Mathematical Notation for Summation, Integration, and Differentiation with Respect to Time

1. Summation (Σ)

Notation:

$$\sum_{i=1}^{n} a_i$$

Meaning: The summation operator Σ represents the addition of a finite sequence of numbers $a_1, a_2, \ldots, a_n$. It combines discrete terms into a single total.

Components:

- i — summation index
- 1 — lower limit (starting index)
- n — upper limit (ending index)
- a_i — the general term being summed

Examples:

1. Simple sum of squares:

$$\sum_{k=1}^{4} k^2 = 1^2 + 2^2 + 3^2 + 4^2 = 30$$

2. Weighted sum of a time-series at time t:

$$y(t) = \sum_{\tau=0}^{N} f(t-\tau)\, w(\tau)$$

   Here, $f(t-\tau)$ represents the value of a discrete (e.g. daily) time-series at time $t-\tau$ (i.e. τ days before the current time), and $w(\tau)$ is a weight applied to values of the time-series at times τ previous to the current discrete step.
   Example Calculation:
   Let $f(t) = [1, 2, 3]$ and $w(t) = [0.5, 0.3, 0.2]$. Then at $t = 2$:

$$y(2) = f(2) \cdot w(0) + f(1) \cdot w(1) + f(0) \cdot w(2)$$

$$y(2) = 3 \cdot 0.5 + 2 \cdot 0.3 + 1 \cdot 0.2$$

$$y(2) = 1.5 + 0.6 + 0.2 = 2.3$$

2. Integration ($\int$)

Notation:

$$\int_a^b f(x)\,dx$$

Meaning: Integration can be viewed as the continuous extension of summation. While summation adds discrete terms, integration accumulates infinitely many infinitesimal contributions of a function over a continuous interval.

Analogy to Summation:

$$\sum_{i=0}^{n} f(x_i)\,\Delta x \quad \longrightarrow \quad \int_a^b f(x)\,dx \quad \text{as } \Delta x \to 0, n \to \infty$$

Components:

- $f(x)$ — the function to integrate
- a — lower limit (i.e. the minimum value of x)
- b — upper limit (i.e. the maximum value of x)
- dx — infinitesimal increment of x

Example: The weighted sum of a time-series at time t:

$$y(t) = \int_0^T f(t-\tau)\,w(\tau)\,d\tau$$

Analogous to the example for summation, $f(t-\tau)$ represents the value of a continuous time-series at time $t-\tau$ (i.e. τ before the current time), and $w(\tau)$ is a weight applied to values of the time-series at times τ previous to the current time.

3. Differentiation with Respect to Time (d/dt)

Notation:

$$\frac{dy}{dt}$$

Meaning: The derivative represents the instantaneous rate of change of a quantity $y(t)$ with respect to time t. It tells us how fast y is changing at any specific moment.
Components:

- $y(t)$ — the function of time

- t — the independent variable (time)
- $\frac{dy}{dt}$ — the rate of change of y with respect to t

Interpretation:

- If $\frac{dy}{dt} > 0$, $y(t)$ is increasing.
- If $\frac{dy}{dt} < 0$, $y(t)$ is decreasing.
- If $\frac{dy}{dt} = 0$, $y(t)$ is momentarily constant.

Applications: Differentiation with respect to time is used in physics, engineering, and finance. Examples include velocity, population growth rates, and chemical reaction rates.

Examples:

1. Polynomial: If $y(t) = t^2$, then
$$\frac{dy}{dt} = 2t$$
At $t = 3$, the rate of change is 6.

2. Exponential growth: For $\frac{dy}{dt} = ky$ with $y(0) = y_0$, the solution is
$$y(t) = y_0 e^{kt}$$
The rate of change of y is proportional to y's current value.