Lissajous Curve

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March 3, 2023

Note for the Readers

Although this notes is especially prepared for IIT-JAM and JEST exams, anyone looking for short notes of lissajous curve can read it.

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1 Introduction

The path of a particle moving under superposition of two mutually perpendicular SHMs simultaneously, is known as a ”Lissajous figure”[1].

The shape of the figure depends upon the ratio of the frequencies (or periods), the individual amplitudes and the phase differences of the two component motions.

For example, if the two motions are of equal frequencies, the Lissajous’ figure is a straight line, ellipse or circle depending upon the amplitudes and phases difference of the motions. Lissajous curves are the family of curves described by the parametric equations

$$x = a\sin(\omega t + \delta) \quad \text{and} \quad y = b\sin\omega t$$

2 Analytical treatment of Lissajous figure

Let two SHMs of amplitudes a and b, angular frequencies of ω_1 and ω_2 and of phase-differnce δ act at right angles to each other. They are represented by the following equations

$$x = a\sin(\omega_1 t + \delta) \tag{1}$$
$$y = b\sin\omega_2 t \tag{2}$$

2.1 Case I (when $\omega_1 = \omega_2 = \omega$)

Then the equations become

$$x = a\sin(\omega t + \delta) \tag{3}$$
$$y = b\sin\omega t \tag{4}$$

After superpositions, the resultant motion can be obtained by eliminating t from Eq. (1) and Eq. (2). Now expanding Eq. (1), we find

$$\frac{x}{a} = \sin\omega t\cos\delta + \cos\omega t\sin\delta.$$

using the identity

$$\sin(A + B) = \sin A\cos B + \cos A\sin B$$

[1]They are sometimes known as Bowditch curves after Nathaniel Bowditch, who studied them in 1815. They were studied in more detail (independently) by Jules-Antoine Lissajous in 1857.

From Eq. (2), we get

$$\sin\omega t = \frac{y}{b}, \Rightarrow \cos\omega t = \sqrt{1-\sin^2\omega t} = \sqrt{1-\frac{y^2}{b^2}}$$

Therefore from Eq. (22), we get

$$\begin{aligned}
&\frac{x}{a} = \frac{y}{b}\cos\delta + \sqrt{1-\frac{y^2}{b^2}}\sin\delta \\
&\Rightarrow \frac{x}{a} - \frac{y}{b}\cos\delta = \sqrt{1-\frac{y^2}{b^2}}\sin\delta. \\
&\Rightarrow \left(\frac{x}{a} - \frac{y}{b}\cos\delta\right)^2 = \left(1-\frac{y^2}{b^2}\right)\sin^2\delta. \\
&\Rightarrow \frac{x^2}{a^2} - 2\frac{xy}{ab}\cos\delta + \frac{y^2}{b^2}\cos^2\delta = \left(1-\frac{y^2}{b^2}\right)\sin^2\delta.
\end{aligned}$$

$$\Rightarrow \frac{x^2}{a^2} - 2\frac{xy}{ab}\cos\delta + \frac{y^2}{b^2} = \sin^2\delta.$$

where we have we used the identity

$$\cos^2\delta + \sin^2\delta = 1$$

This is the general equation of an ellipse enclosed inside a rectangle with sides $2a$ and $2b$. Hence the resultant vibration will be elliptical. It also depends on the individual amplitudes a, b and phase difference δ. Thus in general the Lissajous' figure is an ellipse. There are, however, a number of special cases where it changes its shape.

2.1.1 Case (i)

If $\delta = 0$, Eq. (28) becomes

$$\frac{x^2}{a^2} - 2\frac{xy}{ab} + \frac{y^2}{b^2} = 0. \quad [\text{ Since } \sin 0^\circ = 0, \cos 0^\circ = 1]$$

$$\Rightarrow \left(\frac{x}{a} - \frac{y}{b}\right)^2 = 0$$

$$\Rightarrow y = \frac{b}{a}x$$

This represents two coincident straight lines coinciding with the diagonal of the rectangle.

2.1.2 Case (ii)

If $\delta = \frac{\pi}{4}$, Eq. (28) becomes

$$\frac{x^2}{a^2} - 2\frac{xy}{ab}\left(\frac{1}{\sqrt{2}}\right) + \frac{y^2}{b^2} = \left(\frac{1}{\sqrt{2}}\right)^2$$

using the value of $\sin\left(\frac{\pi}{4}\right) = \cos\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$.
This represents an oblique ellipse.

2.1.3 Case (iii)

If $\delta = \frac{\pi}{2}$, Eq. (28) becomes

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

which is an ellipse whose axes coincide with the coordinate axes.

If in addition $a = b$, the path of the particle becomes a circle, $x^2 + y^2 = a^2$.

2.1.4 Case (iv)

If $\delta = \frac{3\pi}{4}$, Eq. (28) becomes

$$\frac{x^2}{a^2} - 2\frac{xy}{ab}\left(\frac{1}{\sqrt{2}}\right) + \frac{y^2}{b^2} = \left(-\frac{1}{\sqrt{2}}\right)^2 = \frac{1}{2}.$$

using the value of $\sin\left(\frac{3\pi}{4}\right) = \frac{1}{\sqrt{2}}$ and $\cos\left(\frac{3\pi}{4}\right) = -\frac{1}{\sqrt{2}}$.
This again represents an oblique ellipse.

2.1.5 Case (v)

If $\delta = \pi$, Eq. (28) becomes

$$\frac{x^2}{a^2} + 2\frac{xy}{ab} + \frac{y^2}{b^2} = 0. \quad [\sin\pi = 0, \cos\pi = -1]$$
$$\Rightarrow \left(\frac{x}{a} + \frac{y}{b}\right)^2 = 0 \Rightarrow y = -\left(\frac{b}{a}\right)x$$

This represents two coincident straight lines coinciding with the other diagonal of the rectangle.

2.1.6 Summary

When the frequencies of the two SHMs are exactly equal, the elliptic path of the particle remains perfectly steady. If, however, there is a slight difference between the frequencies, the relative phase δ of the two component motion slowly changes and accordingly the form of the ellipse changes. See Fig 1.

1. Starting with $\delta = 0$, the ellipse coincides with the diagonal $y = \left(\frac{b}{a}\right)x$ of the rectangle of sides $2a$ and $2b$.
2. As δ increases from 0 to $\frac{\pi}{2}$, the ellipse opens out to the form $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, afterwards closing up again and ultimately coinciding with the other diagonal $y = -\left(\frac{b}{a}\right)x$, as δ increases from $\frac{\pi}{2}$ to π.
3. As δ changes from π to 2π, the reverse process takes place until the ellipse again coincides with the first diagonal.

2.2 Case II (when $\omega_1 \neq \omega_2$)

When the frequencies of the two perpendicular SHMs are not equal, the resulting motion becomes more complicated. Let us suppose that the displacements of the two mutually perpendicular oscillations are given by

$$x = A_1 \sin\omega_1 t$$
$$y = A_2 \sin(\omega_2 t + \delta)$$

The phase difference between them at any instant t, is given by

$$\Delta(t) = (\omega_2 t + \delta) - \omega_1 t = (\omega_2 - \omega_1)\, t + \delta$$

Therefore,

$$x = A_1 \sin\omega_1 t$$
$$y = A_2 \sin(\omega_1 t + \Delta(t))$$

Since the superimposed orthogonal oscillations are of different frequencies, one of them changes faster than the other and will gain in phase over the other. As a result, the resultant motion passes through different phases.The phase

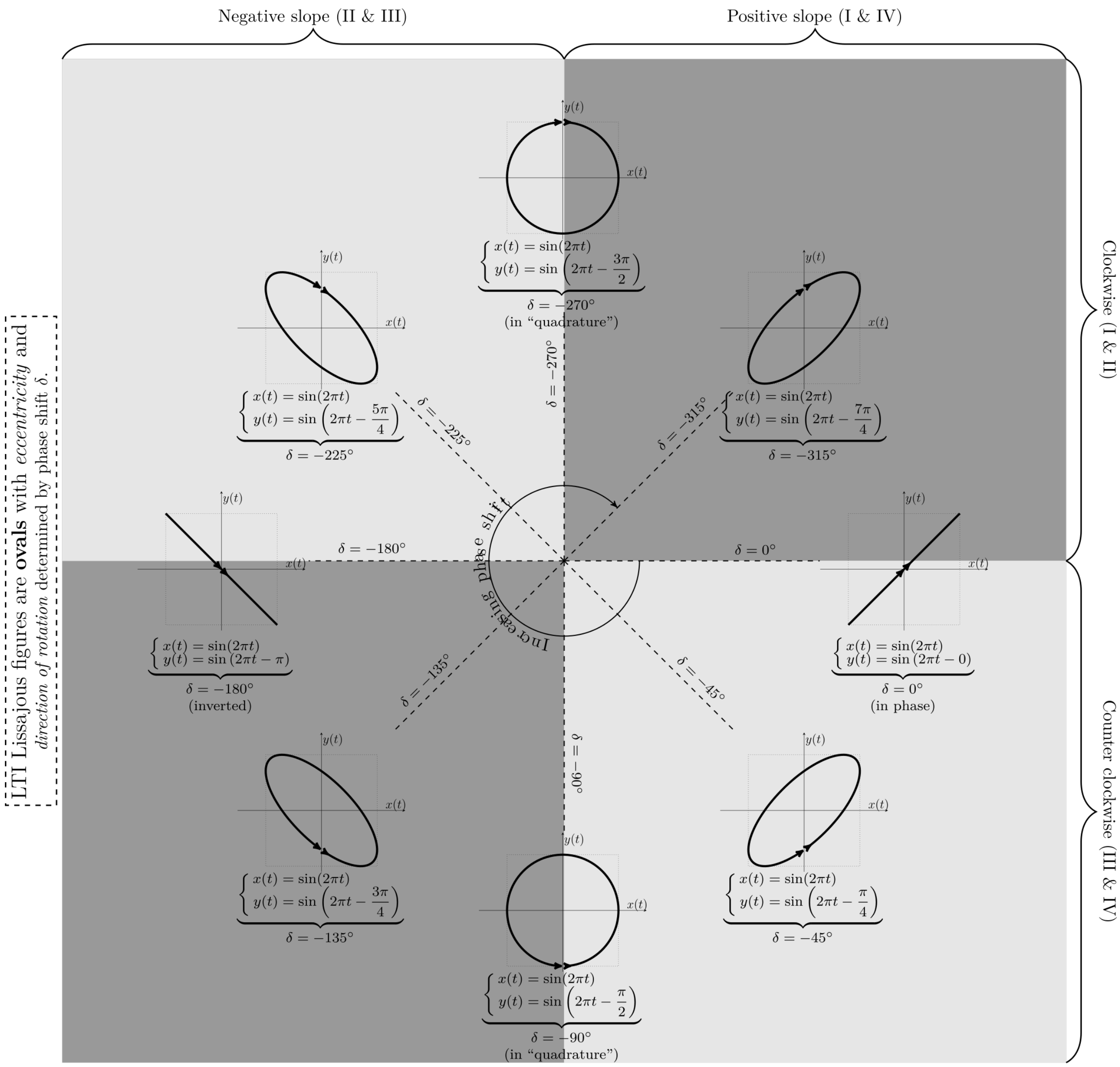

Figure 1: Caption

of resultant changes with time due to the change in the phase difference of superimposed oscillations.However, the motion is confined within a rectangle of sides $2A_1$ and $2A_2$.

$$x = A_1 \cos \omega_1 t$$
$$y = A_2 \cos (\omega_2 t + \delta)$$

If you try to trace the trajectory of the motion of the particle using analytical methods to find y in terms of x, eliminating t, calculation becomes very cumbersome if the phase difference unless $\delta = 0$. In those cases we need the help of plotting tools.

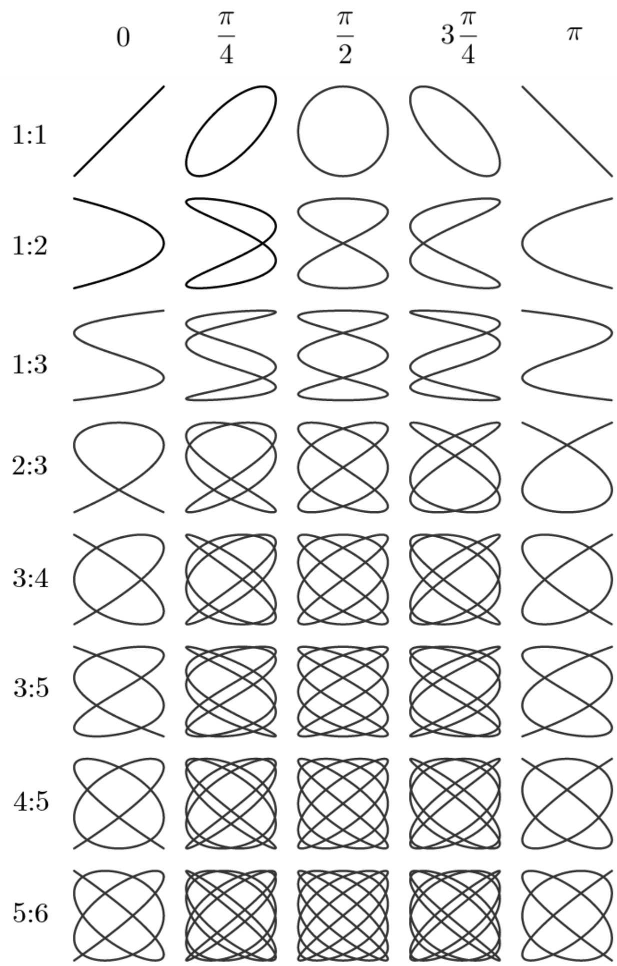

Figure 2: Lissajous figures for $\delta = 0$. ω_1 and ω_2 varying in the ratio $1 : 1$ (both equal), $1 : 2$ ($\omega_2 = 2\omega_1$), $1 : 3$ ($\omega_2 = 3\omega_1$). Here amplitudes are taken equal ($A_1 = A_2$). Fig taken from `https://commons.wikimedia.org/wiki/File:Lissajous_relaciones.png`

3 Visualisation of Lissajous curve in Cathode Ray Oscilloscope (CRO)

Lissajous figures can be seen by using a cathode ray oscilloscope (CRO). Here, two rectangular oscillations are simultaneously imposed upon a beam of cathode ray by connecting two sources of electrical oscillations to horizontal plates XX and vertical plates YY of the oscilloscope. We then see the trace of the resultant effect in the form of an electron beam on the fluorescent screen. By adjusting the phases, amplitudes and the ratio of the frequencies of the applied voltage, we obtain various curves. Lissajous figures may be used to compare two nearly equal frequencies. If the frequencies of two component oscillations are not exactly equal, the Lissajous figure will change gradually.

If you don't have access to CRO, visit the website `https://demonstrations.wolfram.com/VirtualOscilloscope/` (installation required).

4 Refernces

Oscillations, Waves and Acoustics By P. K. Mittal