An Application of Martingales

Components of Erdos-Renyi Graphs

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June 1, 2022

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Asymptotically Almost Surely

We would like to study discrete stochastic processes and characterize their behavior asymptotically almost surely.

Definition

Let $(X_n : \Omega \to \mathbb{R})_{n \geq 1}$ be a stochastic process and let $(x_n)_{n \geq 1} \subset \mathbb{R}$. Then $X_n = x_n$ **asymptotically almost surely** if there exists $(\epsilon_n)_{n \geq 1} \subset \mathbb{R}$ so that $\epsilon_n = o(1)$ and

$$\mathbb{P}((1 - \epsilon_n)x_n \leq X_n \leq (1 + \epsilon_n)x_n) \to 1$$

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We can also say events $\mathcal{E}_n$ a.a.s. if $\mathbb{P}(\mathcal{E}_n) \to 1$

Some examples of a.a.s. events

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Theorem

Let $\kappa \geq 1$ and $c \in \mathbb{R}^{\geq 0}$. Then a.a.s. the number of connected components of order k, for $1 \leq k \leq \kappa$, in $G_{n,\lfloor cn \rfloor}$ is

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This theorem is the one we'll focus on proving.

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- Sigma Field $\mathcal{F}_i$ generated by all the possible first i choices of edges. These will form a filtration.
- $G_{n,i}$, the (random) graph we observe after drawing i edges.
- $Y_k(i)$, the (random) number of components in $G_{n,i}$

Basic Idea

We'll approximate discrete Y_k with continuous $ny_k \in C^2[0,1]$. That is,

$$n(y_k(t_i) + \epsilon_n(t_i)) = Y_k(i)$$

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To do this we need

- good guesses for y_k and ϵ_n
- a way to show the that Y_k will a.a.s. not stray far from our approximation.

Azuma-Hoeffding

We will construct a supermartingale out of our deviation of Y_k from our approximation. Then we can apply

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Theorem (Azuma-Hoeffding's Inequality)

Let $(X_n)_{n\geq 0}$ be a supermartingale and suppose there exists real $C \geq 0$ $|X_{n+1} - X_n| \leq C$ a.s. for $n \geq 0$. Then, for all $\lambda \in \mathbb{R}^{\geq 0}$ and $m \geq 1$,

$$\mathbb{P}(X_m - X_0 \geq \lambda) \leq \exp\left(-\frac{\lambda^2}{2C^2 m}\right)$$

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$$\mathbb{P}(X_m - X_0 \geq \lambda) \leq \exp\left(-\frac{\lambda^2}{2C^2 m}\right)$$

Since our $m = O(n)$, we only need a $\lambda = \omega(n^{1/2})$. This observation will help guide our choice for ϵ_n.

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A step from i to $i+1$ amounts to a step of $\frac{1}{n}$ in t, so $b-a$ will be $\frac{1}{n}$. In our computations, this will render the 2nd derivative term negligible.

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Let
$$\nu_n^{\pm} := \inf\{i \geq 0 : \pm(Y_k(i) - n(y_k(t_i)) > \epsilon_n(t_i))\}.$$
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Define

$$Y_k^{\pm}(i) = Y_k(i \wedge \nu_n^{\pm}) - n(y_k(t_{i \wedge \nu_n^{\pm}}) \pm \epsilon_n(t_{i \wedge \nu_n^{\pm}})).$$

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With the right choice of y_k and ϵ_n, Y_k^{+} will be a supermartingale.

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4. Otherwise, $\Delta Y_k(i) = 0$.

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Choosing an edge not yet picked has probability

$$\frac{1}{\binom{n}{2} - i} = \frac{2}{n^2 + n - 2i} \leq \frac{2}{n^2}\left(\frac{n^2}{n^2 - n}\right) = \frac{2}{n^2}\left(1 + O(\frac{1}{n})\right)$$

Changes to Component Counts

First looking at the contribution of cases 1 and 2 to $\mathbb{E}(\Delta Y_k(i)|\mathcal{F}_i)$

$$-\frac{2}{n^2}\left(1+O(\frac{1}{n})\right)\left[nkY_k(i)-k^2Y_k(i)^2+k^2Y_k(i)^2-k^2Y_k(i)\right]$$

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We make use of the fact that $Y_k(i)\leq n$ and $k\leq\kappa$.

Changes to Component Counts

Next, we look at the contribution of case 3:

$$\frac{2}{n^2}\left(1+O(\frac{1}{n})\right)\frac{1}{2}\sum_{j=1}^{k-1} jY_j(i)(k-j)Y_{k-j}(i)$$

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&= \left(1+O(\frac{1}{n})\right)\sum_{j=1}^{k-1} j(k-j)\frac{Y_j(i)}{n}\frac{Y_{k-j}(i)}{n} \\
&\leq \sum_{j=1}^{k-1}\left[j(k-j)\frac{Y_j(i)}{n}\frac{Y_{k-j}(i)}{n}\right] + \kappa^3 O(\frac{1}{n}) \\
&= \sum_{j=1}^{k-1}\left[j(k-j)\frac{Y_j(i)}{n}\frac{Y_{k-j}(i)}{n}\right] + O(\frac{1}{n})
\end{aligned}
$$

where we again use the fact that $Y_j(i), Y_{k-j}(i) \leq n$ and $j, k-j \leq \kappa$.

Changes to Component Counts

Assuming $i < \nu_n^+$, we have that

$$\begin{aligned}
&\mathbb{E}(\Delta Y_k(i) | \mathcal{F}_i, [i < \nu_n^+]) \\
&\quad = -2k(y_k(t_i) + \epsilon_n(t_i)) \\
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using the fact that $y_k(t) \leq 1$ and $0 \leq \epsilon_n(t) \leq 1$ for sufficiently large n.

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We'll sort out the $\omega(\frac{1}{n})$ term later when we apply Azuma-Hoeffding.

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Claim

$y_k(t) = \frac{k^{k-2}}{k!}(2t)^{k-1}e^{-2kt}$ is the unique solution to the ODE system.

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So in total $\sum_{j=1}^{k-1}\left[\binom{k}{j} j^{j-2}(k-j)^{(k-j)-2} j(k-j)\right]$ choices. After completing all 4 choices, we get a labeled tree on $[k]$.

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Recap

So far we've shown that if

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then $(Y_k^+(i))_{i=0}^{\lfloor cn \rfloor}$ is a supermartingale.

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First we need to show $|\Delta Y_k^+(i)| = O(1)$ a.s. $\forall 0 \leq i \leq \lfloor cn \rfloor$.

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Otherwise, Y_k^+ is frozen, so $|\Delta Y_k^+(i)| = 0$

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&\quad = \mathbb{P}(Y_k^+(m) - Y_k^+(0) > n\epsilon_n(0)) \\
&\quad \leq \exp\left(-\frac{n^2\epsilon_n(0)^2}{2C^2 m}\right)
\end{aligned}$$

Applying Azuma-Hoeffding

For step $m = \lfloor cn \rfloor$ we want to bound $\mathbb{P}(m \geq \nu_n^+) = \mathbb{P}(Y_k^+(m) > 0)$. We have

$$Y_k^+(0) = Y_k(0) - ny_k(0) - n\epsilon_n(0) = -n\epsilon_n(0)$$

so

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where we absorbed c from $m = \lfloor cn \rfloor$ into C^2.

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Choosing $\epsilon_n(0) = \omega(n^{-1/2})$, such as $\epsilon_n(t) = n^{-1/3}e^{4\kappa^3 t}$, should do the trick. So $\mathbb{P}(m \geq \nu_n^+) \to 0$

Symmetric Calculation and Wrap-Up

A symmetric calculation for Y_k^- gets us that $\mathbb{P}(m \geq \nu_n^-) \to 0$ as well.

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Finally, applying a union bound, we get that

$$\mathbb{P}(m \geq \nu_n^+ \text{ or } m \geq \nu_n^-) \leq \mathbb{P}(m \geq \nu_n^+) + \mathbb{P}(m \geq \nu^-) = o(1) + o(1)$$