Linear regression and modelling problems are presented along with their solutions at the bottom of the page. Also a linear regression calculator and grapher may be used to check answers and create more opportunities for practice.

Review

If the plot of n pairs of data (x , y) for an experiment appear to indicate a "linear relationship" between y and x, then the method of least squares may be used to write a linear relationship between x and y.
The least squares regression line is the line that minimizes the sum of the squares (d1 + d2 + d3 + d4) of the vertical deviation from each data point to the line (see figure below as an example of 4 points).

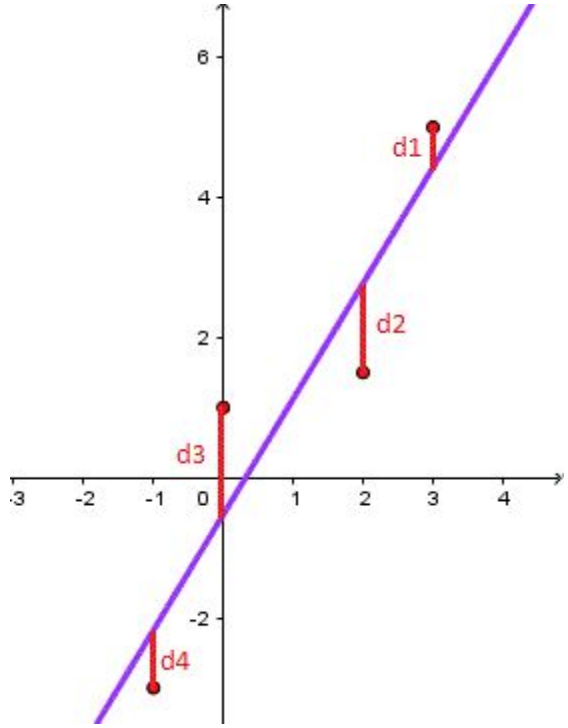


Figure 1. Linear regression where the sum of vertical distances d1 + d2 + d3 + d4 between observed and predicted (line and its equation) values is minimized.

The least square regression line for the set of n data points is given by the equation of a line in slope intercept form:

$$y = a x + b$$

where a and b are given by

$$a = \frac{n\sum_{i=1}^{n} x_i y_i - \sum_{i=1}^{n} x_i \sum_{i=1}^{n} y_i}{n\sum_{i=1}^{n} x_i^2 - (\sum_{i=1}^{n} x_i)^2}$$

$$b = \frac{1}{n}(\sum_{i=1}^{n} y_i - a\sum_{i=1}^{n} x_i)$$

Figure 2. Formulas for the constants a and b included in the linear regression .

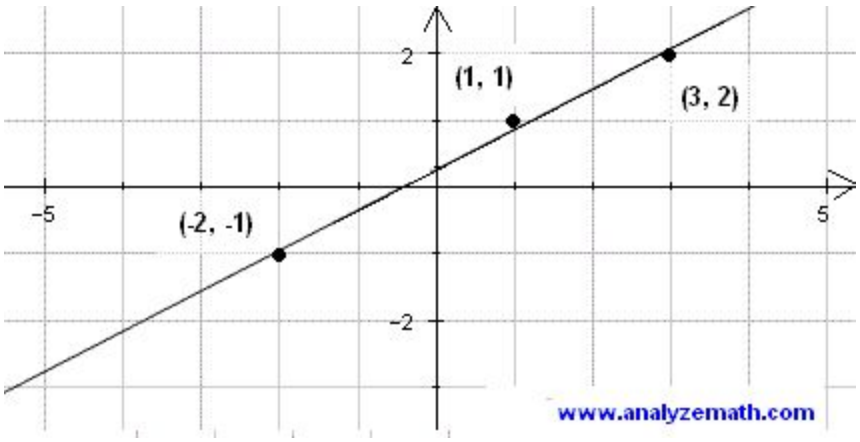
Problem 1

Consider the following set of points: {(-2 , -1) , (1 , 1) , (3 , 2)}
a) Find the least square regression line for the given data points.
b) Plot the given points and the regression line in the same rectangular system of axes.

1. a) Let us organize the data in a table.

x	y	x y	x^2
-2	-1	2	4
1	1	1	1
3	2	6	9
$\Sigma x = 2$	$\Sigma y = 2$	$\Sigma xy = 9$	$\Sigma x^2 = 14$

We now use the above formula to calculate a and b as follows
$a = (n\Sigma x\, y - \Sigma x \Sigma y) / (n\Sigma x^2 - (\Sigma x)^2) = (3*9 - 2*2) / (3*14 - 2^2) = 23/38$
$b = (1/n)(\Sigma y - a\, \Sigma x) = (1/3)(2 - (23/38)*2) = 5/19$
b) We now graph the regression line given by y = a x + b and the given points.



Problem 2

a) Find the least square regression line for the following set of data
{(-1 , 0),(0 , 2),(1 , 4),(2 , 5)}

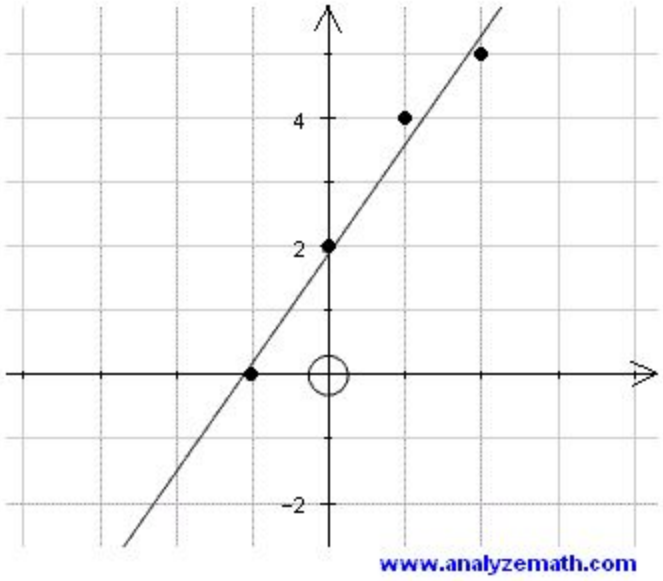
b) Plot the given points and the regression line in the same rectangular system of axes.

a) We use a table as follows

x	y	x y	x^2
-1	0	0	1
0	2	0	0
1	4	4	1
2	5	10	4
$\Sigma x = 2$	$\Sigma y = 11$	$\Sigma x\, y = 14$	$\Sigma x^2 = 6$

We now use the above formula to calculate a and b as follows
$a = (n\Sigma x\, y - \Sigma x \Sigma y) / (n\Sigma x^2 - (\Sigma x)^2) = (4*14 - 2*11) / (4*6 - 2^2) = 17/10 = 1.7$
$b = (1/n)(\Sigma y - a\, \Sigma x) = (1/4)(11 - 1.7*2) = 1.9$
b) We now graph the regression line given by y = ax + b and the given points.

1.



Problem 3

The values of y and their corresponding values of y are shown in the table below

x	0	1	2	3	4
y	2	3	5	4	6

a) Find the least square regression line y = a x + b.
b) Estimate the value of y when x = 10.

a) We use a table to calculate a and b.

x	**y**	**x y**	**x^2**
0	2	0	0
1	3	3	1
2	5	10	4
3	4	12	9
4	6	24	16
$\Sigma x = 10$	$\Sigma y = 20$	$\Sigma x\, y = 49$	$\Sigma x^2 = 30$

We now calculate a and b using the least square regression formulas for a and b.
$a = (n\Sigma x\, y - \Sigma x \Sigma y) / (n\Sigma x^2 - (\Sigma x)^2) = (5*49 - 10*20) / (5*30 - 10^2) = 0.9$
$b = (1/n)(\Sigma y - a\, \Sigma x) = (1/5)(20 - 0.9*10) = 2.2$
b) Now that we have the least square regression line y = 0.9 x + 2.2, substitute x by 10 to find the value of the corresponding y.
y = 0.9 * 10 + 2.2 = 11.2

Problem 4

The sales of a company (in million dollars) for each year are shown in the table below.

x (year)	2005	2006	2007	2008	2009
y (sales)	12	19	29	37	45

a) Find the least square regression line y = a x + b.
b) Use the least squares regression line as a model to estimate the sales of the company in 2012.

a) We first change the variable x into t such that t = x - 2005 and therefore t represents the number of years after 2005. Using t instead of x makes the numbers smaller and therefore manageable. The table of values becomes.

t (years after 2005)	0	1	2	3	4
y (sales)	12	19	29	37	45

We now use the table to calculate a and b included in the least regression line formula.

t	**y**	**t y**	**t^2**
0	12	0	0
1	19	19	1
2	29	58	4
3	37	111	9
4	45	180	16
$\Sigma x = 10$	$\Sigma y = 142$	$\Sigma xy = 368$	$\Sigma x^2 = 30$

We now calculate a and b using the least square regression formulas for a and b.
$a = (n\Sigma t\, y - \Sigma t \Sigma y) / (n\Sigma t^2 - (\Sigma t)^2) = (5*368 - 10*142) / (5*30 - 10^2) = 8.4$
$b = (1/n)(\Sigma y - a\, \Sigma x) = (1/5)(142 - 8.4*10) = 11.6$
b) In 2012, t = 2012 - 2005 = 7
The estimated sales in 2012 are: y = 8.4 * 7 + 11.6 = 70.4 million dollars.