Axis/Angle Representation

Special Orthogonal Matrices

$$\{R \in \mathbb{R}^{3\times 3} \mid R^T R = R R^T = I,\ \det R = 1\}$$

Special Orthogonal group in 3 dimensions

- Coordinates for $SO(3)$
  1 Rotation matrices
  2 Euler angles
  3 Axis angle parameterization
  4 Exponential coordinates
  5 Quaternions

Euler's Theorem

Rotations

Any displacement of a rigid body such that a point on the rigid body, say O, remains fixed, is equivalent to a rotation about a fixed axis through the point O.

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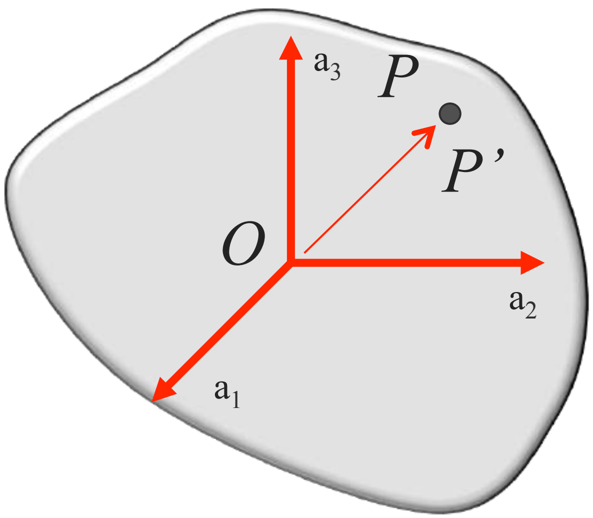
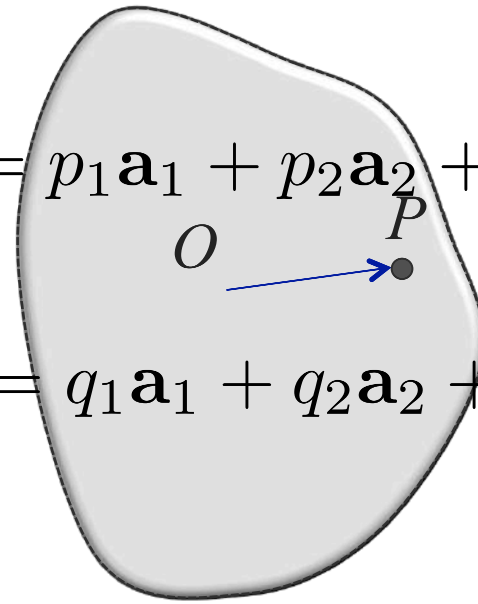
Rotation with O fixed



$$\overrightarrow{OP} = p_1\mathbf{a}_1 + p_2\mathbf{a}_2 + p_3\mathbf{a}_3$$



$$\overrightarrow{OP'} = q_1\mathbf{a}_1 + q_2\mathbf{a}_2 + q_3\mathbf{a}_3$$

$$\underbrace{\begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix}}_{\mathbf{q}} = \begin{bmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{bmatrix} \underbrace{\begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix}}_{\mathbf{p}}$$

$$\mathbf{q} = R\mathbf{p}$$

Proof of Euler's Theorem

$$\mathbf{q} = R\mathbf{p}$$

Is there a point **p** that maps onto itself?

If there were such a point **p** ...

$$\mathbf{p} = R\mathbf{p}$$

Solve eigenvalue problem

$$R\mathbf{p} = \lambda\mathbf{p}$$

Verify λ=1 is an eigenvalue for any *R*

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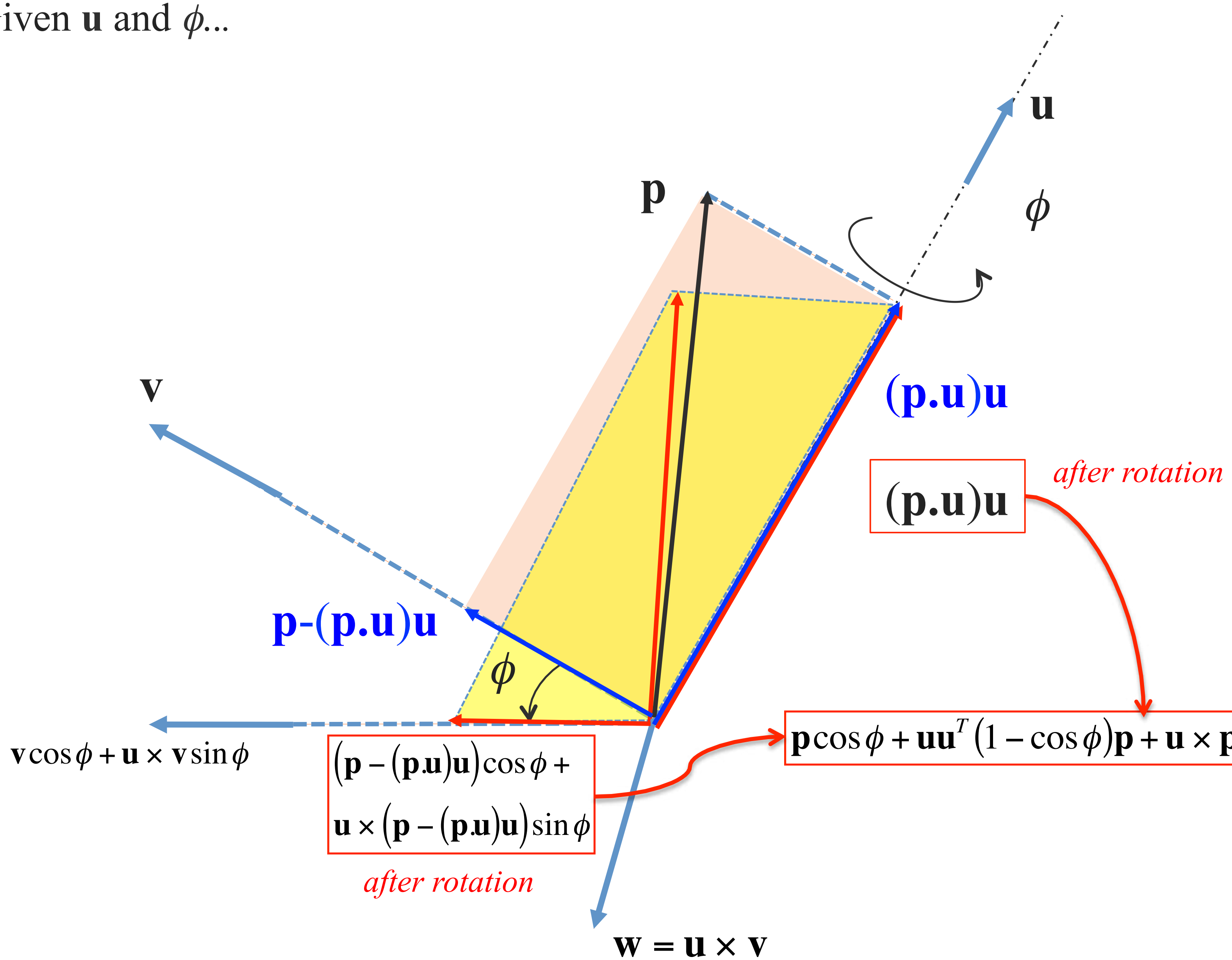
How does one find the rotation matrix for a general axis and angle of rotation?

Note we already know the answer if the axis of rotation is one of the coordinate axes.

Given **u** and ϕ...



1-1 correspondence between any 3×1 vector and a 3×3 skew symmetric matrix

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \times \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ -a_2 b_1 + a_1 b_2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

a

linear operator

For any vector **b**

$$\mathbf{a} \times \mathbf{b} = \mathbf{A}_{3x3}\, \mathbf{b}$$

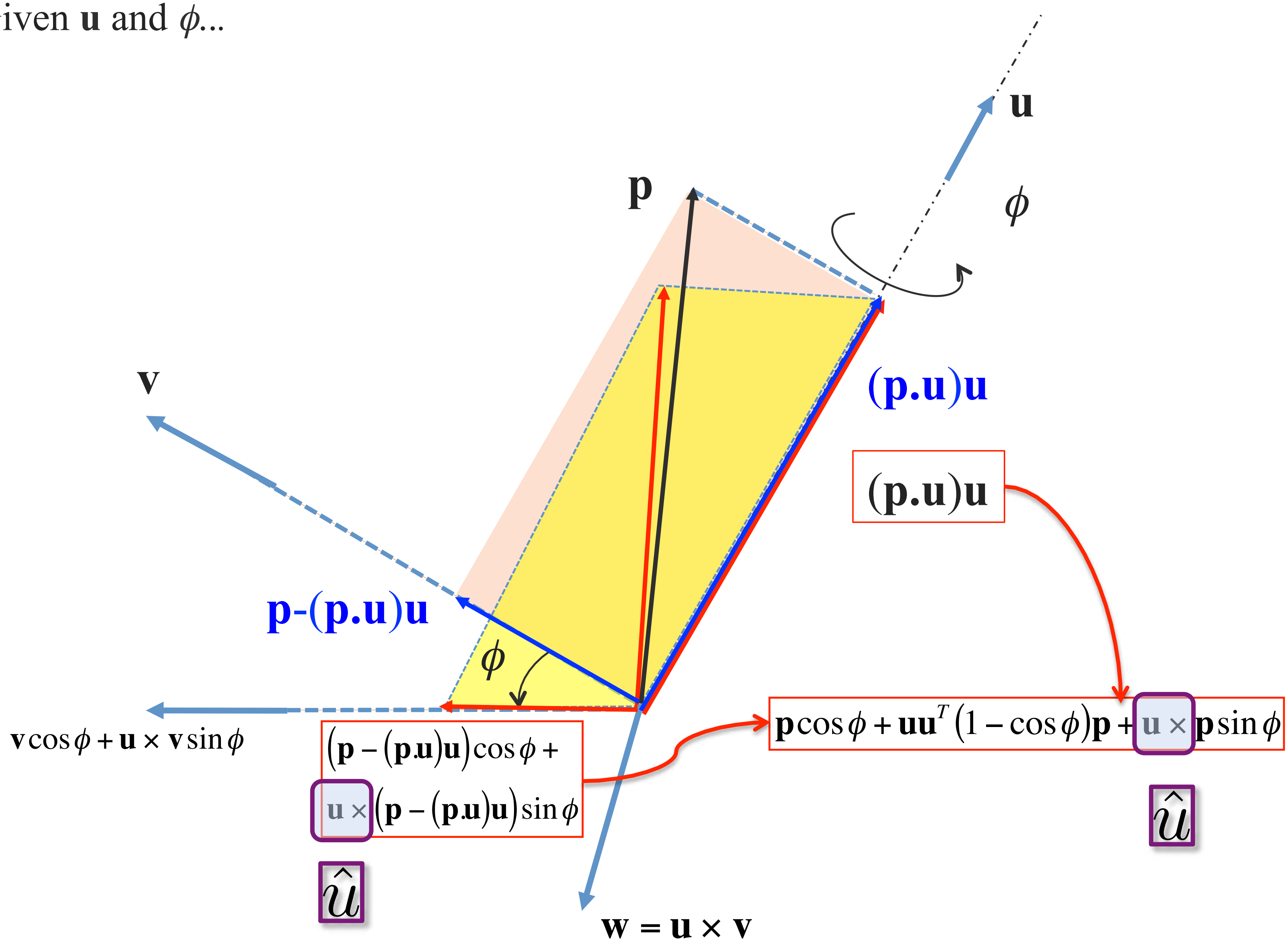
Notation

$\mathbf{A}$

$\mathbf{a}$^

$\hat{\mathbf{a}}$

Given **u** and ϕ...



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Axis/Angle to Rotation Matrix

Rotation of a generic vector p about u through ϕ

$$Rp = p\cos\phi + uu^T(1-\cos\phi)p + \hat{u}p\sin\phi$$

Axis of rotation	u
Rotation angle	ϕ

Rodrigues' formula

$$Rot(u,\phi) = I\cos\phi + uu^T(1-\cos\phi) + \hat{u}\sin\phi$$

Image from wikipedia

1. Set u to be a unit vector along x (or y or z). Verify result is the same as $Rot(x, \phi)$.

2. Is the (axis, angle) to rotation matrix map *onto*? 1-1?

Euler's theorem

$Rot(u,\phi)$ and $Rot(-u, 2\pi-\phi)$?

restrict ϕ to the interval $[0,\pi]$?

Axis/Angle to Rotation Matrix

Rotation of a generic vector p about u through ϕ

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Axis of rotation	u
Rotation angle	ϕ

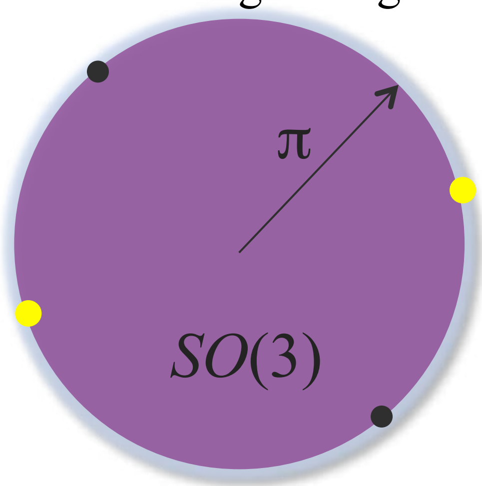
Rodrigues' formula

$$Rot(u,\phi) = I\cos\phi + uu^T(1-\cos\phi) + \hat{u}\sin\phi$$

Lets extract the axis and the angle from the rotation matrix, R

Verify

$$\cos\phi = \frac{\tau - 1}{2} \qquad \hat{u} = \frac{1}{2\sin\phi}(R - R^T)$$ (u, without solving for eigenvector)



1. (axis, angle) to rotation matrix map is many to 1

2. restricting angle to the interval [0,π] makes it 1-1 except for

$\tau = 3 \Rightarrow \phi = 0 \Rightarrow$ *no unique axis*

$\tau = -1 \Rightarrow \phi = \pi \Rightarrow$ *u* or -*u*

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