# Rotations and Angular Velocities

Penn Engineering

# Time Derivatives of Rotations

Rotation matrix

$$R(t)$$

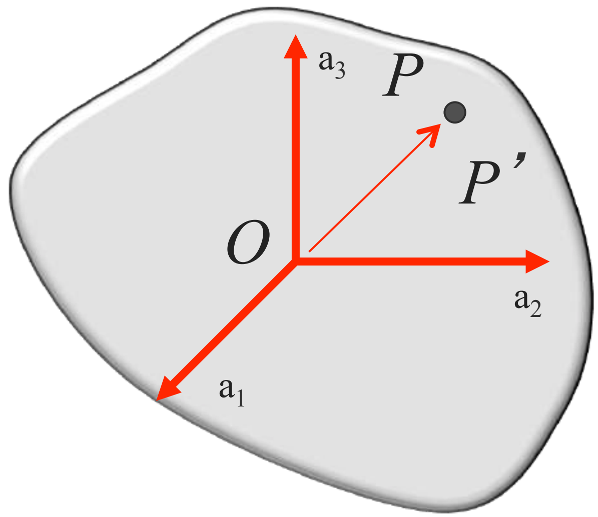
Orthogonality

$$R^T(t)R(t) = I \qquad \xrightarrow{\frac{d}{dt}(\,.\,)} \qquad \dot{R}^T R + R^T \dot{R} = 0$$

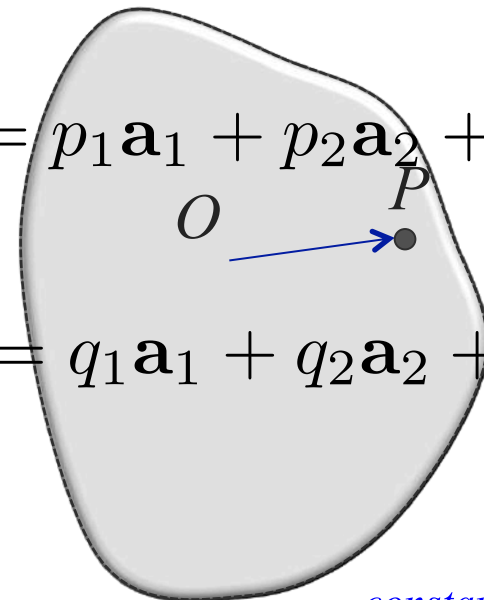
$$R(t)R^T(t) = I \qquad\qquad R\dot{R}^T + \dot{R}R^T = 0$$

$R^T \dot{R}$ and $\dot{R}R^T$ are skew symmetric

# Rotation with $O$ fixed




$$\overrightarrow{OP} = p_1\mathbf{a}_1 + p_2\mathbf{a}_2 + p_3\mathbf{a}_3$$




$$\overrightarrow{OP'} = q_1\mathbf{a}_1 + q_2\mathbf{a}_2 + q_3\mathbf{a}_3$$

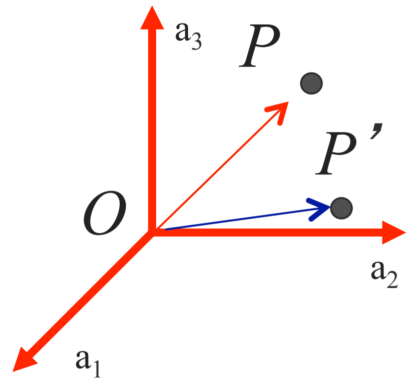
$$\underbrace{\begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix}}_{q} = \begin{bmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{bmatrix} \underbrace{\begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix}}_{p}$$

*constant coordinates of P in body-fixed frame*

$$\boxed{q(t) = R(t)p}$$

*changing coordinates of P as the rigid body rotates*

# Rotation with $O$ fixed




$$\begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix}$$

$$\boxed{q(t) = R(t)p}$$

$$\dot{q} = \dot{R}p$$

*velocity in inertial frame*

*position in body-fixed frame*

$$R^T \dot{q} = \boxed{R^T \dot{R}}p \qquad \hat{\omega}^b$$

*velocity in body-fixed frame*

*encodes angular velocity in body-fixed frame*

$$\dot{q} = \boxed{\dot{R}R^T}q$$

*velocity in inertial frame*

*encodes angular velocity in inertial frame*

$$\hat{\omega}^s$$

# Exercise

What is the angular velocity for a rotation about the *z* axis?

$$R = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R^T = \begin{bmatrix} \cos(\theta) & \sin(\theta) & 0 \\ -\sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \dot{R} = \begin{bmatrix} -\sin(\theta) & -\cos(\theta) & 0 \\ \cos(\theta) & -\sin(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \dot{\theta}$$

# Angular velocity for a rotation about the *z*-axis

$$R = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R^T \dot{R} = \begin{bmatrix} \cos(\theta) & \sin(\theta) & 0 \\ -\sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -\sin(\theta) & -\cos(\theta) & 0 \\ \cos(\theta) & -\sin(\theta) & 0 \\ 0 & 0 & 0 \end{bmatrix} \dot{\theta}$$

$$= \dot{R}R^T = \dot{\theta} \begin{bmatrix} -\sin(\theta) & -\cos(\theta) & 0 \\ \cos(\theta) & -\sin(\theta) & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \cos(\theta) & \sin(\theta) & 0 \\ -\sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \dot{\theta} = \boxed{\widehat{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}} \dot{\theta}}$$

# Two Rotations

$$R = R_z(\theta)R_x(\phi)$$

$$\begin{aligned}\hat{\omega}^b = R^T\dot{R} &= (R_zR_x)^T(\dot{R}_zR_x + R_z\dot{R}_x) \\ &= R_x^TR_z^T\dot{R}_zR_x + R_x^T\dot{R}_x\end{aligned}$$

$$\begin{aligned}\hat{\omega}^s = \dot{R}R^T &= (\dot{R}_zR_x + R_z\dot{R}_x)(R_zR_x)^T \\ &= \dot{R}_zR_z^T + R_z\dot{R}_xR_x^TR_z^T\end{aligned}$$