# Naive Set Theory

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## §1 Notion of Sets

A *set* is a collection of objects which are known as elements or members. Elements of a set can be anything, such as numbers, lines, students, fishes, and even sets.[1] Just as a box can be empty, a set can be empty as well. We denote *empty set* as $\phi$. A set containing single element called *singleton set*.

**Definition 1.1.** We shall use uppercase letters to label sets and lowercase letters to label elements in a set. And, let $A$ be set and $x$ be an object. Then, we write:

- $x \in A$ if $x$ is an element of $A$.

- $x \notin A$ if $x$ is not an element of $A$.

**Remark 1.2.** Note that, $\phi \neq \{\phi\}$ and $a \neq \{a\}$.

## §2 Basic Terminologies

Since a set is defined by its elements, we write a set by listing/defining its elements by two ways.

1. **Roster Method**

2. **Set-Builder Method**

**Definition 2.1** (Empty Set)**.** The set that has no element is called empty set. We denote the empty set as $\phi$. Furthermore, we call a set $A$ non-empty, if $A \neq \phi$, if A has atleast one elementin it.

**Remark 2.2.** Same as Remark 1.2.

**Definition 2.3** (**Subset**)**.** Let $A, B$ be two sets. We say that $A$ is a subset of $B$, denoted by $A \subseteq B$, if every element of $A$ is also an element of $B$.

**Definition 2.4** (**Equality of Sets**)**.** Let $A, B$ be two sets. We say that $A$ and $B$ are equal, denoted by $A = B$, if $A \subseteq B$ and $B \subseteq A$, i.e., every element of $A$ is also an element of $B$ and vice versa.

**Definition 2.5** (**Proper Subset**)**.** Let $A, B$ be two sets. We say that $A$ is a proper subset of $B$, denoted by $A \subset B$, if $A \subseteq B$ and $A \neq B$.

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[1] Here we are not going to Axiomatic Definition of Sets, rather taking "Naive" approach. To be precise, ZFC axioms can eliminate paradoxes and ambiguities.

# §3 Operations on Sets

In this section, we introduce various set operations that allow us generate new sets.

**Definition 3.1.** Let $A, B$ be two sets. Then,

- **Union.** The union of $A$ and $B$, denoted by $A \cup B$, is defined as:

$$A \cup B \overset{\text{def}}{=} \{x : x \in A \lor x \in B\}$$

- **Intersection.** The intersection of $A$ and $B$, denoted by $A \cap B$, is defined as:

$$A \cap B \overset{\text{def}}{=} \{x : x \in A \land x \in B\}$$

- **Difference.** The intersection of $A$ and $B$, denoted by $A \setminus B$, is defined as:

$$A \setminus B \overset{\text{def}}{=} \{x : x \in A \land x \notin B\}$$

- **Symmetric Difference.** The intersection of $A$ and $B$, denoted by $A \triangle B$, is defined as:

$$A \triangle B \overset{\text{def}}{=} (A \setminus B) \cup (B \setminus A)$$

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**Example 3.2**

Let $A = \{1, 2, 3, 4, 5\}$ and $B = \{1, 4, 6, 9\}$. Then:

- $A \cup B = \{1, 2, 3, 4, 5, 6, 9\}$

- $A \cap B = \{1, 4\}$

- $A \setminus B = \{2, 3, 5\}$

- $B \setminus A = \{6, 9\}$

- $A \triangle B = \{2, 3, 5, 6, 9\}$

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**Theorem 3.3**

Let $A, B$ and $C$ be three sets. Then:

1. *Associativity*: $A \cup (B \cup C) = (A \cup B) \cup C$ and $A \cap (B \cap C) = (A \cap B) \cap C$

2. *Distributivity*: $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

3. *Commutativity*: $A \cup B = B \cup A$ and $A \cap B = B \cap A$

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*Proof.* Left as an exercise to the reader.                                    □

**Definition 3.4** (**Disjoint Sets**)**.** Let $A, B$ be two sets. We say that $A, B$ are disjoint if $A \cap B = \phi$, i.e., $A, B$ have no element in common.

**Definition 3.5** (**Complement**)**.** Let $X$ be *Universal Set* and let $A \subseteq X$. The complement of $A$ (relative to $X$) is defined as:

$$A^c \overset{\text{def}}{=} X \setminus A = \{x \in X : x \notin A\}$$

**Remark 3.6.** It is important to define the universal set unambiguously in order to be able to talk about the complement of a set.

Also, note that, $(A^c)^c = A$.

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**Theorem 3.7** (De Morgan's Law)

Let $X$ be the universal set and let $A, B \subseteq X$. Then:

1. $(A \cup B)^c = A^c \cap B^c$

2. $(A \cap B)^c = A^c \cup B^c$

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*Proof.*     1. Let $x \in (A \cup B)^c = X \setminus (A \cup B)$. Then, $x \in X$ and $x \notin A \cup B$. Therefore, $x \notin A$ and $x \notin B \implies x \in X \setminus A$ and $x \in X \setminus B \implies x \in A^c$ and $x \in B^c \implies x \in A^c \cap B^c$. Therefore, $(A \cup B)^c \subseteq A^c \cap B^c$.

Conversely, let $y \in A^c \cap B^c$. Then, $y \in X$ and $y \notin A$ and $y \notin B$. Therefore, $y \notin A \cup B$. Hence, $y \in (A \cup B)^c \implies A^c \cap B^c \subseteq (A \cup B)^c$.

By the definition of equality of set, we conclude,

$$(A \cup B)^c = A^c \cap B^c$$

2. Using (1), we have, $(A^c \cup B^c)^c = (A^c)^c \cap (B^c)^c \implies (A^c \cup B^c)^c = A \cap B \implies [(A^c \cup B^c)^c]^c$
Which implies,
$$A^c \cup B^c = (A \cap B)^c$$

$\square$

**Definition 3.8** (**Power Set**). Let $A$ be a set. The *power set of* $A$, denoted by $\mathcal{P}(A)$ is the set of subsets of $A$.

**Remark 3.9.** Notice that,

- The power set of $A$ is also defined by $2^A$.

- $\mathcal{P}(A) \neq \phi$ for any set $A$, as *phi* $\in \mathcal{P}(A)$.

- Let $n \in \mathbb{N}$ and let $A$ be set of $n$ elements. Then, $\mathcal{P}(A)$ has exactly $2^n$ elements.

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**Example 3.10**

Look at the examples as follows:

- Let $A = \phi$. Then, $\mathcal{P}(A) = \{\phi\}$.

- Let $A = \{\phi\}$. Then, $\mathcal{P}(A) = \{\phi, \{\phi\}\} = \{\phi, A\}$.

- Let $A = \{a, b\}$. Then, $\mathcal{P}(A) = \{\phi, \{a\}, \{b\}, A\}$.

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We now introduce the notion of the cartesian product of two sets. Let us begin with the definition of an ordered pair.

**Definition 3.11** (**Ordered Pair**, *informal*). Let $A, B$ be two non-empty sets and let $a \in A, b \in B$. The *ordered pair* $(a, b)$ is a notation specifying $a$ and $b$, in that *order*.

> **Remark 3.12.** The definition is unsatisfactory because it is only descriptibe and based on intuitive sense of order (we didn't define order rigorously).
>
>     We shall now give a formal definition of the notion of an ordered pair that captures the main property of an ordered pair, which is described in Lemma 3.14.
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>     There are several ways to formally define ordered pair. We here state the one given by *Kuratowski (1921)*.

**Definition 3.13** (**Ordered Pair**, *formal*)**.** Let $A, B$ be non-empty sets and let $a \in A, b \in B$. The *ordered pair* $(a, b)$ is defined as

$$(a, b) \stackrel{\text{def}}{=} \{\{a\}, \{a, b\}\}$$

> **Lemma 3.14**
>
> Let $A, B$ be non-empty sets and let $a, x \in A$ and $b, y \in B$. Then:
>
> $$(a, b) = (x, y) \iff a = x \land b = y$$

*Proof.* Let us suppose that, $(a, b) = (x, y)$. We claim that $a = x \land b = y$. We consider two cases.

**Case I:** $(\mathbf{a} = \mathbf{b})$ Then, $(a, b) = \{\{a\}, \{a, b\}\} = \{\{a\}, \{a\}\} = \{\{a\}\}$.
    Therefore, $\{x, y\} \in \{\{x\}, \{x, y\}\} = (x, y) = (a, b) = \{\{a\}\}$.
    Which implies that, $\{x, y\} = a$. Hence, $x = y = a$, i.e., $x = a \land y = a = b$.

**Case II:** $(\mathbf{a} \neq \mathbf{b})$ In this case, first, we prove the claim:

> **Claim 3.15** — $x \neq y$ for $a \neq b$.

*Proof.* Suppose to the contrary, $x = y$. Then, $\{a, b\} \in (a, b) = (x, y) = \{\{x\}\}$. Which implies that $a = b = x$, which contradicts the hypothesis. Therefore, $x \neq y$. $\qquad\square$

Next, as $\{x\} \in (x, y) = (a, b) = \{\{a\}, \{a, b\}\}$, we have, $\{x\} = \{a\}$ as $\{x\} \neq \{a, b\}$ because $a \neq b$. Hence, $x = a$.

It remains to show that $y = b$. We have $\{x, y\} \in (x, y) = (a, b)$. Therefore, $\{x, y\} = \{a, b\}$ as, $\{x, y\} \neq \{a\}$ because, $x \neq y$. This implies that, $y = b$ as $y \neq a$ because $x \neq y$.

Therefore, $x = a$ and $y = b$.

Conversely, it's trivial that if $a = x \land b = y$, then $(a, b) = \{\{a\}, \{a, b\}\} = \{\{x\}, \{x, y\}\} = (x, y)$

$\qquad\square$

**Definition 3.16** (**Cartesian Product**)**.** Let $A, B$ be two sets. The *cartesian product* of $A$ and $B$, denoted by $A \times B$, is defined as:

$$A \times B \stackrel{\text{def}}{=} \begin{cases} \{(a, b) : a \in A, b \in B\} & \text{if} \quad A \neq \phi \land B \neq \phi \\ \phi, & \text{otherwise} \end{cases}$$

> **Example 3.17**
>
> Look at the examples as follows:
>
> - Let $A = \{1, 2\}, B = \{a, b, c\}$. Then,
>
> $$A \times B = \{(1, a), (1, b), (1, c), (2, a), (2, b), (2, c)\}$$
>
> - Let $A = \{a, b\}$. Then,
> $$A \times A = \{(a, a), (a, b), (b, a), (b, b)\}$$