Stochastic Processes and Branching Brownian Motion

by

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B.A., Rice University, 2019

THESIS

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Defense Committee:
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TODO: A brief dedication to someone you care about. For example, "Dedicated to my cats, Neo and Trinity, who are purrfect in every way.".

An example of "Dedication" can be found on page 15 of the thesis manual[1].

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[1] http://grad.uic.edu/sites/default/files/pdfs/ThesisManual_rev_06Oct2016.pdf

ACKNOWLEDGMENT

TODO: A page or two so of shout-outs to people you appreciate. Don't forget your advisor and committee members!

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CONTRIBUTIONS OF AUTHORS

This Masters Thesis is a culmination of my studies on various topics of Stochastic Processes, Martingales, and Branching Brownian Motion. It is by no means a comprehensive study, though it hopefully can serve as a resource to others who wish to learn more about these topics.

TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

FIGURE		PAGE

LIST OF FIGURES (Continued)

SUMMARY

We will work to first build up the basic theory of Stochastic Processes and Martingales. We will establish conditional expectation, but take most of the basic results and relevant measure theoretic results for granted. From there, we will turn to martingales and start with studying "strategies" on martingales and how they help us prove convergence. From there we will look into decomposition of processes in terms of martingales and a handful of important examples.

With an understanding of martingales, we will then turn to Markov Processes. [[TODO]]

Brownian Motion is both a martingale and markov process, so our prior work will help elucidate some of its properties. Unlike in the previous sections, however, we will not study brownian motion in isolation. Instead, we will examine its deep relationship with a deterministic object, the heat equation.

Finally, we will put together our work from the previous 3 chapters towards a study of Branching Brownian Motion. Branching Brownian motion involves Brownian motions with lifetimes determined by a standard exponential distribution. Upon death, a Brownian motion will split into two Brownian motions, and so on. Like with regular Brownian motion, we will find an interesting connection with diffusion partial differential equations. With a basic understanding of this connection in hand, we will use it to study the distribution of the maximal point of a standard branching Brownian motion.

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CHAPTER 1

INTRODUCTION

1.1 Basics

TODO: Do I need much of an introduction? This can introduce the interesting points.

The Ramones are a great band from New York City (NYC). Probably even better than you remember. There have been books written about them, like "Hey Ho Let's Go: The Story of the Ramones: The Story of The Ramones" (1) and "Jews, punk and the Holocaust: from the Velvet Underground to the Ramones–the Jewish-American story" (2).
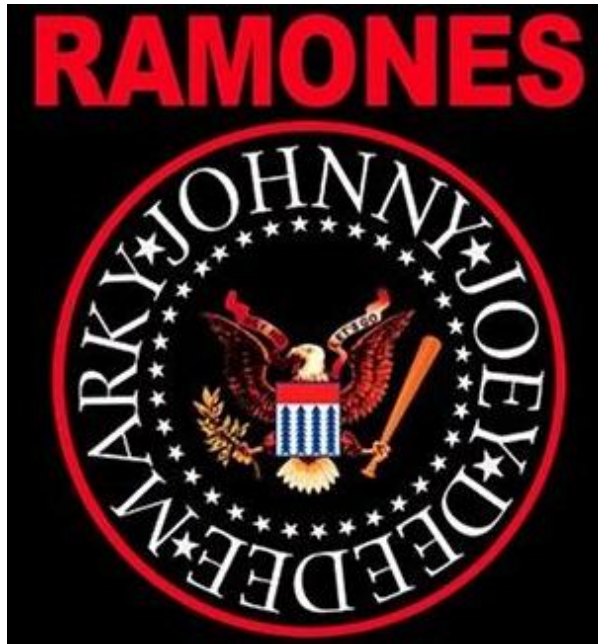

Figure 1: The Ramones band logo, the best dumb band ever.

Band Member	Instrument	Num Albums
Johnny Ramone	Guitar	14
Dee Dee Ramone	Bass	11
Joey Ramone	Vocals	14
Tommy Ramone	Drums	3
Marky Ramone	Drums	9
Richie Ramone	Drums	2
Elvis Ramone	Drums	0
C. J. Ramone	Drums	4

TABLE I: TABLE SHOWING HOW MANY RECORDS EACH MEMBER OF THE RA-MONES WAS INVOLVED IN.

As Figure 4 shows, they had a great logo too. And they wrote amazing albums, like Rocket to Russia (RTR) and End of the Century (EOFC), and a couple more. They even did three (!) jingles for Steel Reserve[1]

1.2 Turmoil

However, things were always smiles-times for The Ramones. As Table V shows, there were lots of members in the band. Many of them were kind crummy, but hey, no one is perfect. Geepers, what a good band.

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[1]The gross malt liquor.

CHAPTER 2

STOCHASTIC PROCESSES AND MARTINGALES

2.1 Conditional Expectation

Suppose we are given a probability space $(\Omega, \mathcal{B}, \mathbb{P})$ and a random variable $X \in L^1(\mathcal{B})$. We would like to describe the operation of "viewing" this random variable from a sub-sigma algebra $\mathcal{G} \subset \mathcal{B}$. We call this operation *conditional expectation* and define it as follows:

Definition 2.1.1. *The conditional expectation of* X *given* $\mathcal{G}$, *denoted* $\mathbb{E}(X|\mathcal{G})$, *is a random variable with the following properties*

1. $\mathbb{E}(X|\mathcal{G}) \in \mathcal{G}$

2. $\int_A X \, d\mathbb{P} = \int_A \mathbb{E}(X|\mathcal{G}) \, d\mathbb{P}$ *for all* $A \in \mathcal{G}$.

The existence and $\mathbb{P}$-a.s. uniqueness of a conditional expectation follows from the following measure-theoretic theorem.

Theorem 2.1.1 (Radon-Nikodym Theorem)**.** *Let* $X, \mathcal{M}$ *be a measure space with sigma-finite nonnegative measure* μ *and sigma-finite signed measure* ν *so that* $\nu \ll \mu$. *Then there exists* μ-*a.e. unique* $f \in L^1(\mathcal{M})$ *so that for all* $A \in \mathcal{M}$,

$$\int_A f \, d\mu = \nu(A).$$

We denote $\frac{d\nu}{d\mu} := f$.

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Note that $\nu(A) = \int_A X d\mathbb{P}$ is a signed measure on $(\Omega, \mathcal{B})$ and $\mathbb{P}$ is a finite measure. Restricting ν to $\mathcal{G}$ does not change this fact, and applying 2.1.1 to $\nu\big|_{\mathcal{G}}$ gives us our $\mathbb{P}$-a.s. unique conditional expectation

$$\mathbb{E}(X|\mathcal{G}) = \frac{d\nu\big|_{\mathcal{G}}}{d\mathbb{P}\big|_{\mathcal{G}}}$$

in $L^1(\mathcal{G})$.

2.2 Turmoil

However, things were always smiles-times for The Ramones. As Table V shows, there were lots of members in the band. Many of them were kind crummy, but hey, no one is perfect. Geepers, what a good band.

CHAPTER 3

INTRODUCTION

3.1   Basics

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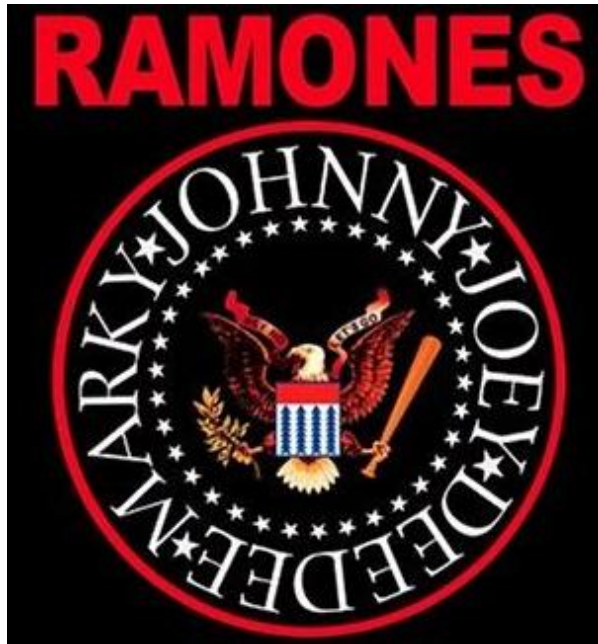

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CHAPTER 4

INTRODUCTION

4.1   Basics

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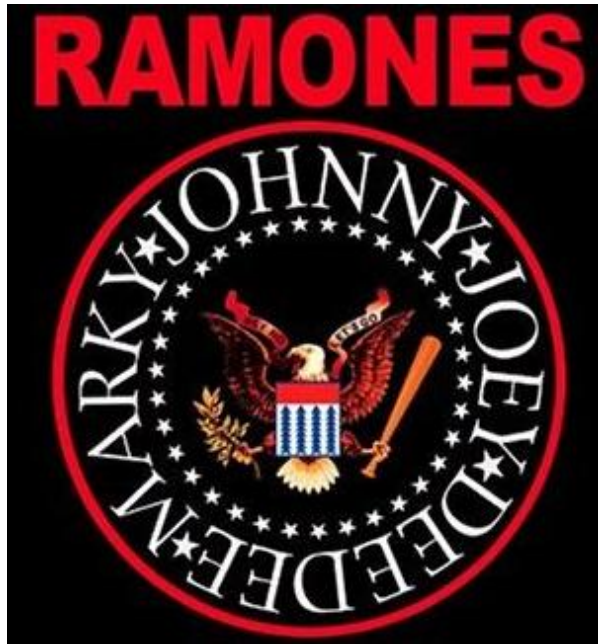

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CHAPTER 5

INTRODUCTION

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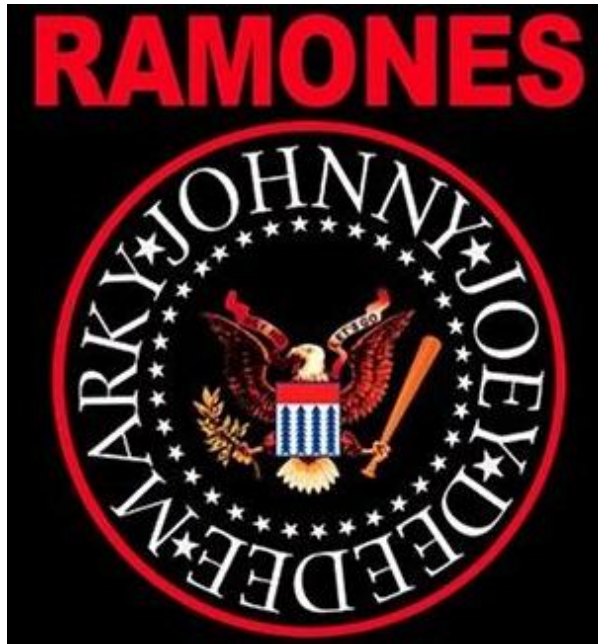

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---

[1]The gross malt liquor.

CITED LITERATURE

1. True, E.: <u>Hey Ho Let's Go: The Story of the Ramones: The Story of The Ramones</u>. Omnibus Press, 2010.

2. Stratton, J.: Jews, punk and the holocaust: from the velvet underground to the ramones–the jewish-american story. <u>Popular music</u>, 24(1):79–105, 2005.

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PUBLICATIONS   Author 1, Author 2, and Author 3. "Paper title 1." In Proceedings of the CONFERENCE NAME (CONFERENCE ABBR, YEAR).

                     Author 1, Author 2, and Author 3. "Paper title 2." In Proceedings of the CONFERENCE NAME (CONFERENCE ABBR, YEAR).