Quotient Group and their Properties

Detail Course 2.0 on Group Theory for IIT JAM '23

Gajendra Purohit • Lesson 8 • Oct 15, 2022


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Coset and normal subgroup

Coset : Let G be a group on which the group operation is multiplication and H be a subgroup of it. Let a be any element of G then the set $Ha = \{ha : h \in H\}$ is called right coset of H in G generated by a.

Similarly, the set $aH = \{ah : h \in H\}$ is called a left coset of H in G generated by a.

Note : If the group operation is addition, then right and left coset of H in G generated by a, as

$$H + a = \{h + a \mid h \in H\}$$

$$a + H = \{a + h \mid h \in H\}$$

$K_4 = \{e, a, b, ab\}$

$a^2 = e$
$b^2 = e$
$(ab)^2 = e$

$\boxed{H = \{e, a\}}$

$eH = \{e, a\} = He = \{e, a\}$

$aH = \{a, e\} = Ha = \{a, e\}$

$bH = \{b, ba\} = Hb = \{b, ab\}$

$(ab)H = \{ab, b\} = H(ab) = \{ab, b\}$

$(a^2)b$

$[K_4 : H] = \frac{o(K_4)}{o(H)} = \frac{4}{2}$

$= 2$

$S_3 = \{e, (12), (13), (23), (123), (132)\}$, $H = \{e, (12)\}$

$eH = \{e, (12)\}$

$(12)H = \{(12), e\}$

$(13)H = \{(13), (123)\}$

$(23)H = \{(23), (132)\}$

$(123)H = \{(123), (13)\}$

$(132)H = \{(132), (23)\}$

$He = \{e, (12)\}$

$H(12) = \{(12), e\}$

$H(13) = \{(13), (132)\}$

$H(23) = \{(23), (123)\}$

$H(123) = \{(123), (23)\}$

$H(132) = \{(132), (13)\}$

$$\frac{o(S_3)}{o(H)} = \frac{6}{2} = 3$$

Conclusion from both example :

Let G be a group and H be its subgroup then right coset and left coset of H in G generated by a are need not be equal

In example (1), Right and left coset are equal.

In example (2), Right and left coset are not equal.

Result :

(1) If H is any subgroup of G and $h \in H$, then $Hh = H = hH$

(2) If a and b are any two elements of a group G & H is a subgroup of G then $Ha = Hb \Leftrightarrow ab^{-1} \in H$ & $aH = bH \Leftrightarrow b^{-1}a \in H$.

(3) Any two right (left) cosets of a subgroup are either disjoint or identical.

(4) Lagrange's theorem : The order of each subgroup of a finite group is a divisor of the order of the group.

Index of a subgroup in a group : The number of distinct coset of a subgroup in a group is called index of subgroup.

Note : Let G be a finite group then index of H in G.

$$= \frac{O(G)}{O(H)}$$

S_4

A_4

$$\frac{O(S_4)}{O(A_4)} = \frac{4!}{4!/2} = 2$$

Normal subgroup : Let G be a group. A subgroup H of G is called a normal subgroup of G iff right coset and left coset are equal i.e. $xH = Hx$; for all $x \in G$.

order 1

$\langle e \rangle$

(e) improper normal subgroup

order 2

$(\mathbb{Z}_2, +)$

simple

$(e, +)$

$\phi\ \mathbb{G}$

order 3

$(\mathbb{Z}_3, +)$

e

$(\mathbb{Z}_3, +)$

order 4

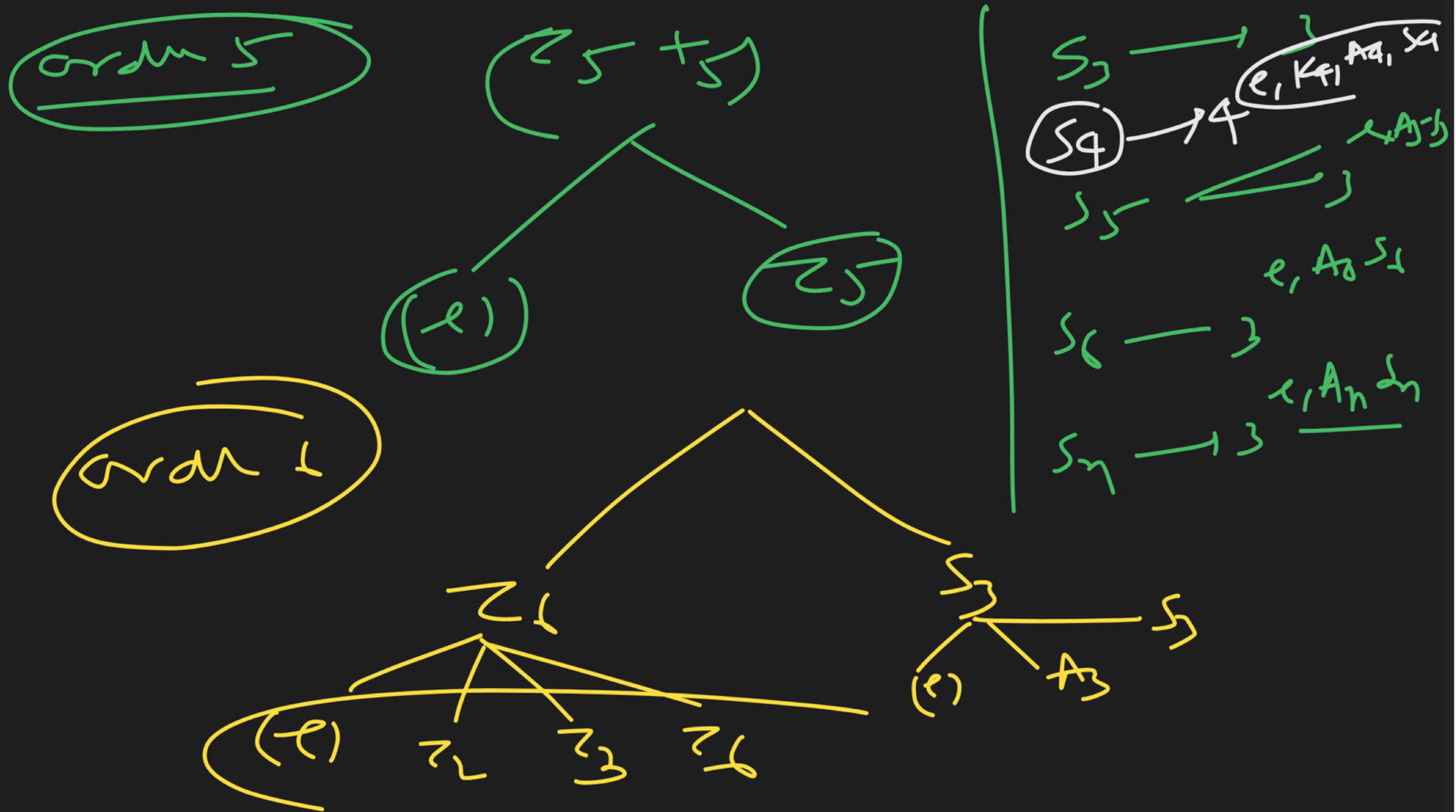
$K_4 = \{e, a, b, ab\}$

$\{e\}$ $\{e, a\}$ $\{e, b\}$ $\{e, ab\}$ K_4

$T(n)$

$(\mathbb{Z}_4, +)$

$\{e\}$ $\mathbb{Z}_2$ $\mathbb{Z}_4$



order 5
(Z5 + 5)
(e)
Z5
order 2
Z2
(e)
z2
z3
z6
S3
(e)
A3
S3
S3 → {e, K4, A4, S4
S4 → 4
e, A5, S5
S5 → {
e, A6, S6
S6 → {
e, An, Sn
Sn → {

order 2

order 8

$\mathbb{Z}_8 \qquad \mathbb{Z}_2 \times \mathbb{Z}_4 \qquad \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \qquad Q_8 \qquad D_4 \qquad 2$

$$Q_8 = \langle \pm 1, \pm i, \pm j, \pm k \rangle$$

$\cancel{H_1 = \langle 1 \rangle}$ $\qquad$ $\cancel{H_4 = \langle \pm 1, \pm j \rangle}$

$\cancel{H_2 = \langle 1, -1 \rangle}$ $\qquad$ $\cancel{H_5 = \langle \pm 1, \pm k \rangle}$

$\cancel{H_3 = \{\pm 1, \pm i\}}$ $\qquad$ $H_6 = Q_8$

ordu 9

2×5

Z_9 $\quad$ $Z_3 \times Z_3$

ordu 10

Z_{10} $\quad$ D_5

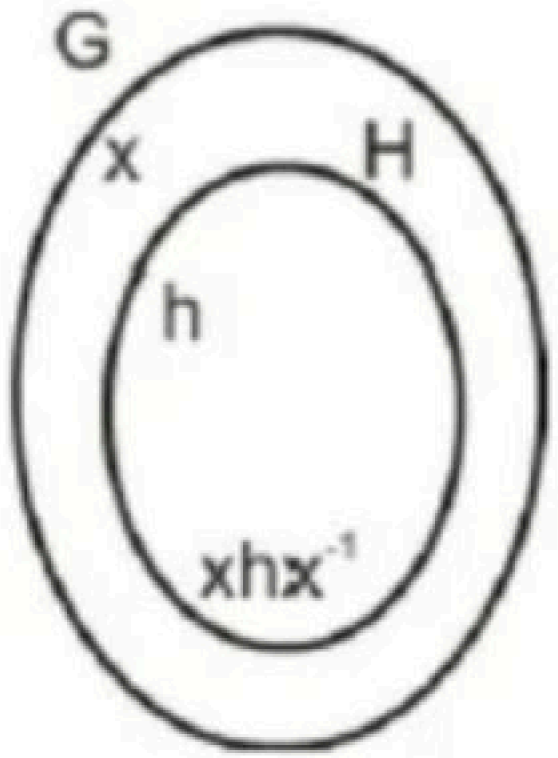
~~4~~ $\quad$ ~~4~~

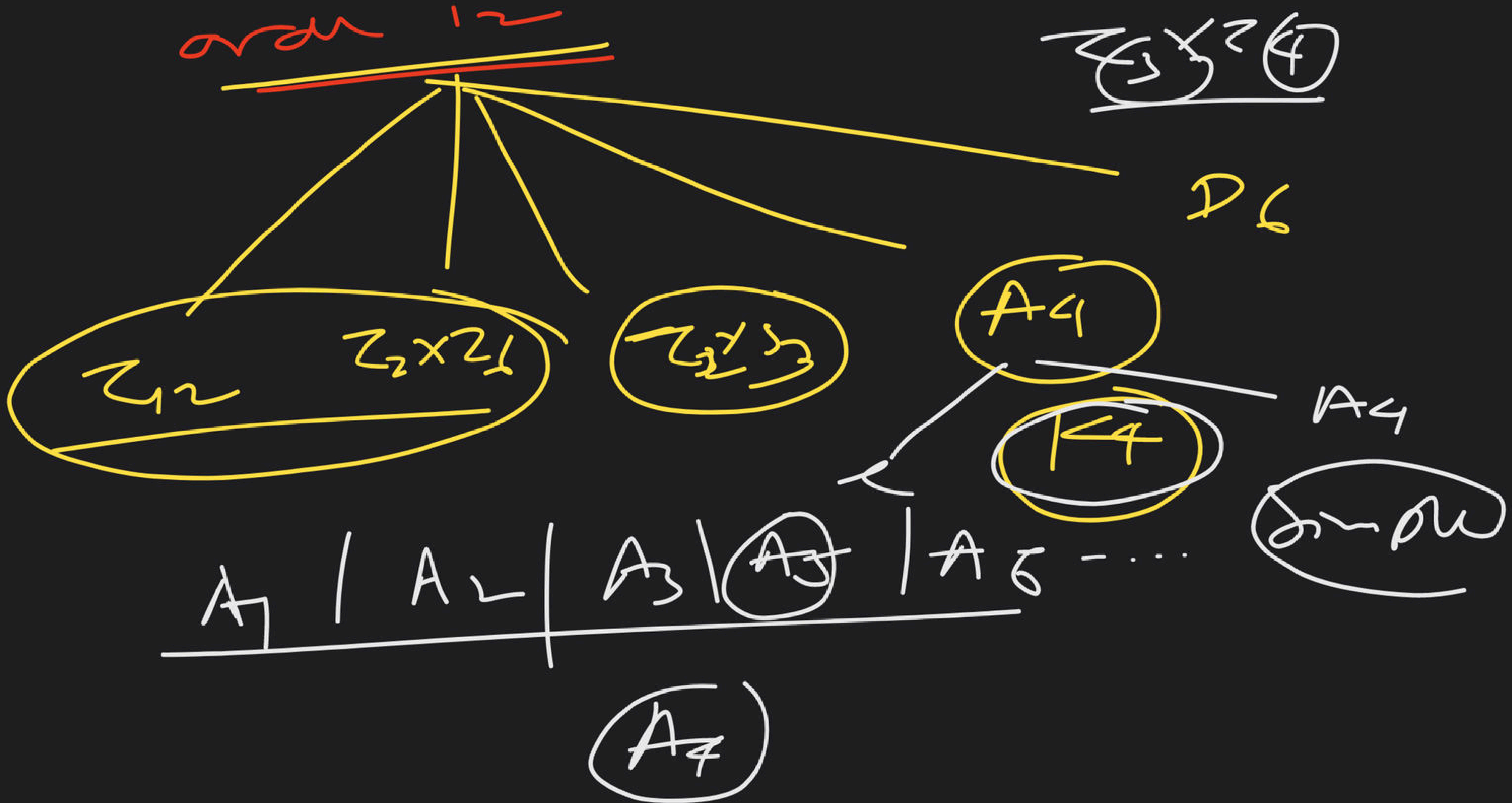
D_3

$\tau(4) \perp \sigma(4)$

Another definition : A subgroup H of a group G is said to be normal subgroup of G if for every $x \in G$ & for every $h \in H$ $xhx^{-1} \in H$

$xhx^{-1} \in H$




order 12
Z₃ x Z₄
D6
Z12
Z2 x Z6
A4
K4
A4
simple
A1 | A2 | A3 | A5 | A6 ....
A4

Q.1. Let G denote the group of all 2×2 invertible matrices with entries from R. Let $H_1 = \{A \in G;\ \det(A) = 1\}$ and $H_2 = \{A \in G,$ A is upper triangular$\}$ Consider the following statements :

$x = \begin{pmatrix} 1 & 2 \\ 3 & 5 \end{pmatrix}$

P : H_1 is a normal subgroup of G

$h = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}$

Q : H_2 is a normal subgroup of G

Then

$x h x^{-1} = -\frac{1}{1}\begin{pmatrix} 1 & 2 \\ 3 & 5 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} 5 & -2 \\ -3 & 1 \end{pmatrix}$

(a) Both P & Q are true

(b) P is true & Q is false

(c) P is false & Q is true

(d) Both P & Q are false.

$\begin{pmatrix} 9 & 3 \\ 0 & c \end{pmatrix} \notin H_2$

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Result :

(1) A subgroup of index 2 in G is normal subgroup of G.

(2) SL(n, F) is normal in GL(n, F)

(3) A_n is normal in S_n.

(4) Every subgroup of abelian group is normal.

(5) Normal Subgroup of $S_n = \begin{cases} I, K_4, A_n, S_n & \text{If } n = 4 \\ I, A_n, S_n & \text{If } n \neq 4 \end{cases}$

(6) Normal Subgroup of Q_8 are {1} ,{-1 , 1} , {1 , -1 ,i , -i},

{1 , -1 ,j , -j} , {1 , -1 ,k , -k} and Q_8

(7) Normal Subgroup of $A_n = \begin{cases} I, K_4, A_n & If\ n = 4 \\ I, A_n & If\ n \neq 4 \end{cases}$

(8) {e} and G are always normal subgroup

Q.2. Let $H = \{ e, (1\ 2)(3\ 4) \}$ and $K = \{ e, (1\ 2)(3\ 4), (1\ 3)(2\ 4), (1\ 4)(2\ 3) \}$ be a subgroup of S_4 where e denotes the identify elements of S_4 then

(a) H and K are normal subgroup of S_4

(b) H is normal in K and K is normal in A_4

(c) H is normal in A_4 and K is not normal in S_4

(d) K is normal in S_4 and H is not

K_4 S_4 A_4 e K_4 A_4 S_3 H

Quotient Group:

Let N be a normal subgroup of G. If a $\in$ G, then Na is a right coset of N in G.

Since N is normal in G, left coset aN will be equal to the right coset Na.

Let $\frac{G}{N}$ be the collection of all distinct coset of N in G i.e. $\frac{G}{N} = \{Na;\ a \in G\}$, then $\frac{G}{N}$ is a group w.r.t. multiplication of coset i.e. (Na).(Nb) = Nab

It is called quotient group or factor group of G by N as the composition in $\frac{G}{N}$.

$G = (\mathbb{Z}, +)$

$H = (2\mathbb{Z}, +)$

$G = \{\ldots -2, -1, 0, 1, 2 \ldots\}$

$H = \{\ldots -4, -2, 0, 2, 4 \ldots\}$

$H + 0 = H = 0 + H$

$H + 1 = \{\ldots -3, -1, 1, 3, 5 \ldots\}$

$(H+a) + (H+b) = H + a + b$

$\frac{G}{H} = \{H, H+1\}$

+	H	H+1
H	H	H+1
H+1	H+1	H

$(\mathbb{Z}_2, +)$

$\frac{\mathbb{Z}}{2\mathbb{Z}} = \mathbb{Z}_2$

$\frac{\mathbb{Z}}{n\mathbb{Z}} = \mathbb{Z}_n$

$G = \mathbb{Z}_6$

$G = \langle 0, 1, 2, 3, 4, 5 \rangle$

$H + 0 = H$

$H + 1 = \langle 1, 4 \rangle$

$H + 2 = \langle 2, 5 \rangle$

$H + 3 = \langle 3, 0 \rangle = H$

$H + 4 = \langle 4, 1 \rangle = H + 1$

$H + 5 = \langle 5, 2 \rangle = H + 2$

$H = \langle 0, 3 \rangle$

$G/H = \langle H, H+1, H+2 \rangle$

$+$	H	$H+1$	$H+2$
$e = H$	H	$H+1$	$H+2$
$H+1$	$H+1$	$H+2$	H
$H+2$	$H+2$	H	$H+1$

Conclusion : Let N be a normal subgroup of a finite group G.

Then index of N in G = $\frac{O(G)}{O(N)}$. If G is finite

Note : Order of Quotient group $O\left(\frac{G}{N}\right) = \frac{O(G)}{O(N)}$. If G is finite

Centre of Group : Let G be a group then $Z(G) = \{ x \in G \mid xa = ax \mid \forall a \in G\}$ is called centre of group

Result :

(1) The quotient group of abelian group is abelian but converse is not true.

(2) The quotient group of cyclic group is cyclic but converse is not true.

K_4

S_3

$e \quad A_3 \quad S_3$

$$\frac{S_3}{\langle e \rangle} \cong S_3$$

$$\frac{6}{3}$$

$$\frac{S_3}{A_3} \cong C_2$$

$$\frac{S_3}{S_3} = \langle e \rangle$$

K_4

$$\frac{K_4}{\langle e \rangle} = K_4$$

(3) Let Z(G) be a centre of a group G, then G is abelian if $\frac{G}{Z(G)}$ is cyclic.

(4) ~~Let G be a cyclic group of order n, then number of factor group of G are τ(n) because number of normal subgroups are τ(n).~~

(5) $\frac{K}{N}$ is a subgroup of $\frac{G}{N}$, if K is a subgroup of G.

$$\frac{K}{N} \triangleleft \frac{G}{N} \quad if \quad K \triangleleft G$$

$\frac{G}{Z(G)}$ — cyclic

$T(10)$

$$\frac{2^{10}}{2} \qquad \frac{2^{10}}{2^{2}} \qquad \frac{2^{10}}{2^{5}} \qquad \frac{2^{10}}{2^{10}}$$

(1) $\frac{Q}{Z}$ is abelian group but not cyclic group.

(2) Number of elements of order n are $\phi(n)$

(3) $\left(\frac{1}{p}+Z\right)\in\frac{Q}{Z}$ *then* $O\left(\frac{1}{p}+Z\right)=p$

$\frac{1}{4}+Z$

And $O\left(\frac{k}{p}+Z\right)=p$ *if* $\gcd(k,p)=1$

$\frac{2}{4}+Z$

$\frac{3}{4}+Z$

(5) There exist a unique cyclic subgroup of each order i.e. For every positive integer n , there is a cyclic subgroup of order n which is unique which is $H=\left\langle\frac{1}{n}+Z\right\rangle$

Q.3. Consider the quotient group of $\frac{Q}{Z}$ of the additive group of rational number, the order of element $\frac{2}{3}+Z$ in $\frac{Q}{Z}$ is

(a) 2 (b) 3 (c) 5
(d) 6

Q.4. Consider the group $G = Q/Z$. Let n be a positive integer. Then there is a cyclic subgroup

of order n

(a) Not necessarily

(b) Yes , a unique

(c) Yes , but not necessarily a unique

(d) Never

Q.5. Consider the following subsets of the group of 2×2 non singular matrices over R

$$G = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} : a, b, d \in R, ad = 1 \right\}$$

$$H = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} : b \in R \right\}$$

Which of the following statements are correct

(a) G form a group under matrix multiplication

(b) H is normal subgroup of G

(c) G/H is well define and is abelian

(d) None of these

Q.6. Let G be a non-abelian group and Z(G) is its centre, then which of the following is cannot be possible of $O\left(\frac{G}{Z(G)}\right)$.

(a) 7 (b) 8 (c) 4

(d) 6

Q.7. If $H \subset K$ are two normal subgroups of a group G and if $[G : H] = 10$ and $[G : K] = 5$, then $[K : H]$ is

(a) 5 (b) 2 (c) 10

(d) 50

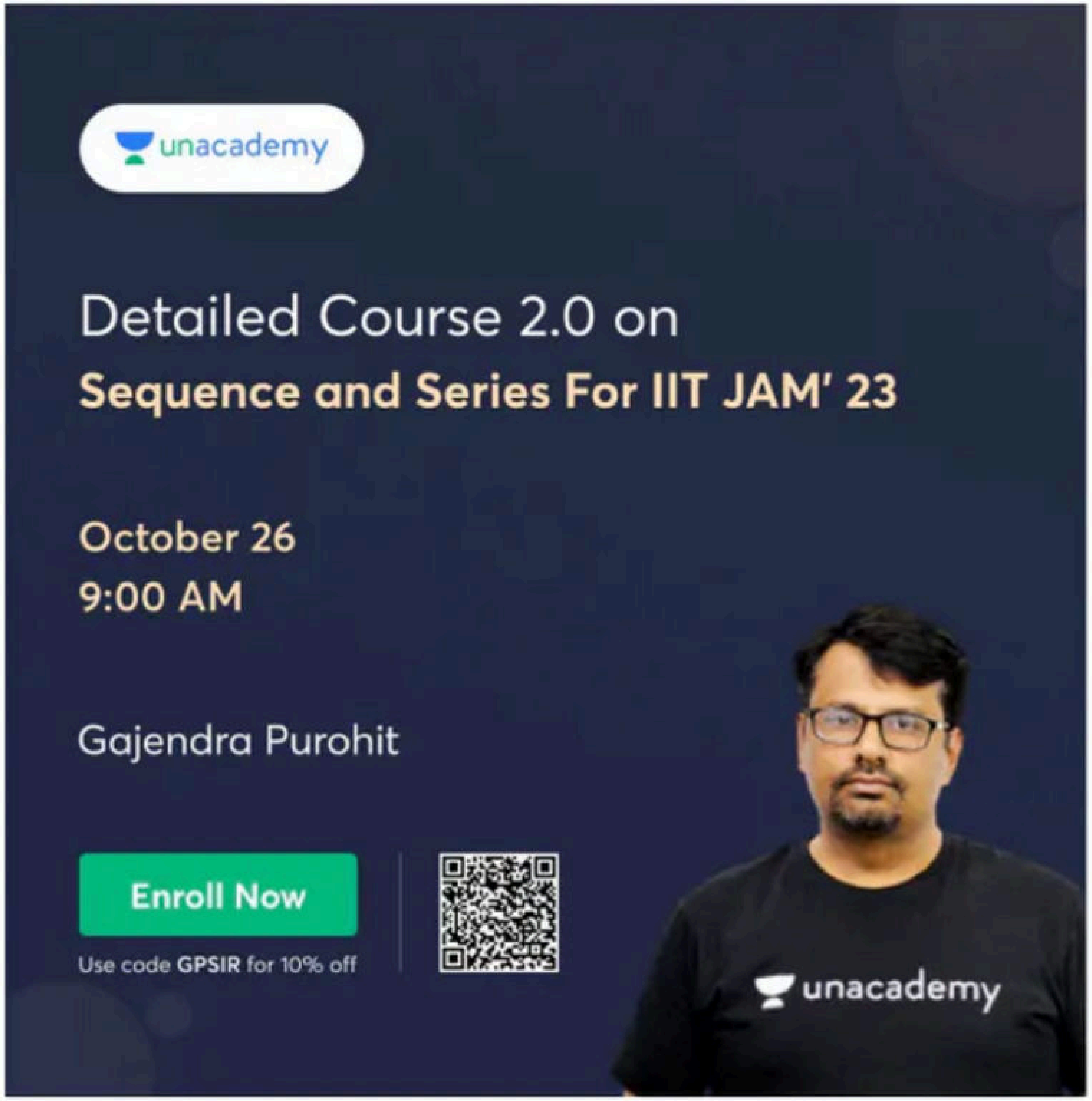
Q.8. Suppose N is a normal subgroup of a group G. Which one of the following is true?

(a) If G is an infinite group, then G/N is an infinite group.

(b) If G is a non-abelian group, then G/N is a non-abelian group.

(c) If G is a cyclic group, then G/N is an abelian group.

(d) If G is an abelian group, then G/N is a cyclic group.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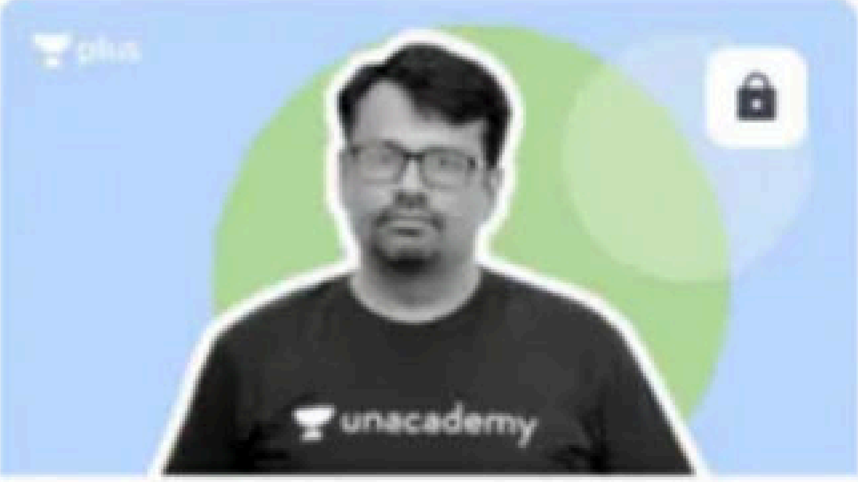

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