

EE5609 Assignment 20

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Abstract—This document solves problem based on Matrix Theory.

Download all solutions from

https://github.com/abhishekt711/EE5609/tree/master/Assignment_20

1 PROBLEM

Consider a Markov Chain with state space $S = \{1, 2, 3\}$ and transition matrix

$$P = \begin{matrix} & \begin{matrix} 1 & 2 & 3 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \end{matrix} & \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix} \end{matrix}$$

Let π be a stationary distribution of the Markov chain and $d(1)$ denote the period of state 1. Which of the following statements are correct?

- a) $d(1) = 1$
- b) $d(1) = 2$
- c) $\pi_1 = \frac{1}{2}$
- d) $\pi_1 = \frac{1}{3}$

2 SOLUTION

a) The period of state 1 i.e, $d(1)$ is given as:

$$d(1) = \text{GCD}\{n : P_{11}^n > 0\} \quad (2.0.1)$$

For $n = 1$,

$$P = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix} \quad (2.0.2)$$

$$\therefore P_{11} = 0 \quad (2.0.3)$$

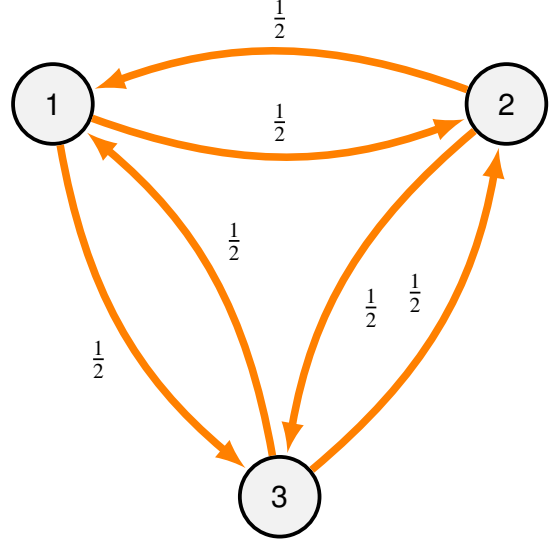


Fig. 1: State transition diagram

For $n = 2$,

$$P^2 = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix} \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix} \quad (2.0.4)$$

$$P^2 = \begin{pmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{pmatrix} \quad (2.0.5)$$

$$\therefore P_{11}^2 > 0 \quad (2.0.6)$$

For $n = 3$,

$$P^3 = P^2 P = \begin{pmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix} \quad (2.0.7)$$

$$P^3 = \begin{pmatrix} \frac{1}{4} & \frac{3}{8} & \frac{3}{8} \\ \frac{3}{8} & \frac{1}{4} & \frac{3}{8} \\ \frac{3}{8} & \frac{3}{8} & \frac{1}{4} \end{pmatrix} \quad (2.0.8)$$

$$\therefore P_{11}^3 > 0 \quad (2.0.9)$$

Similarly, $P_{11}^4, P_{11}^5, \dots \geq 0$

$$d(1) = \text{GCD}\{2, 3, 4, 5, \dots\} \quad (2.0.10)$$

$$\therefore d(1) = 1 \quad (2.0.11)$$

Thus statement a is correct

- b) As calculated above in 2.0.11, $d(1) = 1$
Thus statement b is incorrect.

- c) For stationary distribution,

$$\sum_{i=1}^{i=n} \pi_i = 1 \quad (2.0.12)$$

$$\pi \mathbf{P} = \pi \quad (2.0.13)$$

$$(\pi_1 \quad \pi_2 \quad \pi_3) \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix} = (\pi_1 \quad \pi_2 \quad \pi_3) \quad (2.0.14)$$

$$\Rightarrow -\pi_1 + \frac{\pi_2}{2} + \frac{\pi_3}{2} = 0 \quad (2.0.15)$$

$$\Rightarrow \frac{\pi_1}{2} + -\pi_2 + \frac{\pi_3}{2} = 0 \quad (2.0.16)$$

$$\Rightarrow \frac{\pi_1}{2} + \frac{\pi_2}{2} + -\pi_3 = 0 \quad (2.0.17)$$

Also using 2.0.12,

$$\Rightarrow \pi_1 + \pi_2 + \pi_3 = 1 \quad (2.0.18)$$

The given equation 2.0.15, 2.0.16, 2.0.17, 2.0.18, can be written as:

$$\begin{pmatrix} -1 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} \pi_1 \\ \pi_2 \\ \pi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \quad (2.0.19)$$

We need to solve the augmented matrix to row reduced echelon form to get the solution,

$$\left(\begin{array}{ccc|c} -1 & \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & -1 & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & -1 & 0 \\ 1 & 1 & 1 & 1 \end{array} \right) \xleftrightarrow{R_4=R_4+R_1} \quad (2.0.20)$$

$$\left(\begin{array}{ccc|c} -1 & \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & -1 & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & -1 & 0 \\ 0 & \frac{3}{2} & \frac{3}{2} & 1 \end{array} \right) \xleftrightarrow{R_1=-R_1} \quad (2.0.21)$$

$$\left(\begin{array}{ccc|c} 1 & -\frac{1}{2} & -\frac{1}{2} & 0 \\ \frac{1}{2} & -1 & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & -1 & 0 \\ 0 & \frac{3}{2} & \frac{3}{2} & 1 \end{array} \right) \xleftrightarrow{R_2=R_2-\frac{R_1}{2}, R_3=R_3-\frac{R_1}{2}} \quad (2.0.22)$$

$$\left(\begin{array}{ccc|c} 1 & -\frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & -\frac{3}{4} & \frac{3}{4} & 0 \\ 0 & -\frac{3}{4} & -\frac{3}{4} & 0 \\ 0 & \frac{3}{2} & \frac{3}{2} & 1 \end{array} \right) \xleftrightarrow{R_3=R_3+R_2, R_4=R_4+2R_2} \quad (2.0.23)$$

$$\left(\begin{array}{ccc|c} 1 & -\frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & -\frac{3}{4} & \frac{3}{4} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 1 \end{array} \right) \xleftrightarrow{R_2=-\frac{4}{3}R_2} \quad (2.0.24)$$

$$\left(\begin{array}{ccc|c} 1 & -\frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 1 \end{array} \right) \xleftrightarrow{R_1=R_1+\frac{1}{2}R_2} \quad (2.0.25)$$

$$\left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 1 \end{array} \right) \xleftrightarrow{R_3 \leftrightarrow R_4} \quad (2.0.26)$$

$$\left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{array} \right) \xleftrightarrow{R_3=\frac{R_3}{3}} \quad (2.0.27)$$

$$\left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & \frac{1}{3} \\ 0 & 0 & 0 & 0 \end{array} \right) \xleftrightarrow{R_1=R_1+R_3, R_2=R_2+R_3} \quad (2.0.28)$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & \frac{1}{3} \\ 0 & 1 & 0 & \frac{1}{3} \\ 0 & 0 & 1 & \frac{1}{3} \\ 0 & 0 & 0 & 0 \end{array} \right) \quad (2.0.29)$$

Hence,

$$\pi_1 = \pi_2 = \pi_3 = \frac{1}{3} \quad (2.0.30)$$

Thus statement c is incorrect

d) As, calculated in 2.0.30, $\pi_1 = \frac{1}{3}$

Thus statement d is correct

Hence, statements a and d are correct.