Stochastic Processes and Branching Brownian Motion

by

Abhijeet Anand Mulgund
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THESIS

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Defense Committee:
William Perkins, Chair and Advisor
Cheng Ouyang
Marcus Michelen

TODO: A brief dedication to someone you care about. For example, “Dedicated to my cats, Neo and Trinity, who are purrfect in every way.”.

An example of “Dedication” can be found on page 15 of the thesis manual[1].

[1] http://grad.uic.edu/sites/default/files/pdfs/ThesisManual_rev_06Oct2016.pdf

ACKNOWLEDGMENT

TODO: A page or two so of shout-outs to people you appreciate. Don't forget your advisor and committee members!

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CONTRIBUTIONS OF AUTHORS

This Masters Thesis is a culmination of my studies on various topics of Stochastic Processes, Martingales, and Branching Brownian Motion. It is by no means a comprehensive study, though it hopefully can serve as a resource to others who wish to learn more about these topics.

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SUMMARY

We will work to first build up the basic theory of Stochastic Processes and Martingales. We will establish conditional expectation, but take most of the basic results and relevant measure theoretic results for granted. From there, we will turn to martingales and start with studying "strategies" on martingales and how they help us prove convergence. From there we will look into decomposition of processes in terms of martingales and a handful of important examples.

With an understanding of martingales, we will then turn to Markov Processes. [[TODO]]

Brownian Motion is both a martingale and markov process, so our prior work will help elucidate some of its properties. Unlike in the previous sections, however, we will not study brownian motion in isolation. Instead, we will examine its deep relationship with a deterministic object, the heat equation.

Finally, we will put together our work from the previous 3 chapters towards a study of Branching Brownian Motion. Branching Brownian motion involves Brownian motions with lifetimes determined by a standard exponential distribution. Upon death, a Brownian motion will split into two Brownian motions, and so on. Like with regular Brownian motion, we will find an interesting connection with diffusion partial differential equations. With a basic understanding of this connection in hand, we will use it to study the distribution of the maximal point of a standard branching Brownian motion.

CHAPTER 1

STOCHASTIC PROCESSES AND MARTINGALES

1.1 Conditional Expectation

Suppose we are given a probability space $(\Omega, \mathcal{B}, \mathbb{P})$ and a random variable $X \in L^1(\mathcal{B})$. We would like to describe the operation of "viewing" this random variable from a sub-sigma algebra $\mathcal{G} \subset \mathcal{B}$. We call this operation *conditional expectation* and define it as follows:

Definition 1.1.1. *The conditional expectation of X given $\mathcal{G}$, denoted $\mathbb{E}(X|\mathcal{G})$, is a random variable with the following properties*

1. $\mathbb{E}(X|\mathcal{G}) \in \mathcal{G}$

2. $\int\limits_A X d\mathbb{P} = \int\limits_A \mathbb{E}(X|\mathcal{G}) d\mathbb{P}$ *for all* $A \in \mathcal{G}$.

The existence and $\mathbb{P}$-a.s. uniqueness of a conditional expectation follows from the following measure-theoretic theorem.

Theorem 1.1.1 (Radon-Nikodym Theorem)**.** *Let $X, \mathcal{M}$ be a measure space with sigma-finite nonnegative measure μ and sigma-finite signed measure ν so that $\nu << \mu$. Then there exists μ-a.e. unique $f \in L^1(\mathcal{M})$ so that for all $A \in \mathcal{M}$,*

$$\int\limits_A f d\mu = \nu(A).$$

We denote $\frac{d\nu}{d\mu} := f$.

Note that $\nu(A) = \int_A X d\mathbb{P}$ is a signed measure on $(\Omega, \mathcal{B})$ and $\mathbb{P}$ is a finite measure. Restricting ν to $\mathcal{G}$ does not change this fact, and applying 1.1.1 to $\nu\big|_{\mathcal{G}}$ gives us our $\mathbb{P}$-a.s. unique conditional expectation

$$\mathbb{E}(X|\mathcal{G}) = \frac{d\nu\big|_{\mathcal{G}}}{d\mathbb{P}\big|_{\mathcal{G}}}$$

in $L^1(\mathcal{G})$.

1.2 Regular Conditional Probabilities

TODO: Standard results 1. existence on "nice" spaces 2. relationship to conditional expectation

1.3 Martingales

TODO: - definition of stochastic process - remark about total ordering - definition of adapted process and filter - definition of martingale (total ordering), super, sub - connection to natural number definition

1.4 Decomposition of Discrete-Time martingales

TODO: - predictable process definition - decomp of adapted process into predictable process and martingale - Doob's Decomposition

1.5 Strategies and Discrete-Time martingales

TODO: - Definition - preservation of fairness - relationship to stopping times

1.6 Convergence of Discrete-time Martingales

TODO: - Upcrossing inequality - convergence of discrete-time martingales - pathological examples

1.7 Cotninuous-time martingales, definitions

- indistinguishable - section 1.1 Karatzas shreve

1.8 Continuous-time martingales, stopping times

-section 1.2 Karatzas shreve

1.9 Convergence of Continuous-time Martingales

- section 1.3 karatzas shreve - submartingale inequalities (Karatzas thm 3.8) - convergence of right-continuous martingales

1.10 Polya's Urn

1.11 Borel-Cantelli Lemma

1.12 Differential Equation Method?

TODO: - Azuma's inequality - 2nd case study with graphs

CHAPTER 2

MARKOV PROCESSES

2.1 Basics

CHAPTER 3

BROWNIAN MOTION AND THE HEAT EQUATION

3.1 Basics

- Definition - Existence Remark - Martingale - Markov Process

3.2 Heat Equation

3.3 Brownian Motion as a Solution

CHAPTER 4

BRANCHING BROWNIAN MOTION

4.1 Basics

TODO: - definition

4.2 F-KPP Equation

- F-KPP equation

4.3 McKean Representation

- Mckean Representation

4.4 Kolmogorov's Result

CITED LITERATURE

VITA

NAME	Sheena The Punk Rocker
EDUCATION	B.A., Composition, Rock n' Roll High School, New York City, NYC, 1976
TEACHING	Music lessons (CS342, Summer 1980)
PUBLICATIONS	Author 1, Author 2, and Author 3. “Paper title 1.” In Proceedings of the CONFERENCE NAME (CONFERENCE ABBR, YEAR).
	Author 1, Author 2, and Author 3. “Paper title 2.” In Proceedings of the CONFERENCE NAME (CONFERENCE ABBR, YEAR).