EE5609 Assignment 7

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***Abstract*—This document solves problem based on Singular Value Decomposition.**

Download all solutions from

https://github.com/abhishekt711/EE5609/tree/master/Assignment_7

1 Problem

Find the foot of the perpendicular from

$$\mathbf{c} = \begin{pmatrix} 3 \\ 3 \\ 5 \end{pmatrix} \tag{1.0.1}$$

to the given plane

$$2x + 3y - 4z + 5 = 0 \tag{1.0.2}$$

2 Solution

The given equation of plane can be represented as

$$\begin{pmatrix} 2 & 3 & -4 \end{pmatrix} \mathbf{x} = -5 \tag{2.0.1}$$

$$\mathbf{n} = \begin{pmatrix} 2 \\ 3 \\ -4 \end{pmatrix} \tag{2.0.2}$$

We need to find two vectors $\mathbf{m_1}$ and $\mathbf{m_2}$ that are $\perp$ to $\mathbf{n}$

$$\implies \begin{pmatrix} 2 & 3 & -4 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0 \tag{2.0.3}$$

Put $a = 1$ and $b = 0$ in (2.0.3),we get,

$$\mathbf{m_1} = \begin{pmatrix} 1 \\ 0 \\ \frac{1}{2} \end{pmatrix} \tag{2.0.4}$$

Put $a = 0$ and $b = 1$ in (2.0.3),we get,

$$\mathbf{m_2} = \begin{pmatrix} 0 \\ 1 \\ \frac{3}{4} \end{pmatrix} \tag{2.0.5}$$

Now, solving the equation

$$\mathbf{M}\mathbf{x} = \mathbf{c} \tag{2.0.6}$$

where,

$$\mathbf{M} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \frac{1}{2} & \frac{3}{4} \end{pmatrix} \tag{2.0.7}$$

$$\mathbf{c} = \begin{pmatrix} 3 \\ 3 \\ 5 \end{pmatrix} \tag{2.0.8}$$

Now, to solve equation (2.0.6), we perform Singular Value Decomposition on $\mathbf{M}$ as follows,

$$\mathbf{M} = \mathbf{U}\mathbf{S}\mathbf{V}^T \tag{2.0.9}$$

Substituting the value of $\mathbf{M}$ from equation (2.0.9) to(2.0.6),

$$\mathbf{U}\mathbf{S}\mathbf{V}^T\mathbf{x} = \mathbf{c} \tag{2.0.10}$$

$$\implies \mathbf{x} = \mathbf{V}\mathbf{S}_+\mathbf{U}^T\mathbf{c} \tag{2.0.11}$$

Where, $\mathbf{S}_+$ is the Moore-Pen-rose Pseudo-Inverse of $\mathbf{S}$. Columns of $\mathbf{V}$ are the eigen vectors of $\mathbf{M}^T\mathbf{M}$, columns of $\mathbf{U}$ are the eigen vectors of $\mathbf{M}\mathbf{M}^T$ and $\mathbf{S}$ is diagonal matrix of singular value of eigenvalues of $\mathbf{M}^T\mathbf{M}$.

$$\mathbf{M}^T\mathbf{M} = \begin{pmatrix} \frac{5}{4} & \frac{3}{8} \\ \frac{3}{8} & \frac{25}{16} \end{pmatrix} \tag{2.0.12}$$

Eigen values corresponding to $\mathbf{M}^T\mathbf{M}$ are given by,

$$\left|\mathbf{M}^T\mathbf{M} - \lambda\mathbf{I}\right| = 0 \tag{2.0.13}$$

$$\implies \left|\begin{pmatrix} \frac{5}{4} - \lambda & \frac{3}{8} \\ \frac{3}{8} & \frac{25}{16} - \lambda \end{pmatrix}\right| = 0 \tag{2.0.14}$$

$$\implies \lambda^2 - \frac{45}{16}\lambda + \frac{29}{16} = 0 \tag{2.0.15}$$

Hence eigen values of $\mathbf{M}^T\mathbf{M}$ are,

$$\lambda_1 = \frac{29}{16} \tag{2.0.16}$$

$$\lambda_2 = 1 \tag{2.0.17}$$

Hence the eigen vectors of $\mathbf{M}^T\mathbf{M}$ are,

$$\mathbf{v}_1 = \begin{pmatrix} \frac{2}{3} \\ 1 \end{pmatrix} \tag{2.0.18}$$

$$\mathbf{v}_2 = \begin{pmatrix} -\frac{3}{2} \\ 1 \end{pmatrix} \tag{2.0.19}$$

Normalizing the eigen vectors, we obtain $\mathbf{V}$ of (2.0.9) as follows,

$$\mathbf{V} = \begin{pmatrix} \frac{2}{\sqrt{13}} & -\frac{3}{\sqrt{13}} \\ \frac{3}{\sqrt{13}} & \frac{2}{\sqrt{13}} \end{pmatrix} \tag{2.0.20}$$

$\mathbf{S}$ of the diagonal matrix of (2.0.9) is:

$$\mathbf{S} = \begin{pmatrix} \frac{\sqrt{29}}{4} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \tag{2.0.21}$$

Now, calculating eigen value of $\mathbf{M}\mathbf{M}^T$,

$$\mathbf{M}\mathbf{M}^T = \begin{pmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & \frac{3}{4} \\ \frac{1}{2} & \frac{3}{4} & \frac{13}{16} \end{pmatrix} \tag{2.0.22}$$

Eigen values corresponding to $\mathbf{M}\mathbf{M}^T$ are given by

$$\left|\mathbf{M}\mathbf{M}^T - \lambda\mathbf{I}\right| = 0 \tag{2.0.23}$$

$$\implies \left|\begin{pmatrix} 1-\lambda & 0 & \frac{1}{2} \\ 0 & 1-\lambda & \frac{3}{4} \\ \frac{1}{2} & \frac{3}{4} & \frac{13}{16}-\lambda \end{pmatrix}\right| = 0 \tag{2.0.24}$$

$$\implies \lambda^3 - \frac{45}{16}\lambda^2 + \frac{29}{16}\lambda = 0 \tag{2.0.25}$$

Hence eigen values of $\mathbf{M}^T\mathbf{M}$ are,

$$\lambda_3 = \frac{29}{16} \tag{2.0.26}$$

$$\lambda_4 = 1 \tag{2.0.27}$$

$$\lambda_5 = 0 \tag{2.0.28}$$

Hence we obtain $\mathbf{U}$ of (2.0.9) as follows,

$$\mathbf{U} = \begin{pmatrix} \frac{8}{\sqrt{377}} & -\frac{3}{\sqrt{13}} & -\frac{2}{\sqrt{29}} \\ \frac{12}{\sqrt{377}} & \frac{2}{\sqrt{13}} & -\frac{3}{\sqrt{29}} \\ \frac{13}{\sqrt{377}} & 0 & \frac{4}{\sqrt{29}} \end{pmatrix} \tag{2.0.29}$$

Finally from (2.0.9) we get the Singular Value Decomposition of $\mathbf{M}$ as follows,

$$\mathbf{M} = \begin{pmatrix} \frac{8}{\sqrt{377}} & -\frac{3}{\sqrt{13}} & -\frac{2}{\sqrt{29}} \\ \frac{12}{\sqrt{377}} & \frac{2}{\sqrt{13}} & -\frac{3}{\sqrt{29}} \\ \frac{13}{\sqrt{377}} & 0 & \frac{4}{\sqrt{29}} \end{pmatrix} \begin{pmatrix} \frac{\sqrt{29}}{4} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{2}{\sqrt{13}} & -\frac{3}{\sqrt{13}} \\ \frac{3}{\sqrt{13}} & \frac{2}{\sqrt{13}} \end{pmatrix}^T \tag{2.0.30}$$

Now, Moore-Penrose Pseudo inverse of $\mathbf{S}$ is given by,

$$\mathbf{S}_+ = \begin{pmatrix} \frac{4}{\sqrt{29}} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \tag{2.0.31}$$

Substituting the values of (2.0.29),(2.0.20),(2.0.31) in (2.0.11) we get,

$$\mathbf{U}^T\mathbf{c} = \begin{pmatrix} \frac{125}{\sqrt{377}} \\ \frac{-3}{\sqrt{13}} \\ \frac{5}{\sqrt{29}} \end{pmatrix} \tag{2.0.32}$$

$$\mathbf{S}_+\mathbf{U}^T\mathbf{c} = \begin{pmatrix} \frac{500}{29\sqrt{13}} \\ -\frac{3}{\sqrt{13}} \end{pmatrix} \tag{2.0.33}$$

$$\mathbf{x} = \mathbf{V}\mathbf{S}_+\mathbf{U}^T\mathbf{c} = \begin{pmatrix} \frac{97}{29} \\ \frac{102}{29} \end{pmatrix} \tag{2.0.34}$$

Verifying the solution of (2.0.34) using,

$$\implies \mathbf{M}^T\mathbf{M}\mathbf{x} = \mathbf{M}^T\mathbf{c} \tag{2.0.35}$$

Evaluating the R.H.S in (2.0.35) we get,

$$\mathbf{M}^T\mathbf{c} = \begin{pmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & \frac{3}{4} \end{pmatrix} \begin{pmatrix} 3 \\ 3 \\ 5 \end{pmatrix} = \begin{pmatrix} \frac{11}{2} \\ \frac{27}{4} \end{pmatrix} \tag{2.0.36}$$

$$\implies \begin{pmatrix} \frac{5}{4} & \frac{3}{8} \\ \frac{3}{8} & \frac{25}{16} \end{pmatrix} \mathbf{x} = \begin{pmatrix} \frac{11}{2} \\ \frac{27}{4} \end{pmatrix} \tag{2.0.37}$$

The augmented matrix of (2.0.37) is,

$$\begin{pmatrix} \frac{5}{4} & \frac{3}{8} & \frac{11}{2} \\ \frac{3}{8} & \frac{25}{16} & \frac{27}{4} \end{pmatrix} \tag{2.0.38}$$

Solving the augmented matrix into Row reduced echelon form of (2.0.38) we get,

$$\begin{pmatrix} \frac{5}{4} & \frac{3}{8} & \frac{11}{2} \\ \frac{3}{8} & \frac{25}{16} & \frac{27}{4} \end{pmatrix} \xleftrightarrow{R_1 \leftarrow \frac{4}{5}R_1} \begin{pmatrix} 1 & \frac{3}{10} & \frac{22}{5} \\ \frac{3}{8} & \frac{25}{16} & \frac{27}{4} \end{pmatrix} \tag{2.0.39}$$

$$\xleftrightarrow{R_2 \leftarrow R_2 - \frac{3}{8}R_1} \begin{pmatrix} 1 & \frac{3}{10} & \frac{22}{5} \\ 0 & \frac{29}{20} & \frac{51}{10} \end{pmatrix} \tag{2.0.40}$$

$$\xleftrightarrow{R_2 \leftarrow \frac{20}{29}R_2} \begin{pmatrix} 1 & \frac{3}{10} & \frac{22}{5} \\ 0 & 1 & \frac{102}{29} \end{pmatrix} \tag{2.0.41}$$

$$\xleftrightarrow{R_1 \leftarrow R_1 - \frac{3}{10}R_2} \begin{pmatrix} 1 & 0 & \frac{97}{29} \\ 0 & 1 & \frac{102}{29} \end{pmatrix} \tag{2.0.42}$$

Therefore,

$$\mathbf{x} = \begin{pmatrix} \frac{97}{29} \\ \frac{102}{29} \end{pmatrix} \tag{2.0.43}$$

Comparing results of $\mathbf{x}$ from (2.0.34) and (2.0.43), Hence, the solution is verified.