MA3105 FINAL PROJECT:
CUBIC SPLINE INTERPOLATION

Shreya Ganguly (22MS110)
Niravra Nag (22MS072)
Abhratanu Ray (22MS052)

Autumn 2024

INTRODUCTION

What is interpolation?

Given a set of $n+1$ points $\{x_0, x_1, ..., x_n\}$ at which the values of a function $f(x_0) = y_0, f(x_1) = y_1, ..., f(x_n) = y_n$ are known, interpolation is the process of finding the function that satisfies the condition $f(x_i) = y_i \forall i \in \{0, 1, 2, ..., n\}$.

Polynomial Interpolation

Polynomial interpolation involves fitting a single polynomial of degree n to a given set of $n+1$ data points. While this approach ensures that the polynomial passes through all the given points, it has notable disadvantages when used for a large number of points.

A primary concern is the occurrence of **Runge's phenomenon**, which is characterized by large oscillations near the endpoints of the interpolation interval, particularly when equally spaced nodes are used. This behavior is caused by the high degree of the polynomial, as the interpolant becomes increasingly sensitive to small changes in the data points. Consequently, the error grows significantly, making the interpolation unreliable.

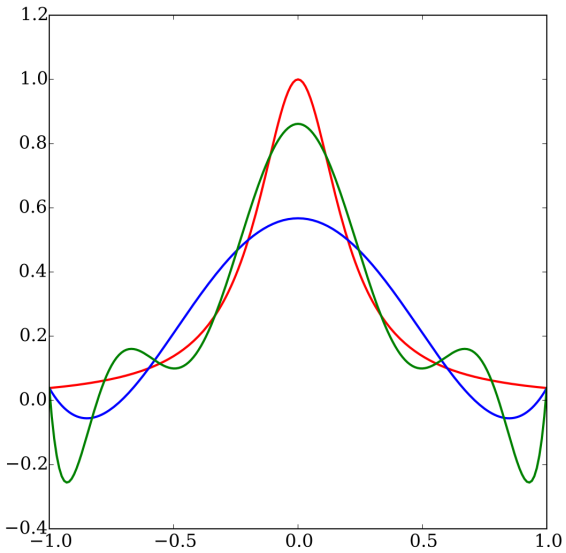


Figure 1: Runge's Phenomenon for $1/(1+x^2)$

Additionally, the computational cost of evaluating and differentiating high-degree polynomials can be prohibitive. These drawbacks highlight the need for an alternative approach that balances accuracy and computational efficiency, such as *piecewise interpolation*.

Piecewise Interpolation

Piecewise interpolation addresses the limitations of polynomial interpolation by dividing the data into smaller intervals and fitting simpler functions, such as linear or quadratic polynomials, within each interval. Piecewise interpolation avoids Runge's phenomenon because it uses low-degree polynomials over small intervals. By limiting each interpolant to a specific subdomain, the method maintains accuracy while preventing large oscillations, especially near the endpoints. Furthermore, the use of lower-degree polynomials reduces computational complexity and improves numerical stability.

Compared to global polynomial interpolation, piecewise methods offer better flexibility and adaptability, making them suitable for applications involving large datasets or unevenly spaced nodes.

Linear Piecewise Interpolation

Linear interpolation fits a straight line between two consecutive data points. For two points (x_i, y_i) and (x_{i+1}, y_{i+1}), the linear interpolant is:

$$L_i(x) = y_i + \frac{y_{i+1} - y_i}{x_{i+1} - x_i}(x - x_i), \quad x \in [x_i, x_{i+1}].$$

This method is computationally efficient and straightforward to implement, but it may not accurately capture the curvature of the underlying function, leading to discontinuities in the first derivative at the data points.

Quadratic Piecewise Interpolation

Piecewise quadratic interpolation divides the entire domain into subintervals, where each subinterval spans three consecutive points. A unique quadratic polynomial is constructed for each subinterval using the Lagrange formula. Let the function be interpolated over a domain divided into n points, where n is a multiple of 3.

For any three consecutive points (x_{i-1}, y_{i-1}), (x_i, y_i), and (x_{i+1}, y_{i+1}), the Lagrange polynomial is expressed as:

$$Q_i(x) = y_{i-1} \cdot \ell_{i-1}(x) + y_i \cdot \ell_i(x) + y_{i+1} \cdot \ell_{i+1}(x),$$

where $\ell_{i-1}(x)$, $\ell_i(x)$, and $\ell_{i+1}(x)$ are the Lagrange basis polynomials defined as:

$$\ell_{i-1}(x) = \frac{(x - x_i)(x - x_{i+1})}{(x_{i-1} - x_i)(x_{i-1} - x_{i+1})},$$

$$\ell_i(x) = \frac{(x - x_{i-1})(x - x_{i+1})}{(x_i - x_{i-1})(x_i - x_{i+1})},$$

$$\ell_{i+1}(x) = \frac{(x - x_{i-1})(x - x_i)}{(x_{i+1} - x_{i-1})(x_{i+1} - x_i)}.$$

The resulting interpolating polynomial $Q_i(x)$ satisfies the following conditions:

- $Q_i(x_{i-1}) = y_{i-1}$,
- $Q_i(x_i) = y_i$,
- $Q_i(x_{i+1}) = y_{i+1}$.

Cubic Spline Interpolation: Constructing the Spline

Given a set of $n+2$ points $x_0 < x_1 < x_2 ... < x_n + 1$ with respective values of the function $y_0, y_1, y_2, ..., y_n + 1$, we construct the spline as

$$S(x) = P_i(x) : x \in [x_i, x_{i+1}], \qquad i = 0, 1, 2, ..., n$$

where $P_i(x)$ are polynomials of degree ≤ 3.
Then, the polynomials P_i satisfy the following conditions:

1. $P_i(x_i) = y_i, \qquad i = 0, 1, 2, ..., n$
2. $P_{i-1}(x_i) = y_i \qquad i = 0, 1, ..., n$
3. $P_i'(x_i) = P_{i-1}'(x_i), \qquad i = 0, 1, 2, ..., n$
4. $P_i''(x_i) = P_{i-1}''(x_i), \qquad i = 0, 1, 2, ..., n$

(These conditions can only be simultaneously satisfied by polynomials of degree ≥ 3, hence the simplest case of spline interpolation is by cubic spline interpolation.)
We note, a general cubic $ax^3 + bx^2 + cx + d$ has 4 coefficients to be determined, giving us, in total, $4n$ equations to be solved to determine the polynomials. However, the conditions above impose a total of $4n - 2$ conditions. There are different approaches to set the remaining 2 conditions.

1. Natural Spline: $P_0''(x_0) = P_{n+1}''(x_{n+1}) = 0$
2. Clamped Spline: $P_0'(x_0) = d_0$ and $P_{n+1}'(x_{n+1}) = d_{n+1}$, where d_0 and d_{n+1} are the derivatives of the function f at the endpoints x_0 and x_{n+1}.
3. Not-a-knot Spline: $P_0'''(x_1) = P_1'''(x_1)$ and $P_{n-1}'''(x_n) = P_n'''(x_n)$

We will be considering the natural boundary conditions. Now, we have $4n$ variables and $4n$ equations to solve. A straightforward approach is to plug in $P_i(x) = a_i x^3 + b_i x^2 + c_i x + d_i$ to obtain

$$P_i'(x) = 3a_i x^2 + 2b_i x + c_i$$

$$P_i''(x) = 6a_i x + 2b_i$$

This gives us the set of equations,

$$a_i x_{i+1}^3 + b_i x_{i+1}^2 + c_i x_{i+1} + d_{i+1} = y_{i+1}$$

$$a_{i+1} x_{i+1}^3 + b_{i+1} x_{i+1}^2 + c_{i+1} x_{i+1} + d_{i+1} = y_{i+1}$$

$$3a_i x_{i+1}^2 + 2b_i x_{i+1} + c_{i+1} - 3a_{i+1} x_{i+1}^2 - 2b_{i+1} x_{i+1} - c_{i+1} = 0$$

$$6a_i x_{i+1} + 2b_i = 6a_{i+1} x_{i+1} + 2b_{i+1}$$

$$6a_0 x_0 + 2b_0 = 0$$

$$6a_n x_{n+1} + 2b_n = 0$$

Then this system of $4n$ equations can be directly solved by Gaussian elimination, to obtain the $4n$ coefficients. However, there are ways to increase the efficiency and reduce the computational expense rather than directly brute-forcing the solution like above.

A more calculated approach is to construct $S''(x)$, and then obtain $S(x)$ by successively integrating it twice. In the following section we consider $P_i(x)$ to be the cubic spline in $[x_{i-1}, x_i]$
Let us denote $P_i''(x_i) = M_i = P_{i+1}''(x_i)$. We note, $M_0 = M_n = 0$. Since P_i are cubics, S'' will be a piecewise linear function which interpolates the set of points $\{(x_i, M_i)\}$. Then, in $[x_{i-1}, x_i]$,

$$S''(x) = P_i''(x) = M_{i-1}\frac{x_i - x}{x_i - x_{i-1}} + M_i\frac{x - x_{i-1}}{x_i - x_{i-1}}$$

.
Integrating, we obtain

$$P_i'(x) = -\frac{M_{i-1}}{2(x_i - x_{i-1})}(x_i - x)^2 + \frac{M_i}{2(x_i - x_{i-1})}(x - x_{i-1})^2 + A_i$$

where A_i is a constant.

$$P_i(x) = \frac{M_{i-1}}{6(x_i - x_{i-1})}(x_i - x)^3 + \frac{M_i}{6(x_i - x_{i-1})}(x - x_{i-1})^3 + A_i x + B_i$$

where A_i, B_i are constants.
We have, $S(x_{i-1}) = y_{i-1}$ and $S(x_i) = y_i$. Using these,

$$P_i(x_{i-1}) = \frac{M_{i-1}}{6(x_i - x_{i-1})}(x_i - x_{i-1})^3 + A_i x_{i-1} + B_i = \frac{M_{i-1}}{6}(x_i - x_{i-1})^2 + A_i x_{i-1} + B_i$$

$$\implies y_{i-1} = \frac{M_{i-1}}{6}(x_i - x_{i-1})^2 + A_i x_{i-1} + B_i$$

$$P_i(x_i) = \frac{M_i}{6(x_i - x_{i-1})}(x_i - x_{i-1})^3 + A_i x_i + B_i = \frac{M_i}{6}(x_i - x_{i-1})^2 + A_i x_i + B_i$$

$$\implies y_i = \frac{M_i}{6}(x_i - x_{i-1})^2 + A_i x_i + B_i$$

We have two equations and two unknowns A_i and B_i. Using these conditions to eliminate these two unknowns, we can simplify the expression to

$$P_i(x) = \frac{(x_i - x)^3 M_{i-1} + (x - x_{i-1})^3 M_i}{6(x_i - x_{i-1})} + \frac{(x_i - x)y_{i-1} + (x - x_{i-1})y_i}{x_i - x_{i-1}} - \frac{1}{6}[(x_i - x)M_{i-1} + (x - x_{i-1})M_i]$$

Further simplifying using the continuity of $S'(x)$, we finally arrive at the final system of equations,

$$\frac{x_i - x_{i-1}}{6}M_{i-1} + \frac{x_{i+1} - x_{i-1}}{3}M_i + \frac{x_{i+1} - x_i}{6}M_{i+1} = \frac{y_{i+1} - y_i}{x_{i+1} - x_i} - \frac{y_i - y_{i-1}}{x_i - x_{i-1}} \qquad i = 1, 2, ..., n-1$$

along with the natural boundary conditions,

$$M_0 = 0$$
$$M_n = 0$$

The system of $n-1$ equations can be solved for $M_1, M_2, ..., M_{n-1}$, from which $S''(x)$ and subsequently $S(x)$ can be determined.

$$\begin{bmatrix} \frac{x_1-x_0}{3} & \frac{x_2-x_1}{6} & 0 & \cdots & 0 & 0 \\ \frac{x_1-x_0}{6} & \frac{x_2-x_0}{3} & \frac{x_3-x_2}{6} & \cdots & 0 & 0 \\ 0 & \frac{x_2-x_1}{6} & \frac{x_3-x_1}{3} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{x_{n-1}-x_{n-2}}{3} & \frac{x_n-x_{n-1}}{6} \\ 0 & 0 & 0 & \cdots & \frac{x_{n-1}-x_{n-2}}{6} & \frac{x_n-x_{n-1}}{3} \end{bmatrix} \begin{bmatrix} M_1 \\ M_2 \\ M_3 \\ \vdots \\ M_{n-1} \\ M_n \end{bmatrix} = \begin{bmatrix} \frac{y_2-y_1}{x_2-x_1} - \frac{y_1-y_0}{x_1-x_0} \\ \frac{y_3-y_2}{x_3-x_2} - \frac{y_2-y_1}{x_2-x_1} \\ \vdots \\ \frac{y_n-y_{n-1}}{x_n-x_{n-1}} - \frac{y_{n-1}-y_{n-2}}{x_{n-1}-x_{n-2}} \end{bmatrix}.$$

This system falls into a class of special matrix forms known as a tridiagonal system, which significantly reduces the computational cost to solve. In the following section, we have discussed an efficient methodology which we follow for our project.

METHODOLOGY

An Efficient Method

In the standard method of finding coefficients for natural spline, our points directly give us $2n$ equations, the first and second derivative continuity give us $2(n-1)$ equations, and we get 2 more from the arbitrary endpoint conditions we set.
So we have $4n$ equations, for the n cubics. Since each cubic has 4 unknowns, we can solve the set of linear equations to get our coefficients.

But it is desirable to simplify this process and try to reduce the number of linear equations, as for large matrices which are not necessarily diagonally dominant, solving via Gaussian elimination becomes computationally expensive.
It is desirable to have a small, diagonally dominant matrix, like we did while constructing the tridiagonal system, so we can apply Jacobi's iteration method, and arrive at a solution efficiently. Such a compact and efficient method is detailed below.

We can write the equation for an interpolating cubic polynomial, given 2 points (x_1, y_1) and (x_2, y_2) in the following Newton divided difference form:

$$P(x) = \big(1 - t(x)\big)\, y_1 + t(x)\, y_2 + t(x)\big(1 - t(x)\big)\Big(\big(1 - t(x)\big)\, a + t(x)\, b\Big) \tag{1}$$

where

$$t(x) = \frac{x - x_1}{x_2 - x_1},$$
$$a = k_1(x_2 - x_1) - (y_2 - y_1),$$
$$b = -k_2(x_2 - x_1) + (y_2 - y_1).$$

Here, we call k_i's our critical values, which are derived using the conditions on the interpolating polynomials.

Consider that,

$$P' = \frac{dP}{dx} = \frac{dP}{dt}\frac{dt}{dx} = \frac{dP}{dt}\frac{1}{x_2 - x_1}.$$

Then we see that,

$$P' = \frac{y_2 - y_1}{x_2 - x_1} + (1 - 2t)\frac{a(1-t) + bt}{x_2 - x_1} + t(1-t)\frac{b - a}{x_2 - x_1},$$

and

$$P'' = 2\frac{b - 2a + (a - b)3t}{(x_2 - x_1)^2}.$$

Setting $t = 0$ and $t = 1$ in these equations, we see that

$$P'(x_1) = k_1, \text{ and } P'(x_2) = k_2.$$

Also,

$$P''(x_1) = 2\frac{b-2a}{(x_2-x_1)^2}, \text{ and } P''(x_2) = 2\frac{a-2b}{(x_2-x_1)^2}.$$

Now, let us have $n+1$ points (x_i, y_i) where $i \in \{0, 1, 2, \ldots, n\}$.

Then, our cubic polynomials take the form

$$P_i = (1-t)\, y_{i-1} + t\, y_i + t(1-t)\big((1-t)\, a_i + t\, b_i\big),$$

where $i \in \{1, 2, \ldots, n\}$ and $t = \frac{x-x_{i-1}}{x_i - x_{i-1}}$.
Each of these n polynomials interpolate y in the intervals $[x_{i-1}, x_i]$, such that $P_i'(x_i) = P_{i+1}'(x_i)$. Then, these cubic polynomials together form a differentiable function in $[x_0, x_n]$. We have

$$a_i = k_{i-1}(x_i - x_{i-1}) - (y_i - y_{i-1}),$$
$$b_i = -k_i(x_i - x_{i-1}) + (y_i - y_{i-1})$$

for $i \in \{1, 2, \ldots, n\}$, where

$$k_0 = P_1'(x_0)$$
$$k_i = P_i'(x_i) = P_{i+1}'(x_i), \text{ when } i \in \{1, 2, \ldots, n-1\}$$
$$k_n = P_n'(x_n).$$

In addition, if $P_i''(x_i) = P_{i+1}''(x_i)$ holds for $i \in \{1, 2, \ldots, n-1\}$, then the resulting function has a continuous second derivative.

From all the imposed conditions and the formulas for a_i and b_i, it is easy to see that this is the case if and only if

$$\frac{k_{i-1}}{x_i - x_{i-1}} + \left(\frac{1}{x_i - x_{i-1}} + \frac{1}{x_{i+1} - x_i}\right) 2k_i + \frac{k_{i+1}}{x_{i+1} - x_i} = 3\left(\frac{y_i - y_{i-1}}{(x_i - x_{i-1})^2} + \frac{y_{i+1} - y_i}{(x_{i+1} - x_i)^2}\right)$$

for $i \in \{1, 2, \ldots, n-1\}$. This gives us $n-1$ linear equations for the $n+1$ unknowns $k_0, k_1, \ldots, k_n$.

The two extra equations we require to solve the unknowns are obtained via imposition of arbitrary conditions on the endpoints. A popular condition, which we will be using, is called the "natural spline" condition. The 2 extra equations are as follows

$$P_1''(x_0) = 2\frac{3(y_1 - y_0) - (k_1 + 2k_0)(x_1 - x_0)}{(x_1 - x_0)^2} = 0,$$

and

$$P_n''(x_n) = -2\frac{3(y_n - y_{n-1}) - (2k_n + k_{n-1})(x_n - x_{n-1})}{(x_n - x_{n-1})^2} = 0.$$

Simplifying further, we get

$$\frac{2}{x_1 - x_0}k_0 + \frac{1}{x_1 - x_0}k_1 = 3\frac{y_1 - y_0}{(x_1 - x_0)^2}$$

and

$$\frac{1}{x_n - x_{n-1}}k_{n-1} + \frac{2}{x_n - x_{n-1}}k_n = 3\frac{y_n - y_{n-1}}{(x_n - x_{n-1})^2}.$$

All that's left to do now is solve for the $n+1$ unknowns using our equations. This will be done using the Jacobi Iteration Method.

Observe that the matrix we get looks like this:

$$\begin{bmatrix} a_{11} & a_{12} & 0 & \ldots & 0 & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 & \ldots & 0 & 0 \\ 0 & a_{32} & a_{33} & a_{34} & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & a_{(n-1)(n-2)} & a_{(n-1)(n-1)} & a_{(n-1)n} \\ 0 & 0 & 0 & \ldots & 0 & a_{n(n-1)} & a_{nn} \end{bmatrix}$$

where $a_{11} = \frac{2}{k_0 + k_1}$, $a_{i(i+1)} = a_{(i+1)i} = \frac{1}{k_i + k_{i-1}}$ and $a_{ii} = 2 \cdot (\frac{1}{k_i + k_{i-1}} + \frac{1}{k_i + k_{i+1}})$. It is easy to see that this matrix is strictly diagonally dominant.

Jacobi Iteration Method

The Jacobi Iteration Method starts with an arbitrary guess for the solution to a set of linear equations written in the matrix form as $Ax = b$.

Rewriting as linear equations, we get:

$$\begin{aligned} x_1 &= \frac{1}{a_{11}} \left(b_1 - a_{12}x_2 - a_{13}x_3 - \cdots - a_{1n}x_n\right) \\ x_2 &= \frac{1}{a_{22}} \left(b_2 - a_{21}x_1 - a_{23}x_3 - \cdots - a_{2n}x_n\right) \\ &\vdots \\ x_n &= \frac{1}{a_{nn}} \left(b_n - a_{n1}x_1 - a_{n2}x_2 - \cdots - a_{n,n-1}x_{n-1}\right) \end{aligned}$$

Then make an initial guess of the solution

$$x^{(0)} = \left(x_1^{(0)}, x_2^{(0)}, x_3^{(0)}, \ldots, x_n^{(0)}\right).$$

Substitute these values into the right-hand side of the rewritten equations to obtain the *first approximation*,

$$\left(x_1^{(1)}, x_2^{(1)}, x_3^{(1)}, \ldots, x_n^{(1)}\right).$$

This accomplishes one *iteration*.
In the same way, the *second approximation*

$$\left(x_1^{(2)}, x_2^{(2)}, x_3^{(2)}, \ldots, x_n^{(2)}\right)$$

is computed by substituting the first approximation's x-values into the right-hand side of the rewritten equations. By repeated iterations, we form a sequence of approximations

$$x^{(k)} = \left(x_1^{(k)}, x_2^{(k)}, x_3^{(k)}, \ldots, x_n^{(k)}\right), \quad k = 1, 2, 3, \ldots$$

We now see that this method will converge for diagonally dominant matrices, such as the one we deal with in our method for cubic spline interpolation.

Consider solving an $n \times n$ size system of linear equations $A\mathbf{x} = \mathbf{b}$ with

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}.$$

We split A into

$$A = \begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix} - \begin{bmatrix} 0 & 0 & \cdots & 0 \\ -a_{21} & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ -a_{n1} & \cdots & -a_{n,n-1} & 0 \end{bmatrix} - \begin{bmatrix} 0 & -a_{12} & \cdots & -a_{1n} \\ 0 & 0 & \cdots & -a_{2n} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}.$$

This can be written as $A = D - L - U$, where

$$D = \begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix}, \quad L = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ -a_{21} & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ -a_{n1} & \cdots & -a_{n,n-1} & 0 \end{bmatrix}, \quad U = \begin{bmatrix} 0 & -a_{12} & \cdots & -a_{1n} \\ 0 & 0 & \cdots & -a_{2n} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}.$$

The equation $A\mathbf{x} = \mathbf{b}$ is transformed into

$$(D - L - U)\mathbf{x} = \mathbf{b}.$$

Assume D^{-1} exists, where

$$D^{-1} = \begin{bmatrix} \frac{1}{a_{11}} & 0 & \cdots & 0 \\ 0 & \frac{1}{a_{22}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{1}{a_{nn}} \end{bmatrix}.$$

Then,

$$\mathbf{x} = D^{-1}(L + U)\mathbf{x} + D^{-1}\mathbf{b}.$$

The matrix form of the Jacobi iterative method is

$$\mathbf{x}^{(k)} = D^{-1}(L + U)\mathbf{x}^{(k-1)} + D^{-1}\mathbf{b}, \quad k = 1, 2, 3, \ldots$$

Define $T := D^{-1}(L + U)$ and $c := D^{-1}\mathbf{b}$. Then we have

$$\mathbf{x}^{(k)} = T\mathbf{x}^{(k-1)} + \mathbf{c}, \quad k = 1, 2, 3, \ldots$$

.

Theorem 1 *If A is a diagonally dominant matrix, then for any choice of $x^{(0)}$, the Jacobi method gives a sequence $\{x^{(k)}\}_{k=0}^{\infty}$ which converges to the unique solution of $Ax = b$.*

This is true because the spectral radius of the matrix will be less than 1, by Gershgorin's Circle Theorem. After that, we can apply the following lemmas.

Lemma 1 If the spectral radius satisfies $\rho(T) < 1$, then $(I - T)^{-1}$ exists, and

$$(I - T)^{-1} = I + T + T^2 + \cdots = \sum_{j=0}^{\infty} T^j.$$

Lemma 2 For any $\mathbf{x}^{(0)} \in \mathbb{R}^n$, the sequence $\{\mathbf{x}^{(k)}\}_{k=0}^{\infty}$ defined by

$$\mathbf{x}^{(k)} = T\mathbf{x}^{(k-1)} + \mathbf{c} \quad \text{for each } k \geq 1$$

converges to the unique solution of $\mathbf{x} = T\mathbf{x} + \mathbf{c}$ if $\rho(T) < 1$. Here, $\rho(T)$ is the spectral radius of T.

Proof.

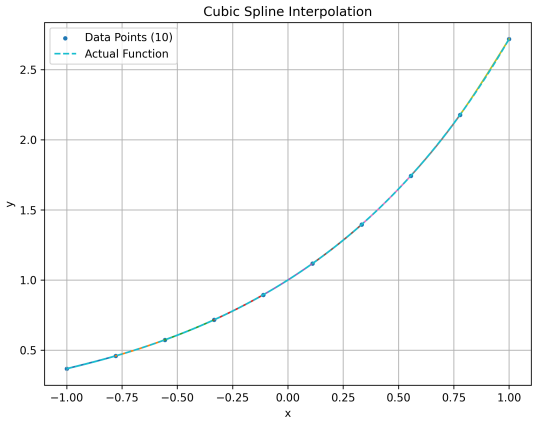
$$\mathbf{x}^{(k)} = T\mathbf{x}^{(k-1)} + \mathbf{c} = T\left(T\mathbf{x}^{(k-2)} + \mathbf{c}\right) + \mathbf{c} = \cdots = T^k\mathbf{x}^{(0)} + (T^{k-1} + \cdots + T + I)\mathbf{c}$$

Since $\rho(T) < 1$, $\lim_{k\to\infty} T^k\mathbf{x}^{(0)} = 0$.

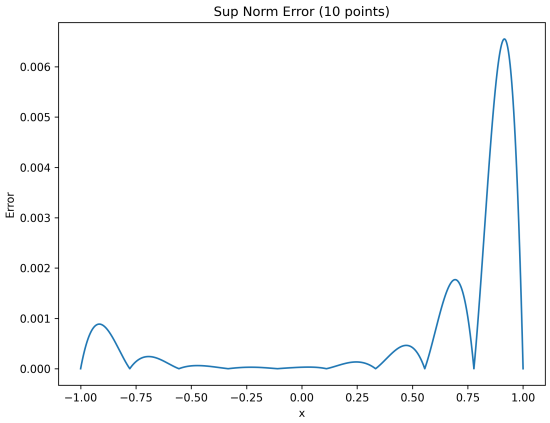
$$\lim_{k\to\infty} \mathbf{x}^{(k)} = 0 + \lim_{k\to\infty} \left(\sum_{j=0}^{k-1} T^j\right)\mathbf{c} = (I - T)^{-1}\mathbf{c}.$$

□

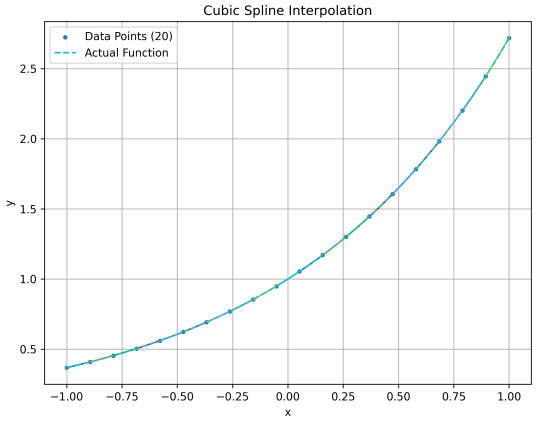
Hence, our convergence is indeed guaranteed.
The rate of convergence of the Jacobi Iteration method is dependent on the spectral radius of the matrix. The closer to 1 it is, the slower the convergence.



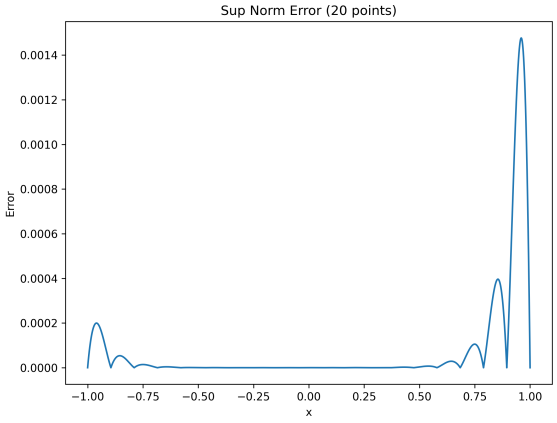
(a) Interpolation using 10 points



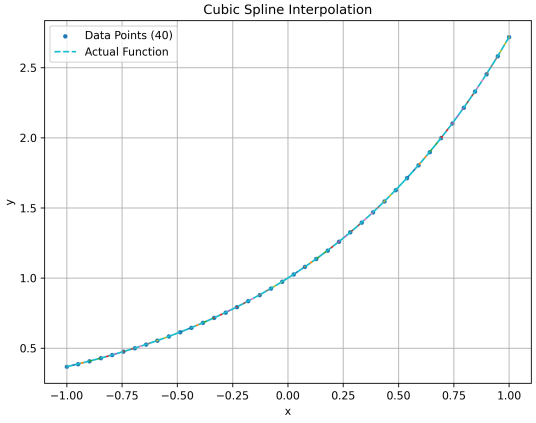
(b) Error plot for 10 points



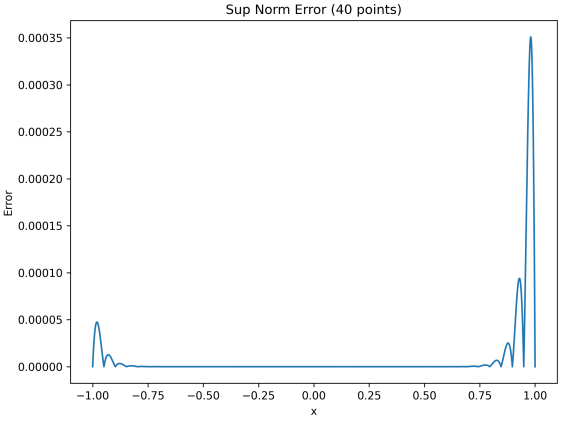
(c) Interpolation using 20 points



(d) Error plot for 20 points



(e) Interpolation using 40 points



(f) Error plot for 40 points

Figure 2: Interpolation and corresponding error plots.

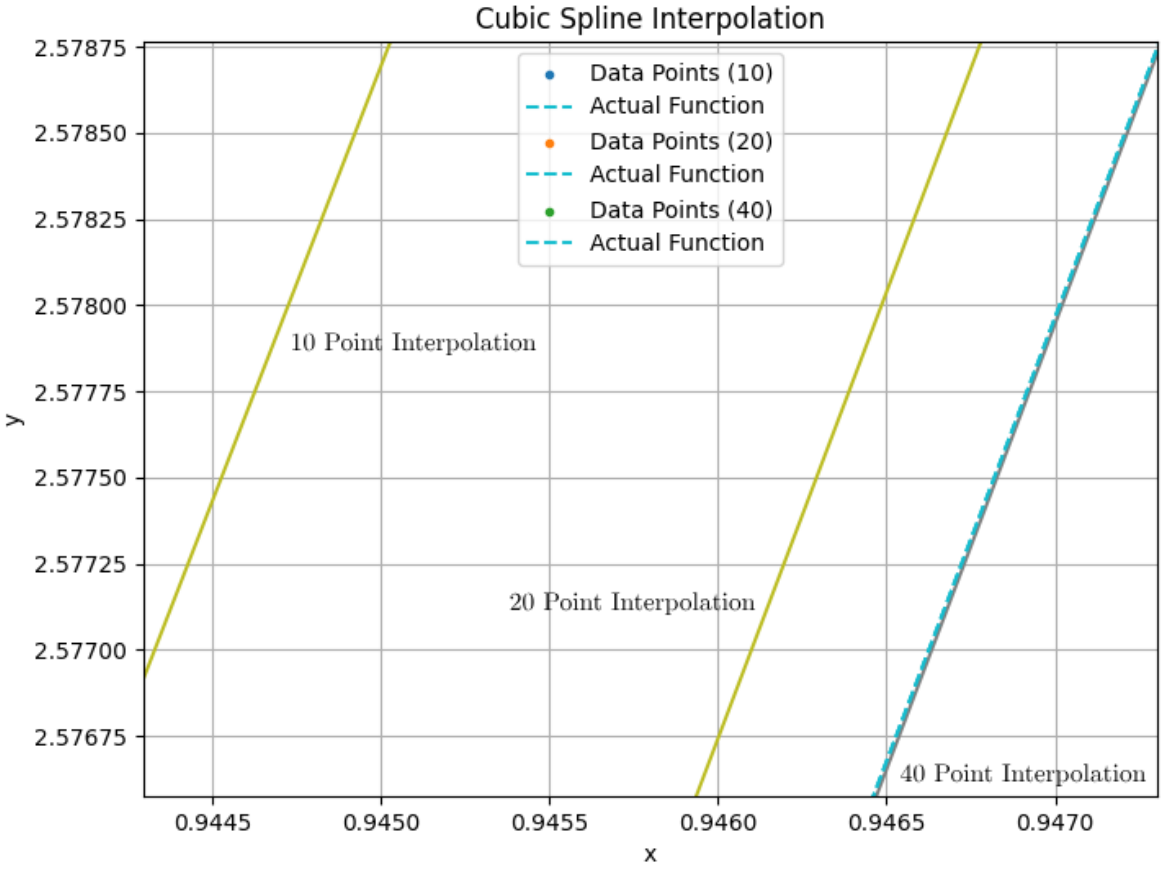
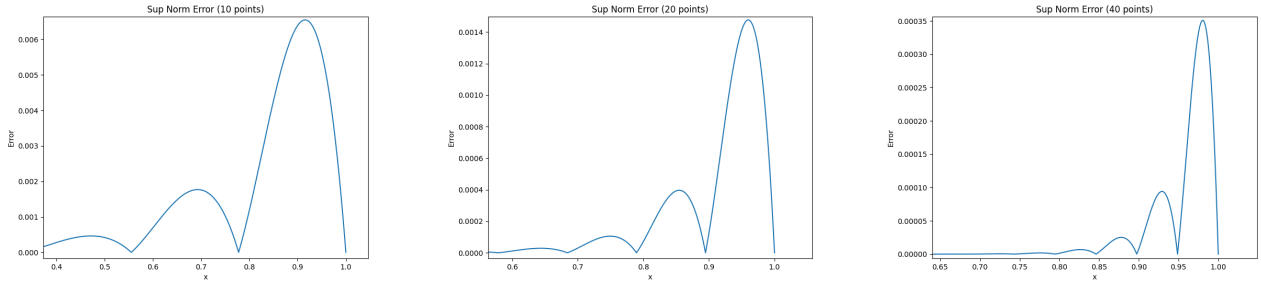


Figure 3: Zoomed in plot of the three interpolation attempts, using 10, 20 and 40 points



(a) Error plot for 10 points (b) Error plot for 20 points (c) Error plot for 40 points

Figure 4: Plots to visualize error differences with increasing number of points.

ERROR ANALYSIS

Error in General Polynomial Interpolation

For a standard polynomial interpolation, for interpolating a function at $n+1$ points by a polynomial of degree n, the error is given by,

$$f(x) - p_n(x) = f[x_0, x_1, ..., x_n, x] \prod_{i=0}^{n} (x - x_i)$$

Lagrange's Form of the Remainder:

Let us assume $f \in C^{n+1}[x_0, x_n]$. Let $f(x) - p_n(x) = R_n$. We define an auxiliary function

$$Y(t) = R_n(t) - R_n(x) \prod_{i=0}^{n} \frac{(t - x_i)}{(x - x_i)}$$

Then,

$$Y^{(n+1)}(t) = R_n^{(n+1)}(t) - R_n(x)(n+1)! \prod_{i=0}^{n} \frac{1}{(x - x_i)}$$

Since, the degree of the interpolating polynomial $p_n(x)$ is $\leq n$, we have

$$R_n^{(n+1)}(t) = f^{(n+1)}(t)$$

and hence,

$$Y^{(n+1)}(t) = f^{(n+1)}(t) - R_n(x)(n+1)! \prod_{i=0}^{n} \frac{1}{(x - x_i)}$$

Since $R_n(x)$ and $\prod_{i=0}^{n}(x - x_i)$ are both 0, we have

$$Y(x) = Y(x_i) = 0$$

that is, Y has at least $n+2$ roots.
By applying Rolle's theorem $n+1$ times, we get that $Y^{(n+1)}(t)$ has at least one root in $[x_0, x_1, ..., x_n]$.

$$\exists\, \xi \in [x_0, x_1, ..., x_n] \text{ such that } Y^{(n+1)}(\xi) = f^{(n+1)}(\xi) - R_n(x)(n+1)! \prod_{i=0}^{n} \frac{1}{(x - x_i)} = 0$$

$$\implies R_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!} \prod_{i=0}^{n} (x - x_i)$$

Error in Cubic Spline Interpolation

At any point $x \in [x_{i-1}, x_i]$, the error is given by

$$\xi(x) = |f(x) - S(x)| = |f(x) - P_i(x)|$$

Now, the general representation of $S(x) = P_i(x)$ is in the form of a cubic, $a_i x^3 + b_i x^2 + c_i x + d_i$. We have interpolated f with S such that $S(x_i) = f(x_i)$, and S' as well as S'' are continuous at all x_i. We assume, for all the following steps, that $f \in C^4$.
Using Taylor's theorem with Lagrange's form of the remainder in $[x_{i-1}, x_i]$, the Taylor expansion of $f(x)$ in $[x_i, x_{i+1}]$ about x_i is given by

$$f(x) = f(x_i) + f'(x_i)(x - x_i) + \frac{f''(x_i)}{2!}(x - x_i)^2 + \frac{f^{(3)}(x_i)}{3!}(x - x_i)^3 + \frac{f^{(4)}(\xi)}{4!}(x - x_i)^4,$$

where $\xi \in [x_i, x_{i+1}]$.
From the error term in the standard polynomial interpolation, we have that the error in cubic spline interpolation will be $\approx O((x - x_i)^4)$.
The error can thus be bounded by the 4^{th} order term in the Taylor expansion. For uniformly spaced nodes, let $h = x_{i+1} - x_i$. On the interval $[x_i, x_{i+1}]$,

$$|\epsilon(x)| \leq \frac{\|f^{(4)}\|_\infty}{24}(x - x_i)^4$$

where $\|F\|_\infty$ denotes the infinity norm or the sup-norm of a function F and is defined as $\max_{i \in \mathbf{R}} |F(i)|$.
The maximum error occurs at $x = x_{i+1}$, where:

$$|\epsilon(x_{i+1})| \approx \frac{\|f^{(4)}\|_\infty}{24} h^4$$

We wish to determine the global bound for the error. The full computation is done by integrating the contributions of the error term over all intervals using the cubic spline basis functions. Let $S_i(x)$ be the cubic spline basis function associated with the node x_i, such that

$$S(x) = \sum_{i=0}^{n} c_i S_i(x)$$

where c_i are elements of the coordinate vector. The contribution to the error from $S_i(x)$ can be expressed as,

$$\int_{x_0}^{x_n} \epsilon(x) S_i(x)\, dx$$

For natural cubic splines, the $S_i(x)$ is symmetric, and its fourth derivative's integral, under the natural boundary conditions $S''(x_0) = S''(x_n) = 0$, evaluates to

$$\int_{x_0}^{x_n} \left(S_i^{(4)}(x)\right)^2 dx = \frac{1}{4!16} = \frac{1}{384}$$

Optimizing over all basis functions, the contribution of $S_i(x)$ evaluates to $\frac{5}{384}$.
Hence, the error for the cubic spline interpolation can be given by,

$$\|\epsilon(x)\|_\infty \leq \frac{5}{384} \|f^{(4)}\|_\infty h^4.$$

Error in Jacobi Iteration Method

From the theorem and lemmas in the section on the acobi iteration method, we note that if $\|T\| < 1$ for any natural matrix norm and $\mathbf{c}$ is a given vector, then the sequence $\{\mathbf{x}^{(k)}\}_{k=0}^{\infty}$ defined by

$$\mathbf{x}^{(k)} = T\mathbf{x}^{(k-1)} + \mathbf{c}$$

converges, for any $\mathbf{x}^{(0)} \in \mathbb{R}^n$, to a vector $\mathbf{x} \in \mathbb{R}^n$, with $\mathbf{x} = T\mathbf{x} + \mathbf{c}$, and the following error bound holds: $\|\mathbf{x} - \mathbf{x}^{(k)}\| \leq \|T\|^k \|\mathbf{x}^{(0)} - \mathbf{x}\|$.

So, the error is approximated as $\|\mathbf{x} - \mathbf{x}^{(k)}\| \approx \rho(T)^k \|\mathbf{x} - \mathbf{x}^{(0)}\|$.

In the worst case, where every element of the matrix is operated in each iteration, the time complexity for each iteration is $\mathcal{O}(n^2)$. The overall time complexity of the algorithm depends on the matrix properties, as shown by the error bound.

Comparison with Linear and Quadratic Interpolation

For interpolating the same set of equally spaced $n+1$ nodes with an n^{th} order polynomial, the error would be given by

$$\frac{f^{(n+1)}(\xi)}{(n+1)!} \prod_{i=0}^{n} (x - x_i) = \frac{f^{(n+1)}(\xi)}{(n+1)!} h^{n+1}$$

Hence, for a piecewise linear interpolation, the error would be bounded by

$$\frac{1}{2} \|f^{(2)}\|_\infty h^2$$

and for piecewise quadratic interpolation, the error would be bounded by

$$\frac{1}{6} \|f^{(3)}\|_\infty h^3.$$

We present some error plots below, which compare linear, quadratic and cubic spline interpolation techniques. We are interpolating e^x over the interval $[-1, 1]$, for 10, 20 and 40 points respectively.

The sup-norm error for cubic spline interpolations was found to be 0.00655 for 10 points, 0.00148 for 20 points and 0.00035 for 40 points. It is clear that the error decreases proportionally with $1/n^2$, where n is the number of points.
Note: The plots for 20 and 40 points have not been included as in Figure 7 as the deviation between the three interpolating functions is difficult to visualize in graph.

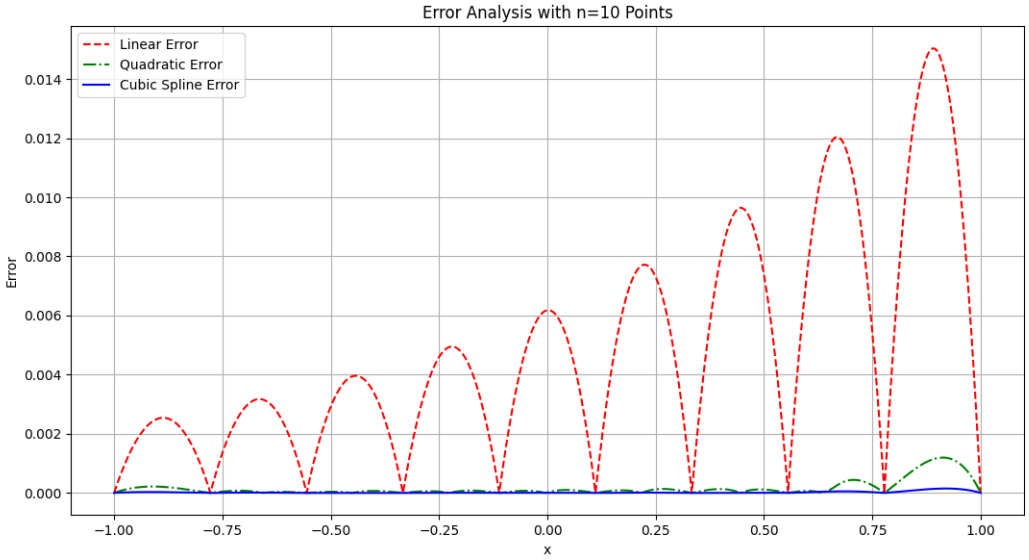


Figure 5: Error comparison for 10 points

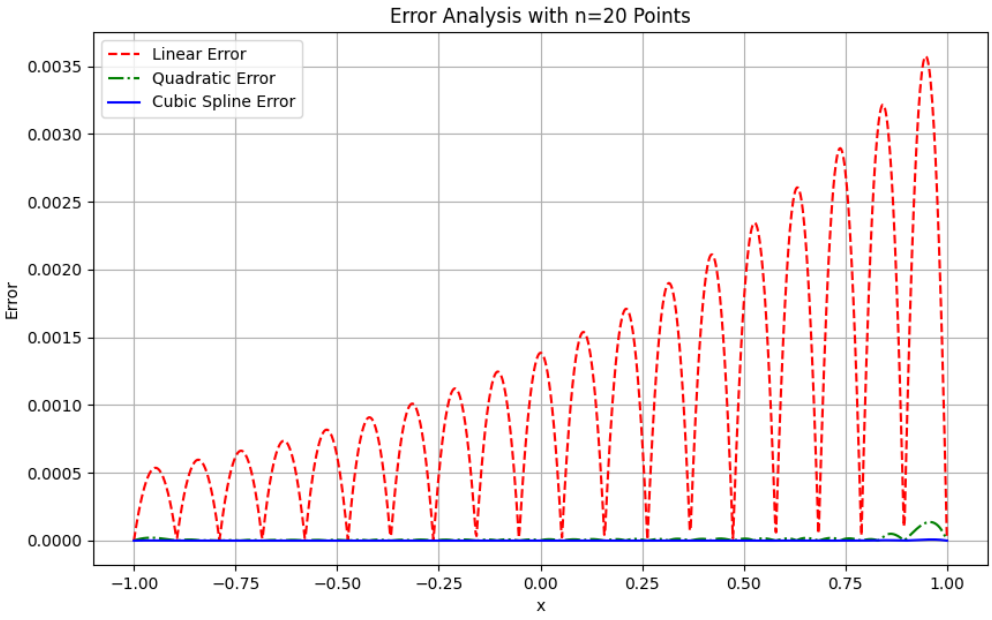


Figure 6: Error comparison for 20 points

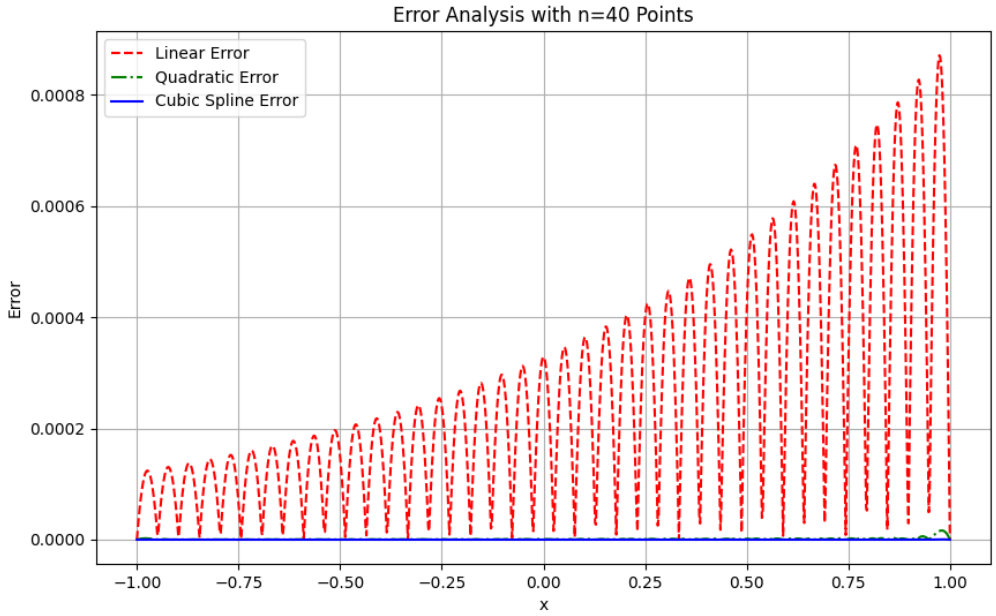


Figure 7: Error comparison for 40 points

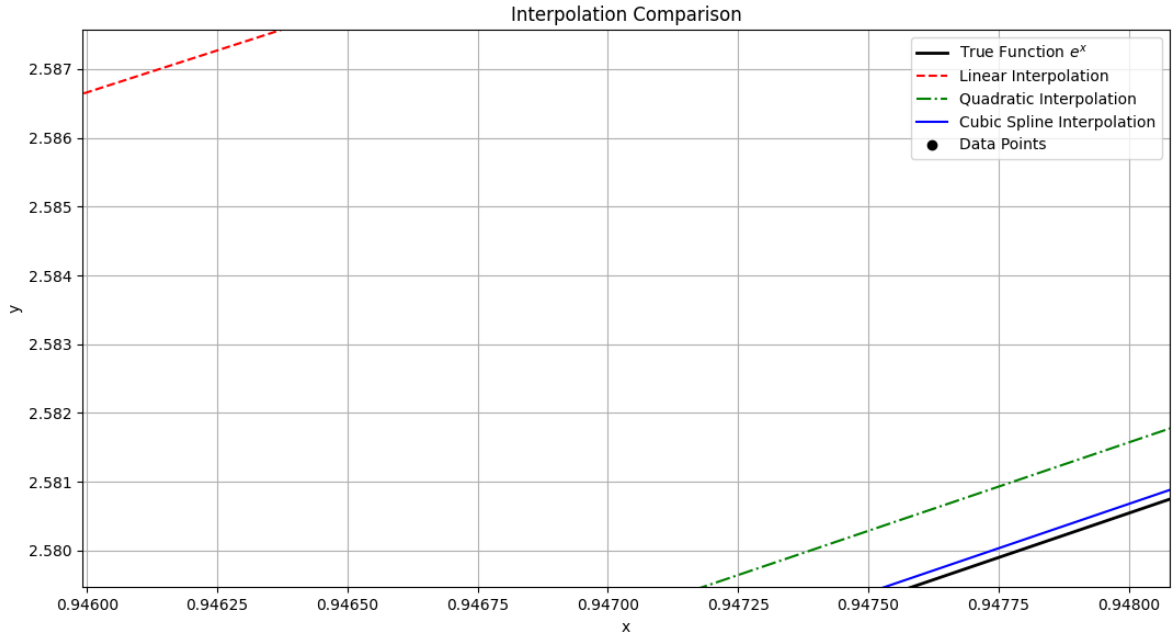


Figure 8: Comparison between the interpolation methods for 10 points

Bibliography

[1] Zhiliang Xu, *Fall 2015 - Numerical Analysis Lecture Notes*. Notre Dame University, 2015.
Available at `https://www3.nd.edu/~zxu2/`.

[2] Wikipedia, *Spline Interpolation*.
Available at `https://en.wikipedia.org/wiki/Spline_interpolation`.

[3] Kendall E. Atkinson, *An Introduction to Numerical Analysis*. Wiley, 1978.

[4] CHARLES A. HALL, W. WESTON MEYER,*Optimal Error Bounds for Cubic Spline Interpolation*

[5] CS412 Lecture Notes, University of Wisconsin, professor Amos Ron

[6] G. BIRKHOFFAND C. DEBCIOR,*Error bounds for spline interpolation* J. Math. Mech 13 (1964), 827-835.