

EE5609 Assignment 9

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Abstract—This document solves problem based on solution of vector space.

Download all solutions from

https://github.com/abhishekt711/EE5609/tree/master/Assignment_9

1 PROBLEM

let $\alpha = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ and $\beta = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ be vectors in \mathbb{R}^2 such that

$$x_1 y_1 + x_2 y_2 = 0; \quad x_1^2 + x_2^2 = y_1^2 + y_2^2 = 1.$$

Prove that $\beta = \{\alpha, \beta\}$ is a basis of \mathbb{R}^2 . Find the coordinates of the vector (a, b) in the ordered basis $\beta = \{\alpha, \beta\}$. (The conditions on α and β say, geometrically, that α and β are perpendicular and each has length 1).

2 SOLUTION

we need to show that α and β are linearly independent in order to prove that $\beta = \{\alpha, \beta\}$ is a basis of \mathbb{R}^2 .

Given in the question are:

$$\alpha^T \beta = 0 \quad (2.0.1)$$

$$\|\alpha\|^2 = \|\beta\|^2 = 1 \quad (2.0.2)$$

Let,

$$\mathbf{A} = (\alpha \ \beta) = \begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \end{pmatrix} \quad (2.0.3)$$

then,

$$\mathbf{A}^T \mathbf{A} = \begin{pmatrix} \|\alpha\|^2 & \alpha^T \beta \\ \alpha^T \beta & \|\beta\|^2 \end{pmatrix} \quad (2.0.4)$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad (2.0.5)$$

$$\therefore \mathbf{A}^T \mathbf{A} = \mathbf{I} \quad (2.0.6)$$

Inverse of \mathbf{A} exist. \mathbf{A}^T is the inverse of \mathbf{A} . Thus, the columns of \mathbf{A} are linearly independent i.e, α and β are linearly independent.

Hence, $\beta = \{\alpha, \beta\}$ is a basis of \mathbb{R}^2 .

To, find the coordinates of the vector (a, b) in the ordered basis $\beta = \{\alpha, \beta\}$.

$$(c_1 \ c_2)(\alpha \ \beta) = \begin{pmatrix} a \\ b \end{pmatrix} \quad (2.0.7)$$

$$c_1 x_1 + c_2 y_1 = a \quad (2.0.8)$$

$$c_1 x_2 + c_2 y_2 = b \quad (2.0.9)$$

We can row-reduce the augmented matrix to find the constant c_1 and c_2 ,

$$\left(\begin{array}{cc|c} x_1 & y_1 & a \\ x_2 & y_2 & b \end{array} \right) \quad (2.0.10)$$

$$\xleftrightarrow{R_1 = \frac{R_1}{x_1}} \left(\begin{array}{cc|c} 1 & \frac{y_1}{x_1} & \frac{a}{x_1} \\ x_2 & y_2 & b \end{array} \right) \quad (2.0.11)$$

$$\xleftrightarrow{R_2 = R_2 - x_2 R_1} \left(\begin{array}{cc|c} 1 & \frac{y_1}{x_1} & \frac{a}{x_1} \\ 0 & y_2 - \frac{x_2 y_1}{x_1} & b - \frac{x_2 a}{x_1} \end{array} \right) \quad (2.0.12)$$

$$= \left(\begin{array}{cc|c} 1 & \frac{y_1}{x_1} & \frac{a}{x_1} \\ 0 & \frac{x_1 y_2 - x_2 y_1}{x_1} & \frac{x_1 b - x_2 a}{x_1} \end{array} \right) \quad (2.0.13)$$

$$\xleftrightarrow{R_2 = \left(\frac{x_1 y_2 - x_2 y_1}{x_1} \right) R_2} \left(\begin{array}{cc|c} 1 & \frac{y_1}{x_1} & \frac{a}{x_1} \\ 0 & 1 & \frac{x_1 b - x_2 a}{x_1 y_2 - x_2 y_1} \end{array} \right) \quad (2.0.14)$$

$$\xleftrightarrow{R_1 = R_1 - \left(\frac{y_1}{x_1} \right) R_2} \left(\begin{array}{cc|c} 1 & 0 & \frac{a y_2 - b y_1}{x_1 y_2 - x_2 y_1} \\ 0 & 1 & \frac{x_1 b - x_2 a}{x_1 y_2 - x_2 y_1} \end{array} \right) \quad (2.0.15)$$

Using 2.0.1 and 2.0.2 and simplifying in 2.0.15,

$$= \left(\begin{array}{cc|c} 1 & 0 & a x_1 + b x_2 \\ 0 & 1 & a y_1 + b y_2 \end{array} \right) \quad (2.0.16)$$

$$\therefore c_1 = a x_1 + b x_2 \quad (2.0.17)$$

$$\therefore c_2 = a y_1 + b y_2 \quad (2.0.18)$$

Hence,

$$\begin{pmatrix} a \\ b \end{pmatrix} = (a x_1 + b x_2)(\alpha) + (a y_1 + b y_2)(\beta) \quad (2.0.19)$$