

EE5609 Assignment 5

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Abstract—This document contains the solution of geometry through linear algebra.

Download latex and python codes from

https://github.com/abhishekt711/EE5609/tree/master/Assignment_5

1 PROBLEM

Prove that the following equations represent two straight lines, find also their point of intersection and the angle between them.

$$6y^2 - xy - x^2 + 30y + 36 = 0.$$

2 EXPLANATION

The given equation can be written as:

$$-x^2 - xy + 6y^2 + 30y + 36 = 0 \quad (2.0.1)$$

$\begin{bmatrix} \mathbf{V} & \mathbf{u} \\ \mathbf{u}^T & f \end{bmatrix}$ of (2.0.1) becomes

$$\begin{bmatrix} -1 & -\frac{1}{2} & 0 \\ -\frac{1}{2} & 6 & 15 \\ 0 & 15 & 36 \end{bmatrix} = 0 \quad (2.0.2)$$

Expanding equation (2.0.2), we get zero.

Hence given equation represents a pair of straight lines.

The general equation second degree is given by

$$ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0 \quad (2.0.3)$$

Let (α, β) be their point of intersection, then

$$\begin{pmatrix} a & h \\ h & b \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} -d \\ -e \end{pmatrix} \quad (2.0.4)$$

Given equation is

$$-x^2 - xy + 6y^2 + 30y + 36 = 0 \quad (2.0.5)$$

Substituting in (2.0.4)

$$\begin{pmatrix} -1 & -\frac{1}{2} \\ -\frac{1}{2} & 6 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 0 \\ -15 \end{pmatrix} \quad (2.0.6)$$

$$\Rightarrow \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \frac{6}{5} \\ -\frac{12}{5} \end{pmatrix} \quad (2.0.7)$$

From, Spectral decomposition,

$$\mathbf{V} = \mathbf{P}\mathbf{D}\mathbf{P}^T \quad (2.0.8)$$

$$\mathbf{V} = \begin{pmatrix} -1 & -\frac{1}{2} \\ -\frac{1}{2} & 6 \end{pmatrix} \quad (2.0.9)$$

$$\mathbf{P} = \begin{pmatrix} 7 - 5\sqrt{2} & 7 + 5\sqrt{2} \\ 1 & 1 \end{pmatrix} \quad (2.0.10)$$

$$\mathbf{D} = \begin{pmatrix} \frac{5+5\sqrt{2}}{2} & 0 \\ 0 & \frac{5-5\sqrt{2}}{2} \end{pmatrix} \quad (2.0.11)$$

P and D are also verified using python code from

codes/diagonalize1.py

Using, (2.0.7), (2.0.10) and (2.0.11) in,

$$u_1(x - \alpha) + u_2(y - \beta) = \pm \sqrt{-\frac{\lambda_2}{\lambda_1}}(v_1(x - \alpha) + v_2(y - \beta)) \quad (2.0.12)$$

$$\Rightarrow (7 - 5\sqrt{2})\left(x - \frac{30}{23}\right) + \left(y + \frac{60}{23}\right)$$

$$= \pm \sqrt{-\frac{\frac{5-5\sqrt{2}}{2}}{\frac{5+5\sqrt{2}}{2}}} \left((7 - 5\sqrt{2})\left(x - \frac{6}{5}\right) + \left(y + \frac{12}{5}\right) \right) \quad (2.0.13)$$

simplifying 2.0.13, we get:

$$-x + 2y + 6 = 0 \text{ and } x + 3y + 6 = 0 \quad (2.0.14)$$

$$\Rightarrow (-x + 2y + 6)(x + 3y + 6) = 0 \quad (2.0.15)$$

$$\therefore -x + 2y = -6, \quad x + 3y = -6 \quad (2.0.16)$$

The augmented matrix for the set of equations represented in (2.0.16) is

$$\begin{pmatrix} -1 & 2 & -6 \\ 1 & 3 & -6 \end{pmatrix} \quad (2.0.17)$$

Row reducing the matrix

$$\begin{pmatrix} -1 & 2 & -6 \\ 1 & 3 & -6 \end{pmatrix} \xrightarrow{R_2 \leftarrow R_2 + R_1} \begin{pmatrix} -1 & 2 & -6 \\ 0 & 5 & -12 \end{pmatrix} \quad (2.0.18)$$

$$\xrightarrow{R_1 \leftarrow R_1 - \frac{2}{5} \times R_2} \begin{pmatrix} -1 & 0 & -\frac{6}{5} \\ 0 & 5 & -12 \end{pmatrix} \quad (2.0.19)$$

$$\xrightarrow{R_1 \leftarrow -1 \times R_1, R_2 \leftarrow \frac{1}{5} \times R_2} \begin{pmatrix} 1 & 0 & \frac{6}{5} \\ 0 & 1 & -\frac{12}{5} \end{pmatrix} \quad (2.0.20)$$

Hence, the intersection point is $\begin{pmatrix} \frac{6}{5} \\ -\frac{12}{5} \end{pmatrix}$

Also, Verified using python code from

codes/Assignment_5.py

Angle between two lines, θ can be given by

$$\mathbf{n}_1 = \begin{pmatrix} -2 \\ -1 \end{pmatrix} \quad (2.0.21)$$

$$\mathbf{n}_2 = \begin{pmatrix} -3 \\ 1 \end{pmatrix} \quad (2.0.22)$$

$$\cos \theta = \frac{\mathbf{n}_1^T \mathbf{n}_2}{\|\mathbf{n}_1\| \|\mathbf{n}_2\|} \quad (2.0.23)$$

$$\cos \theta = \frac{(-2 \quad -1) \begin{pmatrix} -3 \\ 1 \end{pmatrix}}{\sqrt{(-2)^2 + (-1)^2} \times \sqrt{(-3)^2 + 1}} = \frac{1}{\sqrt{2}} \quad (2.0.24)$$

$$\Rightarrow \theta = 45^\circ \quad (2.0.25)$$

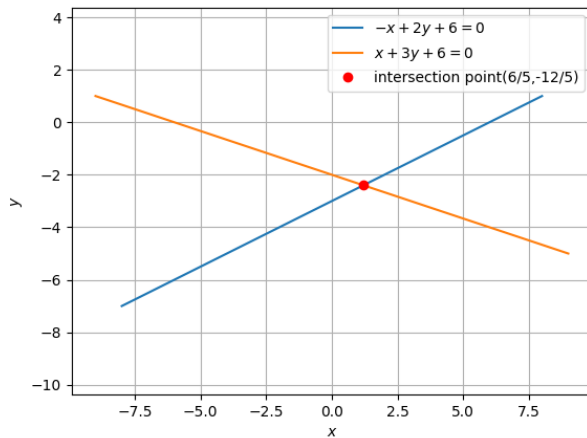


Fig. 0: plot showing intersection of lines.