Minimum Velocity Trajectories

Minimum Velocity Trajectory

Why is the minimum velocity curve also the shortest distance curve?

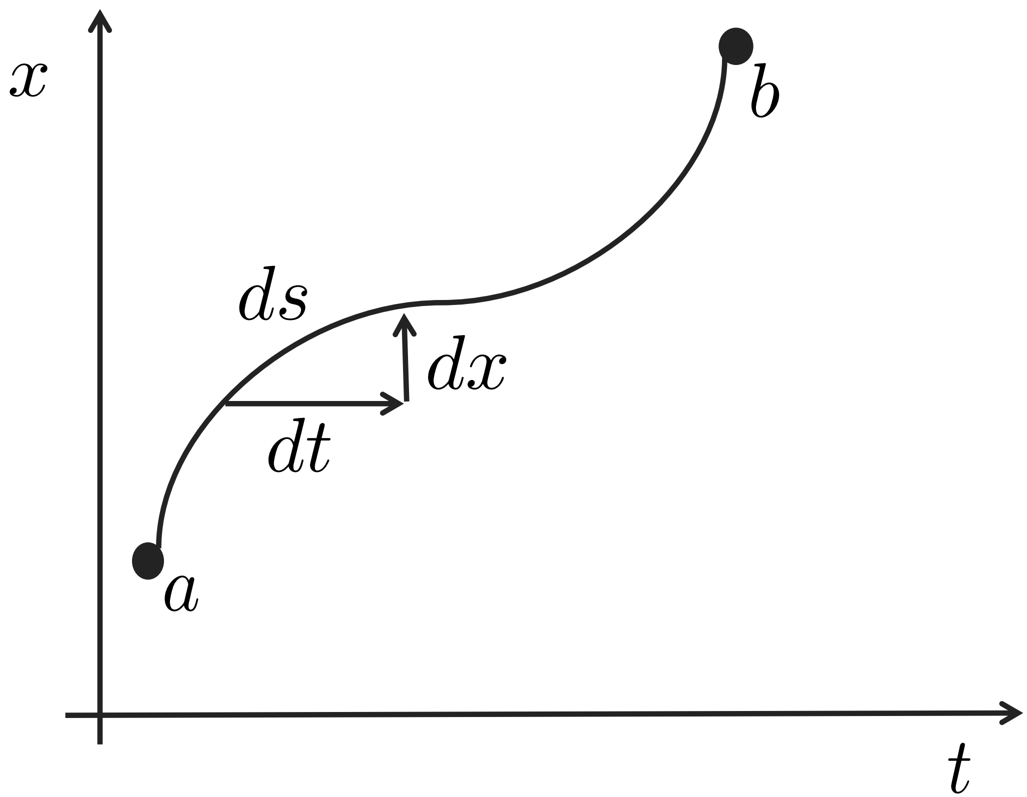
To get the minimum velocity trajectory, we solved:

$$x^\star(t) = \underset{x(t)}{\operatorname{argmin}} \int_0^T \dot{x}^2 dt$$

From the Euler-Lagrange equations, the solution is:

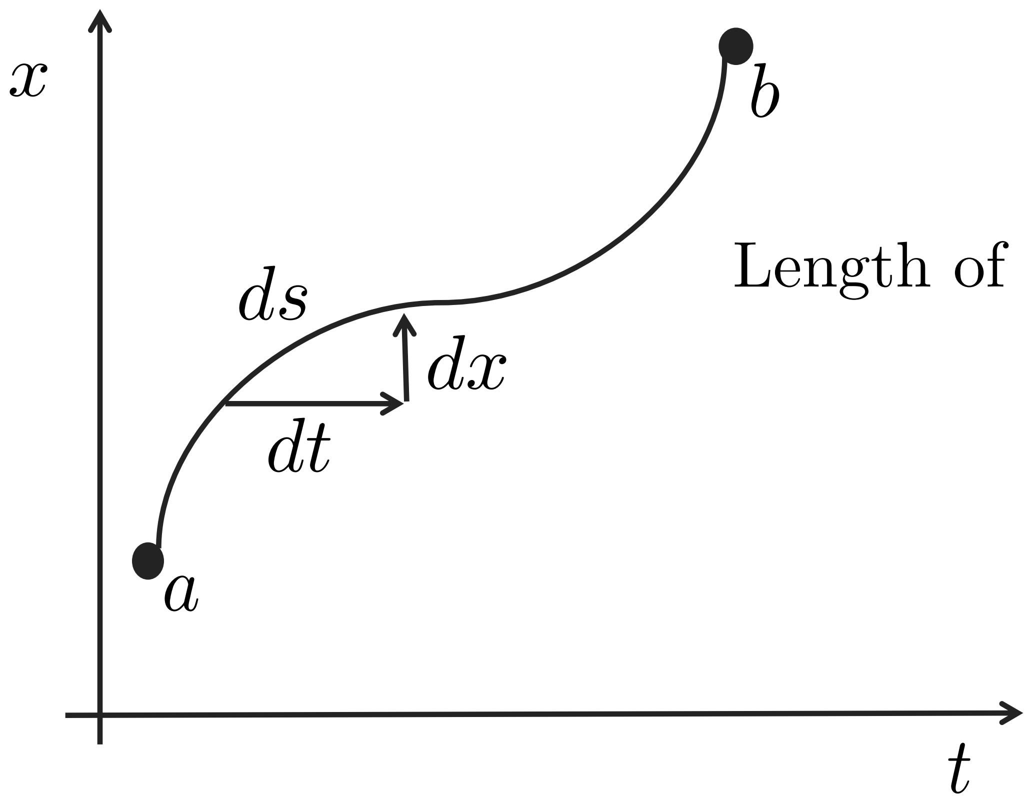
$$x(t) = c_1 t + c_0$$

Minimum Distance Trajectory



$$
\begin{aligned}
ds &= \sqrt{dt^2 + dx^2} \\
&= \sqrt{1 + \left(\frac{dx}{dt}\right)^2}\, dt \\
&= \sqrt{1 + \dot{x}^2}\, dt
\end{aligned}
$$

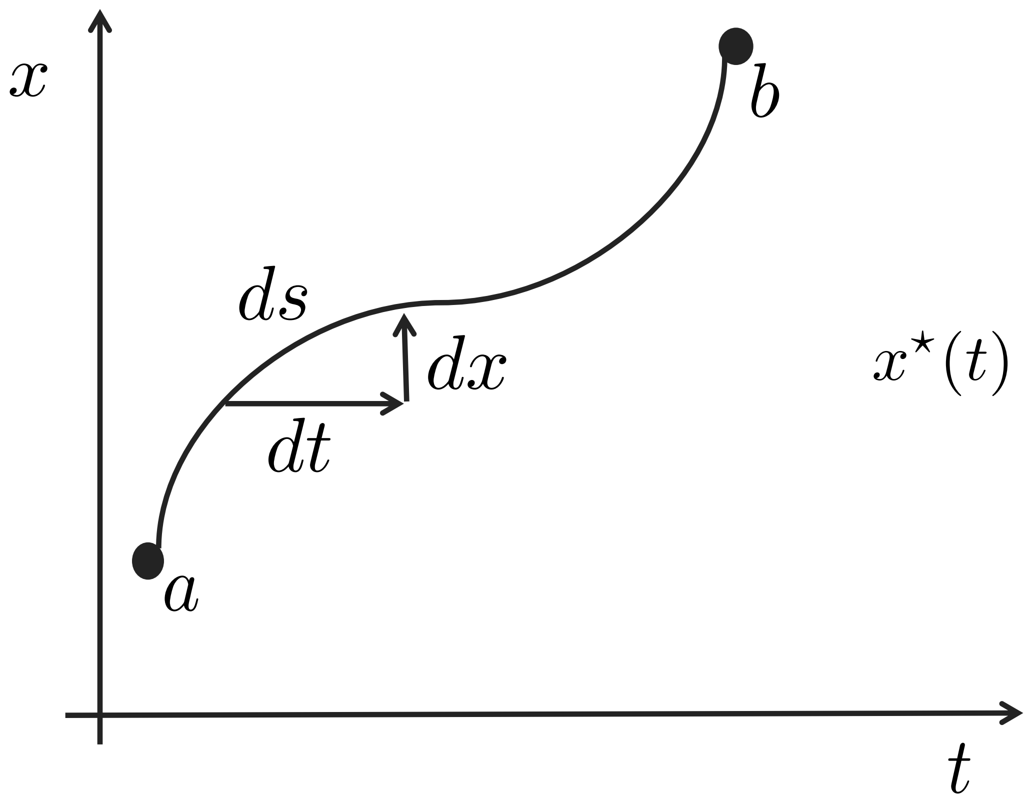
Minimum Distance Trajectory



$$\text{Length of curve} = \int ds$$

$$= \int_0^T \sqrt{1 + \dot{x}^2} dt$$

Minimum Distance Trajectory



$$x^\star(t) = \underset{x(t)}{\operatorname{argmin}} \int_0^T \sqrt{1+\dot{x}^2}dt$$

Minimum Distance Trajectory

$$x^\star(t) = \underset{x(t)}{\operatorname{argmin}} \int_0^T \sqrt{1 + \dot{x}^2} dt$$

$$\mathcal{L}(\dot{x}, x, t) = \sqrt{1 + \dot{x}^2}$$

Euler-Lagrange equation:

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) - \frac{\partial \mathcal{L}}{\partial x} = 0$$

Minimum Distance Trajectory

Euler-Lagrange equation:

$$\boxed{\frac{d}{dt}}\boxed{\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right)} - \boxed{\frac{\partial \mathcal{L}}{\partial x}} = 0$$

Cost-function: $\mathcal{L}(\dot{x}, x, t) = \sqrt{1 + \dot{x}^2}$

Euler-Lagrange terms:

$$\left(\frac{\partial \mathcal{L}}{\partial x}\right) = 0$$

← No x appears in $\mathcal{L}$

$$\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) = \frac{\dot{x}}{\sqrt{1 + \dot{x}^2}}$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) = \frac{d}{dt}\left(\frac{\dot{x}}{\sqrt{1 + \dot{x}^2}}\right)$$

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Minimum Distance Trajectory

Euler-Lagrange equation:

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) - \frac{\partial \mathcal{L}}{\partial x} = 0$$

Euler-Lagrange terms: $\frac{d}{dt}\left(\frac{\dot{x}}{\sqrt{1+\dot{x}^2}}\right) = 0$

Integrate to get velocity: $\frac{\dot{x}}{\sqrt{1+\dot{x}^2}} = K \longrightarrow \dot{x} = \sqrt{\frac{K^2}{1-K^2}} = c_1$

Integrate to get position: $x(t) = c_1 t + c_0 \longleftarrow$ Same as minimum velocity solution