

EE5609 Assignment 5

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Abstract—This document contains the solution of geometry through linear algebra.

Download latex and python codes from

https://github.com/abhishekt711/EE5609/tree/master/Assignment_5

1 PROBLEM

Prove that the following equations represent two straight lines, find also their point of intersection and the angle between them.

$$6y^2 - xy - x^2 + 30y + 36 = 0.$$

2 EXPLANATION

The given equation can be written as:

$$-x^2 - xy + 6y^2 + 30y + 36 = 0 \quad (2.0.1)$$

$\begin{bmatrix} \mathbf{V} & \mathbf{u} \\ \mathbf{u}^T & f \end{bmatrix}$ of (2.0.1) becomes

$$\begin{vmatrix} -1 & -\frac{1}{2} & 0 \\ -\frac{1}{2} & 6 & 15 \\ 0 & 15 & 36 \end{vmatrix} = 0 \quad (2.0.2)$$

Expanding equation (2.0.2), we get zero.

Hence given equation represents a pair of straight lines.

The general equation second degree is given by

$$ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0 \quad (2.0.3)$$

Let (α, β) be their point of intersection, then

$$\begin{pmatrix} a & h \\ h & b \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} -d \\ -e \end{pmatrix} \quad (2.0.4)$$

Under Affine transformation,

$$\mathbf{x} = \mathbf{M}\mathbf{y} + \mathbf{c} \quad (2.0.5)$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} + \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \quad (2.0.6)$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} X + \alpha \\ Y + \beta \end{pmatrix} \quad (2.0.7)$$

(2.0.3) under transformation (2.0.7) will become,

$$aX^2 + 2bXY + cY^2 = 0 \quad (2.0.8)$$

$$\begin{pmatrix} X & Y \end{pmatrix} \begin{pmatrix} a & h \\ h & b \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} = 0 \quad (2.0.9)$$

$$\begin{pmatrix} X & Y \end{pmatrix} \begin{pmatrix} u_1 & v_1 \\ u_2 & v_2 \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} u_1 & u_2 \\ v_1 & v_2 \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} = 0 \quad (2.0.10)$$

$$\begin{pmatrix} X' & Y' \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} X' \\ Y' \end{pmatrix} = 0 \quad (2.0.11)$$

$$\text{where } X' = Xu_1 + Yv_1 \text{ and } Y' = Xu_2 + Yv_2 \quad (2.0.12)$$

$$\Rightarrow \lambda_1(X')^2 + \lambda_2(Y')^2 = 0 \quad (2.0.13)$$

(2.0.13) is called *Spectral decomposition* of matrix

$$\Rightarrow X' = \pm \sqrt{-\frac{\lambda_2}{\lambda_1}} Y' \quad (2.0.14)$$

$$u_1X + u_2Y = \pm \sqrt{-\frac{\lambda_2}{\lambda_1}} (v_1X + v_2Y) \quad (2.0.15)$$

$$u_1(x - \alpha) + u_2(y - \beta) = \pm \sqrt{-\frac{\lambda_2}{\lambda_1}} (v_1(x - \alpha) + v_2(y - \beta)) \quad (2.0.16)$$

Given equation is

$$-x^2 - xy + 6y^2 + 30y + 36 = 0 \quad (2.0.17)$$

Substituting in (2.0.4)

$$\begin{pmatrix} -1 & -\frac{1}{2} \\ -\frac{1}{2} & 6 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 0 \\ -15 \end{pmatrix} \quad (2.0.18)$$

$$\Rightarrow \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \frac{30}{23} \\ -\frac{60}{23} \end{pmatrix} \quad (2.0.19)$$

From, Spectral decomposition,

$$\mathbf{V} = \mathbf{P}\mathbf{D}\mathbf{P}^T \quad (2.0.20)$$

$$\mathbf{V} = \begin{pmatrix} -1 & \frac{-1}{2} \\ \frac{-1}{2} & 6 \end{pmatrix} \quad (2.0.21)$$

$$\mathbf{P} = \begin{pmatrix} 7 - 5\sqrt{2} & 7 + 5\sqrt{2} \\ 1 & 1 \end{pmatrix} \quad (2.0.22)$$

$$\mathbf{D} = \begin{pmatrix} \frac{5+5\sqrt{2}}{2} & 0 \\ 0 & \frac{5-5\sqrt{2}}{2} \end{pmatrix} \quad (2.0.23)$$

P and D are also verified using python code from

codes/diagonalize1.py

Substituting (2.0.19), (2.0.22) and (2.0.23) in (2.0.16),

$$\begin{aligned} &\Rightarrow (7 - 5\sqrt{2})\left(x - \frac{30}{23}\right) + \left(y + \frac{60}{23}\right) \\ &= \pm \sqrt{-\frac{\frac{5-5\sqrt{2}}{2}}{\frac{5+5\sqrt{2}}{2}}} \left((7 - 5\sqrt{2})\left(x - \frac{30}{23}\right) + \left(y + \frac{60}{23}\right) \right) \end{aligned} \quad (2.0.24)$$

simplifying 2.0.24, we get:

$$-x + 2y + 6 = 0 \text{ and } x + 3y + 6 = 0 \quad (2.0.25)$$

$$\Rightarrow (-x + 2y + 6)(x + 3y + 6) = 0 \quad (2.0.26)$$

Hence, the intersection point is $\left(\frac{6}{5}, -\frac{12}{5}\right)$

Verified using python code from

codes/Assignment_5.py

Angle between two lines, θ can be given by

$$\mathbf{n}_1 = \begin{pmatrix} -2 \\ -1 \end{pmatrix} \quad (2.0.27)$$

$$\mathbf{n}_2 = \begin{pmatrix} -3 \\ 1 \end{pmatrix} \quad (2.0.28)$$

$$\cos \theta = \frac{\mathbf{n}_1^T \mathbf{n}_2}{\|\mathbf{n}_1\| \|\mathbf{n}_2\|} \quad (2.0.29)$$

$$\cos \theta = \frac{(-2 \quad -1) \begin{pmatrix} -3 \\ 1 \end{pmatrix}}{\sqrt{(-2)^2 + (-1)^2} \times \sqrt{+(-3)^2 + 1}} = \frac{1}{\sqrt{2}} \quad (2.0.30)$$

$$\Rightarrow \theta = 45^\circ \quad (2.0.31)$$

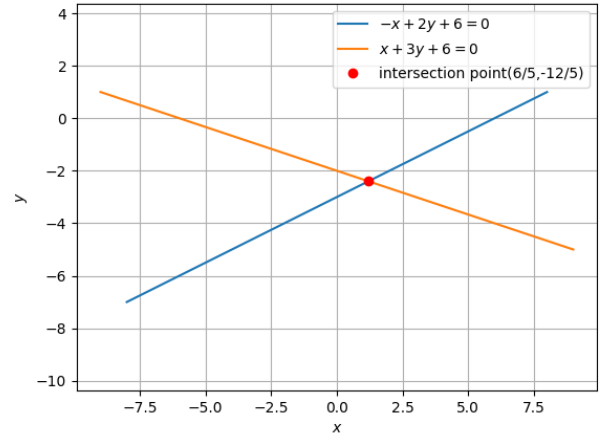


Fig. 0: plot showing intersection of lines.