EE5609 Assignment 9

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Abstract—**This document solves problem based on solution of vector space.**

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1 PROBLEM

let $\alpha = (x_1, x_2)$ and $\beta = (y_1, y_2)$ be vectors in $\mathbb{R}^2$ such that
$x_1 y_1 + x_2 y_2 = 0; \quad x_1^2 + x_2^2 = y_1^2 + y_2^2 = 1.$
Proove that ß $= \{\alpha, \beta\}$ is a basis of $\mathbb{R}^2$. Find the coordinates of the vector (a, b) in the ordered basis ß $= \{\alpha, \beta\}$. (The conditions on α and β say, geometrically, that α and β are perpendicular and each has length 1).

2 SOLUTION

we need to show that α and β are independent in order to proove that ß $= \{\alpha, \beta\}$ is a basis of $\mathbb{R}^2$.
Given in the question are:

$$\alpha \beta^T = 0 \tag{2.0.1}$$

$$\|\alpha\|^2 = \|\beta\|^2 = 1 \tag{2.0.2}$$

2.0.2 show that α and β are non zero vector.
Let suppose,

$$\beta = c\alpha \tag{2.0.3}$$

$$\implies \alpha\beta^T = c\alpha\alpha^T = c\|\alpha\|^2 = c \tag{2.0.4}$$

The given equation 2.0.1 and 2.0.4 contradicts for any non-zero value of c. Equation 2.0.1 and 2.0.4 satisfy only when $c = 0$. Thus, α and β are independent.
Hence, ß $= \{\alpha, \beta\}$ is a basis of $\mathbb{R}^2$.
To, find the coordinates of the vector (a, b) in the ordered basis ß $= \{\alpha, \beta\}$. We can row-reduce the augmented matrix,

$$\begin{pmatrix} x_1 & y_1 & a \\ x_2 & y_2 & b \end{pmatrix} \tag{2.0.5}$$

$$\xleftarrow{R_1 = \frac{R_1}{x_1}} \begin{pmatrix} 1 & \frac{y_1}{x_1} & \frac{a}{x_1} \\ x_2 & y_2 & b \end{pmatrix} \tag{2.0.6}$$

$$\xleftarrow{R_2 = R_2 - x_2 R_1} \begin{pmatrix} 1 & \frac{y_1}{x_1} & \frac{a}{x_1} \\ 0 & y_2 - \frac{x_2 y_1}{x_1} & b - \frac{x_2 a}{x_1} \end{pmatrix} \tag{2.0.7}$$

$$= \begin{pmatrix} 1 & \frac{y_1}{x_1} & \frac{a}{x_1} \\ 0 & \frac{x_1 y_2 - x_2 y_1}{x_1} & \frac{x_1 b - x_2 a}{x_1} \end{pmatrix} \tag{2.0.8}$$

$$\xleftarrow{R_2 = \left(\frac{x_1}{x_1 y_2 - x_2 y_1}\right) R_2} \begin{pmatrix} 1 & \frac{y_1}{x_1} & \frac{a}{x_1} \\ 0 & 1 & \frac{x_1 b - x_2 a}{x_1 y_2 - x_2 y_1} \end{pmatrix} \tag{2.0.9}$$

$$\xleftarrow{R_1 = R_1 - \left(\frac{y_1}{x_1}\right) R_2} \begin{pmatrix} 1 & 0 & \frac{a y_2 - b y_1}{x_1 y_2 - x_2 y_1} \\ 0 & 1 & \frac{x_1 b - x_2 a}{x_1 y_2 - x_2 y_1} \end{pmatrix} \tag{2.0.10}$$

Using 2.0.1 and 2.0.2 and simplifying in 2.0.10,

$$= \begin{pmatrix} 1 & 0 & ax_1 + bx_2 \\ 0 & 1 & ay_1 + by_2 \end{pmatrix} \tag{2.0.11}$$

Hence,

$$\implies \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \alpha & \beta \end{pmatrix} \begin{pmatrix} ax_1 + bx_2 \\ ay_1 + by_2 \end{pmatrix} \tag{2.0.12}$$

$$\therefore \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \end{pmatrix} \begin{pmatrix} ax_1 + bx_2 \\ ay_1 + by_2 \end{pmatrix} \tag{2.0.13}$$