Kirk optimal control theory solution manual

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JAMES S. DEMETRY

Optimal Control Theory

AN INTRODUCTION

SOLUTIONS TO SELECTED PROBLEMS

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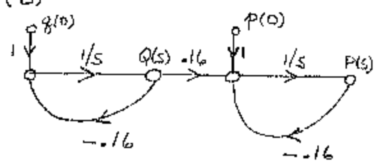
CHAPTER 1

1-1

(a) $$\frac{dq(t)}{dt} = -\frac{8}{50}\, q(t)$$

$$\frac{dp(t)}{dt} = \frac{8}{50}\, q(t) - \frac{8}{50}\, p(t).$$

(b)



(c)

$$\underset{\sim}{A} = \begin{bmatrix} -.16 & 0 \\ .16 & -.16 \end{bmatrix} \rightarrow \left[s\underset{\sim}{I} - \underset{\sim}{A}\right]^{-1} = \underset{\sim}{\Phi}(s) = \frac{\begin{bmatrix} s+.16 & 0 \\ .16 & s+.16 \end{bmatrix}}{[s+.16]^2}$$

Taking the inverse Laplace transform of each element of $\underset{\sim}{\Phi}(s)$ yields

$$\underset{\sim}{\varphi}(t) = \begin{bmatrix} \epsilon^{-.16t} & 0 \\ .16t\epsilon^{-.16t} & \epsilon^{-.16t} \end{bmatrix}.$$

(d) $$\begin{bmatrix} q(t) \\ p(t) \end{bmatrix} = \underset{\sim}{\varphi}(t)\begin{bmatrix} q(0) \\ p(0) \end{bmatrix} = \begin{bmatrix} 60\epsilon^{-.16t} \\ 9.6t\epsilon^{-.16t} \end{bmatrix}$$

1-2

(a) Application of Kirchhoff's voltage law gives

$$e(t) = R i_L(t) + L\frac{di_L(t)}{dt} + v_c(t), \qquad (\mathrm{I})$$

in addition

$$dv_c(t)/dt = \frac{1}{C}\, i_L(t). \qquad (\mathrm{II})$$

Solving (I) for $di_L(t)/dt$, we then have

$$\frac{di_L(t)}{dt} = -\frac{R}{L} i_L(t) - \frac{1}{L} v_C(t) + \frac{1}{L} e(t)$$

$$\frac{dv_C(t)}{dt} = \frac{1}{C} i_L(t).$$

(b)
$$\underset{\sim}{A} = \begin{bmatrix} -\frac{R}{L} & -\frac{1}{L} \\ \frac{1}{C} & 0 \end{bmatrix} \qquad [s\underset{\sim}{I} - \underset{\sim}{A}]^{-1} = \underset{\sim}{\Phi}(s) = \frac{\begin{bmatrix} s & -1 \\ 2 & s+3 \end{bmatrix}}{\det[s\underset{\sim}{I} - \underset{\sim}{A}]}$$

$$\underset{\sim}{\varphi}(t) = \begin{bmatrix} -\epsilon^{-t} + 2\epsilon^{-2t} & \epsilon^{-2t} - \epsilon^{-t} \\ 2[\epsilon^{-t} - \epsilon^{-2t}] & 2\epsilon^{-t} - \epsilon^{-2t} \end{bmatrix}.$$

(c) since initial conditions are zero,

$$\begin{bmatrix} I_L(s) \\ V_C(s) \end{bmatrix} = \underset{\sim}{\Phi}(s)\underset{\sim}{B}\, U(s) = \begin{bmatrix} \frac{s}{[s+1][s+2]} \\ \frac{2}{[s+1][s+2]} \end{bmatrix} \frac{2}{s}\left[\epsilon^{-s} - \epsilon^{-2s}\right]$$

$$i_L(t) = 2\left[\epsilon^{-[t-1]} - \epsilon^{-2[t-1]}\right]\mathbb{1}(t-1) - 2\left[\epsilon^{-[t-2]} - \epsilon^{-2[t-2]}\right]\mathbb{1}(t-2)$$

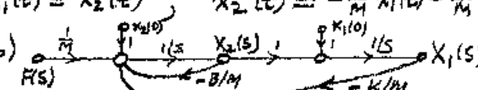
$$v_C(t) = 4\left[\tfrac{1}{2} - \epsilon^{-[t-1]} + \tfrac{1}{2}\epsilon^{-2[t-1]}\right]\mathbb{1}(t-1) - 4\left[\tfrac{1}{2} - \epsilon^{-[t-2]} + \tfrac{1}{2}\epsilon^{-2[t-2]}\right]\cdot\mathbb{1}(t-2) \quad ; \quad \mathbb{1}(z) \triangleq \begin{cases} 0 & z < 0 \\ 1 & z \geq 0 \end{cases}.$$

1-3

(a) $M\ddot{y}(t) = f(t) - Ky(t) - B\dot{y}(t)$; letting $x_1(t) \triangleq y(t)$, $x_2(t) \triangleq \dot{y}(t)$, we have

$$\dot{x}_1(t) = x_2(t), \qquad \dot{x}_2(t) = -\frac{K}{M}x_1(t) - \frac{B}{M}x_2(t) + \frac{1}{M}f(t).$$

(b)



(c) $\underset{\sim}{A} = \begin{bmatrix} 0 & 1 \\ -\frac{K}{M} & -\frac{B}{M} \end{bmatrix} \rightarrow [s\underset{\sim}{I} - \underset{\sim}{A}]^{-1} = \underset{\sim}{\Phi}(s) = \dfrac{\begin{bmatrix} s+2 & 1 \\ -2 & s \end{bmatrix}}{[s^2+2s+2]}$

$$\underset{\sim}{\varphi}(t) = \mathcal{L}^{-1}[\underset{\sim}{\Phi}(s)] = \begin{bmatrix} \sqrt{2}\, e^{-t}\cos(t-\pi/4) & e^{-t}\sin(t) \\ -2\, e^{-t}\sin(t) & \sqrt{2}\, e^{-t}\cos(t+\pi/4) \end{bmatrix}.$$

(d) Forced response is $\underset{\sim}{\Phi}(s)\underset{\sim}{B}\, U(s)$ (in s domain)

$$\underset{\sim}{\Phi}(s)\,\underset{\sim}{B}\,U(s) = \begin{bmatrix} \dfrac{2}{[s+2][s^2+2s+2]} \\[2ex] \dfrac{2s}{[s+2][s^2+2s+2]} \end{bmatrix}$$

$$\mathcal{L}^{-1}\left\{\underset{\sim}{\Phi}(s)\,\underset{\sim}{B}\,U(s)\right\} = \begin{bmatrix} e^{-2t} + \sqrt{2}\, e^{-t}\cos(t-3\pi/4) \\ -2e^{-2t} + 2e^{-t}\cos(t) \end{bmatrix}$$

The initial-condition response is given by $\underset{\sim}{x}(t) = \underset{\sim}{\varphi}(t)\,\underset{\sim}{x}(0)$, so the total response is

$$y(t) = 0.2\sqrt{2}\, e^{-t}\cos(t-\pi/4) + e^{-2t} + \sqrt{2}\, e^{-t}\cos(t-3\pi/4)$$

$$\dot{y}(t) = -0.4\, e^{-t}\sin(t) - 2e^{-2t} + 2e^{-t}\cos(t).$$

1-4 Applying Kirchhoff's voltage law (KVL) to the loop containing L, R_2, C gives

$$L\frac{di_L(t)}{dt} + R_2 i_L(t) = v_c(t) \rightarrow \frac{di_L(t)}{dt} = -\frac{R_2}{L} i_L(t) + \frac{1}{L} v_c(t)$$

Applying Kirchhoff's current law (KCL) at the junction of R_1, R_2, C gives

$$C\frac{dv_c(t)}{dt} + i_L(t) = \frac{e(t) - v_c(t)}{R_1} \rightarrow \frac{dv_c(t)}{dt} = -\frac{1}{C} i_L(t) - \frac{1}{R_1 C} v_c(t) + \frac{1}{R_1 C} e(t).$$

1-5

$$I\,\frac{d^2\theta(t)}{dt} = \lambda(t) - B\,\frac{d\theta(t)}{dt} - K\,\theta(t).$$

Letting $x_1(t) \triangleq \theta(t)$, $x_2(t) \triangleq \dot{\theta}(t)$, $u(t) \triangleq \lambda(t)$

$$\frac{d}{dt}\big(\theta(t)\big) = \dot{\theta}(t) \qquad \text{or} \qquad \dot{x}_1(t) = x_2(t)$$

$$\frac{d}{dt}\big(\dot{\theta}(t)\big) = -\frac{K}{I}\,\theta(t) - \frac{B}{I}\dot{\theta}(t) + \frac{1}{I}\lambda(t), \text{ or}$$

$$\dot{x}_2(t) = -\frac{K}{I}\,x_1(t) - \frac{B}{I}\,x_2(t) + \frac{1}{I}\,u(t).$$

1-6

$$\frac{dh_1(t)}{dt} = \frac{w_1(t)}{\alpha_1} + \frac{m(t)}{\alpha_1} - \frac{k}{\alpha_1}\big[h_1(t) - h_2(t)\big]$$

$$\frac{dh_2(t)}{dt} = \frac{w_2(t)}{\alpha_2} + \frac{k}{\alpha_2}\big[h_1(t) - h_2(t)\big]$$

$$\frac{dv_1(t)}{dt} = \begin{cases} m(t) - \dfrac{v_1(t)\,k}{\alpha_1 h_1(t)}\big[h_1(t) - h_2(t)\big] & \text{for } h_1(t) \geq h_2(t) \\ m(t) + \dfrac{v_2(t)\,k}{\alpha_2\,h_2(t)}\big[h_2(t) - h_1(t)\big] & \text{for } h_2(t) \geq h_1(t) \end{cases}$$

$$\frac{dv_2(t)}{dt} = \begin{cases} \dfrac{v_1(t)\,k}{\alpha_1\,h_1(t)}\big[h_1(t) - h_2(t)\big] & \text{for } h_1(t) \geq h_2(t) \\ -\dfrac{v_2(t)\,k}{\alpha_2\,h_2(t)}\big[h_2(t) - h_1(t)\big] & \text{for } h_2(t) \geq h_1(t) \end{cases}$$

1-7

$$K_a\,e(t) = R_f\,i_f(t) + L_f\,\frac{d}{dt}\big(i_f(t)\big)$$

$$\lambda(t) = K_t\,i_f(t) = I\,\frac{d\omega(t)}{dt} + B\,\omega(t)$$

$$\frac{di_f(t)}{dt} = -\frac{R_f}{L_f}\,i_f(t) + \frac{K_a}{L_f}\,e(t)$$

$$\frac{d\omega(t)}{dt} = \frac{K_t}{I}\,i_f(t) - \frac{B}{I}\,\omega(t).$$

1-8

Mass M_1: $f(t) = K_2 y_1(t) + B_1 \dot{y}_1(t) + K_1[y_1(t) - y_2(t)] + B_2[\dot{y}_1(t) - \dot{y}_2(t)]$
$+ M_1 \frac{d}{dt}(\dot{y}_1(t))$

Mass M_2: $0 = M_2 \frac{d}{dt}(\dot{y}_2(t)) + K_1[y_2(t) - y_1(t)] + B_2[\dot{y}_2(t) - \dot{y}_1(t)]$.

Letting $y_1(t), \dot{y}_1(t), y_2(t), \dot{y}_2(t)$ be the states we have

$$\frac{d}{dt}(y_1(t)) = \dot{y}_1(t)$$

$$\frac{d}{dt}(\dot{y}_1(t)) = -\frac{1}{M_1}[K_1 + K_2]y_1(t) - \frac{[B_1 + B_2]}{M_1}\dot{y}_1(t) + \frac{K_1}{M_1}y_2(t) + \frac{B_2}{M_1}\dot{y}_2(t) + \frac{1}{M_1}f(t)$$

$$\frac{d}{dt}(y_2(t)) = \dot{y}_2(t)$$

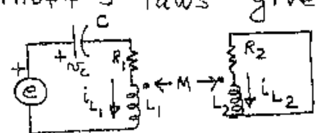
$$\frac{d}{dt}(\dot{y}_2(t)) = \frac{K_1}{M_2}y_1(t) + \frac{B_2}{M_2}\dot{y}_1(t) - \frac{K_1}{M_2}y_2(t) - \frac{B_2}{M_2}\dot{y}_2(t)$$

or, using $x_1 \triangleq y_1$, $x_2 \triangleq \dot{y}_1$, $x_3 \triangleq y_2$, $x_4 \triangleq \dot{y}_2$, $u \triangleq f$

$$\dot{\underset{\sim}{x}}(t) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{-[K_1+K_2]}{M_1} & -\frac{[B_1+B_2]}{M_1} & \frac{K_1}{M_1} & \frac{B_2}{M_1} \\ 0 & 0 & 0 & 1 \\ \frac{K_1}{M_2} & \frac{B_2}{M_2} & -\frac{K_1}{M_2} & -\frac{B_2}{M_2} \end{bmatrix} \underset{\sim}{x}(t) + \begin{bmatrix} 0 \\ 1/M_1 \\ 0 \\ 0 \end{bmatrix} u(t).$$

1-9

Application of Kirchhoff's laws gives



$$C\frac{d}{dt}\left(v_C(t)\right) = i_{L_1}(t)$$

$$L_1\frac{d}{dt}\left(i_{L_1}(t)\right) + M\frac{d}{dt}\left(i_{L_2}(t)\right) + R_1 i_{L_1}(t) + v_C(t) = e(t)$$

$$L_2\frac{d}{dt}\left(i_{L_2}(t)\right) + M\frac{d}{dt}\left(i_{L_1}(t)\right) + R_2 i_{L_2}(t) = 0.$$

Letting v_C, i_{L_1}, and i_{L_2} be the states, algebraic manipulation of these equations gives

$$\begin{bmatrix} \frac{d}{dt}\left(v_C(t)\right) \\ \frac{d}{dt}\left(i_{L_1}(t)\right) \\ \frac{d}{dt}\left(i_{L_2}(t)\right) \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{C} & 0 \\ -\frac{L_2}{k^2} & -\frac{R_1 L_2}{k^2} & \frac{MR_2}{k^2} \\ \frac{M}{k^2} & \frac{MR_1}{k^2} & -\frac{R_2}{L_2} - \frac{M^2 R_2}{L_2 k^2} \end{bmatrix} \begin{bmatrix} v_C(t) \\ i_{L_1}(t) \\ i_{L_2}(t) \end{bmatrix}$$

$$+ \begin{bmatrix} 0 \\ \frac{L_2}{k^2} \\ -\frac{M}{k^2} \end{bmatrix} e(t), \text{ where } k^2 \triangleq L_1 L_2 - M^2.$$

1-10

$$\left.\begin{aligned} R_2(t)\, i_L(t) + L\frac{di_L(t)}{dt} &= v_C(t) \\ \frac{-v_C(t) + e(t)}{R_1} &= C\frac{dv_C(t)}{dt} + f\left(v_C(t)\right) + i_L(t) \end{aligned}\right\} \text{from KVL \& KCL}$$

1-10 (continued)

$$\frac{di_L(t)}{dt} = -\frac{R_2(t)}{L} i_L(t) + \frac{1}{L} v_c(t)$$

$$\frac{dv_c(t)}{dt} = -\frac{1}{C} i_L(t) - \frac{1}{R_1 C} v_c(t) - \frac{f(v_c(t))}{C} + \frac{1}{R_1 C} e(t).$$

1-11

(i) $\underset{\sim}{x}(t) = \underset{\sim}{\varphi}(t, t_\alpha)\underset{\sim}{x}(t_\alpha)$, let $t_\alpha = t$, then $\underset{\sim}{x}(t) = \underset{\sim}{\varphi}(t,t)\underset{\sim}{x}(t)$, which implies $(\Longrightarrow)$ $\underset{\sim}{\varphi}(t,t) = \underset{\sim}{I}$ for all t.

(ii) Consider the arbitrary times t_0, t_1, t_2, and the arbitrary states $\underset{\sim}{x}(t_0), \underset{\sim}{x}(t_1), \underset{\sim}{x}(t_2)$, then we have

$\underset{\sim}{x}(t_1) = \underset{\sim}{\varphi}(t_1, t_0)\underset{\sim}{x}(t_0)$, $\underset{\sim}{x}(t_2) = \underset{\sim}{\varphi}(t_2, t_1)\underset{\sim}{x}(t_1)$,

and $\underset{\sim}{x}(t_2) = \underset{\sim}{\varphi}(t_2, t_0)\underset{\sim}{x}(t_0)$. From the first two of these equations,

$\underset{\sim}{x}(t_2) = \underset{\sim}{\varphi}(t_2, t_1)\underset{\sim}{x}(t_1) = \underset{\sim}{\varphi}(t_2, t_1)\underset{\sim}{\varphi}(t_1, t_0)\underset{\sim}{x}(t_0)$,

but $\underset{\sim}{x}(t_2) = \underset{\sim}{\varphi}(t_2, t_0)\underset{\sim}{x}(t_0)$, therefore

$\underset{\sim}{\varphi}(t_2, t_0) = \underset{\sim}{\varphi}(t_2, t_1)\underset{\sim}{\varphi}(t_1, t_0)$ for all t_0, t_1, t_2.

(iii) From (ii) with $t_0 = t_2$

$\underset{\sim}{\varphi}(t_2, t_2) = \underset{\sim}{\varphi}(t_2, t_1)\underset{\sim}{\varphi}(t_1, t_2) = \underset{\sim}{I}$ from (i), so

$\underset{\sim}{\varphi}^{-1}(t_2, t_1)\,\underset{\sim}{I} = \underset{\sim}{\varphi}^{-1}(t_2, t_1) = \underset{\sim}{\varphi}(t_1, t_2)$, for all t_1, t_2.

1-11 (cont.)

(iv) $\underset{\sim}{x}(t) = \underset{\sim}{\varphi}(t, t_0)\, \underset{\sim}{x}(t_0)$

$$\frac{d}{dt}\, \underset{\sim}{x}(t) = \dot{\underset{\sim}{x}}(t) = \underset{\sim}{\varphi}(t, t_0) \cancelto{0}{\frac{d\underset{\sim}{x}(t_0)}{dt}} + \left[\frac{d}{dt}\, \underset{\sim}{\varphi}(t, t_0)\right] \underset{\sim}{x}(t_0),$$

but $\dot{\underset{\sim}{x}}(t) = \underset{\sim}{A}(t)\, \underset{\sim}{x}(t)$, so

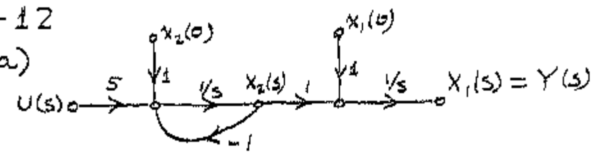
$$\underset{\sim}{A}(t)\, \underset{\sim}{x}(t) = \frac{d\underset{\sim}{\varphi}(t, t_0)}{dt}\, \underset{\sim}{x}(t_0) = \underset{\sim}{A}(t)\, \underset{\sim}{\varphi}(t, t_0)\, \underset{\sim}{x}(t_0).$$

since this holds for all $\underset{\sim}{x}(t_0)$,

$$\frac{d\, \underset{\sim}{\varphi}(t, t_0)}{dt} = \underset{\sim}{A}(t)\, \underset{\sim}{\varphi}(t, t_0).$$

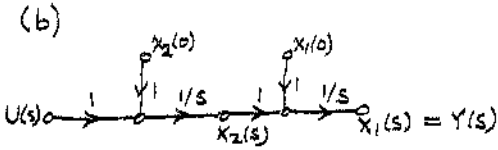
1-12

(a)



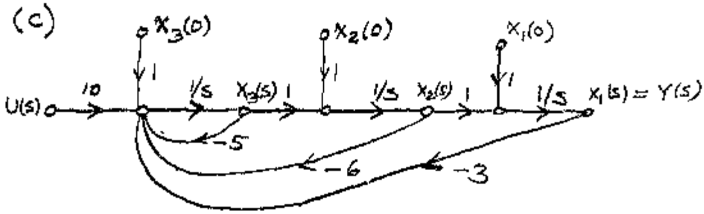
$$\dot{x}_1(t) = x_2(t), \quad \dot{x}_2(t) = -x_2(t) + 5u(t), \quad y(t) = x_1(t).$$

(b)



$$\dot{x}_1(t) = x_2(t), \quad \dot{x}_2(t) = u(t), \quad y(t) = x_1(t).$$

(c)

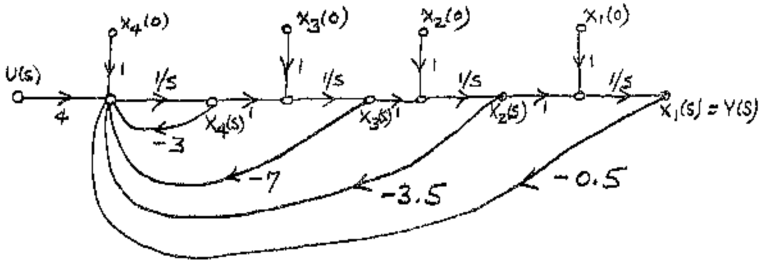


1-12 (c) (cont.)

$\dot{x}_1(t) = x_2(t)$, $\dot{x}_2(t) = x_3(t)$, $\dot{x}_3(t) = -3x_1(t) - 6x_2(t) - 5x_3(t) + 10u(t)$

$y(t) = x_1(t)$.

(d)

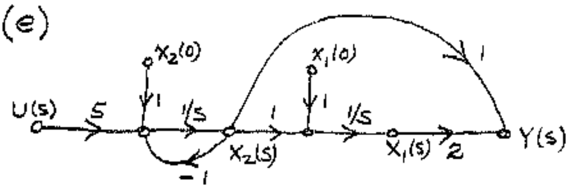


$\dot{x}_1(t) = x_2(t)$, $\dot{x}_2(t) = x_3(t)$, $\dot{x}_3(t) = x_4(t)$,

$\dot{x}_4(t) = -.5x_1(t) - 3.5x_2(t) - 7x_3(t) - 3x_4(t) + 4u(t)$,

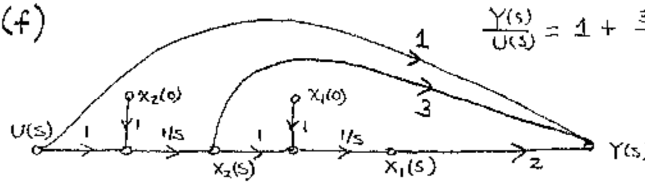
$y(t) = x_1(t)$.

(e)



$\dot{x}_1(t) = x_2(t)$, $\dot{x}_2(t) = -x_2(t) + 5u(t)$, $y(t) = 2x_1(t) + x_2(t)$; compare with part (a).

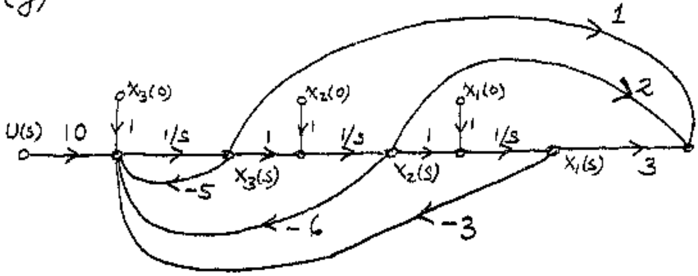
(f) $\qquad \dfrac{Y(s)}{U(s)} = 1 + \dfrac{3s+2}{s^2}$



1-12 (f) (cont.)

$\dot{x}_1(t) = x_2(t)$, $\dot{x}_2(t) = u(t)$, $y(t) = 2x_1(t) + 3x_2(t) + u(t)$.

(g)

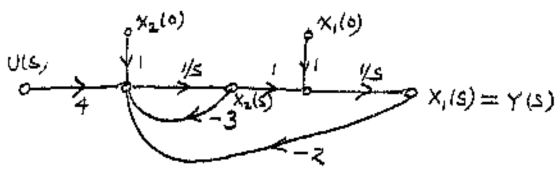


compare with part (c).

$\dot{x}_1(t) = x_2(t)$, $\dot{x}_2(t) = x_3(t)$, $\dot{x}_3(t) = -3x_1(t) - 6x_2(t) - 5x_3(t) + 10u(t)$,

$y(t) = 3x_1(t) + 2x_2(t) + x_3(t)$.

(h) *Method 1 : "phase-variable" form*

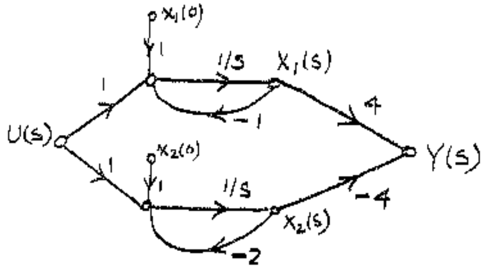


$\dot{x}_1(t) = x_2(t)$, $\dot{x}_2(t) = -2x_1(t) - 3x_2(t) + 4u(t)$,

$y(t) = x_1(t)$.

Method 2 : "canonical" or "de-coupled" form

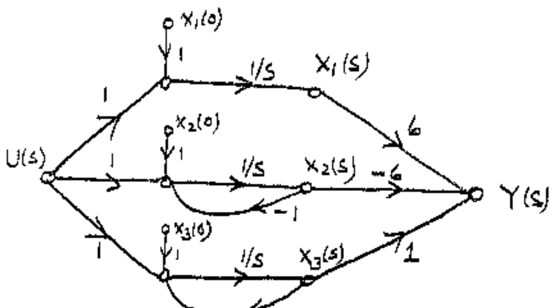
$$Y(s) = \frac{4}{s+1}\,U(s) + \frac{-4}{s+2}\,U(s)$$



For this choice of state variables

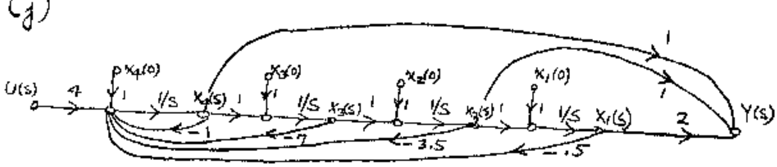
$$\dot{x}_1(t) = -x_1(t) + u(t), \quad \dot{x}_2(t) = -2x_2(t) + u(t),$$
$$y(t) = 4x_1(t) - 4x_2(t).$$

(i) $Y(s) = \left[\frac{6}{s} + \frac{-6}{s+1} + \frac{1}{s+2}\right] U(s)$



$$\dot{x}_1(t) = u(t), \ \dot{x}_2(t) = -x_2(t) + u(t), \ \dot{x}_3(t) = -2x_3(t) + u(t),$$
$$y(t) = 6x_1(t) - 6x_2(t) + x_3(t).$$

(j)



1-12 (j) (cont.)

$$\dot{x}_1(t) = x_2(t),\ \dot{x}_2(t) = x_3(t),\ \dot{x}_3(t) = x_4(t),$$

$$\dot{x}_4(t) = -.5\,x_1(t) - 3.5\,x_2(t) - 7x_3(t) - x_4(t) + 4u(t),$$

$$y(t) = 2x_1(t) + x_2(t) + x_4(t);$$

Compare with part (d).

1-13

(a) $\underset{\sim}{A} = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} \quad [s\underset{\sim}{I} - \underset{\sim}{A}] = \begin{bmatrix} s & -1 \\ 0 & s+1 \end{bmatrix}$

$$[s\underset{\sim}{I} - \underset{\sim}{A}]^{-1} = \underset{\sim}{\Phi}(s) = \begin{bmatrix} 1/s & \frac{1}{s(s+1)} \\ 0 & \frac{1}{s+1} \end{bmatrix}$$

$$\underset{\sim}{\varphi}(t) = \mathcal{L}^{-1}\left[\underset{\sim}{\Phi}(s)\right] = \begin{bmatrix} 1 & 1-\epsilon^{-t} \\ 0 & \epsilon^{-t} \end{bmatrix}.$$

(b) $\underset{\sim}{\Phi}(s) = \begin{bmatrix} \frac{1}{s} & \frac{1}{s^2} \\ 0 & \frac{1}{s} \end{bmatrix} \quad \underset{\sim}{\varphi}(t) = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}.$

(e) $\underset{\sim}{\Phi}(s)$ and $\underset{\sim}{\varphi}(t)$ are the same as in (a).

(f) $\underset{\sim}{\varphi}(t)$ same as in part (b).

(h) $\underset{\sim}{\Phi}(s) = \begin{bmatrix} \frac{1}{s+1} & 0 \\ 0 & \frac{1}{s+2} \end{bmatrix},\ \underset{\sim}{\varphi}(t) = \begin{bmatrix} \epsilon^{-t} & 0 \\ 0 & \epsilon^{-2t} \end{bmatrix}.$

This solution applies for the canonical form of the state equations.

(i) The homogeneous part of each state equation is of the form

$\dot{x}_i(t) = a_i \, x_i(t)$,

hence, the unforced solution has the form

$x_i(t) = \epsilon^{a_i t} \, x_i(0) \qquad i = 1, 2, 3.$

Therefore,

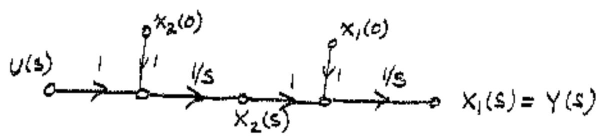
$$\underset{\sim}{\varphi}(t) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \epsilon^{-t} & 0 \\ 0 & 0 & \epsilon^{-2t} \end{bmatrix}.$$

1-14 $\underset{\sim}{E}$ and $\underset{\sim}{G}$ refer to the text definitions in Section 1.2.

(a)

$\underset{\sim}{E} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $\det(\underset{\sim}{E}) \neq 0 \Rightarrow$ system <u>controllable</u>.

$\underset{\sim}{G} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\det(\underset{\sim}{G}) \neq 0 \Rightarrow$ system <u>observable</u>.

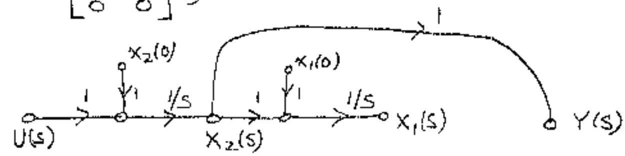


(b)

$\underset{\sim}{E} =$ same as in part (a) $\Rightarrow$ <u>controllable</u>.

1-14 (b) (cont.)

$$\underset{\sim}{G} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \det(\underset{\sim}{G}) = 0 \Rightarrow \underline{\text{not observable}}.$$



From the flowgraph, it is apparent that there is no way for $x_1(t)$ to influence the output $y(t)$.

(c) $M=0$

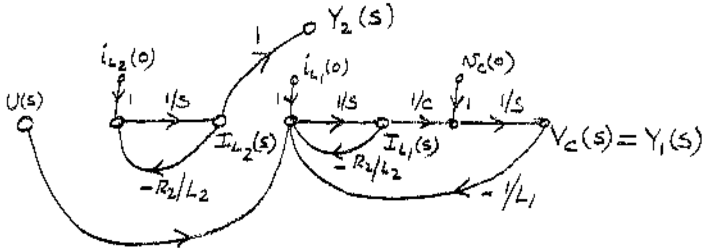
$$\underset{\sim}{E} = \begin{bmatrix} 0 & \frac{1}{CL_1} & -\frac{R_1}{L_1^2 C} \\ \frac{1}{L_1} & \frac{-R_1}{L_1^2} & \frac{R_1^2}{L_1^2} \\ 0 & 0 & 0 \end{bmatrix} \leftarrow \text{singular}$$

$\Rightarrow$ <u>not controllable</u>.

$$\underset{\sim}{G} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{C_1} & 0 \\ 0 & 1 & 0 & -\frac{R_2}{L_2} \end{bmatrix}.$$

The matrix formed by taking the first 3 columns is nonsingular; therefore $\underset{\sim}{G}$ has rank 3 and the system is <u>observable</u>.

1-14 (c) (cont.)



clearly, the control cannot influence $i_{L_2}(t)$.

(d) with the numbers specified

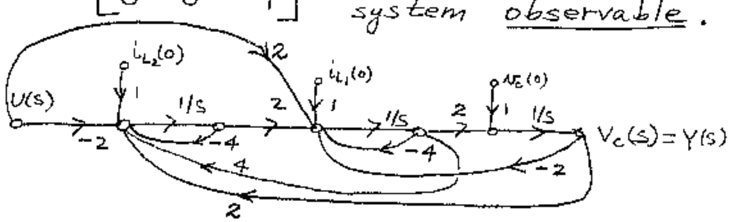
$$\underset{\sim}{A} = \begin{bmatrix} 0 & 2 & 0 \\ -2 & -4 & 2 \\ 2 & 4 & -4 \end{bmatrix} \qquad \underset{\sim}{b} = \begin{bmatrix} 0 \\ 2 \\ -2 \end{bmatrix}$$

$$\underset{\sim}{E} = \begin{bmatrix} 0 & 4 & -24 \\ 2 & -12 & 72 \\ -2 & 16 & -104 \end{bmatrix}, \quad \det(\underset{\sim}{E}) \neq 0 \Rightarrow$$

system <u>controllable</u>.

$$\underset{\sim}{G} = \begin{bmatrix} 1 & 0 & -4 \\ 0 & 2 & -8 \\ 0 & 0 & 4 \end{bmatrix}, \quad \det(\underset{\sim}{G}) \neq 0 \Longrightarrow$$

system <u>observable</u>.

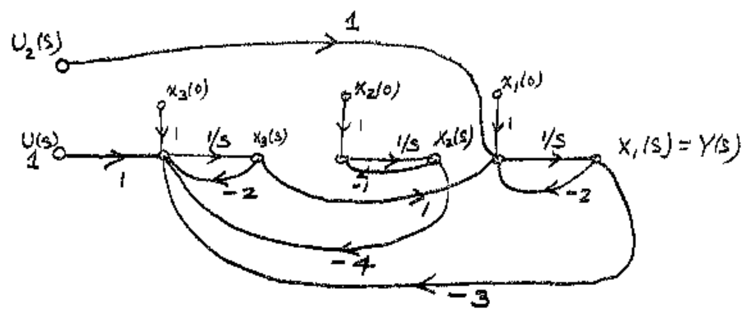


1-14 (e)

$$\underset{\sim}{E} = \begin{bmatrix} 0 & 1 & 1 & -2 & -4 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -2 & -3 & 1 & 12 \end{bmatrix}.$$

By inspection, the rank of $\underset{\sim}{E}$ is 2; therefore, the system is <u>not</u> <u>controllable</u>

$$\underset{\sim}{G} = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 0 & -4 \\ 0 & 1 & -4 \end{bmatrix}$$, Nonsingular $\Rightarrow$ <u>observable</u>.



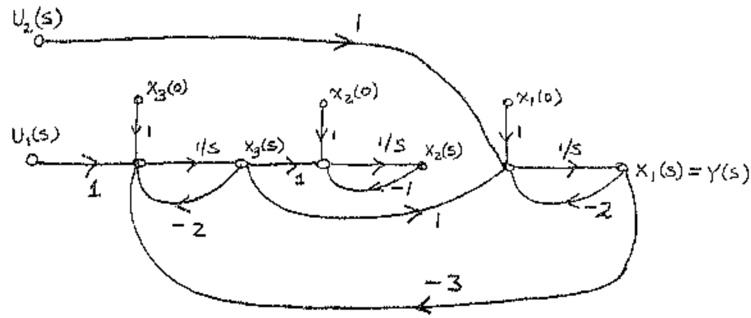
(f)

$$\underset{\sim}{E} = \begin{bmatrix} 0 & 1 & 1 & -2 & -4 & 1 \\ 0 & 0 & 1 & 0 & -3 & -3 \\ 1 & 0 & -2 & -3 & 1 & 12 \end{bmatrix}.$$

The 3x3 submatrix consisting of the first 3 rows and columns is nonsingular; therefore, the rank is 3, and the system is <u>controllable</u>.

$$\underset{\sim}{G} = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & -4 \end{bmatrix} \leftarrow \text{singular} \Rightarrow$$

<u>not</u> <u>observable</u>.



(g)

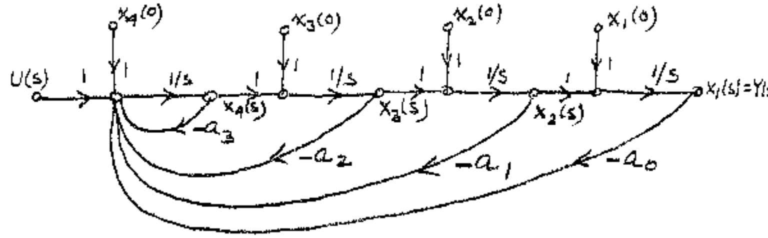
$$\underset{\sim}{E} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -a_3 \\ 0 & 1 & -a_3 & (a_3^2-a_2) \\ 1 & -a_3 & (a_3^2-a_2) & (-a_1-a_3^3+2a_2a_3) \end{bmatrix}.$$

Expanding the determinant by minors according to the elements of the first column yields

$\det(\underset{\sim}{E}) = 1 \Longrightarrow$ <u>controllable</u> <u>for all a_0, a_1, a_2, a_3.</u>

$$\underset{\sim}{G} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \text{nonsingular} \Longrightarrow \underline{\text{observable}}$$

for all a_0, a_1, a_2, a_3.



1-15

The system given in this problem is in the canonical, or normal, form. In the article "Controllability and Observability in Multivariable Control Systems", by E. G. Gilbert, in the <u>Journal of the Society for Industrial and Applied Mathematics</u> (SIAM), Series A, Vol. 1, No. 2, 1963, the definitions of controllability and observability are given in terms of a system in this canonic form. The definitions are:

1) The system [defined in Example 1-15] is controllable if $\underset{\sim}{b}$ has no zero rows, i.e., $b_i \neq 0$, $i = 1, 2, 3, 4$.

2) The system [defined in Example 1-15] is observable if the matrix $\underset{\sim}{C} = [c_1 \ c_2 \ c_3 \ c_4]$ has no zero columns, i.e., $c_i \neq 0$, $i = 1, 2, 3, 4$.

1-15 (cont.)

If the $\underset{\sim}{E}$ matrix is formed

$$\underset{\sim}{E} = \begin{bmatrix} b_1 & \lambda_1 b_1 & \lambda_1^2 b_1 & \lambda_1^3 b_1 \\ b_2 & \lambda_2 b_2 & \lambda_2^2 b_2 & \lambda_2^3 b_2 \\ b_3 & \lambda_3 b_3 & \lambda_3^2 b_3 & \lambda_3^3 b_3 \\ b_4 & \lambda_4 b_4 & \lambda_4^2 b_4 & \lambda_4^3 b_4 \end{bmatrix}$$

it is readily seen that if any of the b_i's is zero, a row of $\underset{\sim}{E}$ is zero, hence $\underset{\sim}{E}$ is singular. Also note that if the eigenvalues are not all distinct -- as assumed -- then the matrix $\underset{\sim}{E}$ is again singular because two or more rows are linearly dependent.

Similar observations apply to the matrix $\underset{\sim}{G}$, which has the same form as $\underset{\sim}{E}$, but with c_i replacing b_i.

CHAPTER 2

NOTE: Since none of the problems here require numerical answers, we have not introduced conversion factors to account for the physical units used.

2-1

(a) $J = \int_0^{1\text{ day}} [v_2(t) - M]^2 dt$, or

$$J = \int_0^{1\text{ day}} |v_2(t) - M| \, dt.$$

(b) state constraints :

(i) $\left.\begin{array}{l} 0 \le h_1(t) \le H_{1\max} \\ 0 \le h_2(t) \le H_{2\max} \end{array}\right\}$ $H_{1\max}$ and $H_{2\max}$ are the depths of the tanks

<u>OR</u>,

(ii) $\left.\begin{array}{l} 0 \le v_1(t) \le V_{1\max} \\ 0 \le v_2(t) \le V_{2\max} \end{array}\right\}$ $V_{1\max}$ and $V_{2\max}$ are the capacities of the tanks.

Notice that $V_{1\max} = \alpha_1 H_{1\max}$, $V_{2\max} = \alpha_2 H_{2\max}$, and that satisfaction of the constraints (i) implies satisfaction of the constraints (ii) and vice-versa; therefore, satisfaction of (i) or (ii) for all $t \in [0, 1]$ is sufficient.

2-1 (b) (cont.)

Control constraints:

$$0 \le w_1(t) \le W_{1max}$$
$$0 \le m(t) \le M_{max} \quad \text{for all}$$
$$0 \le w_2(t) \le W_{2max} \quad t \in [0, T]$$

$W_{1max}, W_{2max}, M_{max}$ determined by maximum flow rates.

2-2

(a) $J = -v_2(t_f)$; $t_f = 1$ day.

The minus sign converts the maximization problem to a minimization problem.

(b) Same as 2-1, but with the additional control constraint

$$\int_0^{1\,\text{day}} m(t)\,dt \le N.$$

2-3

(a) Letting $i_f(t)$ and $\omega(t)$ be the states, the state equations are

$$\frac{di_f(t)}{dt} = -\frac{R_f}{L_f}\, i_f(t) + \frac{1}{L_f}\, e(t)$$

$$\frac{d\omega(t)}{dt} = \frac{K_t}{I}\, i_f(t) - \frac{B}{I}\,\omega(t) - \frac{1}{I}\,\lambda_L(t).$$

(b) State constraints

$|i_f(t)| \le I_{f_{max}}$ Protection against overheating.

$|\omega(t)| \le \Omega_{max}$ speed limit corresponding to no load -- $\lambda_L(t) = 0$.

control constraints:

$|e(t)| \le E_{max}$ voltage limitation on regulator.

$|\lambda_L(t)| \le \lambda_{max}$ slope limitation on hills that can be safely climbed by the vehicle.

(c) (i) Power supplied by voltage regulating system proportional to $e^2(t)$:

$$J = \int_0^{t_f} \left[[k\omega(t) - 5]^2 + \mu e^2(t) \right] dt.$$

μ is a weighting factor, k is inserted to illustrate a conversion factor.

(ii) Power supplied by voltage regulating system is equal to $e(t)\, i_f(t)$:

$$J = \int_0^{t_f} \left[[k\omega(t) - 5]^2 + \mu\, e(t)\, i_f(t) \right] dt.$$

μ is a weighting factor.

2-4

(a) State constraints:

$14.9° \le \theta(30) \le 15.1°$ end point constraint.

control constraints:

$|u(t)| \le U_{max}$ limited thrust available.

2-4 (cont.)

(b) $J = \int_0^{30} |u(t)|\, dt.$

Rate of fuel expenditure is proportional to $|u(t)|$.

2-5

(a) State constraints:

$14.9^\circ \leq \theta(t_f) \leq 15.1^\circ$

Control constraints:

$|u(t)| \leq U_{max}$

There might also be a constraint on the total amount of fuel available to perform the maneuver; if so, this constraint would be

$$\int_0^{t_f} |u(t)|\, dt \leq M,$$

where M is a specified real number.

(b) $J = \int_0^{t_f} dt$ $\quad t_f$ is free --

the first time the constraint

$14.9^\circ \leq \theta(t_f) \leq 15.1^\circ$

is satisfied.

2-6

(a) (Inherent physical) constraints:

state -- $0 \leq x_1(t)$ assuming surface of the earth at zero elevation and a flat earth approximation.

$M_{min} \leq x_5(t) \leq m(t_0)$;

this is a fuel-expended constraint and could alternatively be expressed in terms of an integral involving the thrust.

Control -- $-\pi \leq u_2(t) \leq \pi$

limation on thrust angle

$0 \leq u_1(t) \leq T_{max}$.

(b) $J = -x_1(t_f)$ (J to be minimized).

An additional state constraint imposed by the problem statement is

$y(t_f) = x_3(t_f) = 3$ miles.

(c) $J = \int_0^{2.5} u_1(t)\,dt$, or $J = -x_5(t_f)$.

Additional state constraints imposed:

$x_1(t_f) = 500$ miles

$x_3(t_f) = 3$ miles.

CHAPTER 3

3-1

(a) $x_1(t+\Delta t) \approx x_1(t) + \Delta t\ x_2(t)$

$$x_2(t+\Delta t) \approx x_2(t) + \Delta t\left[-x_1(t) + [1-x_1^2(t)]x_2(t)\right] + \Delta t\ u(t).$$

Collecting terms and defining $x_1(t) = x_1(k)$, $x_2(t) = x_2(k)$, $x_1(t+\Delta t) = x_1(k+1)$, etc.

$$x_1(k+1) = x_1(k) + 0.01\, x_2(k)$$

$$x_2(k+1) = -0.01\, x_1(k) + \left[1. + 0.01[1. - x_1^2(k)]\right]x_2(k) + 0.01\, u(k)$$

$$J = [x_1(T) - 5]^2 + \int_0^{N\Delta t}[x_2^2(t) + 20[x_1(t)-5]^2 + u^2(t)]dt$$

$$= [x_1(T)-5]^2 + \int_0^{\Delta t}[\quad]dt + \int_{\Delta t}^{2\Delta t}[\quad]dt + \cdots + \int_{[N-1]\Delta t}^{N\Delta t}[\quad]dt$$

$$= [x_1(N) - 5]^2 + 0.01\sum_{k=0}^{N-1}\left\{x_2^2(k) + 20[x_1(k)-5]^2 + u^2(k)\right\},$$

$$N = 10/0.01 = 1000.$$

b. No computational adjustments required.

3-2

Following the procedure outlined in Section 3.7, we begin with a zero-stage process.

State value $x(2)$	"Minimum" Cost $J^*_{2,2}(x(2)) = \lvert x(2) \rvert$
.2	.2
.1	.1
0.	0.
-.1	.1
-.2	.2

Next, consider a one-stage process with $J^*_{1,2}(x(1)) = \min\limits_{u(1)} \left\{ \lvert x(1) \rvert + J^*_{2,2}(x(2)) \right\}$.

The trial control values are $u(1) = -.1$, $0.$, $+.1$.

State Value $x(1)$	Control value $u(1)$	Next state $x(2)$	Cost $C^*_{1,2}(x(1),u(1)) = \lvert x(1) \rvert + J^*_{2,2}(x(2))$	Min. Cost $J^*_{1,2}(x(1))$	opt. Control $u^*(x(1),1)$
.2	-.1	-.2	.2 + .2 = .4		
	0.	-.1	.2 + .1 = .3		$u^*(.2,1)$
	.1	0.	.2 + 0. = .2	.2	= .1
.1	-.1	-.15	.1 + .15 = .25		$u^*(.1,1) =$
	0.	-.05	.1 + .05 = .15	.15	0. or
	.1	.05	.1 + .05 = .15		.1
0.	-.1	-.1	0. + .1 = .1		$u^*(0,1) =$
	0.	0.	0. + 0. = 0.	0.	0.
	.1	.1	0. + .1 = .1		

$-.1$	$-.1$ $0.$ $.1$	$-.05$ $.05$ $.15$	$.1 + \begin{Bmatrix} .05 \\ .05 \\ .15 \end{Bmatrix} = \begin{matrix} .15 \\ .15 \\ .25 \end{matrix}$	$.15$	$u^*(-.1,1) = \begin{cases} -.1 \text{ or} \\ 0. \end{cases}$
$-.2$	$-.1$ $0.$ $.1$	$0.$ $.1$ $.2$	$.2 + \begin{Bmatrix} 0. \\ .1 \\ .2 \end{Bmatrix} = \begin{matrix} .2 \\ .3 \\ .4 \end{matrix}$	$.2$	$u^*(-.2,1) = -.1$

Next, we move backward one more stage, and consider the 2-stage problem with

$$J^*_{0,2}(x(0)) = \min_{u(0)} \left\{ |x(0)| + J^*_{1,2}(x(1)) \right\}.$$

State value $x(0)$	Control value $u(0)$	Next state $x(1)$	Min Cost assuming $u(0)$ applied $C^*_{0,2}(x(0),u(0))$	Min Cost $J^*_{0,2}(x(0))$	Opt. Control $u^*(x(0),0)$
$.2$	$-.1$ $0.$ $.1$	$-.2$ $-.1$ $0.$	$.2 + \begin{Bmatrix} .2 \\ .15 \\ 0. \end{Bmatrix} = \begin{matrix} .4 \\ .35 \\ .2 \end{matrix}$	$.2$	$.1$
$.1$	$-.1$ $0.$ $.1$	$-.15$ $-.05$ $.05$	$.1 + \begin{Bmatrix} .175 \\ .075 \\ .075 \end{Bmatrix} = \begin{matrix} .275 \\ .175 \\ .175 \end{matrix}$	$.175$	$0.$ or $.1$
$0.$	$-.1$ $0.$ $.1$	$-.1$ $0.$ $.1$	$0. + \begin{Bmatrix} .15 \\ 0. \\ .15 \end{Bmatrix} = \begin{matrix} .15 \\ 0. \\ .15 \end{matrix}$	$0.$	$0.$
$-.1$	$-.1$ $0.$ $.1$	$-.05$ $.05$ $.15$	$.1 + \begin{Bmatrix} .075 \\ .075 \\ .175 \end{Bmatrix} = \begin{matrix} .175 \\ .175 \\ .275 \end{matrix}$	$.175$	$-.1$ or $0.$
$-.2$	$-.1$ $0.$ $.1$	$0.$ $.1$ $.2$	$.2 + \begin{Bmatrix} 0. \\ .15 \\ .2 \end{Bmatrix} = \begin{matrix} .2 \\ .35 \\ .4 \end{matrix}$	$.2$	$-.1$

(b) $x(0) = .2 \rightarrow \underline{u^*(.2,0) = .1} \rightarrow x(1) = 0. \rightarrow \underline{u^*(0.,1) = 0.}$

If $x(2) \neq 0$, a trajectory is not admissible. One way to handle this computationally is to make $J^*_{2,2}(x(2))$ a very large number, say 10^{25}, if $x(2) \neq 0$. It will be assumed for simplicity that only the quantized control values are available.

Begin with a zero-stage process.

State Value $x(2)$	Cost $J^*_{2,2}(x(2))$
2.	10^{25}
1.	10^{25}
0.	0
−1.	10^{25}
−2.	10^{25}

Next, consider a one-stage process with $J^*_{1,2}(x(1)) = \min_{u(1)} \{ |x(1)| + 5|u(1)| + J^*_{2,2}(x(2)) \}$.

State Value $x(1)$	Control Value $u(1)$	Next state $x(2)$	Min cost assuming $u(1)$ applied $C^*_{1,2}(x(1), u(1))$	Min Cost $J^*_{1,2}(x(1))$	Opt. Control $u^*(x(1),1)$
3.	1. .5 0. −.5 −1.	2.5 2.0 1.5 1.0 0.5	$3. + 5 \times \begin{Bmatrix} 1. \\ .5 \\ 0. \\ .5 \\ 1. \end{Bmatrix} + \begin{Bmatrix} 10^{25} \\ 10^{25} \\ 10^{25} \\ 10^{25} \\ 10^{25} \end{Bmatrix}$	$10^{25}+$	None.
2.	1. .5 0. −.5 −1.	2. 1.5 1.0 0.5 0.0	$2. + 5 \times \begin{Bmatrix} 1. \\ .5 \\ 0. \\ .5 \\ 1. \end{Bmatrix} + \begin{Bmatrix} 10^{25} \\ 10^{25} \\ 10^{25} \\ 10^{25} \\ 0 \end{Bmatrix}$	7.	−1.

1.	1. .5 0. −.5 −1.	1.5 1.0 0.5 0.0 −0.5	$1. + \begin{Bmatrix} 5. \\ 2.5 \\ 0. \\ 2.5 \\ 5. \end{Bmatrix} + \begin{Bmatrix} 10^{25} \\ 10^{25} \\ 10^{25} \\ 0. \\ 10^{25} \end{Bmatrix}$	3.5	−.5
0.	1. .5 0. −.5 −1.	1. 0.5 0. −0.5 −1.	$0. + \begin{Bmatrix} 5. \\ 2.5 \\ 0. \\ 2.5 \\ 5. \end{Bmatrix} + \begin{Bmatrix} 10^{25} \\ 10^{25} \\ 0. \\ 10^{25} \\ 10^{25} \end{Bmatrix}$	0.	0.
−1.	BY SYMMETRY			3.5	.5
−2.				7.	1.
−3.				$10^{25}+$	None

Next, we consider the 2-stage process with $J^*_{0,2}(x(0)) = \min\limits_{u(0)} \left\{ |x(0)| + 5\,|u(0)| + J^*_{1,2}(x(1)) \right\}$.

State value $x(0)$	Control value $u(0)$	Next State $x(1)$	Min Cost assuming $u(0)$ applied $C^*_{0,2}(x(0), u(0))$	Min. Cost $J^*_{0,2}(x(0))$	Opt. Control $u^*(x(0),0)$
3.	1. .5 0. −.5 −1.	2.5 2.0 1.5 1.0 0.5	Not admissible (NA) $5 \times .5 + 7 + 3 = 12.5$ NA $5 \times .5 + 3.5 + 3 = 9.0$ NA	9.0	−.5
2.	1. .5 0. −.5 −1.	2. 1.5 1.0 0.5 0.0	$5 \times 1 + 7 + 2 = 14$ NA $5 \times 0 + 3.5 + 2 = 5.5$ NA $5 \times 1 + 0 + 2 = 7$	5.5	0.
1.	1. .5 0. −.5 −1.	1.5 1.0 0.5 0. −0.5	NA $5 \times .5 + 3.5 + 1 = 7$ NA $5 \times .5 + 0 + 1 = 3.5$ NA	3.5	−.5

0.	1. .5 0. −.5 −1.	1.0 0.5 0. −0.5 −1.	5×1 + 3.5 + 0 = 8.5 NA 5×0 + 0 + 0 = 0 NA 5×1 + 3.5 + 0 = 8.5	0.	0.
−1.	BY SYMMETRY			3.5	.5
−2.				5.5	0.
−3.				9.0	.5

(b) $x(0) = -2. \rightarrow u^*(-2, 0) = 0. \rightarrow x(1) = -1.$
$\rightarrow u^*(-1, 1) = 0.5.$

3-4

Start with zero-stage process, $J^*_{2,2}(x(2)) = 4|x(2)|$.

State value $x(2)$	Cost $J^*_{2,2}(x(2))$
0.0	0.0
0.5	2.0
1.0	4.0

Next, consider the 1-stage process with $J^*_{1,2}(x(1)) = \min_{u(1)} \{|u(1)| + J^*_{2,2}(x(2))\}$.

State value $x(1)$	Control value $u(1)$	Next state $x(2)$	Min Cost assuming $u(1)$ applied $C^*_{1,2}(x(1), u(1))$	Min. Cost $J^*_{1,2}(x(1))$	Opt. Control $u^*(x(1),1)$
0.	.4 .2 0. −.2 −.4	.4 .2 0. Not admissible (NA)	.4 + 1.6 = 2.0 .2 + .8 = 1.0 0. + 0. = 0. =	0.	0.

0.5	.4 .2 0. −.2 −.4	.8 .6 .4 .2 0.	$\begin{Bmatrix} .4 \\ .2 \\ 0. \\ .2 \\ .4 \end{Bmatrix} + \begin{Bmatrix} 3.2 \\ 2.4 \\ 1.6 \\ 0.8 \\ 0.0 \end{Bmatrix} = \begin{matrix} 3.6 \\ 2.6 \\ 1.6 \\ 1.0 \\ 0.4 \end{matrix}$	0.4	−.4
1.0	.4 .2 0. −.2 −.4	1.0 .8 .6 .4 .2	$\begin{Bmatrix} .4 \\ .2 \\ 0. \\ .2 \\ .4 \end{Bmatrix} + \begin{Bmatrix} 4.0 \\ 3.2 \\ 2.4 \\ 1.6 \\ 0.8 \end{Bmatrix} = \begin{matrix} 4.4 \\ 3.4 \\ 2.4 \\ 1.8 \\ 1.2 \end{matrix}$	1.2	−.4

Next, calculate $J^*_{0,2}(x(0)) = \min_{u(0)} \left\{ |u(0)| + J^*_{1,2}(x(1)) \right\}$.

State value $x(0)$	Control value $u(0)$	Next state $x(1)$	Min Cost assuming $u(0)$ applied $C^*_{0,2}(x(0),u(0))$	Min. Cost $J^*_{0,2}(x(1))$	opt. control $u^*(x(0),0)$
0.	.4 .2 0. −.2 −.4	.4 .2 0. −.2 NA −.4 NA	$\begin{Bmatrix} .4 \\ .2 \\ 0. \\ NA \\ NA \end{Bmatrix} + \begin{Bmatrix} .32 \\ .16 \\ 0. \\ \\ \end{Bmatrix} = \begin{matrix} .36 \\ .36 \\ 0. \\ — \\ — \end{matrix}$	0.	0.
0.5	.4 .2 0. −.2 −.4	.8 .6 .4 .2 0.	$\begin{Bmatrix} .4 \\ .2 \\ 0. \\ .2 \\ .4 \end{Bmatrix} + \begin{Bmatrix} .88 \\ .56 \\ .32 \\ .16 \\ 0. \end{Bmatrix} = \begin{matrix} 1.28 \\ .76 \\ .32 \\ .36 \\ .4 \end{matrix}$	.32	0.
1.0	.4 .2 0. −.2 −.4	1.0 .8 .6 .4 .2	$\begin{Bmatrix} .4 \\ .2 \\ 0. \\ .2 \\ .4 \end{Bmatrix} + \begin{Bmatrix} 1.2 \\ .88 \\ .56 \\ .32 \\ .16 \end{Bmatrix} = \begin{matrix} 1.6 \\ 1.08 \\ .56 \\ .52 \\ .56 \end{matrix}$	.52	−.2

(b) $x(0) = 1.0 \rightarrow u^*(1.0, 0) = -.2 \rightarrow x(1) = 0.4$

$\rightarrow u^*(0.4, 1) = -.32$ (using linear interpolation).

First, we consider a zero-stage process. If the point $x(2)$ does not lie in the target set, we assign a very large cost, say 10^{25}, to this point.

State Value $x(2)$	Cost $J^*_{2,2}(x(2))$
6.0	10^{25}
4.0	10^{25}
2.0	0.
0.0	0.

Next, consider a 1-stage process with $J^*_{1,2}(x(1)) = u^2(1)$. Looking at the state equations, it is apparent that positive control values will move the system away from the target set and add to the cost; therefore, only non-positive values of control will be tried. Of course, if a computer were being used, it might be just as well to include the positive control values, rather than complicate the programming required.

State value $x(1)$	Control value $u(1)$	Next state $x(2)$	Min Cost assuming $u(1)$ applied $C^*_{1,2}(x(1), u(1))$	Min. Cost $J^*_{1,2}(x(1))$	Opt. Control $u^*(x(1),1)$
6.0	0.0 −.5 −1.0	4.5 4.0 3.5	$\begin{Bmatrix} 0. \\ .25 \\ 1. \end{Bmatrix} + \begin{Bmatrix} 10^{25} \\ 10^{25} \\ 10^{25} \end{Bmatrix} = 10^{25}+$	—	None, target set not reachable
4.0	0.0 −.5 −1.0	3.0 2.5 2.0	$\begin{Bmatrix} 0. \\ .25 \\ 1. \end{Bmatrix} + \begin{Bmatrix} 10^{25} \\ 10^{25} \\ 0 \end{Bmatrix} = \begin{matrix} 10^{25}+ \\ 10^{25}+ \\ 1. \end{matrix}$	1.	−1.

2.0	0.0 -.5 -1.0	1.5 1.0 0.5	$\begin{Bmatrix} 0. \\ .25 \\ 1. \end{Bmatrix} + \begin{Bmatrix} 0. \\ 0. \\ 0 \end{Bmatrix} = \begin{matrix} 0. \\ .25 \\ 1. \end{matrix}$	0.	0.
0.0	0.0 -.5 -1.0	0.0 NA NA	$\begin{Bmatrix} 0. \\ — \\ — \end{Bmatrix} + \begin{Bmatrix} 0. \\ \\ \end{Bmatrix} = 0.$	0.	0.

Next, we calculate $J^*_{0,2}(x(0))$

$x(0)$	$u(0)$	$x(1)$	$C^*_{0,2}(x(0),u(0))$	$J^*_{0,2}(x(0))$	$u^*(x(0),0)$
6.0	0.0 -.5 -1.0	4.5(NA) 4.0 3.5	$\begin{Bmatrix} — \\ .25 \\ 1. \end{Bmatrix} + \begin{Bmatrix} \\ 1. \\ .75 \end{Bmatrix} = \begin{matrix} \\ 1.25 \\ 1.75 \end{matrix}$	1.25	-.5
4.0	0.0 -.5 -1.0	3.0 2.5 2.0	$\begin{Bmatrix} 0. \\ .25 \\ 1. \end{Bmatrix} + \begin{Bmatrix} .5 \\ .25 \\ 0. \end{Bmatrix} = \begin{matrix} .5 \\ .5 \\ 1. \end{matrix}$	.5	0. or -.5
2.0	0.0 -.5 -1.0	1.5 1.0 0.5	$\begin{Bmatrix} 0. \\ .25 \\ 1. \end{Bmatrix} + \begin{Bmatrix} 0. \\ 0. \\ 0. \end{Bmatrix} = \begin{matrix} 0. \\ .25 \\ 1. \end{matrix}$	0.	0.
0.0	0.0 -.5 -1.0	0. NA NA	$\begin{Bmatrix} 0. \\ .25 \\ 1. \end{Bmatrix} + \begin{Bmatrix} 0. \\ — \\ — \end{Bmatrix} = \begin{matrix} 0. \\ — \\ — \end{matrix}$	0.	0.

$x(0) = 6. \longrightarrow u^*(6.,0) = -.5 \longrightarrow x(1) = 4.0$

$\longrightarrow u^*(4.0,1) = -1. \longrightarrow x(2) = 2.0.$

3-6

In this problem solution we will illustrate that consideration of a zero-stage process is simply a computational convenience. This will be done by using an alternative approach. Let

$$J^*_{1,2}(x(1)) = \min_{u(1)} \left\{ 2\,|x(2) - .4| + |u(1)| \right\}.$$

NA = Not Admissible

$x(1)$	$u(1)$	$x(2)$	$C^*_{1,2}(x(1),u(1))$	$J^*_{1,2}(x(1))$	$u^*(x(1),1)$
0.	.2 .1 0. −.1 −.2	.2 .1 0. NA NA	$2\begin{Bmatrix}.2\\.3\\.4\end{Bmatrix}+\begin{Bmatrix}.2\\.1\\0.\end{Bmatrix}=\begin{matrix}.6\\.7\\.8\end{matrix}$	.6	.2
.1	.2 .1 0. −.1 −.2	.3 .2 .1 0. NA	$2\begin{Bmatrix}.1\\.2\\.3\\.4\end{Bmatrix}+\begin{Bmatrix}.2\\.1\\0.\\.1\end{Bmatrix}=\begin{matrix}.4\\.5\\.6\\.9\end{matrix}$	.4	.2
.2	.2 .1 0. −.1 −.2	.4 .3 .2 .1 0.	$2\begin{Bmatrix}0.\\.1\\.2\\.3\\.4\end{Bmatrix}+\begin{Bmatrix}.2\\.1\\0.\\.1\\.2\end{Bmatrix}=\begin{matrix}.2\\.3\\.4\\.7\\1.\end{matrix}$	.2	.2
.3	.2 .1 0. −.1 −.2	NA .4 .3 .2 .1	$2\begin{Bmatrix}0.\\.1\\.2\\.3\end{Bmatrix}+\begin{Bmatrix}.1\\0.\\.1\\.2\end{Bmatrix}=\begin{matrix}.1\\.2\\.5\\.8\end{matrix}$	.1	.1
.4	.2 .1 0. −.1 −.2	NA NA .4 .3 .2	$2\begin{Bmatrix}0.\\.1\\.2\end{Bmatrix}+\begin{Bmatrix}0.\\.1\\.2\end{Bmatrix}=\begin{matrix}0.\\.3\\.6\end{matrix}$	0.	0.

Next, compute $J^*_{0,2}(x(0))=\min\limits_{u(0)}\left\{2|x(1)-.1|+|u(0)|+J^*_{1,2}(x(1))\right\}.$

$x(0)$	$u(0)$	$x(1)$	$C^*_{0,2}(x(0),u(0))$	$J^*_{0,2}(x(0))$	$u^*(x(0),0)$
0.	.2 .1 0. −.1 −.2	.2 .1 0. NA NA	$2\begin{Bmatrix}.1\\0.\\.1\end{Bmatrix}+\begin{Bmatrix}.2\\.1\\0.\end{Bmatrix}+\begin{Bmatrix}.2\\.4\\.6\end{Bmatrix}=\begin{matrix}.6\\.5\\.8\end{matrix}$	.5	.1
.1	.2 .1 0. −.1 −.2	.3 .2 .1 0. NA	$2\begin{Bmatrix}.2\\.1\\0.\\.1\end{Bmatrix}+\begin{Bmatrix}.2\\.1\\0.\\.1\end{Bmatrix}+\begin{Bmatrix}.1\\.2\\.4\\.6\end{Bmatrix}=\begin{matrix}.7\\.5\\.4\\.9\end{matrix}$	.4	0.

.2	.2 .1 0. −.1 −.2	.4 .3 .2 .1 0.	$2\begin{Bmatrix} .3 \\ .2 \\ .1 \\ 0. \\ .1 \end{Bmatrix} + \begin{Bmatrix} .2 \\ .1 \\ 0. \\ .1 \\ .2 \end{Bmatrix} + \begin{Bmatrix} 0. \\ .1 \\ .2 \\ .4 \\ .6 \end{Bmatrix} = \begin{matrix} .8 \\ .6 \\ .4 \\ .5 \\ 1. \end{matrix}$	.4	0.
.3	.2 .1 0. −.1 −.2	NA .4 .3 .2 .1	$2\begin{Bmatrix} - \\ .3 \\ .2 \\ .1 \\ 0. \end{Bmatrix} + \begin{Bmatrix} .1 \\ 0. \\ .1 \\ .2 \end{Bmatrix} + \begin{Bmatrix} 0. \\ .1 \\ .2 \\ .4 \end{Bmatrix} = \begin{matrix} - \\ .7 \\ .5 \\ .5 \\ .6 \end{matrix}$	.5	0. or −.1
.4	.2 .1 0. −.1 −.2	NA NA .4 .3 .2	$2\begin{Bmatrix} .3 \\ .2 \\ .1 \end{Bmatrix} + \begin{Bmatrix} 0. \\ .1 \\ .2 \end{Bmatrix} + \begin{Bmatrix} 0. \\ .1 \\ .2 \end{Bmatrix} = \begin{matrix} .6 \\ .6 \\ .6 \end{matrix}$	.6	0. or −.1 or −.2

Note 1: Notice that the state value $x(0)$ is not included in the performance measure; however, if it was, only the minimum costs, not the optimal control law, would be altered.

Note 2: If the "standard" approach had been used in the solution, the only difference would have been in the intermediate costs; $J^*_{0,2}(x(0))$ and the optimal control law would be as found above.

(b) $x(0) = .2 \rightarrow u^*(.2, 0) = 0. \rightarrow x(1) = .2$

$\rightarrow u^*(.2, 1) = .2 \rightarrow x(2) = .4.$

The function

$$\mathcal{H} = \frac{1}{2}\left[g_1 x_1^2(t) + g_2 x_2^2(t) + u^2(t)\right] + \frac{\partial J^*}{\partial x_1}(\mathbf{x}(t),t)\, x_2(t)$$

$$+ \frac{\partial J^*}{\partial x_2}(\mathbf{x}(t),t)\left[-x_1(t) + x_2(t) + u(t)\right]$$

is to be minimized with respect to $u(t)$ for all $t \in [0,T]$ and $u(t)$ satisfying $|u(t)| \le 1$. The terms in $\mathcal{H}$ involving $u(t)$ are $\frac{1}{2}u^2(t) + \partial J^*/\partial x_2 \cdot u(t)$; therefore, for $|u(t)| < 1$

$$\frac{\partial}{\partial u}\left[\frac{1}{2}u^2(t) + \frac{\partial J^*}{\partial x_2}\; u(t)\right] \overset{\text{MUST}}{=} 0.$$

Solving this for $u(t)$ gives

$$u^*(t) = -J^*_{x_2}(\mathbf{x}(t),t).$$

Since $|u^*(t)| < 1$, this is valid for $|J^*_{x_2}| < 1$. If $J^*_{x_2} \ge 1$, $\mathcal{H}$ is minimized with respect to $u(t)$ by $u^*(t) = -1$; for $J^*_{x_2} \le -1$, $u^*(t) = +1$ is the minimizing choice. In summary,

$$u^*(t) = \begin{cases} -1, & J^*_{x_2}(\mathbf{x}(t),t) > 1 \\ -J^*_{x_2}(\mathbf{x}(t),t), & |J^*_{x_2}(\mathbf{x}(t),t)| \le 1 \\ +1, & J^*_{x_2}(\mathbf{x}(t),t) < -1. \end{cases}$$

3-8

$$\mathcal{H} = \frac{1}{4}x^2(t) + \frac{1}{2}u^2(t) + J^*_x(x(t),t)\cdot\left[-10x(t) + u(t)\right].$$

Minimization with respect to $u(t)$ gives

$$\frac{\partial \mathcal{H}}{\partial u} \overset{\text{set}}{=} 0 = u^*(t) + J_x^*(x(t),t) \Rightarrow u^*(t) = -J_x^*(x(t),t),$$

therefore, the H-J-B Equation is (omitting the arguments of J^*)

$$J_t^* + \frac{1}{4}x^2 + \frac{1}{2}[J_x^*]^2 - 10xJ_x^* - [J_x^*]^2 = 0.$$

Guessing a solution of the form $J^* = \frac{1}{2}K(t)x^2(t)$ (since this is a linear regulator problem) gives $J_t^* = \frac{1}{2}\dot{K}(t)x^2(t)$, $J_x^* = K(t)x(t)$, which when substituted into the H-J-B Eq. yields

$$\frac{1}{2}\dot{K}(t)x^2(t) - \frac{1}{2}K^2(t)x^2(t) - \frac{1}{4}x^2(t) - 10K(t)x^2(t) = 0.$$

Since this must be satisfied for all $x(t)$,

$$\dot{K}(t) - K^2(t) + \frac{1}{2} - 20K(t) = 0.$$

Separating variables and solving gives

$$\frac{1}{\sqrt{402}} \log_\epsilon \left[\frac{2K(t) + 20 - \sqrt{402}}{2K(t) + 20 + \sqrt{402}}\right] = t + c_1;$$

c_1 is a constant of integration. From Eq. (3.12-15) the boundary condition is $K(0.04) = 1$. Solving for c_1 leads to the final result

$$K(t) = \frac{0.421 + 0.025\,\epsilon^{-\sqrt{402}\,t}}{-0.021 + \epsilon^{-\sqrt{402}\,t}},$$

and $u^*(t) = K(t)\,x(t)$.

3-11

Augment the original state equations by

defining $x_{n+1}(t) = \int_{t_0}^{t_f} u^2(t)\,dt + x_{n+1}(t_0)$ and let $x_{n+1}(t_0)$ be zero. The $(n+1)$ st state equation then is

$$\dot{x}_{n+1}(t) = u^2(t), \qquad x_{n+1}(t_0) = 0.$$

Solve the original problem with the additional state included and the constraint

$$0 \le x_{n+1}(t) \le M \quad \text{for all } t \in [t_0, t_f].$$

3-12

(a) An appropriate recurrence equation is

$$c_{ij}^{(k+1)} = \min_{\substack{l \\ l \ne i}} \left\{ c_{il}^{(0)} + c_{lj}^{(k)} \right\}.$$

(b)
$$c_{ab}^{(1)} = \min\{1+0, 5+6, 10+3, 2+9\} = 1$$
$$c_{ac}^{(1)} = \min\{1+6, 5+0, 10+2, 2+15\} = 5$$
$$c_{ad}^{(1)} = \min\{1+3, 5+2, 10+0, 2+4\} = 4$$
$$c_{ae}^{(1)} = \min\{1+9, 5+15, 10+4, 2+0\} = 2$$

$c_{ba}^{(1)} = c_{ab}^{(1)}$ by symmetry

$$c_{bc}^{(1)} = \min\{1+5, 6+0, 3+2, 9+15\} = 5$$
$$c_{bd}^{(1)} = \min\{1+10, 6+2, 3+0, 9+4\} = 3$$
$$c_{be}^{(1)} = \min\{1+2, 6+15, 3+4, 9+0\} = 3$$

$c_{ca}^{(1)} = c_{ac}^{(1)}$, $c_{cb}^{(1)} = c_{bc}^{(1)}$ by symmetry

$$c_{cd}^{(1)} = \min\{5+10, 6+3, 2+0, 15+4\} = 2$$

$c_{ce}^{(1)} = \min\{5+2, 6+9, 2+4, 15+0\} = 6$

$c_{da}^{(1)} = c_{ad}^{(1)}$, $c_{db}^{(1)} = c_{bd}^{(1)}$, $c_{dc}^{(1)} = c_{cd}^{(1)}$ by symmetry

$c_{de}^{(1)} = \min\{10+2, 3+9, 2+15, 4+0\} = 4$

$$\underset{\sim}{C}^{(1)} = \begin{array}{c} \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \begin{bmatrix} 0 & 1 & 5 & 4 & 2 \\ 1 & 0 & 5 & 3 & 3 \\ 5 & 5 & 0 & 2 & 6 \\ 4 & 3 & 2 & 0 & 4 \\ 2 & 3 & 6 & 4 & 0 \end{bmatrix} \end{array} \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \end{array}$$

Next, find $\underset{\sim}{C}^{(2)}$:

$c_{ab}^{(2)} = \min\{1+0, 5+5, 10+3, 2+3\} = 1$

$c_{ac}^{(2)} = \min\{1+5, 5+0, 10+2, 2+6\} = 5$

$c_{ad}^{(2)} = \min\{1+3, 5+2, 10+0, 2+4\} = 4$

$c_{ae}^{(2)} = \min\{1+3, 5+6, 10+4, 2+0\} = 2$

$c_{ba}^{(2)} = c_{ab}^{(2)}$

$c_{bc}^{(2)} = \min\{1+5, 6+0, 3+2, 9+6\} = 5$

$c_{bd}^{(2)} = \min\{1+4, 6+2, 3+0, 9+4\} = 3$

$c_{be}^{(2)} = \min\{1+2, 6+6, 3+4, 9+0\} = 3$

$c_{ca}^{(2)} = c_{ac}^{(2)}$, $c_{cb}^{(2)} = c_{bc}^{(2)}$

$c_{cd}^{(2)} = \min\{5+4, 6+3, 2+0, 15+4\} = 2$

$c_{ce}^{(2)} = \min\{5+2, 6+3, 2+4, 15+0\} = 6$

$c_{da}^{(2)} = c_{ad}^{(2)}$, $c_{db}^{(2)} = c_{bd}^{(2)}$, $c_{dc}^{(2)} = c_{cd}^{(2)}$

$c_{de}^{(2)} = \min\{10+2, 3+3, 2+6, 4+0\} = 4$

$$\underset{\sim}{C}^{(2)} = \begin{array}{c} \begin{array}{ccccc} a & b & c & d & e \end{array} & \\ \begin{bmatrix} 0 & 1 & 5 & 4 & 2 \\ 1 & 0 & 5 & 3 & 3 \\ 5 & 5 & 0 & 2 & 6 \\ 4 & 3 & 2 & 0 & 4 \\ 2 & 3 & 6 & 4 & 0 \end{bmatrix} & \begin{array}{c} a \\ b \\ c \\ d \\ e \end{array} \end{array}$$

Comparing $\underset{\sim}{C}^{(1)}$ and $\underset{\sim}{C}^{(2)}$, element by element we see that $\underset{\sim}{C}^{(1)} = \underset{\sim}{C}^{(2)}$, hence all optimal paths pass through <u>at most</u> one intermediate node.

(c) Since $\underset{\sim}{C}^{(2)} = \underset{\sim}{C}^{(1)}$ the optimal paths are not improved by allowing the possibility of additional nodes.

(d) $c_{ij}^{(k+1)} \leq c_{ij}^{(k)}$ for all $i, j = 1, 2, \ldots, 4$ and for all $k = 0, 1, 2$.

(e) Essentially the same procedure is followed, but the below-diagonal elements of $\underset{\sim}{C}^{(k)}$ would also have to be computed.

In part (b), another matrix containing information sufficient to generate the optimal routes would usually be stored. In this case, the appropriate matrix, which we shall designate by $\underset{\sim}{P}$, is

$$\underset{\sim}{P} = \begin{array}{c} \begin{array}{ccccc} a & b & c & d & e \end{array} & \\ \begin{bmatrix} a & b & c & b & e \\ a & b & d & d & a \\ a & d & c & d & d \\ b & b & c & d & e \\ a & a & d & d & e \end{bmatrix} & \begin{array}{c} a \\ b \\ c \\ d \\ e \end{array} \end{array} .$$

The ijth element of $\underset{\sim}{P}$ is the first node on the optimal path from i to j which is encountered after leaving i. For example, the optimal path from a to d begins by moving from a to b. To find the sequence of nodes on the optimal path between any two nodes we use $\underset{\sim}{P}$ and the principle of optimality. For example, the optimal path from a to d begins by going to b. If the path a-b-?-?-d is to be optimal, then the segment b-?-?-d must be optimal. Looking at $\underset{\sim}{P}$ we see that the optimal path from b to d is to go directly from b to d, hence, a-b-d is the optimal path from a to d.

3-13

(a) $J_1^*(W) \triangleq w_1 v_1, \quad w_1 \leq W.$

$$
\begin{aligned}
J_2^*(W) &= \max_{\substack{w_1, w_2 \geq 0 \\ w_1 + w_2 \leq W}} \left\{ w_2 v_2 + w_1 v_1 \right\} \\
&= \max_{0 \leq w_2 \leq W} \; \max_{0 \leq w_1 \leq W - w_2} \left\{ w_2 v_2 + w_1 v_1 \right\} \\
&= \max_{0 \leq w_2 \leq W} \left\{ w_2 v_2 + \max_{0 \leq w_1 \leq W - w_2} \left[w_1 v_1 \right] \right\} \\
&= \max_{0 \leq w_2 \leq W} \left\{ w_2 v_2 + J_1^*(W - w_2) \right\}.
\end{aligned}
$$

$$J_3^*(W) = \max_{\substack{w_1, w_2, w_3 \geq 0 \\ w_1 + w_2 + w_3 \leq W}} \left\{ w_3 v_3 + w_2 v_2 + w_1 v_1 \right\}$$

$$= \max_{0 \leq w_3 \leq W} \left\{ w_3 v_3 + \max_{\substack{w_1, w_2 \\ w_1 + w_2 \leq W - w_3}} \left[w_2 v_2 + w_1 v_1 \right] \right\}$$

$$= \max_{0 \leq w_3 \leq W} \left\{ w_3 v_3 + J_2^*(W - w_3) \right\},$$

etc.

(b) In carrying out the computations for the three-activity process, we must allow the argument of J_i^* to be a variable -- at least for $i = 1, 2$ -- in the range $[0, W]$, where $W = 11{,}000$ lb.

Let automobiles be activity no. 1:

$$J_1^*(\alpha) = 0, \quad \text{for } 0 \leq \alpha < 4000$$

$$= 3000, \quad 4000 \leq \alpha < 8000$$

$$= 6000, \quad 8000 \leq \alpha \leq 11{,}000.$$

Let refrigerators be activity no. 2:

$$J_2^*(\alpha) = \max_{0 \leq w_2 \leq W} \left\{ w_2 v_2 + J_1^*(\alpha - w_2) \right\}$$

$$= \left\lceil \frac{\alpha}{400} \right\rceil \cdot 280, \quad \text{for } 0 \leq \alpha < 4000$$

$$= 3000 + \left\lceil \frac{\alpha - 4000}{400} \right\rceil \cdot 280, \quad 4000 \leq \alpha < 8000$$

$$= 6000 + \left\lceil \frac{\alpha - 8000}{400} \right\rceil \cdot 280, \quad 8000 \le \alpha \le 11{,}000,$$

where $\lceil x \rceil$ means the largest integer less than, or equal to, x.

Let sinks be activity no. 3, the maximization can be performed by looking at

$$\max_{0 \le w_3 \le 11{,}000} \left\{ w_3 v_3 + J_2^*(11{,}000 - w_3) \right\}$$

for a few "critical" values of w_3, with the result

$$J_3^*(11{,}000) = \$8060, \quad w_3 = 200$$

$$\Longrightarrow w_2 = 2800, \; w_1 = 8000.$$

(c)
$$J_1^*(\alpha) = \begin{cases} 0, & 0 \le \alpha < 4000 \\ 3000, & 4000 \le \alpha < 8000 \\ 5500, & 8000 \le \alpha < 11{,}000 \end{cases}$$

$$J_2^*(\alpha) = \left\lceil \frac{\alpha}{400} \right\rceil \cdot 280, \quad 0 \le \alpha < 4000$$

$$= 3000 + \left\lceil \frac{\alpha - 4000}{400} \right\rceil \cdot 280, \quad 4000 \le \alpha < 8000$$

$$= 5800 + \left\lceil \frac{\alpha - 8000}{400} \right\rceil \cdot 250, \quad 8000 \le \alpha \le 11{,}000.$$

Note: 1 car + 10 refrig. better than 2 cars.

Again, by inspecting a few critical values $J_3^*(W)$ is found:

$J_3^*(11{,}000) = \$7650$ and value

$w_1 = 4000$		\$3000
$w_2 = 6800$		\$4550
$w_3 = 200$		\$ 100
11,000 lb.		\$ 7650

3-14

A sampling of the results found is tabulated below. The definitions used were

$$J^*_{N-1,N}\left(x(N-1)\right) = \min_{u(N-1)} \left\{ x^2(N) + \lambda \Delta t \;\; u^2(N-1) \right\}$$

$$J^*_{N-K,N}\left(x(N-K)\right) = \min_{u(N-K)} \left\{ \lambda \Delta t \; u^2(N-K) + J^*_{N-K+1,N}\left(x(N-K+1)\right) \right\}.$$

(a)

$x(0)$	$J^*_{0,2}(x(0))$	$u^*(x(0),0)$
1.5	1.125	−0.38
1.2	0.718	−0.30
0.9	0.405	−0.22
0.6	0.180	−0.14
0.3	0.045	−0.08
0.0	0.000	0.00

$x(1)$	$J^*_{1,2}(x(1))$	$u^*(x(1),1)$
1.5	1.500	−0.50
1.2	0.960	−0.40
0.9	0.540	−0.30
0.6	0.240	−0.20
0.3	0.060	−0.10
0.0	0.000	0.00

(b)

$x(k)$	$k=0$: $J^*_{0,3}(x(0))$	$k=0$: $u^*(x(0),0)$	$k=1$: $J^*_{1,3}(x(1))$	$k=1$: $u^*(x(1),1)$	$k=2$: $J^*_{2,3}(x(2))$	$k=2$: $u^*(x(2),2)$
1.5	0.900	−0.30	1.125	−0.38	1.500	−0.50
1.2	0.576	−0.24	0.720	−0.30	0.960	−0.40
0.9	0.324	−0.18	0.405	−0.22	0.540	−0.30
0.6	0.144	−0.12	0.180	−0.14	0.240	−0.20
0.3	0.036	−0.06	0.045	−0.08	0.060	−0.10
0.0	0.000	0.00	0.000	0.00	0.000	0.00

(c)

$x(k)$	$k=0$: $J^*_{0,2}(x(0))$	$k=0$: $u^*(x(0),0)$	$k=1$: $J^*_{1,2}(x(1))$	$k=1$: $u^*(x(1),1)$
1.5	1.501	−0.24	1.800	−0.30
1.2	0.960	−0.20	1.152	−0.24
0.9	0.541	−0.14	0.648	−0.18
0.6	0.240	−0.10	0.288	−0.12
0.3	0.061	−0.06	0.072	−0.06
0.0	0.000	0.00	0.000	0.00

(d)

$x(k)$	$k=0$: $J^*_{0,2}(x(0))$	$k=0$: $u^*(x(0),0)$	$k=1$: $J^*_{1,2}(x(1))$	$k=1$: $u^*(x(1),1)$
1.5	0.450	−0.60	0.750	−1.00
1.2	0.288	−0.48	0.480	−0.80
0.9	0.162	−0.36	0.270	−0.60
0.6	0.072	−0.24	0.120	−0.40
0.3	0.018	−0.12	0.030	−0.20
0.0	0.000	0.00	0.000	0.00

3-15

(a)

$x(k)$	$J^*_{0,2}(x(0))$	$u^*(x(0),0)$	$J^*_{1,2}(x(1))$	$u^*(x(1),1)$
	$k=0$		$k=1$	
3.0	4.500	−0.74	6.000	−1.00
2.5	3.125	−0.62	4.167	−0.84
2.0	2.000	−0.50	2.667	−0.66
1.5	1.125	−0.38	1.500	−0.50
1.0	0.500	−0.24	0.667	−0.34
0.5	0.125	−0.12	0.167	−0.16
0.0	0.000	0.00	0.000	0.00

(b)

$x(k)$	$J^*_{0,3}(x(0))$	$u^*(x(0),0)$	$J^*_{1,3}(x(1))$	$u^*(x(1),1)$	$J^*_{2,3}(x(2))$	$u^*(x(2),2)$
	$k=0$		$k=1$		$k=2$	
3.0	3.600	−0.60	4.500	−0.74	6.000	−1.00
2.5	2.500	−0.50	3.125	−0.62	4.167	−0.84
2.0	1.600	−0.40	2.000	−0.50	2.667	−0.66
1.5	0.900	−0.30	1.125	−0.38	1.500	−0.50
1.0	0.400	−0.20	0.500	−0.24	0.667	−0.34
0.5	0.100	−0.10	0.125	−0.12	0.167	−0.16
0.0	0.000	0.00	0.000	0.00	0.000	0.00

(c)

$x(k)$	$J^*_{0,2}(x(0))$	$u^*(x(0),0)$	$J^*_{1,2}(x(1))$	$u^*(x(1),1)$
	$k=0$		$k=1$	
3.0	6.000	−0.50	7.200	−0.60
2.5	4.167	−0.42	5.000	−0.50
2.0	2.667	−0.34	3.200	−0.40
1.5	1.501	−0.24	1.800	−0.30
1.0	0.667	−0.16	0.800	−0.20
0.5	0.167	−0.08	0.200	−0.10
0.0	0.000	0.00	0.000	0.00

(d)

$x(k)$	$J^*_{0,2}(x(0))$ ($k=0$)	$u^*(x(0),0)$ ($k=0$)	$J^*_{1,2}(x(1))$ ($k=1$)	$u^*(x(1),1)$ ($k=1$)
3.0	2.000	−1.00	4.500	−1.00
2.5	1.250	−1.00	2.750	−1.00
2.0	0.800	−0.80	1.500	−1.00
1.5	0.450	−0.60	0.750	−1.00
1.0	0.200	−0.40	0.333	−0.66
0.5	0.050	−0.20	0.083	−0.34
0.0	0.000	0.00	0.000	0.00

3-16

(a)

$x(k)$	$J^*_{0,2}(x(0))$ ($k=0$)	$u^*(x(0),0)$ ($k=0$)	$J^*_{1,2}(x(1))$ ($k=1$)	$u^*(x(1),1)$ ($k=1$)
1.5	0.174	−0.10	0.540	−0.30
1.2	0.111	−0.08	0.346	−0.24
0.9	0.063	−0.06	0.194	−0.18
0.6	0.028	−0.04	0.086	−0.12
0.3	0.007	−0.02	0.022	−0.06
0.0	0.000	0.00	0.000	0.00

(b)

$x(k)$	$J^*_{0,3}(x(0))$ ($k=0$)	$u^*(x(0),0)$ ($k=0$)	$J^*_{1,3}(x(1))$ ($k=1$)	$u^*(x(1),1)$ ($k=1$)	$J^*_{2,3}(x(2))$ ($k=2$)	$u^*(x(2),2)$ ($k=2$)
1.5	0.061	−0.04	0.174	−0.10	0.540	−0.30
1.2	0.039	−0.02	0.111	−0.08	0.346	−0.24
0.9	0.022	−0.02	0.063	−0.06	0.194	−0.18
0.6	0.010	−0.02	0.028	−0.04	0.086	−0.12
0.3	0.003	0.00	0.007	−0.02	0.022	−0.06
0.0	0.000	0.00	0.000	0.00	0.000	0.00

3-16

(c)

$x(k)$	$J^*_{0,2}(x(0))$	$u^*(x(0),0)$	$J^*_{1,2}(x(1))$	$u^*(x(1),1)$
	$k=0$		$k=1$	
1.5	0.218	−0.06	0.648	−0.18
1.2	0.140	−0.04	0.415	−0.14
0.9	0.078	−0.04	0.234	−0.10
0.6	0.035	−0.02	0.104	−0.08
0.3	0.009	−0.02	0.026	−0.04
0.0	0.000	0.00	0.000	0.00

(d)

$x(k)$	$J^*_{0,2}(x(0))$	$u^*(x(0),0)$	$J^*_{1,2}(x(1))$	$u^*(x(1),1)$
	$k=0$		$k=1$	
1.5	0.079	−0.18	0.270	−0.60
1.2	0.050	−0.14	0.173	−0.48
0.9	0.028	−0.10	0.097	−0.36
0.6	0.013	−0.06	0.043	−0.24
0.3	0.003	−0.04	0.011	−0.12
0.0	0.000	0.00	0.000	0.00

3-17

(a)

$x(k)$	$J^*_{0,2}(x(0))$	$u^*(x(0),0)$	$J^*_{1,2}(x(1))$	$u^*(x(1),1)$
	$k=0$		$k=1$	
3.0	0.694	−0.20	2.160	−0.60
2.5	0.482	−0.16	1.500	−0.50
2.0	0.309	−0.12	0.960	−0.40
1.5	0.174	−0.10	0.540	−0.30
1.0	0.077	−0.06	0.240	−0.20
0.5	0.020	−0.04	0.060	−0.10
0.0	0.000	0.00	0.000	0.00

(b)

$x(k)$	$J^*_{0,3}(x(0))$	$u^*(x(0),0)$	$J^*_{1,3}(x(1))$	$u^*(x(1),1)$	$J^*_{2,3}(x(2))$	$u^*(x(2),2)$
	$k=0$		$k=1$		$k=2$	
3.0	0.241	−0.06	0.694	−0.20	2.160	−0.60
2.5	0.167	−0.06	0.482	−0.16	1.500	−0.50
2.0	0.107	−0.04	0.309	−0.12	0.960	−0.40
1.5	0.061	−0.04	0.174	−0.10	0.540	−0.30
1.0	0.027	−0.02	0.077	−0.06	0.240	−0.20
0.5	0.007	−0.02	0.020	−0.04	0.060	−0.10
0.0	0.000	0.00	0.000	0.00	0.000	0.00

(c)

$x(k)$	$J^*_{0,2}(x(0))$	$u^*(x(0),0)$	$J^*_{1,2}(x(1))$	$u^*(x(1),1)$
	$k=0$		$k=1$	
3.0	0.870	−0.12	2.592	−0.36
2.5	0.605	−0.10	1.800	−0.30
2.0	0.387	−0.08	1.152	−0.24
1.5	0.218	−0.06	0.648	−0.18
1.0	0.097	−0.04	0.288	−0.12
0.5	0.024	−0.02	0.072	−0.06
0.0	0.000	0.00	0.000	0.00

(d)

$x(k)$	$J^*_{0,2}(x(0))$	$u^*(x(0),0)$	$J^*_{1,2}(x(1))$	$u^*(x(1),1)$
	$k=0$		$k=1$	
3.0	0.314	−0.34	1.140	−1.00
2.5	0.218	−0.30	0.750	−1.00
2.0	0.139	−0.24	0.480	−0.80
1.5	0.079	−0.18	0.270	−0.60
1.0	0.035	−0.12	0.120	−0.40
0.5	0.009	−0.06	0.030	−0.20
0.0	0.000	0.00	0.000	0.00

Defining a performance measure

$$J = x^2(1),$$

which will be zero if $x(1)=0$ and positive otherwise, we find

$x(0)$	$J^*_{0,1}(x(0))$	$u^*(x(0),0)$
3.0	4.000	——
2.5	2.250	——
2.0	1.000	——
1.5	0.250	——
1.0	0.000	−1.00
0.5	0.000	−0.50
0.0	0.000	0.00

Of course this problem can be worked by inspection.

3-19

The optimal control sequences and the resulting final states for various values of λ are shown below.

λ	$u^*(0)$	$u^*(1)$	$x(2)$
0.5	−0.60	−0.60	0.30
2.0	−0.38	−0.38	0.74
4.0	−0.24	−0.26	1.00

(a) As λ increases — indicating more concern over control effort — the control magnitudes decrease.

(b) As λ increases the increased premium on control effort causes less concern for the final state value, hence the magnitude of $x(2)$ increases.

3-20

a	$J^*_{0,2}$	$u^*(0)$	$u^*(1)$	$x(2)$
0.0	1.125	−0.38	−0.38	0.74
−0.4	0.174	−0.10	−0.16	0.32

(a) With $a = -0.4$ the system moves toward the origin (the desired state) even with no control applied. Thus, the system is easier to control -- the final state is closer to zero with less expenditure of control effort.

(b) With $a = 0.4$ the system difference equation is

$$x(k+1) = 1.4\,x(k) + u(k).$$

This system is unstable and will move away from the origin if no control is applied. The cost of controlling this system will be greater than for $a = 0$ or $a = -0.4$ and the final value of x should be greater with $a = 0.4$.

3-21

(a) $u^*(0) = -0.5 \rightarrow x(1) = 2.0 \rightarrow u^*(1) = -0.5 \rightarrow$
$x(2) = 1.5 \rightarrow u^*(2) = -0.5 \rightarrow x(3) = 1.0$

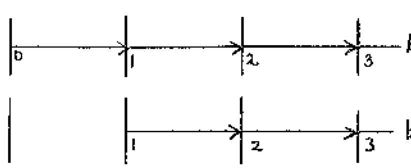
$J^*_{0,3}(2.5) = 2.5.$

(b) $u^*(0) = -0.5 \rightarrow x(1)$ should be 2.0, but actually is $2.1 \rightarrow u^*(1) = -0.52 \rightarrow x(2) = 1.58$ $\rightarrow u^*(2) = -0.52 \rightarrow x(3) = 1.06$.

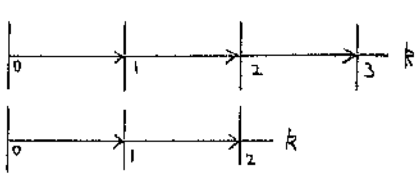
3-22

(a) The optimal controls and minimum costs for the final two stages of the three-stage process in 3-14(b) are identical with those of the two-stage process in 3-14(a); hence, the solution of 3-14(a) is contained, or "imbedded" in the solution of 3-14(b).

(b) Only if certain conditions are met. For example, the solution of a two-stage process beginning at $k=1$ is imbedded in the solution of a three-stage process beginning at $k=0$, as indicated by the sketches below.



Imbedding principle holds for time-varying process



Imbedding principle doesn't hold for time-varying process

CHAPTER 4

4-1

This theorem can be proved by following the argument used in proving the fundamental theorem of the calculus of variations in section 4.1.

4-2

Suppose that $h(t)$ is non-zero, say positive, at some point in the interval $[t_0, t_f]$. Since h is a continuous function, then there exists an interval $[t_1, t_2]$ contained in $[t_0, t_f]$ in which h is positive. Let us select

$$\delta x(t) = \begin{cases} (t-t_1)(t_2-t) & \text{for } t \in [t_1, t_2] \\ 0 & \text{for other } t \in [t_0, t_f]. \end{cases}$$

Clearly this choice for δx is a continuous function. Now, for this choice of δx

$$\int_{t_0}^{t_f} h(t)\,\delta x(t)\,dt = \int_{t_1}^{t_2} h(t)\,\delta x(t)\,dt$$

which is positive because the integrand of the right side is always positive, except at t_0 and t_1, where it is zero. Thus, if $h(t)$

is not identically zero in $[t_0, t_1]$, then

$$\int_{t_0}^{t_f} h(t)\,\delta x(t)\,dt$$

will not be zero for all continuous functions δx.

4-3

(a) $f(t+\Delta t) - f(t) = 4(t+\Delta t)^3 + \dfrac{5}{t+\Delta t}$
$\qquad - 4t^3 - 5/t$

$$(t+\Delta t)^3 = t^3 + 3t^2\,\Delta t + 3t\,(\Delta t)^2 + (\Delta t)^3$$

$$\frac{5}{t+\Delta t} = \frac{5}{t}\cdot\frac{1}{1+\frac{\Delta t}{t}} = \frac{5}{t}\left[1 - \frac{\Delta t}{t} + \frac{(\Delta t)^2}{t^2} - \cdots\right]$$

$$\therefore\ f(t+\Delta t) - f(t) = 12t^2\,\Delta t + 12t\,(\Delta t)^2 + 4(\Delta t)^3 - 5\Delta t/t^2 + \cdots.$$

Separating terms that are linear in Δt

$$df(t,\Delta t) = \left[12t^2 - 5/t^2\right]\Delta t.$$

(b) $f(\underset{\sim}{q} + \Delta \underset{\sim}{q}) - f(\underset{\sim}{q}) = 5[q_1 + \Delta q_1]^2$
$\quad + 6[q_1+\Delta q_1][q_2+\Delta q_2] + 2[q_2+\Delta q_2]^2$
$\quad - 5q_1^2 - 6q_1 q_2 - 2q_2^2$

$= 5[q_1^2 + 2q_1\Delta q_1 + (\Delta q_1)^2] + 6[q_1 q_2 + q_2\Delta q_1$
$\quad + q_1\Delta q_2 + \Delta q_1 \Delta q_2] + 2[q_2^2 + 2q_2\Delta q_2 + (\Delta q_2)^2]$
$\quad - 5q_1^2 - 6q_1 q_2 - 2q_2^2.$

Retaining only the terms that are linear in $\Delta q_1, \Delta q_2$ gives

$$df(\mathbf{q},\Delta\mathbf{q}) = [10q_1 + 6q_2]\Delta q_1 + [6q_1 + 4q_2]\Delta q_2.$$

(c) Following the same procedure as in (a) and (b) gives

$$df(\mathbf{q},\Delta\mathbf{q}) = [2q_1 + 5q_2q_3 + 2q_2]\Delta q_1 + [2q_2 + 5q_1q_3 + 2q_1]\Delta q_2 + [5q_1q_2 + 3]\Delta q_3.$$

4-4

(a) Expanding the integrand of $J(x+\delta x) - J(x)$ and retaining only the linear terms yields

$$\delta J(x,\delta x) = \int_{t_0}^{t_f}\{[3x^2(t) - 2x(t)\dot{x}(t)]\delta x(t) - x^2(t)\,\delta\dot{x}(t)\}\,dt$$

NOTE: The same result is obtained by using Eq. (4.2-7).

(b) Again expanding the integrand of ΔJ,

$$\delta J(\mathbf{x},\delta\mathbf{x}) = \int_{t_0}^{t_f}\{[2x_1(t) + x_2(t)]\delta x_1(t) + [x_1(t) + 2x_2(t)]\delta x_2(t) + [2\dot{x}_2(t)]\delta\dot{x}_1(t) + [2\dot{x}_1(t)]\delta\dot{x}_2(t)\}\,dt.$$

(c)

$$\Delta J(x,\delta x) = \int_{t_0}^{t_f}[\epsilon^{x(t)+\delta x(t)} - \epsilon^{x(t)}]\,dt$$

$$= \int_{t_0}^{t_f}[\epsilon^{\delta x(t)}\epsilon^{x(t)} - \epsilon^{x(t)}]\,dt = \int_{t_0}^{t_f}\{[1 + \delta x(t) + (\delta x(t))^2 + \cdots]\epsilon^{x(t)} - \epsilon^{x(t)}\}\,dt$$

$$\therefore\ \delta J(x,\delta x) = \int_{t_0}^{t_f}\epsilon^{x(t)}\,\delta x(t)\,dt.$$

This result is also obtained by using a Taylor series expansion.

4-5

$$J(x^*+\epsilon\eta) = \int_{t_0}^{t_f} g(x^*(t)+\epsilon\eta(t), \dot{x}^*(t)+\epsilon\dot{\eta}(t), t)\,dt.$$

$$\frac{dJ(x^*+\epsilon\eta)}{d\epsilon} = \int_{t_0}^{t_f}\Big[\frac{\partial g}{\partial x}(x^*(t)+\epsilon\eta(t), \dot{x}^*(t)+\epsilon\dot{\eta}(t), t)\,\eta(t)$$

$$+\frac{\partial g}{\partial \dot{x}}(x^*(t)+\epsilon\eta(t), \dot{x}^*(t)+\epsilon\dot{\eta}(t), t)\,\dot{\eta}(t)\Big]dt.$$

Letting $\epsilon = 0$ gives

$$\left[\frac{dJ(x^*+\epsilon\eta)}{d\epsilon}\right]_{\epsilon=0} = \int_{t_0}^{t_f}\Big[\frac{\partial g}{\partial x}(x^*(t), \dot{x}^*(t), t)\,\eta(t)$$

$$+\frac{\partial g}{\partial \dot{x}}(x^*(t), \dot{x}^*(t), t)\,\dot{\eta}(t)\Big]dt.$$

Integrating by parts, using the requirement that $\eta(t_0) = \eta(t_f) = 0$, and invoking the fundamental lemma (see problem 4-2), gives the Euler equation.

4-6

$$J_d \approx \Delta t \sum_{k=0}^{N-1} g\left(x(k), \frac{x(k+1)-x(k)}{\Delta t}, k\right).$$

$$\frac{\partial J_d}{\partial x(k)} = \Delta t\Big[\frac{\partial g}{\partial x}\left(x^*(k), \frac{x^*(k+1)-x^*(k)}{\Delta t}, k\right)$$

$$-\frac{1}{\Delta t}\frac{\partial g}{\partial \dot{x}}\left(x^*(k), \frac{x^*(k+1)-x^*(k)}{\Delta t}, k\right)$$

4-6 (cont.)

$$+\frac{1}{\Delta t}\frac{\partial g}{\partial \dot{x}}\left(x^*(k-1), \frac{x^*(k)-x^*(k-1)}{\Delta t}, k-1\right)\Bigg] = 0$$

$k = 1, 2, \ldots, N-1$.

The last two terms are a finite-difference approximation to the derivative with respect to time of $\partial g/\partial \dot{x}$; therefore, taking the limit as $\Delta t \to 0$ gives the Euler equation

$$\frac{\partial g}{\partial x}(x^*(t), \dot{x}^*(t), t) - \frac{d}{dt}\left[\frac{\partial g}{\partial \dot{x}}(x^*(t), \dot{x}^*(t), t)\right] = 0.$$

4-7

(a) $df = [t^2 + 3t + 2]\Delta t = [t+1][t+2]\Delta t \overset{set}{=} 0$.

$t_1 = -1$, $t_2 = -2$

$\frac{\partial^2 f}{\partial t^2} = 2t + 3 \qquad \frac{\partial^2 f}{\partial t^2}(t_1) = +1 \Rightarrow$

t_1 is a relative (and absolute) min

$\frac{\partial^2 f}{\partial t^2}(t_2) = -1 \Rightarrow t_2$ is a relative (and absolute) max.

(b) $\frac{\partial f}{\partial t} \overset{set}{=} 0 = \epsilon^{-2t} - 2t\epsilon^{-2t}$

$\Rightarrow 1 - 2t = 0$ or $t = 1/2$

$\frac{\partial^2 f}{\partial t^2} = -2\epsilon^{-2t} - 2\epsilon^{-2t} + 4t\epsilon^{-2t}$

$= (-4 + 4t)\epsilon^{-2t}$

$\frac{\partial^2 f}{\partial t^2}(1/2)$ is negative, therefore,

$t=1/2$ is a maximum (the global maximum); $t=0$ and $t\to\infty$ are global minima.

(c) $df = [2q_1+9+q_2]\Delta q_1 + [4q_2-1+q_1]\Delta q_2 \overset{\text{set}}{=} 0.$

Since Δq_1 & Δq_2 are independent

$2q_1^*+9+q_2^*=0, \quad 4q_2^*-1+q_1^*=0$

$\Rightarrow q_1^* = -37/7, \quad q_2^* = 11/7$

$\frac{\partial^2 f}{\partial q^2} = \begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix}$ a positive-definite matrix, so q_1^*, q_2^* is a minimum.

4-8

(a) The Euler equation is $2x^*(t) - 2\ddot{x}^*(t) = 0.$ The form of the solution -- obtained by classical methods, or Laplace transforms -- is $x^*(t) = c_1\epsilon^{-t} + c_2\epsilon^{t}.$

Now, $x^*(0) = 0 \Rightarrow c_1 + c_2 = 0$, and $x^*(1) = 1 \Rightarrow c_1\epsilon^{-1} + c_2\epsilon^{1} = 0$. Solving these two equations gives

$$x^*(t) = \frac{1}{\epsilon^{-1}-\epsilon^{1}}\left[\epsilon^{-t} - \epsilon^{t}\right].$$

(b) The Euler equation is $2x^*(t) + 2\dot{x}^*(t) - \frac{d}{dt}\left[2x^*(t) + 2\dot{x}^*(t)\right] = 0,$

4-8 (cont.)

or $\ddot{x}^*(t) - x^*(t) = 0 \Rightarrow x^*(t) = c_1\epsilon^{t} + c_2\epsilon^{-t}$.

$x^*(0) = 1 \Rightarrow c_2 = 1 - c_1$; $x^*(2) = -3 = c_1\epsilon^{2} + c_2\epsilon^{-2}$.

Solving for c_1 and c_2 gives

$$c_1 = \frac{-3-\epsilon^{-2}}{\epsilon^{2}-\epsilon^{-2}} \qquad c_2 = \frac{\epsilon^{2}+3}{\epsilon^{2}-\epsilon^{-2}}.$$

(c) The Euler equations are

$$\ddot{x}_1^*(t) - x_2^*(t) = 0, \quad -x_1^*(t) + \ddot{x}_2^*(t) = 0.$$

Differentiating the first equation twice, and adding to the second eq. gives

$$\frac{d^4 x_1^*(t)}{dt^4} - x_1^*(t) = 0.$$

This differential equation has the characteristic equation $s^4 - 1 = 0$, which has roots $s = \pm 1, \pm j1$. The solution has the form

$$x_1^*(t) = c_1\epsilon^{-t} + c_2\epsilon^{t} + c_3\cos t + c_4 \sin t.$$

Differentiating this twice and substituting into the first Euler equation yields

$$x_2^*(t) = c_1\epsilon^{-t} + c_2\epsilon^{t} - c_3\cos t - c_4\sin t.$$

From the specified boundary conditions

$x_1^*(0) = 0 \Rightarrow c_1 + c_2 + c_3 = 0$, $x_2^*(0) = 0 \Rightarrow c_1 + c_2 - c_3 = 0$

$\therefore$ $c_3 = 0$, $c_1 + c_2 = 0$.

$x_1^*(\pi/2) = 1 \Rightarrow c_1\epsilon^{-\pi/2} + c_2\epsilon^{\pi/2} + c_4 = 1$

$x_2^*(\pi/2) = c_1\epsilon^{-\pi/2} + c_2\epsilon^{\pi/2} - c_4 = 1$,

4-8 (cont.)

which imply that $c_4 = 0$. Solving for c_1 and c_2 gives $c_1 = \frac{1}{2\sinh(\pi/2)}$ and $c_2 = \frac{1}{2\sinh(\pi/2)}$; therefore,

$$x_1^*(t) = \frac{\sinh(t)}{\sinh(\pi/2)} \qquad x_2^*(t) = \frac{\sinh(t)}{\sinh(\pi/2)}.$$

4-9

The Euler equation reduces to

$$\ddot{x}^*(t) - 4x^*(t) - 4 = 0.$$

The form of the solution is

$$x^*(t) = c_1 \epsilon^{2t} + c_2 \epsilon^{-2t} + c_3.$$

c_3 can be found by substituting this solution into the original differential equation with the result that $c_3 = -1$. Using the given boundary conditions, we find that $c_2 = \frac{5 - 2\epsilon^2}{\epsilon^{-2} - \epsilon^2}$ and $c_1 = 2 - c_2$.

4-10

(a) The Euler eq. is the same as in Prob. 4-8(a), so $x^*(t) = c_1 \epsilon^{-t} + c_2 \epsilon^{t}$. At $t_f = 1$, the natural b.c. is

$$\frac{\partial g}{\partial \dot{x}}(x^*(1), \dot{x}^*(1)) = 2\dot{x}^*(1) = 0.$$

$$\therefore \quad \dot{x}^*(1) = 0 \Longrightarrow -c_1 \epsilon^{-1} + c_2 \epsilon^{1} = 0$$
$$x^*(0) = 1 \Longrightarrow \quad c_1 + c_2 = 1,$$

and the solution is

$$\frac{1}{\epsilon^{-1}+\epsilon^{1}}\left[\epsilon^{(1-t)}+\epsilon^{-(1-t)}\right]=x^*(t).$$

(b) The Euler eq. reduces to

$\ddot{x}^*(t)=1$, which when integrated twice gives $x^*(t)=\frac{1}{2}t^2+c_1t+c_2$.

The natural boundary condition is

$$\frac{\partial g}{\partial \dot{x}}(x^*(1),\dot{x}^*(1))=\dot{x}^*(1)+x^*(1)+1=0, \text{ so}$$

$1+c_1+\frac{1}{2}+c_1+c_2+1=0$; also $x^*(0)=\frac{1}{2}=c_2$.

Solving for c_1 and c_2 yields

$$x^*(t)=\frac{1}{2}t^2-\frac{3}{2}t+\frac{1}{2}.$$

(c) The Euler eqs. and their solutions are the same as in Prob. 4-8 (c); therefore,

$$x_1^*(t)=c_1\epsilon^{-t}+c_2\epsilon^{t}+c_3\cos t+c_4\sin t$$
$$x_2^*(t)=c_1\epsilon^{-t}+c_2\epsilon^{t}-c_3\cos t-c_4\sin t.$$

Three of the four b.c. from 4-8(c) also hold so

$$\left.\begin{aligned}c_1+c_2+c_3&=0\\ c_1+c_2-c_3&=0\end{aligned}\right\}\Rightarrow c_3=0,\ c_1+c_2=0$$

$$c_1\epsilon^{-\pi/2}+c_2\epsilon^{\pi/2}-c_4=1.$$

The fourth relationship is

$$\frac{\partial g}{\partial \dot{x}_1}(x^*(\pi/2),\dot{x}^*(\pi/2))=2\dot{x}_1^*(\pi/2)=0, \text{ so}$$

4-10 (cont.)

$$-c_1 \epsilon^{-\pi/2} + c_2 \epsilon^{\pi/2} = 0.$$

Solving for c_1, c_2, and c_4 leads to

$$x_1^*(t) = -\sin(t) \qquad x_2^*(t) = \sin(t).$$

4-11

Introducing variations $\delta x, \delta \dot{x}, \delta \ddot{x}, \ldots, \delta \overset{(r)}{x}$ gives

$$\delta J = \int_{t_0}^{t_f} \left[\frac{\partial g}{\partial x} \delta x + \frac{\partial g}{\partial \dot{x}} \delta \dot{x} + \cdots + \frac{\partial g}{\partial \overset{(r)}{x}} \delta \overset{(r)}{x} \right] dt$$

(omitting arguments).

Integrating by parts the terms containing derivatives of δx gives

$$\delta J = \frac{\partial g}{\partial \dot{x}} \delta x \Big|_{t_0}^{t_f} + \frac{\partial g}{\partial \ddot{x}} \delta \dot{x} \Big|_{t_0}^{t_f} + \cdots + \frac{\partial g}{\partial \overset{(r)}{x}} \delta \overset{(r-1)}{x} \Big|_{t_0}^{t_f}$$

$$+ \int_{t_0}^{t_f} \left\{ \frac{\partial g}{\partial x} \delta x - \left(\frac{d}{dt} \left[\frac{\partial g}{\partial \dot{x}} \right] \right) \delta x - \cdots - \left(\frac{d}{dt} \left[\frac{\partial g}{\partial \overset{(r)}{x}} \right] \right) \cdot \delta \overset{(r-1)}{x} \right\} dt.$$

The specified boundary conditions imply that $\delta x, \delta \dot{x}, \ldots, \delta \overset{(r-1)}{x}$ are zero at $t = t_0$ and at $t = t_f$, so the terms outside the integral are all zero. Continuing to integrate by parts, using the specified boundary conditions,

4-11 (cont.)

applying the fundamental theorem, and calling on the fundamental lemma yields the Euler eq. of rth order

$$\sum_{k=0}^{r} (-1)^k \frac{d^k}{dt^k}\left[\frac{\partial g}{\partial x^{(k)}}\left(x^*(t),\dots,\frac{d^r x^*}{dt^r}(t),t\right)\right] = 0.$$

4-12

(a) The Euler eq. is

$$\frac{d^4 x^*}{dt^4}(t) = 0 \text{ ; therefore}$$

$$x^*(t) = c_1 t^3 + c_2 t^2 + c_3 t + c_4$$

$$x^*(0)=0 \implies c_4 = 0$$
$$\dot{x}^*(0)=1 \implies c_3 = 1$$
$$x^*(1)=2 \implies c_1 + c_2 + 1 = 2$$
$$\dot{x}^*(1)=4 \implies 3c_1 + 2c_2 + 1 = 4.$$

After solving these, we have

$$x^*(t) = t^3 + t.$$

(b) The Euler eq.

$$\frac{d^4 x^*}{dt^4}(t) - 2\ddot{x}^*(t) + x^*(t) = 0$$

has the characteristic equation

$$s^4 - 2s^2 + 1$$

and characteristic roots $s = +1, +1, -1, -1.$

The solution has the form

$$x^*(t) = c_1 \epsilon^{t} + c_2 t \epsilon^{t} + c_3 \epsilon^{-t} + c_4 t \epsilon^{-t}.$$

4-12 (cont.)

Since $x^*(\infty) = \dot{x}^*(\infty) = 0$, $c_1 = 0$, $c_2 = 0$.

$x^*(0) = 1 \Longrightarrow c_3 = 1$

$\dot{x}^*(0) = 2 \Rightarrow -c_3 + c_4 = 2$.

Solving yields

$$x^*(t) = \epsilon^{-t} + 3t\,\epsilon^{-t}.$$

4-13

The solution of the Euler eq.

$\ddot{x}^*(t) = 0$

is $\quad x^*(t) = c_1 t + c_2$.

$x^*(0) = 2 \Longrightarrow c_2 = 2$,

$x^*(t_f) = c_1 t_f + 2 = -4t_f + 5$,

and, from the transversality condition Eq. (4.2-72)

$$\frac{\dot{x}^*(t_f)}{[1+\dot{x}^{*2}(t_f)]^{1/2}}\left\{-4 - \dot{x}^*(t_f)\right\} + \left[1+\dot{x}^{*2}(t_f)\right]^{1/2} = 0,$$

we obtain $\quad \dot{x}^*(t_f) = c_1 = 1/4$

$\therefore \quad x^*(t) = \frac{1}{4}t + 2$.

4-14

After simplification, the Euler eq. becomes

$$x^*(t)\,\ddot{x}^*(t) + \dot{x}^{*2}(t) = -1.$$

The left side of this nonlinear d.e. is the exact derivative of

$x^*(t)\, \dot{x}^*(t)$; therefore,

$$\frac{d}{dt}\left[x^*(t)\,\dot{x}^*(t)\right] = -1 \Longrightarrow x^*(t)\,\dot{x}^*(t) = -t + c_1.$$

But

$$\frac{d}{dt}\left[\tfrac{1}{2}x^{*2}(t)\right] = x^*(t)\ \dot{x}^*(t), \text{ so}$$

$$\tfrac{1}{2}x^{*2}(t) = -\tfrac{1}{2}t^2 + c_1 t + c_2.$$

$$x^*(0) = 0 \Longrightarrow c_2 = 0;\ x^*(t_f) = t_f - 5$$

$\Longrightarrow \frac{1}{2}(t_f - 5)^2 = -\frac{1}{2}t_f^2 + c_1 t_f$. Completing the square yields

$$t_f^2 - (5 + c_1)t_f + \tfrac{25}{2} = 0.$$

The transversality condition leads to

$$1 + \dot{x}^*(t_f) = 0.$$

Solving for c_1 and t_f gives $c_1 = 5$, $t_f = 5 \pm \sqrt{25/2}$; and

$$x^{*2}(t) = -t^2 + 10t.$$

4-15

The *Euler* eq. reduces to $\ddot{x}^*(t) = 0$, hence, $x^*(t) = c_1 t + c_2$, and $x^*(0) = 5$ implies that $c_2 = 5$.

In finding c_1, the problem is that in Eq. (4.2-65) δx_f and δt_f are related;

let us now show two ways of finding the appropriate relationship. Let

$$m(x(t),t) \triangleq x^2(t) + (t-5)^2 - 4 = 0.$$

(i) $m(x^*(t_f), t_f)$ must equal zero, as must $m(x^*(t_f)+\delta x_f, t_f+\delta t_f)$ (to first-order in δx_f, and δt_f). Expanding $m(x^*(t_f)+\delta x_f, t_f+\delta t_f)$ gives

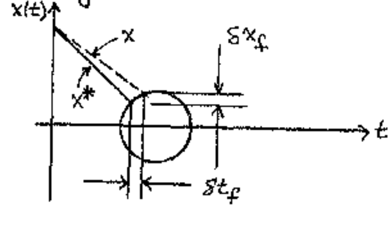
$$x^{*2}(t_f) + 2x^*(t_f)\delta x_f + [\delta x_f]^2 + t_f^2 + 2t_f\delta t_f$$
$$-10t_f + [\delta t_f]^2 - 10\delta t_f + 21.$$

Using the fact that $m(x^*(t_f), t_f) = 0$ and dropping second-order terms we have

$$2x^*(t_f)\delta x_f + 2(t_f - 5)\delta t_f = 0,$$

which is the required relationship.

(ii) Let us now consider a geometric approach; the curve $m(x(t),t)=0$, a possible extremal curve x^*, and a neighboring curve x are shown below.



From the diagram it is seen that, to first order, $\delta x_f = \alpha \delta t_f$, where α is the slope of the tangent to the circle at the point $x^*(t_f), t_f$.

Then, to find this slope we note that

$$dm = 0 = \frac{\partial m}{\partial x} \delta x_f + \frac{\partial m}{\partial t} \delta t_f,$$

or

$$2x^*(t_f)\, \delta x_f + 2(t_f - 5)\, \delta t_f = 0.$$

Either approach leads to the result

$$\delta x_f = \frac{-(t_f - 5)}{x^*(t_f)} \delta t_f.$$

We next substitute this for δx_f in the equation (4.3-18)

$$\frac{\partial g}{\partial \dot{x}}(*, t_f)\, \delta x_f + \left[g(*, t_f) - \frac{\partial g}{\partial \dot{x}}(*, t_f)\, \dot{x}^*(t_f)\right] \delta t_f$$
$$= 0.$$

Performing algebraic simplification leads to

$$(5 - t_f)\, c_1 + c_1 t_f + 5 = 0 \Rightarrow c_1 = -1$$

So,

$$x^*(t) = -t + 5.$$

4-16

From Prob. 4-14, the form of the solution to the Euler eq. is

$$\tfrac{1}{2}x^{*2}(t) = -\frac{t^2}{2} + c_1 t + c_2.$$

Since $x^*(0) = 0$, we have $c_2 = 0$.
Using the approach given in Prob. 4-15, we find that at the final time the relationship

$$2(t_f - 9)\,\delta t_f + 2x^*(t_f)\,\delta x_f = 0$$

must be satisfied. Solving this equation for δx_f, substituting into the boundary-condition equation

$$0 = \frac{\partial g}{\partial \dot{x}}(*, t_f)\,\delta x_f + \left[g(*, t_f) - \dot{x}^*(t_f)\frac{\partial g}{\partial \dot{x}}(*, t_f)\right]\delta t_f,$$

and simplifying gives

$$-\dot{x}^*(t_f)(t_f - 9) + x^*(t_f) = 0. \qquad (\mathrm{I})$$

To obtain $\dot{x}^*(t_f)$, we differentiate

$$\tfrac{1}{2}x^{*2}(t) = -\frac{t^2}{2} + c_1 t \qquad (\mathrm{II})$$

with respect to time to obtain

$$x^*(t)\,\dot{x}^*(t) = -t + c_1. \qquad (\mathrm{III})$$

Solving (III) for $\dot{x}^*(t)$, letting $t = t_f$, and substituting in (I) leads to

$$(t_f - c_1)(t_f - 9) + x^{*2}(t_f) = 0. \qquad (\mathrm{IV})$$

Using (II) in (IV) and in the constraint equation, we obtain

4-16 (cont.)

$$c_1 t_f - 9t_f + 9c_1 = 0$$

and

$$18t_f - 2c_1 t_f - 72 = 0.$$

Solving these simultaneously yields $c_1 = 4$, $t_f = 7.2$, so

$$x^{*2}(t) = -t^2 + 8t.$$

4-17

(a) From 4-8(c)

$$x_1^*(t) = c_1 \epsilon^{-t} + c_2 \epsilon^{t} + c_3 \cos t + c_4 \sin t \quad \text{(I)}$$

$$x_2^*(t) = c_1 \epsilon^{-t} + c_2 \epsilon^{t} - c_3 \cos t - c_4 \sin t. \quad \text{(II)}$$

From the specified initial conditions:

$$0 = c_1 + c_2 + c_3 \quad \text{(III)}$$

$$0 = c_1 + c_2 - c_3. \quad \text{(IV)}$$

From the specified $\underset{\sim}{\theta}(t)$:

$$5t_f + 3 = c_1 \epsilon^{-t_f} + c_2 \epsilon^{t_f} + c_3 \cos t_f + c_4 \sin t_f \quad \text{(V)}$$

$$\tfrac{1}{2} t_f^2 = c_1 \epsilon^{-t_f} + c_2 \epsilon^{t_f} - c_3 \cos t_f - c_4 \sin t_f. \quad \text{(VI)}$$

From Entry 5. of Table 4-1:

$$\left\{ g + \left[\frac{\partial g}{\partial \dot{\underset{\sim}{x}}}\right]^T \left[\frac{d\underset{\sim}{\theta}}{dt} - \dot{\underset{\sim}{x}}\right] \right\}_{\underset{\sim}{x}^*(t_f),\, \dot{\underset{\sim}{x}}^*(t_f),\, t_f} = 0.$$

Substituting $\left[\frac{\partial g}{\partial \dot{\underset{\sim}{x}}}\right]^T = \begin{bmatrix} 2\dot{x}_1 & 2\dot{x}_2 \end{bmatrix}$ and $\frac{d\underset{\sim}{\theta}}{dt} = \begin{bmatrix} 5 \\ t \end{bmatrix}$, and simplifying, yields

$$-\dot{x}_1^{*2}(t_f) - \dot{x}_2^{*2}(t_f) + 2x_1^*(t_f)\, x_2^*(t_f) + 10\dot{x}_1^*(t_f) + 2t_f\, \dot{x}_2^*(t_f) = 0 \quad \text{(VII)}$$

4-17 (cont.)

$\dot{x}_1^*$ and $\dot{x}_2^*$ are found by differentiating (I) and (II) with respect to time. Equations (III) through (VII) must be solved for the five unknowns c_1, c_2, c_3, c_4, t_f.

(b) Equations (III) and (IV) from part (a) must again be satisfied.

From the specified terminal constraint:

$$x_1^*(t_f) + 3x_2^*(t_f) + 5t_f = 15. \qquad \text{(VIII)}$$

Using the approach discussed in Prob. 4-15, we find that

$$\delta x_{1_f} + 3\delta x_{2_f} + 5\,\delta t_f = 0$$

$$\Rightarrow \delta x_{1_f} = -3\,\delta x_{2_f} - 5\,\delta t_f.$$

Substituting this expression into

$$\left\{\left[\frac{\partial g}{\partial \dot{\mathbf{x}}}\right]^T \delta \mathbf{x}_f + \left[g - \left(\frac{\partial g}{\partial \dot{\mathbf{x}}}\right)^T \dot{\mathbf{x}}\right]\delta t_f\right\}_{\mathbf{x}^*(t_f),\, \dot{\mathbf{x}}^*(t_f),\, t_f} = 0$$

and collecting terms gives an equation of the form

$$0 = f_1(\mathbf{x}^*(t_f), \dot{\mathbf{x}}^*(t_f), t_f)\,\delta x_{2_f} + f_2(\mathbf{x}^*(t_f), \dot{\mathbf{x}}^*(t_f), t_f)\,\delta t_f.$$

Since δx_{2_f} and δt_f are independently arbitrary,

$$f_1(\mathbf{x}^*(t_f), \dot{\mathbf{x}}^*(t_f), t_f) = 0 \qquad \text{(IX)}$$

$$f_2(\mathbf{x}^*(t_f), \dot{\mathbf{x}}^*(t_f), t_f) = 0. \qquad \text{(X)}$$

4-17 (cont.)

Equations (III), (IV), (VIII), (IX), and (X) must then be solved for c_1, c_2, c_3, c_4, and t_f.

NOTE: The decision to solve the eq. $\delta x_{1_f} + 3\,\delta x_{2_f} + 5\,\delta t_f = 0$ for δx_{1_f} is arbitrary, we could have solved for δx_{2_f}, or δt_f and proceeded in the same way.

4-18

The solution of the Euler eq. is

$$x^*(t) = c_1 t + c_2, \quad t \in [-2, t_1]$$
$$= c_3 t + c_4, \quad t \in [t_1, 1].$$

The boundary conditions are

$$x^*(-2) = 0 = -2c_1 + c_2$$
$$x^*(1) = 0 = c_3 + c_4$$

$$c_1 t_1 + c_2 = t_1^2 + 2$$
$$c_3 t_1 + c_4 = t_1^2 + 2$$

and, from the Weierstrass–Erdmann corner conditions,

$$\frac{2t_1 c_1 + 1}{[1 + c_1^2]^{1/2}} = \frac{2t_1 c_3 + 1}{[1 + c_3^2]^{1/2}}.$$

Solving these five eqs. simultaneously gives

4-18 (cont.)

$$x^*(t) = \begin{cases} 1.039t + 2.077, & t \in [-2, -.0696] \\ -1.874t + 1.874, & t \in [-.0696, 1]. \end{cases}$$

4-19

From the Weierstrass-Erdmann corner condition (4.4-5a)

$$\frac{\partial g}{\partial \dot{x}}\left(x^*(t_1), \dot{x}^*(t_1^-), t_1\right) = \frac{\partial g}{\partial \dot{x}}\left(x^*(t_1), \dot{x}^*(t_1^+), t_1\right)$$

$$2a\dot{x}^*(t_1^-) + bx^*(t_1) = 2a\dot{x}^*(t_1^+) + bx^*(t_1)$$

Since $a \neq 0$, this implies that

$$\dot{x}^*(t_1^-) = \dot{x}^*(t_1^+),$$

hence there can be no corners.

4-20

Since g depends only on $\dot{x}(t)$, the solution of the Euler eq. is (see Appendix 3, Case 1)

$$x^*(t) = c_1 t + c_2, \quad t \in [0, t_1]$$

$$x^*(0) = 0 \Rightarrow c_2 = 0$$

$$x^*(t) = c_3 t + c_4, \quad t \in [t_1, 4]$$

$$x^*(4) = 2 \Rightarrow 4c_3 + c_4 = 2.$$

Writing out the Weierstrass-Erdmann corner conditions, we see that

$$\dot{x}^*(t_1^-) = \pm 1 \text{ and } \dot{x}^*(t_1^+) = \pm 1$$

satisfy these conditions.

(i) Suppose $\dot{x}^*(t_1^-)=1$, $\dot{x}^*(t_1^+)=-1$, then

$$c_1 = 1, \qquad c_3 = -1, \quad 4c_3+c_4=2,$$

and $c_1t_1 = c_3t_1 + c_4$

$$\Rightarrow \quad t_1 = -t_1 + 2+4, \text{ or } 2t_1 = 6, \; t_1 = 3.$$

$$\therefore \; x^*(t) = \begin{cases} t, & t \in [0,3] \\ -t+6, & t \in [3,4]. \end{cases}$$

(ii) Suppose $\dot{x}^*(t_1^-) = -1$, $\dot{x}^*(t_1^+) = +1$, then

$$c_1 = -1, \; c_3 = +1, \; 4c_3 + c_4 = 2, \text{ and}$$

$$c_1 t_1 = c_3 t_1 + c_4$$

$$\Longrightarrow \quad -t_1 = t_1 + 2 - 4, \text{ or } t_1 = 1.$$

$$\therefore \; x^*(t) = \begin{cases} -t, & t \in [0,1] \\ t-2, & t \in [1,4]. \end{cases}$$

Both of these curves yield $J^* = 0$, which is clearly the absolute minimum of J.

4-21

$$f_a \triangleq y_1^2 + y_2^2 + p y_2 - p y_1^2 + 4.5p.$$

setting $df_a = 0$ gives

$$2y_1^* - 2y_1^* p^* = 0 \qquad (\mathrm{I})$$

$$2y_2^* + p^* = 0 \qquad (\mathrm{II})$$

$$y_2^* - y_1^{*2} + 4.5 = 0, \qquad (\mathrm{III})$$

where we have used the argument in the text to equate the coefficients of Δy_1, Δy_2, and Δp to zero.

Eq. (I) implies that $y_1^*[1-p^*]=0$, and either $y_1^*=0$, or $p^*=1$ (or both).

(i) If $y_1^*=0$, (III) gives $y_2^*=-4.5$, and from (II) $p^*=9$ (not necessary to find p^*).

(ii) If $p^*=+1$, (II) gives $y_2^*=-\frac{1}{2}$, and from (III) $y_1^{*2}=4$, $y_1^*=\pm 2$.

For (i) $f=[4.5]^2=20.25$, and for (ii)

$$f = 4+\tfrac{1}{4} = 4.25.$$

Therefore, the minimizing points are

$$y_1^* = \pm 2,\quad y_2^* = -\tfrac{1}{2}.$$

4-22

$$f_a = y_1^2+y_2^2+y_3 + p_1[y_1+y_2+y_3-5] + p_2[y_1^2+y_2^2+y_3-9].$$

From $df_a=0$, we obtain the necessary conditions

$$2y_1^*[1+p_2^*]+p_1^*=0 \qquad \text{(I)}$$

$$2y_2^*[1+p_2^*]+p_1^*=0 \qquad \text{(II)}$$

$$2y_3^*+p_1^*+p_2^*=0 \qquad \text{(III)}$$

$$y_1^* + y_2^* + y_3^* - 5 = 0 \qquad \text{(IV)}$$

$$y_1^{*2} + y_2^{*2} + y_3^* - 9 = 0. \qquad \text{(V)}$$

(I) and (II) imply $y_1^* = y_2^*$, or $p_2^* = -1$.

(i) Suppose $y_1^* = y_2^*$.

Substituting in (IV) and (V) gives

$2y_1^* + y_3^* = +5$, $2y_1^{*2} + y_3^* = 9$.

Eliminating y_3^* yields

$y_1^{*2} - y_1^* - 2 = 0 \implies y_1^* = +2$, or -1;

therefore, $y_2^* = +2$, or -1, and

$y_3^* = 5 - 2\left\{\begin{matrix} 2 \\ -1 \end{matrix}\right\} = +1$, or $+7$, and

$$\underset{\sim}{y}^* = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}, \text{ or } \underset{\sim}{y}^* = \begin{bmatrix} -1 \\ -1 \\ 7 \end{bmatrix}.$$

(ii) Suppose $p_2^* = -1$, $\implies p_1^* = 0$

$2y_3^* + p_1^* = +1$, or $y_3^* = 1/2$

$y_1^* + y_2^* = 4.5$

$y_1^{*2} + y_2^{*2} = 8.5$.

Eliminating y_1^* gives

$(4.5 - y_2^*)^2 + y_2^{*2} = 8.5$

$20.25 - 9y_2^* + 2y_2^{*2} = 8.5$.

It turns out that there are no real roots to this equation, so the only possible solutions are given in (i).
By inspection of the alternatives, $\underset{\sim}{y}^* = [2 \;\; 2 \;\; 1]^T$ is the closest point to the origin that lies on the specified constraints.

4-23

Integrating the terms in Eq. (4.5-39) that contain $\delta\dot{\underset{\sim}{w}}(t)$, we obtain (considering an extremal curve)

$$\left\{\left[\frac{\partial g}{\partial \dot{\underset{\sim}{w}}}^T(*,t) + \underset{\sim}{p}^{*T}(t)\,\frac{\partial \underset{\sim}{f}}{\partial \dot{\underset{\sim}{w}}}(*,t)\right]\delta\underset{\sim}{w}(t)\right\}_{t_0}^{t_f}$$

+ the integral term in Eq. (4.5-40).
$\delta\underset{\sim}{w}(t_0) = \underset{\sim}{0}$, so the terms at t_0 vanish.
Allowing t_f to be free introduces the additional terms

$$\left\{g(*,t_f) + \underset{\sim}{p}^{*T}(t_f)\left[\underset{\sim}{f}(*,t_f)\right]\right\}\delta t_f .$$

Finally, using the relationship

$$\delta\underset{\sim}{w}(t_f) \doteq \delta\underset{\sim}{w}_f - \dot{\underset{\sim}{w}}^*(t_f)\,\delta t_f ,$$

4-23 (cont.)

and collecting terms yields

$$\left\{\frac{\partial g^T}{\partial \dot{\mathbf{w}}}(*,t_f) + \mathbf{p}^{*T}(t_f)\frac{\partial \mathbf{f}}{\partial \dot{\mathbf{w}}}(*,t_f)\right\} \delta \mathbf{w}_f$$

$$+\left\{ g(*,t_f) + \mathbf{p}^{*T}(t_f)\, \mathbf{f}(*,t_f) - \left[\frac{\partial g^T}{\partial \dot{\mathbf{w}}}(*,t_f) + \mathbf{p}^{*T}(t_f)\frac{\partial \mathbf{f}}{\partial \dot{\mathbf{w}}}(*,t_f)\right] \dot{\mathbf{w}}^*(t_f)\right\} \delta t_f$$

+ the integral in (4.5-40) evaluated on an extremal $= \delta J_a$.

4-24

(a) $$g_a = w_1^2(t) + w_1(t)w_2(t) + w_2^2(t) + w_3^2(t) + p_1(t)\left[w_2(t) - \dot{w}_1(t)\right] + p_2(t)\left[-w_1(t) + \left(1 - w_1^2(t)\right)w_2(t) + w_3(t) - \dot{w}_2(t)\right]$$

$$2w_1^*(t) + w_2^*(t) + p_2^*(t)\left[-1 - 2w_1^*(t)w_2^*(t)\right] = -\dot{p}_1^*(t)$$

$$w_1^*(t) + 2w_2^*(t) + p_1^*(t) + p_2^*(t)\left[1 - w_1^{*2}(t)\right] = -\dot{p}_2^*(t)$$

$$2w_3^*(t) + p_2^*(t) = 0$$

and the original differential conditions are necessary conditions for optimality.

(b) $$g_a = \lambda + w_3^2(t) + p_1(t)\left[w_2(t) - \dot{w}_1(t)\right] + p_2(t)\left[w_3(t) - \dot{w}_2(t)\right]$$

$$\dot{p}_1^*(t) = 0$$

$\dot{p}_2^*(t) = -p_1^*(t)$

$2w_3^*(t) + p_2^*(t) = 0$

and the original d.e. are necessary for optimality.

(c) $g_a = \lambda + w_3^2(t) + p_1(t)\,[w_2(t) - \dot{w}_1(t)]$
$\qquad + p_2(t)\left[-w_2(t)\,|w_2(t)| + w_3(t) - \dot{w}_2(t)\right]$

$\dot{p}_1^*(t) = 0$

$\dot{p}_2^*(t) = -p_1^*(t) + 2p_2^*(t)\,|w_2^*(t)|$

$0 \;=\; -2w_3^*(t) - p_2^*(t)$

and the original d.e. equation constraints must be satisfied by an optimal trajectory.

4-25

$g_a = \dot{x}^2(t) + t^2 + p(t)\,[x^2(t) - \dot{z}(t)]$

$2x^*(t)\,p^*(t) - 2\ddot{x}^*(t) = 0$

$\dot{p}^*(t) = 0 \Longrightarrow p^*(t) = c_1$

are necessary for optimality; hence,

$\ddot{x}^*(t) - c_1 x^*(t) = 0.$

The solution of this d.e. is one of the two forms

$x^*(t) = c_3\cos(c_2 t) + c_4\sin(c_2 t),\ c_2 = \sqrt{|c_1|}$

or,

$$x^*(t) = c_2' e^{-c_2 t} + c_4' e^{c_2 t}, \quad c_2 = \sqrt{c_1}$$

where the first solution corresponds to $c_1 < 0$, and the second solution corresponds to $c_1 > 0$.

It can be shown that if $c_1 > 0$, the boundary conditions $x^*(0) = x^*(1) = 0$ require that $c_2' = c_4' = 0$, or $x^*(t) = 0$. Clearly, this solution cannot satisfy

$$\int_0^1 x^{*2}(t)\,dt = 2,$$

so c_1 must be less than zero.

Thus, $x^*(t) = c_3 \cos(c_2 t) + c_4 \sin(c_2 t)$

$x^*(0) = 0 \Rightarrow c_3 = 0$

$x^*(1) = 0 \Rightarrow c_4 \sin(c_2) = 0$, which requires that

$$c_2 = \pm n\pi, \quad n = 1, 2, \ldots$$

and

$$\int_0^1 c_4^2 \sin^2(n\pi t)\,dt = 2$$

$$= c_4^2 \left[\frac{t}{2} - \frac{\sin(2n\pi t)}{4n\pi}\right]_0^1 = \frac{c_4^2}{2} = 2$$

$c_4^2 = 4 \quad c_4 = \pm 2$

$$x^*(t) = \pm 2 \sin(n\pi t), \quad n = 1, 2, \ldots.$$

4-26

$$J = \frac{1}{2}\int_0^T \left[\ddot{w}_1^2(t) + \ddot{w}_2^2(t) + \ddot{w}_3^2(t)\right] dt.$$

$$g_a = \frac{1}{2}\left[\ddot{w}_1^2(t) + \ddot{w}_2^2(t) + \ddot{w}_3^2(t)\right]$$
$$+ p(t)\ f(w_1(t), w_2(t), w_3(t))$$

$$\ddot{w}_1^*(t) = p^*(t)\frac{\partial f}{\partial w_1}(*, t)$$
$$\ddot{w}_2^*(t) = p^*(t)\frac{\partial f}{\partial w_2}(*, t)$$
$$\ddot{w}_3^*(t) = p^*(t)\frac{\partial f}{\partial w_3}(*, t).$$

Therefore,

$$p^*(t) = \frac{\ddot{w}_1^*(t)}{\frac{\partial f}{\partial w_1}(*, t)} = \frac{\ddot{w}_2^*(t)}{\frac{\partial f}{\partial w_2}(*, t)} = \frac{\ddot{w}_3^*(t)}{\frac{\partial f}{\partial w_3}(*, t)}.$$

CHAPTER 5

5-1

(a) Let the x and y coordinates of the boat be the state variables, then

$$\dot{x}(t) = V\cos\beta(t)$$
$$\dot{y}(t) = -V\sin\beta(t) + s(x(t)).$$

(b) $\mathcal{H} = 1 + p_1(t)\,V\cos\beta(t) - p_2(t)V\sin\beta(t)$
$\qquad + p_2(t)\,s(x(t))$

$$\dot{p}_1^*(t) = -p_2^*(t)\,\frac{\partial s}{\partial x}\left(x^*(t)\right)$$

$$\dot{p}_2^*(t) = 0$$

$$0 = \frac{\partial\mathcal{H}}{\partial\beta}(*,t) = -p_1^*(t)\,V\sin\beta^*(t) - p_2^*(t)\,V\cos\beta^*(t)$$

and the state differential eqs. are necessary conditions for optimal control.

(c) $V > s(x(t))$

(d) $\dot{p}_1^*(t) = 0 \Rightarrow p_1^*(t) = c_1$

$\qquad \dot{p}_2^*(t) = 0 \Rightarrow p_2^*(t) = c_2,$

and $\partial\mathcal{H}/\partial\beta = 0 \Rightarrow -c_1\sin\beta^*(t) = c_2\cos\beta^*(t),$

therefore,

$$\frac{\sin\beta^*(t)}{\cos\beta^*(t)} = \text{a constant} \Rightarrow \beta^*(t) = \text{a const.}$$

5-2

(a) $\mathcal{H} = x_1^2(t) + \frac{1}{2}x_2^2(t) + \frac{1}{2}u^2(t) + p_1(t)x_2(t) - p_2(t)x_1(t)$
$+ p_2(t)[1 - x_1^2(t)]x_2(t) + p_2(t)u(t)$

$$\dot{p}_1^*(t) = -\frac{\partial\mathcal{H}}{\partial x_1}(*,t) = -\left[2x_1^*(t) - p_2^*(t) - 2p_2^*(t)x_2^*(t)x_1^*(t)\right]$$

$$\dot{p}_2^*(t) = -\frac{\partial\mathcal{H}}{\partial x_2}(*,t) = -\left[x_2^*(t) + p_1^*(t) + p_2^*(t)[1 - x_1^{*2}(t)]\right].$$

(b) (i) $u(t)$ not bounded, $\mathcal{H}$ quadratic in u:

$$\frac{\partial\mathcal{H}}{\partial u}(*,t) = 0 = u^*(t) + p_2^*(t), \quad \boxed{u^*(t) = -p_2^*(t).}$$

(ii) $|u(t)| \leq 1$

For $|p_2^*(t)| \leq 1$, the minimizing value of $u(t)$ is as given in part (i). For $|p_2^*(t)| > 1$, select $u(t)$ to make $\mathcal{H}$ as negative as possible (treating $x^*(t)$, $p^*(t)$ as being fixed), that is, minimize $\frac{1}{2}u^2(t) + p_2^*(t)\,u(t)$ with respect to admissible values of $u(t)$. Performing this minimization gives

$$u^*(t) = \begin{cases} -1, & p_2^*(t) > 1 \\ -p_2^*(t), & -1 \leq p_2^*(t) \leq 1 \\ +1, & p_2^*(t) < -1. \end{cases}$$

5-3

(a) $\mathcal{H} = \frac{1}{2}u^2(t) + p_1(t)x_2(t) - p_2(t)x_1(t)$
$+ p_2(t)[1 - x_1^2(t)]x_2(t) + p_2(t)u(t)$

$$\dot{p}_1^*(t) = -\frac{\partial \mathcal{H}}{\partial x_1}(*,t) = p_2^*(t) + 2\,p_2^*(t)\,x_1^*(t)\,x_2^*(t)$$

$$\dot{p}_2^*(t) = -\frac{\partial \mathcal{H}}{\partial x_2}(*,t) = -p_1^*(t) - p_2^*(t)\left[1 - x_1^{*2}(t)\right].$$

(b)(i) $\frac{\partial \mathcal{H}}{\partial u}(*,t) = 0$ since u not bounded, and $\mathcal{H}$ quadratic in u. This gives

$$u^*(t) = -p_2^*(t).$$

(ii) using the same reasoning as in (b) (ii) of Prob. 5-2 gives

$$u^*(t) = \begin{cases} -1, & p_2^*(t) > 1 \\ -p_2^*(t), & -2 \le p_2^*(t) \le 1 \\ +2, & p_2^*(t) < -2. \end{cases}$$

(c) t_f free, $m(\mathbf{x}(t), t) = 15x_1(t) + 20x_2(t) + 12t - 60$
See Entry 8. in Table 5-1.

$$-\mathbf{p}^*(t_f) = d\begin{bmatrix} 15 \\ 20 \end{bmatrix}$$

$$15x_1^*(t_f) + 20\,x_2^*(t_f) + 12t_f - 60 = 0$$

$\mathcal{H}(*, t_f) \triangleq \mathcal{H}(\mathbf{x}^*(t_f), \mathbf{p}^*(t_f), \mathbf{u}^*(t_f))$, using this definition

$$\mathcal{H}(*, t_f) = 12d.$$

5-4

(a) Costate eqs. same as in Prob. 5-3.

(b) The terms in $\mathcal{H}$ that contain $u(t)$ are

$p_2(t)\, u(t) + |u(t)|$.

Since $\mathcal{H}$ is piecewise linear in $u(t)$, we know that the minimizing control occurs on a boundary. Performing the minimization yields

$$u^*(t) = \begin{cases} -1, & 1 < p_2^*(t) \\ 0, & -1 < p_2^*(t) < 1 \\ +1, & p_2^*(t) < -1. \end{cases}$$

If $p_2^*(t) = +1$, $-1 \le u^*(t) \le 0$,

if $p_2^*(t) = -1$, $0 \le u^*(t) \le 1$.

(c) $[x_1^*(t_f) - 4]^2 + [x_2^*(t_f) - 5]^2 + [t_f - 2]^2 - 9 = 0$

$$-\underset{\sim}{p}^*(t_f) = 2d \begin{bmatrix} x_1^*(t_f) - 4 \\ x_2^*(t_f) - 5 \end{bmatrix}$$

$$\mathcal{H}(*, t_f) = 2d\,[t_f - 2].$$

5-5

$$\mathcal{H}(\underset{\sim}{x}(t), \underset{\sim}{u}(t), \underset{\sim}{p}(t)) = g(\underset{\sim}{x}(t), \underset{\sim}{u}(t)) + \underset{\sim}{p}^T(t)\, \underset{\sim}{a}(\underset{\sim}{x}(t), \underset{\sim}{u}(t)).$$

From the boundary conditions we have

$\mathcal{H}(\underset{\sim}{x}^*(t_f), \underset{\sim}{u}^*(t_f), \underset{\sim}{p}^*(t_f)) = 0$.

The total derivative of $\mathcal{H}$ with respect to time is

$p_2(t)\, u(t) + |u(t)|$.

Since $\mathcal{H}$ is piecewise linear in $u(t)$, we know that the minimizing control occurs on a boundary. Performing the minimization yields

$$u^*(t) = \begin{cases} -1, & 1 < p_2^*(t) \\ 0, & -1 < p_2^*(t) < 1 \\ +1, & p_2^*(t) < -1. \end{cases}$$

If $p_2^*(t) = +1$, $-1 \le u^*(t) \le 0$,

if $p_2^*(t) = -1$, $0 \le u^*(t) \le 1$.

(c) $[x_1^*(t_f) - 4]^2 + [x_2^*(t_f) - 5]^2 + [t_f - 2]^2 - 9 = 0$

$$-\mathbf{p}^*(t_f) = 2d \begin{bmatrix} x_1^*(t_f) - 4 \\ x_2^*(t_f) - 5 \end{bmatrix}$$

$$\mathcal{H}(*, t_f) = 2d\,[t_f - 2].$$

5-5

$\mathcal{H}(\mathbf{x}(t), \mathbf{u}(t), \mathbf{p}(t)) = g(\mathbf{x}(t), \mathbf{u}(t)) + \mathbf{p}^T(t)\, \mathbf{a}(\mathbf{x}(t), \mathbf{u}(t))$.

From the boundary conditions we have

$\mathcal{H}(\mathbf{x}^*(t_f), \mathbf{u}^*(t_f), \mathbf{p}^*(t_f)) = 0$.

The total derivative of $\mathcal{H}$ with respect to time is is $p^*(T)=0$, and the solution of the state-costate d.e. has the form

$$x^*(T) = \varphi_{11}(T-t)\,x^*(t) + \varphi_{12}(T-t)\,p^*(t) \quad \text{(I)}$$

$$p^*(T) = 0 = \varphi_{21}(T-t)\,x^*(t) + \varphi_{22}(T-t)\,p^*(t), \quad \text{(II)}$$

where the φ's are the components of the state-costate transition matrix for the linear, homogeneous, const. coef. system

$$\left.\begin{cases} \dot{x}^*(t) = x^*(t) - p^*(t) \\ \dot{p}^*(t) = -3x^*(t) - p^*(t) \end{cases}\right\} \begin{bmatrix} \dot{x}^*(t) \\ \dot{p}^*(t) \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -3 & -1 \end{bmatrix} \begin{bmatrix} x^*(t) \\ p^*(t) \end{bmatrix}.$$

Actually, we only need to calculate φ_{21} and φ_{22} because from (II) above

$$p^*(t) = -\varphi_{22}^{-1}(T-t)\,\varphi_{21}(T-t)\,x^*(t).$$

By standard methods we find that

$$\varphi_{21}(t) = \tfrac{3}{4}\left(\epsilon^{-2t} - \epsilon^{2t}\right)$$

$$\varphi_{22}(t) = \tfrac{1}{4}\left(3\epsilon^{-2t} + \epsilon^{2t}\right),$$

therefore,

$$u^*(t) = \frac{3\left[\epsilon^{-2(T-t)} - \epsilon^{2(T-t)}\right]}{\left[3\epsilon^{-2(T-t)} + \epsilon^{2(T-t)}\right]}\,x(t)$$

(we drop the $*$ on $x^*(t)$ because the optimal control law applies for all $x(t)$).

5-6 (cont.)

(b) As $T \to \infty$, $u^*(t) \longrightarrow -3x(t)$.

5-7

A problem of the linear regulator type.

(a) $\mathcal{H} = u^2(t) + p(t)\,u(t) - p(t)\,a\,x(t)$

$$\dot{p}^*(t) = a\,p^*(t)$$

$$\frac{\partial \mathcal{H}}{\partial u}(*,t) = 0 = 2u^*(t) + p^*(t), \text{ or } u^*(t) = -\frac{p^*(t)}{2}$$

$$\begin{bmatrix} \dot{x}^*(t) \\ \dot{p}^*(t) \end{bmatrix} = \begin{bmatrix} -a & -\tfrac{1}{2} \\ 0 & a \end{bmatrix} \begin{bmatrix} x^*(t) \\ p^*(t) \end{bmatrix}.$$

The $\underset{\sim}{\varphi}$ matrix for this system is

$$\underset{\sim}{\varphi}(t) = \begin{bmatrix} \epsilon^{-at} & \frac{1}{4a}\left[\epsilon^{-at} - \epsilon^{at}\right] \\ 0 & \epsilon^{at} \end{bmatrix}.$$

Hence, we have

$$x^*(\tau) = \varphi_{11}(\tau)\,x^*(0) + \varphi_{12}(\tau)\,p^*(0). \quad (I)$$

Letting $\tau = T$ gives

$$x^*(T) = 0 = \varphi_{11}(T)\,x^*(0) + \varphi_{12}(T)\,p^*(0), \text{ or}$$

$$p^*(0) = -\varphi_{12}^{-1}(T)\,\varphi_{11}(T)\,x^*(0).$$

Letting $\tau = t$ in (II) gives

$$x^*(t) = \varphi_{11}(t)\,x^*(0) + \varphi_{12}(t)\,p^*(0), \text{ or}$$

$$x^*(t) = \left[\varphi_{11}(t) - \varphi_{12}(t)\,\varphi_{12}(T)\,\varphi_{11}(T)\right] x^*(0).$$

Substituting in this expression gives

$$x^*(t) = \left[\epsilon^{-at} - \left(\frac{\epsilon^{-at} - \epsilon^{at}}{\epsilon^{-aT} - \epsilon^{aT}}\right)\epsilon^{-aT}\right] x(0)$$

which applies for all $x^*(0)$, so we write $x(0)$.

(b) $p^*(t) = \varphi_{22}(t)\,p^*(0)$

$$= -\varphi_{22}(t)\,\varphi_{12}^{-1}(T)\,\varphi_{11}(T)\,x(0)$$

and

$$u^*(t) = \tfrac{1}{2}\,\varphi_{22}(t)\,\varphi_{12}^{-1}(T)\,\varphi_{11}(T)\,x(0)$$

$$= \tfrac{1}{2}\epsilon^{at}\left(\frac{4a}{\epsilon^{-aT} - \epsilon^{aT}}\right)\epsilon^{-aT}\,x(0)$$

$$u^*(t) = \frac{2a\,\epsilon^{-a(T-t)}}{\epsilon^{-aT} - \epsilon^{aT}}\;x(0).$$

(c) From the solution of the state-costate equations,

$$x^*(T) = 0 = \varphi_{11}(T-t)\,x^*(t) + \varphi_{12}(T-t)\,p^*(t),$$

or

$$p^*(t) = -\varphi_{12}^{-1}(T-t)\,\varphi_{11}(T-t)\,x^*(t);$$

hence

$$u^*(t) = \frac{2a\,\epsilon^{-a(T-t)}}{\epsilon^{-a(T-t)} - \epsilon^{a(T-t)}}\;x(t)$$

(we drop the * from $x^*(t)$ because the expression for $u^*(t)$ is valid for all state values).

(d) As $t \to T$, $F(t, T; a) \to \infty$

$$u^*(t) \to \frac{2a\, x(0)}{\epsilon^{-aT} - \epsilon^{aT}}.$$

This is reasonable physically since for the specified plant to reach the origin control must be applied as $t \to T$, but since $x(T) \to 0$, the feedback gain matrix must approach infinity to generate a non-zero control value.

As $T \to \infty$, $F(t, T, a) \to \frac{2a\epsilon^{-a(T-t)}}{-\epsilon^{a(T-t)}}$.

Which is small except that when $t \to T$ we have the same result as before, that is $F \to \infty$, $u^*(T) \neq 0$.

5-8

$$\mathcal{H} = \tfrac{1}{2}\,\underset{\sim}{x}^T(t)\,\underset{\sim}{Q}(t)\,\underset{\sim}{x}(t) + \tfrac{1}{2}\,\underset{\sim}{u}^T(t)\,\underset{\sim}{R}(t)\,\underset{\sim}{u}(t)$$

$$+\,\underset{\sim}{p}^T(t)\left[\underset{\sim}{A}(t)\,\underset{\sim}{x}(t) + \underset{\sim}{B}(t)\,\underset{\sim}{u}(t)\right]$$

$$\frac{\partial\mathcal{H}}{\partial\underset{\sim}{u}}(*,t) = \underset{\sim}{0} = \underset{\sim}{R}(t)\,\underset{\sim}{u}^*(t) + \underset{\sim}{B}^T(t)\,\underset{\sim}{p}^*(t)$$

$$\frac{\partial^2\mathcal{H}}{\partial\underset{\sim}{u}^2}(*,t) = \underset{\sim}{R}(t).$$

Since $\mathcal{H}$ is quadratic in $\underset{\sim}{u}$ and $\partial^2\mathcal{H}/\partial\underset{\sim}{u}^2$ is positive definite (because $\underset{\sim}{R}(t)$ is positive definite by assumption), the control

$$\underset{\sim}{u}^*(t) = -\underset{\sim}{R}^{-1}(t)\,\underset{\sim}{B}^T(t)\,\underset{\sim}{p}^*(t)$$

does (globally) minimize $\mathcal{H}$.

5-9

(a) Differentiating $\underset{\sim}{p}^*(t) = \underset{\sim}{K}(t)\,\underset{\sim}{x}^*(t)$ gives

$$\dot{\underset{\sim}{p}}^*(t) = \dot{\underset{\sim}{K}}(t)\,\underset{\sim}{x}^*(t) + \underset{\sim}{K}(t)\,\dot{\underset{\sim}{x}}^*(t), \qquad (\mathrm{I})$$

but

$$\dot{\underset{\sim}{x}}^*(t) = \underset{\sim}{A}(t)\,\underset{\sim}{x}^*(t) + \underset{\sim}{B}(t)\,\underset{\sim}{u}^*(t)$$

$$= \underset{\sim}{A}(t)\,\underset{\sim}{x}^*(t) - \underset{\sim}{B}(t)\,\underset{\sim}{R}^{-1}(t)\,\underset{\sim}{B}^T(t)\,\underset{\sim}{p}^*(t).$$

Thus,

$$\dot{\mathbf{x}}^*(t) = \left[\mathbf{A}(t) - \mathbf{B}(t)\mathbf{R}^{-1}(t)\mathbf{B}^T(t)\mathbf{K}(t)\right]\mathbf{x}^*(t),$$

and

$$\dot{\mathbf{p}}^*(t) = -\mathbf{Q}(t)\mathbf{x}^*(t) - \mathbf{A}^T(t)\,\mathbf{p}^*(t)$$

$$= \left[-\mathbf{Q}(t) - \mathbf{A}^T(t)\mathbf{K}(t)\right]\mathbf{x}^*(t).$$

Substituting for $\dot{\mathbf{x}}^*(t)$ and $\dot{\mathbf{p}}^*(t)$ in (I) gives the result

$$\left[-\mathbf{Q}(t) - \mathbf{A}^T(t)\mathbf{K}(t)\right]\mathbf{x}^*(t) = \left[\dot{\mathbf{K}}(t) + \mathbf{K}(t)\mathbf{A}(t) - \mathbf{K}(t)\mathbf{B}(t)\mathbf{R}^{-1}(t)\mathbf{B}^T(t)\mathbf{K}(t)\right]\mathbf{x}^*(t).$$

Since this equation must hold for all $\mathbf{x}^*(t)$, we have

$$\dot{\mathbf{K}}(t) = -\mathbf{K}(t)\mathbf{A}(t) - \mathbf{A}^T(t)\mathbf{K}(t) - \mathbf{Q}(t) + \mathbf{K}(t)\mathbf{B}(t)\mathbf{R}^{-1}(t)\mathbf{B}^T(t)\mathbf{K}(t),$$

and

$$\mathbf{p}^*(t_f) = \mathbf{K}(t_f)\mathbf{x}^*(t_f) = \mathbf{H}\,\mathbf{x}^*(t_f),$$

$$\therefore\ \mathbf{K}(t_f) = \mathbf{H}.$$

(b) Taking the transpose of the Riccati equation gives

$$[\dot{\underset{\sim}{K}}]^T = [-\underset{\sim}{K}\underset{\sim}{A} - \underset{\sim}{A}^T\underset{\sim}{K} - \underset{\sim}{Q} + \underset{\sim}{K}\underset{\sim}{B}\underset{\sim}{R}^{-1}\underset{\sim}{B}^T\underset{\sim}{K}]^T$$

(the argument t has been omitted for simplicity). By using the matrix identity $[\underset{\sim}{C}\underset{\sim}{D}]^T = \underset{\sim}{D}^T\underset{\sim}{C}^T$, we have

$$[\dot{\underset{\sim}{K}}]^T = -\underset{\sim}{A}^T\underset{\sim}{K} - \underset{\sim}{K}^T\underset{\sim}{A} - \underset{\sim}{Q}^T + \underset{\sim}{K}^T\underset{\sim}{B}[\underset{\sim}{R}^{-1}]^T\underset{\sim}{B}^T\underset{\sim}{K}^T. \quad (\mathrm{I})$$

But $\underset{\sim}{Q}$ is symmetric so $\underset{\sim}{Q}^T = \underset{\sim}{Q}$; $\underset{\sim}{R}$ is symmetric, so $\underset{\sim}{R}^{-1}$ is symmetric and $[\underset{\sim}{R}^{-1}]^T = \underset{\sim}{R}^{-1}$. Therefore,

$$[\dot{\underset{\sim}{K}}]^T = -\underset{\sim}{A}^T\underset{\sim}{K}^T - \underset{\sim}{K}^T\underset{\sim}{A} - \underset{\sim}{Q} + \underset{\sim}{K}^T\underset{\sim}{B}\underset{\sim}{R}^{-1}\underset{\sim}{B}^T\underset{\sim}{K}^T. \quad (\mathrm{II})$$

We know that $\left[\frac{d}{dt}(\underset{\sim}{K}(t))\right]^T = \frac{d}{dt}\left[\underset{\sim}{K}^T(t)\right]$, i.e., that operations of transposition and differentiation are commutative; hence, the left side of (II) can be replaced by $\frac{d}{dt}[\underset{\sim}{K}^T]$. The boundary condition for (II) is $\underset{\sim}{K}^T(t_f) = \underset{\sim}{H}^T = \underset{\sim}{H}$. In (II) let $\underset{\sim}{K}^T \triangleq \underset{\sim}{M}$, then

5-9 (cont.)

$$\dot{\underset{\sim}{M}} = -\underset{\sim}{A}^T\underset{\sim}{M} - \underset{\sim}{M}\underset{\sim}{A} - \underset{\sim}{Q} + \underset{\sim}{M}\underset{\sim}{B}\underset{\sim}{R}^{-1}\underset{\sim}{B}^T\underset{\sim}{M}$$

and

$$\underset{\sim}{M}(t_f) = \underset{\sim}{H}.$$

$\underset{\sim}{M}(t)$ and $\underset{\sim}{K}(t)$ satisfy the same differential equation with the same boundary conditions; therefore, since the solution is unique, we conclude that

$$\underset{\sim}{M}(t) \triangleq \underset{\sim}{K}^T(t) = \underset{\sim}{K}(t).$$

Since $\underset{\sim}{K}$ is symmetric we need solve only for the elements of $\underset{\sim}{K}$ which lie on or above the main diagonal. The number of terms involved is $1+2+3+\cdots+n$. This summation is formed from the elements of an arithmetic progression. The sum is equal to $n(n+1)/2$.

(c) If $\underset{\sim}{x}(t_f) = \underset{\sim}{0}$, $\underset{\sim}{K}(t_f)$ does not exist (contains infinite-valued elements), so modifications must be made. For two ideas on this subject, see the <u>IEEE Trans. on Automatic Control</u>, June 1967, p. 303 and p. 323.

5-10

From Eq. (5.2-12)

$$\underset{\sim}{x}^*(t_f) = \underset{\sim}{\varphi}_{11}(t_f, t)\,\underset{\sim}{x}^*(t) + \underset{\sim}{\varphi}_{12}\,\underset{\sim}{p}^*(t).$$

With $\underset{\sim}{x}^*(t_f) = \underset{\sim}{0}$ we obtain

$$\underset{\sim}{p}^*(t) = -\underset{\sim}{\varphi}_{12}^{-1}(t_f, t)\,\underset{\sim}{\varphi}_{11}(t_f, t)\,\underset{\sim}{x}^*(t).$$

Since

$$\underset{\sim}{u}^*(t) = -\underset{\sim}{R}^{-1}(t)\,\underset{\sim}{B}^T(t)\,\underset{\sim}{p}^*(t),$$

$$\underset{\sim}{u}^*(t) = \underset{\sim}{R}^{-1}(t)\,\underset{\sim}{B}^T(t)\,\underset{\sim}{\varphi}_{12}^{-1}(t_f, t)\,\underset{\sim}{\varphi}_{11}(t_f, t)\,\underset{\sim}{x}(t)$$

(valid for all $\underset{\sim}{x}^*(t)$).

5-11

(a) The state-costate equations

$$\begin{bmatrix} \dot{x}^*(t) \\ \dot{p}^*(t) \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} x^*(t) \\ p^*(t) \end{bmatrix}$$

5-11 (cont.)

have the transition matrix

$$\underset{\sim}{\varphi}(t) = \begin{bmatrix} \frac{3}{4}\epsilon^{-2t} + \frac{1}{4}\epsilon^{2t} & \frac{1}{4}\epsilon^{-2t} - \frac{1}{4}\epsilon^{2t} \\ \frac{3}{4}\epsilon^{-2t} - \frac{3}{4}\epsilon^{2t} & \frac{1}{4}\epsilon^{-2t} + \frac{3}{4}\epsilon^{2t} \end{bmatrix}.$$

From the solution of Problem 5-10

$$u^*(t) = \frac{3\epsilon^{-2(t_f - t)} + \epsilon^{2(t_f - t)}}{\epsilon^{-2(t_f - t)} - \epsilon^{2(t_f - t)}}\; x(t).$$

(b) With $x(1)$ free, using Eq. (5.2-14) gives

$$p^*(t) = \left[\varphi_{22}(t_f - t)\right]^{-1}\left[-\varphi_{21}(t_f - t)\right] x^*(t),$$

thus,

$$u^*(t) = \frac{3\left[\epsilon^{-2(t_f - t)} - \epsilon^{2(t_f - t)}\right]}{\epsilon^{-2(t_f - t)} + 3\epsilon^{2(t_f - t)}}\; x(t).$$

5-12

(a) The form of the solution for $\underset{\sim}{x}^*(t)$, $\underset{\sim}{p}^*(t)$ is the same as given by Eq. (5.1-68). The boundary condition equations are

$$x_1^*(2) + 5\, x_2^*(2) = 15 \qquad (\mathrm{I})$$

and from entry 3 of Table 5-1

$$\begin{bmatrix} x_1^*(2) - 5 \\ x_2^*(2) - 2 \end{bmatrix} - p^*(2) = d \begin{bmatrix} 1 \\ 5 \end{bmatrix},$$

therefore,

$$p_1^*(2) = x_1^*(2) - 5 - d \qquad (\mathrm{II})$$

$$p_2^*(2) = x_2^*(2) - 2 - 5d. \qquad (\mathrm{III})$$

From (II)

$$d = x_1^*(2) - 5 - p_1^*(2),$$

hence

$$p_2^*(2) = x_2^*(2) - 2 \quad 5\left[x_1^*(2) - 5 - p_1^*(2)\right],$$

or

$$p_2^*(2) = x_2^*(2) - 5x_1^*(2) + 5p_1^*(2) + 23. \quad (\mathrm{III}a)$$

As in Example 5.1-1 $c_1 = c_2 = 0$; to find c_3 and c_4 we substitute from Eq. (5.1-69) (with $t_f = 2 = t$) into (I) and (IIIa) and solve for c_3 and c_4; the result is

$$c_3 = -2.598 \qquad c_4 = -2.637$$

$$u^*(t) = -p_2^*(t) = -c_3 + c_3\epsilon^{t} - c_4\epsilon^{t}.$$

5-12 (cont.)

(b) The cost of control is

$$\int_0^2 \tfrac{1}{2} u^{*2}(t)\, dt = \tfrac{1}{2}\Big[c_3^2 t - 2c_3 c_5 \epsilon^{t} + \tfrac{1}{2} c_5^2 \epsilon^{2t}\Big]_0^2,$$

(where $c_5 \triangleq c_3 - c_4$).

$$\int_0^2 \tfrac{1}{2} u^{*2}(t)\, dt = c_3^2 - c_3 c_5 \epsilon^2 + c_3 c_5 + \tfrac{1}{4} c_5^2 \epsilon^4 - \tfrac{1}{4} c_5^2 .$$

Substituting the values found for c_3 and c_4 gives

$J_u = 16.58$ (iv) Part (a) of Example 5.1-1

$J_u = 3.55$ (v) Part (b) of Example 5.1-1

$J_u = 7.42$ (vi) Part (a) of Problem 5-12,

where J_u denotes the cost of control.

The point is that in (iv) above the final state is most restricted, $\underline{x}(2)$ must be exactly $[5 \;\; 2]^T$; thus, the required control energy is largest. In (v) above we only require that $\underline{x}(2)$ be "close" to $[5 \;\; 2]^T$. Finally, in (vi), $\underline{x}(2)$

5-12 (cont.)

is to be "close" to $[5\ 2]^T$ and lie on the line $x_1(2)+5x_2(2)=15$; this case is less restrictive than (iv) (hence smaller J_u for vi) but more restrictive than (v).

5-13

(a)

$$\mathcal{H} = -x(t) - .1p(t)\,x(t) + p(t)\,u(t)$$

from which

$$\dot{p}^*(t) = 1 + 0.1\,p^*(t).$$

This differential equation has a solution of the form

$$p^*(t) = c_1 + c_2\,\epsilon^{.1t}.$$

Substituting this solution in the D.E. and equating coefficients of like powers of ϵ gives $c_1=-10$. Since the final state is free and there is no term in $x(t_f)$ in J,

$$p^*(100)=0 \Rightarrow c_2 = 10\epsilon^{-10};$$

thus,

$$p^*(t) = -10 + 10\epsilon^{-.1(100-t)}.$$

5-13 (cont.)

Minimization of $\mathcal{H}$ with respect to u gives

$$u^*(t) = \begin{cases} 0 & \text{for } p^*(t) > 0 \\ M & \text{for } p^*(t) < 0. \end{cases}$$

The solution found above for $p^*(t)$ is less than zero for all $t \in [0, 100]$; hence,

$$u^*(t) = M, \quad t \in [0, 100].$$

(b) Let $\dot{x}_2(t) \triangleq u(t)$, $x_2(0) = 0$, $x_2(100) = K$

$$\mathcal{H} = -x_1(t) - .1\, p_1(t) x_1(t) + p_1(t) u(t) + p_2(t) u(t),$$

where it is easily shown that $p_2^*(t)$ is a constant. The D.E. and solution for $p_1^*(t)$ are as found in part (a), that is,

$$p_1^*(t) = -10 + 10\epsilon^{-.1(100-t)}.$$

Minimizing $\mathcal{H}$ with respect to u gives

$$u^*(t) = \begin{cases} 0 & \text{for } p_1^*(t) + p_2^*(t) > 0 \\ M & \text{for } p_1^*(t) + p_2^*(t) < 0, \end{cases}$$

5-13 (cont.)

$$p_1^*(t) + p_2^*(t) = -10 + 10\epsilon^{-.1(100-t)} \Big|^{100} + c_3.$$

Examining the possible forms for $p_1^*(t) + p_2^*(t)$ we see that the sign of $p_1^*(t) + p_2^*(t)$ can change at most once; therefore, $u^*(t)$ must be of one of the following forms

$$u^*(t) = \underset{(I)}{\{M\}},\ \underset{(II)}{\{0\}},\ \underset{(III)}{\{M, 0\}},\ \underset{(IV)}{\{0, M\}}.$$

(IV) is impossible because $p_1^*(t) + p_2^*(t)$ cannot change from + to −.

(II) is impossible because it will not satisfy $\int_0^{100} u(t)\,dt = K$ unless $K = 0$ (a trivial case).

(I) is not possible unless $\int_0^{100} M\,dt = K$, a very special circumstance -- we will assume that this is not the case, which means that

$$u^*(t) = \begin{cases} M, & t \in [0, t_1] \\ 0, & t \in (t_1, 100]. \end{cases}$$

Since $\int_0^{t_1} M\,dt = K$, $t_1 = K/M$.

5-13 (cont.)

(c) $\mathcal{H} = -.1\,p_1(t)x_1(t) + [p_1(t)+p_2(t)]\,u(t)$

$\dot{p}_1^*(t) = .1\,p_1^*(t) \quad \therefore\ p_1^*(t) = c_1\epsilon^{.1t}$

$p_1^*(100) = -1 \Rightarrow \quad c_1 = -\epsilon^{-10}$

and

$p_1^*(t) = -\epsilon^{-.1(100-t)}, \quad p_2^*(t) = c_2.$

We again find that $p_1^*(t)+p_2^*(t)$ can change sign at most once, and that if a sign change occurs it is from + to −.

Using the same reasoning as in part (b), we rule out all possibilities except

$$u^*(t) = \begin{cases} 0, & t \in [0, 100-K/M) \\ M, & t \in [100-K/M, 100]. \end{cases}$$

5-14

(a) The Riccati equation with $\dot{k}=0$ is

$0 = -2ak - 2q + k^2/2r$

$k^2 - 4ark - 4qr = 0;$

the solution is

5-14 (cont.)

$k = 2ar \pm \sqrt{4a^2r^2 + 4qr}$.

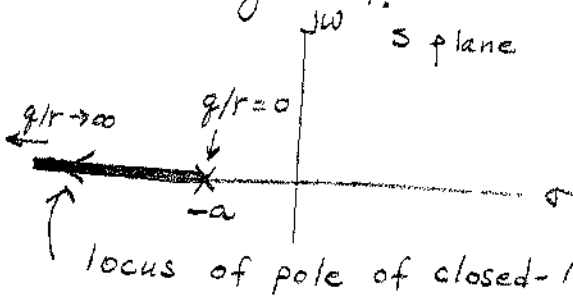
The feedback gain is

$f = -a \mp \sqrt{a^2 + q/r}$.

The D.E. for the closed-loop system is

$$\dot{x}(t) = a\,x(t) + \left[-a \mp \sqrt{a^2+q/r}\right] x(t)$$
$$= \mp \sqrt{a^2+q/r}\; x(t).$$

The − sign is the optimal solution; the + sign gives an unstable system.



locus of pole of closed-loop system

(b) Expanding the terms in the Riccati eq. with $\dot{\underline{K}}(t) = \underline{0}$ gives

$$k_{12}^2 + 16\,k_{12} - 80 = 0 \quad (\mathrm{I})$$

$$k_{12}k_{22} + 8\,k_{22} + 8\,k_{12} - 2k_{11} = 0 \quad (\mathrm{II})$$

5-14 (cont.)

$$k_{22}^2 + 16k_{22} - 4k_{12} - 20 = 0. \quad \text{(III)}$$

Noting that $u^*(t) = -R^{-1}\underset{\sim}{B}^T\underset{\sim}{K}\,\underset{\sim}{x}(t)$, and that $-R^{-1}\underset{\sim}{B}^T\underset{\sim}{K} \triangleq \underset{\sim}{F} = \left[-\frac{k_{12}}{2} \quad -\frac{k_{22}}{2}\right]$, it is necessary only to solve for k_{12} and k_{22}.

From (I)

$$k_{12} = -8 \pm \sqrt{64+80} = +4, -20.$$

From (III)

$$k_{22}^2 + 16k_{22} - 20 + \begin{Bmatrix} -16 \\ +80 \end{Bmatrix} = 0$$

$$k_{22} = -8 \pm \sqrt{64+20+\begin{Bmatrix} 16 \\ -80 \end{Bmatrix}} = -8 \pm \sqrt{\begin{Bmatrix} 100 \\ 4 \end{Bmatrix}}$$

$$= \underbrace{+2, -18}_{k_{12}=+4}; \quad \underbrace{-6, -10}_{k_{12}=-20}.$$

The poles of the closed-loop system are the roots of

$$\det\left[s\underset{\sim}{I} - \underset{\sim}{A} - \underset{\sim}{B}\underset{\sim}{F}\right] = \begin{vmatrix} s & -1 \\ 4+\frac{k_{12}}{2} & s+4+\frac{k_{22}}{2} \end{vmatrix}$$

$$= s^2 + 4s + \frac{k_{22}}{2}s + 4 + \frac{k_{12}}{2}.$$

For stability it is necessary and

5-14 (cont.)

sufficient that

$$4 + \frac{k_{22}}{2} > 0 \implies k_{22} > -8$$

and

$$4 + \frac{k_{12}}{2} > 0 \implies k_{12} > -8$$

$$\therefore \quad k_{12} = +4, \quad k_{22} = +2.$$

The characteristic equation for the closed-loop system is

$$s^2 + 5s + 6 = (s+3)(s+2)$$

$\implies$ poles at $s = -3, -2$.

The poles of the open-loop system are the roots of

$$\det[s\underset{\sim}{I} - \underset{\sim}{A}] = \begin{vmatrix} s & -1 \\ 4 & s+4 \end{vmatrix} = s^2 + 4s + 4$$

$$= (s+2)(s+2).$$

The optimal controller moves one of the poles to -3.

5-15

Expanding the Riccati equation with $\dot{\underset{\sim}{K}}(t) = 0$ gives

5-15 (cont.)

$$0 = -2 + \frac{1}{2} k_{12}^2 \qquad (\mathrm{I})$$

$$0 = k_{12} - k_{11} + \frac{1}{2} k_{12} k_{22} \qquad (\mathrm{II})$$

$$0 = 2k_{22} - 2k_{12} - 2 + \frac{1}{2} k_{22}^2 . \qquad (\mathrm{III})$$

Since the feedback gain matrix $\underset{\sim}{F}$ is

$$\underset{\sim}{F} = -\underset{\sim}{R}^{-1} \underset{\sim}{B}^T \underset{\sim}{K} = -\frac{1}{2} \begin{bmatrix} k_{12} & k_{22} \end{bmatrix},$$

we only need to solve for k_{12} and k_{22}. From (I), $k_{12} = \pm 2$; substituting this into (III) yields

$$k_{22}^2 + 4k_{22} \mp 8 - 4 = 0$$

$$k_{22} = -2 \pm \sqrt{4 \pm 8 + 4} = -2 \pm \begin{Bmatrix} 4 \\ 0 \end{Bmatrix}$$

$$= \begin{Bmatrix} +2, -6 \\ -2, -2 \end{Bmatrix} .$$

The poles of the closed-loop system are the roots of

$$\det[s\underset{\sim}{I} - \underset{\sim}{A} - \underset{\sim}{F}] = s^2 + s + \frac{k_{22}}{2} s + \frac{k_{12}}{2} .$$

For stability it is necessary

5-15 (cont.)

and sufficient that

$$1+\frac{k_{22}}{2}>0 \quad \Rightarrow \quad k_{22}>-2$$

and

$$\frac{k_{12}}{2}>0\ ;$$

therefore, $k_{12}=+2$ and $k_{22}=+2$. Also, $\underset{\sim}{F}=[-1\quad -1]$, or

$u^*(t)=-x_1(t)-x_2(t)$.

5-16

(a) Replacing $\underset{\sim}{y}$ by $\underset{\sim}{C}\underset{\sim}{x}$ in J gives

$$J=\frac{1}{2}\underset{\sim}{x}^T(t_f)\underset{\sim}{C}^T(t_f)\underset{\sim}{H}\,\underset{\sim}{C}(t_f)\underset{\sim}{x}(t_f)+\frac{1}{2}\int_{t_0}^{t_f}\Big[\underset{\sim}{x}^T(t)\underset{\sim}{C}^T(t)\underset{\sim}{Q}(t)\underset{\sim}{C}(t)\underset{\sim}{x}(t)$$
$$+\underset{\sim}{u}^T(t)\underset{\sim}{R}(t)\underset{\sim}{u}(t)\Big]dt.$$

This is the same form as the performance measure for the linear regulator problem with $\underset{\sim}{H}$ replaced by

$$\underset{\sim}{H}'=\underset{\sim}{C}^T(t_f)\underset{\sim}{H}\,\underset{\sim}{C}(t_f)$$

and $\underset{\sim}{Q}(t)$ replaced by

$$\underset{\sim}{Q}'(t) = \underset{\sim}{C}^T(t)\,\underset{\sim}{Q}(t)\,\underset{\sim}{C}(t).$$

Notice that symmetry of $\underset{\sim}{H}$ implies that $\underset{\sim}{H}'$ is symmetric; similarly $\underset{\sim}{Q}(t)$ symmetric $\Rightarrow \underset{\sim}{Q}'(t)$ is symmetric. It is not difficult to show that since $\underset{\sim}{H}$ and $\underset{\sim}{Q}$ are positive semi-definite, $\underset{\sim}{H}'$ and $\underset{\sim}{Q}'$ are also; hence, the problem has been reduced to a linear (state) regulator problem.

(b) The optimal control law follows directly from the results given in Section 5.2, i.e.

$$\underset{\sim}{u}^*(t) = -\underset{\sim}{R}^{-1}(t)\,\underset{\sim}{B}^T(t)\,\underset{\sim}{K}(t)\,\underset{\sim}{x}(t)$$

where $\underset{\sim}{K}$ is the solution of the Riccati equation with $\underset{\sim}{Q}$ replaced by $\underset{\sim}{Q}'$ and with the boundary condition $\underset{\sim}{K}(t_f) = \underset{\sim}{H}'$.

Notice that the optimal control law is linear feedback of the system states. The assumption that the system is completely observable ensures that the state vector can be computed from knowledge of the system outputs.

5-17

(a) Let $\tilde{x}(t) \triangleq x(t) - r(t)$, then

$$J = \frac{1}{2}\int_0^{t_f} \left[q\, \tilde{x}^2(t) + u^2(t)\right] dt.$$

The D.E. for $\tilde{x}(t)$ is

$$\dot{\tilde{x}}(t) = \dot{x}(t) - \dot{r}(t) = -x(t) + u(t) - \left[-\alpha \epsilon^{-t}\right]$$

$$= -x(t) + u(t) + r(t)$$

$$\dot{\tilde{x}}(t) = -\tilde{x}(t) + u(t).$$

Thus, we have a linear regulator problem. Notice that the optimal control law is

$$u^*(t) = f(t)\ \tilde{x}(t) = f(t)\left[x(t) - r(t)\right],$$

that is, linear feedback of the error between the actual and desired state.

(b) Defining the system states as suggested gives

$x_1(t) = y(t) - r(t)$, $x_2(t) = \dot{y}(t) - \dot{r}(t)$,
$\ldots$, $x_n(t) = \frac{d^{n-1}}{dt^{n-1}}[y(t) - r(t)]$, which
makes

$$J = \frac{1}{2}\int_0^{t_f}\left[q\, x_1^2(t) + u^2(t)\right]dt.$$

The state equations become

$$\dot{x}_1(t) = \dot{y}(t) - \dot{r}(t) = x_2(t)$$
$$\dot{x}_2(t) = x_3(t)$$
$$\vdots$$
$$\dot{x}_n(t) = \frac{d^n}{dt^n}y(t) - \frac{d^n}{dt^n}r(t).$$

Because of the D.E. satisfied by $y(t)$ and $r(t)$, the last equation becomes

$$\dot{x}_n(t) = -a_{n-1}\frac{d^{n-1}}{dt^{n-1}}y(t) - \ldots - a_0 y(t) + u(t)$$
$$-\left\{-a_{n-1}\frac{d^{n-1}}{dt^{n-1}}r(t) - \ldots - a_0 r(t)\right\}$$
$$= -a_{n-1}\frac{d^{n-1}}{dt^{n-1}}[y(t) - r(t)] - \ldots - a_0[y(t) - r(t)] + u(t)$$
$$= -a_{n-1}x_n(t) - \ldots - a_0 x_1(t) + u(t).$$

Hence, the problem has been reduced to one of the linear regulator type.

5-19

(a) Let t_1 be the time required for the missile required to reach b,

$$a + .1\, t_1^3 = b \implies t_1 = (10[b-a])^{1/3}.$$

(b) For interception to occur at $t = t_2$

$$a + .1\, t_2^3 = b_1 \quad (\mathrm{I}) \text{ and}$$

$$\tfrac{1}{2} t_2^2 = b_1 . \quad (\mathrm{II})$$

Solving (II) for t_2 gives $t_2 = (2b_1)^{1/2}$; substituting this in (I) yields

$$a + .1\,(2b_1)^{3/2} = b_1 .$$

Rearranging and squaring gives

$$8b_1^3 - 100b_1^2 + 200ab_1 - 100a^2 = 0. \quad (\mathrm{III})$$

(c) We already know from the text example that interception can occur only if $a \le 1.85$. To determine the values of a and b for which interception is accomplished before the missile reaches b, we solve (III) for a

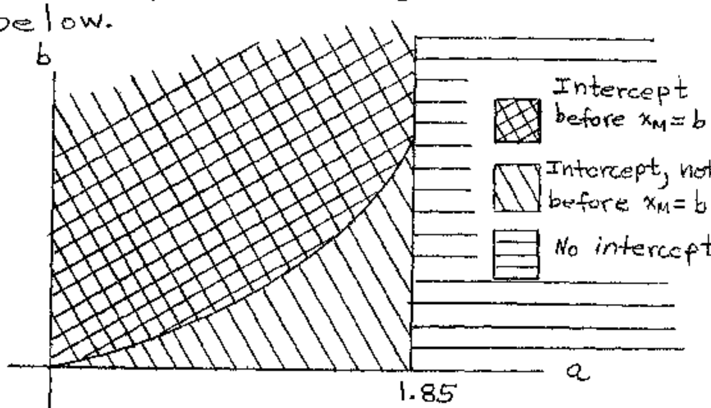
and b_1. Doing this gives

$$a = b_1 \pm \sqrt{.08 b_1^3}.$$

A few tabulated values are shown below (only feasible solutions are included).

b_1	a
0.	0.
0.5	0.4
1.0	0.72
2.0	1.2
3.0	1.53
4.0	1.74
5.0	1.84
5.55	1.85

The required diagram is shown below.



$$\mathcal{H} = 1 + p_1(t)\, x_2(t) + p_2(t)\, u_1(t) + p_2(t)\, u_2(t)$$

$$\dot{p}_1^*(t) = 0 \Rightarrow p_1^*(t) = c_1 \quad (\text{a constant})$$

$$\dot{p}_2^*(t) = -p_1^*(t) = -c_1 \Rightarrow p_2^*(t) = -c_1 t + c_2 .$$

Minimizing $\mathcal{H}$ with respect to $\underline{u}(t)$ gives

$$u_1^*(t) = \begin{cases} 0, & p_2^*(t) > 0 \\ M, & p_2^*(t) < 0 \end{cases}$$

$$u_2^*(t) = \begin{cases} 0, & p_2^*(t) > 0 \\ -M, & p_2^*(t) < 0 . \end{cases}$$

Since $p_2^*(t)$ can change sign at most once, the optimal controls have the possible forms

$$u_1^* = \{0, M\}, \quad u_2^* = \{-M, 0\} \qquad (\text{I})$$

or

$$u_1^* = \{M, 0\}, \quad u_2^* = \{0, -M\} \qquad (\text{II})$$

or

$$u_1^* = \{0\}, \quad u_2^* = \{-M\} \qquad (\text{III})$$

or

$$u_1^* = \{+M\}, \quad u_2^* = \{0\} . \qquad (\text{IV})$$

(IV) is eliminated because the boundary condition $x_2(t_f) = 0$ could not be satisfied.

(III) can be eliminated because of the constraint $0 \le x_2(t)$ — (I) can be eliminated for the same reason. If the braking system of the car is mechanical, (I) and (III) could also be eliminated because there would be no decelerating force without a velocity having been acquired and (I) and (III) would not follow velocity acquisition with braking. For braking by deflection of thrusting gases, (I) and (III) would be eliminated by the constraint $0 \le x_2(t)$.

This leaves only (II), so

$$u_1^*(t) = \begin{cases} M, & t \in [t_0, t_1] \\ 0, & t \in (t_1, t_f] \end{cases}$$

$$u_2^*(t) = \begin{cases} 0, & t \in [t_0, t_1) \\ -M, & t \in [t_1, t_f] \end{cases}$$

t_1 is the time where switching occurs ($p_2^*(t_1) = 0$). To find t_1 we note that

$$x_2(t_1) = 0 + \int_{t_0}^{t_1} M\,dt = M(t_1 - t_0)$$

5-20 (cont.)

and

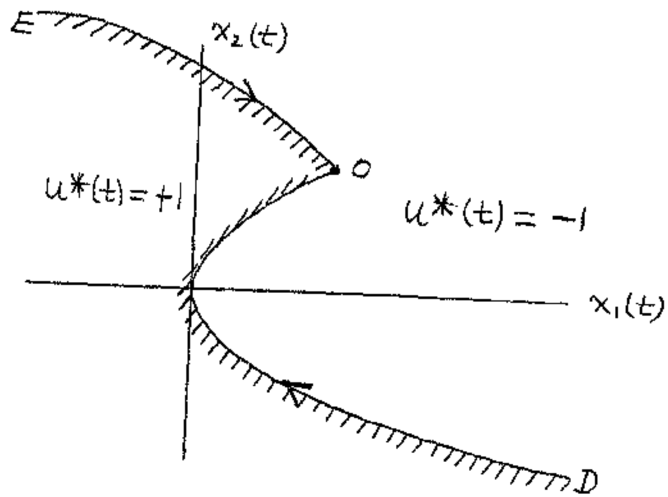
$$x_2(t_f) = 0 = x_2(t_1) + \int_{t_1}^{t_f} -M\,dt$$

$$= M(t_1 - t_0) - M(t_f - t_1)$$

$$2Mt_1 = M(t_f + t_0), \quad t_1 = \frac{t_0 + t_f}{2}.$$

5-21

The optimal trajectories must be composed of segments of the parabolas shown in Fig. 5-20 and must intersect at the point $\underline{x} = [2 \quad 2]^T$. The switching curve is shown below.



To the left of the curve E-O-D the optimal control is $u^*(t) = +1$; to the right of E-O-D $u^*(t) = -1$.

To find the expression for E-O-D we use Eqs. (5.4-39) and (5.4-40). From (5.4-39)

$$2 = \tfrac{1}{2}(4) + c_5 \Rightarrow c_5 = 0.$$

From (5.4-40)

$$2 = -\tfrac{1}{2}(4) + c_6 \Rightarrow c_6 = 4.$$

Hence the equation of E-O-D is

$$x_1(t) = \tfrac{1}{2}x_2^2(t), \quad x_2(t) \le 2$$

$$x_1(t) = -\tfrac{1}{2}x_2^2(t) + 4, \quad x_2(t) \ge 2.$$

5-22

Since part (b) can be done as easily as part (a), we will show the solution for (b) only.

$$\begin{aligned}\mathcal{H} &= 1 + p_1(t)x_1(t) + a_1 p_1(t)u(t) + a_2 p_2(t)x_2(t) \\ &\quad + a_2 p_2(t)u(t) \\ &= 1 + a_1 p_1(t)x_1(t) + a_2 p_2(t)x_2(t) \\ &\quad + [a_1 p_1(t) + a_2 p_2(t)]u(t).\end{aligned}$$

Minimizing $\mathcal{H}$ with respect to u gives

$$u^*(t) = \begin{cases} -1, & a_1 p_1^*(t) + a_2 p_2^*(t) > 0 \\ +1, & a_1 p_1^*(t) + a_2 p_2^*(t) < 0 \\ \text{Undetermined for } a_1 p_1^*(t) + a_2 p_2^*(t) = 0. \end{cases}$$

The costate equations are

$$\dot{p}_1^*(t) = -\frac{\partial \mathcal{H}}{\partial x_1}(*, t) = -a_1 p_1^*(t)$$

$$\dot{p}_2^*(t) = -\frac{\partial \mathcal{H}}{\partial x_2}(*, t) = -a_2 p_2^*(t);$$

hence,

$$p_1^*(t) = c_1 \epsilon^{-a_1 t}, \quad p_2^*(t) = c_2 \epsilon^{-a_2 t}.$$

We note that a singular interval cannot exist because

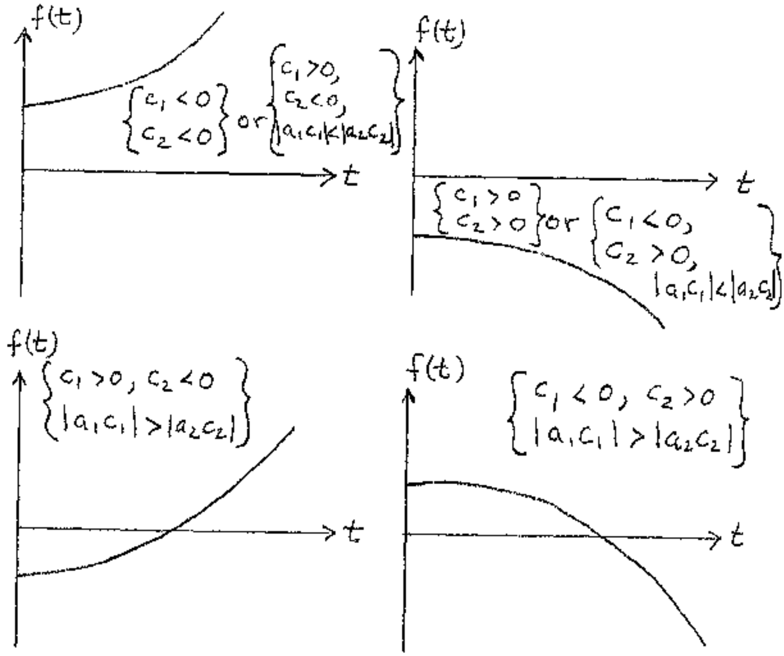
$a_1 p_1^*(t) + a_2 p_2^*(t)$ can only be zero for a finite time interval if $c_1 = c_2 = 0$. But if $c_1 = c_2 = 0$, $p_1^*(t) = p_2^*(t) = 0$ for all $t \in [t_0, t_f]$, and this implies that $\mathcal{H}(*, t) = 1$ for $t \in [t_0, t_f]$. Since this is a free final time problem with $\mathcal{H}$ not explicitly dependent on t, it is necessary that $\mathcal{H}(*, t) = 0$, $t \in [t_0, t_f]$.

Hence, $a_1 p_1^*(t) + a_2 p_2^*(t)$ cannot be zero for a finite time interval.

The possible forms for

$$a_1 p_1^*(t) + a_2 p_2^*(t) = a_1 c_1 \epsilon^{-a_1 t} + a_2 c_2 \epsilon^{-a_2 t} \triangleq f(t)$$

are shown below.



We see that the optimal control can switch at most once -- this also follows from Theorem 5.4-3; hence, the possible forms for the optimal controls are

$$u^* = \{+1\}, \{-1\}, \{-1, +1\}, \text{or } \{+1, -1\}.$$

5-22 (cont.)

Integrating the state equations with $u(t) = \pm 1$ gives

$$x_1(t) = \mp 1 + c_3 \epsilon^{a_1(t-t_f)}, \quad x_2(t) = \mp 1 + c_4 \epsilon^{a_2(t-t_f)}.$$

At $t = t_f$ $x_1(t_f) = x_2(t_f) = 0$

$$\Rightarrow \quad c_3 = \pm 1, \quad c_4 = \pm 1$$

$$x_1(t) = \mp 1 \pm \epsilon^{a_1(t-t_f)} \quad (\mathrm{I})$$

$$x_2(t) = \mp 1 \pm \epsilon^{a_2(t-t_f)}. \quad (\mathrm{II})$$

Solving for $(t-t_f)$ in (I) yields

$$t - t_f = \frac{1}{a_1} \ln(\pm x_1(t) + 1).$$

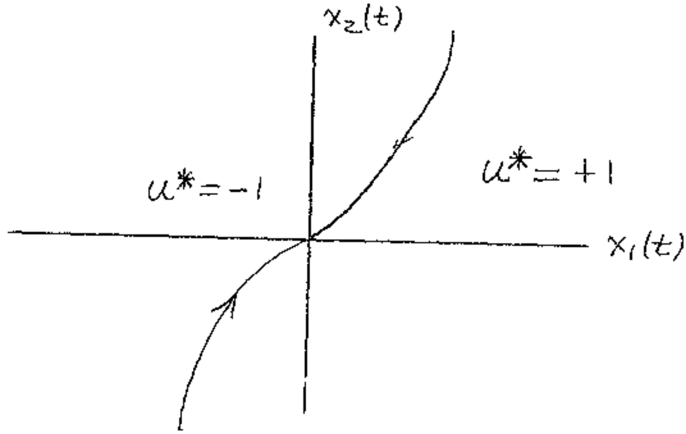
Substituting this into (II) gives

$$x_2(t) = \mp 1 \pm \epsilon^{\frac{a_2}{a_1} \ln(\pm x_1(t) + 1)}$$

$$= \mp 1 \pm \epsilon^{\ln(\pm x_1(t) + 1)^{a_2/a_1}}$$

$$\therefore \quad x_2(t) = \mp 1 \pm (\pm x_1(t) + 1)^{a_2/a_1}.$$

The upper signs correspond to $u^*(t) = +1$. This is the equation of the switching curve; the curve and the optimal control law are shown below.



In this case the regions where $u^* = +1$ and $u^* = -1$ can be determined (as in Example 5.4-4) by examining a few trajectories which begin off the switching curve and have $u = \pm 1$.

5-23

(a) Let the states be $x_1 \triangleq x$, $x_2 \triangleq \dot{x}$, $x_3 \triangleq y$, $x_4 \triangleq \dot{y}(t)$, and the control is $u \triangleq \beta$. The state equations are

$$\dot{x}_1(t) = x_2(t)$$

$$\dot{x}_2(t) = \frac{T}{M} \cos u(t)$$

$$\dot{x}_3(t) = x_4(t)$$

$$\dot{x}_4(t) = \frac{T}{M} \sin u(t).$$

(b) $\mathcal{H} = 1 + p_1(t)\,x_2(t) + p_2(t)\left[\frac{T}{M}\right]\cos u(t)$
$$+ p_3(t)\,x_4(t) + p_4(t)\left[\frac{T}{M}\right]\sin u(t).$$

The costate equations are

$$\dot{p}_1^*(t) = 0 \Rightarrow p_1^*(t) = c_1$$
$$\dot{p}_2^*(t) = -p_1^*(t) \Rightarrow p_2^*(t) = -c_1 t + c_2$$
$$\dot{p}_3^*(t) = 0 \Rightarrow p_3^*(t) = c_3$$
$$\dot{p}_4^*(t) = -p_3^*(t) \Rightarrow p_4^*(t) = -c_3 t + c_4.$$

The specified final states are

$$x_2^*(t_f) = V, \quad x_3^*(t_f) = D, \quad x_4^*(t_f) = 0.$$

Since the final state $x_1(t_f)$ is free and there is no term in J involving $x_1(t_f)$, $p_1^*(t_f) = 0$. The other boundary condition is $\mathcal{H}(*, t_f) = 0$.

(c) The control which minimizes $\mathcal{H}$ is found from

$$\frac{\partial \mathcal{H}}{\partial u}(*, t) = 0 = \frac{T}{M}\left\{ p_2^*(t)\left[-\sin u^*(t)\right] + p_4^*(t)\left[\cos u^*(t)\right]\right\}$$

$$\Rightarrow \quad \frac{\sin u^*(t)}{\cos u^*(t)} = \frac{p_4^*(t)}{p_2^*(t)},$$

or $$\sin u^*(t) = \frac{p_4^*(t)}{\left[p_2^{*2}(t) + p_4^{*2}(t)\right]^{1/2}}$$

$$\cos u^*(t) = \frac{p_2^*(t)}{\left[p_2^{*2}(t) + p_4^{*2}(t)\right]^{1/2}}.$$

From (b), since $p_1^*(t_f) = 0$, $c_1 = 0$; this implies that $p_2^*(t) = c_2$ (a const.). To determine c_2, c_3 and c_4 we would have to substitute for $u^*(t)$ in the state equations (in terms of c_2, c_3, c_4), integrate and set

$$x_2^*(t_f) = V,\ x_3^*(t_f) = D,\ x_4^*(t_f) = 0.$$

Then we must solve these equations (which will be nonlinear algebraic equations) to obtain c_2, c_3 and c_4.

(d) Now $J = -x_1(t_f)$. The costate equations and the optimal control (in terms of $p_2^*(t)$ and $p_4^*(t)$) will be as found earlier. The boundary conditions are

$$x_3^*(t_f) = D \quad p_1^*(t_f) = -1,\ p_2^*(t_f) = p_4^*(t_f) = 0.$$

With these boundary conditions
$c_1 = -1$, $c_2 = -t_f$, $-c_3 t_f + c_4 = 0$.
The procedure to solve for c_3 (or c_4), since t_f is known, is to substitute for $u^*(t)$ in the state equations (in terms of c_3), integrate, set

$$x_3^*(t_f) = D,$$

and solve for c_3.

5-24

(a) The solution of the state equation is

$$x(t) = \varphi(t)\left[x_0 + \int_0^t \varphi(-\tau)\, u(\tau)\, d\tau\right],$$

where $\varphi(t) = \epsilon^{2t}$. Suppose that a control is found which makes $x(T) = 0$, then

$$0 = x_0 + \int_0^T \varphi(-\tau)\, u(\tau)\, d\tau,$$

or

$$-x_0 = \int_0^T \varphi(-\tau)\, u(\tau)\, d\tau$$

which implies that

$$|x_0| = \left| \int_0^T \varphi(-\tau)\, u(\tau)\, d\tau \right|.$$

But,

$$\left| \int_0^T \varphi(-\tau) u(\tau)\, d\tau \right| \le \int_0^T |\varphi(-\tau)|\, |u(\tau)|\, d\tau,$$

and since $|u(t)| \le 1$,

$$|x_0| \le \int_0^T |\varphi(-\tau)|\, d\tau.$$

Substituting $\epsilon^{-2\tau}$ for $\varphi(-\tau)$ and integrating gives

$$|x_0| \le \tfrac{1}{2}\left[1 - \epsilon^{-2T}\right],$$

or

$$2|x_0| - 1 \le -\epsilon^{-2T} \Rightarrow \epsilon^{-2T} \le 1 - 2|x_0|.$$

Since $\epsilon^{-2T} > 0$ for all finite T, this can only be satisfied if

$$|x_0| < 1/2.$$

An alternative approach is the following:

Assume that $x_0 < 0$. By inspection of the state equation it is clear that if x is to be transferred to zero, then it must be possible to make $\dot{x}(t) > 0$. In fact, if $\dot{x}(0) < 0$ then $x(t)$ will be getting more negative; hence, if the control cannot make $\dot{x}(0)>0$, the control cannot bring the system to the origin. Therefore,

$\dot{x}(0) > 0$, or $0<2x_0+u(0)$,

and this implies that

$-2x_0 < u(0)$.

Since $x_0 < 0$, we have

$$-2x_0 < 1 \qquad \text{or} \qquad x_0 > -1/2.$$

Similar reasoning for $x_0 > 0$ leads to the conclusion that $x_0 < 1/2$; therefore,

$-1/2 < x_0 < 1/2$, or $|x_0| < 1/2$.

Both solutions indicate that if

$$|x_0| \geq 1/2$$

there is no admissible control which transfers the system to the origin.

(b) We now have n uncoupled state equations which have the solution

$$x_i(t) = \epsilon^{a_i t}\left[x_{i_0} + \int_0^t \epsilon^{-a_i \tau} b_i\, u(\tau)\, d\tau\right],$$

for $i = 1, 2, \dots, n$. Let us suppose that there is a control which makes $x_i(T) = 0$, $i = 1, 2, \dots, n$, then

$$0 = x_{i_0} + \int_0^T \epsilon^{-a_i \tau} b_i u(\tau) d\tau, \text{ or}$$

$$|x_{i_0}| = \left| \int_0^T \epsilon^{-a_i \tau} b_i u(\tau) d\tau \right|$$

$$\leq \int_0^T |\epsilon^{-a_i \tau}| |b_i| |u(\tau)| d\tau.$$

Since $|u(t)| \leq 1$, $\epsilon^{-a_i \tau} \geq 0$,

$$|x_{i_0}| \leq \left[\int_0^T \epsilon^{-a_i \tau} d\tau \right] |b_i|, \text{ or}$$

$$|x_{i_0}| \leq -\frac{|b_i|}{a_i} \left[\epsilon^{-a_i T} - 1 \right].$$

If $a_i < 0$, and T finite, this implies $|x_{i_0}| < \infty$. Now suppose that the largest a_i is a_1 and that $a_1 > 0$. In this case

$$\frac{a_1 |x_{1_0}|}{|b_1|} \leq 1 - \epsilon^{-a_1 T}, \text{ or}$$

$$\epsilon^{-a_1 T} \leq 1 - \frac{a_1 |x_{1_0}|}{|b_1|}.$$

Since $\epsilon^{-a_1T} > 0$ for all finite T, this implies that

$$\frac{a_1 |x_{1_0}|}{|b_1|} < 1$$

, or that only those initial states which satisfy

$$|x_{1_0}| < \frac{|b_1|}{a_1}$$

can be brought to the origin.

In summary, if all of the eigenvalues $(a_1, a_2, \ldots, a_n)$ are negative, all initial states can be transferred to the origin; if $a_1, a_2, \ldots, a_j$ are positive then initial states for which

$$|x_{i_0}| \geq \frac{|b_i|}{a_i} \quad , i = 1, 2, \ldots, j$$

cannot be transferred to the origin.

5-25

(a) $\det[s\underset{\sim}{I} - \underset{\sim}{A}] = \det\begin{bmatrix} s & -1 \\ 2 & s+2 \end{bmatrix}$

$$= s^2 + 2s + 2.$$

The roots are $s = -1 \pm j1$.

(b) $\mathcal{H} = 1 + p_1(t)\,x_2(t) - 2p_2(t)\,x_1(t)$
$- 2p_2(t)\,x_2(t) + p_2(t)\,u(t)$.

Therefore,

$$u^*(t) = \begin{cases} +1, & p_2^*(t) < 0 \\ -1, & p_2^*(t) > 0 \\ \text{undetermined}, & p_2^*(t) = 0. \end{cases}$$

To show that a singular interval, i.e. an interval $[t_1, t_2]$ such that $p_2^*(t) = 0$, cannot exist, we observe that it is necessary that $\mathcal{H}(*, t) = 0$ for $t \in [t_0, t_f]$. The costate equations are

$\dot{p}_1^*(t) = 2p_2^*(t)$

$\dot{p}_2^*(t) = -p_1^*(t) + 2p_2^*(t)$.

If there is a singular interval, $[t_1, t_2]$ such that $p_2^*(t) = 0$ for $t \in [t_1, t_2]$, then $\dot{p}_2^*(t) = 0$ for $t \in [t_1, t_2]$. But, if this is the case, $p_1^*(t) = 0$ for $t \in [t_1, t_2]$ (by inspection of the second costate equation). However, if $p_2^*(t) = p_1^*(t) = 0$ for $t \in [t_1, t_2]$, then $\mathcal{H}(*, t) = 1$ for $t \in [t_1, t_2]$. Since this violates a necessary condition for optimality, a singular interval cannot exist. If, therefore, an optimal control exists, it is bang-bang.

(c) The eigenvalues for the costate system of equations are the roots of $s^2 - 2s + 2$. These eigenvalues are $\lambda_1 = 1 - j1$, $\lambda_2 = 1 + j1$. The transition matrix for the costate system is of the form

$$\begin{bmatrix} \alpha_1 \epsilon^t \cos(t+\beta_1) & \alpha_3 \epsilon^t \cos(t+\beta_3) \\ \alpha_2 \epsilon^t \cos(t+\beta_2) & \alpha_4 \epsilon^t \cos(t+\beta_4) \end{bmatrix},$$

and

$$p_2^*(t) = p_1^*(0)\alpha_2 \epsilon^t \cos(t+\beta_2) + p_2^*(0)\alpha_4 \epsilon^t \cos(t+\beta_4).$$

5-25 (cont.)

$\alpha_2, \alpha_4, \beta_2$ and β_4 are constants. Since the cosine terms are periodic, $p_2^*(t)$ can change sign an unlimited number of times.

5-26

In the paper "Design of Quasi-Optimal Minimum-Time Controllers", which appeared in the IEEE Trans. Automatic Control, Vol. AC-11, No. 1, Jan. 1966, F.W. Smith states that the switching surface is given by

$$x_1 = -\left[\frac{x_3^3}{3} + d\, x_2 x_3 + d\left(\frac{x_3^2}{2} - d\, x_2\right)^{3/2}\right],$$

where $d = \operatorname{sign}\left[x_2 + \frac{x_3|x_3|}{2}\right]$.

5-27

Only the equations which differ from those in the example will be given.

(a) Let $x_5(t) \triangleq M(t)$, $u_2(t) \triangleq \dot{M}(t)$

$$\dot{x}_3(t) = x_4^2(t)/x_1(t) - g_0R^2/x_1^2(t) - \frac{k u_2(t)}{x_5(t)} \sin u_1(t)$$

$$\dot{x}_4(t) = -x_3(t)x_4(t)/x_1(t) - \frac{k u_2(t)}{x_5(t)} \cos u_1(t)$$

$$\dot{x}_5(t) = u_2(t).$$

(b) $$\dot{p}_5^*(t) = \frac{-k p_3^*(t) u_2^*(t)}{x_5^{*2}(t)} \sin u_1^*(t)$$

$$- \frac{k p_4^*(t) u_2^*(t)}{x_5^{*2}(t)} \cos u_1^*(t).$$

(c)

$$\frac{\partial \mathcal{H}}{\partial u_1}(*,t) = 0 = \frac{-k p_3^*(t) u_2^*(t)}{x_5^*(t)} \cos u_1^*(t)$$

$$+ \frac{k p_4^*(t) u_2^*(t)}{x_5^*(t)} \sin u_1^*(t).$$

To minimize $\mathcal{H}$ with respect to u_2, let

$$s(\underset{\sim}{x}(t), \underset{\sim}{p}(t), u_1(t)) \triangleq \frac{-k p_3(t)}{x_5(t)} \sin u_1(t)$$

$$- \frac{k p_4(t)}{x_5(t)} \cos u_1(t) + p_5(t).$$

Then

$$s(\underset{\sim}{x}^*(t), \underset{\sim}{p}^*(t), u_1^*(t)) u_2^*(t) \leq s(\underset{\sim}{x}^*(t), \underset{\sim}{p}^*(t), u_1(t)) u_2(t)$$

which means that

$$u_2^*(t) = \begin{cases} -\eta, & s(\underset{\sim}{x}^*(t), \underset{\sim}{p}^*(t), u_1^*(t)) > 0 \\ 0, & s(\underset{\sim}{x}^*(t), \underset{\sim}{p}^*(t), u_1^*(t)) < 0 \\ \text{undetermined}, & s(\underset{\sim}{x}^*(t), \underset{\sim}{p}^*(t), u_1^*(t)) = 0. \end{cases}$$

5-27 (cont.)

(d) Boundary conditions are the same as in Example 5.1-2 except

$$p_5^*(t_f) = 0$$

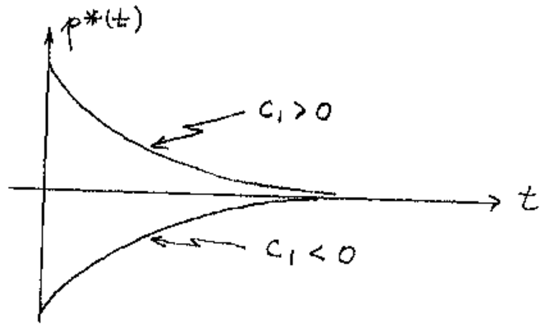
because $M(t_f)$ is free.

5-28

(a) $\mathcal{H} = 1 + p(t)\,a\,x(t) + p(t)\,u(t)$

$\dot{p}^*(t) = -a\,p^*(t) \Longrightarrow p^*(t) = c_1 \epsilon^{-at}.$

The possible forms for $p^*(t)$ are shown below.



Minimizing $\mathcal{H}$ with respect to u gives

$$u^*(t) = \begin{cases} -1, & p^*(t) > 0 \\ +1, & p^*(t) < 0 \\ \text{undetermined}, & p^*(t) = 0. \end{cases}$$

It is easily shown that $p^*(t)$

cannot equal zero for a finite time interval; hence, a singular interval cannot exist.

The possible forms for the optimal control are

$$\mathcal{U} = \{-1\}, \{1\}.$$

By inspection of the state equation it is clear that the time-optimal control law is

$$u^*(t) = \begin{cases} -1, & x(t) > 0 \\ +1, & x(t) < 0 \\ 0, & x(t) = 0 \end{cases}.$$

(b) $\mathcal{H} = |u(t)| + p(t)\,a\,x(t) + p(t)\,u(t)$.

The costate equations are the same as in part (a); hence, the costate solutions have the form shown previously.

Minimizing $\mathcal{H}$ with respect to u gives

$$u^*(t) = \begin{cases} -1, & 1 < p^*(t) \\ 0, & -1 < p^*(t) < 1 \\ +1, & p^*(t) < -1 \\ \text{undetermined}, & p^*(t) = \pm 1. \end{cases}$$

It can be shown that there are no singular intervals.

Since $p^*(t)$ cannot change sign, the possible forms for the optimal control are

$$u = \{-1, 0\}, \{1, 0\}, \{0\}, \{1\}, \{-1\}.$$

The first three possibilities cannot be optimal because the system cannot reach the origin at the end of an interval of zero control. This is seen to be the case from the state equation with $u(t) = 0$; the solution has the form

$$x(t) = x(t_1)\,\epsilon^{a(t-t_1)},$$

and since $a > 0$, $|x(t)| > |x(t_1)|$ for $t > t_1$ (the system moves further from the origin when $u = 0$). Hence, by inspection the optimal control law is

$$u^*(t) = \begin{cases} -1, & \text{for } x(t) > 0 \\ +1, & \text{for } x(t) < 0 \\ 0, & \text{for } x(t) = 0, \end{cases}$$

where it is assumed that x_0 is such that admissible controls can transfer the state to zero (see Problem 5-24).

5-28 (cont.)

(c) For those states for which an optimal control exists when $a>0$ the time-optimal solutions are the same for $a>0$ and for $a<0$.

For those states for which an optimal control exists when $a>0$, the fuel-optimal solution for $a>0$ is identical to the time optimal solution for $a>0$ and for $a<0$. This fuel-optimal solution is quite different than the results obtained for the fuel-optimal problem when $a<0$ (see Examples 5.5-2, 5.5-3, and 5.5-4). The difference is caused by the fact that with $a>0$ the system is unstable and moves away from the origin when no control is applied; thus, coasting increases the fuel required.

5-29

(a) $\mathcal{H} = |u(t)| + p_1(t)\,x_2(t) + p_2(t)\,x_1(t)$

$$+ p_2(t)\,x_2(t)\,u(t).$$

$$\dot{p}_1^*(t) = -\frac{\partial \mathcal{H}}{\partial x_1}(*, t) = -p_2^*(t)$$

5-29 (cont.)

$$\dot{p}_2^*(t) = -\frac{\partial \mathcal{H}}{\partial x_2}(*, t) = -p_1^*(t) - p_2^*(t)\,u^*(t).$$

(b) The form of the state equations is as assumed in Eq. (5.5-5) of the text; hence, the optimal control is given by Eq. (5.5-14), that is,

$$u^*(t) = \begin{cases} 1, & p_2^*(t)\,x_2^*(t) < -1 \\ 0, & -1 < p_2^*(t)\,x_2^*(t) < 1 \\ -1, & 1 < p_2^*(t)\,x_2^*(t) \\ \text{Undetermined}, & p_2^*(t)\,x_2^*(t) = \pm 1. \end{cases}$$

5-30

$$\mathcal{H} = \lambda + \sum_{i=1}^{m} r_{ii}\,u_i^2(t) + \mathbf{p}^T(t)\,\mathbf{a}(\mathbf{x}(t), t)$$

$$+ \mathbf{p}^T(t)\,\mathbf{B}(\mathbf{x}(t), t)\,\mathbf{u}(t).$$

$\mathbf{p}^T(t)\,\mathbf{B}(\mathbf{x}(t), t)$ can be written

$$\left[\mathbf{p}^T(t)\,\mathbf{b}_1(\mathbf{x}(t), t) \,\middle|\, \mathbf{p}^T(t)\,\mathbf{b}_2(\mathbf{x}(t), t) \,\middle|\, \cdots \,\middle|\, \mathbf{p}^T(t)\,\mathbf{b}_m(\mathbf{x}(t), t)\right],$$

where $\mathbf{b}_i$ is the ith column of $\mathbf{B}$. Then the terms in $\mathcal{H}$ which involve $\mathbf{u}$ are

$$\sum_{i=1}^{m}\left[r_{ii}\,u_i^2(t) + \mathbf{p}^T(t)\,\mathbf{b}_i(\mathbf{x}(t), t)\,u_i(t)\right].$$

If the controls are not constrained, the extremal controls are found from

$$0 = \frac{\partial \mathcal{H}}{\partial u_i}(*, t) = 2r_{ii}u_i^*(t) + \underset{\sim}{p}^{*T}(t)\,\underset{\sim}{b}_i(\underset{\sim}{x}^*(t), t),$$

or,

$$u_i^*(t) = -\frac{1}{2r_{ii}}\,\underset{\sim}{p}^{*T}(t)\,\underset{\sim}{b}_i(\underset{\sim}{x}^*(t), t),$$

$i = 1, 2, \ldots, m.$

If the controls are constrained by

$$|u_i(t)| \leq 1, \quad i = 1, 2, \ldots, m,$$

then,

$$u_i^*(t) = \begin{cases} -\frac{1}{2r_{ii}}\underset{\sim}{p}^{*T}(t)\underset{\sim}{b}_i(\underset{\sim}{x}^*(t), t), & \left|\frac{\underset{\sim}{p}^{*T}(t)\underset{\sim}{b}_i}{2r_{ii}}\right| < 1 \\ +1, & \frac{1}{2r_{ii}}\underset{\sim}{p}^{*T}(t)\,\underset{\sim}{b}_i \leq -1 \\ -1, & 1 \leq \frac{1}{2r_{ii}}\underset{\sim}{p}^{*T}(t)\,\underset{\sim}{b}_i \end{cases}.$$

5-31

(a) $\mathcal{H} = 1 + p_1(t)x_2(t) - p_2(t)g - \frac{k p_2(t)}{x_3(t)}u(t) + p_3(t)u(t)$

$$\dot{p}_1^*(t) = -\frac{\partial \mathcal{H}}{\partial x_1}(*, t) = 0$$

$$\dot{p}_2^*(t) = -\frac{\partial \mathcal{H}}{\partial x_2}(*, t) = -p_1^*(t)$$

$$\dot{p}_3^*(t) = -\frac{\partial \mathcal{H}}{\partial x_3}(*,t) = \frac{-k p_2^*(t)}{x_3^{*2}(t)}\, u^*(t).$$

The boundary conditions are

$$x_1^*(t_f) = 0,\ x_2^*(t_f) = 0,\ p_3^*(t_f) = 0,\ \mathcal{H}(*,t_f) = 0.$$

(b) $\mathcal{H} = |u(t)| + p_1(t)x_2(t) - p_2(t)g - \frac{k p_2(t)}{x_3(t)} u(t) + p_3(t)u(t)$

The costate equations and boundary conditions are as given in part (a). The control which minimizes $\mathcal{H}$ is given by

$$u^*(t) = \begin{cases} -M, & 1 < p_3^*(t) - k p_2^*(t)/x_3^*(t) \\ 0, & p_3^*(t) - k p_2^*(t)/x_3^*(t) < 1 \\ \text{undetermined}, & p_3^*(t) - k p_2^*(t)/x_3^*(t) = 1. \end{cases}$$

An alternative formulation is to let $J = -x_3(t_f)$; doing this we have

$$\mathcal{H} = p_1(t)x_2(t) - p_2(t)g - \frac{k p_2(t)}{x_3(t)} u(t) + p_3(t)u(t).$$

The costate equations are unchanged; however, the expression for the optimal control is as given in Eq. (5.4-25) of the text. The boundary conditions are

$$x_1^*(t_f) = 0,\ x_2^*(t_f) = 0,\ p_3^*(t_f) = -1,\ \mathcal{H}(*,t_f) = 0.$$

(c) $\frac{d}{dt}[\dot{x}(t)] = -g - k\frac{d}{dt}[\ln(m(t))]$.

Integrating both sides from 0 to t gives

$$\dot{x}(t) = \dot{x}(0) - gt - k\ln(m(t)/m(0)).$$

Letting $t = t_f$, and solving for $m(t_f)$ gives

5-31 (cont.)

$$m(t_f) = m(0)\,\epsilon^{\frac{1}{k}(\dot{x}(0) - g t_f)}$$
$$= m(0)\,\epsilon^{\dot{x}(0)/k}\,\epsilon^{-g t_f/k}$$
$$m(t_f) = c_1\,\epsilon^{-g t_f/k}. \qquad (\mathrm{I})$$

In obtaining this result we have used the fact that $\dot{x}(t_f) = 0$.

Eq. (I) clearly shows that the larger t_f is, the smaller the final mass is. This implies that fuel expenditure is a monotone increasing function of the final time; hence, the minimum-time and minimum-fuel solutions are identical.

5-32

For the minimum-time case there are at most $(n-1)$ switchings; hence, the functions

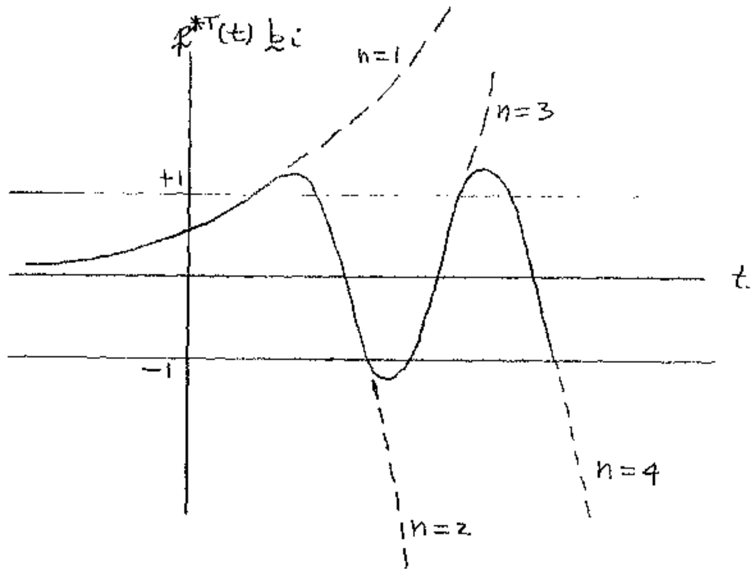
$$\underset{\sim}{p}^{*T}(t)\,\underset{\sim}{b}_i, \quad i = 1, 2, \ldots, m$$

have at most $(n-1)$ zeros. In the minimum-fuel case we are interested in how many times

$$\underset{\sim}{p}^{*T}(t)\,\underset{\sim}{b}_i$$

passes through the values ± 1. Since $\underset{\sim}{p}^{*T}(t)\,\underset{\sim}{b}_i$ has only $(n-1)$ zeros, it is seen that $\underset{\sim}{p}^{*T}(t)\,\underset{\sim}{b}_i$ also has at most $(n-1)$ maxima and minima (see sketch below). The dashed curves show

how the functions must behave as $t \to \infty$ for various values of n.



It is not difficult to convince oneself that the curve shown for $n=1$ yields the maximum possible number of switchings (1). Similarly, the other possibilities shown represent situations in which the maximum number of crossings of the ± 1 points occur. Counting the maximum number of switchings for the values of n shown leads to $(2n-1)$ as the upper bound on the number of control switchings. It should also be clear that this upper bound also applies for larger values of n.

5-33

(a) $\mathcal{H}_1 = \lambda + \sum_{i=1}^{m} |u_i(t)| + \mathbf{p}^T(t)\left[\mathbf{a}(\mathbf{x}(t),t) + \mathbf{B}(\mathbf{x}(t),t)\,\mathbf{u}(t)\right]$

$\mathcal{H}_2 = 1 + \mathbf{p}^T(t)\left[\mathbf{a}(\mathbf{x}(t),t) + \mathbf{B}(\mathbf{x}(t),t)\,\mathbf{u}(t)\right]$

$\mathcal{H}_3 = \sum_{i=1}^{m} |u_i(t)| + \mathbf{p}^T(t)\left[\mathbf{a}(\mathbf{x}(t),t) + \mathbf{B}(\mathbf{x}(t),t)\,\mathbf{u}(t)\right].$

(b) In all three cases the costate equations are given by

$$\dot{p}_i^*(t) = -\frac{\partial}{\partial x_i}\left[\mathbf{p}^{*T}(t)\left\{\mathbf{a}(\mathbf{x}^*(t),t) + \mathbf{B}(\mathbf{x}^*(t),t)\,\mathbf{u}^*(t)\right\}\right].$$

5-34

(a) $\mathcal{H} = \lambda + |u(t)| + p_1(t)x_2(t) - a p_2(t) x_2(t) + p_2(t)u(t)$

$$\dot{p}_1^*(t) = -\frac{\partial \mathcal{H}}{\partial x_1}(*,t) = 0 \Rightarrow p_1^*(t) = c_1$$

$$\dot{p}_2^*(t) = -\frac{\partial \mathcal{H}}{\partial x_2}(*,t) = -p_1^*(t) + a p_2^*(t)$$
$$= -c_1 + a p_2^*(t)$$

$$u^*(t) = \begin{cases} -1, & p_2^*(t) > 1 \\ 0, & -1 < p_2^*(t) < 1 \\ +1, & p_2^*(t) < -1 \\ \text{Undetermined}, & p_2^*(t) = \pm 1. \end{cases}$$

(b) The solution of the second costate equation is of the form

$$p_2^*(t) = c_2 + c_3 \epsilon^{at}. \qquad \text{(I)}$$

By sketching the possible forms for $p_2^*(t)$ it is found that the candidates for optimal controls have the forms

$$u = \{-1\}, \{0\}, \{+1\}, \{0,-1\}, \{0,+1\}, \{-1,0,+1\}, \{+1,0,-1\}.$$

(c) Since t_f is free and $\mathcal{H}$ does not depend explicitly on t, $\mathcal{H}$ is identically zero on an extremal trajectory. We also know that if $p_2^*(t) = +1$, then $u^*(t)$, although undetermined, is non-positive (see Eq. (5.5-14) of the text). It also can be shown that in (I) [the expression for $p_2^*(t)$] the value for c_2 is c_1/a. Using these relationships and assuming that $p_2^*(t) = 1$ for a finite time interval, we have

$$\mathcal{H} = \lambda + |u^*(t)| + c_1 x_1^*(t) - a x_2^*(t) + u^*(t).$$

But, since $p_2^*(t) = 1$ for a finite time interval, $c_3 = 0$, $c_2 = c_1/a = 1 \Rightarrow c_1 = a$; thus,

$$\mathcal{H} = \lambda + |u^*(t)| + u^*(t).$$

But we know that for $p_2^*(t) = 1$, $u^*(t) \le 0$; therefore $|u^*(t)| = -u^*(t)$ and $\mathcal{H} = \lambda \neq 0$, a contradiction.

It can be shown in a similar manner that $p_2^*(t)$ cannot be equal to -1 for a non-zero time interval; thus, singular intervals cannot exist.

(d) The procedure here is essentially the same as that used in solving Example 5.5-5 in the text. First, consider only those trajectories which terminate with an interval of $u=-1$, that is, $u = \{-1\}, \{0,-1\}, \{1,0,-1\}$. The results obtained in Example 5.4-5 of the text can be used to give

$$x_1(t) = \frac{1}{a^2}\ln\left(1+a\,x_2(t)\right) - \frac{1}{a}x_2(t) \qquad (\text{II})$$

as the equation for the $u=-1$ trajectory which passes through the origin. Eq. (II) also describes the switching curve where the control changes from 0 to -1. Next, we need to determine the curve where u switches from $+1$ to 0. First, we solve the state equations using $u=0$ to obtain

$$x_1(t_2) = x_1(t_1) + \frac{1}{a}\left[1-\epsilon^{-a(t_2-t_1)}\right]x_2(t_1) \qquad (\text{III})$$

$$x_2(t_2) = x_2(t_1)\,\epsilon^{-a(t_2-t_1)}. \qquad (\text{IV})$$

In the expression (I) for $p_2^*(t)$ it can be shown that $c_2 = c_1/a$;

therefore, we have

$$p_2^*(t_1) = \frac{c_1}{a} + c_3 \epsilon^{a t_1} = -1 \qquad \text{(V)}$$

$$p_2^*(t_2) = \frac{c_1}{a} + c_3 \epsilon^{a t_2} = +1; \qquad \text{(VI)}$$

t_1 refers to the time when u switches from the value $+1$ to 0, and t_2 is the time when u switches from 0 to -1. Since t_f is free and $\mathcal{H}$ does not depend on t explicitly, we know that $\mathcal{H}$ must be zero on an extremal trajectory. Using this knowledge, Eqs. (V) and (VI), and the knowledge that $u^*(t_1) \geq 0$ and $u^*(t_2) \leq 0$, we obtain

$$\lambda + c_1 x_2(t_1) + a x_2(t_1) = 0 \qquad \text{(VII)}$$

$$\lambda + c_1 x_2(t_2) - a x_2(t_2) = 0. \qquad \text{(VIII)}$$

Our objective is to solve for $x_1(t_1)$ in terms of $x_2(t_1)$. We begin by solving (V) and (VI) for $(t_2 - t_1)$ with the result

$$t_2 - t_1 = \frac{1}{a} \ln\left(\frac{c_1 - a}{c_1 + a}\right).$$

To eliminate c_1, we solve (VII) to obtain

$$c_1 = \frac{-\lambda - a x_2(t_1)}{x_2(t_1)}.$$

5-34 (cont.)

Eliminating c_1 gives

$$(t_2 - t_1) = \frac{1}{a} \ln\left(\frac{\lambda + 2a\,x_2(t_1)}{\lambda}\right),$$

which when substituted in (III) and (IV) gives

$$x_1(t_2) = x_1(t_1) + \frac{x_2(t_1)}{a} - \frac{\lambda\, x_2(t_1)}{a[\lambda + 2a x_2(t_1)]}$$

$$x_2(t_2) = \frac{\lambda\, x_2(t_1)}{\lambda + 2a\, x_2(t_1)}.$$

Using (II) we obtain then

$$x_1(t_2) = \frac{1}{a^2} \ln\left(1 + \frac{a\lambda\, x_2(t_1)}{\lambda + 2a\, x_2(t_1)}\right) - \frac{1}{a}\,\frac{\lambda\, x_2(t_1)}{\lambda + 2a x_2(t_1)}$$

$$= x_1(t_1) + \frac{x_2(t_1)}{a}\left[1 - \frac{\lambda}{\lambda + 2a x_2(t_1)}\right],$$

or,

$$x_1(t_1) = -\frac{1}{a}\,x_2(t_1) + \frac{1}{a^2} \ln\left(1 + \frac{a\lambda\, x_2(t_1)}{\lambda + 2a\, x_2(t_1)}\right).$$

This equation, which applies for $x_2 > 0$, is the sought after result.

Using similar reasoning, it can be shown that u switches from -1 to 0 where

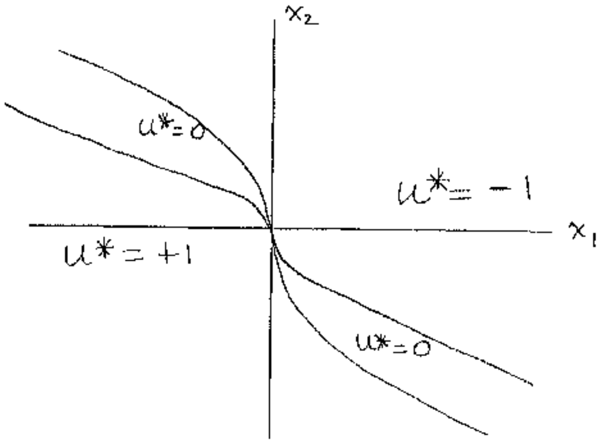
$$x_1(t_2') = -\frac{1}{a}\,x_2(t_2') - \frac{1}{a^2} \ln\left(1 - a x_2(t_2')\right),$$

and that switching from 0 to $+1$ occurs where

5-34 (cont.)

$$x_1(t_1') = -\frac{x_2(t_1')}{a} - \frac{1}{a^2}\ln\left(1 - \frac{a\lambda x_2(t_1')}{\lambda - 2a x_2(t_1')}\right).$$

These last two switching curves apply for $x_2 < 0$. The switching curves and the optimal control law are shown below.



5-35

(a) $$\mathcal{H} = 1 + p_1(t)\,x_2(t) - \alpha p_2(t)\,x_2(t)\left[x_2^2(t) + x_4^2(t)\right]^{1/2}$$
$$+ p_2(t)\,u_1(t) + p_3(t)\,x_4(t) - \alpha p_4(t)\,x_4(t)\left[x_2^2(t) + x_4^2(t)\right]^{1/2}$$
$$+ p_4(t)\,u_2(t) \quad ; \text{ Let } \beta(t) \triangleq \left[x_2^2(t) + x_4^2(t)\right]$$

$$\dot{p}_1^*(t) = -\frac{\partial \mathcal{H}}{\partial x_1}(*, t) = 0$$

$$\dot{p}_2^*(t) = -\frac{\partial \mathcal{H}}{\partial x_2}(*, t) = -p_1^*(t) + \alpha p_2^*(t)\left\{\left[\beta^*(t)\right]^{1/2}\right.$$
$$\left. + x_2^{*2}(t)\left[\beta^*(t)\right]^{-1/2}\right\} + \alpha p_4^*(t)\,x_2^*(t)\,x_4^*(t)\left[\beta^*(t)\right]^{-1/2}$$

5-35 (cont.)

$$\dot{p}_3^*(t) = -\frac{\partial \mathcal{H}}{\partial x_3}(*, t) = 0$$

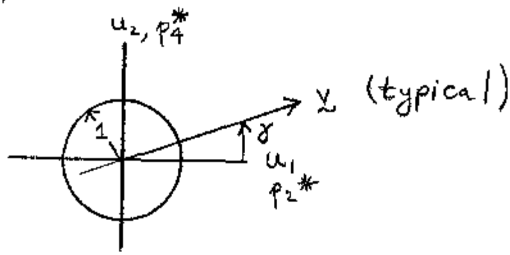
$$\dot{p}_4^*(t) = -\frac{\partial \mathcal{H}}{\partial x_4}(*, t) = -p_3^*(t) + \alpha p_4^*(t) \left\{ \frac{x_2^{*2}(t) + 2x_4^{*2}(t)}{[\beta^*(t)]^{1/2}} \right\}$$

$$+ \frac{\alpha\, p_2^*(t)\, x_2^*(t)\, x_4^*(t)}{[\beta^*(t)]^{1/2}} .$$

(b) $x_1^*(t_f) = e_1$, $p_2^*(t_f) = 0$, $x_3^*(t_f) = e_3$

$p_4^*(t_f) = 0$, $\mathcal{H}(*, t_f) = 0$.

(c)



Let $\underset{\sim}{v} = [p_2^*(t) \quad p_4^*(t)]^T$; we seek the admissible $\underset{\sim}{u}^*$ which minimizes

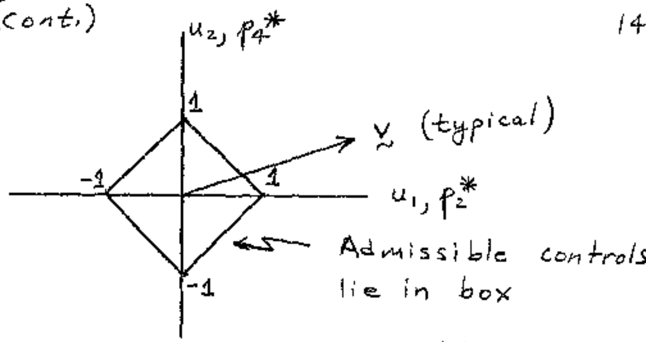
$\underset{\sim}{v}^T \underset{\sim}{u}$.

Assuming $\underset{\sim}{v} \neq \underset{\sim}{0}$, the admissible control that minimizes $\underset{\sim}{v}^T \underset{\sim}{u}$ has the largest possible amplitude, i.e. $u_1^2(t) + u_2^*(t) = 1$, and is directed oppositely to $\underset{\sim}{v}$; therefore,

$u_1^*(t) = -\cos \gamma(t)$, $u_2^*(t) = -\sin \gamma(t)$,

where $$\gamma(t) = \tan^{-1}\left[\frac{p_4^*(t)}{p_2^*(t)}\right].$$

(d)

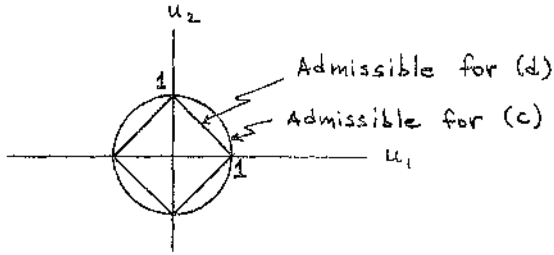


By inspection it is seen that (except when the angle of $\underset{\sim}{v}$ is $45°$, $135°$, $225°$, and $315°$) the maximum projection of a vector $\underset{\sim}{u}$ onto $\underset{\sim}{v}$ occurs when $\underset{\sim}{u}$ lies at one of the vertices; the $\underset{\sim}{u}$ that minimizes $\underset{\sim}{v}^T\underset{\sim}{u}$ lies at the opposite vertex. The minimizing control is

$$\underset{\sim}{u}^*(t) = \begin{cases} [-1 \quad 0]^T, & |p_4^*(t)/p_2^*(t)| < 1,\ p_2^*(t) > 0 \\ [1 \quad 0]^T, & |p_4^*(t)/p_2^*(t)| < 1,\ p_2^*(t) < 0 \\ [0 \quad -1]^T, & |p_4^*(t)/p_2^*(t)| > 1,\ p_4^*(t) > 0 \\ [0 \quad 1]^T, & |p_4^*(t)/p_2^*(t)| > 1,\ p_4^*(t) < 0. \end{cases}$$

(e) Part (c) controls must be at least as effective as Part (d) controls; as seen in the sketch below, every admissible control for part (d) is also admissible for part (c) — the reverse is not true, however. Thus,

$$J_c \le J_d.$$



5-36

We will assume, as in Example 5.5-5, that $\underset{\sim}{x}(t_f)=\underset{\sim}{0}$.

(a) $\mathcal{H} = |u(t)| + p_1(t)\,x_2(t) + p_2(t)\,u(t)$

$$\dot{p}_1^*(t) = -\frac{\partial \mathcal{H}}{\partial x_1}(*,t) = 0 \Rightarrow p_1^*(t) = c_1$$

$$\dot{p}_2^*(t) = -\frac{\partial \mathcal{H}}{\partial x_2}(*,t) = -p_1^*(t) \Rightarrow p_2^*(t) = -c_1 t + c_2.$$

(b) If there are no singular intervals,

$$u^*(t) = \begin{cases} -1, & p_2^*(t) > 1 \\ 0, & |p_2^*(t)| < 1 \\ +1, & p_2^*(t) < -1. \end{cases}$$

To show the existence of non-unique optimal controls we first find a lower bound on the fuel required to transfer the system to the origin. From the second state equation

5-36 (cont.)

$$x_2(t) = x_{20} + \int_0^t u(t)\,dt\,, \qquad (\mathrm{I})$$

but, we have assumed that $x_2(t_f)=0$, hence,

$$|x_{20}| = \left| \int_0^{t_f} u(t)\,dt \right| \le \int_0^{t_f} |u(t)|\,dt.$$

Thus, if we can construct a control signal which satisfies

$$|x_{20}| = \int_0^{t_f} |u(t)|\,dt, \qquad (\mathrm{II})$$

then this control signal is fuel optimal. To determine such a control we notice in (I) that if $u(t)$ does not change sign, then (II) will be satisfied. Note that t_f is not specified.

Actually, it is not difficult to generate many optimal controls. For example, if we assume that

$$u(t) = \begin{cases} 0<M_1, & t \in [0, t_1] \\ 0, & t \in (t_1, t_2) \\ 0<M_2, & t \in [t_2, t_f] \end{cases}$$

it can be shown that

$$x_1(t_f) = 0 = x_{10} - \tfrac{1}{2} M_1 t_1^2 + x_{20} t_f + M_1 t_1 t_f - M_2 t_2 t_f + \tfrac{1}{2} M_2 t_2^2 + \tfrac{1}{2} M_2 t_f^2 \qquad (\mathrm{III})$$

$$x_2(t_f) = 0 = x_{20} + M_1 t_1 + M_2 t_f - M_2 t_2\,. \qquad (\mathrm{IV})$$

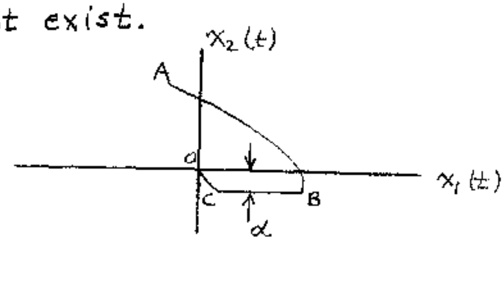
5-36 (cont.)

Suppose $\underline{x}_0 = [1 \;\; -1]^T$. If we let $M_1 = M_2 = .5$ and $t_1 = 1$, (III) and (IV) can be solved for $t_2 = 7$ and $t_f = 8$. Similarly, if $M_1 = M_2 = .25$, $t_1 = 2$, solving (III) and (IV) gives $t_2 = 6$, $t_f = 8$. Both of these control signals (and many more) require the minimum fuel possible, and hence are optimal.

(c) Consider the trajectory shown below; the initial segment corresponds to $u(t) = -1$. The segment B-C results from $u(t) = 0$, and C-O results from $u(t) = +1$. It is easily verified that the fuel required is equal to

$$|x_{20}| + 2\alpha.$$

In the limit as $\alpha \to 0$, the required fuel approaches the lower bound on fuel expenditure; however, as $\alpha \to 0$, $t_f \to \infty$. Thus, the fuel required can be arbitrarily close to, but never attain, the lower bound. This means that a minimum-fuel control does not exist.



(d) Singular intervals can exist if $p_2^*(t) = \pm 1$ for a finite time interval. This occurs if $c_1 = 0$, $c_2 = \pm 1$, and, if so, the condition that $\mathcal{H}(*,t) = 0$ is satisfied. This indicates that singular intervals can exist; the controls generated in part (b) are in fact singular and optimal.

5-38

(a) $\mathcal{H} = \frac{1}{2}x_1^2(t) + p_1(t)x_2(t) + p_1(t)u(t) - p_2(t)u(t)$

$$\dot{p}_1^*(t) = -\frac{\partial \mathcal{H}}{\partial x_1}(*,t) = -x_1^*(t) \qquad (\mathrm{I})$$

$$\dot{p}_2^*(t) = -\frac{\partial \mathcal{H}}{\partial x_2}(*,t) = -p_1^*(t). \qquad (\mathrm{II})$$

The boundary conditions are $\mathbf{x}(t_f) = \mathbf{0}$.

(b) Minimizing $\mathcal{H}$ with respect to u gives

$$u^*(t) = \begin{cases} -1, & 0 < p_1^*(t) - p_2^*(t) \\ +1, & p_1^*(t) - p_2^*(t) < 0 \\ \text{undetermined}, & p_1^*(t) - p_2^*(t) = 0. \end{cases}$$

A singular interval can exist if

$$p_1^*(t) - p_2^*(t) = 0, \text{ for } t \in [t_1, t_2]. \qquad (\mathrm{III})$$

Combined with the necessary condition that $\mathcal{H}(*,t) = 0$ for $t \in [0, t_f]$, Eq. (III) implies that

$$\frac{1}{2}x_1^{*2}(t) + p_1^*(t)\,x_2^*(t) = 0, \text{ for } t\in[t_1, t_2]. \quad (\text{IV})$$

Differentiating (IV) and setting the result equal to zero gives

$$x_1^*(t)\,\dot{x}_1^*(t) + p_1^*(t)\,\dot{x}_2^*(t) + \dot{p}_1^*(t)\,x_2^*(t) = 0\,. \quad (\text{V})$$

Substituting the expression $p_1^*(t) = -x_1^{*2}(t)/2x_2^*(t)$ from Eq. (IV) and the right sides of the state and costate equations for $\dot{x}_1^*(t)$, $\dot{x}_2^*(t)$ and $\dot{p}_1^*(t)$ into Eq. (V), and rearranging yields

$$2x_1^*(t)\,x_2^*(t) + x_1^{*2}(t) = 0\;,\; t\in[t_0, t_1]$$

or

$$x_1^*(t)\left[2x_2^*(t) + x_1^*(t)\right] = 0\,.$$

Hence, the candidates are

$$x_1^*(t) = 0\;, \quad x_1^*(t) = -2x_2^*(t)\,.$$

From the state equations

$$x_1^*(t) = 0 \Rightarrow \dot{x}_1^*(t) = 0 \Rightarrow u^*(t) = -x_2^*(t),$$

but this trajectory always moves away from the desired final state (the origin) along the vertical axis, hence it cannot be optimal.

On the other singular segment, $x_1^*(t) = -2x_2^*(t)$, the state equations indicate that

$$x_1^*(t) = -2x_2^*(t) \Rightarrow \dot{x}_1^*(t) = -2\dot{x}_2^*(t) = 2u^*(t)$$
$$= u^*(t) + x_2^*(t).$$

5-38 (cont.)

Thus,

$$u^*(t) = x_2^*(t).$$

Whether or not this singular trajectory segment is a part of any optimal trajectories must be determined by further investigation.

5-40

$$\mathcal{H} = 1 + p_1(t)\,x_2(t) - g\,p_2(t) - \frac{k\,p_2(t)}{x_3(t)}\,u(t) + p_3(t)u(t)$$

$$\dot{p}_1^*(t) = -\frac{\partial\mathcal{H}}{\partial x_1}(*,t) = 0 \;\Rightarrow\; p_1^*(t) = c_1$$

$$\dot{p}_2^*(t) = -\frac{\partial\mathcal{H}}{\partial x_2}(*,t) = -p_1^*(t) \;\Rightarrow\; p_2^*(t) = -c_1 t + c_2$$

$$\dot{p}_3^*(t) = -\frac{\partial\mathcal{H}}{\partial x_3}(*,t) = -k\,p_2^*(t)\,u^*(t)\,/\,x_3^{*2}(t)\;.$$

A singular interval exists if

$$p_3^*(t) - \frac{k\,p_2^*(t)}{x_3^*(t)} = 0\;,\quad t\in[t_1,t_2]. \qquad (\mathrm{I})$$

Recall that $k>0$, $x_3^*(t)>0$. Since t_f is free and t does not appear explicitly in $\mathcal{H}$, $\mathcal{H}(*,t)=0$, $t\in[t_0,t_f]$. Thus, if (I) is satisfied

$$1 + p_1^*(t)\,x_2^*(t) - g\,p_2^*(t) = 0,\; t\in[t_1,t_2]. \quad (\mathrm{II})$$

Since (II) holds for a finite time interval, the derivative with respect to time of the left side of (II) must also be zero for $t\in[t_1,t_2]$, hence

$$\dot{p}_1^*(t)\,x_2^*(t) + p_1^*(t)\,\dot{x}_2^*(t) - g\,\dot{p}_2^*(t) = 0,\; t\in[t_1,t_2].$$

Substituting $\dot{p}_1^*(t)=0$, $\dot{p}_2^*(t)=-c_1$, and $\dot{x}_2^*(t)=-g-ku^*(t)/x_3^*(t)$ gives

$$c_1\left[-g-\frac{ku^*(t)}{x_3^*(t)}\right]+gc_1=0 \Rightarrow c_1ku^*(t)=0$$

because $x_3^*(t)>0$. There are then two subcases to consider.

(i) Suppose $c_1=0$, then (II) implies that $p_2^*(t)=1/g$. Since $x_3^*(t_f)$ is free, $p_3^*(t_f)=0$. Using the fact that $u^*(t)\le 0$ and $p_2^*(t)=1/g$ in the third costate equation gives $\dot{p}_3^*(t)\ge 0$, $t\in[t_0,t_f]$. This implies that $p_3^*(t)\le 0$ for $t\in[t_0,t_f]$. Looking at $p_3^*(t)-[kp_2^*(t)/x_3^*(t)]$ we see that since $p_3^*(t)\le 0$, $p_2^*(t)>0$, $x_3^*(t)>0$, $-kp_2^*(t)/x_3^*(t)<0$ and hence

$$p_3^*(t)-\frac{kp_2^*(t)}{x_3^*(t)}<0$$

; a contradiction which implies that a singular interval cannot exist.

(ii) Suppose $u^*(t)=0 \Rightarrow \dot{p}_3^*(t)=0$ and $\dot{x}_3^*(t)=0$ which in turn implies that $p_3^*(t)=c_3$ and $x_3^*(t)=c_4$ for $t\in[t_1,t_2]$. Hence, (I) can hold only if $p_2^*(t)=$ a constant $=c_2$, but this implies that $p_1^*(t)=c_1=0$ for $t\in[t_1,t_2]$ and we have already shown that a singular interval cannot exist if $c_1=0$.

Our conclusion is then that a singular interval cannot exist.

5-41

(a) $\mathcal{H} = \lambda + |u(t)| + p_1(t)x_2(t) + p_2(t)u(t)$

A singular interval exists if $p_2^*(t) = \pm 1$.
The costate equations are

$$\dot{p}_1^*(t) = -\frac{\partial \mathcal{H}}{\partial x_1}(*,t) = 0 \Rightarrow p_1^*(t) = c_1$$

$$\dot{p}_2^*(t) = -\frac{\partial \mathcal{H}}{\partial x_2}(*,t) = -p_1^*(t) \Rightarrow p_2^*(t) = -c_1 t + c_2 .$$

If $p_2^*(t) = \pm 1$, $t \in [t_1, t_2]$, then $c_1 = 0$, $c_2 = \pm 1 \Rightarrow p_1^*(t) = 0$, $t \in [t_0, t_f]$. But this implies that $\mathcal{H}(*,t) = \lambda > 0$ which contradicts the necessary condition that $\mathcal{H}(*,t) = 0$; therefore, a singular interval cannot exist.

(b) $\mathcal{H} = \lambda + |u(t)| + \underline{p}^T(t)\underline{A}\,\underline{x}(t) + \underline{p}^T(t)\underline{b}\,u(t)$

If $\underline{p}^{*T}(t)\underline{b} = \pm 1$, a singular interval exists. If $\underline{p}^{*T}(t)\underline{b} = \pm 1 \Rightarrow \underline{p}^{*T}(t)\underline{A} = \underline{0}$ (as in part (a) of this problem) then we have $\mathcal{H}(*,t) = \lambda > 0$ which contradicts the necessary condition that $\mathcal{H}(*,t) = 0$, in which case a singular interval cannot exist. It does not seem possible to show that in general $\underline{p}^{*T}(t)\underline{b} = \pm 1$ $\Rightarrow \underline{p}^{*T}(t)\underline{A} = \underline{0}$. We can, however, show as in Section 5.6 that a singular interval can exist only if either $\underset{\sim}{A}$ or $[\underset{\sim}{b} \mid \underset{\sim}{A}\underset{\sim}{b} \mid \cdots \mid \underset{\sim}{A}^{n-1}\underset{\sim}{b}]$, or both, are singular. Notice that this is not a sufficient condition for the existence of a singular interval -- in part (a) $\underset{\sim}{A}$ is singular, but a singular interval does not exist.

CHAPTER 6

6-1

(a) The desired value of $\underset{\sim}{p}(t_f)$ is $\underset{\sim}{d}$. Noting that $\underset{\sim}{p}(t_f)$ is a function of $\underset{\sim}{p}(t_0)$ we can write

$$\underset{\sim}{p}(t_f) = \underset{\sim}{f}(\underset{\sim}{p}(t_0)).$$

Expanding in a Taylor series and retaining terms of up to first order gives

$$\underset{\sim}{p}^{(i+1)}(t_f) = \underset{\sim}{p}^{(i)}(t_f) + \left[\frac{\partial \underset{\sim}{f}}{\partial \underset{\sim}{p}}(\underset{\sim}{p}^{(i)}(t_0))\right]\left[\underset{\sim}{p}^{(i+1)}(t_0) - \underset{\sim}{p}^{(i)}(t_0)\right]. \qquad (\mathrm{I})$$

The matrix $\partial \underset{\sim}{f}/\partial \underset{\sim}{p}$ is the same as the matrix denoted by $\underset{\sim}{P}_p(\underset{\sim}{p}^{(i)}(t_0), t_f)$. To save writing, we will shorten this to $\underset{\sim}{P}_p^{(i)}$.

Since we desire to make $\underset{\sim}{p}(t_f) = \underset{\sim}{d}$, we set $\underset{\sim}{p}^{(i+1)}(t_f) = \underset{\sim}{d}$ in (I) and solve for $\underset{\sim}{p}^{(i+1)}(t_0)$ with the result

$$\underset{\sim}{p}^{(i+1)}(t_0) = \underset{\sim}{p}^{(i)}(t_0) - \left[\underset{\sim}{P}_p^{(i)}\right]^{-1}\left[\underset{\sim}{p}^{(i)}(t_f) - \underset{\sim}{d}\right].$$

(b) In this case, the desired value of $\underset{\sim}{p}(t_f)$ is $2\underset{\sim}{H}\,\underset{\sim}{x}(t_f)$, so we set $\underset{\sim}{p}^{(i+1)}(t_f) = 2\underset{\sim}{H}\,\underset{\sim}{x}^{(i+1)}(t_f)$. For the dependence of $\underset{\sim}{x}(t_f)$ on $\underset{\sim}{p}(t_0)$ we have the equation analogous to (I),

$$\underset{\sim}{x}^{(i+1)}(t_f) = \underset{\sim}{x}^{(i)}(t_f) + \left[\underset{\sim}{P}_x^{(i)}\right]\left[\underset{\sim}{p}^{(i+1)}(t_0) - \underset{\sim}{p}^{(i)}(t_0)\right]. \qquad (\mathrm{II})$$

Replacing $\underset{\sim}{p}^{(i+1)}(t_f)$ in (I) by $2\underset{\sim}{H}\,\underset{\sim}{x}^{(i+1)}(t_f)$ and using the right side of (II) gives

$$2\underline{H}\,\underline{x}^{(i)}(t_f) + 2\underline{H}\left[\underline{P}_x^{(i)}\right]\left[\underline{p}^{(i+1)}(t_0) - \underline{p}^{(i)}(t_0)\right]$$

$$= \underline{p}^{(i)}(t_f) + \left[\underline{P}_p^{(i)}\right]\left[\underline{p}^{(i+1)}(t_0) - \underline{p}^{(i)}(t_0)\right].$$

Solving for $\underline{p}^{(i+1)}(t_0)$ gives

$$\underline{p}^{(i+1)}(t_0) = \underline{p}^{(i)}(t_0) + \left[2\underline{H}\,\underline{P}_x^{(i)} - \underline{P}_p^{(i)}\right]^{-1}\left[\underline{p}^{(i)}(t_f) - 2\underline{H}\,\underline{x}^{(i)}(t_f)\right].$$

(c) Here the desired final costate is $\frac{\partial h}{\partial \underline{x}}(\underline{x}(t_f))$, so we set

$$\underline{p}^{(i+1)}(t_f) = \frac{\partial h}{\partial \underline{x}}(\underline{x}^{(i+1)}(t_f)).$$

In general this will be a nonlinear relationship in $\underline{x}^{(i+1)}(t_f)$. Although we could substitute the right side of (II) for $\underline{x}^{(i+1)}(t_f)$, we will first simplify matters by linearizing $\frac{\partial h}{\partial \underline{x}}(\underline{x}^{(i+1)}(t_f))$ about the point $\underline{x}^{(i)}(t_f)$; doing this gives

$$\underline{p}^{(i+1)}(t_f) = \frac{\partial h}{\partial \underline{x}}(\underline{x}^{(i)}(t_f)) + \left[\frac{\partial^2 h}{\partial \underline{x}^2}(\underline{x}^{(i)}(t_f))\right]\left[\underline{x}^{(i+1)}(t_f) - \underline{x}^{(i)}(t_f)\right].$$

Substituting the right side of (II) for $\underline{x}^{(i+1)}(t_f)$ and using (I) yields

$$\underline{p}^{(i)}(t_f) + \underline{P}_p^{(i)}\left[\underline{p}^{(i+1)}(t_0) - \underline{p}^{(i)}(t_0)\right] = \frac{\partial h}{\partial \underline{x}}(\underline{x}^{(i)}(t_f))$$

$$+ \left[\frac{\partial^2 h}{\partial \underline{x}^2}(\underline{x}^{(i)}(t_f))\right]\underline{P}_x^{(i)}\left[\underline{p}^{(i+1)}(t_0) - \underline{p}^{(i)}(t_0)\right].$$

Solving for $\underline{p}^{(i+1)}(t_0)$ we obtain

$$\underline{p}^{(i+1)}(t_0) = \underline{p}^{(i)}(t_0) + \left\{\left[\frac{\partial^2 h}{\partial \underline{x}^2}(\underline{x}^{(i)}(t_f))\right]\underline{P}_x^{(i)} - \underline{P}_p^{(i)}\right\}^{-1}\left[\underline{p}^{(i)}(t_f) - \frac{\partial h}{\partial \underline{x}}(\underline{x}^{(i)}(t_f))\right]. \quad \text{(III)}$$

6-1 (cont.)

(d) To obtain the result of part (a) substitute $\frac{\partial h}{\partial \mathbf{x}} = \mathbf{d}$, $\frac{\partial^2 h}{\partial \mathbf{x}^2} = \mathbf{0}$ in (IV).

To obtain the result given in part (b) substitute $\frac{\partial h}{\partial \mathbf{x}} = 2\mathbf{H}\mathbf{x}$, $\frac{\partial^2 h}{\partial \mathbf{x}^2} = 2\mathbf{H}$ in (IV).

To obtain Eq. (6.3-18) of the text set $\frac{\partial h}{\partial \mathbf{x}} = \mathbf{0}$, $\frac{\partial^2 h}{\partial \mathbf{x}^2} = \mathbf{0}$ in (IV).

6-2

For a regulator problem with $\mathbf{H} = \mathbf{0}$, $\mathbf{p}(t_f) = \mathbf{0}$ and the reduced D.E. are

$$\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{D}(t)\mathbf{p}(t)$$
$$\dot{\mathbf{p}}(t) = \mathbf{E}(t)\mathbf{x}(t) + \mathbf{G}(t)\mathbf{p}(t), \qquad \text{(I)}$$

where
$$\mathbf{D}(t) \triangleq -\mathbf{B}(t)\mathbf{R}^{-1}(t)\mathbf{B}^T(t)$$
$$\mathbf{E}(t) \triangleq -\mathbf{Q}(t)$$
$$\mathbf{G}(t) \triangleq -\mathbf{A}^T(t).$$

The solution of (I) has the form

$$\begin{bmatrix} \mathbf{x}(t) \\ \hline \mathbf{p}(t) \end{bmatrix} = \left[\begin{array}{c|c} \boldsymbol{\varphi}_{11}(t) & \boldsymbol{\varphi}_{12}(t) \\ \hline \boldsymbol{\varphi}_{21}(t) & \boldsymbol{\varphi}_{22}(t) \end{array}\right] \begin{bmatrix} \mathbf{x}(t_0) \\ \hline \mathbf{p}(t_0) \end{bmatrix},$$

hence $\mathbf{p}(t_f) = \boldsymbol{\varphi}_{21}(t_f)\,\mathbf{x}(t_0) + \boldsymbol{\varphi}_{22}(t_f)\,\mathbf{p}(t_0)$,

and
$$\mathbf{P}_p(t_f) \triangleq \frac{\partial \mathbf{p}(t_f)}{\partial \mathbf{p}(t_0)} = \boldsymbol{\varphi}_{22}(t_f). \qquad \text{(II)}$$

Substituting (II) into Eq. (6.3-18) of the text gives

$$\mathbf{p}^{(1)}(t_0) = -\left[\boldsymbol{\varphi}_{22}(t_f)\right]^{-1} \boldsymbol{\varphi}_{21}(t_f)\, \mathbf{x}(t_0). \quad \text{(III)}$$

Using (III) in (6.3-18) to find $\mathbf{p}^{(2)}(t_0)$ gives

$$\mathbf{p}^{(2)}(t_0) = -\left[\boldsymbol{\varphi}_{22}(t_f)\right]^{-1} \boldsymbol{\varphi}_{21}(t_f)\, \mathbf{x}(t_0),$$

but this equals $\mathbf{p}^{(1)}(t_0)$, hence the procedure converges in one iteration regardless of the choice of $\mathbf{p}^{(0)}(t_0)$.

6-3

Noting that $\mathbf{x}(t_f)$ is a fcn. of $\mathbf{p}(t_0)$ we write

$$\mathbf{x}(t_f) = \mathbf{f}(\mathbf{p}(t_0)).$$

Expanding in a Taylor series about $\mathbf{x}^{(i)}(t_f)$ and retaining terms of up to first order, we have

$$\mathbf{x}^{(i+1)}(t_f) = \mathbf{x}^{(i)}(t_f) + \left[\frac{\partial \mathbf{f}}{\partial \mathbf{p}}(\mathbf{p}^{(i)}(t_0))\right]\left[\mathbf{p}^{(i+1)}(t_0) - \mathbf{p}^{(i)}(t_0)\right]$$

$$\triangleq \mathbf{x}^{(i)}(t_f) + \left[\mathbf{P}_x^{(i)}\right]\left[\mathbf{p}^{(i+1)}(t_0) - \mathbf{p}^{(i)}(t_0)\right].$$

Since the desired final state value is $\mathbf{x}_f$, we set $\mathbf{x}^{(i+1)}(t_f) = \mathbf{x}_f$ and solve for $\mathbf{p}^{(i+1)}(t_0)$ to obtain

$$\mathbf{p}^{(i+1)}(t_0) = \mathbf{p}^{(i)}(t_0) - \left[\mathbf{P}_x^{(i)}\right]^{-1}\left[\mathbf{x}^{(i)}(t_f) - \mathbf{x}_f\right].$$

6-4

Suppose that in the performance measure $h = 0$, and that

$x_1(t_f) = x_{1_f}, \ldots, x_r(t_f) = x_{r_f}$ are specified, Then $p_{r+1}(t_f) = 0, \ldots, p_n(t_f) = 0$. Noting that the state and costate values depend on $p(t_0)$ we write (also using a Taylor series)

$$x_1(t_f) = f_1(p(t_0)) \Rightarrow x_1^{(i+1)}(t_f) = x_1^{(i)}(t_f) + \left[\frac{\partial f_1}{\partial p}(p^{(i)}(t_0))\right]^T \left[p^{(i+1)}(t_0) - p^{(i)}(t_0)\right]$$

Hereafter, we omit $p^{(i)}(t_0)$ in $\partial f_j(p^{(i)}(t_0))/\partial p$)

$$x_2(t_f) = f_2(p(t_0)) \Rightarrow x_2^{(i+1)}(t_f) = x_2^{(i)}(t_f) + \left[\frac{\partial f_2}{\partial p}\right]_i^T \left[p^{(i+1)}(t_0) - p^{(i)}(t_0)\right]$$

$$\vdots$$

$$x_r(t_f) = f_r(p(t_0)) \Rightarrow x_r^{(i+1)}(t_f) = x_r^{(i)}(t_f) + \left[\frac{\partial f_r}{\partial p}\right]_i^T \left[p^{(i+1)}(t_0) - p^{(i)}(t_0)\right]$$

$$p_{r+1}(t_f) = f_{r+1}(p(t_0)) \Rightarrow p_{r+1}^{(i+1)}(t_f) = p_{r+1}^{(i)}(t_f) + \left[\frac{\partial f_{r+1}}{\partial p}\right]_i^T \left[p^{(i+1)}(t_0) - p^{(i)}(t_0)\right]$$

$$\vdots$$

$$p_n(t_f) = f_n(p(t_0)) \Rightarrow p_n^{(i+1)}(t_f) = p_n^{(i)}(t_f) + \left[\frac{\partial f_n}{\partial p}\right]_i^T \left[p^{(i+1)}(t_0) - p^{(i)}(t_0)\right].$$

Let $z^{(i)} \triangleq \left[x_1^{(i)}(t_f), \ldots, x_r^{(i)}(t_f), p_{r+1}^{(i)}(t_f), \ldots, p_n^{(i)}(t_f)\right]$,

then

$$z_d = \left[x_{1_f}, \ldots, x_{r_f}, 0, \ldots, 0\right] = \text{desired value of } z.$$

From the series expansions written in matrix form we have

$$z^{(i+1)} = z^{(i)} + D^{(i)}\left[p^{(i+1)}(t_0) - p^{(i)}(t_0)\right],$$

Where the first r rows of $D^{(i)}$ are the first r rows of the matrix $P_x^{(i)}$ and the last (n-r) rows of $D^{(i)}$ are

6-4 (cont.)

the last (n-r) rows of $\underset{\sim}{P}_p^{(i)}$. Setting $\underset{\sim}{z}^{(i+1)} = \underset{\sim}{z}_d$ and solving for $\underset{\sim}{p}^{(i+1)}(t_0)$ gives

$$\underset{\sim}{p}^{(i+1)}(t_0) = \underset{\sim}{p}^{(i)}(t_0) - \left[\underset{\sim}{D}^{(i)}\right]^{-1}\left[\underset{\sim}{z}^{(i)} - \underset{\sim}{z}_d\right].$$

Thus, solving this problem is simply a matter of using some entries of $\underset{\sim}{P}_p^{(i)}$ and some entries of $\underset{\sim}{P}_x^{(i)}$.

6-5

(a) Integrating from t_f to t_0 we can use the dependence of $\underset{\sim}{x}(t_0)$ on $\underset{\sim}{x}(t_f)$ to derive the iteration equation. This dependence is indicated by

$$\underset{\sim}{x}(t_0) = \underset{\sim}{f}(\underset{\sim}{x}(t_f)).$$

Expanding in a Taylor series and retaining only terms of up to first order, we have

$$\underset{\sim}{x}^{(i+1)}(t_0) = \underset{\sim}{x}^{(i)}(t_0) + \left[\frac{\partial \underset{\sim}{f}}{\partial \underset{\sim}{x}}(\underset{\sim}{x}^{(i)}(t_f))\right]\left[\underset{\sim}{x}^{(i+1)}(t_f) - \underset{\sim}{x}^{(i)}(t_f)\right].$$

Setting $\underset{\sim}{x}^{(i+1)}(t_0) = \underset{\sim}{x}_0$, the specified (and hence desired) value of $\underset{\sim}{x}(t_0)$, and solving for $\underset{\sim}{x}^{(i+1)}(t_f)$ yields

$$\underset{\sim}{x}^{(i+1)}(t_f) = \underset{\sim}{x}^{(i)}(t_f) - \left[\frac{\partial \underset{\sim}{f}}{\partial \underset{\sim}{x}}(\underset{\sim}{x}^{(i)}(t_f))\right]^{-1}\left[\underset{\sim}{x}^{(i)}(t_0) - \underset{\sim}{x}_0\right]. \quad (*)$$

To generate the nxn matrix $\frac{\partial \underset{\sim}{f}}{\partial \underset{\sim}{x}}(\underset{\sim}{x}^{(i)}(t_f))$ we start with the reduced differential equations

$$\dot{\underset{\sim}{x}}(t) = \frac{\partial \mathcal{H}}{\partial \underset{\sim}{p}}(\underset{\sim}{x}(t), \underset{\sim}{p}(t), t)$$

$$\dot{\underset{\sim}{p}}(t) = -\frac{\partial \mathcal{H}}{\partial \underset{\sim}{x}}(\underset{\sim}{x}(t), \underset{\sim}{p}(t), t)$$

and take the partial derivatives of both sides with respect to $\underset{\sim}{x}(t_f)$. Interchanging the order of differentiation on the left and using the chain rule on the right gives

$$\frac{d}{dt}\left[\frac{\partial \underset{\sim}{x}(t)}{\partial \underset{\sim}{x}(t_f)}\right] = \left[\frac{\partial^2 \mathcal{H}}{\partial \underset{\sim}{x} \partial \underset{\sim}{p}}\right]\frac{\partial \underset{\sim}{x}(t)}{\partial \underset{\sim}{x}(t_f)} + \left[\frac{\partial^2 \mathcal{H}}{\partial \underset{\sim}{p}^2}\right]\frac{\partial \underset{\sim}{p}(t)}{\partial \underset{\sim}{x}(t_f)} \qquad (\mathrm{II})$$

$$\frac{d}{dt}\left[\frac{\partial \underset{\sim}{p}(t)}{\partial \underset{\sim}{x}(t_f)}\right] = -\left[\frac{\partial^2 \mathcal{H}}{\partial \underset{\sim}{x}^2}\right]\frac{\partial \underset{\sim}{x}(t)}{\partial \underset{\sim}{x}(t_f)} - \left[\frac{\partial^2 \mathcal{H}}{\partial \underset{\sim}{x} \partial \underset{\sim}{p}}\right]\frac{\partial \underset{\sim}{p}(t)}{\partial \underset{\sim}{x}(t_f)}.$$

The arguments of the second partials of $\mathcal{H}$ have been omitted. The boundary conditions are

$$\frac{\partial \underset{\sim}{x}(t_f)}{\partial \underset{\sim}{x}(t_f)} = \underset{\sim}{I}, \quad \frac{\partial \underset{\sim}{p}(t_f)}{\partial \underset{\sim}{x}(t_f)} = \underset{\sim}{0}$$ (assuming $h = 0$ in the performance measure). The matrix $\partial \underset{\sim}{f}/\partial \underset{\sim}{x}$ appearing in (I) is $\partial \underset{\sim}{x}(t_0)/\partial \underset{\sim}{x}(t_f)$, the solution of (II) at $t = t_0$.

(b) Select $\underset{\sim}{x}^{(0)}(t_f) = \underset{\sim}{0}$ because J indicates a desire to bring the state near the origin. If the system is controllable and the penalty on control effort expenditure is not too severe, we would hope that $\underset{\sim}{x}(t_f)$ would at east be close to the origin.

6-6

Substitute the given $\underset{\sim}{y}(t)$ into the D.E. and see if it reduces to an identity, that is,

$$\frac{d}{dt}\left[\underset{\sim}{y}(t)\right] \overset{?}{=} \underset{\sim}{D}(t)\,\underset{\sim}{y}(t) + \underset{\sim}{f}(t). \qquad (I)$$

Performing the differentiation indicated on the left gives

$$\frac{d}{dt}\left[\underset{\sim}{y}(t)\right] = c_1 \frac{d}{dt}\left[\underset{\sim}{z}^{H1}(t)\right] + \cdots + c_q \frac{d}{dt}\left[\underset{\sim}{z}^{Hq}(t)\right] + \frac{d}{dt}\left[\underset{\sim}{z}^{P}(t)\right].$$

But, by definition of $\underset{\sim}{z}^{H1}(t), \ldots, \underset{\sim}{z}^{P}(t)$,

$$\frac{d}{dt}\left[\underset{\sim}{z}^{Hi}(t)\right] = \underset{\sim}{D}(t)\,\underset{\sim}{z}^{Hi}(t), \quad i = 1, 2, \ldots, q$$

$$\frac{d}{dt}\left[\underset{\sim}{z}^{P}(t)\right] = \underset{\sim}{D}(t)\,\underset{\sim}{z}^{P}(t) + \underset{\sim}{f}(t).$$

Therefore,

$$\begin{aligned}\frac{d}{dt}\left[\underset{\sim}{y}(t)\right] &= c_1 \underset{\sim}{D}(t)\,\underset{\sim}{z}^{H1}(t) + \cdots + c_q \underset{\sim}{D}(t)\,\underset{\sim}{z}^{Hq}(t) + \underset{\sim}{D}(t)\,\underset{\sim}{z}^{P}(t) + \underset{\sim}{f}(t)\\ &= \underset{\sim}{D}(t)\left[c_1 \underset{\sim}{z}^{H1}(t) + \cdots + c_q \underset{\sim}{z}^{Hq}(t) + \underset{\sim}{z}^{P}(t)\right] + \underset{\sim}{f}(t)\\ &\triangleq \underset{\sim}{D}(t)\,\underset{\sim}{y}(t) + \underset{\sim}{f}(t),\end{aligned}$$

which is the right side of (I); hence, the specified $\underset{\sim}{y}(t)$ is a solution of the D.E.

6-7

We generate the n homogeneous solutions and the particular solution to the linearized reduced D.E. as usual. Then,

$$\underset{\sim}{x}(t) = c_1 \underset{\sim}{x}^{H1}(t) + c_2 \underset{\sim}{x}^{H2}(t) + \dots + c_n \underset{\sim}{x}^{Hn}(t) + \underset{\sim}{x}^{P}(t) \quad (\mathrm{I})$$

$$\underset{\sim}{p}(t) = c_1 \underset{\sim}{p}^{H1}(t) + c_2 \underset{\sim}{p}^{H2}(t) + \dots + c_n \underset{\sim}{p}^{Hn}(t) + \underset{\sim}{p}^{P}(t). \quad (\mathrm{II})$$

If $\underset{\sim}{p}(t_f)$ were known we would solve for $c_1, c_2, \dots, c_n$ and obtain the result given in Eq. (6.4-35) of the text. Here, since $\underset{\sim}{x}(t_f) = \underset{\sim}{x}_f$ is specified, we have from (I)

$$\underset{\sim}{x}(t_f) = \underset{\sim}{x}_f = \left[\underset{\sim}{x}^{H1}(t_f) \,\vdots\, \underset{\sim}{x}^{H2}(t_f) \,\vdots\, \dots \,\vdots\, \underset{\sim}{x}^{Hn}(t_f)\right] \underset{\sim}{c} + \underset{\sim}{x}^{P}(t_f).$$

Solving for $\underset{\sim}{c}$, we obtain

$$\underset{\sim}{c} = \left[\underset{\sim}{x}^{H1}(t_f) \,\vdots\, \dots \,\vdots\, \underset{\sim}{x}^{Hn}(t_f)\right]^{-1} \left[\underset{\sim}{x}_f - \underset{\sim}{x}^{P}(t_f)\right].$$

6-8

We generate the n homogeneous solutions and the particular solution as usual to obtain (I) and (II) given in the solution to Problem 6-7. Suppose that

$$x_1(t_f) = x_{1_f},\ x_2(t_f) = x_{2_f},\ \dots,\ x_r(t_f) = x_{r_f}$$

$$p_{r+1}(t_f) = p_{r+1_f},\ \dots,\ p_n(t_f) = p_{n_f}$$

are specified values. Let

$$\underset{\sim}{z}(t) \triangleq \left[x_1(t) \;\; x_2(t) \dots x_r(t) \;\; p_{r+1}(t) \dots p_n(t)\right]^T,$$

then n of the equations (I) and (II) can be written at $t = t_f$ as

$$\underset{\sim}{z}(t_f) = c_1 \underset{\sim}{z}^{H1}(t_f) + \dots + c_n \underset{\sim}{z}^{Hn}(t_f) + \underset{\sim}{z}^{P}(t_f).$$

Everything on the right side is known except the c's; on the left side $\underset{\sim}{z}(t_f) = \underset{\sim}{z}_f$ (the specified value). Solving for $\underset{\sim}{c}$ gives

$$\underset{\sim}{c} = \left[\underset{\sim}{z}^{H1}(t_f) \,\vdots\, \underset{\sim}{z}^{H2}(t_f) \,\vdots\, \cdots \,\vdots\, \underset{\sim}{z}^{Hn}(t_f)\right]^{-1}\left[\underset{\sim}{z}_f - \underset{\sim}{z}^{P}(t_f)\right].$$

6-9

For simplicity assume $\underset{\sim}{H} = \underset{\sim}{0}$, then $\underset{\sim}{p}(t_f) = \underset{\sim}{0}$ since the final states are free.
In this case the reduced D.E. are linear and homogeneous, that is, they are of the form

$$\dot{\underset{\sim}{x}}(t) = \underset{\sim}{A}(t)\,\underset{\sim}{x}(t) + \underset{\sim}{D}(t)\,\underset{\sim}{p}(t) \qquad (\mathrm{I})$$

$$\dot{\underset{\sim}{p}}(t) = \underset{\sim}{E}(t)\,\underset{\sim}{x}(t) + \underset{\sim}{G}(t)\,\underset{\sim}{p}(t). \qquad (\mathrm{II})$$

Using the reasoning of Problem 6-6 we can show that

$$c_1\underset{\sim}{x}^{H1}(t) + \cdots + c_n\underset{\sim}{x}^{Hn}(t) + \underset{\sim}{x}^{H(n+1)}(t)$$

is a solution of (I) and

$$c_1\underset{\sim}{p}^{H1}(t) + \cdots + c_n\underset{\sim}{p}^{Hn}(t) + \underset{\sim}{p}^{H(n+1)}(t) \qquad (\dagger)$$

is a solution of Eq.(II).
$\underset{\sim}{x}^{H(n+1)}(t)$, $\underset{\sim}{p}^{H(n+1)}(t)$ is the solution of (I) and (II) that satisfies the boundary conditions $\underset{\sim}{x}^{H(n+1)}(t_0) = \underset{\sim}{x}_0$ and $\underset{\sim}{p}^{H(n+1)}(t_0) = \underset{\sim}{0}$. The other n homogeneous solutions are generated

as usual. Now, since $\mathbf{p}(t_f) = \mathbf{0}$ we can set the quantity (†) equal to $\mathbf{0}$ and solve for $\mathbf{c}$, which gives

$$\mathbf{c} = -\left[\mathbf{p}^{H1}(t_f) \,\vdots\, \cdots \,\vdots\, \mathbf{p}^{Hn}(t_f)\right] \mathbf{p}^{H(n+1)}(t_f).$$

For these values of the c's the specified boundary conditions are satisfied by a solution of (I) and (II); hence, a solution has been obtained in one iteration.

6-10

The starting point is the equations

$$\mathbf{x}(t_f) = c_1 \mathbf{x}^{H1}(t_f) + \cdots + c_n \mathbf{x}^{Hn}(t_f) + \mathbf{x}^{P}(t_f) \quad \text{(I)}$$

$$\mathbf{p}(t_f) = c_1 \mathbf{p}^{H1}(t_f) + \cdots + c_n \mathbf{p}^{Hn}(t_f) + \mathbf{p}^{P}(t_f). \quad \text{(II)}$$

The iteration index is understood to be $(i+1)$. We will assume that $\mathbf{x}(t_f)$ is free.

(a) Since $\mathbf{x}(t_f)$ is free, the final costate is

$$\mathbf{p}^{(i+1)}(t_f) = 2\mathbf{H}\mathbf{x}^{(i+1)}(t_f).$$

Multiplying the right side of (I) by $2\mathbf{H}$ and equating this to the right side of (II) gives (after rearranging terms)

$$c_1\left[\mathbf{p}^{H1}(t_f) - 2\mathbf{H}\mathbf{x}^{H1}(t_f)\right] + \cdots + c_n\left[\mathbf{p}^{Hn}(t_f) - 2\mathbf{H}\mathbf{x}^{Hn}(t_f)\right] = 2\mathbf{H}\mathbf{x}^{P}(t_f) - \mathbf{p}^{P}(t_f).$$

Solving for $\mathbf{c} \triangleq [c_1\ c_2 \ldots c_n]^T$ gives

$$\mathbf{c} = \left[\mathbf{p}^{H1}(t_f) - 2\mathbf{H}\mathbf{x}^{H1}(t_f) \,\vdots\, \ldots \,\vdots\, \mathbf{p}^{Hn}(t_f) - 2\mathbf{H}\mathbf{x}^{Hn}(t_f)\right]^{-1}\left[2\mathbf{H}\mathbf{x}^{P}(t_f) - \mathbf{p}^{P}(t_f)\right].$$

(b) Here we have

$$\mathbf{p}^{(i+1)}(t_f) = \frac{\partial h}{\partial \mathbf{x}}\left(\mathbf{x}^{(i+1)}(t_f)\right) \triangleq \frac{\partial h}{\partial \mathbf{x}}^{(i+1)}$$

As in the solution to Problem 6-1, we expand $\mathbf{p}(t_f)$ in a Taylor series to obtain

$$\mathbf{p}^{(i+1)}(t_f) = \frac{\partial h}{\partial \mathbf{x}}\left(\mathbf{x}^{(i)}(t_f)\right) + \left[\frac{\partial^2 h}{\partial \mathbf{x}^2}\left(\mathbf{x}^{(i)}(t_f)\right)\right]\left[\mathbf{x}^{(i+1)}(t_f) - \mathbf{x}^{(i)}(t_f)\right]$$

$$\triangleq \frac{\partial h^{(i)}}{\partial \mathbf{x}} + \mathbf{M}\left[\mathbf{x}^{(i+1)}(t_f) - \mathbf{x}^{(i)}(t_f)\right]$$

$$= \frac{\partial h^{(i)}}{\partial \mathbf{x}} - \mathbf{M}\mathbf{x}^{(i)}(t_f) + \mathbf{M}\mathbf{x}^{(i+1)}(t_f). \qquad \text{(III)}$$

Substituting the right side of (I) for $\mathbf{x}^{(i+1)}(t_f)$ in (III) and equating the right side of (III) to the right side of (II) yields

$$c_1\mathbf{p}^{H1}(t_f) + \ldots + c_n\mathbf{p}^{Hn}(t_f) + \mathbf{p}^{P}(t_f) = \frac{\partial h^{(i)}}{\partial \mathbf{x}} - \mathbf{M}\mathbf{x}^{(i)}(t_f)$$

$$+ \mathbf{M}\left[c_1\mathbf{x}^{H1}(t_f) + \ldots + c_n\mathbf{x}^{Hn}(t_f) + \mathbf{x}^{P}(t_f)\right].$$

Rearranging terms and solving for $\mathbf{c}$ gives

$$\mathbf{c} = \left[\mathbf{p}^{H1}(t_f) - \mathbf{M}\mathbf{x}^{H1}(t_f) \,\vdots\, \ldots \,\vdots\, \mathbf{p}^{Hn}(t_f) - \mathbf{M}\mathbf{x}^{Hn}(t_f)\right]^{-1}\left[\frac{\partial h^{(i)}}{\partial \mathbf{x}} - \mathbf{M}\mathbf{x}^{(i)}(t_f) + \mathbf{M}\mathbf{x}^{P}(t_f) - \mathbf{p}^{P}(t_f)\right], \qquad \text{(IV)}$$

6-10 (cont.)

which is Eq. (6.4-38) of the text.

(c) To obtain the result obtained in part (a), substitute

$$\underset{\sim}{M} = 2\underset{\sim}{H}\ , \ \frac{\partial h^{(i)}}{\partial \underset{\sim}{x}} = 2\underset{\sim}{H}\,\underset{\sim}{x}^{(i)}(t_f)$$

into (IV). To obtain Eq. (6.4-35) of the text, substitute

$$\underset{\sim}{v} = \underset{\sim}{p}_f \text{ for } \frac{\partial h^{(i)}}{\partial \underset{\sim}{x}} \quad (\text{since } h = \underset{\sim}{v}^T \underset{\sim}{x}(t_f))$$

and $\underset{\sim}{M} = \underset{\sim}{0}$ into Eq. (IX).

6-11

We have a two-point boundary-value problem of the form

$$\dot{\underset{\sim}{x}}(t) = \underset{\sim}{A}(t)\,\underset{\sim}{x}(t) + \underset{\sim}{D}(t)\,\underset{\sim}{p}(t) + \underset{\sim}{m}_1(t) \qquad (\text{I})$$

$$\dot{\underset{\sim}{p}}(t) = \underset{\sim}{E}(t)\,\underset{\sim}{x}(t) + \underset{\sim}{G}(t)\,\underset{\sim}{p}(t) + \underset{\sim}{m}_2(t) \qquad (\text{II})$$

which is similar to the equations encountered in the linear tracking problem. We postulate that

$$\underset{\sim}{p}(t) = \underset{\sim}{K}(t)\,\underset{\sim}{x}(t) + \underset{\sim}{s}(t), \qquad (\text{III})$$

where $\underset{\sim}{K}$ is a $n \times n$ real symmetric matrix and $\underset{\sim}{s}(t)$ is an n vector.

Differentiating (III) with respect to time yields

$$\dot{\underset{\sim}{p}}(t) = \dot{\underset{\sim}{K}}(t)\,\underset{\sim}{x}(t) + \underset{\sim}{K}\,\dot{\underset{\sim}{x}}(t) + \dot{\underset{\sim}{s}}(t).$$

Substituting for $\dot{\underset{\sim}{p}}(t)$ and $\dot{\underset{\sim}{x}}(t)$ from (I) and (II), eliminating $\underset{\sim}{p}(t)$ by substitution of (III), and collecting terms yields

$$\underset{\sim}{0} = \left[\underset{\sim}{E}(t) - \dot{\underset{\sim}{K}}(t) - \underset{\sim}{K}(t)\underset{\sim}{A}(t) - \underset{\sim}{K}(t)\underset{\sim}{D}(t)\underset{\sim}{K}(t)\right]\underset{\sim}{x}(t) + \underset{\sim}{m}_2(t)$$
$$- \underset{\sim}{K}(t)\underset{\sim}{m}_1(t) - \dot{\underset{\sim}{s}}(t) + \left[\underset{\sim}{G}(t) - \underset{\sim}{K}(t)\underset{\sim}{D}(t)\right]\underset{\sim}{s}(t). \quad (\text{IV})$$

Since this equation must hold for all $\underset{\sim}{x}(t)$, $\underset{\sim}{m}_1(t)$ and $\underset{\sim}{m}_2(t)$, the coefficient of $\underset{\sim}{x}(t)$ must be zero, and the other terms must add to zero; thus,

$$\dot{\underset{\sim}{K}}(t) = \underset{\sim}{E}(t) - \underset{\sim}{K}(t)\underset{\sim}{A}(t) + \underset{\sim}{G}(t)\underset{\sim}{K}(t) - \underset{\sim}{K}(t)\underset{\sim}{D}(t)\underset{\sim}{K}(t) \quad (\text{V})$$

and

$$\dot{\underset{\sim}{s}}(t) = \left[\underset{\sim}{G}(t) - \underset{\sim}{K}(t)\underset{\sim}{D}(t)\right]\underset{\sim}{s}(t) - \underset{\sim}{K}(t)\underset{\sim}{m}_1(t) + \underset{\sim}{m}_2(t). \quad (\text{VI})$$

Assuming that $\underset{\sim}{x}(t_f)$ is free, the boundary conditions for (I) and (II) are

$$\underset{\sim}{p}^{(i+1)}(t_f) = \frac{\partial h}{\partial \underset{\sim}{x}}\left(\underset{\sim}{x}^{(i)}(t_f)\right) - \underset{\sim}{M}\,\underset{\sim}{x}^{(i)}(t_f) + \underset{\sim}{M}\,\underset{\sim}{x}^{(i+1)}(t_f) \quad (\text{VII})$$

(see Problem 6-10).

From (VII) and (III) the boundary conditions for (V) and (VI) are

$$\underset{\sim}{K}(t_f) = \underset{\sim}{M} \triangleq \frac{\partial^2 h}{\partial \underset{\sim}{x}^2}\left(\underset{\sim}{x}^{(i)}(t_f)\right) \quad (\text{VIII})$$

and

$$\underset{\sim}{s}(t_f) = \frac{\partial h}{\partial \underset{\sim}{x}}\left(\underset{\sim}{x}^{(i)}(t_f)\right) - M\,\underset{\sim}{x}^{(i)}(t_f). \quad (\text{IX})$$

To obtain $\underset{\sim}{p}(t_0)$ integrate (V) and (VI) from t_f to t_0 using the boundary conditions (VIII) and (IX). Then, from (III),

$$\underset{\sim}{p}(t_0) = \underset{\sim}{K}(t_0)\underset{\sim}{x}(t_0) + \underset{\sim}{s}(t_0).$$

The reduced D.E. are of the form

$$\dot{\underset{\sim}{x}}(t) = \underset{\sim}{A}(t)\,\underset{\sim}{x}(t) + \underset{\sim}{D}(t)\,\underset{\sim}{p}(t)$$

$$\dot{\underset{\sim}{p}}(t) = \underset{\sim}{E}(t)\,\underset{\sim}{x}(t) + \underset{\sim}{G}(t)\,\underset{\sim}{p}(t),$$

or, defining $\underset{\sim}{z}(t) \triangleq [\underset{\sim}{x}^T(t) \mid \underset{\sim}{p}^T(t)]^T$, we have

$$\dot{\underset{\sim}{z}}(t) = \underset{\sim}{\mathcal{a}}(t)\,\underset{\sim}{z}(t) \qquad (\mathrm{I})$$

which has a solution of the form

$$\underset{\sim}{z}(t) = \underset{\sim}{\Psi}(t, t_0)\,\underset{\sim}{z}(t_0), \qquad (\mathrm{II})$$

where $\underset{\sim}{\Psi}(t, t_0)$ is the transition matrix for the system (I). $\underset{\sim}{\Psi}(t, t_0)$ can be determined by integrating

$$\dot{\underset{\sim}{\Psi}}(t) = \underset{\sim}{\mathcal{a}}(t)\,\underset{\sim}{\Psi}(t)$$

with the initial condition $\underset{\sim}{\Psi}(t_0) = \underset{\sim}{I}$.

We also know that

$$\underset{\sim}{p}(t) = \underset{\sim}{K}(t)\,\underset{\sim}{x}(t) \qquad (\mathrm{III})$$

for all t; $\underset{\sim}{K}(t)$ is the solution of the Riccati equation. At $t = t_0$ we have

$$\underset{\sim}{p}(t_0) = \underset{\sim}{K}(t_0)\,\underset{\sim}{x}(t_0).$$

If we select $\underset{\sim}{x}^{(1)}(t_0) = [1\ 0\ 0 \ldots 0]^T$, then

$$\underset{\sim}{p}^{(1)}(t_0) = \begin{bmatrix} k_{11}(t_0) \\ k_{21}(t_0) \\ \vdots \\ k_{n1}(t_0) \end{bmatrix}.$$

More generally, for $\underset{\sim}{x}^{(i)}(t_0) = [0\ 0 \ldots 1\ 0 \ldots 0]$ ($\longleftarrow i \longrightarrow$ entries)

we have

$$\underset{\sim}{p}^{(i)}(t_0) = \begin{bmatrix} k_{1i}(t_0) \\ k_{2i}(t_0) \\ \vdots \\ k_{ni}(t_0) \end{bmatrix}, \text{ the } i\text{th column of } \underset{\sim}{K}(t) \text{ at } t = t_0.$$

If we can find the $\underset{\sim}{p}^{(i)}(t_0)$ which corresponds to $\underset{\sim}{x}^{(i)}(t_0)$, the matrix $\underset{\sim}{K}(t)$ will be known at $t = t_0$.

From Eq. (III) with $t = t_f$ we have

$$\begin{bmatrix} \underset{\sim}{x}(t_f) \\ \hline \underset{\sim}{p}(t_f) \end{bmatrix} = \left[\begin{array}{c|c} \underset{\sim}{\Psi}_{11}(t_f, t_0) & \underset{\sim}{\Psi}_{12}(t_f, t_0) \\ \hline \underset{\sim}{\Psi}_{21}(t_f, t_0) & \underset{\sim}{\Psi}_{22}(t_f, t_0) \end{array}\right] \begin{bmatrix} \underset{\sim}{x}(t_0) \\ \hline \underset{\sim}{p}(t_0) \end{bmatrix},$$

which implies that for the specified value of $\underset{\sim}{x}(t_f)$ (i.e. $\underset{\sim}{x}_f$),

$$\underset{\sim}{x}_f = \underset{\sim}{\Psi}_{11}(t_f, t_0)\, \underset{\sim}{x}^{(i)}(t_0) + \underset{\sim}{\Psi}_{12}(t_f, t_0)\, \underset{\sim}{p}^{(i)}(t_0).$$

Solving for $\underset{\sim}{p}^{(i)}(t_0)$ gives

$$\underset{\sim}{p}^{(i)}(t_0) = \underset{\sim}{\Psi}_{12}^{-1}(t_f, t_0)\left[\underset{\sim}{x}_f - \underset{\sim}{\Psi}_{11}(t_f, t_0)\, \underset{\sim}{x}^{(i)}(t_0)\right]. \quad \text{(IV)}$$

Thus, we can use Eq. (IV) to determine $\underset{\sim}{K}(t_0)$; the Riccati equation can then be solved for $t_0 \le t \le t_1 < t_f$ by integrating from t_0 to t_1 (t_1 is the time when the solution for $\underset{\sim}{K}(t)$ begins to exceed the allowable range of numbers for the computer used to

6-12 (cont.)
generate the solution).

6-14

The constraints can be written as

$$\begin{bmatrix} 1 & 0 & 0 & -1 & 1 & 1 \\ 0 & 1 & 0 & -1 & 1 & -2 \\ 0 & 0 & 1 & -1 & -1 & 0 \end{bmatrix}^T \underset{\sim}{y} - \begin{bmatrix} 0 \\ 0 \\ 0 \\ -1 \\ 0 \\ 0 \end{bmatrix} \geq \underset{\sim}{0} \ ;$$

hence, in the normalized version of these constraints

$$\underset{\sim}{N}_L = \begin{bmatrix} 1 & 0 & 0 & -.5774 & .5774 & .4472 \\ 0 & 1 & 0 & -.5774 & .5774 & -.8944 \\ 0 & 0 & 1 & -.5774 & -.5774 & 0 \end{bmatrix}$$

and

$$\underset{\sim}{v}_L = [0 \ \ 0 \ \ 0 \ \ -.5774 \ \ 0 \ \ 0]^T.$$

(i) If $\underset{\sim}{y}^{(0)} = [2.0 \ \ 50.0 \ \ 17.0]^T$ (not an admissible point), the algorithm first corrects back to the admissible point $\underset{\sim}{y}'^{(0)} = [0.666 \ \ 0.3333 \ \ 0.0]^T$; the next point found is

$$\underset{\sim}{y}^{(1)} = \underset{\sim}{y}^* = [0.3333 \ \ 0.1666 \ \ 0.5000]^T.$$

(ii) For $\underset{\sim}{y}^{(0)} = [50.0 \ \ 2.0 \ \ -2.0]^T$ the algorithm first corrects back to the admissible point

$$\underset{\sim}{y}'^{(0)} = [1.0 \ \ 0.0 \ \ 0.0]^T$$

and then finds

$$\underset{\sim}{y}^{(1)} = [0.5 \ \ 0.0 \ \ 0.5]^T$$

6-14 (cont.)

and

$$\underset{\sim}{y}^{(2)} = \underset{\sim}{y}^* = [0.3333 \quad 0.1666 \quad 0.5000]^T.$$

For both (i) and (ii) $f(\underset{\sim}{y}^*) = 2.167$.

6-16

The constraints are

$$\begin{bmatrix} & 0 & -1 & 0 & 0 & 0 \\ & 0 & -1 & 0 & .5 & .3333 \\ & 0 & 0 & 0 & -1 & 0 \\ & 0 & 0 & 0 & -1 & 0 \\ \underset{\sim}{I}_9 & 0 & 0 & -1 & 0 & 0 \\ & 0 & 0 & -1 & .375 & .625 \\ & 0 & 0 & 0 & 0 & -1 \\ & 0 & 0 & 0 & 0 & -1 \\ & -1 & .5 & .25 & 0 & 0 \end{bmatrix}^T \underset{\sim}{y} - \begin{bmatrix} \underset{\sim}{0}_9 \\ \hline -1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \geq \underset{\sim}{0}$$

where $\underset{\sim}{I}_9$ is the 9x9 identity matrix and $\underset{\sim}{0}_9$ is the 9-dimensional zero vector.

The computer program normalizes these constraints before beginning the computations.

(i) For $\underset{\sim}{y}^{(0)} = \underset{\sim}{0}$, which is an admissible point, the next point is

$$\underset{\sim}{y}^{(1)} = \underset{\sim}{y}^* = [0. \ .5 \ 0. \ .25 \ .25 \ 0. \ .1666 \ 0. \ 1.]^T.$$

(ii) For $\underset{\sim}{y}^{(0)} = [1. \ 2. \ 0. \ 0. \ 5. \ 4. \ 0. \ 0. \ 1.]^T$ (not an admissible point), the algorithm first corrects to the admissible point $\underset{\sim}{y}'^{(0)} = [0. \ .5 \ 0. \ 0. \ 5. \ 4. \ 0. \ 0. \ 1.]^T$.

Then, the next point generated is

6-16 (cont.)

$$\underset{\sim}{y}^{(1)} = [.0217 \;\; .4732 \;\; .0485 \;\; .0606 \;\; .2474 \;\; 0. \;\; .0970 \;\; .0606 \;\; .9898]^T$$

and

$$\underset{\sim}{y}^{(2)} = [.0323 \;\; .4600 \;\; .1022 \;\; .1278 \;\; .2462 \;\; 0. \;\; .1150 \;\; .0383 \;\; .9848]^T,$$

$$\underset{\sim}{y}^{(3)} = [.0279 \;\; .4655 \;\; .0905 \;\; .1422 \;\; .2467 \;\; 0. \;\; .1551 \;\; 0. \;\; .9868]^T$$

$$\underset{\sim}{y}^{(5)} = \underset{\sim}{y}^* = [0. \;\; .5 \;\; 0. \;\; .25 \;\; .25 \;\; 0. \;\; .1666 \;\; 0. \;\; 1.]^T;$$

$$f(\underset{\sim}{y}^*) = 0.1458.$$

6-17

In addition to the two linear constraints we have a nonlinear constraint of the form $g(\underset{\sim}{y}) \geq 0$ which we linearize by expanding g in a Taylor series about the point $\underset{\sim}{y}^{(i)}$ (which is known) and retaining terms of up to first order, that is,

$$g(\underset{\sim}{y}^{(i+1)}) = g(\underset{\sim}{y}^{(i)}) + \left[\frac{\partial g}{\partial \underset{\sim}{y}}(\underset{\sim}{y}^{(i)})\right]^T [\underset{\sim}{y}^{(i+1)} - \underset{\sim}{y}^{(i)}].$$

Performing the indicated operations and collecting terms we obtain

$$\begin{bmatrix} -4y_1^{(i)} \\ -6y_2^{(i)} \end{bmatrix}^T \underset{\sim}{y}^{(i+1)} - \left[-2[y_1^{(i)}]^2 - 3[y_2^{(i)}]^2 - 6\right] \geq 0;$$

hence, the constraints are

6-17 (cont.)

$$\begin{bmatrix} 1 & 0 & -4y_1^{(i)} \\ 0 & 1 & -6y_2^{(i)} \end{bmatrix} \underset{\sim}{y}^{(i+1)} - \begin{bmatrix} 0 \\ 0 \\ -2[y_1^{(i)}]^2 - 3[y_2^{(i)}]^2 - 6 \end{bmatrix} \geq \underset{\sim}{0}.$$

At the end of each minimization $\underset{\sim}{y}^{(i)}$, which appears in the above constraints, is replaced by the new value of $\underset{\sim}{y}$, the constraints are normalized, and the algorithm begins a new iteration. The initial guess was $[10 \quad 2]^T$. Then

$$\left.\begin{aligned} \underset{\sim}{y}'^{(0)} &= [5.275 \quad .5826]^T \\ \underset{\sim}{y}^{(1)} &= [.9415 \quad 1.089]^T \\ \underset{\sim}{y}^{(2)} &= [2.114 \quad 11.11]^T \\ \underset{\sim}{y}^{(3)} &= [0.0 \quad 18.16]^T \end{aligned}\right\} \text{Iteration 1}$$

$$\left.\begin{aligned} \underset{\sim}{y}'^{(3)} &= [0.0 \quad 9.138]^T \\ \underset{\sim}{y}^{(4)} &= [1.0 \quad 9.138]^T \end{aligned}\right\} \text{Iteration 2}$$

$$\left.\begin{aligned} \underset{\sim}{y}'^{(4)} &= [.6737 \quad 4.666]^T \\ \underset{\sim}{y}^{(5)} &= [.9635 \quad 4.644]^T \end{aligned}\right\} \text{Iteration 3}$$

$$\left.\begin{aligned} \underset{\sim}{y}'^{(5)} &= [.6686 \quad 2.512]^T \\ \underset{\sim}{y}^{(6)} &= [.9308 \quad 2.476]^T \end{aligned}\right\} \text{Iteration 4}$$

$$\left.\begin{aligned} \underset{\sim}{y}'^{(6)} &= [.7067 \quad 1.581]^T \\ \underset{\sim}{y}^{(7)} &= [.8747 \quad 1.539]^T \end{aligned}\right\} \text{Iteration 5}$$

6-17 (cont.)

$$\left.\begin{array}{l} \underset{\sim}{y}'^{(7)} = [.7801 \quad 1.289]^T \\ \underset{\sim}{y}^{(8)} = [.8106 \quad 1.278]^T \end{array}\right\} \text{Iteration } 6$$

$$\left.\begin{array}{l} \underset{\sim}{y}'^{(8)} = [.8006 \quad 1.254]^T \\ \underset{\sim}{y}^{(8)} = [.7885 \quad 1.259]^T \end{array}\right\} \text{Iteration } 7$$

$$\left.\begin{array}{l} \underset{\sim}{y}'^{(8)} = [.7886 \quad 1.259]^T \\ \underset{\sim}{y}^{(9)} = [.7913 \quad 1.258]^T \end{array}\right\} \text{Iteration } 8$$

$$\underset{\sim}{y}^{(10)} = [.7904 \quad 1.258]^T \Big\} \text{Iteration } 9$$

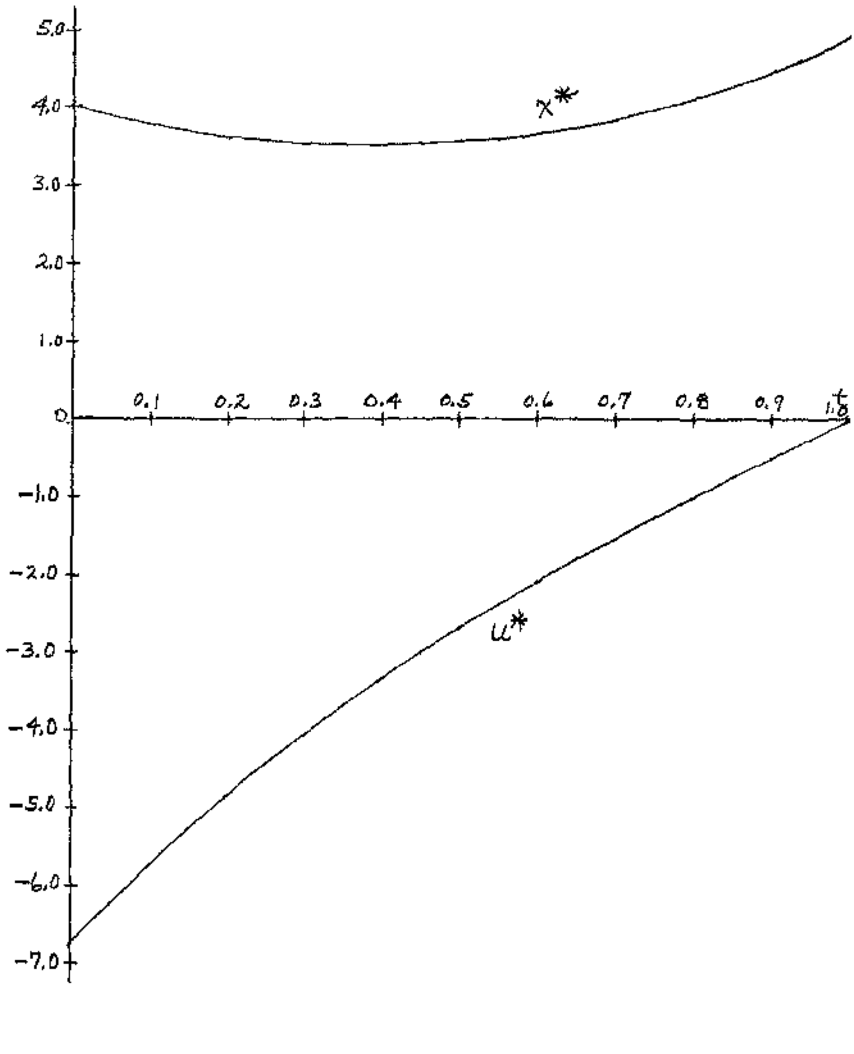
$$\underset{\sim}{y}^{(11)} = \underset{\sim}{y}^{*} = [.7907 \quad 1.258]^T$$

$$f(\underset{\sim}{y}^{*}) = 2.2147.$$

6-19 through 6-22

The curves of the optimal trajectory and control are given on the following page.

$J^* = 13.5160$



6-23 through 6-26

The optimal trajectory and control are as shown below.