EE5609 Assignment 8

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***Abstract*—This document solves problem based on solution of vector space.**

Download all solutions from

https://github.com/abhishekt711/EE5609/tree/master/Assignment_9

1 Problem

let $\alpha = \left(x_1, x_2\right)$ and $\beta = \left(y_1, y_2\right)$ be vectors in $\mathbb{R}^2$ such that
$x_1 y_1 + x_2 y_2 = 0; \quad x_1^2 + x_2^2 = y_1^2 + y_2^2 = 1.$
Proove that ß = $\{\alpha, \beta\}$ is a basis of $\mathbb{R}^2$. Find the coordinates of the vector $\left(a, b\right)$ in the ordered basis ß = $\{\alpha, \beta\}$. (The conditions on α and β say, geometrically, that α and β are perpendicular and each has length 1).

2 Solution

we need to show that α and β are independent in order to proove that ß = $\{\alpha, \beta\}$ is a basis of $\mathbb{R}^2$

$$x_1 y_1 + x_2 y_2 = 0 \tag{2.0.1}$$
$$x_1^2 + x_2^2 = y_1^2 + y_2^2 = 1 \tag{2.0.2}$$

2.0.2 show that α and β are non zero vector.
Let suppose,

$$\beta = c\alpha \tag{2.0.3}$$
$$\implies y_1 = cx_1; \quad y_2 = cx_2 \tag{2.0.4}$$
$$\implies x_1 y_1 + x_2 y_2 = 0 \tag{2.0.5}$$
$$\implies x_1 y_1 + x_2 y_2 = cx_1^2 + cx_2^2 = c(x_1^2 + x_2^2) = c \tag{2.0.6}$$

The given equation 2.0.5 and 2.0.6 contradicts the equation 2.0.3 for any non-zero value of c. Equation 2.0.5 and 2.0.6 satisfy only when $c = 0$. Thus, α and β are independent.
Hence, ß = $\{\alpha, \beta\}$ is a basis of $\mathbb{R}^2$.
To, find the coordinates of the vector $\left(a, b\right)$ in the ordered basis ß = $\{\alpha, \beta\}$. We can row-reduce the augmented matrix

$$\left(\begin{array}{cc|c} x_1 & y_1 & a \\ x_2 & y_2 & b \end{array}\right) \tag{2.0.7}$$
$$\xleftrightarrow{R_1 = \frac{R_1}{x_1}} \left(\begin{array}{cc|c} 1 & \frac{y_1}{x_1} & \frac{a}{x_1} \\ x_2 & y_2 & b \end{array}\right) \tag{2.0.8}$$
$$\xleftrightarrow{R_2 = R_2 - x_2 R_1} \left(\begin{array}{cc|c} 1 & \frac{y_1}{x_1} & \frac{a}{x_1} \\ 0 & y_2 - \frac{x_2 y_1}{x_1} & b - \frac{x_2 a}{x_1} \end{array}\right) \tag{2.0.9}$$
$$= \left(\begin{array}{cc|c} 1 & \frac{y_1}{x_1} & \frac{a}{x_1} \\ 0 & \frac{x_1 y_2 - x_2 y_1}{x_1} & \frac{x_1 b - x_2 a}{x_1} \end{array}\right) \tag{2.0.10}$$
$$\xleftrightarrow{R_2 = \left(\frac{x_1}{x_1 y_2 - x_2 y_1}\right) R_2} \left(\begin{array}{cc|c} 1 & \frac{y_1}{x_1} & \frac{a}{x_1} \\ 0 & 1 & \frac{x_1 b - x_2 a}{x_1 y_2 - x_2 y_1} \end{array}\right) \tag{2.0.11}$$
$$\xleftrightarrow{R_1 = R_1 - \left(\frac{y_1}{x_1}\right) R_2} \left(\begin{array}{cc|c} 1 & 0 & \frac{a y_2 - b y_1}{x_1 y_2 - x_2 y_1} \\ 0 & 1 & \frac{x_1 b - x_2 a}{x_1 y_2 - x_2 y_1} \end{array}\right) \tag{2.0.12}$$

Using 2.0.1 and 2.0.2 in 2.0.12,

$$= \left(\begin{array}{cc|c} 1 & 0 & ax_1 + bx_2 \\ 0 & 1 & ay_1 + by_2 \end{array}\right) \tag{2.0.13}$$

Hence,

$$\implies \left(a, b\right) = (ax_1 + bx_2)\alpha + (ay_1 + by_2)\beta \tag{2.0.14}$$
$$\therefore \left(a, b\right) = (ax_1 + bx_2)\left(x_1, x_2\right) + (ay_1 + by_2)\left(y_1, y_2\right) \tag{2.0.15}$$