EE5609 Assignment 5

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Abstract—**This document contains the solution of geometry through linear algebra.**

Download latex and python codes from

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https://github.com/abhishekt711/EE5609/tree/
    master/Assignment_5
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1 Problem

Prove that the following equations represent two straight lines, find also their point of intersection and the angle between them.
$6y^2 - xy - x^2 + 30y + 36 = 0$.

2 Explanation

The given equation can be written as:

$$-x^2 - xy + 6y^2 + 30y + 36 = 0 \tag{2.0.1}$$

$\begin{vmatrix} \mathbf{V} & \mathbf{u} \\ \mathbf{u}^T & f \end{vmatrix}$ of (2.0.1) becomes

$$\begin{vmatrix} -1 & -\frac{1}{2} & 0 \\ \frac{-1}{2} & 6 & 15 \\ 0 & 15 & 36 \end{vmatrix} = 0 \tag{2.0.2}$$

Expanding equation (2.0.2), we get zero.
Hence given equation represents a pair of straight lines.

The general equation second degree is given by

$$ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0 \tag{2.0.3}$$

Let (α, β) be their point of intersection, then

$$\begin{pmatrix} a & h \\ h & b \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} -d \\ -e \end{pmatrix} \tag{2.0.4}$$

Under *Affine transformation*,

$$\mathbf{x} = \mathbf{M}\mathbf{y} + c \tag{2.0.5}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} + \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \tag{2.0.6}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} X + \alpha \\ Y + \beta \end{pmatrix} \tag{2.0.7}$$

(2.0.40) under transformation (2.0.7) will become,

$$aX^2 + 2bXY + cY^2 = 0 \tag{2.0.8}$$

$$\begin{pmatrix} X & Y \end{pmatrix} \begin{pmatrix} a & h \\ h & b \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} = 0 \tag{2.0.9}$$

$$\begin{pmatrix} X & Y \end{pmatrix} \begin{pmatrix} u_1 & v_1 \\ u_2 & v_2 \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} u_1 & u_2 \\ v_1 & v_2 \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} = 0 \tag{2.0.10}$$

$$\begin{pmatrix} X' & Y' \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} X' \\ Y' \end{pmatrix} = 0 \tag{2.0.11}$$

$$\text{where } X' = Xu_1 + Yu_2 \text{ and } Y' = Xv_1 + Yv_2 \tag{2.0.12}$$

$$\implies \lambda_1 (X')^2 + \lambda_2 (Y')^2 = 0 \tag{2.0.13}$$

(2.0.13) is called *Spectral decomposition* of matrix

$$\implies X' = \pm \sqrt{-\frac{\lambda_2}{\lambda_1}} Y' \tag{2.0.14}$$

$$u_1 X + u_2 Y = \pm \sqrt{-\frac{\lambda_2}{\lambda_1}} (v_1 X + v_2 Y) \tag{2.0.15}$$

$$u_1(x - \alpha) + u_2(y - \beta) = \pm \sqrt{-\frac{\lambda_2}{\lambda_1}} (v_1(x - \alpha) + v_2(y - \beta)) \tag{2.0.16}$$

Given equation is

$$-x^2 - xy + 6y^2 + 30y + 36 = 0 \tag{2.0.17}$$

Substituting in (2.0.4)

$$\begin{pmatrix} -1 & \frac{-1}{2} \\ \frac{-1}{2} & 6 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 0 \\ -15 \end{pmatrix} \tag{2.0.18}$$

$$\implies \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \frac{30}{23} \\ \frac{-60}{23} \end{pmatrix} \tag{2.0.19}$$

$$\text{From } \textit{Spectral theorem, } \mathbf{V} = \mathbf{P}\mathbf{D}\mathbf{P}^T \tag{2.0.20}$$

$$\mathbf{V} = \begin{pmatrix} -1 & \frac{-1}{2} \\ \frac{-1}{2} & 6 \end{pmatrix} \tag{2.0.21}$$

P and D are found using python code from

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codes/diagonalize1.py
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Substituting and simplifying in 2.0.16.

$$-x + 2y + 6 = 0 \text{ and } x + 3y + 6 = 0 \tag{2.0.22}$$
$$\implies (-x + 2y + 6)(x + 3y + 6) = 0 \tag{2.0.23}$$

Hence, the intersection point is $\begin{pmatrix} \frac{6}{5} \\ -\frac{12}{5} \end{pmatrix}$
Verified using python code from

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codes/Assignment_5.py
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Also, from the given equation:

$$-x^2 - xy + 6y^2 + 30y + 36 = 0 \tag{2.0.24}$$

Slopes of the individual lines are roots of equation

$$cm^2 + 2bm + a = 0 \tag{2.0.25}$$
$$\implies 6m^2 - m - 1 = 0 \tag{2.0.26}$$
$$\text{Solving, } m = \frac{1}{2}, -\frac{1}{3} \tag{2.0.27}$$

The normal vectors of the lines then become

$$\mathbf{n_1} = \begin{pmatrix} -1 \\ 2 \end{pmatrix} \tag{2.0.28}$$
$$\mathbf{n_2} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \tag{2.0.29}$$

Equations of the lines can therefore be written as

$$\begin{pmatrix} -1 & 2 \end{pmatrix} \mathbf{x} = c_1 \tag{2.0.30}$$
$$\begin{pmatrix} 1 & 3 \end{pmatrix} \mathbf{x} = c_2 \tag{2.0.31}$$
$$\implies \left[\begin{pmatrix} -1 & 2 \end{pmatrix} \mathbf{x} - c_1\right]\left[\begin{pmatrix} 1 & 3 \end{pmatrix} \mathbf{x} - c_2\right] \tag{2.0.32}$$

represents the equation specified in (2.0.1)
Comparing the equations, we have

$$\begin{pmatrix} -1 & 1 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} c_2 \\ c_1 \end{pmatrix} = \begin{pmatrix} 0 \\ -30 \end{pmatrix} \tag{2.0.33}$$
$$\tag{2.0.34}$$

Row reducing the augmented matrix

$$\begin{pmatrix} -1 & 1 & 0 \\ 2 & 3 & -30 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow R_2 + 2 \times R_1} \begin{pmatrix} -1 & 1 & 0 \\ 0 & 5 & -30 \end{pmatrix} \tag{2.0.35}$$
$$\xleftrightarrow{R_2 \leftarrow R_1 - R_2 \times \frac{1}{5}} \begin{pmatrix} -1 & 0 & 6 \\ 0 & 5 & -30 \end{pmatrix} \tag{2.0.36}$$
$$\xleftrightarrow{R_1 \leftarrow -1 \times R_1, R_2 \leftarrow \frac{1}{5} \times R_2} \begin{pmatrix} 1 & 0 & -6 \\ 0 & 1 & -6 \end{pmatrix} \tag{2.0.37}$$
$$\implies c_1 = -6 \text{ and } c_2 = -6 \tag{2.0.38}$$

The individual line equations therefore become

$$\begin{pmatrix} -1 & 2 \end{pmatrix} \mathbf{x} = -6, \tag{2.0.39}$$
$$\begin{pmatrix} 1 & 3 \end{pmatrix} \mathbf{x} = -6 \tag{2.0.40}$$

Note that the convolution of the normal vectors, should satisfy the below condition

$$\begin{pmatrix} -1 \\ 2 \end{pmatrix} * \begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} a \\ 2b \\ c \end{pmatrix} \tag{2.0.41}$$

The LHS part of (2.0.40) can be rewritten using toeplitz matrix as

$$\begin{pmatrix} -1 & 0 \\ 2 & -1 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ 6 \end{pmatrix} = \begin{pmatrix} a \\ 2b \\ c \end{pmatrix} \tag{2.0.42}$$

The augmented matrix for the set of equations represented in (2.0.38), (2.0.39) is

$$\begin{pmatrix} -1 & 2 & -6 \\ 1 & 3 & -6 \end{pmatrix} \tag{2.0.43}$$

Row reducing the matrix

$$\begin{pmatrix} -1 & 2 & -6 \\ 1 & 3 & -6 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow R_2 + R_1} \begin{pmatrix} -1 & 2 & -6 \\ 0 & 5 & -12 \end{pmatrix} \tag{2.0.44}$$
$$\xleftrightarrow{R_1 \leftarrow R_1 - \frac{2}{5} \times R_2} \begin{pmatrix} -1 & 0 & -\frac{6}{5} \\ 0 & 5 & -12 \end{pmatrix} \tag{2.0.45}$$
$$\xleftrightarrow{R_1 \leftarrow -1 \times R_1, R_2 \leftarrow \frac{1}{5} \times R_2} \begin{pmatrix} 1 & 0 & \frac{6}{5} \\ 0 & 1 & -\frac{12}{5} \end{pmatrix} \tag{2.0.46}$$
$$\text{Hence, the intersection point is} \begin{pmatrix} \frac{6}{5} \\ -\frac{12}{5} \end{pmatrix} \tag{2.0.47}$$

Angle between two lines θ can be given by

$$\cos\theta = \frac{\mathbf{n_1}^T \mathbf{n_2}}{\|\mathbf{n_1}\| \, \|\mathbf{n_2}\|} \tag{2.0.48}$$

$$\cos\theta = \frac{\begin{pmatrix} -1 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \end{pmatrix}}{\sqrt{(-1)^2 + (2)^2} \times \sqrt{1 + (3)^2}} = \frac{1}{\sqrt{2}} \tag{2.0.49}$$
$$\implies \theta = 45^\circ \tag{2.0.50}$$

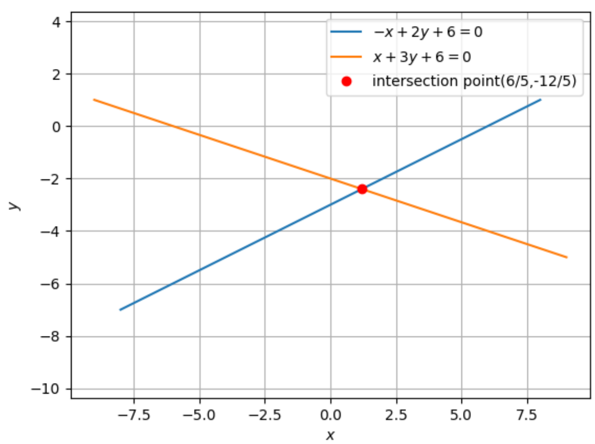

Fig. 0: plot showing intersection of lines.