Definite integral

Definition : If $\frac{d}{dx}[f(x)] = \phi(x)$ and a & b are constant, then

$$\int_a^b \phi(x)dx = [f(x)]_a^b = f(b) - f(a)$$

is called definite integration of $\phi(x)$ within limit a & b.

Note : This is also called fundamental theorem of calculus.

Basic properties of definite integrals.

(1) $$\int_a^b f(t)dt = \int_a^b f(x)dx$$

(2) $$\int_a^b f(x)dx = -\int_b^a f(x)dx$$

(3) $$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

For any c $\in$ (a, b)

(4) $$\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$$

(5) $$\int_{-a}^{a} f(x)dx = \begin{cases} 2\int_0^a f(x)dx & if\ f(x)\ is\ even \\ 0 & if\ f(x)\ is\ odd \end{cases}$$

(6) $$\int_0^{2a} f(x)dx = \begin{cases} 2\int_0^a f(x)dx; & \text{if } f(2a-x) = f(x) \\ 0; & \text{if } f(2a-x) = -f(x) \end{cases}$$

Q1. The value of the integral $\int_{-\pi}^{\pi} |x| \cos nx dx, n \geq 1$ is

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(a) 0, when n is even

(b) 0, when n is odd

(c) $-\frac{4}{n^2}$, when n is even

(d) $-\frac{4}{n^2}$, when n is odd

Q.2. The value of $\int_0^{\pi/4} \log(1+\tan\theta)\,d\theta$ is

(a) $\frac{\pi}{8}$

(b) $\log 2$

(c) $\frac{\pi}{8}\log 2$

(d) $\frac{\pi}{8}\log 3$

Q.3. Evaluate : $\int_0^{\pi} \frac{x\,dx}{a^2\cos^2 x + b^2\sin^2 x}$

(a) $\frac{\pi^2}{2a^2b^2}$

(b) $\frac{\pi^2}{2ab}$

(c) $\frac{2\pi^2}{a^2b}$

(d) π^2

Definite integral as the limit of a sum :

$$\int_0^1 f(x)dx = \lim_{n\to\infty}\frac{1}{n}\sum_{r=0}^{n} f\left(\frac{r}{n}\right)$$

Where f(x) is continuous function on closed interval [0, 1]

Q.4. The limit of sum $\frac{1}{n}+\frac{1}{n+1}+\frac{1}{n+2}+.....+\frac{1}{3n}$, when $n \to \infty$ is

(a) $\log 3$ (b) $\log 2$

(c) 0 (d) 1

Q5. The limit when n $\to\infty$ of the product

$$\left\{\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)\left(1+\frac{3}{n}\right)\ldots\ldots\left(1+\frac{n}{n}\right)\right\}^{1/n}$$

(a) 2/e

(b) 4/e

(c) 5/e

(d) e

Q6. The value : $\lim_{n\to\infty}\left[\frac{n!}{n}\right]^{1/n}$ is

(a) 1/e (b) 2/e

(c) 3/e (d)4/e

Gamma Function:

If m and n are non-negative integers, then

$$\int_0^{\pi/2} \sin^m x \cos^n x dx = \frac{\Gamma\left(\frac{m+1}{2}\right)\Gamma\left(\frac{n+1}{2}\right)}{2\Gamma\left(\frac{m+n+2}{2}\right)}$$

where $\Gamma(n)$ is called gamma function which satisfied the following properties

$\Gamma(n+1) = n\Gamma(n) = n!$ *i.e.* $\Gamma(1) = 1$ and $\Gamma(1/2) = \sqrt{\pi}$

In place of gamma function, we can also use the following formula :

$$\int_0^{\pi/2} \sin^m x \cos^n x dx = \frac{(m-1)(m-3).....(2 \text{ or } 1)(n-1)(n-3).....(2 \text{ or } 1)}{(m+n)(m+n-2)....(2 \text{ or } 1)}$$

It is important to note that we multiply by $(\pi/2)$; when both m and n are even.

Q.7. Evaluate $\int_0^{\infty} x^{1/4} e^{-\sqrt{x}}\, dx$

(a) $2\sqrt{\pi}$

(b) $\frac{3}{2}\pi$

(c) $\sqrt{\pi}$

(d) $\frac{3}{2}\sqrt{\pi}$

Q.8. Let a,b be positive real numbers such that a < b Given that $\lim_{n\to\infty}\int_0^n e^{-t^2}\,dt = \frac{\sqrt{\pi}}{2}$ Then value of $\lim_{n\to\infty}\int_0^n \frac{1}{t^2}\left(e^{-at^2} - e^{-bt^2}\right)dt$ is

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(a) $\sqrt{\pi}\left(\sqrt{b} - \sqrt{a}\right)$

(b) $\sqrt{\pi}\left(\sqrt{b} + \sqrt{a}\right)$

(c) $-\sqrt{\pi}\left(\sqrt{b} - \sqrt{a}\right)$

(d) $\sqrt{\pi}\left(-\sqrt{b} + \sqrt{a}\right)$

Q9. The value of $\int_0^{\pi/2} \sin^4 x \cos^6 x dx$

(a) $3\pi/312$

(b) $5\pi/512$

(c) $3\pi/512$

(d) $5\pi/312$

Q10. If $I_n = \int_0^\infty e^{-x} x^{n-1} dx$, then $\int_0^\infty e^{-\lambda x} x^{n-1} dx$ is equal to

(a) λI_n

(b) $\frac{1}{\lambda} I_n$

(c) $\frac{I_n}{\lambda^n}$

(d) $\lambda^n I_n$

Q11. $\int_0^{\pi/2} \sin^7 x dx$ has value

(a) $\frac{37}{184}$ (b) $\frac{17}{45}$

(c) $\frac{16}{35}$ (d) $\frac{16}{45}$