1 Expected value of kth element of an ordered sequence of elements from some interval

1.1 Question:

Let each ordered sequence of n unique elements $X_1, \ldots, X_n$ from the interval (a, b) be equally probable. What is the expected value of the element X_k?

1.2 Answer:

Consider random variables $Y_1, \ldots, Y_n$ sampled from a uniform distribution over (a, b) such that no two Y_i are equal. First, we show that any ordered sequence made from $Y_1, \ldots, Y_n$ is equally probable.

Let $X_1 = Y_{i_1}, \ldots, X_n = Y_{i_n}$ be the ordered sequence made from our random variables $Y_1, \ldots, Y_n$. Then we see the probability density function, f, at a specific random sequence is

$$\begin{aligned} f_X([X_1, \ldots, X_n]) &= \sum_{Y_1,\ldots,Y_n \in (a,b)^n; Y_i \text{ unique}} P(Y_1, \ldots, Y_n) \\ &= \sum_{Y_1,\ldots,Y_n \in (a,b)^n; Y_i \text{ unique}} f_{Y_1}(Y_1) \cdots f_{Y_n}(Y_n) \end{aligned}$$

Note that the value of a uniform probability density does not change if we introduce a finite number of holes, thus:

$$f_{Y_i}(Y_i) = \begin{cases} 1, & \text{if } Y_i \in (a, b) \text{ and } Y_i \neq Y_j \text{ for } j \neq i \\ 0, & \text{otherwise} \end{cases} \tag{1}$$

To see this