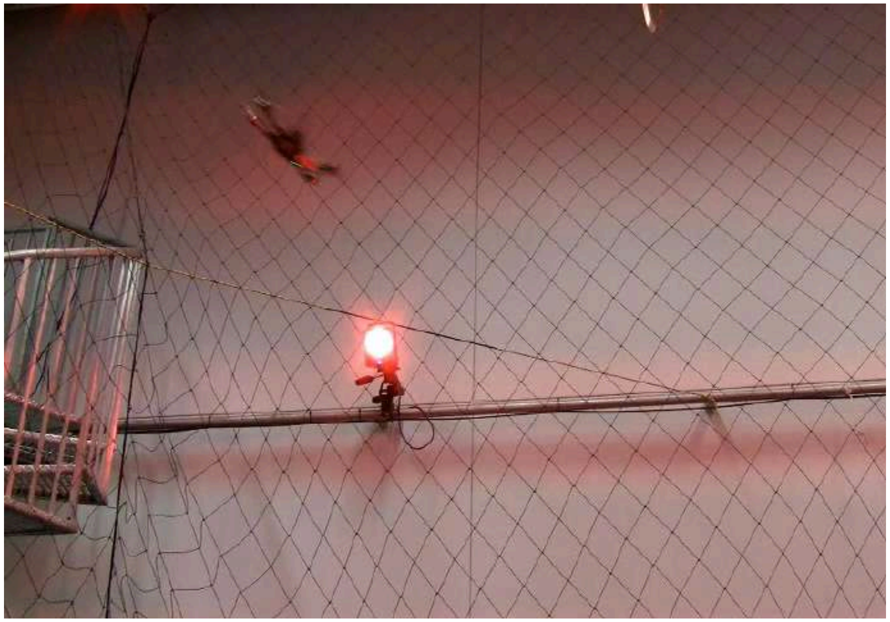
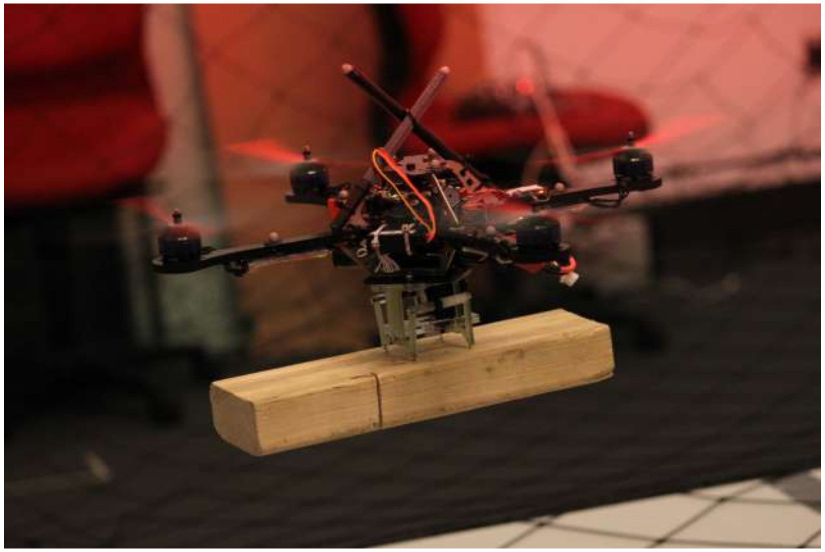
Nonlinear Control

Limitations of Linear Control

- Assumption: roll and pitch angles, and all velocities are close to zero





Nonlinear Control

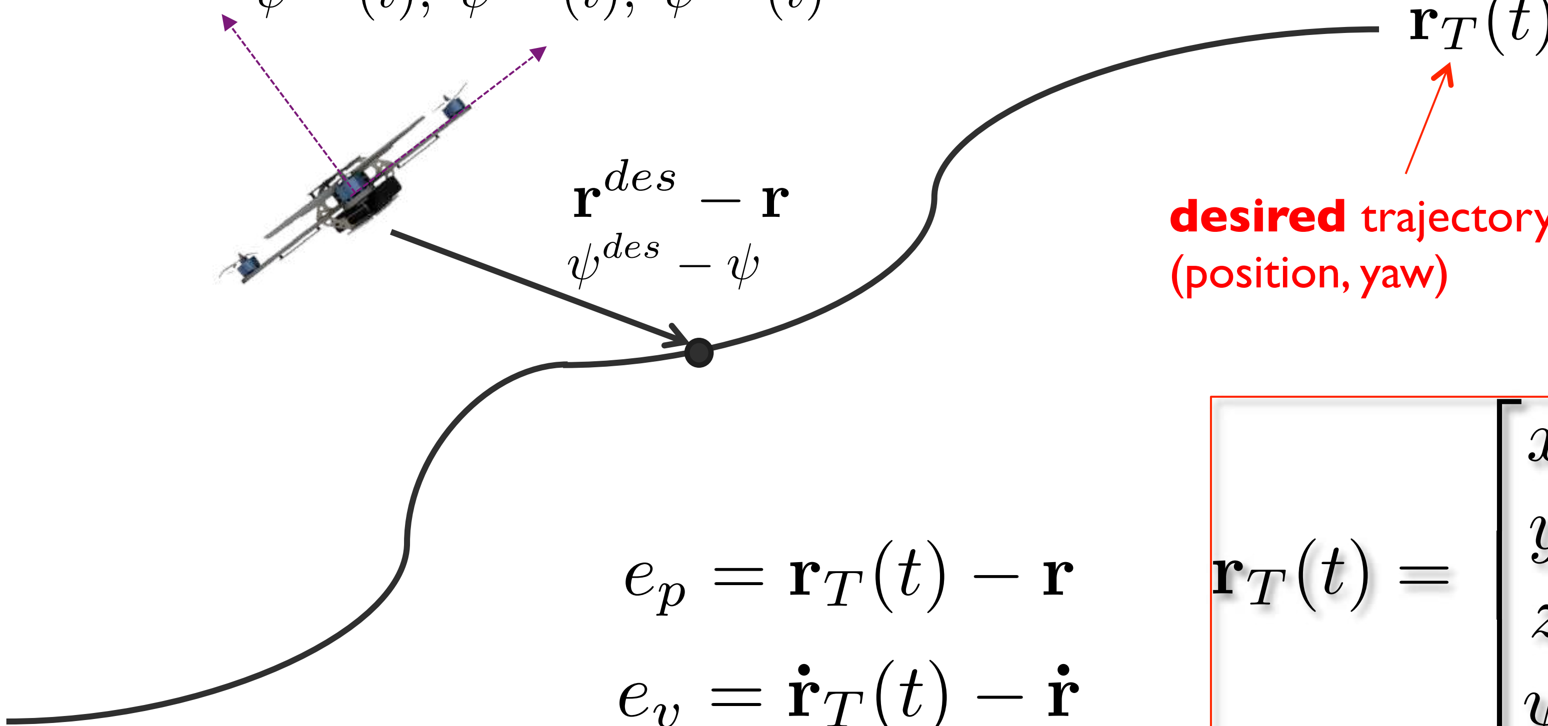
Control the robot at states far away from the equilibrium (hover) state

Trajectory Tracking

Given $\mathbf{r}_T(t), \dot{\mathbf{r}}_T(t), \ddot{\mathbf{r}}_T(t)$

$$\mathbf{r}^{des}(t),\ \dot{\mathbf{r}}^{des}(t),\ \ddot{\mathbf{r}}^{des}(t)$$

$$\psi^{des}(t),\ \dot{\psi}^{des}(t),\ \ddot{\psi}^{des}(t)$$



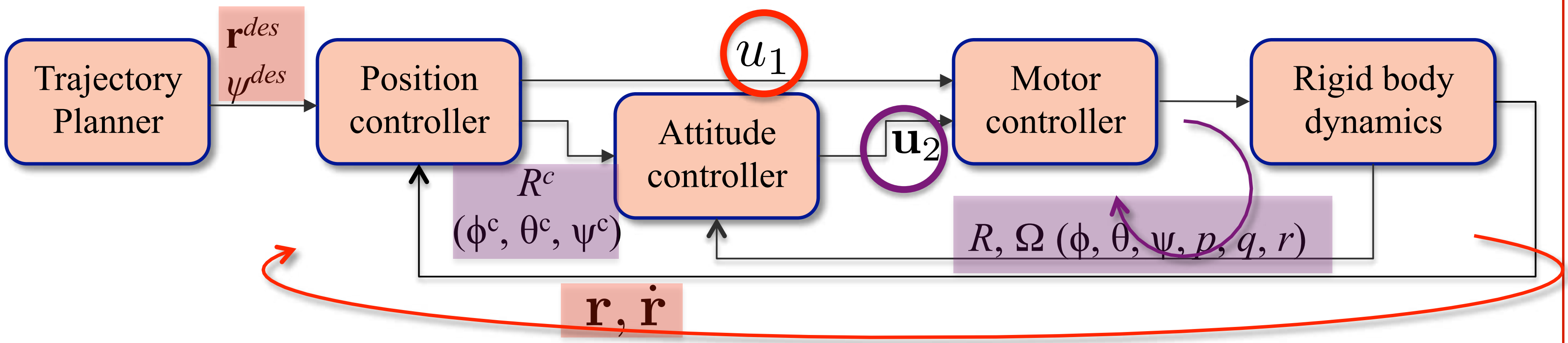
$$e_p = \mathbf{r}_T(t) - \mathbf{r}$$

$$e_v = \dot{\mathbf{r}}_T(t) - \dot{\mathbf{r}}$$

$$\boxed{\mathbf{r}_T(t) = \begin{bmatrix} x(t) \\ y(t) \\ z(t) \\ \psi(t) \end{bmatrix}}$$

Want $(\ddot{\mathbf{r}}_T(t) - \ddot{\mathbf{r}}_c) + k_{d,x} e_v + k_{p,x} e_p = 0$

Commanded acceleration, calculated by the controller



$$m\ddot{\mathbf{r}} = \begin{bmatrix} 0 \\ 0 \\ -mg \end{bmatrix} + R \begin{bmatrix} 0 \\ 0 \\ \boxed{F_1 + F_2 + F_3 + F_4} \end{bmatrix}$$

u_1

$$I \begin{bmatrix} \dot{p} \\ \dot{q} \\ \dot{r} \end{bmatrix} = \boxed{\begin{bmatrix} L(F_2 - F_4) \\ L(F_3 - F_1) \\ M_1 - M_2 + M_3 - M_4 \end{bmatrix}} - \begin{bmatrix} p \\ q \\ r \end{bmatrix} \times I \begin{bmatrix} p \\ q \\ r \end{bmatrix}$$

$\mathbf{u}_2$

Trajectory Tracking

$$r_T(t) = \begin{bmatrix} x^{des}(t) \\ y^{des}(t) \\ z^{des}(t) \\ \psi^{des}(t) \end{bmatrix}$$

$$u_1 = \overbrace{\left(\ddot{r}^{des} + K_v \mathbf{e}_{\dot{r}} + K_p \mathbf{e}_r + mg\mathbf{a}_3\right)}^{\mathbf{t}} \cdot \mathbf{R}\mathbf{b}_3$$

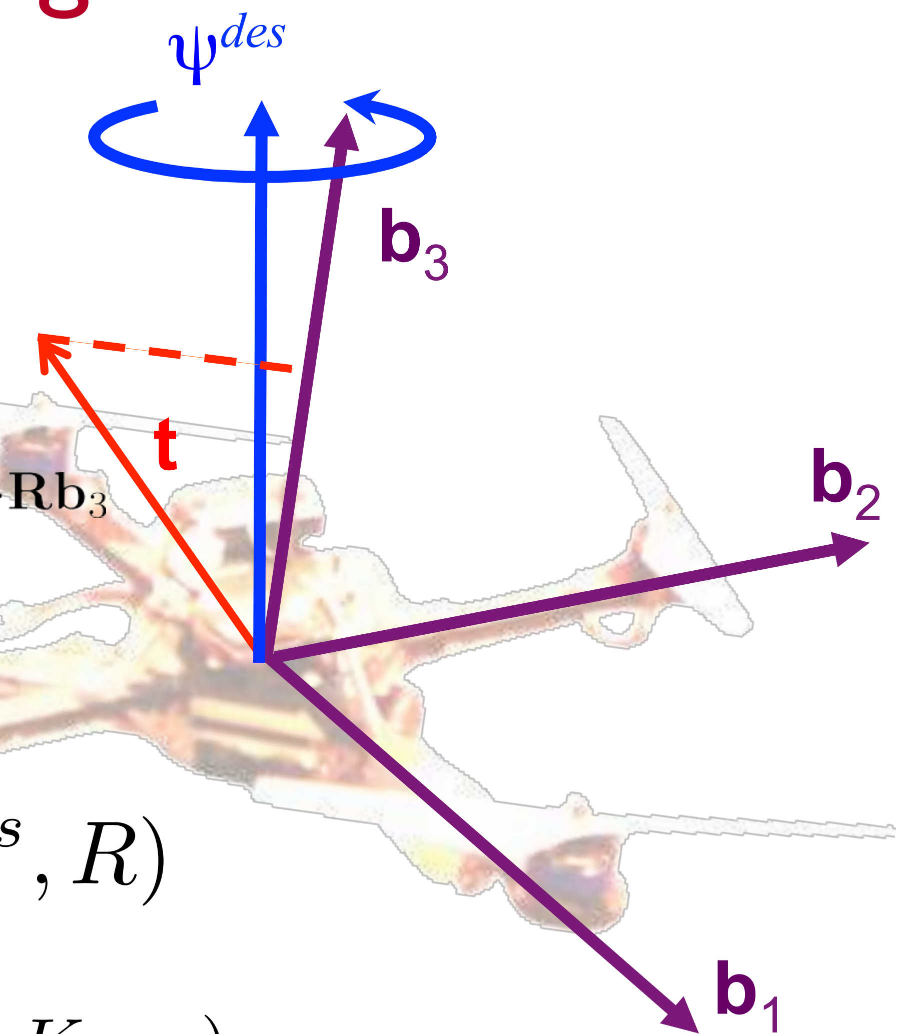
$$R^{des}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\mathbf{R}^{des}\mathbf{b}_3 = \frac{\mathbf{t}}{\|\mathbf{t}\|}$$

$$\psi = \psi^{des}$$

$$e_R(R^{des}, R)$$

$$\mathbf{u}_2 = \omega \times \mathbf{I}\omega + \mathbf{I}\left(-K_R \mathbf{e}_R - K_\omega \mathbf{e}_\omega\right)$$



How to determine $\mathbf{R}^{des}$?

You are given two pieces of information

$$\mathbf{R}^{des}\mathbf{b}_3 = \frac{\mathbf{t}}{\|\mathbf{t}\|}$$

$$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\psi = \psi^{des}$$

You know that the rotation matrix has the form

$$\mathbf{R} = \begin{bmatrix} c\psi c\theta - s\phi s\psi s\theta & -c\phi s\psi & c\psi s\theta + c\theta s\phi s\psi \\ c\theta s\psi + c\psi s\phi s\theta & c\phi c\psi & s\psi s\theta - c\theta s\phi c\psi \\ -c\phi s\theta & s\phi & c\phi c\theta \end{bmatrix}$$

You should be able to find the roll and pitch angles.

How to calculate the error $\mathbf{e}_R(\mathbf{R}^{des}, \mathbf{R})$?

- Cannot simply take the difference of two rotation matrices

What is the magnitude of the rotation required to go from the current orientation to the desired orientation?

$$\mathbf{R} \rightarrow \mathbf{R}^{des}$$

The required rotation is

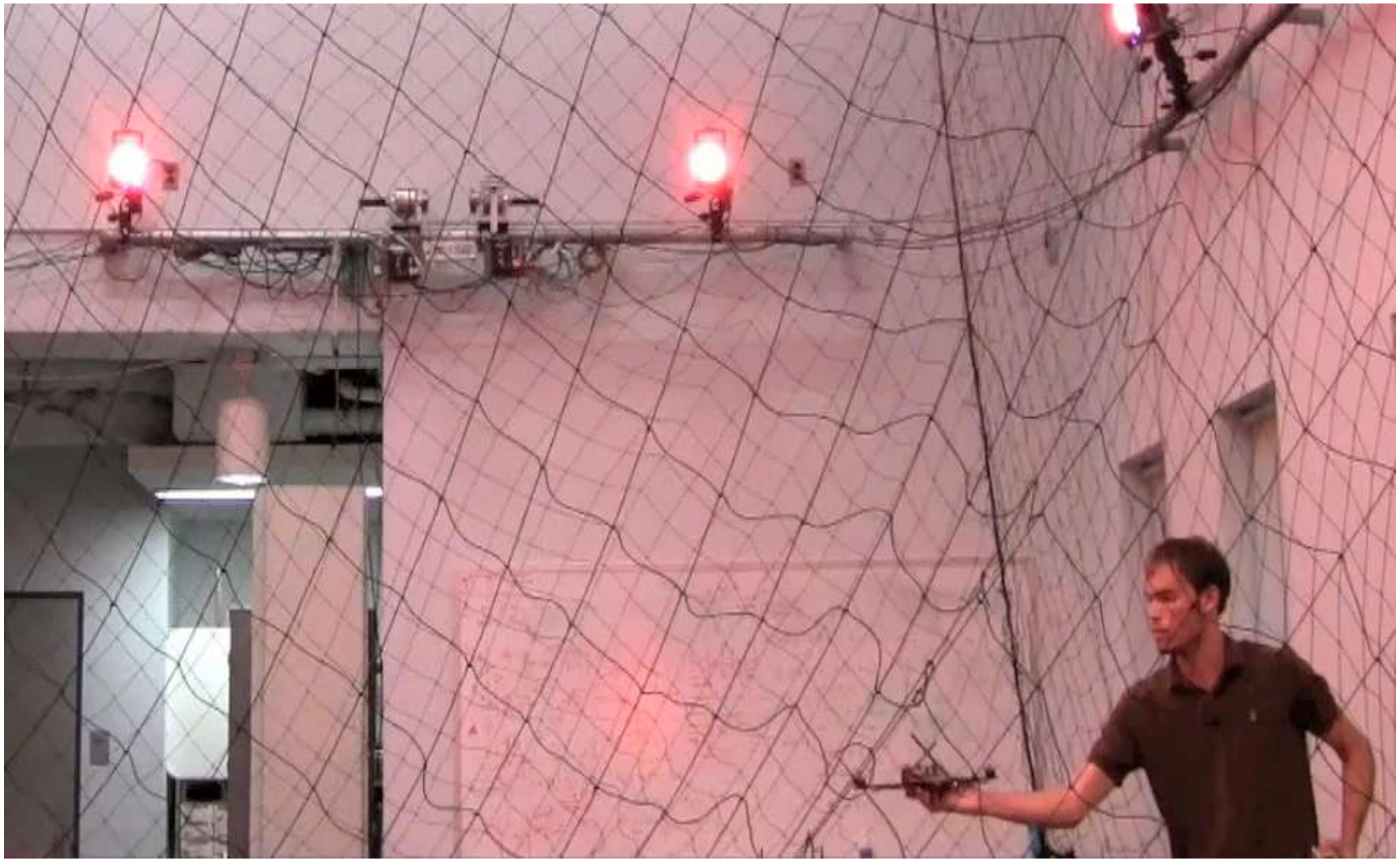
$$\Delta R = \mathbf{R}^T \mathbf{R}^{des}$$

The angle and axis of rotation can be determined using Rodrigues formula

Stability

Large basin of attraction*

$$tr[I - (R^{des})^T R] < 2 \qquad \|e_\omega(0)\|^2 \leq \frac{2}{\boxed{\lambda_{min}(I)}} k_R \left(1 - \frac{1}{2} tr\left[I - (R^{des})^T R\right]\right)$$

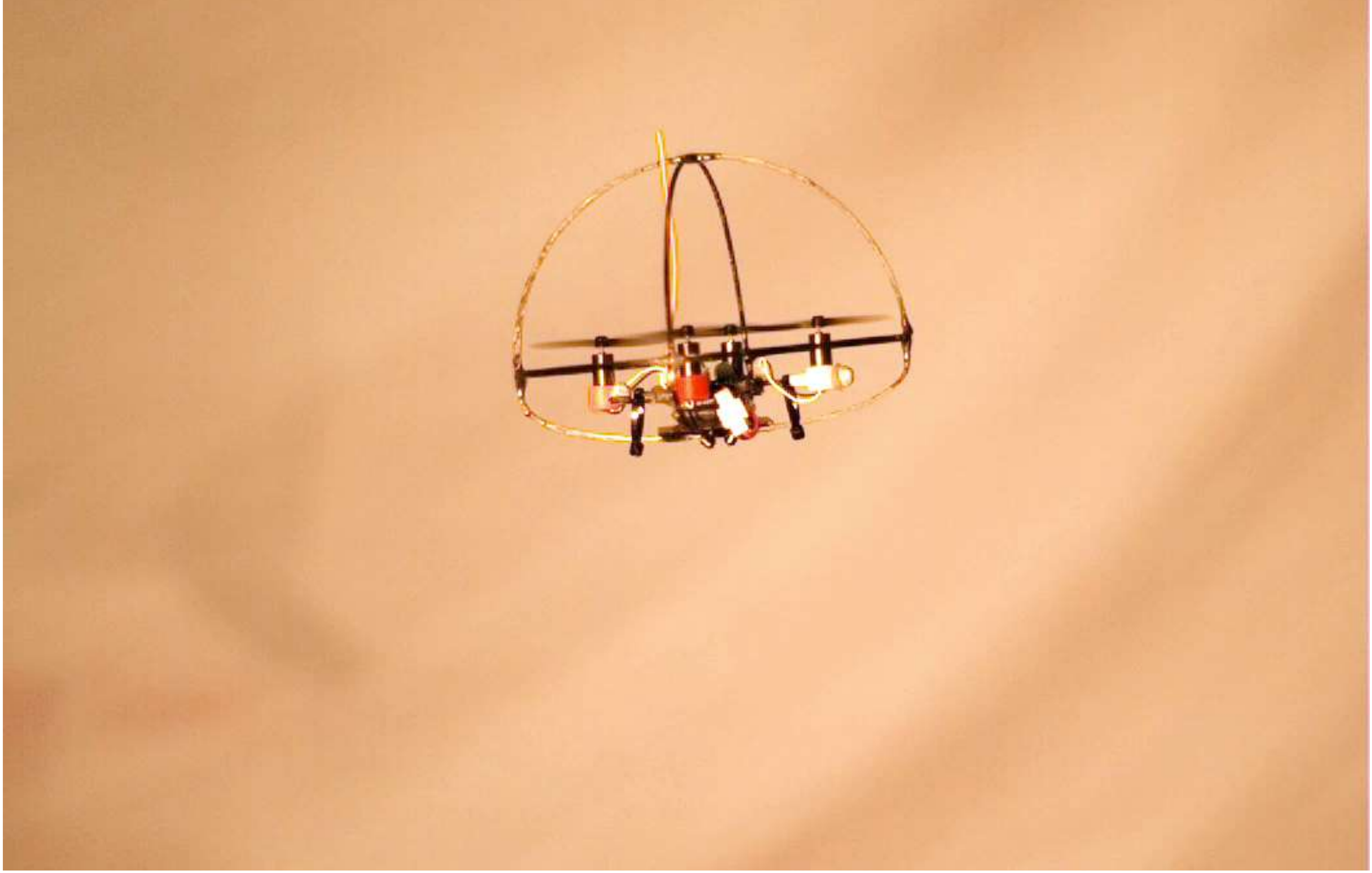


*T. Lee, M. Leoky, and N. H. McClamroch, Geometric tracking control of a quadrotor UAV on SE(3), *IEEE Conference on Decision and Control*, 2010.

Smaller, safer …

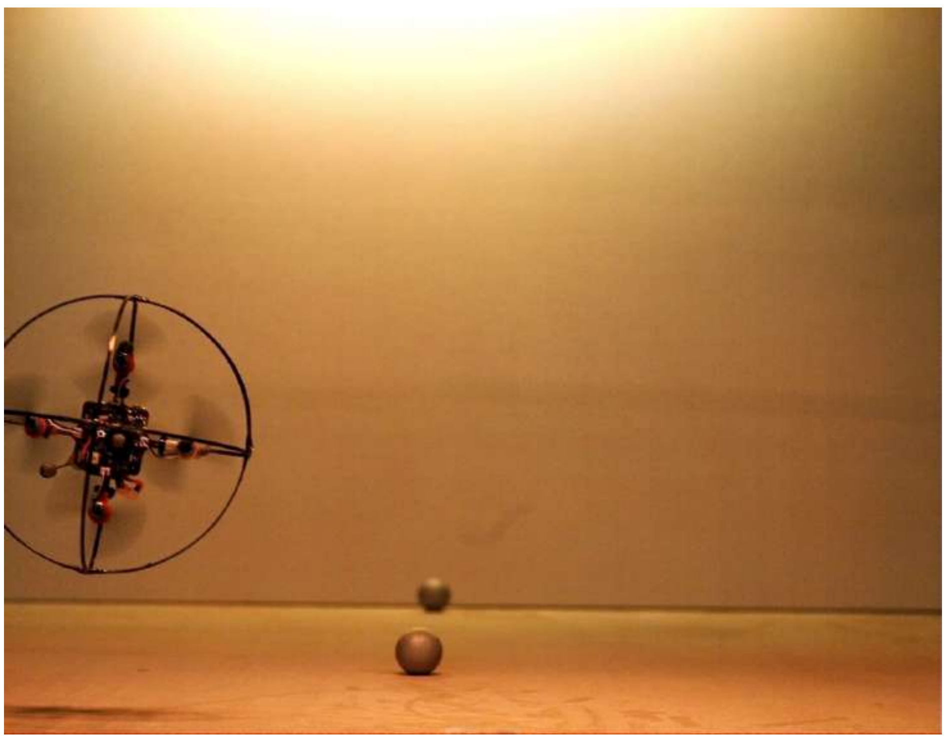
Pico Quadrotor
11 cm
20 g,
6.5 Watts
Max speed 6m/s
Safer
More maneuverable
Y. Mulgaonkar, G. Cross and V. Kumar, "Design of small, safe and robust quadrotor swarms," IEEE International Conference on Robotics and Automation (ICRA), Seattle WA, May 2015.

Recovery from mid air collisions



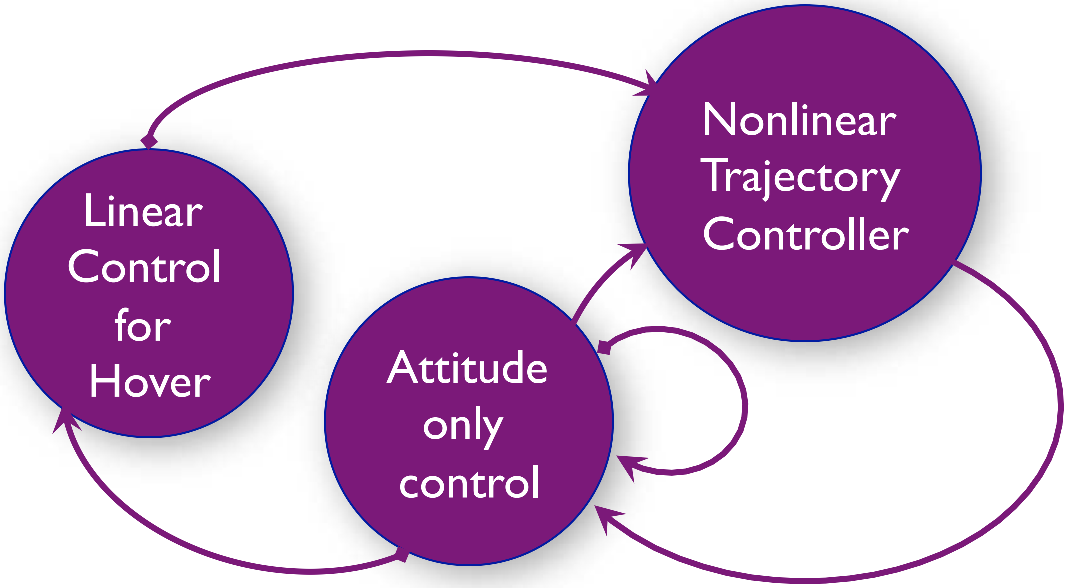
basin of attraction $\sim \dfrac{1}{L^{\frac{5}{2}}}$

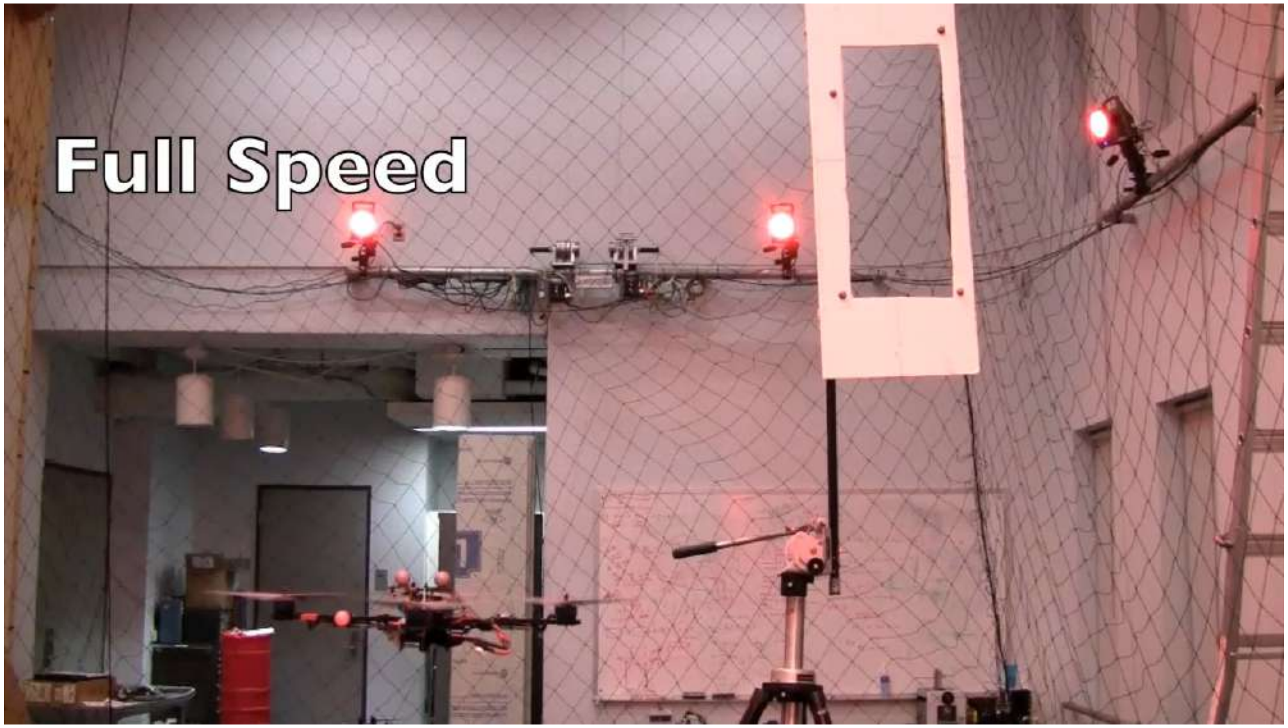
D. Mellinger and V. Kumar, "Minimum Snap Trajectory Generation and Control for Quadrotors," *Proc. IEEE International Conference on Robotics and Automation*. Shanghai, China, May, 2011.



Y. Mulgaonkar, G. Cross and V. Kumar, "Design of small, safe and robust quadrotor swarms," in IEEE International Conference on Robotics and Automation (ICRA), Seattle WA, May 2015.

Sequential Composition





Trajectory Planning

Inputs

$u_1, \mathbf{u}_2$

$$u_1 = \sum_{i=1}^{4} F_i \qquad \mathbf{u}_2 = L \begin{bmatrix} 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ \mu & -\mu & \mu & -\mu \end{bmatrix} \begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{bmatrix}$$

State

$(\mathbf{q}, \dot{\mathbf{q}})$

position

yaw

jerk

snap

yaw

$$\begin{bmatrix} \mathbf{r} \\ \mathbf{v} \\ \mathbf{a} \\ \mathbf{j} \\ \mathbf{s} \\ \psi \\ \dot{\psi} \\ \ddot{\psi} \end{bmatrix} \leftrightarrow \begin{bmatrix} \mathbf{q} \\ \dot{\mathbf{q}} \\ u_1 \\ \dot{u}_1 \\ \ddot{u}_1 \\ \mathbf{u_2} \end{bmatrix}$$

acceleration in the vertical direction

jerk in the vertical direction

$$u_1 = m(a_3 - \mathbf{g}.\mathbf{b}_3)$$

$$\dot{u}_1 = m j_3$$

$$\ddot{u}_1 = m s_3 + u_1(q^2 + rp)$$

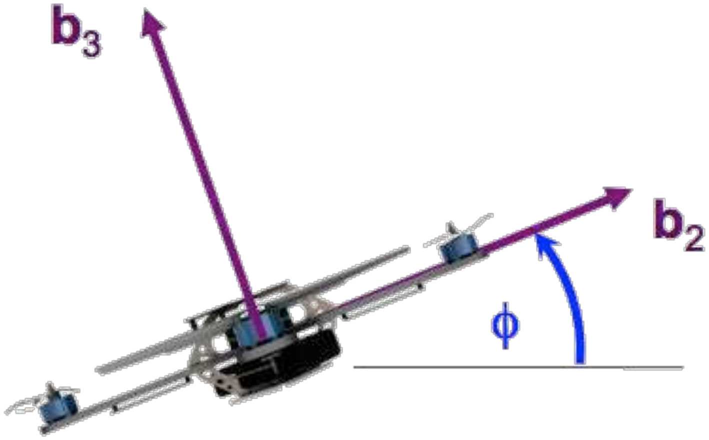
$$p = \frac{-m j_2}{u_1}$$

$$q = \frac{m j_1}{u_1}$$

snap in the vertical direction

$$\omega = \begin{bmatrix} p \\ q \\ r \end{bmatrix}$$

Planar Quadrotor



Inputs

u_1, u_2

$$u_1 = F_2 + F_4$$

$$u_2 = (F_2 - F_4)L$$

$$\mathbf{q} = \begin{bmatrix} y \\ z \\ \phi \end{bmatrix}$$

State

$(\mathbf{q}, \dot{\mathbf{q}})$

Equations of motion

$$\begin{bmatrix} \ddot{y} \\ \ddot{z} \\ \ddot{\phi} \end{bmatrix} = \begin{bmatrix} 0 \\ -g \\ 0 \end{bmatrix} + \begin{bmatrix} -\frac{1}{m}\sin\phi & 0 \\ \frac{1}{m}\cos\phi & 0 \\ 0 & \frac{1}{I_{xx}} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

Differential Flatness

All state variables and the inputs can be written as smooth functions of $\boxed{\textit{flat outputs}}$ and their derivatives (and the other way around)

Planar Quadrotor

$$\begin{bmatrix} \boxed{\begin{matrix} y \\ z \end{matrix}} \\ \dot{y} \\ \dot{z} \\ \ddot{y} \\ \ddot{z} \\ \dddot{y} \\ \dddot{z} \\ y^{(iv)} \\ z^{(iv)} \end{bmatrix} \rightleftarrows \begin{bmatrix} y \\ z \\ \phi \\ \dot{y} \\ \dot{z} \\ \dot{\phi} \\ u_1 \\ \dot{u}_1 \\ \ddot{u}_1 \\ u_2 \end{bmatrix}$$

Planar Quadrotor

The flat outputs and their derivatives can be written as a function of the state, the inputs, and their derivatives

Flat outputs / State / Input

$$\begin{bmatrix} y \\ z \end{bmatrix} \qquad \begin{bmatrix} y \\ z \\ \phi \\ \dot{y} \\ \dot{z} \\ \dot{\phi} \end{bmatrix} \qquad \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$\begin{bmatrix} \ddot{y} \\ \ddot{z} \end{bmatrix} = \begin{bmatrix} -\frac{1}{m}\sin\phi \\ \frac{1}{m}\cos\phi \end{bmatrix} u_1$$

$$\begin{bmatrix} y^{(iii)} \\ z^{(iii)} \end{bmatrix} = \frac{1}{m}\begin{bmatrix} -u_1\dot{\phi}\cos\phi - \dot{u}_1\sin\phi \\ -u_1\dot{\phi}\sin\phi + \dot{u}_1\cos\phi \end{bmatrix}$$

$$\begin{bmatrix} y^{(iv)} \\ z^{(iv)} \end{bmatrix} = \frac{1}{m}\begin{bmatrix} -\sin\phi & -\frac{u_1}{I_{xx}}\cos\phi \\ \cos\phi & -\frac{u_1}{I_{xx}}\sin\phi \end{bmatrix}\begin{bmatrix} \ddot{u}_1 \\ u_2 \end{bmatrix} + \frac{1}{m}\begin{bmatrix} -2\dot{u}_1\dot{\phi}\cos\phi + u_1\dot{\phi}^2\sin\phi \\ -2\dot{u}_1\dot{\phi}\sin\phi - u_1\dot{\phi}^2\cos\phi \end{bmatrix}$$

Planar Quadrotor

The state, the inputs, and their derivatives can be written as a function of the flat outputs and their derivatives

Flat outputs	State	Input
$\begin{bmatrix} y \\ z \end{bmatrix}$	$\begin{bmatrix} y \\ z \\ \phi \\ \dot{y} \\ \dot{z} \\ \dot{\phi} \end{bmatrix}$	$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$

$$u_1 = m\left(\ddot{y}^2 + \ddot{z}^2\right)$$

$$\phi = \operatorname{atan2}\left(-\frac{m\ddot{y}}{u_1}, \frac{m\ddot{z}}{u_1}\right)$$

$$\dot{u}_1 = m(-y^{(iii)} \sin\phi + z^{(iii)} \cos\phi)$$

$$\dot{\phi} = \frac{-m}{u_1}\left(y^{(iii)} \cos\phi + z^{(iii)} \sin\phi\right)$$

$$\ddot{u}_1 = \ldots$$

$$\ddot{\phi} = \ldots$$

$$u_2 = \ldots$$

Differential Flatness

All state variables and the inputs can be written as smooth functions of *flat outputs* and their derivatives (and the other way around)

Planar Quadrotor

$$\begin{bmatrix} \boxed{\begin{matrix} y \\ z \end{matrix}} \\ \dot{y} \\ \dot{z} \\ \ddot{y} \\ \ddot{z} \\ \dddot{y} \\ \dddot{z} \\ y^{(iv)} \\ z^{(iv)} \end{bmatrix} \rightleftarrows \begin{bmatrix} y \\ z \\ \phi \\ \dot{y} \\ \dot{z} \\ \dot{\phi} \\ u_1 \\ \dot{u}_1 \\ \ddot{u}_1 \\ u_2 \end{bmatrix}$$

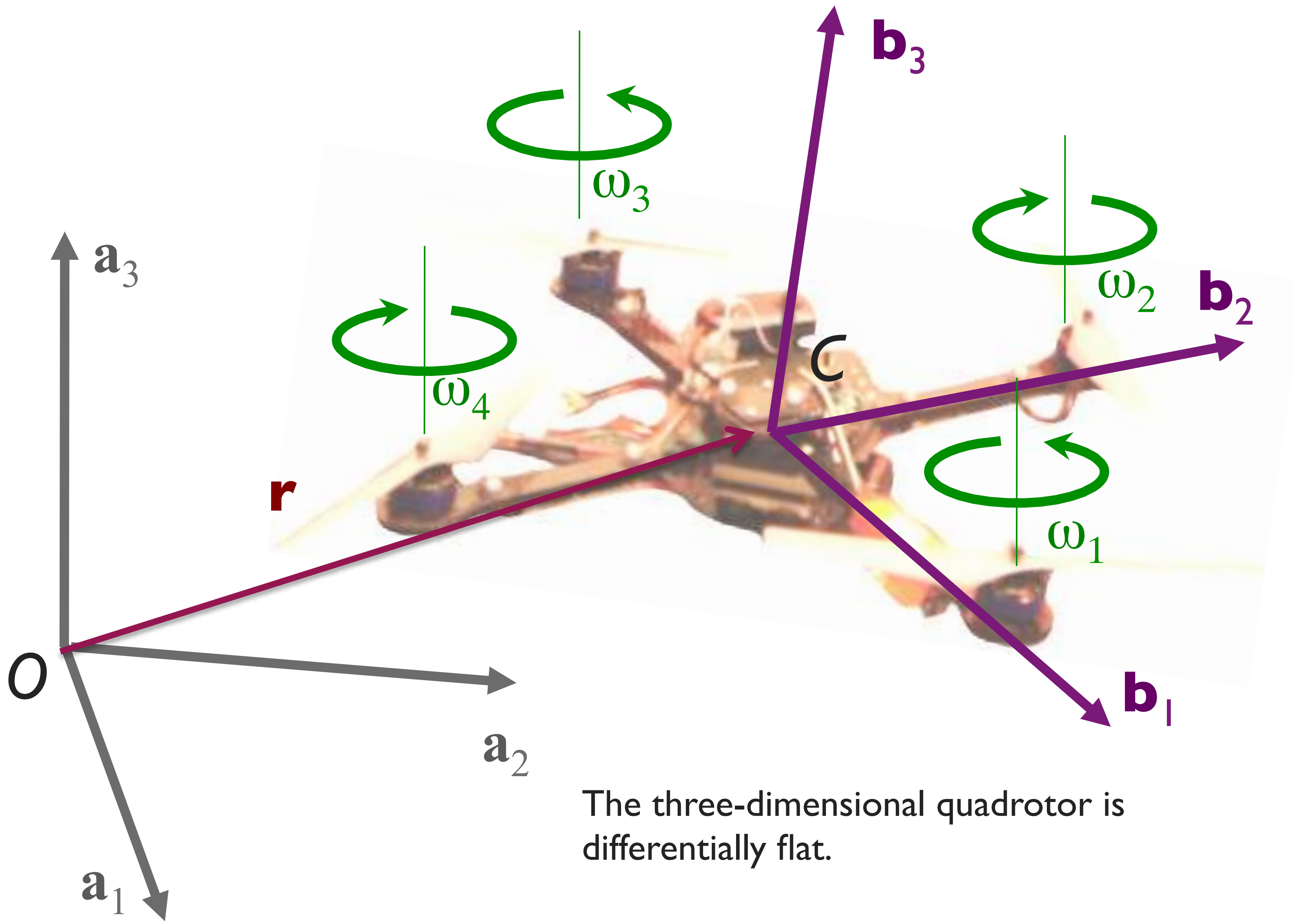
Diffeomorphism

Differential Flatness

All state variables and the inputs can be written as smooth functions of *flat outputs* and their derivatives

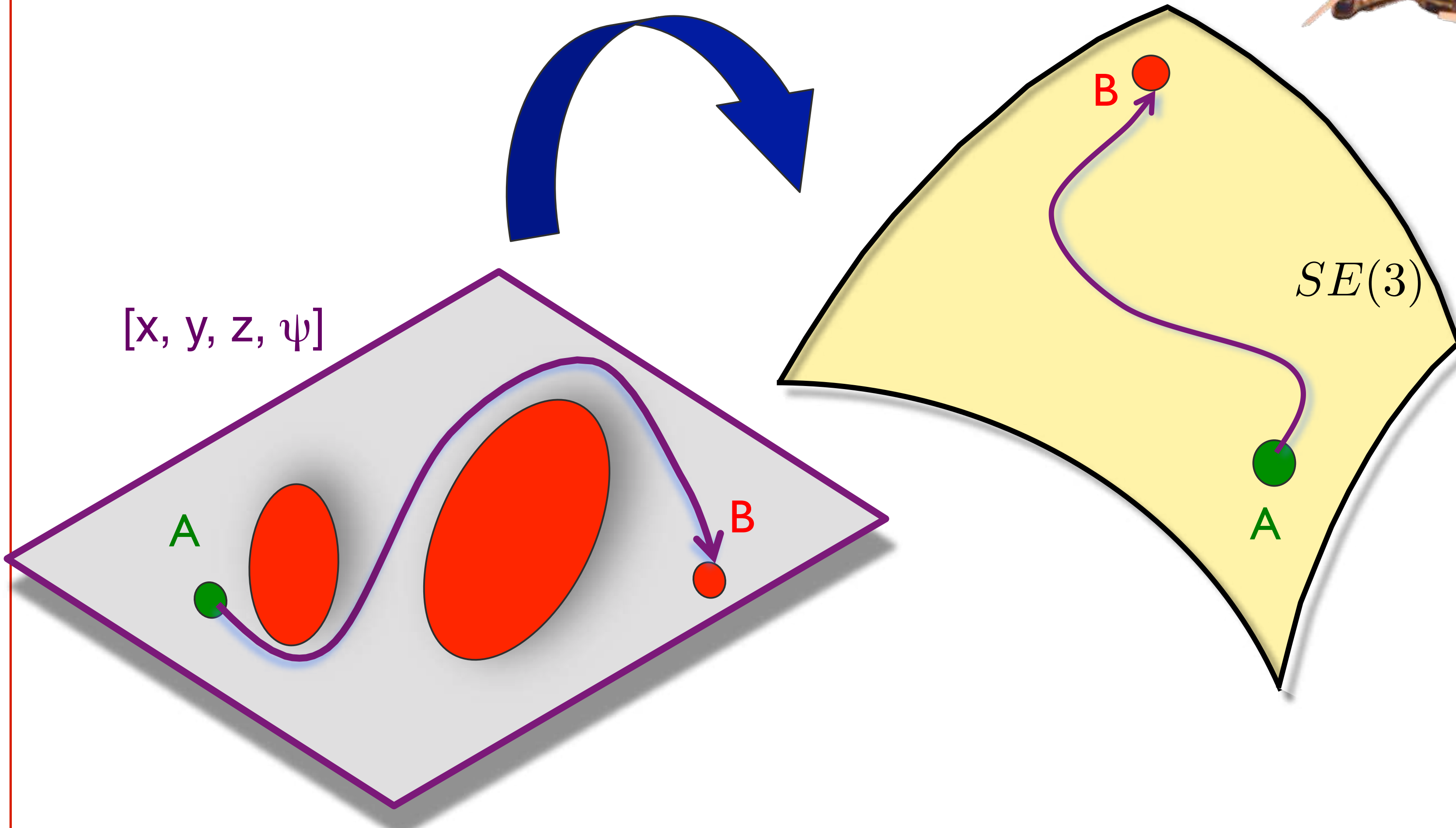
<u>3-D Quadrotor</u>

$$\begin{bmatrix} \boxed{\mathbf{r}} \\ \mathbf{v} \\ \mathbf{a} \\ \mathbf{j} \\ \mathbf{s} \\ \boxed{\psi} \\ \dot{\psi} \\ \ddot{\psi} \end{bmatrix} \leftrightarrow \begin{bmatrix} \mathbf{q} \\ \dot{\mathbf{q}} \\ u_1 \\ \dot{u}_1 \\ \ddot{u}_1 \\ \mathbf{u_2} \end{bmatrix}$$



The three-dimensional quadrotor is differentially flat.

Trajectory Planning



$$\begin{bmatrix} x \\ y \\ z \\ \theta \\ \phi \\ \psi \\ \dot{x} \\ \dot{y} \\ \dot{z} \\ \dot{\theta} \\ \dot{\psi} \\ \dot{\phi} \end{bmatrix}$$

Minimum snap trajectory

$$\min_{\sigma(t)} \int_0^T \alpha \| \ddddot{\mathbf{r}}(t) \|^2 \; + \beta \ddot{\psi}(t)^2 dt$$


Minimum Snap Trajectory