Dynamical Systems in State-Space Form

Dynamical Systems

Systems where the effects of actions do not occur immediately.

Evolution of the system's states is governed by a set of ordinary differential equations.

Ordinary differential equations are often rearranged into *state-space form*.

$$\dot{\mathbf{x}} = f(\mathbf{x}, \mathbf{u})$$

Matrices

State-Space Form

Given an ordinary differential equation:

1. Identify the order, n, of the system
2. Define the states $x_1 = y(t), x_2 = \dot{y}(t), \ldots, x_n = y^{(n-1)}(t)$

   (n-1)st derivative
3. Create the *state vector* $\mathbf{x} = \begin{bmatrix} x_1 & x_2 & \ldots & x_n \end{bmatrix}^T = \begin{bmatrix} y & \dot{y} & \ldots & y^{(n-1)} \end{bmatrix}^T$
4. Write the coupled first-order differential equations:

$$\frac{d}{dt}x_1 = \frac{d}{dt}y = \dot{y} = x_2$$

$$\frac{d}{dt}x_2 = \frac{d}{dt}\dot{y} = \ddot{y} = x_3$$

$$\ldots$$

$$\frac{d}{dt}x_n = \frac{d}{dt}y^{(n-1)} = g(y, \dot{y}, \ldots, y^{(n-1)}, \mathbf{u}) = g(x_1, x_2, \ldots, x_n, \mathbf{u})$$

From governing ODEs

State-Space Form

5. Write system of first-order differential equations as matrix:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} x_2 \\ x_3 \\ \dots \\ g(x_1, x_2, \dots, x_n, \mathbf{u}) \end{bmatrix}$$

$$\dot{\mathbf{x}} = f(\mathbf{x}, \mathbf{u})$$

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Example 1: Mass-Spring System

$$m\ddot{y}(t) + ky(t) = u(t)$$

1. Identify n = 2
2. Define states $x_1 = y, x_2 = \dot{y}$
3. Create the state vector $\mathbf{x} = \begin{bmatrix} x_1 & x_2 \end{bmatrix}^T = \begin{bmatrix} y & \dot{y} \end{bmatrix}^T$
4. Write the coupled first-order differential equations:

$$\frac{d}{dt}x_1 = \frac{d}{dt}y = \dot{y} = x_2$$

$$\frac{d}{dt}x_2 = \frac{d}{dt}\dot{y} = \ddot{y} = \frac{u(t) - ky(t)}{m} = \frac{u(t) - kx_1}{m}$$

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Example 1: Mass-Spring System

$$m\ddot{y}(t) + ky(t) = u(t)$$

5. Write system of first-order differential equations as matrix:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ \frac{u(t) - kx_1}{m} \end{bmatrix}$$

This system is actually *linear*:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} u(t)$$

$$\dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{u}$$

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Example 2: Planar Quadrotor Model

$$m\ddot{y} = -\sin(\phi)u_1$$
$$m\ddot{z} = \cos(\phi)u_1 - mg$$
$$I_{xx}\ddot{\phi} = u_2$$

1. Identify n = 2
2. Define states $x_1 = y, x_2 = z, x_3 = \phi, x_4 = \dot{y}, x_5 = \dot{z}, x_6 = \dot{\phi}$
3. Define the state vector

$$\mathbf{x} = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \end{bmatrix}^T = \begin{bmatrix} y & z & \phi & \dot{y} & \dot{z} & \dot{\phi} \end{bmatrix}^T$$

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$$m\ddot{y} = -\sin(\phi)u_1$$

$$m\ddot{z} = \cos(\phi)u_1 - mg$$

$$I_{xx}\ddot{\phi} = u_2$$

4. Define the system of first-order differential equations:

$$\frac{d}{dt}x_1 = \frac{d}{dt}y = \dot{y} = x_4$$

$$\frac{d}{dt}x_2 = \frac{d}{dt}z = \dot{z} = x_5$$

$$\frac{d}{dt}x_3 = \frac{d}{dt}\phi = \dot{\phi} = x_6$$

$$\frac{d}{dt}x_4 = \frac{d}{dt}\dot{y} = \ddot{y} = \frac{-\sin(\phi)u_1}{m} = \frac{-\sin(x_3)u_1}{m}$$

$$\frac{d}{dt}x_5 = \frac{d}{dt}\dot{z} = \ddot{z} = \frac{\cos(\phi)u_1}{m} - g = \frac{\cos(x_3)u_1}{m} - g$$

$$\frac{d}{dt}x_6 = \frac{d}{dt}\dot{\phi} = \ddot{\phi} = \frac{u_2}{I_{xx}}$$

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$$m\ddot{z} = \cos(\phi)u_1 - mg$$

$$I_{xx}\ddot{\phi} = u_2$$

5. Write system of first-order differential equations as matrix:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ \dot{x}_5 \\ \dot{x}_6 \end{bmatrix} = \begin{bmatrix} x_4 \\ x_5 \\ x_6 \\ \frac{-\sin(x_3)u_1}{m} \\ \frac{\cos(x_3)u_1}{m} - g \\ \frac{u_2}{I_{xx}} \end{bmatrix}$$

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