Solving for Coefficients of Minimum Jerk Trajectories

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Minimum Jerk Trajectory

$$x^\star(t) = \underset{x(t)}{\operatorname{argmin}} \int_0^T \mathcal{L}\left(\dddot{x}, \ddot{x}, \dot{x}, x, t\right) dt = \underset{x(t)}{\operatorname{argmin}} \int_0^T \dddot{x}^2 dt$$

We can solve the Euler-Lagrange equation:

$$\frac{\partial \mathcal{L}}{\partial x} - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) + \frac{d^2}{dt^2}\left(\frac{\partial \mathcal{L}}{\partial \ddot{x}}\right) - \frac{d^3}{dt^3}\left(\frac{\partial \mathcal{L}}{\partial x^{(3)}}\right) = 0$$

to get the condition:

$$x^{(6)} = 0$$

Thus, we want a trajectory of the form:

$$x(t) = c_5 t^5 + c_4 t^4 + c_3 t^3 + c_2 t^2 + c_1 t + c_0$$

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Solving for Coefficients

Boundary conditions:

	Position	Velocity	Acceleration
t = 0	a	0	0
t = T	b	0	0

Position constraints: $x(t) = c_5 t^5 + c_4 t^4 + c_3 t^3 + c_2 t^2 + c_1 t + c_0$

$$x(0) = c_0 = a$$

$$x(T) = c_5 (T)^5 + c_4 (T)^4 + c_3 (T)^3 + c_2 (T)^2 + c_1 (T) + c_0 = b$$

Solving for Coefficients

Position constraints in matrix form:

$$x(0) = c_0 = a$$

$$\underbrace{\underbrace{\begin{bmatrix}0 & 0 & 0 & 0 & 0 & 1\end{bmatrix}}_{1 \times 6}\underbrace{\begin{bmatrix}c_5\\ c_4\\ c_3\\ c_2\\ c_1\\ c_0\end{bmatrix}}_{6 \times 1}}_{1 \times 1} = a$$

Solving for Coefficients

Position constraints in matrix form:

$$x(0) = c_0 = a$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_5 \\ c_4 \\ c_3 \\ c_2 \\ c_1 \\ c_0 \end{bmatrix} = a$$

Solving for Coefficients

Position constraints in matrix form:

$$x(T) = c_5(T)^5 + c_4(T)^4 + c_3(T)^3 + c_2(T)^2 + c_1(T) + c_0 = b$$

$$\begin{bmatrix} T^5 & T^4 & T^3 & T^2 & T & 1 \end{bmatrix} \begin{bmatrix} c_5 \\ c_4 \\ c_3 \\ c_2 \\ c_1 \\ c_0 \end{bmatrix} = b$$

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Solving for Coefficients

Boundary conditions:

	Position	Velocity	Acceleration
t = 0	a	0	0
t = T	b	0	0

Velocity constraints: $\dot{x}(t) = 5c_5 t^4 + 4c_4 t^3 + 3c_3 t^2 + 2c_2 t + c_1$

$$\dot{x}(0) = c_1 = 0$$

$$\dot{x}(T) = 5c_5(T)^4 + 4c_4(T)^3 + 3c_3(T)^2 + 2c_2(T) + c_1 = 0$$

Solving for Coefficients

Velocity constraints in matrix form:

$$\dot{x}(0) = c_1 = 0$$

$$\dot{x}(T) = 5c_5(T)^4 + 4c_4(T)^3 + 3c_3(T)^2 + 2c_2(T) + c_1 = 0$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} c_5 \\ c_4 \\ c_3 \\ c_2 \\ c_1 \\ c_0 \end{bmatrix} = 0$$

$$\begin{bmatrix} 5T^4 & 4T^3 & 3T^2 & 2T & 1 & 0 \end{bmatrix} \begin{bmatrix} c_5 \\ c_4 \\ c_3 \\ c_2 \\ c_1 \\ c_0 \end{bmatrix} = 0$$

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Solving for Coefficients

Boundary conditions:

	Position	Velocity	Acceleration
t = 0	a	0	0
t = T	b	0	0

Acceleration constraints: $\ddot{x}(t) = 20c_5t^3 + 12c_4t^2 + 6c_3t^2 + 2c_2$

$$\ddot{x}(0) = 2c_2 = 0$$

$$\ddot{x}(T) = 20c_5(T)^3 + 12c_4(T)^2 + 6c_3(T)^2 + 2c_2 = 0$$

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Solving for Coefficients

Acceleration constraints in matrix form:

$$\ddot{x}(0) = 2c_2 = 0$$

$$\ddot{x}(T) = 20c_5(T)^3 + 12c_4(T)^2 + 6c_3(T)^2 + 2c_2 = 0$$

$$\begin{bmatrix} 0 & 0 & 0 & 2 & 0 & 0 \end{bmatrix} \begin{bmatrix} c_5 \\ c_4 \\ c_3 \\ c_2 \\ c_1 \\ c_0 \end{bmatrix} = 0$$

$$\begin{bmatrix} 20T^3 & 12T^2 & 6T & 2 & 0 & 0 \end{bmatrix} \begin{bmatrix} c_5 \\ c_4 \\ c_3 \\ c_2 \\ c_1 \\ c_0 \end{bmatrix} = 0$$

Solving for Coefficients

Boundary conditions:

	Position	Velocity	Acceleration
t = 0	a	0	0
t = T	b	0	0

Combine constraints into one matrix expression:

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ T^5 & T^4 & T^3 & T^2 & T & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 5T^4 & 4T^3 & 3T^2 & 2T & 1 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 20T^3 & 12T^2 & 6T & 2 & 0 & 0 \end{bmatrix} \begin{bmatrix} c_5 \\ c_4 \\ c_3 \\ c_2 \\ c_1 \\ c_0 \end{bmatrix} = \begin{bmatrix} a \\ b \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Example 1: Find the Minimum Jerk Trajectory

Find the minimum jerk trajectory with boundary conditions:

	Position	Velocity	Acceleration
t = 0	a = 0	0	0
t = T = 1	b = 5	0	0

$$Ax = b$$

$$x = A^{-1}b$$

$$\underbrace{\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 5 & 4 & 3 & 2 & 1 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 20 & 12 & 6 & 2 & 0 & 0 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} c_5 \\ c_4 \\ c_3 \\ c_2 \\ c_1 \\ c_0 \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} 0 \\ 5 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{b}$$

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Example 1: Find the Minimum Jerk Trajectory

$$x = \begin{bmatrix} 30 \\ -75 \\ 50 \\ 0 \\ 0 \\ 0 \end{bmatrix} \longleftrightarrow \begin{bmatrix} c_5 \\ c_4 \\ c_3 \\ c_2 \\ c_1 \\ c_0 \end{bmatrix}$$

$$x(t) = 30t^5 - 75t^4 + 50t^3$$

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Example 1: Find the Minimum Jerk Trajectory

We can verify that this trajectory does in fact satisfy all boundary conditions:

	Position	Velocity	Acceleration
t = 0	a = 0 ✓	0	0
t = T = 1	b = 5 ✓	0	0

$$x(t) = 30t^5 - 75t^4 + 50t^3$$

$$x(0) = 0$$

$$x(1) = 30(1)^5 - 75(1)^4 + 50(1)^3 = 5$$

Example 1: Find the Minimum Jerk Trajectory

We can verify that this trajectory does in fact satisfy all boundary conditions:

	Position	Velocity	Acceleration
t = 0	a = 0 ✓	0 ✓	0
t = T = 1	b = 5 ✓	0 ✓	0

$$\dot{x}(t) = 150t^4 - 300t^3 + 150t^2$$

$$\dot{x}(0) = 0$$

$$\dot{x}(1) = 150 - 300 + 150 = 0$$

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Example 1: Find the Minimum Jerk Trajectory

We can verify that this trajectory does in fact satisfy all boundary conditions:

	Position	Velocity	Acceleration
t = 0	a = 0 ✓	0 ✓	0 ✓
t = T = 1	b = 5 ✓	0 ✓	0 ✓

$$\ddot{x}(t) = 600t^3 - 900t^2 + 300t$$

$$\ddot{x}(0) = 0$$

$$\ddot{x}(1) = 600 - 900 + 300 = 0$$

Find the Minimum Jerk Trajectory

The order of coefficients in the matrix of unknowns matters.

$$\begin{bmatrix} T^5 & T^4 & T^3 & T^2 & T & 1 \end{bmatrix} \begin{bmatrix} c_5 \\ c_4 \\ c_3 \\ c_2 \\ c_1 \\ c_0 \end{bmatrix} = b$$

Find the Minimum Jerk Trajectory

The order of coefficients in the matrix of unknowns matters.

$$\begin{bmatrix} 1 & T & T^2 & T^3 & T^4 & T^5 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \end{bmatrix} = b$$

Find the Minimum Jerk Trajectory

The order of coefficients in the matrix of unknowns matters.

$$\begin{bmatrix} T^4 & T & T^2 & T^5 & T^3 & 1 \end{bmatrix} \begin{bmatrix} c_4 \\ c_1 \\ c_2 \\ c_5 \\ c_3 \\ c_0 \end{bmatrix} = b$$