EE5609 Assignment 9

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***Abstract*—This document solves problem based on solution of vector space.**

Download all solutions from

https://github.com/abhishekt711/EE5609/tree/master/Assignment_9

1 Problem

let $\alpha = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ and $\beta = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ be vectors in $\mathbb{R}^2$ such that
$x_1y_1 + x_2y_2 = 0; \quad x_1^2 + x_2^2 = y_1^2 + y_2^2 = 1$.
Proove that ß $= \{\alpha,\beta\}$ is a basis of $\mathbb{R}^2$. Find the coordinates of the vector $\begin{pmatrix} a, b \end{pmatrix}$ in the ordered basis ß $= \{\alpha,\beta\}$. (The conditions on α and β say, geometrically, that α and β are perpendicular and each has length 1).

2 Solution

we need to show that α and β are linearly independent in order to proove that ß $= \{\alpha,\beta\}$ is a basis of $\mathbb{R}^2$.

Given in the question are:

$$\alpha^T\beta = 0 \tag{2.0.1}$$

$$||\alpha||^2 = ||\beta||^2 = 1 \tag{2.0.2}$$

Let,

$$\mathbf{A} = \begin{pmatrix} \alpha & \beta \end{pmatrix} = \begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \end{pmatrix} \tag{2.0.3}$$

then,

$$\mathbf{A}^T\mathbf{A} = \begin{pmatrix} ||\alpha||^2 & \alpha^T\beta \\ \alpha^T\beta & ||\beta||^2 \end{pmatrix} \tag{2.0.4}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{2.0.5}$$

$$\therefore \mathbf{A}^T\mathbf{A} = \mathbf{I} \tag{2.0.6}$$

Inverse of $\mathbf{A}$ exist. $\mathbf{A}^T$ is the inverse of $\mathbf{A}$. Thus, the columns of $\mathbf{A}$ are linearly independent i.e, α and β are linearly independent.

Hence, ß $= \{\alpha,\beta\}$ is a basis of $\mathbb{R}^2$.

To, find the coordinates of the vector $\begin{pmatrix} a, b \end{pmatrix}$ in the ordered basis ß $= \{\alpha,\beta\}$.

$$c_1\begin{pmatrix} \alpha \end{pmatrix} + c_2\begin{pmatrix} \beta \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} \tag{2.0.7}$$

$$c_1x_1 + c_2y_1 = a \tag{2.0.8}$$

$$c_1x_2 + c_2y_2 = b \tag{2.0.9}$$

We can row-reduce the augmented matrix to find the constant c_1 and c_2,

$$\left(\begin{array}{cc|c} x_1 & y_1 & a \\ x_2 & y_2 & b \end{array}\right) \tag{2.0.10}$$

$$\xleftrightarrow{R_1 = \frac{R_1}{x_1}} \left(\begin{array}{cc|c} 1 & \frac{y_1}{x_1} & \frac{a}{x_1} \\ x_2 & y_2 & b \end{array}\right) \tag{2.0.11}$$

$$\xleftrightarrow{R_2 = R_2 - x_2R_1} \left(\begin{array}{cc|c} 1 & \frac{y_1}{x_1} & \frac{a}{x_1} \\ 0 & y_2 - \frac{x_2y_1}{x_1} & b - \frac{x_2a}{x_1} \end{array}\right) \tag{2.0.12}$$

$$= \left(\begin{array}{cc|c} 1 & \frac{y_1}{x_1} & \frac{a}{x_1} \\ 0 & \frac{x_1y_2 - x_2y_1}{x_1} & \frac{x_1b - x_2a}{x_1} \end{array}\right) \tag{2.0.13}$$

$$\xleftrightarrow{R_2 = \left(\frac{x_1}{x_1y_2 - x_2y_1}\right)R_2} \left(\begin{array}{cc|c} 1 & \frac{y_1}{x_1} & \frac{a}{x_1} \\ 0 & 1 & \frac{x_1b - x_2a}{x_1y_2 - x_2y_1} \end{array}\right) \tag{2.0.14}$$

$$\xleftrightarrow{R_1 = R_1 - \left(\frac{y_1}{x_1}\right)R_2} \left(\begin{array}{cc|c} 1 & 0 & \frac{ay_2 - by_1}{x_1y_2 - x_2y_1} \\ 0 & 1 & \frac{x_1b - x_2a}{x_1y_2 - x_2y_1} \end{array}\right) \tag{2.0.15}$$

Using 2.0.1 and 2.0.2 and simplifying in 2.0.16,

$$= \left(\begin{array}{cc|c} 1 & 0 & ax_1 + bx_2 \\ 0 & 1 & ay_1 + by_2 \end{array}\right) \tag{2.0.16}$$

$$\therefore c_1 = ax_1 + bx_2 \tag{2.0.17}$$

$$\therefore c_2 = ay_1 + by_2 \tag{2.0.18}$$

Hence,

$$\begin{pmatrix} a \\ b \end{pmatrix} = (ax_1 + bx_2)\begin{pmatrix} \alpha \end{pmatrix} + (ay_1 + by_2)\begin{pmatrix} \beta \end{pmatrix} \tag{2.0.19}$$