

# EE5609 Assignment 9

Abhishek Thakur

**Abstract**—This document solves problem based on solution of vector space.

Download all solutions from

[https://github.com/abhishekt711/EE5609/tree/master/Assignment\\_9](https://github.com/abhishekt711/EE5609/tree/master/Assignment_9)

## 1 PROBLEM

let  $\alpha = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$  and  $\beta = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$  be vectors in  $\mathbb{R}^2$  such that

$$x_1 y_1 + x_2 y_2 = 0; \quad x_1^2 + x_2^2 = y_1^2 + y_2^2 = 1.$$

Proove that  $\beta = \{\alpha, \beta\}$  is a basis of  $\mathbb{R}^2$ . Find the coordinates of the vector  $(a, b)$  in the ordered basis  $\beta = \{\alpha, \beta\}$ . (The conditions on  $\alpha$  and  $\beta$  say, geometrically, that  $\alpha$  and  $\beta$  are perpendicular and each has length 1).

## 2 SOLUTION

we need to show that  $\alpha$  and  $\beta$  are linearly independent in order to prove that  $\beta = \{\alpha, \beta\}$  is a basis of  $\mathbb{R}^2$ .

Given in the question are:

$$\alpha^T \beta = 0 \quad (2.0.1)$$

$$\|\alpha\|^2 = \|\beta\|^2 = 1 \quad (2.0.2)$$

Let,

$$\mathbf{A} = (\alpha \ \beta) = \begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \end{pmatrix} \quad (2.0.3)$$

then,

$$\mathbf{A}^T \mathbf{A} = \begin{pmatrix} \|\alpha\|^2 & \alpha^T \beta \\ \alpha^T \beta & \|\beta\|^2 \end{pmatrix} \quad (2.0.4)$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad (2.0.5)$$

$$\therefore \mathbf{A}^T \mathbf{A} = \mathbf{I} \quad (2.0.6)$$

Inverse of  $\mathbf{A}$  exist.  $\mathbf{A}^T$  is the inverse of  $\mathbf{A}$ . Thus, the columns of  $\mathbf{A}$  are linearly independent i.e,  $\alpha$  and  $\beta$  are linearly independent.

Hence,  $\beta = \{\alpha, \beta\}$  is a basis of  $\mathbb{R}^2$ .

To, find the coordinates of the vector  $(a, b)$  in the ordered basis  $\beta = \{\alpha, \beta\}$ .

$$(\alpha \ \beta) \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} \quad (2.0.7)$$

$$c_1 x_1 + c_2 y_1 = a \quad (2.0.8)$$

$$c_1 x_2 + c_2 y_2 = b \quad (2.0.9)$$

We can row-reduce the augmented matrix to find the constant  $c_1$  and  $c_2$ ,

$$\left( \begin{array}{cc|c} x_1 & y_1 & a \\ x_2 & y_2 & b \end{array} \right) \quad (2.0.10)$$

$$\xleftrightarrow{R_1 = \frac{R_1}{x_1}} \left( \begin{array}{cc|c} 1 & \frac{y_1}{x_1} & \frac{a}{x_1} \\ x_2 & y_2 & b \end{array} \right) \quad (2.0.11)$$

$$\xleftrightarrow{R_2 = R_2 - x_2 R_1} \left( \begin{array}{cc|c} 1 & \frac{y_1}{x_1} & \frac{a}{x_1} \\ 0 & y_2 - \frac{x_2 y_1}{x_1} & b - \frac{x_2 a}{x_1} \end{array} \right) \quad (2.0.12)$$

$$= \left( \begin{array}{cc|c} 1 & \frac{y_1}{x_1} & \frac{a}{x_1} \\ 0 & \frac{x_1 y_2 - x_2 y_1}{x_1} & \frac{x_1 b - x_2 a}{x_1} \end{array} \right) \quad (2.0.13)$$

$$\xleftrightarrow{R_2 = \left( \frac{x_1 y_2 - x_2 y_1}{x_1} \right) R_2} \left( \begin{array}{cc|c} 1 & \frac{y_1}{x_1} & \frac{a}{x_1} \\ 0 & 1 & \frac{x_1 b - x_2 a}{x_1 y_2 - x_2 y_1} \end{array} \right) \quad (2.0.14)$$

$$\xleftrightarrow{R_1 = R_1 - \left( \frac{y_1}{x_1} \right) R_2} \left( \begin{array}{cc|c} 1 & 0 & \frac{a y_2 - b y_1}{x_1 y_2 - x_2 y_1} \\ 0 & 1 & \frac{x_1 b - x_2 a}{x_1 y_2 - x_2 y_1} \end{array} \right) \quad (2.0.15)$$

Using 2.0.1 and 2.0.2 and simplifying in 2.0.15,

$$= \left( \begin{array}{cc|c} 1 & 0 & a x_1 + b x_2 \\ 0 & 1 & a y_1 + b y_2 \end{array} \right) \quad (2.0.16)$$

$$\therefore c_1 = a x_1 + b x_2 \quad (2.0.17)$$

$$\therefore c_2 = a y_1 + b y_2 \quad (2.0.18)$$

Hence,

$$\begin{pmatrix} a \\ b \end{pmatrix} = (a x_1 + b x_2) (\alpha) + (a y_1 + b y_2) (\beta) \quad (2.0.19)$$