1 Expected value of kth element of an ordered sequence of elements from some interval

1.1 Question:

Let each ordered sequence of n unique elements $X_1, \ldots, X_n$ from the interval (a, b) be equally probable. What is the expected value of the element X_k?

1.2 Answer:

Consider random variables $Y_1, \ldots, Y_n$ sampled from a uniform distribution over (a, b) such that no two Y_i are equal. First, we show that any ordered sequence made from $Y_1, \ldots, Y_n$ is equally probable.

Let $X_1 = Y_{i_1}, \ldots, X_n = Y_{i_n}$ be the ordered sequence made from our random variables $Y_1, \ldots, Y_n$. Then we see the probability density function, f, at a specific random sequence is

$$
\begin{aligned}
f_X([X_1, \ldots, X_n]) &= \sum_{Y_1,\ldots,Y_n \in \text{ Permuations}(X_1,\ldots,X_n)} f_{Y_1,\ldots,Y_n}(Y_1, \ldots, Y_n) \\
&= \sum_{Y_1,\ldots,Y_n \in \text{ Permuations}(X_1,\ldots,X_n)} f_{Y_1}(Y_1) \cdots f_{Y_n}(Y_n)
\end{aligned}
$$

Note that the value of a uniform probability density does not change if we introduce a finite number of holes (as the integral over the pdf does not change in value), thus:

$$
f_{Y_i}(Y_i) = \begin{cases} \frac{1}{b-a}, & \text{if } Y_i \in (a, b) \text{ and } Y_i \neq Y_j \text{ for } j \neq i \\ 0, & \text{otherwise} \end{cases}
$$

NOTE: *I think I can rewrite it this way, because all I'm doing is reorganizing the points of the composite Y pdf into groups that produce the same ordered sequence. So when I integrate over the pdf of the ordered sequences, it's integrating over the same region as the composite Y pdf.* **But the multiplier of $n!$ can't be right or the pdf for f_X will integrate to greater than 1!**

Therefore we can rewrite f_X:

$$
\begin{aligned}
f_X([X_1, \ldots, X_n]) &= \sum_{Y_1,\ldots,Y_n \in \text{ Permuations}(X_1,\ldots,X_n)} \frac{1}{(b-a)^n} \\
&= \frac{n!}{(b-a)^n}
\end{aligned}
$$