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# UGP: ESSAYS ON DYNAMIC PROGRAMMING

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## §1. A Multi-Stage Allocation Process

### 1.1. Introduction

#### Notation

The standard Dynamic Programming problem described throughout will be in metric space  $\mathbb{X}$ .

Given,  $\Phi : \mathbb{X} \times \mathbb{X} \rightarrow \mathbb{R}$  and  $\Gamma : \mathbb{X} \rightrightarrows \mathbb{R}$

For a fixed  $x^0$ , find a sequence  $x_m = x^1, \dots, x^m, \dots$  such that

$$\max \Phi(x^0, x^1) + \sum_{i=1}^{\infty} \delta^i \Phi(x^i, x^{i+1}), \text{ where } \delta \in (0, 1) \text{ and } x^1 \in \Gamma(x^0), x^{i+1} \in \Gamma(x^i), m = 1, 2, 3, \dots$$

Here  $\delta$  is called the discount factor.

#### Functional Equations

Considering a space of Real numbers  $\mathbb{R}$ , we can define a functional equation as follows:

$$f(x) = \max_{0 \leq y \leq x} [g(y) + h(x - y) + f(ay + b(x - y))]$$

where  $a, b \in \mathbb{R}$  and  $g, h$  are given functions.

This can be seen as an **insurance problem** where functions  $g$  and  $h$  represent the different places where and initial premium  $x$  is invested. The function  $f$  represents the value of the policy at the end of the period.

We can define,  $R_1(x, y) = g(y) + h(x - y)$ , initially the aim is to determine  $\max_{0 \leq y \leq x} R_1(x, y)$ . The value of  $x$  and  $y$  obtained from this maximization can be used to generate the next element of the sequence  $x_m$ .

i.e.  $x_1 = ay + b(x - y) = y_1 + (x_1 - y_1)$

Further we define  $R_2(x, y, y_1) = g(y) + h(x - y) + g(y_1) + h(x_1 - y_1)$  where  $(y, y_1) \in [0, x] \times [0, x_1]$

This can be generalized to  $R_m(x, y, y_1, \dots, y_{m-1}) = g(y) + h(x - y) + g(y_1) + h(x_1 - y_1) + g(y_2) + h(x_2 - y_2) + \dots + g(y_{m-1}) + h(x_{m-1} - y_{m-1})$

and the maximization problem becomes

$$\max R_m(x, y, y_1, \dots, y_{m-1}) \text{ such that}$$

$$\begin{array}{ll} ay + b(x - y) = x_1 & 0 \leq y \leq x \\ ay_1 + b(x_1 - y_1) = x_2 & 0 \leq y_1 \leq x_1 \\ \vdots & \vdots \\ ay_{m-1} + b(x_{m-1} - y_{m-1}) = x_m & 0 \leq y_{m-1} \leq x_{m-1} \end{array}$$

Here,  $f_N(x) = \max_{t, y_i} R_N(x, y_1, \dots, y_N)$  and  $f_N(x)$  can be interpreted as the maximum return obtained from an  $N$ -stage decision with initial premium  $x$  and  $x \geq 0$ .

$$\therefore f_N(x) = \max_{0 \leq y \leq x} [g(y) + h(x - y) + f_{N-1}(ay + b(x - y))]$$

or

$$f(x) = \max_{0 \leq y \leq x} [g(y) + h(x - y) + f(ay + b(x - y))] \text{ for } N \rightarrow \infty$$

### 1.2. Existence and Uniqueness Theorems

Under the following assumptions, the functional equation has a unique solution, which is continuous at  $x = 0$  and has a value 0 at  $x = 0$ . Moreover the solution is continuous at  $x \geq 0$ .

The assumptions are:

1.  $g$  and  $h$  are continuous on  $[0, \infty)$  and  $g(0) = h(0) = 0$
2. If  $m(x) = \max_{0 \leq y \leq x} [\max\{|g(y)|, |h(y)|\}]$  and  $c = \max\{a, b\}$ , then  $\sum_{n=0}^{\infty} m(c^n x) < \infty \forall x \geq 0$

*Proof.* We define a mapping  $T(f, y) = g(y) + h(x - y) + f(ay + b(x - y))$  and  $f_{N+1}(x) = \max_{0 \leq y \leq x} T(f_N, y)$ , assuming  $g$  and  $h$  to be non-negative.

The sequence  $\{f_N(x)\}$  is an increasing sequence of functions such that  $f_N \rightarrow f$   
Proving that  $T$  is bounded above would suffice.

**Method 1:** We know that  $x_1 \leq cx$

$$\begin{aligned} \therefore ax_1 + b(x_1 - y) &\leq cx_1 \leq c^2x \\ \implies T(f_N, y) &\leq 2(m(x) + m(cx) + \dots + m(c^N x)) \end{aligned}$$

Which proves that  $T$  is bounded above by a constant.

**Method 2:**

$$\begin{aligned} f_{N+1}(x) &= \sup_{0 \leq y \leq x} T(f_N, y) \\ f(x) &\geq \sup_{0 \leq y \leq x} T(f_N, y) \\ f(x) &\geq T(f_N, y) \forall y \in [0, x] \\ f(x) &\geq T(f, y) \text{ as } N \rightarrow \infty \\ \therefore f(x) &= \sup_{0 \leq y \leq x} T(f, y) \end{aligned}$$

Also,

$$\begin{aligned} T(f_N, y) &= g(y) + h(x - y) + f_N(ay + b(x - y)) \\ &\leq g(y) + h(x - y) + f(ay + b(x - y)) \\ &= T(f, y) \end{aligned}$$

$$\begin{aligned} f_{N+1}(x) &= \sup_{0 \leq y \leq x} T(f_N, y) \\ &\leq \sup_{0 \leq y \leq x} T(f, y) \\ f_{N+1}(x) &\leq \sup_{0 \leq y \leq x} T(f, y) \end{aligned}$$

As  $N \rightarrow \infty$ ,

$$f(x) \leq \sup_{0 \leq y \leq x} T(f, y)$$

$$\therefore f(x) = \sup_{0 \leq y \leq x} T(f, y) \text{ using the above two inequalities}$$

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### Existence and Uniqueness with $g$ and $h$ not necessarily non-negative

We can write,

$$\begin{aligned} f_1(x) &= \max_{0 \leq y \leq x} [g(y) + h(x - y)] \\ &= \max_{y \in [0, x]} [g(y) + h(x - y) + 0] \\ &= \max_{y \in \Gamma(x)} [g(y) + h(x - y)] \end{aligned}$$

$\Gamma(x) = [0, x]$  a set valued map

### Digression

We will use a generalized form of Berge Maximum Theorem in the following proof, which is described here.

**Berge Maximum Theorem (Generalized Form)**

Let  $\Theta$  and  $\mathbb{X}$  be two metric spaces and  $\Gamma : \Theta \rightrightarrows \mathbb{X}$  is a compact valued correspondence. Let  $\phi : \mathbb{X} \times \Theta \rightarrow \mathbb{R}$  be a continuous function.

Let  $\sigma(\theta) = \operatorname{argmax}\{\phi(x, \theta) : x \in \Gamma(\theta)\}$  and let  $\phi^*(\theta) = \max_x \{\phi(x, \theta) : x \in \Gamma(\theta)\}$

Let  $\Gamma : \Theta \rightrightarrows \mathbb{X}$  is continuous at some  $\theta \in \Theta$ . Then,

1.  $\sigma : \Theta \rightrightarrows \mathbb{X}$  is compact-valued, upper semi-continuous and closed at  $\theta$ .
2.  $\phi^* : \Theta \rightarrow \mathbb{R}$  is continuous at  $\theta$ .

Some definitions that follow from above are:

1. **Upper semi-continuous Map:** Let  $\Gamma : \mathbb{X} \rightrightarrows \mathbb{Y}$ , then  $\Gamma$  is upper semi-continuous at  $x \in \mathbb{X}$  if for any  $\{x^m\} \in \mathbb{X}$  and  $\{y^m\} \in \mathbb{Y}$  with  $x_0^m \rightarrow x$  and  $y^m \in \Gamma(x^m)$ , the sequence  $\{y^m\}$  has a convergent subsequence such that its limit is in  $\Gamma(x)$ .
2. **Compact valued Map:**  $\Gamma : \mathbb{X} \rightrightarrows \mathbb{Y}$  is a compact valued correspondence which is upper semi-continuous if for a compact subset  $S \subset \mathbb{X}$ , the set  $\Gamma(S)$  is compact.
3. **Closed Map:**  $\Gamma : \mathbb{X} \rightrightarrows \mathbb{Y}$  is closed at  $x \in \mathbb{X}$  if for any  $\{x^m\} \in \mathbb{X}$  with  $x^m \rightarrow x$  and  $\{y^m\} \in \mathbb{Y}$  with  $y^m \rightarrow y$  such that  $y^m \in \Gamma(x^m)$  for each  $m$ , we have  $y \in \Gamma(x)$ .

By applying the generalized form of Berge Maximum Theorem, we can show that  $f_1(x)$  is continuous.

$$T(f_N, y) = g(y) + h(x - y) + f_N(ay + b(x - y)) \text{ and } f_{N+1}(x) = \max_{0 \leq y \leq x} T(f_N, y).$$

$$\begin{aligned} f_{N+1}(x) &= T(f_N, y_N) \geq T(f_N, y_N + 1) \\ f_{N+2}(x) &= T(f_N + 1, y_N + 1) \geq T(f_N + 1, y_N) \end{aligned}$$

Define,

$$u_N(x) = \sup_{0 \leq z \leq x} |f_N(z) - f_N + 1(z)|$$

and

$$\begin{aligned} f_{N+1}(x) - f_{N+2}(x) &\geq T(f_N, y_{N+1}) - T(f_N + 1, y_{N+1}) \\ f_{N+1}(x) - f_{N+2}(x) &\leq T(f_N, y_N) - T(f_N + 1, y_N) \end{aligned}$$

From these two inequalities, we get

$$|f_{N+1}(x) - f_{N+2}(x)| \leq \max\{|T(f_N, y_{N+1}) - T(f_N + 1, y_{N+1})|, |T(f_N, y_N) - T(f_N + 1, y_N)|\}$$

$$|T(f_N, y_{N+1}) - T(f_N + 1, y_{N+1})| = |f_N(ay_{N+1} + b(x - y_{N+1})) - f_{N+1}(ay_{N+1} + b(x - y_{N+1}))| \leq u_N(cx)$$

$$|T(f_N, y_N) - T(f_N + 1, y_N)| = |f_N(ay_N + b(x - y_N)) - f_{N+1}(ay_N + b(x - y_N))| \leq u_N(cx)$$

$$\therefore ay + b(x - y) \leq cx \text{ where } c = \max\{a, b\}$$

$$\therefore \boxed{u_{N+1}(x) \leq u_N(cx)} \text{ for } N \in \mathbb{N}$$

We now need to estimate  $u_1(x)$

$$\begin{aligned} f_2(x) &= T(f_1, y_1) \\ f_1(x) - f_2(x) &\geq g(y_1) + h(x - y_1) - T(f_1, y_1) \\ f_2(x) - f_1(x) &\leq f_1(ay_1 + b(x - y_1)) \end{aligned}$$

Let  $y_0$  be the maximizing point to obtain  $f_1(x)$  i.e.  $f_1(x) = g(y_0) + h(x - y_0)$  and  $f_2(x) \geq T(f_1, y_0)$

$$\begin{aligned} f_1(x) - f_2(x) &\geq g(y_0) + h(x - y_0) - T(f_1, y_0) \\ &= -f_1(ay_0 + b(x - y_0)) \\ f_2(x) - f_1(x) &\geq f_1(ay_0 + b(x - y_0)) \end{aligned}$$

$$|f_2(x) - f_1(x)| \leq \max\{f_1(ay_1 + b(x - y_1)), |f_1(ay_0 + b(x - y_0))|\} \implies |f_2(x) - f_1(x)| \leq 2m(cx) \forall x \in [0, x]$$

$$\therefore \boxed{u_1(x) \leq 2m(cx)}$$

$$u_{N+1}(x) \leq u_N(cx) \leq u_{N-1}(c^2x) \leq \dots \leq u_1(c^Nx) \leq 2m(c^{N+1}x)$$

$$\therefore u_{N+1}(x) \leq 2m(c^{N+1}x) \text{ and } u_1(x) \leq 2m(cx) \text{ for } N \in \mathbb{N}$$

$$\implies \boxed{u_N(x) \leq 2m(c^Nx) \text{ for } N \in \mathbb{N}}$$

From the above result we can say that,

$$\sum_{N=1}^{\infty} u_N(x) \leq \sum_{N=1}^{\infty} 2m(c^Nx)$$

that is

$$\sum_{N=1}^{\infty} u_N(x) \leq \infty (\text{convergent})$$

$$\implies u_N(x) \rightarrow 0 \text{ as } N \rightarrow \infty$$

$\implies f_N(x)$  is a cauchy sequence (which then implies that  $f_N - f_M$  can be expressed as a sum of differences).

$$\therefore f(x) = \lim_{N \rightarrow \infty} f_N(x) \text{ exists and is continuous } \forall x \geq 0$$

Now we know that,

$$\begin{aligned} f_{N+1}(x) &\geq T(f_N, y) \\ &\geq g(y) + h(x - y) + f_N(ay + b(x - y)) \text{ for } y \in [0, x] \end{aligned}$$

For large  $N$ ,

$$\begin{aligned} f(x) &\geq g(y) + h(x - y) + f(ay + b(x - y)) \\ &\geq \max_{0 \leq y \leq x} T(f, y) \end{aligned} \tag{A}$$

For a bounded sequence  $\{y_N\}$  with  $y_N \rightarrow y^*$  for some  $y^* \in [0, x]$  and  $N \rightarrow \infty$  we have,

$$\begin{aligned}
 f(x) &= T(f, y^*) \\
 &\leq \max_{0 \leq y \leq x} T(f, y)
 \end{aligned} \tag{B}$$

using (A) and (B),

$$f(x) = \max_{0 \leq y \leq x} T(f, y)$$

### Uniqueness

By proving the existence, we know that,

$$f(x) = \max_{0 \leq y \leq x} T(f, y)$$

Assume on the contrary that there are two such solutions  $f$  &  $F$

$$\begin{aligned}
 f(x) &= T(f, y) \geq T(f, w) \\
 F(x) &= T(F, w) \geq T(F, y)
 \end{aligned}$$

Now,

$$\begin{aligned}
 |f(x) - F(x)| &\leq \max\{|T(f, y) - T(F, y)|, |T(f, w) - T(F, w)|\} \\
 &\leq \max\{|f(ay + b(x - y)) - F(ay + b(x - y))|, |f(aw + b(x - w)) - F(aw + b(x - w))|\}
 \end{aligned}$$

From previous definition,  $u(x) = \sup_{0 \leq z \leq x} f(z) - F(z)$  We know that  $u(x) \geq 0$  and  $u(x) \rightarrow 0$  as  $x \rightarrow \infty$ .

$$\begin{aligned}
 \therefore 0 &\geq u(x) \geq |f(x) - F(x)| \\
 \implies |f(x) - F(x)| &\rightarrow 0 \text{ as } x \rightarrow \infty \\
 \implies f(x) &= F(x) \text{ for all } x \geq 0
 \end{aligned}$$

$\therefore$  The solution obtained is unique.

[Bel57]

## References

[Bel57] Richard Bellman. *Dynamic Programming*. Dover Publications, 1957.