UGP: Essays on Dynamic Programming

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§1. A Multi-Stage Allocation Process

1.1. Introduction

1.1.1. Notation

The standard Dynamic Programming problem desribed throughout will be in metric space $\mathbb{X}$.
Given, $\Phi : \mathbb{X} \times \mathbb{X} \to \mathbb{R}$ and $\Gamma : \mathbb{X} \rightrightarrows \mathbb{R}$
For a fixed x^0, find a sequence $x_m = x^1, ..., x^m, ...$ such that

$$\max \Phi(x^0, x^1) + \sum_{i=1}^{\infty} \delta^i \Phi(x^i, x^{i+1}), \text{ where } \delta \in (0,1) \text{ and } x^1 \in \Gamma(x^0), x^{i+1} \in \Gamma(x^i), m = 1, 2, 3, ...$$

Here δ is called the discount factor.

1.1.2. Functional Equations

Considering a space of Real numbers $\mathbb{R}$, we can define a functional equation as follows:

$$f(x) = \max_{0 \le y \le x} [g(y) + h(x-y) + f(ay + b(x-y))]$$

where $a, b \in \mathbb{R}$ and g, h are given functions.
This can be seen as an **insurance problem** where functions g and h represent the different places where and initial premium x is invested. The function f represents the value of the policy at the end of the period.

We can define, $R_1(x, y) = g(y) + h(x-y)$, initially the aim is to determine $\max_{0 \le y \le x} R_1(x, y)$. The value of x and y obtained from this maximization can be used to generate the next element of the sequence x_m.
i.e. $x_1 = ay + b(x-y) = y_1 + (x_1 - y_1)$
Further we define $R_2(x, y, y_1) = g(y) + h(x-y) + g(y_1) + h(x_1 - y_1)$ where $(y, y_1) \in [0, x] \times [0, x_1]$

This can be generalized to $R_m(x, y, y_1, ..., y_{m-1}) = g(y) + h(x-y) + g(y_1) + h(x_1 - y_1) + g(y_2) + h(x_2 - y_2) + ... + g(y_{m-1}) + h(x_{m-1} - y_{m-1})$
and the maximization problem becomes

$$\max R_m(x, y, y_1, ..., y_{m-1}) \text{ such that}$$

$$\begin{array}{ll}
ay + b(x-y) = x_1 & 0 \le y \le x \\
ay_1 + b(x_1 - y_1) = x_2 & 0 \le y_1 \le x_1 \\
\vdots & \vdots \\
ay_{m-1} + b(x_{m-1} - y_{m-1}) = x_m & 0 \le y_{m-1} \le x_{m-1}
\end{array}$$

Here, $f_N(x) = \max_{t, y_i} R_N(x, y_1, ..., y_N)$ and $f_N(x)$ can be interpreted as the maximum return obtained from an N-stage decision with initial premium x and $x \ge 0$.

$$\therefore f_N(x) = \max_{0 \le y \le x} [g(y) + h(x-y) + f_{N-1}(ay + b(x-y))]$$

or

$$f(x) = \max_{0 \le y \le x} [g(y) + h(x-y) + f(ay + b(x-y))] \text{ for } N \to \infty$$

1.2. Existence and Uniqueness Theorems

Under the following assumptions, the functional equation has a unique solution, which is continuous at $x = 0$ and has a value 0 at $x = 0$. Moreover the solution is continuous at $x \ge 0$.

The assumptions are:

1. g and h are continuous on $[0, \infty)$ and g(0) = h(0) = 0
2. If $m(x) = \max_{0 \le y \le x}[\max\{|g(y)|, |h(y)|\}]$ and $c = \max\{a, b\}$, then $\sum_{n=0}^{\infty} m(c^n x) < \infty \forall x \ge 0$

Proof. We define a mapping $T(f, y) = g(y)+h(x-y)+f(ay+b(x-y))$ and $f_{N+1}(x) = \max_{0 \le y \le x} T(f_N, y)$, assuming g and h to be non-negative.
The sequence $\{f_N(x)\}$ is an increasing sequence of functions such that $f_N \to f$
Prooving that T is bounded above would suffice.
Method 1: We know that $x_1 \le cx$

$$\therefore ax_1 + b(x_1 - y) \le cx_1 \le c^2 x$$
$$\implies T(f_N, y) \le 2(m(x) + m(cx) + ... + m(c^N x))$$

Which proves that T is bounded above by a constant.
Method 2:

$$f_{N+1}(x) = \sup_{0 \le y \le x} T(f_N, y)$$
$$f(x) \ge \sup_{0 \le y \le x} T(f_N, y)$$
$$f(x) \ge T(f_N, y) \forall y \in [0, x]$$
$$f(x) \ge T(f, y) \text{ as } N \to \infty$$
$$\therefore \boxed{f(x) = \sup_{0 \le y \le x} T(f, y)}$$

Also,

$$\begin{aligned} T(f_N, y) &= g(y) + h(x - y) + f_N(ay + b(x - y)) \\ &\le g(y) + h(x - y) + f(ay + b(x - y)) \\ &= T(f, y) \end{aligned}$$

$$\begin{aligned} f_{N+1}(x) &= \sup_{0 \le y \le x} T(f_N, y) \\ &\le \sup_{0 \le y \le x} T(f, y) \\ f_{N+1}(x) &\le \sup_{0 \le y \le x} T(f, y) \end{aligned}$$

As $N \to \infty$,

$$\boxed{f(x) \le \sup_{0 \le y \le x} T(f, y)}$$

$$\therefore f(x) = \sup_{0 \le y \le x} T(f, y) \text{ using the above two inequalities}$$

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1.2.1. Existence and Uniqueness with g and h not necessarily non-negative

We can write,

$$\begin{aligned} f_1(x) &= \max_{0 \le y \le x} [g(y) + h(x - y)] \\ &= \max_{y \in [0,x]} [g(y) + h(x - y) + 0] \\ &= \max_{y \in \Gamma(x)} [g(y) + h(x - y)] \qquad \Gamma(x) = [0, x] \text{ a set valued map} \end{aligned}$$

Digression

We will use a generalized form of Berge Maximum Theorem in the following proof, which is described here.
Berge Maximum Theorem (Generalized Form)
Let Θ and $\mathbb{X}$ be two metric spaces and $\Gamma : \Theta \rightrightarrows \mathbb{X}$ is a compact valued correspondence. Let $\phi : \mathbb{X} \times \Theta \to \mathbb{R}$ be a continuous function.
Let $\sigma(\theta) = argmax\{\phi(x, \theta) : x \in \Gamma(\theta)\}$ and let $\phi^*(\theta) = \max_x\{\phi(x, \theta) : x \in \Gamma(\theta)\}$

Let $\Gamma : \Theta \rightrightarrows \mathbb{X}$ is continuous at some $\theta \in \Theta$. Then,

1. $\sigma : \Theta \rightrightarrows \mathbb{X}$ is compact-valued, upper semi-continuous and closed at θ.
2. $\phi^* : \Theta \to \mathbb{R}$ is continuous at θ.

Some definitions that follow from above are:

1. **Upper semi-continuous Map:** Let $\Gamma : \mathbb{X} \rightrightarrows \mathbb{Y}$, then Γ is upper semi-continuous at $x \in \mathbb{X}$ if for any $\{x^m\} \in \mathbb{X}$ and $\{y^m\} \in \mathbb{Y}$ with $x_0^m \to x$ and $y^m \in \Gamma(x^m)$, the sequence $\{y^m\}$ has a convergent subsequence such that its limit is in $\Gamma(x)$.
2. **Compact valued Map:** $\Gamma : \mathbb{X} \rightrightarrows \mathbb{Y}$ is a compact valued correspondence which is upper semi-continuous if for a compact subset $S \subset \mathbb{X}$, the set $\Gamma(S)$ is compact.
3. **Closed Map:** $\Gamma : \mathbb{X} \rightrightarrows \mathbb{Y}$ is closed at $x \in \mathbb{X}$ if for any $\{x^m\} \in \mathbb{X}$ with $x^m \to x$ and $\{y^m\} \in \mathbb{Y}$ with $y^m \to y$ such that $y^m \in \Gamma(x^m)$ for each m, we have $y \in \Gamma(x)$.

By applying the generalized form of Berge Maximum Theorem, we can show that $f_1(x)$ is continuous.

$T(f_N, y) = g(y) + h(x - y) + f_N(ay + b(x - y))$ and $f_{N+1}(x) = \max\limits_{0 \le y \le x} T(f_N, y)$.

$$\begin{aligned} f_{N+1}(x) &= T(f_N, y_N) \ge T(f_N, y_N + 1) \\ f_{N+2}(x) &= T(f_N + 1, y_N + 1) \ge T(f_N + 1, y_N) \end{aligned}$$

Define,

$$u_N(x) = \sup_{0 \le z \le x} |f_N(z) - f_N + 1(z)|$$

and

$$\begin{aligned} f_{N+1}(x) - f_{N+2}(x) &\ge T(f_N, y_{N+1}) - T(f_N + 1, y_{N+1}) \\ f_{N+1}(x) - f_{N+2}(x) &\le T(f_N, y_N) - T(f_N + 1, y_N) \end{aligned}$$

From these two inequalities, we get

$$|f_{N+1}(x) - f_{N+2}(x)| \le \max\{|T(f_N, y_{N+1}) - T(f_N + 1, y_{N+1})|, |T(f_N, y_N) - T(f_N + 1, y_N)|\}$$

$$|T(f_N, y_{N+1}) - T(f_N + 1, y_{N+1})| = |f_N(ay_N + b(x - y_N)) - f_{N+1}(ay_{N+1} + b(x - y_{N+1})) \le u_N(cx)$$

$$|T(f_N, y_N) - T(f_N + 1, y_N)| = |f_N(ay_N + b(x - y_N)) - f_{N+1}(ay_N + b(x - y_N)) \le u_N(cx)$$

$\because ay + b(x - y) \le cx$ where $c = \max\{a, b\}$

$$\therefore \boxed{u_{N+1}(x) \le u_N(cx)} \text{ for } N \in \mathbb{N}$$

We now need to estimate $u_1(x)$

$$\begin{aligned} f_2(x) &= T(f_1, y_1) \\ f_1(x) - f_2(x) &\geq g(y_1) + h(x - y_1) - T(f_1, y_1) \\ f_2(x) - f_1(x) &\leq f_1(ay_1 + b(x - y_1)) \end{aligned}$$

Let y_0 be the maximizing point to obtain $f_1(x)$ i.e. $f_1(x) = g(y_0) + h(x - y_0)$ and $f_2(x) \geq T(f_1, y_0)$

$$\begin{aligned} f_1(x) - f_2(x) &\geq g(y_0) + h(x - y_0) - T(f_1, y_0) \\ &= -f_1(ay_0 + b(x - y_0)) \\ f_2(x) - f_1(x) &\geq f_1(ay_0 + b(x - y_0)) \end{aligned}$$

$$|f_2(x) - f_1(x)| \leq \max\{f_1(ay_1 + b(x - y_1)), |f_1(ay_0 + b(x - y_0))|\} \implies |f_2(x) - f_1(x)| \leq 2m(cx) \forall z \in [0, x]$$

$$\therefore \boxed{u_1(x) \leq 2m(cx)}$$

$$u_{N+1}(x) \leq u_N(cx) \leq u_{N-1}(c^2x) \leq ... \leq u_1(c^N x) \leq 2m(c^{N+1}x)$$

$$\therefore u_{N+1}(x) \leq 2m(c^{N+1}x) \text{ and } u_1(x) \leq 2m(cx) \text{ for } N \in \mathbb{N}$$

$$\implies \boxed{u_N(x) \leq 2m(c^N x) \text{ for } N \in \mathbb{N}}$$

From the above result we can say that,

$$\sum_{N=1}^{\infty} u_N(x) \leq \sum_{N=1}^{\infty} 2m(c^N x)$$

that is

$$\sum_{N=1}^{\infty} u_N(x) \leq \infty (\text{convergent})$$

$$\implies u_N(x) \to 0 \text{ as } N \to \infty$$

$\implies$ $f_N(x)$ is a cauchy sequence (which then implies that $f_N - f_M$ can be expressed as a sum of differences).

$$\therefore f(x) = \lim_{N \to \infty} f_N(x) \text{ exists and is continuous } \forall x \geq 0$$

Now we know that,

$$\begin{aligned} f_{N+1}(x) &\geq T(f_N, y) \\ &\geq g(y) + h(x - y) + f_N(ay + b(x - y)) \text{ for } y \in [0, x] \end{aligned}$$

For large N,

$$\begin{aligned} f(x) &\geq g(y) + h(x - y) + f(ay + b(x - y)) \\ &\geq \max_{0 \leq y \leq x} T(f, y) \end{aligned} \tag{A}$$

For a bounded sequence $\{y_N\}$ with $y_N \to y^*$ for some $y^* \in [0, x]$ and $N \to \infty$ we have,

$$\begin{aligned} f(x) &= T(f, y^*) \\ &\leq \max_{0 \leq y \leq x} T(f, y) \qquad (B) \end{aligned}$$

using (A) and (B),

$$\boxed{f(x) = \max_{0 \leq y \leq x} T(f, y)}$$

1.2.2. Uniqueness

By proving the existence, we know that,

$$f(x) = \max_{0 \leq y \leq x} T(f, y)$$

Assume on the contrary that there are two such solutions $f \& F$

$$\begin{aligned} f(x) = T(f, y) \geq T(f, w) \\ F(x) = T(F, w) \geq T(F, y) \end{aligned}$$

Now,

$$\begin{aligned} |f(x) - F(x)| &\leq \max\{|T(f, y) - T(F, y)|, |T(f, w) - T(F, w)|\} \\ &\leq \max\{|f(ay + b(x - y) - F(ay + b(x - y))|, |f(aw + b(x - w) - F(aw + b(x - w))|\} \end{aligned}$$

From previous definition, $u(x) = \sup\limits_{0 \leq z \leq x} f(z) - F(z)$ We know that $u(x) \geq 0$ and $u(x) \to 0$ as $x \to \infty$.

$$\begin{aligned} \therefore 0 \geq u(x) \geq |f(x) - F(x)| \\ \implies |f(x) - F(x)| \to 0 \text{ as } x \to \infty \\ \implies f(x) = F(x) \text{ for all } x \geq 0 \end{aligned}$$

$\therefore$ The solution obtained is unique.
[Bel57]

References

[Bel57] Richard Bellman. *Dynamic Programming*. Dover Publications, 1957.