

Minimum Velocity Trajectories from the Euler-Lagrange Equations

Minimum Velocity Trajectory

Find the function $x(t)$ such that:

$$\begin{aligned} x^*(t) &= \operatorname{argmin}_{x(t)} \int_0^T \mathcal{L}(\dot{x}, x, t) dt \\ &= \operatorname{argmin}_{x(t)} \int_0^T \dot{x}^2 dt \end{aligned}$$

Euler-Lagrange equation:

$$\frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{x}} \right) - \frac{\partial \mathcal{L}}{\partial x} = 0$$

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Cost function:

$$\mathcal{L}(\dot{x}, x, t) = (\dot{x})^2$$

Euler-Lagrange terms:

$$\left(\frac{\partial \mathcal{L}}{\partial x} \right) = 0 \quad \leftarrow \text{No } x \text{ appears in } \mathcal{L}$$

$$\left(\frac{\partial \mathcal{L}}{\partial \dot{x}} \right) = 2\dot{x}$$

$$\frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{x}} \right) = \frac{d}{dt} (2\dot{x}) = 2\ddot{x}$$

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Euler-Lagrange equation:

$$\frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{x}} \right) - \frac{\partial \mathcal{L}}{\partial x} = 0$$

Euler-Lagrange terms: $2\ddot{x} - 0 = 0 \rightarrow 2\ddot{x} = 0 \rightarrow \ddot{x} = 0$

Integrate to get the velocity: $\dot{x} = c_1$

Integrate to get position: $x(t) = c_1 t + c_0$