EE5609 Assignment 16

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***Abstract*—This document solves problem based on Matrix Theory.**

Download all solutions from

https://github.com/abhishekt711/EE5609/tree/master/Assignment_16

1 Problem

Let A be an $n \times n$ self-adjoint matrix with eigenvalues $\lambda_1, \cdots, \lambda_n$.
Let $\|X\|_2 = \sqrt{|X_1|^2 + \cdots + |X_n|^2}$ for $X = (x_1, \cdots, x_n) \in \mathbb{C}^n$. If $\rho(A) = a_0 I + a_1 A + \cdots + a_n A^n$ then $sup_{\|X\|_2=1} \|\rho(A)X\|_2$ is equal to.

2 Definition

A matrix $\mathbf{A}$ for which $\mathbf{A}^H = \mathbf{A}$ is said to be a self-adjoint matrix, here conjugate transpose is denoted as $\mathbf{A}^H$. If a matrix is self-adjoint, It is said to be hermitian. Every eigen value of a self-adjoint matrix are real.

3 solution

For $\mathbf{A}$ be an $n \times n$ self-adjoint matrix with eigenvalues $\lambda_1, \cdots, \lambda_n$.

$$\therefore \lambda_j \in \mathbb{R} : 1 \le j \le n \tag{3.0.1}$$

$$\implies \mathbf{A}X = \lambda_j X \tag{3.0.2}$$

$$\implies \mathbf{A}^n X = \lambda_j^n X \tag{3.0.3}$$

Also given,

$$\rho(\mathbf{A}) = a_0 I + a_1 \mathbf{A} + \cdots + a_n \mathbf{A}^n \tag{3.0.4}$$

$$\implies (\rho(\mathbf{A}))X = (a_0 I + a_1 \mathbf{A} + \cdots + a_n \mathbf{A}^n)X \tag{3.0.5}$$

$$(\rho(\mathbf{A}))X = a_0 X + a_1 \mathbf{A}X + \cdots + a_n \mathbf{A}^n X \tag{3.0.6}$$

Using 3.0.3 in 3.0.6,

$$(\rho(\mathbf{A}))X = a_0 X + a_1 \lambda_j X + \cdots + a_n \lambda_j^n X \tag{3.0.7}$$

$$\implies \|\rho(\mathbf{A})X\|_2 = \left\|(a_0 + a_1\lambda_j + \cdots + a_n\lambda_j^n)X\right\|_2 \tag{3.0.8}$$

All the eigen values of self-adjoint matrix are real. So, $(a_0 + a_1\lambda_j + \cdots + a_n\lambda_j^n) \in \mathbb{R}$. Equation 3.0.8 can be written as,

$$= |(a_0 + a_1\lambda_j + \cdots + a_n\lambda_j^n)| \, \|X\|_2 \tag{3.0.9}$$

Hence,

$$sup_{\|X\|_2=1} \|\rho(\mathbf{A})X\|_2 = \max\{|(a_0 + a_1\lambda_j + \cdots + a_n\lambda_j^n)| : 1 \le j \le n\} \tag{3.0.10}$$