Galerkin Technique

AS6320

08th April, 2024

$$\gamma M \frac{\partial u'}{\partial t} + \frac{\partial p'}{\partial x} = 0$$

$$\frac{\partial p'}{\partial t} + \gamma M \frac{\partial u'}{\partial x} = K\left[\sqrt{\left|\frac{1}{3} + u'_f(t-\tau)\right|} - \sqrt{\frac{1}{3}}\right]\delta(x - x_f)$$

Change PDE into ODE

ODEs are much easier to solve than PDEs

Galerkin Technique
or
Modal Expansion

1. Expand variables (u', p') in terms of basis functions

2. Project PDEs onto the basis functions

3. Individual equations for all the terms

Projecting along basis functions??

$$\vec{F} = m\,\vec{a} = m\,\frac{d^2\mathrm{x}}{dt^2}$$

$$F_x\,\vec{\imath} + F_y\vec{\jmath} + F_z\vec{k} = m\left(\frac{d^2x}{dt^2}\vec{\imath} + \frac{d^2y}{dt^2}\,\vec{\jmath} + \frac{d^2z}{dt^2}\vec{k}\right)$$

$$\vec{F} = F_x\,\vec{\imath} + F_y\vec{\jmath} + F_z\vec{k}$$

$$F_x = m\frac{d^2x}{dt^2} \qquad F_y = m\frac{d^2y}{dt^2} \qquad F_z = m\frac{d^2z}{dt^2}$$

$$\vec{X} = x\,\vec{\imath} + y\,\vec{\jmath} + z\,\vec{k}$$

How does one “mathematically” arrive at this statement ?

Projecting along basis functions??

$$F_x\vec{i} + F_y\vec{j} + F_z\vec{k} = m\left(\frac{d^2x}{dt^2}\vec{i} + \frac{d^2y}{dt^2}\vec{j} + \frac{d^2z}{dt^2}\vec{k}\right)$$

$$\vec{i}\,.\left[F_x\vec{i} + F_y\vec{j} + F_z\vec{k} = m\left(\frac{d^2x}{dt^2}\vec{i} + \frac{d^2y}{dt^2}\vec{j} + \frac{d^2z}{dt^2}\vec{k}\right)\right]$$

$$\vec{i}\,.\left[F_x\vec{i} + F_y\vec{j} + F_z\vec{k} = m\left(\frac{d^2x}{dt^2}\vec{i} + \frac{d^2y}{dt^2}\vec{j} + \frac{d^2z}{dt^2}\vec{k}\right)\right]$$

$$F_x = m\frac{d^2x}{dt^2}$$

Projecting along basis functions??

$$F_x\,\vec{i} + F_y\vec{j} + F_z\vec{k} = m\left(\frac{d^2x}{dt^2}\vec{i} + \frac{d^2y}{dt^2}\,\vec{j} + \frac{d^2z}{dt^2}\vec{k}\right)$$

$$\vec{j}\,.\left[F_x\,\vec{i} + F_y\vec{j} + F_z\vec{k} = m\left(\frac{d^2x}{dt^2}\vec{i} + \frac{d^2y}{dt^2}\,\vec{j} + \frac{d^2z}{dt^2}\vec{k}\right)\right]$$

$$F_y = m\frac{d^2y}{dt^2}$$

Projecting along basis functions??

$$F_x\,\vec{i} + F_y\vec{j} + F_z\vec{k} = m\left(\frac{d^2x}{dt^2}\vec{i} + \frac{d^2y}{dt^2}\,\vec{j} + \frac{d^2z}{dt^2}\vec{k}\right)$$

$$\vec{k}.\left[F_x\,\vec{i} + F_y\vec{j} + F_z\vec{k} = m\left(\frac{d^2x}{dt^2}\vec{i} + \frac{d^2y}{dt^2}\,\vec{j} + \frac{d^2z}{dt^2}\vec{k}\right)\right]$$

$$F_z = m\frac{d^2z}{dt^2}$$

Projecting along basis functions

$$\vec{k}.\left[F_x\vec{i}+F_y\vec{j}+F_z\vec{k}=m\left(\frac{d^2x}{dt^2}\vec{i}+\frac{d^2y}{dt^2}\vec{j}+\frac{d^2z}{dt^2}\vec{k}\right)\right]$$

$$\gamma M \frac{\partial u'}{\partial t} + \frac{\partial p'}{\partial x} = 0 \qquad \frac{\partial p'}{\partial t} + \gamma M \frac{\partial u'}{\partial x} = K\left[\sqrt{\left|\frac{1}{3} + u'_f(t-\tau)\right|} - \sqrt{\frac{1}{3}}\right]\delta(x - x_f)$$

Duct open at both ends: $p'(x = 0) = 0, p'(x = 1) = 0$

$$p'(x,t) = \sum_{j=1}^{N} a_j(t)\sin(j\pi x)$$

$$\gamma M \frac{\partial u'}{\partial t} + \frac{\partial p'}{\partial x} = 0$$

Duct open at both ends: $p'(x = 0) = 0, p'(x = 1) = 0$

$$p'(x,t) = \sum_{j=1}^{N} a_j(t) \sin(j\pi x)$$

$$a_j(t) = -\frac{\gamma M}{j\pi} \dot{\eta}_j(t)$$

$$\gamma M \frac{\partial u'}{\partial t} + \frac{\partial p'}{\partial x} = 0 \qquad \frac{\partial p'}{\partial t} + \gamma M \frac{\partial u'}{\partial x} = K\left[\sqrt{\left|\frac{1}{3} + u'_f(t-\tau)\right|} - \sqrt{\frac{1}{3}}\right]\delta(x - x_f)$$

Duct open at both ends: $p'(x = 0) = 0, p'(x = 1) = 0$

$$p'(x,t) = \sum_{j=1}^{N} -\gamma M \frac{\dot{\eta}_j(t)}{j\pi} \sin(j\pi x)$$

$$\gamma M \frac{\partial u'}{\partial t} + \frac{\partial p'}{\partial x} = 0 \qquad \frac{\partial p'}{\partial t} + \gamma M \frac{\partial u'}{\partial x} = K\left[\sqrt{\left|\frac{1}{3} + u'_f(t-\tau)\right|} - \sqrt{\frac{1}{3}}\right]\delta(x - x_f)$$

Duct open at both ends: $p'(x=0) = 0, p'(x=1) = 0$

$$p'(x,t) = \sum_{j=1}^{N} -\gamma M \frac{\dot{\eta}_j(t)}{j\pi} \sin(j\pi x)$$

$$\gamma M \frac{\partial u'}{\partial t} + \frac{\partial p'}{\partial x} = 0 \qquad \frac{\partial p'}{\partial t} + \gamma M \frac{\partial u'}{\partial x} = K \left[\sqrt{\left| \frac{1}{3} + u'_f(t-\tau) \right|} - \sqrt{\frac{1}{3}} \right] \delta(x - x_f)$$

Duct open at both ends: $p'(x=0) = 0, p'(x=1) = 0$

$$p'(x,t) = \sum_{j=1}^{N} -\gamma M \frac{\dot{\eta}_j(t)}{j\pi} \sin(j\pi x)$$

$$u'(x,t) = \sum_{j=1}^{N} \eta_j(t) \cos(j\pi x)$$

$$\gamma M \frac{\partial u'}{\partial t} + \frac{\partial p'}{\partial x} = 0 \qquad \frac{\partial p'}{\partial t} + \gamma M \frac{\partial u'}{\partial x} = K \left[\sqrt{\left| \frac{1}{3} + u'_f(t-\tau) \right|} - \sqrt{\frac{1}{3}} \right] \delta(x - x_f)$$

Duct open at both ends: $p'(x = 0) = 0, p'(x = 1) = 0$

$$p'(x,t) = \sum_{j=1}^{N} -\gamma M \frac{\dot{\eta}_j(t)}{j\pi} \sin(j\pi x)$$

$$u'(x,t) = \sum_{j=1}^{N} \eta_j(t) \cos(j\pi x)$$

$$\gamma M \frac{\partial u'}{\partial t} + \frac{\partial p'}{\partial x} = 0 \qquad \frac{\partial p'}{\partial t} + \gamma M \frac{\partial u'}{\partial x} = K\left[\sqrt{\left|\frac{1}{3} + u'_f(t-\tau)\right|} - \sqrt{\frac{1}{3}}\right]\delta(x - x_f)$$

Duct open at both ends: $p'(x=0) = 0, p'(x=1) = 0$

$$p'(x,t) = \sum_{j=1}^{N} -\gamma M \frac{\dot{\eta}_j(t)}{j\pi} \sin(j\pi x) \qquad \frac{\partial p'(x,t)}{\partial t} = -\sum_{j=1}^{N} \frac{\gamma M}{j\pi} \frac{d}{dt}(\dot{\eta}_j) \sin(j\pi x)$$

$$u'(x,t) = \sum_{j=1}^{N} \eta_j(t) \cos(j\pi x) \qquad \frac{\partial u'(x,t)}{\partial x} = -\sum_{j=1}^{N} j\pi \quad (\eta_j) \sin(j\pi x)$$

$$\gamma M \frac{\partial u'}{\partial t} + \frac{\partial p'}{\partial x} = 0$$

$$\frac{\partial p'}{\partial t} + \gamma M \frac{\partial u'}{\partial x} = K \left[\sqrt{\left| \frac{1}{3} + u'_f(t-\tau) \right|} - \sqrt{\frac{1}{3}} \right] \delta(x - x_f)$$

Duct open at both ends: $p'(x=0) = 0, p'(x=1) = 0$

$$p'(x,t) = \sum_{j=1}^{N} -\gamma M \frac{\dot{\eta}_j(t)}{j\pi} \sin(j\pi x)$$

$$\frac{\partial p'(x,t)}{\partial t} = -\sum_{j=1}^{N} \frac{\gamma M}{j\pi} \frac{d}{dt}(\dot{\eta}_j) \sin(j\pi x)$$

$$u'(x,t) = \sum_{j=1}^{N} \eta_j(t) \cos(j\pi x)$$

$$\frac{\partial u'(x,t)}{\partial x} = -\sum_{j=1}^{N} j\pi\, \eta_j \sin(j\pi x)$$

$$\frac{\partial p'(x,t)}{\partial t} = -\sum_{j=1}^{N} \frac{\gamma M}{j\pi} \frac{d}{dt}\left(\dot{\eta}_j\right) \sin(j\pi x)$$

$$\frac{\partial u'(x,t)}{\partial x} = -\sum_{j=1}^{N} j\pi\, \eta_j \sin(j\pi x)$$

$$\frac{\partial p'}{\partial t} + \gamma M \frac{\partial u'}{\partial x} = K\left[\sqrt{\left|\frac{1}{3} + u'_f(t-\tau)\right|} - \sqrt{\frac{1}{3}}\right]\delta(x - x_f)$$

$$\frac{\partial p'(x,t)}{\partial t} = -\sum_{j=1}^{N} \frac{\gamma M}{j\pi} \frac{d}{dt}(\dot{\eta}_j) \sin(j\pi x)$$

$$\frac{\partial p'}{\partial t} + \gamma M \frac{\partial u'}{\partial x} = K\left[\sqrt{\left|\frac{1}{3} + u'_f(t-\tau)\right|} - \sqrt{\frac{1}{3}}\right]\delta(x - x_f)$$

$$\frac{\partial u'(x,t)}{\partial x} = -\sum_{j=1}^{N} j\pi\, \eta_j \sin(j\pi x)$$

$$-\left[\sum_{j=1}^{N} \frac{\gamma M}{j\pi} \frac{d}{dt}(\dot{\eta}_j) + \gamma M\, j\pi\, \eta_j\right] \sin(j\pi x) = K\left[\sqrt{\left|\frac{1}{3} + u'_f(t-\tau)\right|} - \sqrt{\frac{1}{3}}\right]\delta(x - x_f)$$

$$\frac{\partial p'(x,t)}{\partial t} = -\sum_{j=1}^{N} \frac{\gamma M}{j\pi} \frac{d}{dt}(\dot{\eta}_j) \sin(j\pi x)$$

$$\frac{\partial p'}{\partial t} + \gamma M \frac{\partial u'}{\partial x} = K\left[\sqrt{\left|\frac{1}{3} + u'_f(t-\tau)\right|} - \sqrt{\frac{1}{3}}\right] \delta(x - x_f)$$

$$\frac{\partial u'(x,t)}{\partial x} = -\sum_{j=1}^{N} j\pi\, \eta_j \sin(j\pi x)$$

$$-\left[\sum_{j=1}^{N} \frac{\gamma M}{j\pi} \frac{d}{dt}(\dot{\eta}_j) + \gamma M\, j\pi\, \eta_j\right] \sin(j\pi x) = K\left[\sqrt{\left|\frac{1}{3} + u'_f(t-\tau)\right|} - \sqrt{\frac{1}{3}}\right] \delta(x - x_f)$$

Multiply both sides by $\frac{j\pi}{\gamma M}$

$$\sum_{j=1}^{N} \left[\frac{d\,\dot{\eta}_j}{dt} + (j\pi)^2 \eta_j\right] \sin(j\pi x) = -\frac{j\pi K}{\gamma M}\left[\sqrt{\left|\frac{1}{3} + u'_f(t-\tau)\right|} - \sqrt{\frac{1}{3}}\right] \delta(x - x_f)$$

$$\frac{\partial p'(x,t)}{\partial t} = -\sum_{j=1}^{N} \frac{\gamma M}{j\pi} \frac{d}{dt}(\dot{\eta}_j) \sin(j\pi x)$$

$$\frac{\partial p'}{\partial t} + \gamma M \frac{\partial u'}{\partial x} = K\left[\sqrt{\left|\frac{1}{3} + u'_f(t-\tau)\right|} - \sqrt{\frac{1}{3}}\right] \delta(x - x_f)$$

$$\frac{\partial u'(x,t)}{\partial x} = -\sum_{j=1}^{N} j\pi\, \eta_j \sin(j\pi x)$$

$$-\left[\sum_{j=1}^{N} \frac{\gamma M}{j\pi} \frac{d}{dt}(\dot{\eta}_j) + \gamma M\, j\pi\, \eta_j\right] \sin(j\pi x) = K\left[\sqrt{\left|\frac{1}{3} + u'_f(t-\tau)\right|} - \sqrt{\frac{1}{3}}\right] \delta(x - x_f)$$

Multiply both sides by $\frac{j\pi}{\gamma M}$

$$\sum_{j=1}^{N} \left[\frac{d\,\dot{\eta}_j}{dt} + (j\pi)^2 \eta_j\right] \sin(j\pi x) = -\frac{j\pi K}{\gamma M}\left[\sqrt{\left|\frac{1}{3} + u'_f(t-\tau)\right|} - \sqrt{\frac{1}{3}}\right] \delta(x - x_f)$$

$k_j = j\pi =$ nondimensionalized wave number

Projecting along the basis function

$$\sum_{j=1}^{N}\left[\frac{d\,\dot{\eta}_j}{dt}+k_j^2\eta_j\right]\sin(j\pi x) = -\frac{j\pi K}{\gamma M}\left[\sqrt{\left|\frac{1}{3}+u'_f(t-\tau)\right|}-\sqrt{\frac{1}{3}}\right]\delta(x-x_f)$$

Multiply both sides by $\sin(j\pi x)$ and integrate along the domain $[0, 1]$

$$\int_{x=0}^{x=1}\sum_{j=1}^{N}\left[\frac{d\,\dot{\eta}_j}{dt}+k_j^2\eta_j\right]\sin(j\pi x)\sin(j\pi x)\;dx = \int_{x=0}^{x=1}-\frac{j\pi K}{\gamma M}\left[\sqrt{\left|\frac{1}{3}+u'_f(t-\tau)\right|}-\sqrt{\frac{1}{3}}\right]\delta(x-x_f)\sin(j\pi x)\,dx$$

$$\int_{x=0}^{x=1} \sum_{j=1}^{N} \left[\frac{d\,\dot{\eta}_j}{dt} + k_j^2 \eta_j\right] \sin(j\pi x)\sin(j\pi x)\ dx = \int_{x=0}^{x=1} -\frac{j\pi K}{\gamma M}\left[\sqrt{\left|\frac{1}{3} + u'_f(t-\tau)\right|} - \sqrt{\frac{1}{3}}\right]\delta(x - x_f)\sin(j\pi x)\,dx$$

$$\int_0^1 \sin(j\pi x)\sin(n\pi x)dx = \frac{1}{2}\delta_{jn}$$

$$\int_0^1 f(x)\delta(x - x_f)dx = f(x_f)$$

$$\int_{x=0}^{x=1} \sum_{j=1}^{N} \left[\frac{d\,\dot{\eta}_j}{dt} + k_j^2 \eta_j\right] \sin(j\pi x)\sin(j\pi x)\ \ dx = \int_{x=0}^{x=1} -\frac{j\pi K}{\gamma M}\left[\sqrt{\left|\frac{1}{3} + u'_f(t-\tau)\right|} - \sqrt{\frac{1}{3}}\right] \delta(x\ - x_f) \sin(j\pi x)\,dx$$

$$\int_{0}^{1} \sin(j\pi x)\sin(n\pi x)dx = \frac{1}{2}\delta_{jn}$$

$$\int_{0}^{1} f(x)\delta(x\ - x_f)dx = f(x_f)$$

$$\int_{x=0}^{x=1} \sum_{j=1}^{N} \left[\frac{d\,\dot{\eta}_j}{dt} + k_j^2 \eta_j\right] \sin(j\pi x)\sin(j\pi x)\;dx = \int_{x=0}^{x=1} -\frac{j\pi K}{\gamma M}\left[\sqrt{\left|\frac{1}{3} + u'_f(t-\tau)\right|} - \sqrt{\frac{1}{3}}\right]\delta(x - x_f)\sin(j\pi x)\,dx$$

$$\int_0^1 \sin(j\pi x)\sin(n\pi x)dx = \frac{1}{2}\delta_{jn}$$

$$\int_0^1 f(x)\delta(x - x_f)dx = f(x_f)$$

$$\left[\frac{d\,\dot{\eta}_j}{dt} + k_j^2 \eta_j\right]\frac{1}{2} = -\frac{j\pi K}{\gamma M}\left[\sqrt{\left|\frac{1}{3} + u'_f(t-\tau)\right|} - \sqrt{\frac{1}{3}}\right]\sin(j\pi x_f)$$

$$\frac{d\,\dot{\eta}_j}{dt} + k_j^2 \eta_j = -\frac{2j\pi K}{\gamma M}\left[\sqrt{\left|\frac{1}{3} + u'_f(t-\tau)\right|} - \sqrt{\frac{1}{3}}\right]\sin(j\pi x_f)$$

$$\frac{d\,\eta_j}{dt} = \dot{\eta}_j$$

Equation 12, Balasubramanian and Sujith, Physics of Fluids, 2008

$$\frac{d\,\dot{\eta}_j}{dt} + 2\zeta_j\omega_j\dot{\eta}_j + k_j^2\eta_j = -\frac{2j\pi K}{\gamma M}\left[\sqrt{\left|\frac{1}{3} + u'_f(t-\tau)\right|} - \sqrt{\frac{1}{3}}\right]\sin(j\pi x_f)$$

$$\frac{d\,\eta_j}{dt} = \dot{\eta}_j$$

Equation 12, Balasubramanian and Sujith, Physics of Fluids, 2008

$\zeta_j =$ frequency dependent damping $= \frac{1}{2\,\pi}\left[c_1\frac{\omega_j}{\omega_1} + c_2\sqrt{\frac{\omega_1}{\omega_j}}\right]$

$$\frac{d\,\dot{\eta}_j}{dt} + 2\zeta_j\omega_j\dot{\eta}_j + k_j^2\eta_j = -\frac{2j\pi K}{\gamma M}\left[\sqrt{\left|\frac{1}{3} + u'_f(t-\tau)\right|} - \sqrt{\frac{1}{3}}\right]\sin(j\pi x_f)$$

$$\frac{d\,\eta_j}{dt} = \dot{\eta}_j$$

Equation 12, Balasubramanian and Sujith, Physics of Fluids, 2008

$\zeta_j =$ frequency dependent damping $= \frac{1}{2\,\pi}\left[c_1\frac{\omega_j}{\omega_1} + c_2\sqrt{\frac{\omega_1}{\omega_j}}\right]$

Numerical solution of ODEs using RK4 scheme

$$\frac{d\,\eta_j}{dt} = \dot{\eta}_j$$

$$\frac{d\,\dot{\eta}_j}{dt} + 2\zeta_j\omega_j\dot{\eta}_j + k_j^2\eta_j = -\frac{2j\pi K}{\gamma M}\left[\sqrt{\left|\frac{1}{3} + u'_f(t-\tau)\right|} - \sqrt{\frac{1}{3}}\right]\sin(j\pi x_f)$$

$$\frac{d\,\eta_j}{dt} = \dot{\eta}_j$$

$$\frac{d\,\dot{\eta}_j}{dt} = -2\zeta_j\omega_j\dot{\eta}_j - k_j^2\eta_j - \frac{2j\pi K}{\gamma M}\left[\sqrt{\left|\frac{1}{3} + u'_f(t-\tau)\right|} - \sqrt{\frac{1}{3}}\right]\sin(j\pi x_f)$$

$$\frac{d\,\eta_j}{dt} = \dot{\eta}_j$$

$$\frac{d\,\dot{\eta}_j}{dt} = -2\zeta_j\omega_j\dot{\eta}_j - k_j^2\eta_j - \frac{2j\pi K}{\gamma M}\left[\sqrt{\left|\frac{1}{3} + u'_f(t-\tau)\right|} - \sqrt{\frac{1}{3}}\right]\sin(j\pi x_f)$$

To simplify, we have the following system of equations:

$$\frac{d\,\eta_j}{dt} = \dot{\eta}_j$$

$$\frac{d\,\dot{\eta}_j}{dt} = -2\zeta_j\omega_j\dot{\eta}_j - k_j^2\eta_j - \frac{2j\pi K}{\gamma M}\left[\sqrt{\left|\frac{1}{3} + u'_f(t-\tau)\right|} - \sqrt{\frac{1}{3}}\right]\sin(j\pi x_f)$$

To simplify, we have the following system of equations:

$$\frac{d\,\eta_j}{dt} = \mathrm{f}(\eta_j, \dot{\eta}_j)$$

for $j = 1, 2, \ldots . N$ Galerkin modes

$$\frac{d\,\dot{\eta}_j}{dt} = \mathrm{g}(\eta_j, \dot{\eta}_j)$$

For a general ordinary first-order differential equation,

$$\frac{dy(t)}{dx} = f(t, y), \qquad y(t_0) = y_0$$

For a general ordinary first-order differential equation,

$$\frac{dy(t)}{dx} = f(t,y), \qquad y(t_0) = y_0$$

We wish to evolve the solution from $t = t_0$ to $t = T$

For a general ordinary first-order differential equation,

$$\frac{dy(t)}{dx} = f(t, y), \qquad y(t_0) = y_0$$

We wish to evolve the solution from $t = t_0$ to $t = T$

Thus, from expanding $y(t)$ in the vicinity of t_0 through Taylor's expansion, we get,

$$y(t_0 + h) = y(t_0) + h\, y'(t_0) + \frac{h^2}{2!} y''(t_0) + \ \ldots$$

For a general ordinary first-order differential equation,

$$\frac{dy(t)}{dx} = f(t, y), \qquad y(t_0) = y_0$$

We wish to evolve the solution from $t = t_0$ to $t = T$

Thus, from expanding $y(t)$ in the vicinity of t_0 through Taylor's expansion, we get,

$$y(t_0 + h) = y(t_0) + h\, y'(t_0) + \frac{h^2}{2!} y''(t_0) + \ \dots$$

We can now neglect higher order terms to get the Euler's solution as

$$y(t_0 + h) = y(t_0) + h\, y'(t_0)$$

For a general ordinary first-order differential equation,

$$\frac{dy(t)}{dx} = f(t, y), \qquad y(t_0) = y_0$$

We wish to evolve the solution from $t = t_0$ to $t = T$

Thus, from expanding $y(t)$ in the vicinity of t_0 through Taylor's expansion, we get,

$$y(t_0 + h) = y(t_0) + h\, y'(t_0) + \frac{h^2}{2!} y''(t_0) + \ \ldots$$

We can now neglect higher order terms to get the Euler's solution as

$$y(t_0 + h) = y(t_0) + h\, y'(t_0)$$

$$y_1 = y_0 + h\, y'(t_0) = y_0 + hf(x_0, y_0)$$

$$y_{n+1} = y_n + h\, y'(x_n, y_n)$$

If we seek higher order accuracy, we must retain higher order terms in the Taylor Expansion. The Runge-Kutta method is an extension of the Euler Method where we retain terms up to h^5. Thus, we have:

$$y_{n+1} = y_n + \frac{1}{6}(k_0 + 2k_1 + 2k_2 + k_3)$$

If we seek higher order accuracy, we must retain higher order terms in the Taylor Expansion. The Runge-Kutta method is an extension of the Euler Method where we retain terms up to h^5. Thus, we have:

$$y_{n+1} = y_n + \frac{1}{6}(k_0 + 2k_1 + 2k_2 + k_3)$$

where,

$$k_0 = hf(t_n, y_n)$$
$$k_1 = hf\left[t_n + \frac{h}{2}, y_n + \frac{k_0}{2}\right]$$
$$k_2 = hf\left[t_n + \frac{h}{2}, y_n + \frac{k_1}{2}\right]$$
$$k_3 = hf[t_n + h, y_n + k_2]$$

Runge-Kutta for the Rijke tube model

In the Rijke tube model, we have:

$$\frac{d\,\eta_j}{dt} = f(t, \eta_j, \dot{\eta}_j) \qquad \frac{d\,\dot{\eta}_j}{dt} = g(t, \eta_j, \dot{\eta}_j)$$

for $j = 1, 2, \ldots . N$ Galerkin modes

Runge-Kutta for the Rijke tube model

In the Rijke tube model, we have:

$$\frac{d\,\eta_j}{dt} = f(t, \eta_j, \dot{\eta}_j) \qquad \frac{d\,\dot{\eta}_j}{dt} = g(t, \eta_j, \dot{\eta}_j)$$

for $j = 1, 2, \dots . N$ Galerkin modes

These two equations need to be solved simultaneously.

Runge-Kutta for the Rijke tube model

In the Rijke tube model, we have:

$$\frac{d\,\eta_j}{dt} = f(t, \eta_j, \dot{\eta}_j) \qquad \frac{d\,\dot{\eta}_j}{dt} = g(t, \eta_j, \dot{\eta}_j) \qquad \text{for } j = 1, 2, \ldots .N \text{ Galerkin modes}$$

These two equations need to be solved simultaneously.

The solution will proceed along the lines of

$$\eta_j^{n+1} = \eta_j^n + \frac{1}{6}[k_0 + 2k_1 + 2k_2 + k_3]$$

$$\dot{\eta}_j^{n+1} = \dot{\eta}_j^n + \frac{1}{6}[l_0 + 2l_1 + 2l_2 + l_3]$$

Runge-Kutta for the Rijke tube model

In the Rijke tube model, we have:

$$\frac{d\,\eta_j}{dt} = f(t, \eta_j, \dot{\eta}_j) \qquad \frac{d\,\dot{\eta}_j}{dt} = g(t, \eta_j, \dot{\eta}_j) \qquad \text{for } j = 1, 2, \dots. N \text{ Galerkin modes}$$

These two equations need to be solved simultaneously.

The solution will proceed along the lines of

$$\eta_j^{n+1} = \eta_j^n + \frac{1}{6}[k_0 + 2k_1 + 2k_2 + k_3]$$

$$\dot{\eta}_j^{n+1} = \dot{\eta}_j^n + \frac{1}{6}[l_0 + 2l_1 + 2l_2 + l_3]$$

Tackling time-delay τ

Let's say the integration of the aforementioned equations takes place from $t = 0 \rightarrow T$ in steps of $\Delta t = \frac{T}{N}$, where $N = 1000$. Let's assume $\tau = 0.2$ s.

Tackling time-delay τ

Let's say the integration of the aforementioned equations takes place from $t = 0 \rightarrow T$ in steps of $\Delta t = \frac{T}{N}$, where $N = 1000$. Let's assume $\tau = 0.2$ s.

$$\frac{d\,\eta_j}{dt} = \dot{\eta}_j$$

$$\frac{d\,\dot{\eta}_j}{dt} = -2\zeta_j\omega_j\dot{\eta}_j - k_j^2\eta_j - \frac{2j\pi K}{\gamma M}\left[\sqrt{\left|\frac{1}{3} + u'_f(t-\tau)\right|} - \sqrt{\frac{1}{3}}\right]\sin(j\pi x_f)$$

Tackling time-delay τ

Let's say the integration of the aforementioned equations takes place from $t = 0 \rightarrow T$ in steps of $\Delta t = \frac{T}{N}$, where $N = 1000$. Let's assume $\tau = 0.2$ s.

$$\frac{d\,\eta_j}{dt} = \dot{\eta}_j$$

$$\frac{d\,\dot{\eta}_j}{dt} = -2\zeta_j\omega_j\dot{\eta}_j - k_j^2\eta_j - \frac{2j\pi K}{\gamma M}\left[\sqrt{\left|\frac{1}{3} + u'_f(t-\tau)\right|} - \sqrt{\frac{1}{3}}\right]\sin(j\pi x_f)$$

$$t - \tau < 0$$

Tackling time-delay τ

Let's say the integration of the aforementioned equations takes place from $t = 0 \rightarrow T$ in steps of $\Delta t = \frac{T}{N}$, where $N = 1000$. Let's assume $\tau = 0.2$ s.

$$\frac{d\,\eta_j}{dt} = \dot{\eta}_j$$

$$\frac{d\,\dot{\eta}_j}{dt} = -2\zeta_j\omega_j\dot{\eta}_j - k_j^2\eta_j - \frac{2j\pi K}{\gamma M}\left[\sqrt{\frac{1}{3}} - \sqrt{\frac{1}{3}}\right]\sin(j\pi x_f)$$

$t - \tau < 0$

Tackling time-delay τ

Let's say the integration of the aforementioned equations takes place from $t = 0 \rightarrow T$ in steps of $\Delta t = \frac{T}{N}$, where $N = 1000$. Let's assume $\tau = 0.2$ s.

$$\frac{d\,\eta_j}{dt} = \dot{\eta}_j$$

$$\frac{d\,\dot{\eta}_j}{dt} = -2\zeta_j\omega_j\dot{\eta}_j - k_j^2\eta_j$$

$t - \tau < 0$

Tackling time-delay τ

Let's say the integration of the aforementioned equations takes place from $t = 0 \rightarrow T$ in steps of $\Delta t = \frac{T}{N}$, where $N = 1000$. Let's assume $\tau = 0.2$ s.

$$\frac{d\,\eta_j}{dt} = \dot{\eta}_j$$

$$\frac{d\,\dot{\eta}_j}{dt} = -2\zeta_j\omega_j\dot{\eta}_j - k_j^2\eta_j$$

For the first $\frac{\tau}{\Delta t}$ steps

$$\frac{d\,\dot{\eta}_j}{dt} = -2\zeta_j\omega_j\dot{\eta}_j - k_j^2\eta_j - \frac{2j\pi K}{\gamma M}\left[\sqrt{\left|\frac{1}{3} + u'_f(t-\tau)\right|} - \sqrt{\frac{1}{3}}\right]\sin(j\pi x_f)$$

After initial $\frac{\tau}{\Delta t}$ steps

Tackling time-delay τ

$$\frac{d\,\eta_j}{dt} = \dot{\eta}_j$$

$$\frac{d\,\dot{\eta}_j}{dt} = -2\zeta_j\omega_j\dot{\eta}_j - k_j^2\eta_j$$

For the first $\frac{\tau}{\Delta t}$ steps

$$\frac{d\,\eta_j}{dt} = \dot{\eta}_j$$

$$\frac{d\,\dot{\eta}_j}{dt} = -2\zeta_j\omega_j\dot{\eta}_j - k_j^2\eta_j - \frac{2j\pi K}{\gamma M}\left[\sqrt{\left|\frac{1}{3} + u'_f(t-\tau)\right|} - \sqrt{\frac{1}{3}}\right]\sin(j\pi x_f)$$

After initial $\frac{\tau}{\Delta t}$ steps

Tackling time-delay τ

$$\frac{d\,\eta_j}{dt} = \dot{\eta}_j$$

$$\frac{d\,\dot{\eta}_j}{dt} = -2\zeta_j\omega_j\dot{\eta}_j - k_j^2\eta_j$$

For the first $\frac{\tau}{\Delta t}$ steps

$$\frac{d\,\eta_j}{dt} = \dot{\eta}_j$$

$$\frac{d\,\dot{\eta}_j}{dt} = -2\zeta_j\omega_j\dot{\eta}_j - k_j^2\eta_j - \frac{2j\pi K}{\gamma M}\left[\sqrt{\left|\frac{1}{3} + u'_f(t-\tau)\right|} - \sqrt{\frac{1}{3}}\right]\sin(j\pi x_f)$$

After initial $\frac{\tau}{\Delta t}$ steps

You will still have to calculate u_f' for the time steps $t = 0, \dots, \tau$ and use these values to find the solution to $\dot{\eta}(t = \tau + 1), \dot{\eta}(t = \tau + 2), \dots$ and so on.

$u_f'(t-\tau)$ inside the square root term

$$\frac{d\,\dot{\eta}_j}{dt} = -2\zeta_j\omega_j\dot{\eta}_j - k_j^2\eta_j - \frac{2j\pi K}{\gamma M}\left[\sqrt{\left|\frac{1}{3} + u'_f(t-\tau)\right|} - \sqrt{\frac{1}{3}}\right]\sin(j\pi x_f)$$

$u_f{}'(t-\tau)$ inside the square root term

$$\frac{d\,\dot{\eta}_j}{dt} = -2\zeta_j\omega_j\dot{\eta}_j - k_j^2\eta_j - \frac{2j\pi K}{\gamma M}\left[\sqrt{\left|\frac{1}{3} + u'_f(t-\tau)\right|} - \sqrt{\frac{1}{3}}\right]\sin(j\pi x_f)$$

$$\sqrt{\left|\frac{1}{3} + u'_f(t-\tau)\right|} - \sqrt{\frac{1}{3}} \approx \frac{\sqrt{3}}{2}u_f'(t-\tau)$$

$u_f{}'(t-\tau)$ inside the square root term

$$\frac{d\,\dot{\eta}_j}{dt} = -2\zeta_j\omega_j\dot{\eta}_j - k_j^2\eta_j - \frac{2j\pi K}{\gamma M}\left[\sqrt{\left|\frac{1}{3} + u'_f(t-\tau)\right|} - \sqrt{\frac{1}{3}}\right]\sin(j\pi x_f)$$

$$\sqrt{\left|\frac{1}{3} + u'_f(t-\tau)\right|} - \sqrt{\frac{1}{3}} \approx \frac{\sqrt{3}}{2}u'_f(t-\tau)$$

$$u'_f(t-\tau) \approx u'_f(t) - \tau\frac{\partial u'_f(t)}{\partial t}$$

${u_f}'(t-\tau)$ inside the square root term

$$\frac{d\,\dot{\eta}_j}{dt} = -2\zeta_j\omega_j\dot{\eta}_j - k_j^2\eta_j - \frac{2j\pi K}{\gamma M}\left[\sqrt{\left|\frac{1}{3} + u'_f(t-\tau)\right|} - \sqrt{\frac{1}{3}}\right]\sin(j\pi x_f)$$

$$\sqrt{\left|\frac{1}{3} + u'_f(t-\tau)\right|} - \sqrt{\frac{1}{3}} \approx \frac{\sqrt{3}}{2}u'_f(t-\tau)$$

$$u'_f(t-\tau) \approx u'_f(t) - \tau\frac{\partial u'_f(t)}{\partial t}$$

$$u'(x_f,t) = \sum_{j=1}^{N} \eta_j(t)\cos(j\pi x_f)$$

Assignment tasks

[Task 1]: Perform the numerical integration of Rijke tube model using 4^{th} order Runge-Kutta method

Assignment tasks

[Task 1]: Perform the numerical integration of Rijke tube model using 4th order Runge-Kutta method

[Task 2]: Reproduce figure 4-6 in Balasubramanian and Sujith, *Physics of Fluids*, 2008.

Assignment tasks

[Task 1]: Perform the numerical integration of Rijke tube model using 4^{th} order Runge-Kutta method

[Task 2]: Reproduce figure 4-6 in Balasubramanian and Sujith, *Physics of Fluids*, 2008.

[Task 3]: Reproduce the bifurcation diagram (figure 6a) shown in Subramanian et al., *IJCSD*, 2010.

Assignment tasks

[Task 1]: Perform the numerical integration of Rijke tube model using 4th order Runge-Kutta method

[Task 2]: Reproduce figure 4-6 in Balasubramanian and Sujith, *Physics of Fluids*, 2008.

[Task 3]: Reproduce the bifurcation diagram (figure 6a) shown in Subramanian et al., *IJCSD*, 2010.

[Task 4]: Prepare project report.

Assignment tasks

[Task 1]: Perform the numerical integration of Rijke tube model using 4th order Runge-Kutta method

[Task 2]: Reproduce figure 4-6 in Balasubramanian and Sujith, *Physics of Fluids*, 2008.

[Task 3]: Reproduce the bifurcation diagram (figure 6a) shown in Subramanian et al., *IJCSD*, 2010.

[Task 4]: Prepare project report.

[Task 5]: (Bonus) Reproduce the 3D bifurcation plot between $|\eta_1|$, non-dimensional heater power K and time delay τ

Your project report must contain the following elements:

Introduction: Introduction to thermoacoustic instability, motivation for studying the present problem, and an overview of your approach

Your project report must contain the following elements:

Introduction: Introduction to thermoacoustic instability, motivation for studying the present problem, and an overview of your approach

Methodology: Governing equations and numerical method used for solving the equations.

Your project report must contain the following elements:

Introduction: Introduction to thermoacoustic instability, motivation for studying the present problem, and an overview of your approach

Methodology: Governing equations and numerical method used for solving the equations.

Results and Discussions: You have to discuss the results (the figures you make according to the previous slide), and possible implications of these results in regards to the study of combustion in rockets and gas-turbine engines.

Your project report must contain the following elements:

Introduction: Introduction to thermoacoustic instability, motivation for studying the present problem, and an overview of your approach

Methodology: Governing equations and numerical method used for solving the equations.

Results and Discussions: You have to discuss the results (the figures you make according to the previous slide), and possible implications of these results in regards to the study of combustion in rockets and gas-turbine engines.

Conclusion

References

Contact

In case of queries, contact:

ae21d014@smail.iitm.ac.in
ae21d017@smail.iitm.ac.in