MA2102 (Linear Algebra I): Internal Assesment

Solution to Presentation Problems: Set 6

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§1 Finding the Dimension

Problem.
Compute the dimension of $S = \{a + b\sqrt{2} + c\sqrt{3} + d\sqrt{6} : a, b, c, d \in \mathbb{Q}\}$ as a vector space over $\mathbb{Q}$.

Solution. We claim that the $\dim_{\mathbb{Q}} S = 4$ by constructing the basis $\beta = \{1, \sqrt{2}, \sqrt{3}, \sqrt{6}\}$.
Clearly β is spanning by definition of the set S. Consider the representation of the $0 \in S$ in β: Suppose there exists $a, b, c, d \in \mathbb{Q}$ such that

$$a + b\sqrt{2} + c\sqrt{3} + d\sqrt{6} = 0$$

which can be expressed as

$$a + b\sqrt{2} + \sqrt{3}(c + d\sqrt{2}) = 0$$

Suppose, $c + d\sqrt{2} \neq 0$. Then then above expression can be rewritten as

$$\sqrt{3} = \frac{a + b\sqrt{2}}{c + d\sqrt{2}}.$$

Since $\mathbb{Q}[\sqrt{2}]$ is a field (we obtained this result in class), we have

$$\sqrt{3} = e + f\sqrt{2},$$

where $e, f \in \mathbb{Q}$. Therefore

$$3 = e^2 + 2f^2 + 2ef\sqrt{2}. \tag{1}$$

Clearly, $e = f = 0$ is not possible. Let $e = 0$. Then

$$2f^2 = 3$$

which is not possible for $f \in \mathbb{Q}$ (we can easily argue by claiming that the power of 2 on the left is odd (we are allowing negative powers as well and prime-factorizing non-zero rationals), while on the right it is 0, i.e., even). Similarly, if $f = 0$,

$$e^2 = 3$$

is not possible for $e \in \mathbb{Q}$. Hence, e, f both are non-zero. Thus we rewrite our Equation (1):

$$\sqrt{2} = \frac{3 - e^2 - 2f^2}{2ef}$$

which is not possible, since $\sqrt{2}$ is irrational.

Therefore, $c + d\sqrt{2} = 0$. Notice that, $d \neq 0$ is not possible for the irrationality of $\sqrt{2}$. Thus if $d = 0$, it is forced to have $c = 0$. Therefore, $a + b\sqrt{2} = 0$. By the similar argument as mentioned, we conclude $b = 0$ and thus $a = 0$. This completes the proof of the fact that β is linealy independent set of vectors in S. Along with the spanning property, β is then a basis, which proves our claim. $\square$

§2 Comparing Eigenvalues

Problem.
Let $T : V \to V$ be an invertible linear transformation. Compare the eigenvalues of T and T^{-1}.

Solution. Let $\mathbb{F} = \mathbb{R}/\mathbb{C}$. Suppose there exists an eigenvector $v \in V$ (by definition, non-zero) of the linear transformation T, associated to the eigenvalue $\lambda \in \mathbb{F}$. Then

$$T(v) = \lambda v.$$

Since T is invertible linear transformation, we have

$$T^{-1}(\lambda v) = v.$$

Then linearity of T^{-1} implies

$$\lambda T^{-1}(v) = v.$$

Claim. (1)
Inveritible $T : V \to V$ will have non-zero eigenvalues, if exists.

Proof. Suppose 0 is an eigenvalue of T. Then there exists a eigenvector (by definition, non-zero) v such that

$$T(v) = 0$$

associated to the eigenvalue 0. This allows us to deduce that $\dim N(T) \geq 1$, contradicting to the injectivity of T. $\square$

Therefore,

$$T^{-1}(v) = \frac{1}{\lambda} v.$$

Hence we conclude $\frac{1}{\lambda} \in \mathbb{F}$ is an eigenvalue of T^{-1}. Similarly, the converse will also follow: if $\mu \in \mathbb{F}$ then $\frac{1}{\mu} \in \mathbb{F}$ is an eigenvalue of T, by similar arguments. Hence, T has the eigenvalue $\lambda \in \mathbb{F}$ if and only if T^{-1} has the eigenvalue $\frac{1}{\lambda} \in \mathbb{F}$. $\square$

§3 Linearly Independent Eigenvectors

Problem.
If the matrix

$$A = \begin{pmatrix} 0 & 0 & 1 \\ x & 1 & y \\ 1 & 0 & 0 \end{pmatrix}$$

has three linearly independent eigenvectors, then show that $x + y = 0$.

Solution. The characteristic polynomial of the matrix is:

$$p(\lambda) = \det(A - \lambda I) = \det \begin{pmatrix} -\lambda & 0 & 1 \\ x & 1-\lambda & y \\ 1 & 0 & -\lambda \end{pmatrix} = -(\lambda+1)(\lambda-1)^2.$$

Therefore the eigenvalues are $\lambda = \pm 1$, and the algebraic multiplicity of $\lambda = 1$ is 2. So we have at least two linearly independent eigenvectors associated to two distinct eigenvalues. Thus the matrix will have three linealy independent eigenvectors if and only if the eigenspace E_1 have dimension 2. To find the eigenvectors

$$v = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \in \mathbb{R}^3$$

associated to eigenvalue 1, we solve $(A - I)v = 0$ for v:

$$\begin{pmatrix} -1 & 0 & 1 \\ x & 0 & y \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0$$

which just translates to two linear equations:

$$a = c; \quad ax + cy = 0.$$

Thus

$$a(x+y) = 0$$

which gives rise to two cases:
Case: 1. $a = 0$. In this case, for any $x, y \in \mathbb{R}$, we have

$$v = \begin{pmatrix} 0 \\ b \\ 0 \end{pmatrix}.$$

Since $b \in \mathbb{R}$ has no restriction, any $v \in E_1$ with 0 in the first coordinate, will be in the span of $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$.

Therefore $\dim E_1 = 1$, which is *not* the required criterion.
Case: 2. $x + y = 0$. In this case, we have no restriction on $a, b \in \mathbb{R}$. So any $v \in E_1$ with $x + y = 0$ can be written as

$$v = \begin{pmatrix} a \\ b \\ a \end{pmatrix} = a \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + b \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

i.e., in the span of 2 linearly independent vectors, which immediately tells us $\dim E_1 = 2$, as desired. □

§4 Geometry in Symmetric Orthogonal Matrices

Problem.

Let T be a linear transformation on $\mathbb{R}^3$ whose matrix (relative to the usual basis for $\mathbb{R}^3$) is both symmetric and orthogonal. Prove that T is either plus or minus the identity, or a rotation by 180° about some axis in $\mathbb{R}^3$, or a reflection about some two-dimensional subspace of $\mathbb{R}^3$.

Solution. We will treat $\mathbb{R}^3$ as the usual inner product space in this problem. Given an usual ordered basis β of $\mathbb{R}^3$, we first notice that

$$([T]_\beta)^2 = I.$$

Suppose there exists an invertible matrix P such that $[T]_\gamma = P[T]_\beta P^{-1}$ is diagonal. So, it follows that

$$([T]_\gamma)^2 = P([T]_\beta)^2 P^{-1} = PP^{-1} = I.$$

By the diagonal property, we can conclude that $[T]_\gamma$ would be equal to one of the 8 matrices of order 3×3, having ± 1 in the diagonal.

Suppose the matrix P is just a change of basis matrix which changes usual basis to another orthonormal basis, then we can observe that, those 8 matrices represent the maps with required properties.

Hence we are left to prove the following proposition:

Proposition 4.1

Every symmetric matrix (3×3 here) is diagonalizable. The diagonal matrix will be the matrix representation of the same map written in another orthonormal basis.

Proof. Since the matrix $[T]_\beta$ is symmetric, the eigenvalues are real. Suppose one of the eigenspace is E_a. Then it contains an eigenvector, say v_1. Consider the subspace

$$W = \operatorname{span}\{v_1\}.$$

This is a T-invariant subspace. Consider the 2-dimensional subspace

$$W^\perp = \{v \in \mathbb{R}^3 : \langle v, v_1 \rangle = 0\}.$$

Observe that, T is a self-adjoint map (since $[T]_\beta$ is symmetric), $\forall w \in W^\perp$ we have

$$\langle [T]_\beta w, v_1 \rangle = \langle w, [T]_\beta v_1 \rangle = \langle w, a v_1 \rangle = a \langle w, v_1 \rangle = 0,$$

i.e., $W^\perp$ is T-invariant. Consider the restriction map of T on $W^\perp$, say L.

Lemma 4.2

The characteristic polynomial of L will divide the characteristic polynomial of T.

Proof. Let $\delta = \{w_1, w_2\}$ be an basis of $W^\perp$. Then it can be extended to a basis of $\mathbb{R}^3$. Then the matrix representation of T will have a block diagonal form (in fact it will be symmetric):

$$[T]_\delta = \begin{pmatrix} t_{11} & t_{12} & 0 \\ t_{12} & t_{22} & 0 \\ 0 & 0 & t_{33} \end{pmatrix}$$

The characteristic polynomial of T is then

$$\det(T - xI) = \det \begin{pmatrix} t_{11} - x & t_{12} & 0 \\ t_{12} & t_{22} - x & 0 \\ 0 & 0 & t_{33} - x \end{pmatrix} = (t_{33} - x) \det \begin{pmatrix} t_{11} - x & t_{12} \\ t_{12} & t_{22} - x \end{pmatrix}$$

which is equal to $\det(L - xI)\det(t_{33} - x)$, as desired. $\square$

By factor theorem, the roots of the characteristic polynomial of L will be the roots of the characteristic polynomial of T. Since all the roots of the characteristic polynomial of T is real, eigenvalues of L would be real. Choose an eigenvector of L, say v_2, that will be an eigenvector of T as well. Moreover, v_2 is perpendicular to v_1. Let

$$X = \text{span}\{v_1, v_2\}$$

and consider $X^{\perp}$, which is again T-invariant subspace. Consider the restriction map of T on $X^{\perp}$, say M. By the similar argument, the eigenvalue of M will be the eigenvalue of T. Therefore, eigenvector v_3 of M will be eigenvector T as well. Moreover,

$$\langle v_3, v_1 \rangle = \langle v_3, v_2 \rangle = 0$$

i.e.,

$$\theta = \{v_1, v_2, v_3\}$$

is an orthogonal basis, which can be made orthonormal by scaling. $\square$

With the change of basis matrix from β to θ, we have the required diagonal matrix, which can be interpreted geometrically to the desired properties in the statement of the problem. $\square$

§5 Linear Independence of Exponentials

Problem.

Let $\alpha_1, \dots, \alpha_n$ be distinct real numbers. Show that the n exponential functions $e^{\alpha_1 x}, \dots e^{\alpha_n x}$ are linearly independent (over the real numbers) in $\mathcal{F}(\mathbb{R}, \mathbb{R})$, the vector space of all functions from $\mathbb{R}$ to $\mathbb{R}$.

Solution. We begin with a statement, which is very easy to prove, stated as a fact:

Fact.

Exponential function $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) = e^{kx}; \quad k \in \mathbb{R}$ is infinitely differentiable.

Now consider the representation of $\mathbf{0} \in \mathcal{F}(\mathbb{R}, \mathbb{R})$ (as map) in terms of the given n exponential functions:

$$c_1 e^{\alpha_1 x} + \cdots + c_n e^{\alpha_n x} = \mathbf{0}$$

where, $c_i \in \mathbb{R}$ for each $i = \{1, \dots, n\}$. Plugging $x = 0$, we obtain

$$\sum_{i=1}^{n} c_i = 0.$$

Differentiating the representation of $\mathbf{0}$ (which is possible due to the stated fact) and plugging $x = 0$, we obtain

$$\sum_{i=1}^{n} c_i \alpha_i = 0.$$

Continuing this process, on the k-th step, we obtain

$$\sum_{i=1}^{n} c_i \alpha_i^{k-1} = 0.$$

We stop at n-th step and write all the equations in $A\mathbf{x} = \mathbf{0}$ form:

$$\begin{pmatrix} c_1 & c_2 & \dots & c_n \\ c_1\alpha_1 & c_2\alpha_2 & \dots & c_n\alpha_n \\ \vdots & \vdots & \ddots & \vdots \\ c_1\alpha_1^{n-1} & c_2\alpha_2^{n-1} & \dots & c_n\alpha_n^{n-1} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

which immediately tells us that the matrix A as linear map from $\mathbb{R}^n \to \mathbb{R}^n$ is not injective, so it's not full rank, which in turn says that

$$\begin{aligned} \det \begin{pmatrix} c_1 & c_2 & \dots & c_n \\ c_1\alpha_1 & c_2\alpha_2 & \dots & c_n\alpha_n \\ \vdots & \vdots & \ddots & \vdots \\ c_1\alpha_1^{n-1} & c_2\alpha_2^{n-1} & \dots & c_n\alpha_n^{n-1} \end{pmatrix} &= \prod_{i=1}^{n} c_i \det \begin{pmatrix} 1 & 1 & \dots & 1 \\ \alpha_1 & \alpha_2 & \dots & \alpha_n \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_1^{n-1} & \alpha_2^{n-1} & \dots & \alpha_n^{n-1} \end{pmatrix} \\ &= \prod_{i=1}^{n} c_i \prod_{1 \le j \le k \le n} (\alpha_k - \alpha_j) = 0 \end{aligned}$$

Since, α_is are distinct, we have at least one $c_j = 0$. After re-ordering, suppose $c_n = 0$. We plug it in every equation we had obtained. First $n - 1$ equations will give rise to a $n - 1 \times n - 1$ matrix, whose determinant for the similar reason, will be zero. Again after renaming, $c_{n-1} = 0$. The process ends after n steps and yields $c_1 = c_2 = \cdots = c_{n-1} = c_n = 0$, proving the linear independence. $\square$

§6 Union becomes Subspace

Problem.
Let W_1, W_2 be subspaces of a vector space V. Determine when $W_1 \cup W_2$ is a subspace of V.

Solution. We claim that $W_1 \cup W_2$ is a subspace of V if and only if $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$.

If $W_1 \subseteq W_2$ then $W_1 \cup W_2 = W_2$ which is a subspace of V. Otherwise $W_2 \subseteq W_1$ then $W_1 \cup W_2 = W_1$ which is a subspace of V.

To show the converse, suppose to the contrary that $W_1 \cup W_2$ is a subspace of V but $W_1 \nsubseteq W_2$ and $W_2 \nsubseteq W_1$. Let $v_1 \in W_1$ and $v_2 \in W_2$ such that $v_1 \notin W_2$ and $v_2 \notin W_1$. Since both of them belong to the subspace $W_1 \cup W_2$, we have $v_1 + v_2 \in W_1 \cup W_2$. Hence either $v_1 + v_2 \in W_1$ or $v_1 + v_2 \in W_2$.

Case: 1. $v_1 + v_2 \in W_1$. Since $v_1 \in W_1$ and W_1 is a subspace, we have $-v_1 \in W_1$. But that implies that $v_1 + v_2 - v_1 = v_2 \in W_1$, which is a contradiction.

Case: 2. $v_1 + v_2 \in W_2$. Since $v_2 \in W_2$ and W_2 is a subspace, we have $-v_2 \in W_2$. But that implies that $v_1 + v_2 - v_2 = v_1 \in W_2$, which is a contradiction. □

§7 Finding 2-dimensional Invariant Subspace

Problem.
Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation, where $n > 1$. Prove that there is a 2-dimensional subspace $V \subseteq \mathbb{R}^n$ such that $T(V) \subseteq V$.

Solution. We break the problem into 3 cases:
Case: 1. If we have at least two distinct real eigenvalues, then we are done by choosing the corresponding linearly independent eigenvectors: Suppose $\lambda_1 \neq \lambda_2 \in \mathbb{R}$ and $v_1, v_2 \in \mathbb{R}^n$ are non-zero vectors such that

$$T(v_1) = \lambda_1 v_1; \quad T(v_2) = \lambda_2 v_2$$

Then consider the subspace

$$W_1 = \text{span}\{v_1, v_2\}$$

and observe the following claim:

Claim. (1)
W_1 is 2 dimensional T-invariant subspace, i.e., $T(W_1) \subseteq W_1$.

Proof. Let $v \in W_1$. Then $v = c_1 v_1 + c_2 v_2$, for some $c_1, c_2 \in \mathbb{R}$. So

$$T(v) = T(c_1 v_1 + c_2 v_2) = c_1 T(v_1) + c_2 T(v_2) = c_1 \lambda_1 v_1 + c_2 \lambda_2 v_2 \in W_1.$$

Therefore W_1 is T-invariant. Also note that, by definition along with the linear independence $\{v_1, v_2\}$ is a basis for W_1, which makes it 2-dimensional, as desired. $\square$

Case: 2. No distinct real eigenvalue and at least one irreducible (over $\mathbb{R}$) quadratic factor (i.e., *the* eigenvalue has multiplicity $< n$) of the characteristic polynomial of T. The existence of this irreducible (over $\mathbb{R}$) quadratic factor is due to the fact that we are working with the real coefficients, so the complex roots comes in conjugate pair.

Suppose, $m(\lambda)$ is a polynomial of smallest degree k such that T satisfies it[1]:

$$m(T) = \mathbf{O}_{n\times n}$$

Consider

$$S = \{j \in \mathbb{N} : \exists p \in \mathcal{P}_j(\mathbb{R}),\ p(T) = \mathbf{O}_{n\times n}\}$$

By Cayley-Hamilton theorem, we can find a polynomial of degree n which is satisfied by the matrix T. So S is non-empty. By well-ordering principle then the smallest degree polynomial $m(\lambda)$ exists.

Claim. (2)
In this case, $m(\lambda)$ will have at least one irreducible quadratic factor and at most one linear factor.

Proof. Suppose to the contrary that $m(\lambda)$ is factored into at least 2 linear factors over $\mathbb{R}$. Then we can write

$$m(\lambda) = (\lambda - a)(\lambda - b)r(\lambda)$$

for some $a, b \in \mathbb{R}$ By definition,

$$m(T) = (T - aI)(T - bI)r(T) = \mathbf{O}_{n\times n}$$

[1]Such polynomial of a matrix is called the minimal polynomial. By division algorithm in polynomials, we observe that minimal polynomial divides characteristic polynomial.

where I is the identity matrix of order n. Observe that $(T - bI)r(T)$ is a $k-1$ degree polynomial evaluated at $T \in M_n(\mathbb{R})$, thus $\neq \mathbf{O}_{n\times n}$. So there must exist non-zero $v \in \mathbb{R}^n$ such that

$$(T - bI)r(T)v \neq \mathbf{0}.$$

Then

$$m(T)v = (T - aI)(T - bI)r(T)v = (T - aI)\big((T - bI)r(T)v\big) = \mathbf{0}.$$

But this shows that $(T - bI)r(T)v$ is an eigenvector associated to the eigenvalue a. Proceeding in similar way and switching brackets (because we are working only with T and I, which are compatible with commutativity), we obtain $(T - aI)r(T)v$ is an eigenvector associated to the eigenvalue b. Which says that a, b both are roots of the characteristic polynomial of A. But the starting assumption of this case forces to conclude that $a = b$. But now this increases the multiplicity of the eigenvalue a to 2, which again contradicts our assumption.

If $m(\lambda)$ contains 0 linear factor, then it must contain at least one quadratic factor, otherwise degree will become 0.

If $m(\lambda)$ contains 1 linear factor $\lambda - a$ and no other factors, then not only a will be an eigenvalue, but note that we also have $m(T) = T - aI = \mathbf{O}_{n\times n} \implies T = aI$, and thus the characteristic polynomial of aI will become $\det(aI - \lambda I) = (a - \lambda)^n$, which contradicts our assumption about multiplicity of eigenvalues. Therefore, it will contain at least one quadratic factor as well. □

Therefore, we can assume

$$m(\lambda) = q(\lambda)r(\lambda)$$

where q, r are polynomials and q is a monic quadratic factor, say $q(\lambda) = \lambda^2 + a\lambda + b$.

By definition of minimality, $r(T) \neq \mathbf{O}_{n\times n}$, unless r is of degree 0. In that case also $r(T)$ turns out to be identity matrix, which is non-zero. Taking a level higher, we can also claim $Tr(T) \neq \mathbf{O}_{n\times n}$ for similar reason. Hence $\exists v \neq \mathbf{0} \in \mathbb{R}^n$ such that

$$Tr(T)v \neq \mathbf{0}.$$

It follows that $r(T)v \neq \mathbf{0}$ as well. We now make the following claim:

Claim. (3)
The subspace $W_2 := \operatorname{span}\{r(T)v, Tr(T)v\}$ is 2-dimensional and T-invariant.

Proof. First we show that W_2 is T-invariant. Consider any $w \in W_2$. Then $\exists c_1, c_2 \in \mathbb{R}$ such that

$$c_1 r(T)v + c_2 Tr(T)v = w.$$

Observe that,

$$\begin{aligned} T(w) = c_1 Tr(T)v + c_2 T^2 r(T)v &= -c_2 br(T)v + (c_1 - c_2 a)Tr(T)v + c_2(T^2 + aT + b)r(T)v \\ &= -c_2 br(T)v + (c_1 - c_2 a)Tr(T)v + c_2 q(T)r(T)v \\ &= -c_2 br(T)v + (c_1 - c_2 a)Tr(T)v \in W_2. \end{aligned}$$

□

To prove it is 2-dimensional, we first consider the restriction of T on W_2, say L. Note that, $\forall w \in W_2$ we have

$$q(T)w = \mathbf{0} \implies q(L)w = \mathbf{0}$$

which means $q(L)$ is 0 map on W_2. This means the minimal polynomial of L divides $q(x)$. But $q(x)$ is irreducible quadratic over $\mathbb{R}$. Hence the minimal polynomial of L is just the quadratic $q(x)$. Now suppose the spanning set of W_2 is linearly dependent. Then there exists $c \neq 0 \in \mathbb{R}$ such that

$$Tr(T)v = cr(T)v$$

which means that $\forall w \in W_2$, we have $(T - cI)w = \mathbf{0}$. Considering the restriction map we have

$$L - cI = 0$$

as a map equality, which violates the minimality of degree of the quadratic $q(x)$. Hence, it is 2-dimensional.
Case: 3. If there is no quadratic factor and same eigenvalue $a \in \mathbb{R}$ is repeated n-times. Then the characteristic polynomial is

$$p(x) = (x - a)^n$$

Therefore, by Cayley-Hamilton theorem,

$$p(T) = (T - aI)^n = \mathbf{O}_{n\times n}$$

Consider the null spaces of the following linear maps:

$$T - aI : \mathbb{R}^n \to \mathbb{R}^n; \quad (T - aI)^2 : \mathbb{R}^n \to \mathbb{R}^n$$

Claim. (4)
If $\dim N(T - aI) \geq 2$, there exists 2-dimensional subspace W_3 of it which is T-invariant.

Proof. Choose any two linearly independent vector $v_1, v_2 \in N(T - aI)$, and consider the subspace W_3 spanned by $\{v_1, v_2\}$. Then

$$(T - aI)T(v_i) = T(T - aI)v_i = T(\mathbf{0}) = \mathbf{0} \in N(T - aI)$$

for $i = 1, 2$, which shows the $T-$invariance. □

Claim. (5)
If $\dim N(T - aI) = 1$ then

$$N(T - aI) \subset N(T - aI)^2.$$

Proof. Notice that, it is clearly evident that $N(T-aI) \subseteq N(T-aI)^2$. We will prove that $N(T-aI)^2 \neq N(T - aI)$. Suppose to the contrary, $N(T - aI)^2 = N(T - aI)$. Let $v \in N(T - aI)^3$ then

$$(T - aI)^3 v = (T - aI)(T - aI)^2 v = \mathbf{0}$$

which means either $(T-aI)^2 v = \mathbf{0}$, in which case, $v \in N(T-aI)^2$, *or,* $(T-aI)^2 v \neq \mathbf{0}$. So, $(T-aI)v \neq \mathbf{0}$ as well. In this case we observe that,

$$(T - aI)^3 v = (T - aI)^2 (T - aI)v = \mathbf{0}$$

i.e., $(T - aI)v \in N(T - aI)^2 = N(T - aI)$, which also implies

$$(T - aI)^2 v = \mathbf{0}$$

which is a contradiction. Therefore by proceeding inductively, we have

$$N(T - aI) = N(T - aI)^2 = \cdots = N(T - aI)^n = \mathbb{R}^n$$

since $p(T) = (T - aI)^n = \mathbf{O}_{n\times n}$, which is impossible since $\dim N(T - aI) = 1 < n = \dim \mathbb{R}^n$. □

Therefore, $\dim N(T - aI)^2 \geq 2$. It follows that there exists 2-dimensional subspace W_4 of it which is T-invariant, similar to Claim 4. □