EE5609 Assignment 20

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***Abstract*—This document solves problem based on Matrix Theory.**

Download all solutions from

https://github.com/abhishekt711/EE5609/tree/master/Assignment_20

1 Problem

Consider a Markov Chain with state space $S = \{1, 2, 3\}$ and transition matrix

$$P = \begin{array}{c c} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix} \end{array}$$

Let π be a stationary distribution of the Markov chain and $d(1)$ denote the period of state 1. Which of the following statements are correct?

a) $d(1) = 1$
b) $d(1) = 2$
c) $\pi_1 = \frac{1}{2}$
d) $\pi_1 = \frac{1}{3}$

2 solution

a) The period of state 1 i.e, $d(1)$ is given as:

$$d(1) = GCD\{n : P_{11}^n > 0\} \tag{2.0.1}$$

For $n = 1$,

$$\mathbf{P} = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix} \tag{2.0.2}$$

(2.0.3)

For $n = 2$,

$$\mathbf{P}^2 = \begin{pmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{pmatrix} \tag{2.0.4}$$

(2.0.5)

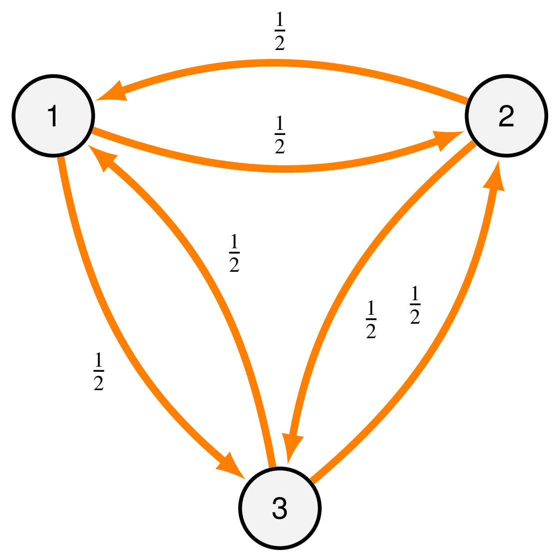

Fig. 1: State transition diagram

For $n = 3$,

$$\mathbf{P}^3 = \begin{pmatrix} \frac{1}{4} & \frac{3}{8} & \frac{3}{8} \\ \frac{3}{8} & \frac{1}{4} & \frac{3}{8} \\ \frac{3}{8} & \frac{3}{8} & \frac{1}{4} \end{pmatrix} \tag{2.0.6}$$

(2.0.7)

For $n = 4$,

$$\mathbf{P}^4 = \begin{pmatrix} \frac{3}{8} & \frac{5}{16} & \frac{5}{16} \\ \frac{5}{16} & \frac{3}{8} & \frac{5}{16} \\ \frac{5}{16} & \frac{5}{16} & \frac{3}{8} \end{pmatrix} \tag{2.0.8}$$

Thus P_{11}^n follows the sequence, that is defined as:

$$P_{11}^n = \begin{cases} 0, & \text{if } n = 1 \\ \frac{1}{2}, & \text{if } n = 2 \\ \frac{1}{2}(P_{11}^{n-1} + P_{11}^{n-2}), & \text{if } n > 2 \end{cases} \tag{2.0.9}$$

Since, for $n > 1$, P_{11}^n are positive

$$d(1) = GCD\{2, 3, 4, 5 \cdots\} \tag{2.0.10}$$

$$\therefore d(1) = 1 \tag{2.0.11}$$

Thus statement a is correct

b) As calucalted above in 2.0.12, $d(1) = 1$
Thus statement b is incorrect.

c) For stationary distribution,

$$\sum_{i=1}^{i=n} \pi_i = 1 \tag{2.0.12}$$

$$\implies \begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} \pi_1 \\ \pi_2 \\ \pi_3 \end{pmatrix} = 1 \tag{2.0.13}$$

Also for a stationary distribution,

$$\pi \mathbf{P} = \pi \tag{2.0.14}$$

$$(\pi \mathbf{P})^T = \pi^T \tag{2.0.15}$$

$$\mathbf{P}^T \pi^T = \pi^T \tag{2.0.16}$$

$$\begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix} \begin{pmatrix} \pi_1 \\ \pi_2 \\ \pi_3 \end{pmatrix} = \begin{pmatrix} \pi_1 \\ \pi_2 \\ \pi_3 \end{pmatrix} \tag{2.0.17}$$

The given equation 2.0.14, 2.0.18 can be written as:

$$\begin{pmatrix} -1 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} \pi_1 \\ \pi_2 \\ \pi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \tag{2.0.18}$$

We need to solve the augmented matrix to row reduced echelon form to get the solution,

$$\left(\begin{array}{ccc|c} -1 & \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & -1 & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & -1 & 0 \\ 1 & 1 & 1 & 1 \end{array}\right) \xleftrightarrow{R_4 = R_4 + R_1} \tag{2.0.19}$$

$$\left(\begin{array}{ccc|c} -1 & \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & -1 & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & -1 & 0 \\ 0 & \frac{3}{2} & \frac{3}{2} & 1 \end{array}\right) \xleftrightarrow{R_1 = -R_1} \tag{2.0.20}$$

$$\left(\begin{array}{ccc|c} 1 & -\frac{1}{2} & -\frac{1}{2} & 0 \\ \frac{1}{2} & -1 & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & -1 & 0 \\ 0 & \frac{3}{2} & \frac{3}{2} & 1 \end{array}\right) \xleftrightarrow{R_2 = R_2 - \frac{R_1}{2}, R_3 = R_3 - \frac{R_1}{2}} \tag{2.0.21}$$

$$\left(\begin{array}{ccc|c} 1 & -\frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & -\frac{3}{4} & \frac{3}{4} & 0 \\ 0 & \frac{3}{4} & -\frac{3}{4} & 0 \\ 0 & \frac{3}{2} & \frac{3}{2} & 1 \end{array}\right) \xleftrightarrow{R_3 = R_3 + R_2, R_4 = R_4 + 2R_2} \tag{2.0.22}$$

$$\left(\begin{array}{ccc|c} 1 & -\frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & -\frac{3}{4} & \frac{3}{4} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 1 \end{array}\right) \xleftrightarrow{R_2 = -\frac{4}{3} R_2} \tag{2.0.23}$$

$$\left(\begin{array}{ccc|c} 1 & -\frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 1 \end{array}\right) \xleftrightarrow{R_1 = R_1 + \frac{1}{2} R_2} \tag{2.0.24}$$

$$\left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 1 \end{array}\right) \xleftrightarrow{R_3 \leftrightarrow R_4} \tag{2.0.25}$$

$$\left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right) \xleftrightarrow{R_3 = \frac{R_3}{3}} \tag{2.0.26}$$

$$\left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & \frac{1}{3} \\ 0 & 0 & 0 & 0 \end{array}\right) \xleftrightarrow{R_1 = R_1 + R_3, R_2 = R_2 + R_3} \tag{2.0.27}$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & \frac{1}{3} \\ 0 & 1 & 0 & \frac{1}{3} \\ 0 & 0 & 1 & \frac{1}{3} \\ 0 & 0 & 0 & 0 \end{array}\right) \tag{2.0.28}$$

Hence,

$$\pi_1 = \pi_2 = \pi_3 = \frac{1}{3} \tag{2.0.29}$$

Thus statement c is incorrect

d) As, calculated in 2.0.30, $\pi_1 = \frac{1}{3}$
Thus statement d is correct

Hence, statements a and d are correct.