Gradient Characterization of Convexity

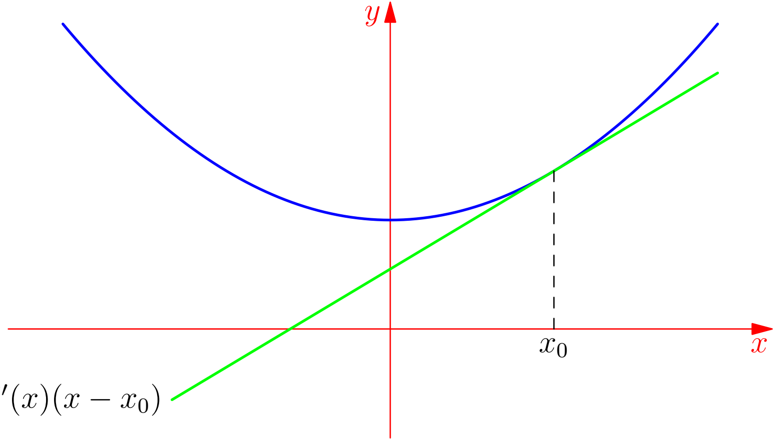
Theorem

Let $f : \mathbb{R}^d \to \mathbb{R}$ be differentiable. Then f is convex iff

$$f(x+v) \geq f(x) + \nabla f(x)^T v$$

hold for all $x, v \in \mathbb{R}^d$.

Gradient Approximation Gives Global Underestimator



$f(x_0) + f'(x)(x - x_0)$

Subgradients

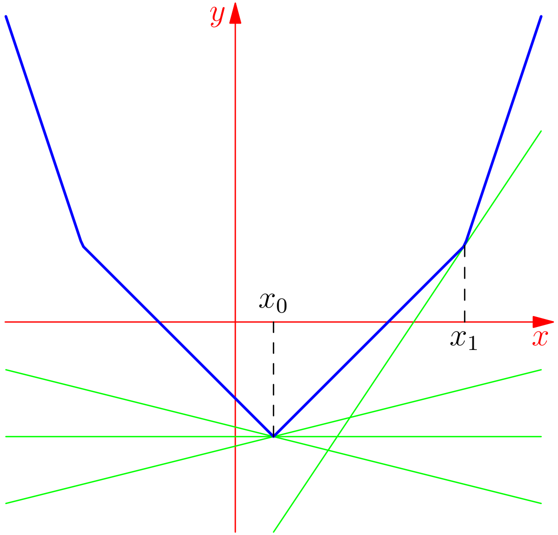
Definition (Subgradient, Subdifferential, Subdifferentiable)

Let $f : \mathbb{R}^d \to \mathbb{R}$. We say that $g \in \mathbb{R}^d$ is a *subgradient* of f at $x \in \mathbb{R}^d$ if

$$f(x + v) \geq f(x) + g^T v$$

for all $v \in \mathbb{R}^d$. The *subdifferential* $\partial f(x)$ is the set of all subgradients of f at x. We say that f is *subdifferentiable* at x if $\partial f(x) \neq \emptyset$ (i.e., if there is at least one subgradient).

Subgradients at x_0 and x_1



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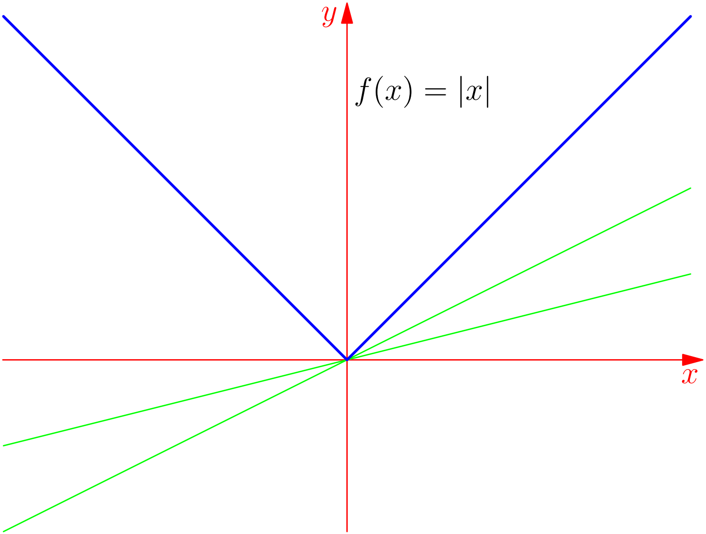
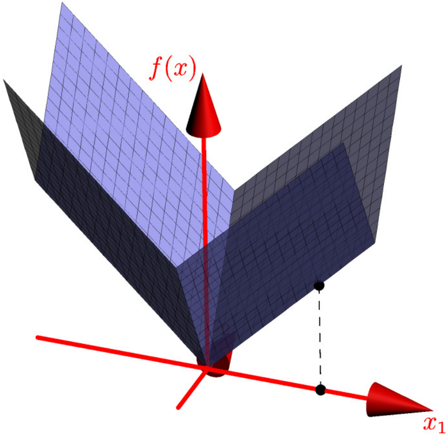
4. If the zero vector is a subgradient of f at x, then x is a global minimum.
5. If g is a subgradient of f at x, then $(g, -1)$ is orthogonal to the underestimating hyperplane $\{(x + v, f(x) + g^T v) \mid v \in \mathbb{R}^d\}$ at $(x, f(x))$.

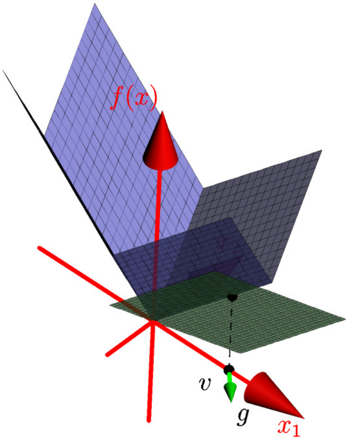
Compute the Subdifferentials of $f(x) = |x|$



Compute $\partial f(3,0)$ For $f(x_1,x_2)=|x_1|+2|x_2|$

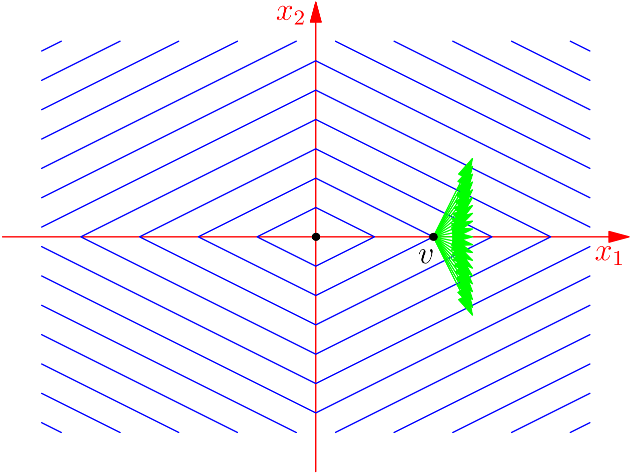


Compute $\partial f(3,0)$ For $f(x_1,x_2) = |x_1| + 2|x_2|$



Compute $\partial f(3,0)$ For $f(x_1,x_2) = |x_1| + 2|x_2|$

$$\partial f(3,0) = \{(1,b)^T \mid b \in [-2,2]\}$$



Gradient Lies Normal To Contours

Theorem

If $f : \mathbb{R}^d \to \mathbb{R}$ *is continuously differentiable and* $x_0 \in \mathbb{R}^d$ *with* $\nabla f(x_0) \neq 0$ *then* $\nabla f(x_0)$ *is normal to the level set* $S = \{x \in \mathbb{R}^d \mid f(x) = f(x_0)\}$.

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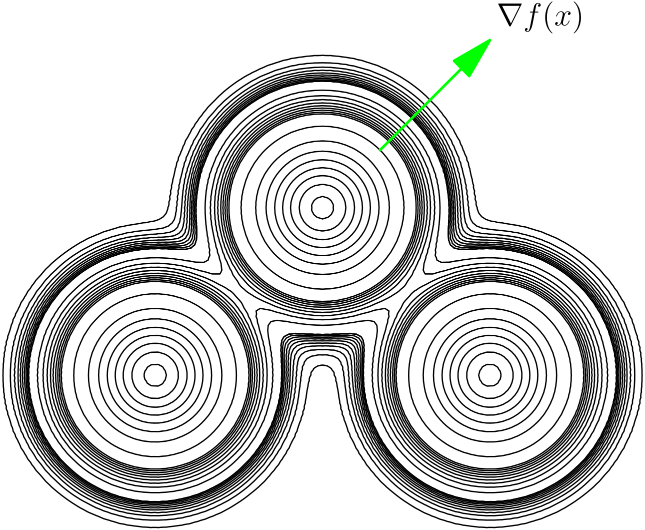
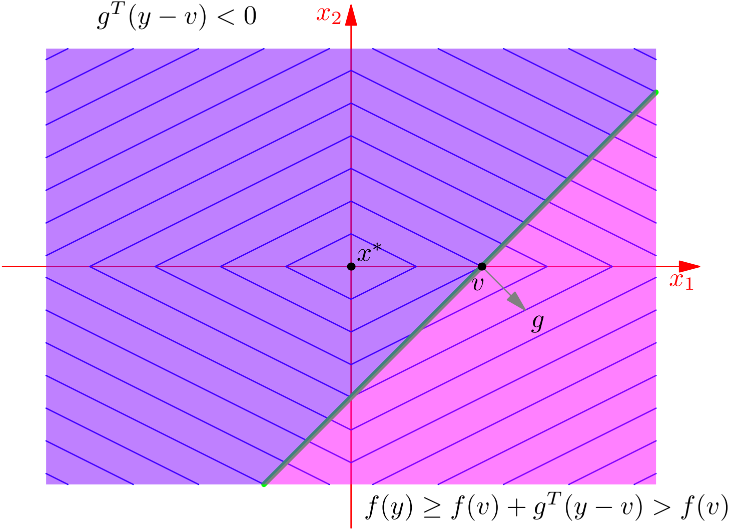


Normal Plane to Subgradient Splits Space



Subgradient Descent

1. Let $x^{(0)}$ denote the initial point.
2. For $k = 1, 2, \ldots$
   1. Assign $x^{(k)} = x^{(k-1)} - \alpha_k g$, where $g \in \partial f(x^{(k-1)})$ and α_k is the step size.
   2. Set $f_{\text{best}}^{(k)} = \min_{i=1,\ldots,k} f(x^{(i)})$. (Used since this isn't a descent method.)

Convergence of Subgradient Descent

Theorem

Let $f : \mathbb{R}^n \to \mathbb{R}$ be convex and Lipschitz with constant G, and let x^* be a minimizer. For a fixed step size t, the subgradient method satisfies:

$$\lim_{k\to\infty} f(x_{best}^{(k)}) \leq f(x^*) + G^2 t/2.$$

For step sizes respecting the Robbins-Monro conditions,

$$\lim_{k\to\infty} f(x_{best}^{(k)}) = f(x^*).$$