O of a to the terminal point B of b is called the sum of vectors a and b and is denoted by**a + b**. This is called the triangle law of addition of vectors.



Parallelogram Law

Let a and b be any two vectors. From the initial point of a, vector b is drawn and parallelogram OACB is completed with OA and OB as adjacent sides. The vector OC is defined as the sum of a and b. This is called the parallelogram law of addition of vectors.

The sum of two vectors is also called their resultant and the process of addition as composition.



Properties of Vector Addition

(i) a + b = b + a (commutativity)

(ii) a + (b + c)= (a + b)+ c (associativity)

(iii) a+ O = a (additive identity)

(iv) a + (— a) = 0 (additive inverse)

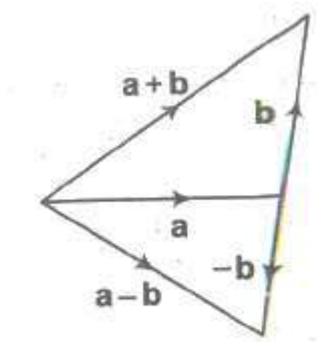
(v) $(k_1 + k_2)$ a = k_1 a + k_2a (multiplication by scalars)

(vi) k(a + b) = k a + k b (multiplication by scalars)

(vii) |a+ b| ≤ |a| + |b| and |a – b| ≥ |a| – |b|

Difference (Subtraction) **of Vectors**

If a and b be any two vectors, then their difference a – b is defined as a + (- b).



Multiplication of a Vector by a Scalar

Let a be a given vector and λ be a scalar. Then, the product of the vector a by the scalar λ is λ a and is called the multiplication of vector by the scalar.

Important Properties

(i) |λ a| = |λ| |a|

(ii) λ O = O

(iii) m (-a) = – ma = – (m a)

(iv) (-m) (-a) = m a

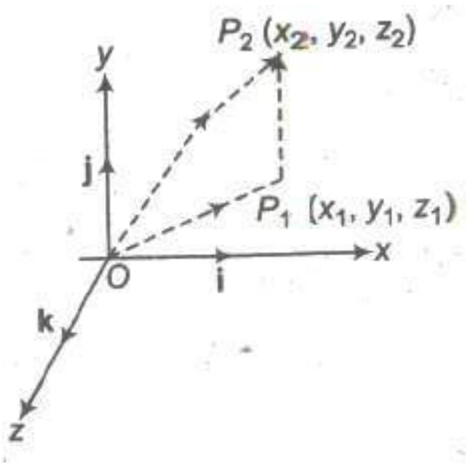
(v) m (n a) = mn a = n(m a)

(vi) (m + n)a = m a+ n a

(vii) m (a+b) = m a + m b

Vector Equation of Joining by Two Points

Let P_1 (x_1, y_1, z_1) and P_2 (x_2, y_2, z_2) are any two points, then the vector joining P_1 and P_2 is the vector P_1 P_2.



The component vectors of P and Q are

OP = $x_1i + y_1j + z_1k$

and OQ = $x_2i + y_2j + z_2k$

i.e., P_1 P_2 = $(x_2i + y_2j + z_2k) – (x_1i + y_1j + z_1k)$

= $(x_2 – x_1)\ i + (y_2 – y_1)\ j + (z_2 – z_1)\ k$

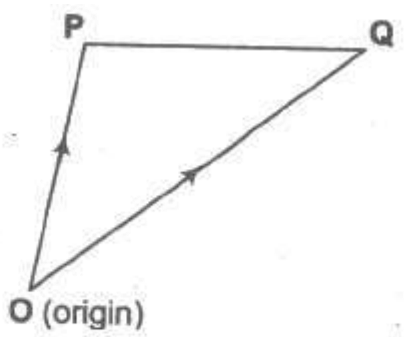
Its magnitude is

P_1 P_2 = $\sqrt{}(x_2 – x_1)^2 + (y_2 – y_1)^2 + (z_2 – z_1)^2$

Position Vector of a Point

The position vector of a point P with respect to a fixed point, say O, is the vector OP. The fixed point is called the origin.

Let PQ be any vector. We have PQ = PO + OQ = — OP + OQ = OQ — OP = Position vector of Q — Position vector of P.



i.e., PQ = PV of Q — PV of P

Collinear Vectors

Vectors a and b are collinear, if a = λb, for some non-zero scalar λ.

Collinear Points

Let A, B, C be any three points.

Points A, B, C are collinear <=> AB, BC are collinear vectors.

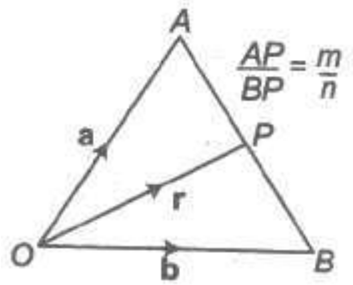
<=> AB = λBC for some non-zero scalar λ.

Section Formula

Let A and B be two points with position vectors a and b, respectively and OP= r.

(i) Let P be a point dividing AB internally in the ratio m : n. Then,

r = m b + n a / m + n



Also, (m + n) OP = m OB + n OA

(ii) The position vector of the mid-point of a and b is a + b / 2.

(iii) Let P be a point dividing AB externally in the ratio m : n. Then,

r = m b + n a / m + n

Position Vector of Different Centre of a Triangle

(i) If a, b, c be PV's of the vertices A, B, C of a ΔABC respectively, then the PV of the centroid G of the triangle is a + b + c / 3.

(ii) The PV of incentre of ΔABC is (BC)a + (CA)b + (AB)c / BC + CA + AB

(iii) The PV of orthocentre of ΔABC is

a(tan A) + b(tan B) + c(tan C) / tan A + tan B + tan C

Scalar Product of Two Vectors

If a and b are two non-zero vectors, then the scalar or dot product of a and b is denoted by a * b and is defined as a * b = |a| |b| cos θ, where θ is the angle between the two vectors and $0 < \theta < \pi$.

(i) The angle between two vectors a and b is defined as the smaller angle θ between them, when they are drawn with the same initial point.

Usually, we take $0 < \theta < \pi$.Angle between two like vectors is O and angle between two unlike vectors is π .

(ii) If either a or b is the null vector, then scalar product of the vector is zero.

(iii) If a and b are two unit vectors, then a * b = cos θ.

(iv) The scalar product is commutative

i.e., a * b= b * a

(v) If i , j and k are mutually perpendicular unit vectors i , j and k, then

i * i = j * j = k * k =1

and i * j = j * k = k * i = 0

(vi) The scalar product of vectors is distributive over vector addition.

(a) a * (b + c) = a * b + a * c (left distributive)
(b) (b + c) * a = b * a + c * a (right distributive)

Note Length of a vector as a scalar product

If a be any vector, then the scalar product

$a * a = |a|\, |a| \cos\theta \Rightarrow |a|^2 = a^2 \Rightarrow a = |a|$

Condition of perpendicularity a * b = 0 <=> a ⊥ b, a and b being non-zero vectors.

Important Points to be Remembered

(i) $(a + b) * (a – b) = |a|^2 2 – |b|^2$

(ii) $|a + b|^2 = |a|^2 2 + |b|^2 + 2 (a * b)$

(iii) $|a - b|^2 = |a|^2 2 + |b|^2 - 2\ (a * b)$

(iv) $|a + b|^2 + |a - b|^2 = (|a|^2 2 + |b|^2)$ and $|a + b|^2 - |a - b|^2 = 4\ (a * b)$

or $a * b = 1 / 4\ [\ |a + b|^2 - |a - b|^2\ ]$

(v) If $|a + b| = |a| + |b|$, then a is parallel to b.

(vi) If $|a + b| = |a| - |b|$, then a is parallel to b.

(vii) $(a * b)^2 \leq |a|^2 2\ |b|^2$

(viii) If $a = a_1 i + a_2 j + a_3 k$, then $|a|^2 = a * a = a_1^2 + a_2^2 + a_3^2$

Or

$|a| = \sqrt{}a_1^2 + a_2^2 + a_3^2$

(ix) **Angle between Two Vectors** If θ is angle between two non-zero vectors, a, b, then we have

$a * b = |a|\ |b| \cos\theta$

$\cos\theta = a * b / |a|\ |b|$

If $a = a_1 i + a_2 j + a_3 k$ and $b = b_1 i + b_2 j + b_3 k$

Then, the angle θ between a and b is given by

$\cos\theta = a * b / |a|\ |b| = a_1 b_1 + a_2 b_2 + a_3 b_3 / \sqrt{}a_1^2 + a_2^2 + a_3^2\ \sqrt{}b_1^2 + b_2^2 + b_3^2$

(x) **Projection and Component of a Vector**

Projection of a on b = $a * b / |a|$

Projection of b on a = $a * b / |a|$

Vector component of a vector a on b

$$= \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{b}|} \cdot \hat{\mathbf{b}} = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{b}|} \cdot \frac{\mathbf{b}}{|\mathbf{b}|} = \frac{(\mathbf{a} \cdot \mathbf{b})}{|\mathbf{b}^2|} \mathbf{b}$$

Similarly, the vector component of b on a = $((a * b) / |a^2|) * a$

(xi) **Work done by a Force**

The work done by a force is a scalar quantity equal to the product of the magnitude of the force and the resolved part of the displacement.

∴ F * S = dot products of force and displacement.

Suppose F_1, F_1,…, F_n are n forces acted on a particle, then during the displacement S of the particle, the separate forces do quantities of work F_1 * S, F_2 * S, F_n * S.

The total work done is $\sum_{i=1}^{n} \mathbf{F}_i \cdot \mathbf{S} = \sum_{i=1}^{n} \mathbf{S} \cdot \mathbf{F}_i = \mathbf{S} \cdot \mathbf{R}$

Here, system of forces were replaced by its resultant R.

Vector or Cross Product of Two Vectors

The vector product of the vectors a and b is denoted by a * b and it is defined as

a * b = (|a| |b| sin θ) n = ab sin θ n …..(i)

where, a = |a|, b= |b|, θ is the angle between the vectors a and b and n is a unit vector which is perpendicular to both a and b, such that a, b and n form a right-handed triad of vectors.

Important Points to be Remembered

(i) Let a = a_1i + a_2j + a_3k and b = b_1i + b_2j + b_3k

Then, $$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

(ii) If a = b or if a is parallel to b, then sin θ = 0 and so a * b = 0.

(iii) The direction of a * b is regarded positive, if the rotation from a to b appears to be anti-clockwise.

(iv) a * b is perpendicular to the plane, which contains both a and b. Thus, the unit vector perpendicular to both a and b or to the plane containing is given by n = a * b / |a * b| = a * b / ab sin θ

(v) Vector product of two parallel or collinear vectors is zero.

(vi) If a * b = 0, then a = O or b = 0 or a and b are parallel on collinear.

(vii) **Vector Product of Two Perpendicular Vectors**

If θ = 900, then sin θ = 1, i.e. , a * b = (ab)n or |a * b| = |ab n| = ab

(viii) **Vector Product of Two Unit Vectors** If a and b are unit vectors, then

a = |a| = 1, b = |b| = 1
∴ a * b = ab sin θ n = (sin theta;).n

(ix) **Vector Product is not Commutative** The two vector products a * b and b * a are equal in magnitude but opposite in direction.

i.e., b * a =- a * b ……..(i)

(x) The vector product of a vector a with itself is null vector, i. e., a * a= 0.

(xi) **Distributive Law** For any three vectors a, b, c

a * (b + c) = (a * b) + (a * c)

(xii) **Area of a Triangle and Parallelogram**

(a) The vector area of a ΔABC is equal to 1 / 2 |AB * AC| or 1 / 2 |BC * BA| or 1 / 2 |CB * CA|.

(b) The area of a ΔABC with vertices having PV's a, b, c respectively, is 1 / 2 |a * b + b * c + c * a|.

(c) The points whose PV's are a, b, c are collinear, if and only if a * b + b * c + c * a

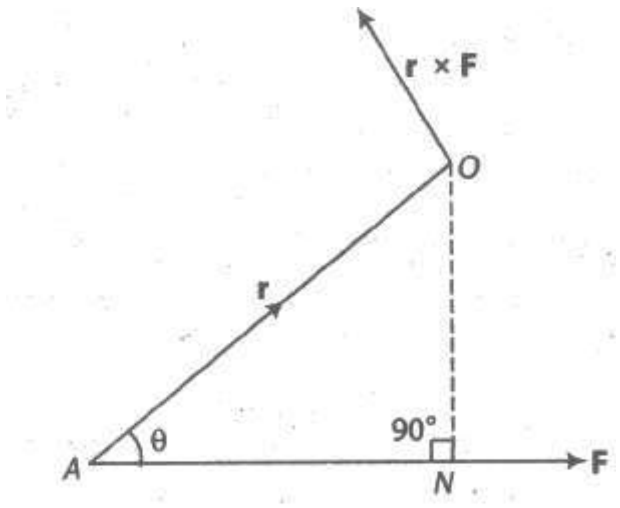
(d) The area of a parallelogram with adjacent sides a and b is |a * b|.

(e) The area of a Parallelogram with diagonals a and b is 1 / 2 |a * b|.

(f) The area of a quadrilateral ABCD is equal to 1 / 2 |AC * BD|.

(xiii) **Vector Moment of a Force about a Point**

The vector moment of torque M of a force F about the point O is the vector whose magnitude is equal to the product of |F| and the perpendicular distance of the point O from the line of action of F.



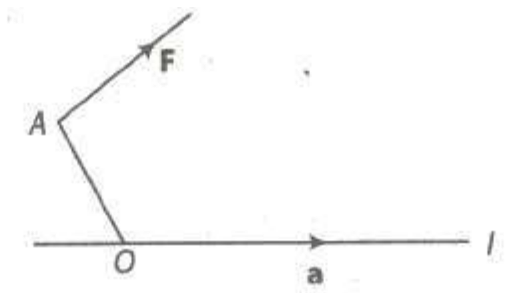
∴ M = r * F

where, r is the position vector of A referred to O.

(a) The moment of force F about O is independent of the choice of point A on the line of action of F.

(b) If several forces are acting through the same point A, then the vector sum of the moments of the separate forces about a point O is equal to the moment of their resultant force about O.

(xiv) **The Moment of a Force about a Line**

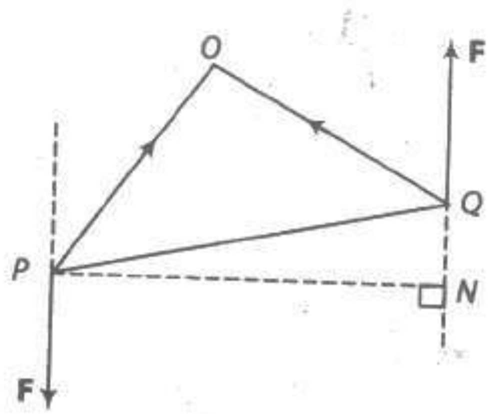


Let F be a force acting at a point A, O be any point on the given line L and a be the unit vector along the line, then moment of F about the line L is a scalar given by (OA x F) * a

(xv) **Moment of a Couple**

(a) Two equal and unlike parallel forces whose lines of action are different are said to constitute a couple.

(b) Let P and Q be any two points on the lines of action of the forces – F and F, respectively.



The moment of the couple = PQ x F

Scalar Triple Product

If a, b, c are three vectors, then (a * b) * c is called scalar triple product and is denoted by [a b c].

∴ [a b c] = (a * b) * c

Geometrical Interpretation of Scalar Triple Product

The scalar triple product (a * b) * c represents the volume of a parallelepiped whose coterminous edges are represented by a, b and c which form a right handed system of vectors.

Expression of the scalar triple product (a * b) * c in terms of components

$a = a_1i + a_1j + a_1k$, $b = a_2i + a_2j + a_2k$, $c = a_3i + a_3j + a_3k$ is

$$[\mathbf{a}\ \mathbf{b}\ \mathbf{c}] = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & b_3 \end{vmatrix}$$

Properties of Scalar Triple Products

1. The scalar triple product is independent of the positions of dot and cross i.e., (a * b) * c = a * (b * c).

2. The scalar triple product of three vectors is unaltered so long as the cyclic order of the vectors remains unchanged.

i.e., (a * b) * c = (b * c) * a= (c * a) * b
or
[a b c] = [b c a] = [c a b].

3. The scalar triple product changes in sign but not in magnitude, when the cyclic order is changed.

i.e., [a b c] = – [a c b] etc.

4. The scalar triple product vanishes, if any two of its vectors are equal.

i.e., [a a b] = 0, [a b a] = 0 and [b a a] = 0.

5. The scalar triple product vanishes, if any two of its vectors are parallel or collinear.

6. For any scalar x, [x a b c] = x [a b c]. Also, [x a yb zc] = xyz [a b c].

7. For any vectors a, b, c, d, [a + b c d] = [a c d] + [b c d]

8. [i j k] = 1

9. $(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d})=\begin{vmatrix}\mathbf{a}\cdot\mathbf{c} & \mathbf{b}\cdot\mathbf{c}\\ \mathbf{a}\cdot\mathbf{d} & \mathbf{b}\cdot\mathbf{d}\end{vmatrix}$

10. $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]\,[\mathbf{u}\ \mathbf{v}\ \mathbf{w}]=\begin{vmatrix}\mathbf{a}\cdot\mathbf{u} & \mathbf{b}\cdot\mathbf{u} & \mathbf{c}\cdot\mathbf{u}\\ \mathbf{a}\cdot\mathbf{v} & \mathbf{b}\cdot\mathbf{v} & \mathbf{c}\cdot\mathbf{v}\\ \mathbf{a}\cdot\mathbf{w} & \mathbf{b}\cdot\mathbf{w} & \mathbf{c}\cdot\mathbf{w}\end{vmatrix}$

11. Three non-zero vectors a, b and c are coplanar, if and only if [a b c] = 0.

12. Four points A, B, C, D with position vectors a, b, c, d respectively are coplanar, if and only if [AB AC AD] = 0.

i.e., if and only if [b — a c— a d— a] = 0.

13. Volume of parallelepiped with three coterminous edges a, b,c is | [a b c] |.

14. Volume of prism on a triangular base with three coterminous edges a, b,c is 1 / 2 | [a b c] |.

15. Volume of a tetrahedron with three coterminous edges a, b,c is 1 / 6 | [a b c] |.

16. If a, b, c and d are position vectors of vertices of a tetrahedron, then

Volume = 1 / 6 [b — a c — a d — a].

Vector Triple Product

If a, b, c be any three vectors, then (a * b) * c and a * (b * c) are known as vector triple product.

∴ a * (b * c)= (a * c)b — (a * b) c

and (a * b) * c = (a * c)b — (b * c) a

Important Properties

(i) The vector r = a * (b * c) is perpendicular to a and lies in the plane b and c.

(ii) a * (b * c) ≠ (a * b) * c, the cross product of vectors is not associative.

(iii) a * (b * c)= (a * b) * c, if and only if and only if (a * c)b — (a * b) c = (a * c)b — (b * c) a, if and only if c = (b * c) / (a * b) * a

Or if and only if vectors a and c are collinear.

Reciprocal System of Vectors

Let a, b and c be three non-coplanar vectors and let

a’ = b * c / [a b c], b’ = c * a / [a b c], c’ = a * b / [a b c]

Then, a’, b’ and c’ are said to form a reciprocal system of a, b and c.

Properties of Reciprocal System

(i) a * a’ = b * b’= c * c’ = 1

(ii) a * b’= a * c’ = 0, b * a’ = b * c’ = 0, c * a’ = c * b’= 0

(iii) [a’, b’, c’] [a b c] = 1 ⇒ [a’ b’ c’] = 1 / [a b c]

(iv) a = b’ * c’ / [a’, b’, c’], b = c’ * a’ / [a’, b’, c’], c = a’ * b’ / [a’, b’, c’]

Thus, a, b, c is reciprocal to the system a’, b’ ,c’.

(v) The orthonormal vector triad i, j, k form self reciprocal system.

(vi) If a, b, c be a system of non-coplanar vectors and a’, b’, c’ be the reciprocal system of vectors, then any vector r can be expressed as r = (r * a’)a + (r * b’)b + (r * c’) c.

Linear Combination of Vectors

Let a, b, c,… be vectors and x, y, z, … be scalars, then the expression x a yb + z c + … is called a linear combination of vectors a, b, c,….

Collinearity of Three Points

The necessary and sufficient condition that three points with PV’s b, c are collinear is that there exist three scalars x, y, z not all zero such that xa + yb + zc ⇒ x + y + z = 0.

Coplanarity of Four Points

The necessary and sufficient condition that four points with PV’s a, b, c, d are coplanar, if there exist scalar x, y, z, t not all zero, such that xa + yb + zc + td = 0 rArr; x + y + z + t = 0.

If r = xa + yb + zc…

Then, the vector r is said to be a linear combination of vectors a, b, c,….

Linearly Independent and Dependent System of Vectors

(i) The system of vectors a, b, c,… is said to be linearly dependent, if there exists a scalars x, y, z, … not all zero, such that xa + yb + zc + … = 0.

(ii) The system of vectors a, b, c, … is said to be linearly independent, if xa + yb + zc + td = 0 rArr; x + y + z + t… = 0.

Important Points to be Remembered

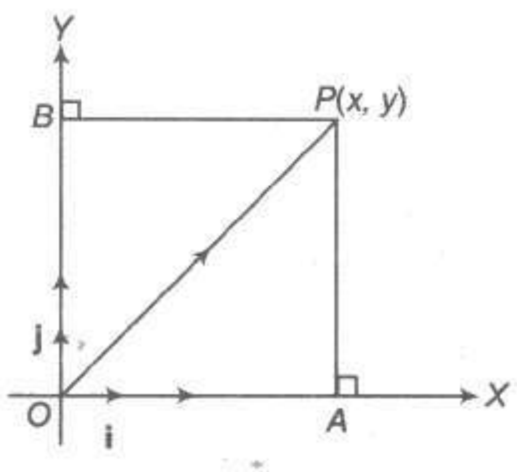
(i) Two non-collinear vectors a and b are linearly independent.

(ii) Three non-coplanar vectors a, b and c are linearly independent.

(iii) More than three vectors are always linearly dependent.

Resolution of Components of a Vector in a Plane

Let a and b be any two non-collinear vectors, then any vector r coplanar with a and b, can be uniquely expressed as r = x a + y b, where x, y are scalars and x a, y b are called components of vectors in the directions of a and b, respectively.



∴ Position vector of $P(x, y) = x\,i + y\,j$.

$OP^2 = OA^2 + AP^2 = |x|^2 + |y|^2 = x^2 + y^2$

$OP = \sqrt{x^2 + y^2}$. This is the magnitude of OP.

where, $x\,i$ and $y\,j$ are also called resolved parts of OP in the directions of i and j, respectively.

Vector Equation of Line and Plane

(i) Vector equation of the straight line passing through origin and parallel to b is given by $r = t\,b$, where t is scalar.

(ii) Vector equation of the straight line passing through a and parallel to b is given by $r = a + t\,b$, where t is scalar.

(iii) Vector equation of the straight line passing through a and b is given by $r = a + t(b - a)$, where t is scalar.

(iv) Vector equation of the plane through origin and parallel to b and c is given by $r = s\,b + t\,c$, where s and t are scalars.

(v) Vector equation of the plane passing through a and parallel to b and c is given by $r = a + sb + t\,c$, where s and t are scalars.

(vi) Vector equation of the plane passing through a, b and c is $r = (1 - s - t)a + sb + tc$, where s and t are scalars.

Bisectors of the Angle between Two Lines

(i) The bisectors of the angle between the lines $r = \lambda a$ and $r = \mu b$ are given by r = &lamba; (a / |a| &plumsn; b / |b|)

(ii) The bisectors of the angle between the lines $r = a + \lambda b$ and $r = a + \mu c$ are given by r = a + &lamba; (b / |b| &plumsn; c / |c|).

VECTOR DIFFERENTIATION

1. Differentiation of vectors: In Cartesian coordinates, the derivative of the vector $\mathbf{a}(u) = a_x i + a_y j + a_z k$ is given by

$$\frac{d\mathbf{a}}{du} = \frac{da_x}{du} i + \frac{da_y}{du} j + \frac{da_z}{du} k$$

If $\mathbf{r}(t) = x(t)\, i + y(t)\, j + z(t)\, k$, the velocity of the particle is given by the vector

$$\mathbf{v}(t) = \frac{d\mathbf{r}}{dt} = \frac{dx}{dt} i + \frac{dy}{dt} j + \frac{dz}{dt} k$$

And the acceleration of the particle is given in by

$$\mathbf{a}(t) = \frac{d\mathbf{v}}{dt} = \frac{d^2x}{dt^2} i + \frac{d^2y}{dt^2} j + \frac{d^2z}{dt^2} k$$

☒ *Differentiation formulas*. If **A**, **B** and **C** are differentiable vector functions of a scalar u, and Φ is a differentiable scalar function of u, then

1. $\frac{d}{du}(\mathbf{A} + \mathbf{B}) = \frac{d\mathbf{A}}{du} + \frac{d\mathbf{B}}{du}$
2. $\frac{d}{du}(\mathbf{A} \cdot \mathbf{B}) = \mathbf{A} \cdot \frac{d\mathbf{B}}{du} + \frac{d\mathbf{A}}{du} \cdot \mathbf{B}$
3. $\frac{d}{du}(\mathbf{A} \times \mathbf{B}) = \mathbf{A} \times \frac{d\mathbf{B}}{du} + \frac{d\mathbf{A}}{du} \times \mathbf{B}$
4. $\frac{d}{du}(\Phi\mathbf{A}) = \Phi \frac{d\mathbf{A}}{du} + \frac{d\Phi}{du} \mathbf{A}$
5. $\frac{d}{du}(\mathbf{A} \cdot \mathbf{B} \times \mathbf{C}) = \mathbf{A} \cdot \mathbf{B} \times \frac{d\mathbf{C}}{du} + \mathbf{A} \cdot \frac{d\mathbf{B}}{du} \times \mathbf{C} + \frac{d\mathbf{A}}{du} \cdot \mathbf{B} \times \mathbf{C}$
6. $\frac{d}{du}(\mathbf{A} \times (\mathbf{B} \times \mathbf{C})) = \mathbf{A} \times (\mathbf{B} \times \frac{d\mathbf{C}}{du}) + \mathbf{A} \times (\frac{d\mathbf{B}}{du} \times \mathbf{C}) + \frac{d\mathbf{A}}{du} \times (\mathbf{B} \times \mathbf{C})$

☒ *Partial derivatives of vectors*. If **A** is a vector depending on more than one scalar variable (x, y, z), then we write $\mathbf{A} = \mathbf{A}(x, y, z)$. The partial derivative of **A** with respect to x, y and z respectively, defined as:

$$\frac{\partial \mathbf{A}}{\partial x} = \lim_{\Delta x \to 0} \frac{\mathbf{A}(x+\Delta x, y, z) - \mathbf{A}(x,y,z)}{\Delta x}$$
$$\frac{\partial \mathbf{A}}{\partial y} = \lim_{\Delta y \to 0} \frac{\mathbf{A}(x, y+\Delta y, z) - \mathbf{A}(x,y,z)}{\Delta y}$$
$$\frac{\partial \mathbf{A}}{\partial z} = \lim_{\Delta z \to 0} \frac{\mathbf{A}(x, y, z+\Delta z) - \mathbf{A}(x,y,z)}{\Delta z}$$

Higher derivatives can be defined as

$$\frac{\partial^2 \mathbf{A}}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial \mathbf{A}}{\partial x}\right), \qquad \frac{\partial^2 \mathbf{A}}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial \mathbf{A}}{\partial y}\right), \qquad \frac{\partial^2 \mathbf{A}}{\partial z^2} = \frac{\partial}{\partial z}\left(\frac{\partial \mathbf{A}}{\partial z}\right)$$

$$\frac{\partial^2 \mathbf{A}}{\partial x\, \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial \mathbf{A}}{\partial y}\right), \qquad \frac{\partial^2 \mathbf{A}}{\partial y\, \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial \mathbf{A}}{\partial x}\right), \qquad \frac{\partial^2 \mathbf{A}}{\partial x\, \partial y} = \frac{\partial^2 \mathbf{A}}{\partial y\, \partial x}$$

Thus if **A** and **B** are functions of *x*,*y*,*z* then,

1. $\frac{\partial}{\partial x}(\mathbf{A} \cdot \mathbf{B}) = \mathbf{A} \cdot \frac{\partial \mathbf{B}}{\partial x} + \frac{\partial \mathbf{A}}{\partial x} \cdot \mathbf{B}$

2. $\frac{\partial}{\partial x}(\mathbf{A} \times \mathbf{B}) = \mathbf{A} \times \frac{\partial \mathbf{B}}{\partial x} + \frac{\partial \mathbf{A}}{\partial x} \times \mathbf{B}$

3. $\frac{\partial^2}{\partial y\, \partial x}(\mathbf{A} \cdot \mathbf{B}) = \frac{\partial}{\partial y}\left\{\frac{\partial}{\partial x}(\mathbf{A} \cdot \mathbf{B})\right\} = \frac{\partial}{\partial y}\left\{\mathbf{A} \cdot \frac{\partial \mathbf{B}}{\partial x} + \frac{\partial \mathbf{A}}{\partial x} \cdot \mathbf{B}\right\}$

$$= \mathbf{A} \cdot \frac{\partial^2 \mathbf{B}}{\partial y\, \partial x} + \frac{\partial \mathbf{A}}{\partial y} \cdot \frac{\partial \mathbf{B}}{\partial x} + \frac{\partial \mathbf{A}}{\partial x} \cdot \frac{\partial \mathbf{B}}{\partial y} + \frac{\partial^2 \mathbf{A}}{\partial y\, \partial x} \cdot \mathbf{B}$$

☒ *Differentials of vectors* follow rules similar to those of elementary calculus.

1. If **A** = $A_1 i + A_2 j + A_3 k$, then $d\mathbf{A} = dA_1 i + dA_2 j + dA_3 k$

2. $d(\mathbf{A} \,.\, \mathbf{B}) = \mathbf{A} \,.\, d\mathbf{B} + d\mathbf{A} \,.\, \mathbf{B}$

3. $d(\mathbf{A} \times \mathbf{B}) = \mathbf{A} \times d\mathbf{B} + d\mathbf{A} \times \mathbf{B}$

4. If **A** = **A**(x,y,z), then

$$d\mathbf{A} = \frac{\partial \mathbf{A}}{\partial x}dx + \frac{\partial \mathbf{A}}{\partial y}dy + \frac{\partial \mathbf{A}}{\partial z}dz, \qquad \Longrightarrow \qquad (total\ differential)$$

$$\frac{d\mathbf{A}}{dx} = \frac{\partial \mathbf{A}}{\partial x} + \left(\frac{\partial \mathbf{A}}{\partial y}\right)\frac{dy}{dx} + \left(\frac{\partial \mathbf{A}}{\partial z}\right)\frac{dz}{dx}\,, \quad \Longrightarrow \quad (total\ derivative\ for\ x\,)$$

2. Vector operators: Central to all these differential operations is the vector operator $\mathbf{\nabla}$, and in Cartesian coordinates is defined by

$$\mathbf{\nabla} = i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}$$

***2.1.The Gradient**:* The gradient of a scalar field $\phi(x, y, z)$ is defined by

$$grad\ \emptyset = \mathbf{\nabla}\emptyset = i\frac{\partial \emptyset}{\partial x} + j\frac{\partial \emptyset}{\partial y} + k\frac{\partial \emptyset}{\partial z}$$

<u>Note</u> that $\mathbf{\nabla}\emptyset$ defines a *vector field*.

Any scalar field $\emptyset$ for which $\mathbf{\nabla}\emptyset = 0$ is said to be *constant*.

***2.2.The Divergence**:* The divergence of a vector field $\boldsymbol{A}(x, y, z) = A_x\, i + A_y\, j + A_z\, k$ is defined by:

$$div\ \mathbf{A} = \mathbf{\nabla} \cdot \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

<u>Note</u> that $\mathbf{\nabla} \cdot \mathbf{A}$ defines a *scalar field*. Also that $\mathbf{\nabla} \cdot \mathbf{A} \neq \mathbf{A} \cdot \mathbf{\nabla}$

Any vector field $\mathbf{A}$ for which $\mathbf{\nabla} \cdot \mathbf{A} = 0$ is said to be *solenoidal*.

***2.3.The Curl**:* The curl of a vector field $\boldsymbol{A}(x, y, z)$ is defined by

$$curl\ \mathbf{A} = \mathbf{\nabla} \times \mathbf{A} = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)i + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)j + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)k$$

$$Also, \quad \mathbf{\nabla} \times \mathbf{A} = \begin{vmatrix} i & j & k \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

Note that $\nabla \times \mathbf{A}$ defines a *vector field*. Any vector field $\mathbf{A}$ for which $\nabla \times \mathbf{A} = 0$ is said to be *irrotational*.

2.4.The Laplacian: The Laplacian of a scalar field $\phi(x, y, z)$ is defined by

$$\nabla^2 = \nabla \cdot \nabla = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

Note that ∇^2 defines a *scalar field*.

☒ *Formulas involving* ∇. If $\mathbf{A}$ and $\mathbf{B}$ are differentiable vector functions, and ϕ and ψ are differentiable scalar functions of position *(x, y, z)*, then

1. $\nabla(\phi + \psi) = \nabla\phi + \nabla\psi$
2. $\nabla \cdot (\mathbf{A} + \mathbf{B}) = \nabla \cdot \mathbf{A} + \nabla \cdot \mathbf{B}$
3. $\nabla \times (\mathbf{A} + \mathbf{B}) = \nabla \times \mathbf{A} + \nabla \times \mathbf{B}$
4. $\nabla \cdot (\phi\mathbf{A}) = (\nabla\phi) \cdot \mathbf{A} + \phi(\nabla \cdot \mathbf{A})$
5. $\nabla(\phi\psi) = \phi\,\nabla\psi + \psi\,\nabla\phi$
6. $\nabla \times (\phi\mathbf{A}) = (\nabla\phi) \times \mathbf{A} + \phi(\nabla \times \mathbf{A})$
7. $\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B})$
8. $\nabla \times (\mathbf{A} \times \mathbf{B}) = (\mathbf{B} \cdot \nabla)\mathbf{A} - \mathbf{B}(\nabla \cdot \mathbf{A}) - (\mathbf{A} \cdot \nabla)\mathbf{B} + \mathbf{A}(\nabla \cdot \mathbf{B})$
9. $\nabla(\mathbf{A} \cdot \mathbf{B}) = (\mathbf{B} \cdot \nabla)\mathbf{A} + (\mathbf{A} \cdot \nabla)\mathbf{B} + \mathbf{B} \times (\nabla \times \mathbf{A}) + \mathbf{A} \times (\nabla \times \mathbf{B})$
10. $\nabla \times (\nabla\phi) = 0$, $\qquad curl\ grad\ \phi = 0$
11. $\nabla \cdot (\nabla \times \mathbf{A}) = 0$, $\qquad div\ curl\ \mathbf{A} = 0$
12. $\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2\mathbf{A}$
13. $\nabla \cdot (\nabla\phi \times \nabla\psi) = 0$