EE5609 Assignment 5

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Abstract—**This document contains the solution of geometry through linear algebra.**

Download latex and python codes from

https://github.com/abhishekt711/EE5609/tree/master/Assignment_5

1 Problem

Prove that the following equations represent two straight lines, find also their point of intersection and the angle between them.
$6y^2 - xy - x^2 + 30y + 36 = 0$.

2 Explanation

The given equation can be written as:

$$-x^2 - xy + 6y^2 + 30y + 36 = 0 \tag{2.0.1}$$

$\begin{vmatrix} \mathbf{V} & \mathbf{u} \\ \mathbf{u}^T & f \end{vmatrix}$ of (2.0.1) becomes

$$\begin{vmatrix} -1 & -\frac{1}{2} & 0 \\ \frac{-1}{2} & 6 & 15 \\ 0 & 15 & 36 \end{vmatrix} = 0 \tag{2.0.2}$$

Expanding equation (2.0.2), we get zero.
Hence given equation represents a pair of straight lines.

The general equation second degree is given by

$$ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0 \tag{2.0.3}$$

Let (α, β) be their point of intersection, then

$$\begin{pmatrix} a & h \\ h & b \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} -d \\ -e \end{pmatrix} \tag{2.0.4}$$

Under *Affine transformation*,

$$\mathbf{x} = \mathbf{M}\mathbf{y} + c \tag{2.0.5}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} + \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \tag{2.0.6}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} X + \alpha \\ Y + \beta \end{pmatrix} \tag{2.0.7}$$

(2.0.3) under transformation (2.0.7) will become,

$$aX^2 + 2bXY + cY^2 = 0 \tag{2.0.8}$$

$$\begin{pmatrix} X & Y \end{pmatrix} \begin{pmatrix} a & h \\ h & b \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} = 0 \tag{2.0.9}$$

$$\begin{pmatrix} X & Y \end{pmatrix} \begin{pmatrix} u_1 & v_1 \\ u_2 & v_2 \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} u_1 & u_2 \\ v_1 & v_2 \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} = 0 \tag{2.0.10}$$

$$\begin{pmatrix} X' & Y' \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} X' \\ Y' \end{pmatrix} = 0 \tag{2.0.11}$$

$$\text{where } X' = Xu_1 + Yu_2 \text{ and } Y' = Xv_1 + Yv_2 \tag{2.0.12}$$

$$\implies \lambda_1 (X')^2 + \lambda_2 (Y')^2 = 0 \tag{2.0.13}$$

(2.0.13) is called *Spectral decomposition* of matrix

$$\implies X' = \pm \sqrt{-\frac{\lambda_2}{\lambda_1}} Y' \tag{2.0.14}$$

$$u_1 X + u_2 Y = \pm \sqrt{-\frac{\lambda_2}{\lambda_1}} (v_1 X + v_2 Y) \tag{2.0.15}$$

$$u_1(x - \alpha) + u_2(y - \beta) = \pm \sqrt{-\frac{\lambda_2}{\lambda_1}} (v_1(x - \alpha) + v_2(y - \beta)) \tag{2.0.16}$$

Given equation is

$$-x^2 - xy + 6y^2 + 30y + 36 = 0 \tag{2.0.17}$$

Substituting in (2.0.4)

$$\begin{pmatrix} -1 & \frac{-1}{2} \\ \frac{-1}{2} & 6 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 0 \\ -15 \end{pmatrix} \tag{2.0.18}$$

$$\implies \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \frac{30}{23} \\ \frac{-60}{23} \end{pmatrix} \tag{2.0.19}$$

From, Spectral decomposition,

$$\mathbf{V} = \mathbf{PDP}^T \tag{2.0.20}$$

$$\mathbf{V} = \begin{pmatrix} -1 & \frac{-1}{2} \\ \frac{-1}{2} & 6 \end{pmatrix} \tag{2.0.21}$$

$$\mathbf{P} = \begin{pmatrix} 7-5\sqrt{2} & 7+5\sqrt{2} \\ 1 & 1 \end{pmatrix} \tag{2.0.22}$$

$$\mathbf{D} = \begin{pmatrix} \frac{5+5\sqrt{2}}{2} & 0 \\ 0 & \frac{5-5\sqrt{2}}{2} \end{pmatrix} \tag{2.0.23}$$

P and D are also verified using python code from

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codes/diagonalize1.py
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Substituting (2.0.19), (2.0.22) and (2.0.23) in (2.0.16),

$$\implies \left(7-5\sqrt{2}\right)\left(x-\frac{30}{23}\right)+\left(y+\frac{60}{23}\right) = \pm\sqrt{-\frac{\frac{5-5\sqrt{2}}{2}}{\frac{5+5\sqrt{2}}{2}}}\left(\left(7-5\sqrt{2}\right)\left(x-\frac{30}{23}\right)+\left(y+\frac{60}{23}\right)\right) \tag{2.0.24}$$

simplifying 2.0.24, we get:

$$-x+2y+6=0 \text{ and } x+3y+6=0 \tag{2.0.25}$$

$$\implies (-x+2y+6)(x+3y+6)=0 \tag{2.0.26}$$

Hence, the intersection point is $\begin{pmatrix} \frac{6}{5} \\ -\frac{12}{5} \end{pmatrix}$

Verified using python code from

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codes/Assignment_5.py
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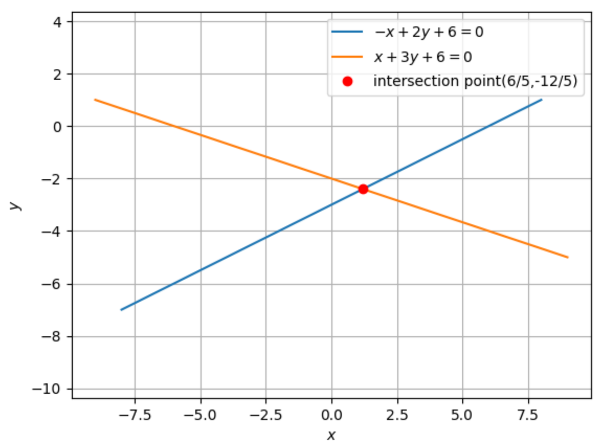

Fig. 0: plot showing intersection of lines.

Angle between two lines, θ can be given by

$$n_1 = \begin{pmatrix} -2 \\ -1 \end{pmatrix} \tag{2.0.27}$$

$$n_2 = \begin{pmatrix} -3 \\ 1 \end{pmatrix} \tag{2.0.28}$$

$$\cos\theta = \frac{\mathbf{n_1}^T\mathbf{n_2}}{\|\mathbf{n_1}\|\,\|\mathbf{n_2}\|} \tag{2.0.29}$$

$$\cos\theta = \frac{\begin{pmatrix} -2 & -1 \end{pmatrix}\begin{pmatrix} -3 \\ 1 \end{pmatrix}}{\sqrt{(-2)^2+(-1)^2}\times\sqrt{+(-3)^2+1}} = \frac{1}{\sqrt{2}} \tag{2.0.30}$$

$$\implies \theta = 45^\circ \tag{2.0.31}$$