Quaternions

Quaternion Definition

Quaternion:

$$q = (q_0, q_1, q_2, q_3)$$

This can be interpreted as a constant + vector:

$$q = (q_0, \mathbf{q})$$

Operations with Quaternions

Quaternion addition/subtraction:

$$p \pm q = (p_0 \pm q_0, \mathbf{p} \pm \mathbf{q})$$

Quaternion multiplication:

$$pq = (p_0 q_0 - \mathbf{p}^T \mathbf{q}, p_0 \mathbf{q} + q_0 \mathbf{p} + \mathbf{p} \times \mathbf{q})$$

Quaternion inverse:

$$q^{-1} = (q_0, -\mathbf{q})$$

Axis-Angle Representation to Quaternion

Quaternions can be used to represent rigid-body rotations.

Recall the axis-angle representation of rotations:

Angle of rotation: ϕ

Axis of rotation: $\mathbf{u}$

The equivalent quaternion is:

$$q = (\cos(\frac{\phi}{2}), u_1 \sin(\frac{\phi}{2}), u_2 \sin(\frac{\phi}{2}), u_3 \sin(\frac{\phi}{2}))$$

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Quaternions to Axis-Angle Representation

Given a quaternion:

$$q = (q_0, q_1, q_2, q_3)$$

The equivalent axis-angle representation is:

Angle of rotation: $2\cos^{-1}(q_0)$

Axis of rotation: $\mathbf{u}_2 = \begin{bmatrix} \frac{q_1}{\sqrt{1-q_0^2}} \\ \frac{q_2}{\sqrt{1-q_0^2}} \\ \frac{q_3}{\sqrt{1-q_0^2}} \end{bmatrix}$

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Vector Rotation with Quaternions

To rotate a vector **p** in R^3 by the quaternion *q*:

1. Define quaternion:

$$p = (0, \mathbf{p})$$

2. The result after rotation is:

$$p' = qpq^{-1} = (0, \mathbf{p}')$$

We can easily compose two rotations:

$$q = q_2 q_1$$

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Properties of Quaternions

- $q = (q_0, q_1, q_2, q_3)$ and $-q = (-q_0, -q_1, -q_2, -q_3)$

  represent the same rotation.

- Compact representation of rotations, with only 4 parameters.
- No singularities
- Quaternion product is more numerically stable than matrix multiplication.