UGP: Essays on Dynamic Programming

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Contents

§1. Introductory Dynamic Programming (Understanding Bellman)

1.1. Introduction

1.1.1. Notation

The standard Dynamic Programming problem desribed throughout will be in metric space $\mathbb{X}$.
Given, $\Phi : \mathbb{X} \times \mathbb{X} \to \mathbb{R}$ and $\Gamma : \mathbb{X} \rightrightarrows \mathbb{R}$
For a fixed x^0, find a sequence $x_m = x^1, ..., x^m, ...$ such that

$$\max \Phi(x^0, x^1) + \sum_{i=1}^{\infty} \delta^i \Phi(x^i, x^{i+1}), \text{ where } \delta \in (0,1) \text{ and } x^1 \in \Gamma(x^0), x^{i+1} \in \Gamma(x^i), m = 1, 2, 3, ...$$

Here δ is called the discount factor.

1.1.2. Functional Equations

Considering a space of Real numbers $\mathbb{R}$, we can define a functional equation as follows:

$$f(x) = \max_{0 \le y \le x} [g(y) + h(x-y) + f(ay + b(x-y))]$$

where $a, b \in \mathbb{R}$ and g, h are given functions.
This can be seen as an **insurance problem** where functions g and h represent the different places where and initial premium x is invested. The function f represents the value of the policy at the end of the period.

We can define, $R_1(x,y) = g(y) + h(x-y)$, initially the aim is to determine $\max_{0 \le y \le x} R_1(x,y)$. The value of x and y obtained from this maximization can be used to generate the next element of the sequence x_m.
i.e. $x_1 = ay + b(x-y) = y_1 + (x_1 - y_1)$
Further we define $R_2(x, y, y_1) = g(y) + h(x-y) + g(y_1) + h(x_1 - y_1)$ where $(y, y_1) \in [0,x] \times [0, x_1]$

This can be generalized to $R_m(x, y, y_1, ..., y_{m-1}) = g(y) + h(x-y) + g(y_1) + h(x_1 - y_1) + g(y_2) + h(x_2 - y_2) + ... + g(y_{m-1}) + h(x_{m-1} - y_{m-1})$
and the maximization problem becomes

$$\max R_m(x, y, y_1, ..., y_{m-1}) \text{ such that}$$

$$\begin{array}{ll} ay + b(x-y) = x_1 & 0 \le y \le x \\ ay_1 + b(x_1 - y_1) = x_2 & 0 \le y_1 \le x_1 \\ \vdots & \vdots \\ ay_{m-1} + b(x_{m-1} - y_{m-1}) = x_m & 0 \le y_{m-1} \le x_{m-1} \end{array}$$

Here, $f_N(x) = \max_{t, y_i} R_N(x, y_1, ..., y_N)$ and $f_N(x)$ can be interpreted as the maximum return obtained from an N-stage decision with initial premium x and $x \ge 0$.

$$\therefore f_N(x) = \max_{0 \le y \le x} [g(y) + h(x-y) + f_{N-1}(ay + b(x-y))]$$

or

$$f(x) = \max_{0 \le y \le x} [g(y) + h(x-y) + f(ay + b(x-y))] \text{ for } N \to \infty$$

1.2. Existence and Uniqueness Theorems

Under the following assumptions, the functional equation has a unique solution, which is continuous at $x = 0$ and has a value 0 at $x = 0$. Moreover the solution is continuous at $x \ge 0$.

The assumptions are:

1. g and h are continuous on $[0, \infty)$ and g(0) = h(0) = 0
2. If $m(x) = \max_{0 \le y \le x}[\max\{|g(y)|, |h(y)|\}]$ and $c = \max\{a, b\}$, then $\sum_{n=0}^{\infty} m(c^n x) < \infty \forall x \ge 0$

Proof. We define a mapping $T(f, y) = g(y) + h(x-y) + f(ay + b(x-y))$ and $f_{N+1}(x) = \max_{0 \le y \le x} T(f_N, y)$, assuming g and h to be non-negative.
The sequence $\{f_N(x)\}$ is an increasing sequence of functions such that $f_N \to f$
Prooving that T is bounded above would suffice.
Method 1: We know that $x_1 \le cx$

$$\therefore ax_1 + b(x_1 - y) \le cx_1 \le c^2 x$$
$$\implies T(f_N, y) \le 2(m(x) + m(cx) + ... + m(c^N x))$$

Which proves that T is bounded above by a constant.
Method 2:

$$f_{N+1}(x) = \sup_{0 \le y \le x} T(f_N, y)$$
$$f(x) \ge \sup_{0 \le y \le x} T(f_N, y)$$
$$f(x) \ge T(f_N, y) \forall y \in [0, x]$$
$$f(x) \ge T(f, y) \text{ as } N \to \infty$$
$$\therefore \boxed{f(x) = \sup_{0 \le y \le x} T(f, y)}$$

Also,

$$\begin{aligned} T(f_N, y) &= g(y) + h(x-y) + f_N(ay + b(x-y)) \\ &\le g(y) + h(x-y) + f(ay + b(x-y)) \\ &= T(f, y) \end{aligned}$$

$$\begin{aligned} f_{N+1}(x) &= \sup_{0 \le y \le x} T(f_N, y) \\ &\le \sup_{0 \le y \le x} T(f, y) \\ f_{N+1}(x) &\le \sup_{0 \le y \le x} T(f, y) \end{aligned}$$

As $N \to \infty$,

$$\boxed{f(x) \le \sup_{0 \le y \le x} T(f, y)}$$

$$\therefore f(x) = \sup_{0 \le y \le x} T(f, y) \text{ using the above two inequalities}$$

■

1.2.1. Existence and Uniqueness with g and h not necessarily non-negative

We can write,

$$\begin{aligned} f_1(x) &= \max_{0 \le y \le x}[g(y) + h(x-y)] \\ &= \max_{y \in [0,x]}[g(y) + h(x-y) + 0] \\ &= \max_{y \in \Gamma(x)}[g(y) + h(x-y)] & \Gamma(x) = [0, x] \text{ a set valued map} \end{aligned}$$

Digression

We will use a generalized form of Berge Maximum Theorem in the following proof, which is described here.
Berge Maximum Theorem (Generalized Form)
Let Θ and $\mathbb{X}$ be two metric spaces and $\Gamma : \Theta \rightrightarrows \mathbb{X}$ is a compact valued correspondence. Let $\phi : \mathbb{X} \times \Theta \to \mathbb{R}$ be a continuous function.
Let $\sigma(\theta) = argmax\{\phi(x,\theta) : x \in \Gamma(\theta)\}$ and let $\phi^*(\theta) = \max_x \{\phi(x,\theta) : x \in \Gamma(\theta)\}$

Let $\Gamma : \Theta \rightrightarrows \mathbb{X}$ is continuous at some $\theta \in \Theta$. Then,

1. $\sigma : \Theta \rightrightarrows \mathbb{X}$ is compact-valued, upper semi-continuous and closed at θ.
2. $\phi^* : \Theta \to \mathbb{R}$ is continuous at θ.

Some definitions that follow from above are:

1. **Upper semi-continuous Map:** Let $\Gamma : \mathbb{X} \rightrightarrows \mathbb{Y}$, then Γ is upper semi-continuous at $x \in \mathbb{X}$ if for any $\{x^m\} \in \mathbb{X}$ and $\{y^m\} \in \mathbb{Y}$ with $x_0^m \to x$ and $y^m \in \Gamma(x^m)$, the sequence $\{y^m\}$ has a convergent subsequence such that its limit is in $\Gamma(x)$.
2. **Compact valued Map:** $\Gamma : \mathbb{X} \rightrightarrows \mathbb{Y}$ is a compact valued correspondence which is upper semi-continuous if for a compact subset $S \subset \mathbb{X}$, the set $\Gamma(S)$ is compact.
3. **Closed Map:** $\Gamma : \mathbb{X} \rightrightarrows \mathbb{Y}$ is closed at $x \in \mathbb{X}$ if for any $\{x^m\} \in \mathbb{X}$ with $x^m \to x$ and $\{y^m\} \in \mathbb{Y}$ with $y^m \to y$ such that $y^m \in \Gamma(x^m)$ for each m, we have $y \in \Gamma(x)$.

By applying the generalized form of Berge Maximum Theorem, we can show that $f_1(x)$ is continuous.

$T(f_N, y) = g(y) + h(x-y) + f_N(ay + b(x-y))$ and $f_{N+1}(x) = \max_{0 \le y \le x} T(f_N, y)$.

$$f_{N+1}(x) = T(f_N, y_N) \ge T(f_N, y_N + 1)$$
$$f_{N+2}(x) = T(f_N + 1, y_N + 1) \ge T(f_N + 1, y_N)$$

Define,

$$u_N(x) = \sup_{0 \le z \le x} |f_N(z) - f_N + 1(z)|$$

and

$$f_{N+1}(x) - f_{N+2}(x) \ge T(f_N, y_{N+1}) - T(f_N + 1, y_{N+1})$$
$$f_{N+1}(x) - f_{N+2}(x) \le T(f_N, y_N) - T(f_N + 1, y_N)$$

From these two inequalities, we get

$$|f_{N+1}(x) - f_{N+2}(x)| \le \max\{|T(f_N, y_{N+1}) - T(f_N + 1, y_{N+1})|, |T(f_N, y_N) - T(f_N + 1, y_N)|\}$$

$$|T(f_N, y_{N+1}) - T(f_N + 1, y_{N+1})| = |f_N(ay_N + b(x - y_N)) - f_{N+1}(ay_{N+1} + b(x - y_{N+1})) \le u_N(cx)$$

$$|T(f_N, y_N) - T(f_N + 1, y_N)| = |f_N(ay_N + b(x - y_N)) - f_{N+1}(ay_N + b(x - y_N)) \le u_N(cx)$$

$\because ay + b(x-y) \le cx$ where $c = \max\{a, b\}$

$$\therefore \boxed{u_{N+1}(x) \le u_N(cx)} \text{ for } N \in \mathbb{N}$$

We now need to estimate $u_1(x)$

$$\begin{aligned} f_2(x) &= T(f_1, y_1) \\ f_1(x) - f_2(x) &\geq g(y_1) + h(x - y_1) - T(f_1, y_1) \\ f_2(x) - f_1(x) &\leq f_1(ay_1 + b(x - y_1)) \end{aligned}$$

Let y_0 be the maximizing point to obtain $f_1(x)$ i.e. $f_1(x) = g(y_0) + h(x - y_0)$ and $f_2(x) \geq T(f_1, y_0)$

$$\begin{aligned} f_1(x) - f_2(x) &\geq g(y_0) + h(x - y_0) - T(f_1, y_0) \\ &= -f_1(ay_0 + b(x - y_0)) \\ f_2(x) - f_1(x) &\geq f_1(ay_0 + b(x - y_0)) \end{aligned}$$

$$|f_2(x) - f_1(x)| \leq \max\{f_1(ay_1 + b(x - y_1)), |f_1(ay_0 + b(x - y_0))|\} \implies |f_2(x) - f_1(x)| \leq 2m(cx) \forall z \in [0, x]$$

$$\therefore \boxed{u_1(x) \leq 2m(cx)}$$

$$u_{N+1}(x) \leq u_N(cx) \leq u_{N-1}(c^2x) \leq ... \leq u_1(c^N x) \leq 2m(c^{N+1}x)$$

$$\therefore u_{N+1}(x) \leq 2m(c^{N+1}x) \text{ and } u_1(x) \leq 2m(cx) \text{ for } N \in \mathbb{N}$$

$$\implies \boxed{u_N(x) \leq 2m(c^N x) \text{ for } N \in \mathbb{N}}$$

From the above result we can say that,

$$\sum_{N=1}^{\infty} u_N(x) \leq \sum_{N=1}^{\infty} 2m(c^N x)$$

that is

$$\sum_{N=1}^{\infty} u_N(x) \leq \infty (\text{convergent})$$

$$\implies u_N(x) \to 0 \text{ as } N \to \infty$$

$\implies$ $f_N(x)$ is a cauchy sequence (which then implies that $f_N - f_M$ can be expressed as a sum of differences).

$$\therefore f(x) = \lim_{N \to \infty} f_N(x) \text{ exists and is continuous } \forall x \geq 0$$

Now we know that,

$$\begin{aligned} f_{N+1}(x) &\geq T(f_N, y) \\ &\geq g(y) + h(x - y) + f_N(ay + b(x - y)) \text{ for } y \in [0, x] \end{aligned}$$

For large N,

$$\begin{aligned} f(x) &\geq g(y) + h(x - y) + f(ay + b(x - y)) \\ &\geq \max_{0 \leq y \leq x} T(f, y) \qquad (A) \end{aligned}$$

For a bounded sequence $\{y_N\}$ with $y_N \to y^*$ for some $y^* \in [0, x]$ and $N \to \infty$ we have,

$$\begin{aligned} f(x) &= T(f, y^*) \\ &\leq \max_{0 \leq y \leq x} T(f, y) \qquad (B) \end{aligned}$$

using (A) and (B),

$$\boxed{f(x) = \max_{0 \leq y \leq x} T(f, y)}$$

1.2.2. Uniqueness

By proving the existence, we know that,

$$f(x) = \max_{0 \leq y \leq x} T(f, y)$$

Assume on the contrary that there are two such solutions $f \& F$

$$\begin{aligned} f(x) = T(f, y) \geq T(f, w) \\ F(x) = T(F, w) \geq T(F, y) \end{aligned}$$

Now,

$$\begin{aligned} |f(x) - F(x)| &\leq \max\{|T(f, y) - T(F, y)|, |T(f, w) - T(F, w)|\} \\ &\leq \max\{|f(ay + b(x - y) - F(ay + b(x - y))|, |f(aw + b(x - w) - F(aw + b(x - w))|\} \end{aligned}$$

From previous definition, $u(x) = \sup\limits_{0 \leq z \leq x} f(z) - F(z)$ We know that $u(x) \geq 0$ and $u(x) \to 0$ as $x \to \infty$.

$$\begin{aligned} \therefore 0 \geq u(x) \geq |f(x) - F(x)| \\ \implies |f(x) - F(x)| \to 0 \text{ as } x \to \infty \\ \implies f(x) = F(x) \text{ for all } x \geq 0 \end{aligned}$$

$\therefore$ The solution obtained is unique.

1.3. Approximations

Theorem 1.1. Let $f_0(x)$ satisfy the following conditions:

- $f_0(x)$ is continuous for $x \geq 0$
- $f_0(0) = 0$

Then if the conditions of existence are fulfilled, the sequence defined by

$$f_{n+1}(x) = \max_{0 \leq y \leq x} T(f_n, y) \text{ for } n \in \mathbb{W}$$

converges to the solution $f(x)$ of the problem, uniformly in any finite interval.

1.3.1. Approximation in the policy space

We call a sequence of allocations, i.e. a sequence of admissible choices of y, a policy and a policy which yields $f(x)$ an optimal policy.
The duality that exists in the theory of dynamic programmingarises from the interconnection between the functions $f(x)$ which measure the maximum return and the policies which yield these maximum returns. Actually a policy is a function, since a policy is a determination of y as a function of x. If the policy is not unique, y will not be a single-valued function of x.
Just as we can approximate in the space of the functions $f(x)$, so we can approximate in the space of policies, $y(x)$.

Theorem 1.2. Let $f_0(x)$ be the result of an initial approximation in the policy space, that is, $f_0(x) = T(f_0, y_0(x))$ where $y_0(x)$ is any continuous function of x satisfying $0 \leq y_0(x) \leq x$, the sequence defined by

$$f_{n+1}(x) = \max_{0\leq y\leq x} T(f_n, y) \text{ for } n \in \mathbb{N}$$

converges to the solution $f(x)$, uniformly and in a finite interval.

Proof. By the definition of $f_n(x)$, we have

$$f_{n+1} \geq f_n \geq f_0 \ \forall \ \mathbb{N}$$

Proving that $f_0(x)$ is continuous, will imply continuity of $f_n(x)$.

$$f_0(x) = g(y_0) + h(x - y_0) + \cdots$$

where g and h are continuous functions.
Hence, $f_0(x)$ which is iteratively obtained from $g(x)$ and $h(x)$ is also continuous in a finite interval. ∎

We can construct examples for such a solution, say

Example. $f(x) = T(f, y(x))$ be an iterative form of the solution.
If the optimal policy consisted of the choice $y = 0$ continually, the solution can be represented by the equation $f(x) = h(x) + f(bx)$, with the value of b such that $h(b^n x) < \infty$
i.e.,

$$\begin{aligned}
f(x) &= h(x) + f(bx) \\
&= h(x) + h(bx) + f(b^2x) \\
&= h(x) + h(bx) + h(b^2x) + f(b^3x) \\
&= \cdots \\
&= \sum_{n=0}^{\infty} h(b^n x)
\end{aligned}$$

Here, $h(b^n x) < \infty$ for all $n \in \mathbb{N}$, so the series converges and $f(x)$ is a continuous function of x.

1.4. Properties of the solution

1.4.1. Convexity

Theorem 1.3. If, along with existence assumptions, we impose the conditions that g and h be convex functions of x, then $f(x)$ will be a convex function, and for each value of x, y will equal 0 or x.

Proof. g and h are convex functions and,

$$f_1(x) = \max_{0\leq y\leq x} (g(y) + h(x - y))$$

In the interval $[0, x]$, g and h are convex, hence $g(y) + h(x - y)$ is also convex.
$\because$ Maximum of a convex function must occur at the endpoints of the interval, i.e. $y = 0$ or $y = x$. We can say that,

$$f_1(x) = \max(g(y), h(x))$$

Then,

$$f_2(x) = \max_{0 \leq y \leq x} (g(y) + h(x - y) + f_1(ay + b(x - y)))$$

similarly follows,

$$f_2(x) = \max(g(x) + f_1(ax), h(x) + f_1(bx))$$

Therefore we can inductively conclude that $f_n(x)$ is convex thus, the limit of the sequence $f(x)$ is also convex. ■

1.4.2. Concavity

Theorem 1.4. [Weak Version] If, along with existence assumptions, we impose the conditions that g and h be strictly concave functions of x, then $f(x)$ will be a strictly concave function, with an optimal policy which is unique.

Lemma 1.5. If $G(x, y)$ is a concave function of x and y for $x, y \geq 0$, then $f(x)$ as defined by $f(x) = \max_{0 \leq y \leq x} G(x, y)$ is a concave function of x for $x > 0$.

Theorem 1.6. [Strong Version] Assuming that,

1. $g(x)$ and $h(x)$ are both strictly concave for $x > 0$, monotone increasing with continuous derivatives and that $g(0) = h(0) = 0$.
2. $\dfrac{g'(0)}{(1-a)} > \dfrac{h'(0)}{(1-b)}, h'(0) > g'(\infty), b > a$.

Then the optimal policy has the following form:

1. $y = x$ for $0 < x < \bar{x}$, where $\bar{x}$ is the root of $h'(0) = g'(x) + (b - a)g'(ax) + (b - a)ag'(a^2x) + \cdots$
2. $y = y(x)$ for $x > \bar{x}$ where $y(x)$ is a function satisfying the inequalities $0 < y(x) < x$, and $y(x)$ is the solution of $\boxed{g'(y) - h'(x - y) + (a - b)f'(ay + b(x - y)) = 0}$

A Result on Concavity

Let $g(x) = \sup_{0 \leq y \leq x} \phi(x, y)$ and $\epsilon > 0$ $\exists y_1 \in [0, x]$ such that,

$$\phi(x_1, y_1) > g(x_1) - \epsilon$$
$$\phi(x_2, y_2) > g(x_2) - \epsilon$$

$$\begin{aligned}
\therefore \lambda(\phi(x_1, y_1)) + (1-\lambda)(\phi(x_2, y_2)) &> \lambda g(x_1) + (1-\lambda)g(x_2) - \epsilon \\
\implies \phi(\lambda(x_1, y_1) + (1-\lambda)(x_2, y_2)) &> \lambda g(x_1) + (1-\lambda)g(x_2) - \epsilon \\
\implies \phi(\lambda x_1 + (1-\lambda)x_2, \lambda y_1 + (1-\lambda)y_2) &> \lambda g(x_1) + (1-\lambda)g(x_2) - \epsilon
\end{aligned}$$

We can also say that,

$$\begin{aligned}
\max_{0 \leq y \leq \lambda x_1 + (1-\lambda)x_2} \phi(\lambda x_1 + (1-\lambda)x_2, y) &\geq \lambda g(x_1) + (1-\lambda)g(x_2) - \epsilon \\
\implies g(\lambda x_1 + (1-\lambda)x_2) &\geq \lambda g(x_1) + (1-\lambda)g(x_2) - \epsilon \\
\implies g \text{ is concave.}
\end{aligned}$$

Proof. It is known that,

$$f_1(x) = \max_{0\le y\le x} (g(y) + h(x-y))$$

$\because g'(0) > h'(0) \implies g'(y) - h'(x-y) > 0$ for $y \in [0, x]$ with the roots at $y = x$ and in the interval $[0, x'], \forall x' > y$
At $y = x$, the value x^* for which $g'(x^*) - h'(0) = 0$ is the root of the equation $g'(x) - h'(0) = 0$.
This equation has precisely one solution due to the fact that g and h are strictly concave.
For $x \ge x^*$, let $y_1(x)$ be the unique solution of $g'(y) = h'(x-y)$. Then, $f_1(x) = g(y_1) + h(x-y_1)$ and $f'(x) = [g'(y_1) - h'(x-y_1)]\frac{dy_1}{dx} + h'(x-y_1) = h'(x-y_1)$ for $x > x_1$.
Further,

$$f_2(x) = \begin{cases} g'(x), & \text{if } 0 \le x \le x_2 \\ h'(x-y_2) + bf_1(ay_2 + b(x-y_2)), & \text{if } x > x_2 \end{cases}$$

where $y_2(x)$ is the unique solution of $g'(y) = h'(x-y) + (b-a)f_1(ay + b(x-y))$ Inductively,

$$f_{n+1}(x) = \max_{0\le y\le x} \left\{g(y) + h(x-y) + f_n(ay + b(x-y))\right\}$$

The solution space hence can be divided into three parts from, $[0, x_2]$, $[x_2, x_1]$ and $[x_1, \infty)$, wherein the inequality $f'_{n+1}(x) \ge f'_n(x)$ holds for all x.
For $n = 1$,

$$f'_2(x) = \frac{bg'(y_2) - ah'(x-y_2)}{b-a} \text{ for } x \ge x_2$$

and

$$f'_1(x) = \frac{bg'(y_1) - ah'(x-y_1)}{b-a} \text{ for } x \ge x_1$$

For $[x_1, \infty)$, and f' is monotonically decreasing and $y_2 < y_1$, $f'_2(x) > f'_1(x)$. In, $[0, x_2]$, $f'_2(x) = f'_1(x)$.

In $[x_2, x_1]$,

$$f'_1(x) = g'(x)$$

$$f'_2(x) = \frac{bg'(y_2) - ah'(x-y_2)}{b-a} \text{ for } x \ge x_2$$

Hence, in the interval, $0 \le y_2 \le x$ for $x \in [x_2, x_1]$,

$$f'_2(x) \ge \frac{bg'(x) - ah'(0)}{b-a} > g'(x)$$

And we assume that, $g'(x) \ge h'(0) \therefore f'_2(x) > f'_1(x)$
Or, for $x_1 > x_2 > x_3 > \cdots > x_n > 0$, $f'_1(x) \le f'_2(x) \le f'_3(x) \le \cdots \le f'_n(x) \le \cdots$

Since $f_n(x)$ converges to $f(x)$, $f'_n(x)$ to $f'(x)$, $y_n(x)$ to $y(x)$ and x_n to $\bar{x}$, the solution has the form,

$$f(x) = \max_{0\le y\le x} \left\{g(y) + h(x-y) + f(ay + b(x-y))\right\}$$

$$\frac{df}{dy} = 0 \implies g'(y) - h'(x-y) + (a-b)f'(ay + b(x-y)) = 0$$

∎

1.5. Examples

Example 1. Show that, The continuous solution of $f(x) = \max[cx^d + f(ax), ex^g + f(bx)], f(0) = 0$, subject to

1. $a, b \in (0, 1)$, $c, d, e, g > 0$,
2. $0 < d < g$

is given by,

$$f(x) = \begin{cases} \frac{cx^d}{1-a^d}, & \text{if } 0 \le x \le \bar{x} \\ ex^g + f(bx), & \text{if } x \ge \bar{x} \end{cases}$$

where, $\bar{x} = \left(\dfrac{c/(1-a^d)}{e/(1-b^d)} \right)^{\frac{1}{g-d}}$.

Solution: Assuming, A as the choice $cx^d + f(ax)$ and B as the choice $ex^g + f(bx)$, we have any solution of the form $A^{k_1^t} B^{k_2^t}$ or the number of times A and B are chosen to reach a solution at the t^{th} step.
At optimal time, represented by say ∞, any choice of A or B would not change the solution. Mathematically,

$$BA^\infty = A^\infty$$

Now,

$$\begin{aligned} A^\infty &= cx^d + f(ax) = cx^d + c(ax)^d + f(a^2x) + f(a^3x) + \cdots \\ &= \frac{cx^d}{(1-a^d)} \end{aligned}$$

Similarly, $BA^\infty \implies f(x) = ex^g + \frac{cb^d x^d}{(1-a^d)}$.
Equating them we get,

$$ex^g + \frac{cb^d x^d}{(1-a^d)} = \frac{cx^d}{(1-a^d)} \implies x = \left(\frac{c/(1-a^d)}{e/(1-b^d)} \right)^{\frac{1}{g-d}}$$

The value of x is the optimal x, $\bar{x}$.
Case I: When $0 \le x \le \bar{x}$ We have to show that $\frac{cx^d}{1-a^d}$ is the solution. At $t = \infty$,

$$\max \left[\frac{cx^d}{(1-a^d)}, ex^g + \frac{cb^d x^d}{(1-a^d)} \right] = \frac{cx^d}{(1-a^d)}$$

This is true for small x when $g > d > 0$ and $b \in (0, 1)$. If we proceed for more values of x, the smallest value at which solution B becomes optimal is attained at $BA^\infty = A^\infty$ which occurs at $x = \bar{x}$ as calculated above.

Case II: When $x \ge \bar{x}$, we have to show that $ex^g + f(bx)$ is the solution. At $t = \infty$,
Define, $f_{AB}(x) = cx^d + ea^g x^g + f(abx)$ and $f_{BA}(x) = ex^g + cb^d x^d + f(abx)$. These are compositions of the two choices A and B. It can be interpreted as the dominant choice at each step, as f_{AB} will signify the choice of A at the first step and B at the second step and so on. Similarly, f_{BA} will signify the choice of B at the first step and A at the second step and so on.

Now, their point of intersection occurs at $p = \left(\frac{c/(1-b^d)}{e/(1-a^g)} \right)^{\frac{1}{g-d}}$.
Now, $p < \bar{x}$ as $g > d$. At this point, $f_{AB}(x) < f_{BA}(x)$ for $x > p$ for $x > \bar{x}$. This arises a contradiction.
$\therefore f(x) = ex^g + f(bx)$ is the optimal choice only when $x \ge \bar{x}$.

Example 2. Let us define the function $f_N(a) = \max_R[x_1 x_2 \cdots x_n]$ where R is the region determined by the conditions

1. $x_1 + x_2 + \cdots + x_n = a, a > 0$
2. $x_i \geq 0$

Prove that $f_N(a)$ satisfies the recurrence relation $f_N(a) = \max\limits_{0 \leq x \leq a} x f_{N-1}(a-x), N \geq 2$ with $f_1(a) = a$.

Solution: We can solve this using mathematical induction.
$N = 1$: $f_1(a) = \max[x_1] = a$ as $x_1 = a$. Hence the relation holds.
Let $N = k$ be true. That is, $f_k(a) = \max\limits_{0 \leq x \leq a} x f_{k-1}(a-x)$.
For $N = k+1$,

$$\begin{aligned} f_{k+1}(x) &= \max[x_1 x_2 \cdots x_{k+1}] \\ &= \max[\max[x_1 x_2 \cdots x_k] x_{k+1}] \\ &= \max[x_{k+1} f_k(a - x_{k+1})] && \text{as } x_1 + x_2 + \cdots + x_k = a - x_{k+1} \end{aligned}$$

$\because$ The relationship hold for all $N \in \mathbb{N}$. The recurrence relation holds.
This completes the proof.

Example 3. For the previous f, show inductively that $f_N(a) = \frac{a^N}{N^N}$, and hence establish the arithmetic-geometric mean inequality, for $x_i > 0$.

Solution: Given, $f_N(a) = \max\limits_{0 \leq x \leq a} x f_{N-1}(a-x)$.
For $N = 1$ $f_1(a) = a = \frac{a^1}{1^1}$ which is true.
Let $f_k(a) = \frac{a^k}{k^k}$ hold as well.
For, $N = k+1$, $f_{k+1}(a) = \max\limits_{0 \leq x \leq a} x f_k(a-x) = \max\limits_{0 \leq x \leq a} x \frac{(a-x)^k}{k^k}$ which is maximized at $x = \frac{a}{k+1}$ resulting in $f_{k+1}(x) = \frac{a^{k+1}}{(k+1)^{k+1}}$
$\because$ The relationship hold for all $N \in \mathbb{N}$. The recurrence relation holds.
This completes the proof.

Example 4. Define a function,

$$f_N(a) = \min_R \sum_{i=1}^{N} x_i^p, p > 0$$

where R is a region defined by,

1. $\sum\limits_{i=1}^{N} x_i \geq a, a > 0$
2. $x_i \geq 0$

Show that $f_N(a)$ satisfies the recurrence relation

$$f_N(a) = \min_{0 \leq x \leq a} [x^p + f_{N-1}(a-x)], N \geq 2 \text{ with } f_1(a) = a^p$$

Solution: We can solve this inductively as,
For $N = 1$, $f_1(a) = a^p$, which is true.
Let for $N = k$, $f_k(a) = \min\limits_{0 \leq x \leq a} [x^p + f_{k-1}(a-x)]$
For, $N = k+1$,

$$
\begin{aligned}
f_{k+1}(a) &= \min_R \sum_{i=1}^{k+1} x_i^p \\
&= \min_R [\min_R \sum_{i=1}^{k} x_i^p + x_{k+1}^p] \\
&= \min_{0 \le x \le a} [x_{k+1}^p + f_k(a-x)]
\end{aligned}
$$

$\because$ The relationship hold for all $N \in \mathbb{N}$. The recurrence relation holds.
This completes the proof.

Example 5. Let $f(x)$ and $F(x)$ be the continuous solutions of the above equations under the assumptions that $u(x,y)$ and $v(x,y)$ are continuous in x and $y \forall x, y > 0$, with $0 < a, b < 1$, and that $\sum_{n=0}^{\infty} m(c^n z) < \infty$ where $m(z) = \max_{0 \le x \le z} [\max_{0 \le y \le x} \max |u(x,y)|, |v(x,y)|]$. If $\max_{0 \le x \le z} \max_{0 \le y \le x} |u(x,y) - v(x,y)| = D(z)$ and $\sum_{n=0}^{\infty} D(c^n z) < \infty$, $c = \max(a,b)$, show that,

$$
|f(x) - F(x)| < \sum_{n=0}^{\infty} D(c^n z)
$$

Solution: We define,

$$
\begin{aligned}
f_1(x) &= \max_{0 \le y \le x} u(x,y) \\
f_{N+1}(x) &= \max_{0 \le y \le x} \max \left\{ u(x,y), f_N(ay + b(x-y)) \right\} \\
F_1(x) &= \max_{0 \le y \le x} v(x,y) \\
F_{N+1}(x) &= \max_{0 \le y \le x} \max \left\{ v(x,y), F_N(ay + b(x-y)) \right\}
\end{aligned}
$$

Also, $\lim_{N \to \infty} f_N(x) = f(x)$ and $\lim_{N \to \infty} F_N(x) = F(x)$.
Now,

$$
|f_1(x) - F_1(x)| \le \max_{0 \le y \le x} |u(x,y) - v(x,y)| \le D(x)
$$

Similarly,

$$
\begin{aligned}
|f_{N+1}(x) - F_{N+1}(x)| &\le \max_{0 \le y \le x} \max \left\{ |u(x,y) - v(x,y)|, |f_N(ay + b(x-y)) - F_N(ay + b(x-y))| \right\} \\
&\le D(x) + \max_{0 \le y \le x} |f_N(ay + b(x-y)) - F_N(ay + b(x-y))|
\end{aligned}
$$

$\therefore$

$$
|f_{N+1}(x) - F_{N+1}(x)| \le \sum_{n=0}^{\infty} D(c^n x)
$$

When, $N \to \infty$

$$
|f(x) - F(x)| \le \sum_{n=0}^{\infty} D(c^n x)
$$

This completes the proof.

[Bel57]

References

[Bel57] Richard Bellman. *Dynamic Programming*. Dover Publications, 1957.