Challenging Question

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***Abstract*—This a simple document that explains how to compute rank of a linear transformation wrt ordered basis.**

Download all latex-tikz codes from

https://github.com/abhishekt711/EE5609/blob/master/Challenging_Ques

1 Problem

Let $\mathbb{C}$ be the complex vector space of 2×2 matrices with complex entries. Let

$$\mathbf{B} = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ -4 & 4 \end{pmatrix} \tag{1.0.1}$$

$\mathbf{P}$ is the matrix of linear transformation $\mathbf{T}$. What is the relationship between $\mathbf{B}$ and $\mathbf{P}$, what is the relationship between their ranks?

2 Solution

An ordered basis for $\mathbb{C}^{2\times 2}$ is given by

$$\mathbf{A_{11}} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad \mathbf{A_{12}} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \tag{2.0.1}$$

$$\mathbf{A_{21}} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \qquad \mathbf{A_{22}} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{2.0.2}$$

Now, we compute

$$\mathbf{T}(\mathbf{A_{11}}) = \mathbf{BA_{11}} \tag{2.0.3}$$

$$\mathbf{T}(\mathbf{A_{11}}) = B_{11}\mathbf{A_{11}} + B_{21}\mathbf{A_{21}} \tag{2.0.4}$$

$$= \begin{pmatrix} 1 & -1 \\ -4 & 4 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{2.0.5}$$

$$= \begin{pmatrix} 1 & 0 \\ -4 & 0 \end{pmatrix} \tag{2.0.6}$$

$$\mathbf{T}(\mathbf{A_{12}}) = \mathbf{BA_{12}} \tag{2.0.7}$$

$$\mathbf{T}(\mathbf{A_{12}}) = B_{11}\mathbf{A_{12}} + B_{21}\mathbf{A_{22}} \tag{2.0.8}$$

$$= \begin{pmatrix} 1 & -1 \\ -4 & 4 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \tag{2.0.9}$$

$$= \begin{pmatrix} 0 & 1 \\ 0 & -4 \end{pmatrix} \tag{2.0.10}$$

$$\mathbf{T}(\mathbf{A_{21}}) = \mathbf{BA_{21}} \tag{2.0.11}$$

$$\mathbf{T}(\mathbf{A_{21}}) = B_{12}\mathbf{A_{11}} + B_{22}\mathbf{A_{21}} \tag{2.0.12}$$

$$= \begin{pmatrix} 1 & -1 \\ -4 & 4 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \tag{2.0.13}$$

$$= \begin{pmatrix} -1 & 0 \\ 4 & 0 \end{pmatrix} \tag{2.0.14}$$

$$\mathbf{T}(\mathbf{A_{22}}) = \mathbf{BA_{22}} \tag{2.0.15}$$

$$\mathbf{T}(\mathbf{A_{22}}) = B_{12}\mathbf{A_{12}} + B_{22}\mathbf{A_{22}} \tag{2.0.16}$$

$$= \begin{pmatrix} 1 & -1 \\ -4 & 4 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{2.0.17}$$

$$= \begin{pmatrix} 0 & -1 \\ 0 & 4 \end{pmatrix} \tag{2.0.18}$$

Now, by (2.0.4), (2.0.8), (2.0.12) and (2.0.16) we write matrix of the linear transformation as follows

$$\mathbf{P} = \begin{pmatrix} B_{11} & 0 & B_{12} & 0 \\ 0 & B_{11} & 0 & B_{12} \\ B_{21} & 0 & B_{22} & 0 \\ 0 & B_{21} & 0 & B_{22} \end{pmatrix} \tag{2.0.19}$$

$$= \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -4 & 0 & 4 & 0 \\ 0 & -4 & 0 & 4 \end{pmatrix} \tag{2.0.20}$$

$$Trace(P) = 2(B_{11} + B_{22}) = 2Trace(B) \tag{2.0.21}$$

We know the property of idempotent matrix that, If $\mathbf{M}$ is an idempotent matrix then,

$$Rank(M) = Trace(M) \tag{2.0.22}$$

Here,

$$\mathbf{B}^2 = \begin{pmatrix} 1 & -1 \\ -4 & 4 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ -4 & 4 \end{pmatrix} = \begin{pmatrix} 5 & -5 \\ -20 & 20 \end{pmatrix} = 5\mathbf{B} \tag{2.0.23}$$

$$\mathbf{P}^2 = 5\mathbf{P} \tag{2.0.24}$$

Using 2.0.22,

$$Rank(P) = \frac{1}{5}Trace(P) \quad (2.0.25)$$

$$Rank(B) = \frac{1}{5}Trace(B) \quad (2.0.26)$$

Using 2.0.21

$$Rank(P) = 2Rank(B) \quad (2.0.27)$$

$$\mathbf{B} = \begin{pmatrix} 1 & -1 \\ -4 & 4 \end{pmatrix} \xleftrightarrow{R_2 = R_2 + 4R_1} \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \quad (2.0.28)$$

$$Rank(B) = 1 \quad (2.0.29)$$

Using 2.0.27

$$Rank(P) = 2 \quad (2.0.30)$$