Skew-Symmetric Matrices and the Hat Operator

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Matrix Transpose

Every matrix has a *transpose*, denoted A^T.

Let A be a n x m matrix and A_{ij} be the element in the i^{th} row and j^{th} column of A.

The transpose is defined by $A_{ij}^T = A_{ji}$, that is, the rows and columns of A are "flipped".

Example 1: Matrix Transpose

Consider:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

The transpose is:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}^T = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$$

Example 2: Matrix Transpose

Consider:

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

2x3 matrix

The transpose is:

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

3x2 matrix

Example 3: Matrix Transpose

Consider:

$$\begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

The transpose is:

$$\begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}^T = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \longleftarrow (A^T)^T = A$$

Example 4: Matrix Transpose

Consider:

$$\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$$

The transpose is:

$$\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}^T = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Matrix Symmetry

A matrix is *symmetric* if:

$$A^T = A$$

A matrix is *skew-symmetric* if:

$$A^T = -A$$

3x3 Skew-Symmetric Matrices

A matrix is *skew-symmetric* if:

$$A^T = -A$$

Consider a 3x3 matrix:

$$A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$$

This matrix is skew-symmetric if:

$$A^T = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix} = - \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} = -A$$

3x3 Skew-Symmetric Matrices

Matching components gives the constraints:

$$
\left.\begin{aligned}
A_{11} &= -A_{11} \\
A_{22} &= -A_{22} \\
A_{33} &= -A_{33}
\end{aligned}\right\} \text{Must be 0}
$$

$$
\begin{aligned}
A_{21} &= -A_{12} \\
A_{13} &= -A_{31} \\
A_{23} &= -A_{32}
\end{aligned}
$$

$$
A = \begin{bmatrix} 0 & A_{12} & A_{13} \\ A_{21} & 0 & A_{23} \\ A_{31} & A_{32} & 0 \end{bmatrix}
$$

3x3 Skew-Symmetric Matrices

Matching components gives the constraints:

$$\begin{aligned} A_{11} &= -A_{11} \\ A_{22} &= -A_{22} \\ A_{33} &= -A_{33} \\ \left.\begin{aligned} A_{21} &= -A_{12} \\ A_{13} &= -A_{31} \\ A_{23} &= -A_{32} \end{aligned}\right\} \end{aligned}$$

$$\Rightarrow \quad A = \begin{bmatrix} 0 & -A_{21} & A_{13} \\ A_{21} & 0 & -A_{32} \\ -A_{13} & A_{32} & 0 \end{bmatrix}$$

A 3x3 skew-symmetric matrix only has 3 independent parameters!

3x3 Skew-Symmetric Matrices

We can concisely represent a skew-symmetric matrix as a 3x1 vector:

$$A = \begin{bmatrix} 0 & -A_{21} & A_{13} \\ A_{21} & 0 & -A_{32} \\ -A_{13} & A_{32} & 0 \end{bmatrix} \Rightarrow a = \begin{bmatrix} A_{32} \\ A_{13} \\ A_{21} \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

We use the *hat operator* to switch between these two representations.

$$\hat{a} = \widehat{\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}} = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}$$

Example: 3x3 Skew-Symmetric Matrices

Consider:

$$\omega = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

The corresponding skew-symmetric matrix is:

$$\hat{\omega} = \begin{bmatrix} 0 & -3 & 2 \\ 3 & 0 & -1 \\ -2 & 1 & 0 \end{bmatrix}$$

Vector Cross Product

The hat operator is also used to denote the cross product between two vectors.

$$\mathbf{u} \times \mathbf{v} = \hat{\mathbf{u}}\mathbf{v}$$

$$= \underbrace{\begin{bmatrix} 0 & -u_3 & u_2 \\ u_3 & 0 & -u_1 \\ -u_2 & u_1 & 0 \end{bmatrix}}_{[u\times]} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

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Representation of Angular Velocities

Recall we defined the angular velocity vectors:

$$\hat{\omega}^b = R^T \dot{R}$$

$$\hat{\omega}^s = \dot{R} R^T$$

$R^T \dot{R}$ and $\dot{R} R^T$ are skew-symmetric.

We are guaranteed to find vectors ω^b, ω^s that satisfy the given definitions of angular velocity.