EE5609 Assignment 5

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Abstract—**This document contains the solution of geometry through linear algebra.**

Download latex and python codes from

1 PROBLEM

Prove that the following equations represent two straight lines, find also their point of intersection and the angle between them.
$6y^2 - xy - x^2 + 30y + 36 = 0$.

2 EXPLANATION

The given equation can be written as:

$$-x^2 - xy + 6y^2 + 30y + 36 = 0 \qquad (2.0.1)$$

$\begin{vmatrix} \mathbf{V} & \mathbf{u} \\ \mathbf{u}^T & f \end{vmatrix}$ of (2.0.1) becomes

$$\begin{vmatrix} -1 & -\frac{1}{2} & 0 \\ \frac{-1}{2} & 6 & 15 \\ 0 & 15 & 36 \end{vmatrix} = 0 \qquad (2.0.2)$$

Expanding equation (2.0.2), we get zero.
Hence given equation represents a pair of straight lines.

The general equation second degree is given by

$$ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0 \qquad (2.0.3)$$

Let (α, β) be their point of intersection, then

$$\begin{pmatrix} a & h \\ h & b \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} -d \\ -e \end{pmatrix} \qquad (2.0.4)$$

Under *Affine transformation*,

$$\mathbf{x} = \mathbf{My} + c \qquad (2.0.5)$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} + \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \qquad (2.0.6)$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} X + \alpha \\ Y + \beta \end{pmatrix} \qquad (2.0.7)$$

(2.0.3) under transformation (2.0.7) will become,

$$aX^2 + 2bXY + cY^2 = 0 \qquad (2.0.8)$$

$$\begin{pmatrix} X & Y \end{pmatrix} \begin{pmatrix} a & h \\ h & b \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} = 0 \qquad (2.0.9)$$

$$\begin{pmatrix} X & Y \end{pmatrix} \begin{pmatrix} u_1 & v_1 \\ u_2 & v_2 \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} u_1 & u_2 \\ v_1 & v_2 \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} = 0 \qquad (2.0.10)$$

$$\begin{pmatrix} X' & Y' \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} X' \\ Y' \end{pmatrix} = 0 \qquad (2.0.11)$$

where $X' = Xu_1 + Yu_2$ and $Y' = Xv_1 + Yv_2$
$$\qquad (2.0.12)$$

$$\implies \lambda_1(X')^2 + \lambda_2(Y')^2 = 0 \qquad (2.0.13)$$

(2.0.13) is called *Spectral decomposition* of matrix

$$\implies X' = \pm\sqrt{-\frac{\lambda_2}{\lambda_1}}Y' \qquad (2.0.14)$$

$$u_1X + u_2Y = \pm\sqrt{-\frac{\lambda_2}{\lambda_1}}(v_1X + v_2Y) \qquad (2.0.15)$$

$$u_1(x - \alpha) + u_2(y - \beta) = \pm\sqrt{-\frac{\lambda_2}{\lambda_1}}(v_1(x - \alpha) + v_2(y - \beta)) \qquad (2.0.16)$$

Given equation is

$$-x^2 - xy + 6y^2 + 30y + 36 = 0 \qquad (2.0.17)$$

Substituting in (2.0.4)

$$\begin{pmatrix} -1 & \frac{-1}{2} \\ \frac{-1}{2} & 6 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 0 \\ -15 \end{pmatrix} \qquad (2.0.18)$$

$$\implies \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \frac{30}{23} \\ \frac{-60}{23} \end{pmatrix} \qquad (2.0.19)$$

From, Spectral decomposition,

$$\mathbf{V} = \mathbf{PDP}^T \tag{2.0.20}$$

$$\mathbf{V} = \begin{pmatrix} -1 & \frac{-1}{2} \\ \frac{-1}{2} & 6 \end{pmatrix} \tag{2.0.21}$$

$$\mathbf{P} = \begin{pmatrix} 7 - 5\sqrt{2} & 7 + 5\sqrt{2} \\ 1 & 1 \end{pmatrix} \tag{2.0.22}$$

$$\mathbf{D} = \begin{pmatrix} \frac{5+5\sqrt{2}}{2} & 0 \\ 0 & \frac{5-5\sqrt{2}}{2} \end{pmatrix} \tag{2.0.23}$$

P and D are also verified using python code from

codes/diagonalize1.py

Substituting (2.0.19), (2.0.22) and (2.0.23) in (2.0.16),

$$\left(7 - 5\sqrt{2}\right)\left(x - \frac{30}{23}\right) + \left(y + \frac{60}{23}\right)$$

$$= \pm \sqrt{-\frac{\frac{5-5\sqrt{2}}{2}}{\frac{5+5\sqrt{2}}{2}}\left(\left(7 - 5\sqrt{2}\right)\left(x - \frac{30}{23}\right) + \left(y + \frac{60}{23}\right)\right)} \tag{2.0.24}$$

simplifying 2.0.24, we get:

$$-x + 2y + 6 = 0 \text{ and } x + 3y + 6 = 0 \tag{2.0.25}$$

$$\implies (-x + 2y + 6)(x + 3y + 6) = 0 \tag{2.0.26}$$

Hence, the intersection point is $\begin{pmatrix} \frac{6}{5} \\ -\frac{12}{5} \end{pmatrix}$
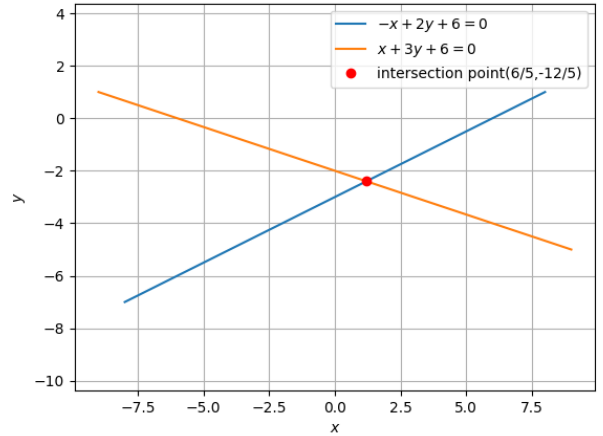
Verified using python code from

codes/Assignment_5.py

Fig. 0: plot showing intersection of lines.

Angle between two lines, θ can be given by

$$n_1 = \begin{pmatrix} -2 \\ -1 \end{pmatrix} \tag{2.0.27}$$

$$n_2 = \begin{pmatrix} -3 \\ 1 \end{pmatrix} \tag{2.0.28}$$

$$\cos\theta = \frac{\mathbf{n_1}^T\mathbf{n_2}}{\|\mathbf{n_1}\|\,\|\mathbf{n_2}\|} \tag{2.0.29}$$

$$\cos\theta = \frac{\begin{pmatrix} -2 & -1 \end{pmatrix}\begin{pmatrix} -3 \\ 1 \end{pmatrix}}{\sqrt{(-2)^2 + (-1)^2} \times \sqrt{+(-3)^2 + 1}} = \frac{1}{\sqrt{2}} \tag{2.0.30}$$

$$\implies \theta = 45° \tag{2.0.31}$$