Eigenvalues and Eigenvectors of Matrices

Determinant

A determinant is a scalar property of square matrices, denoted $\det(A)$ or $|A|$.

- Think of rows of an n x n matrix as n vectors in R^n.
- The determinant represents the "space contained" by these vectors.

In this course, we will be working with 2x2 or 3x3 matrices.

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Determinant (2x2 Matrix)

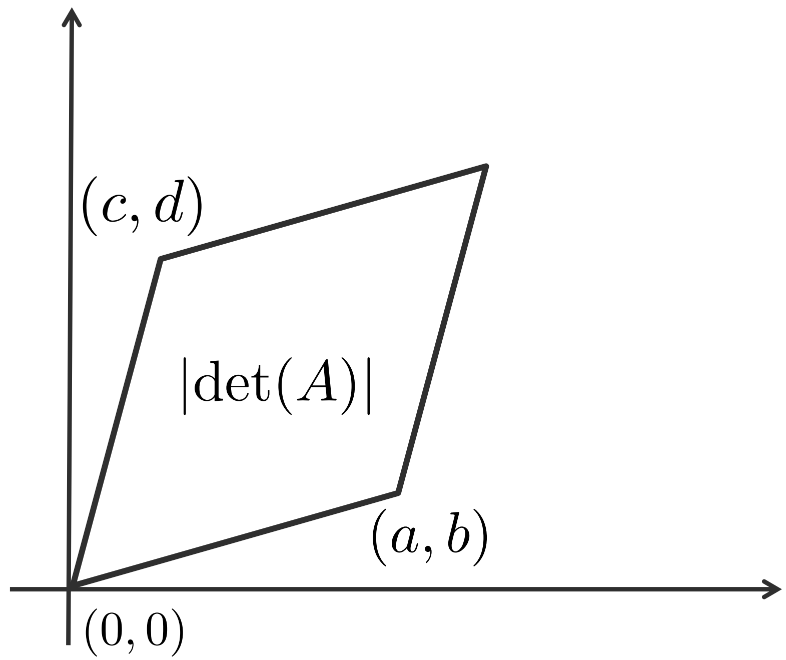
Consider:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Determinant:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

Area of parallelogram defined by the rows.



Example: Determinant (2x2 Matrix)

Consider:

$$\begin{bmatrix} 1 & 3 \\ 4 & 1 \end{bmatrix}$$

Determinant:

$$\begin{vmatrix} 1 & 3 \\ 4 & 1 \end{vmatrix} = (1)(1) - (3)(4) = -11$$

Determinant (3x3 Matrix)

Consider:

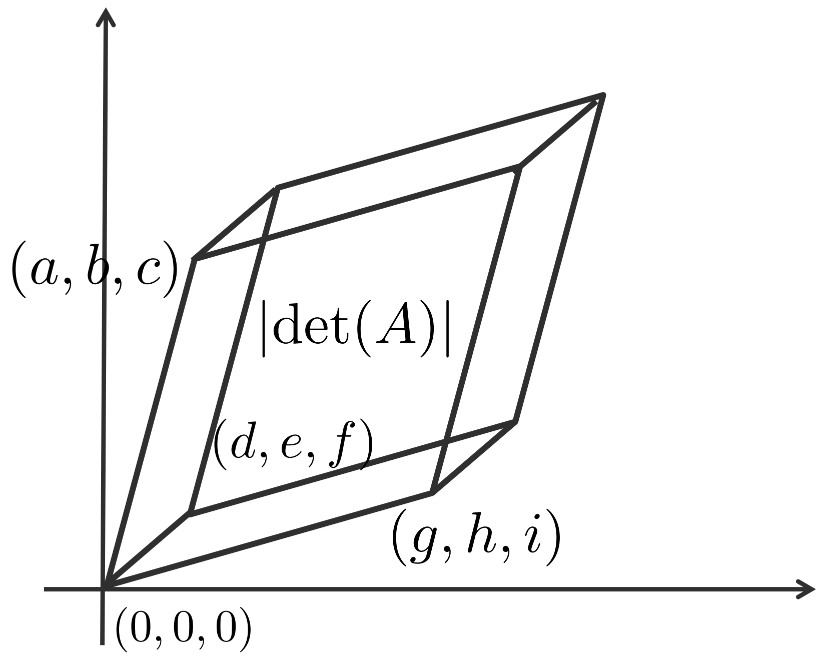
$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

Determinant:

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = aei + bfg + cdh$$

$$- ceg - bdi - afh$$

Volume of parallelepiped defined by the rows.



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Example: Determinant (3x3 Matrix)

Consider:

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -4 & 1 \\ 0 & 3 & -1 \end{bmatrix}$$

Determinant:

$$\begin{aligned} &\begin{vmatrix} 1 & 2 & 3 \\ 0 & -4 & 1 \\ 0 & 3 & -1 \end{vmatrix} \\ &= (1)(-4)(-1) + (2)(1)(0) + (3)(0)(3) \\ &\quad - (3)(-4)(0) - (2)(0)(-1) - (1)(1)(3) \\ &= 1 \end{aligned}$$

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Eigenvalues and Eigenvectors

A matrix is a transformation.

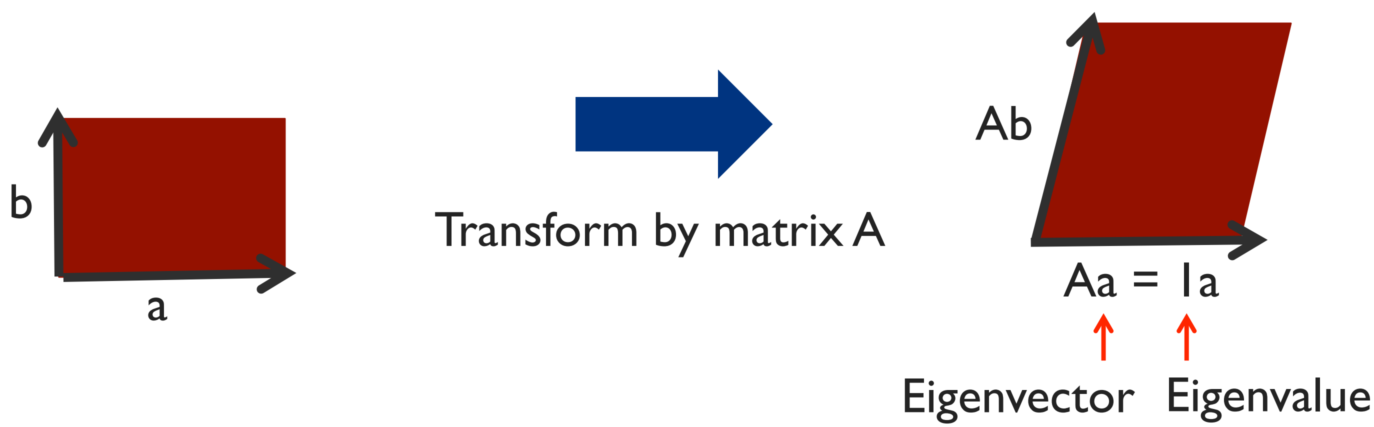
$$y = Ax$$

Eigenvalues and Eigenvectors

Eigenvectors are vectors associated by a square matrix that do not change in direction when multiplied by the matrix.

Eigenvalues are scalar values representing how much each eigenvector changes in length.

$$A\mathbf{v} = \lambda\mathbf{v}$$



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Finding Eigenvalues

1. Calculate:

$$\det(A - \lambda \mathbf{I})$$

2. Find solutions to:

$$\det(A - \lambda \mathbf{I}) = 0$$

There will be n eigenvalues for an n x n matrix, but not all of them have to be distinct or real values.

Example: Finding Eigenvalues

Consider:

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

1. Calculate $\det(A - \lambda \mathbf{I})$

$$\begin{aligned}
\det(A - \lambda \mathbf{I}) &= \det \left(\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \\
&= \det \left(\begin{bmatrix} 2 - \lambda & 1 \\ 1 & 2 - \lambda \end{bmatrix} \right) \\
&= (2 - \lambda)^2 - 1 \\
&= \lambda^2 - 4\lambda + 3
\end{aligned}$$

Example: Finding Eigenvalues

Consider:

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

2. Find solutions to $\det(A - \lambda \mathbf{I}) = 0$

$$\lambda^2 - 4\lambda + 3 = 0$$

$$(\lambda - 1)(\lambda - 3) = 0$$

$$\lambda_1 = 1, \lambda_2 = 3$$

2 eigenvalues for a 2 x 2 matrix

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Finding Eigenvectors

1. For each eigenvalue, solve the equation:

$$A\mathbf{v} = \lambda\mathbf{v}$$

or:

$$(A - \lambda\mathbf{I})\,\mathbf{v} = \mathbf{0}$$

Notice that if $\mathbf{v}$ is an eigenvector, then $\alpha\mathbf{v}$ is also an eigenvector, where α is any scalar.

Thus, we typically think about **linearly independent** eigenvectors.

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Eigenvectors

n vectors $\{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_n\}$ are **linearly independent (LI)** if the only solution to the equation:

$$a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + ... + a_n\mathbf{v}_n = 0$$

is $a_1 = a_2 = ... = a_n = 0$.

There will be at least one LI eigenvector for each eigenvalue.

If eigenvalues are repeated, there might be multiple LI eigenvectors for that eigenvalue.

Example: Finding Eigenvectors

Consider:

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

1. For $\lambda_1 = 1$:

$$\begin{aligned} &(A - \lambda_1 \mathbf{I})\, \mathbf{v}_1 \\ &= \left(\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} - 1 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \mathbf{v}_1 \\ &= \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \mathbf{v}_1 \\ &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{aligned}$$

$$\Rightarrow$$

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\mathbf{v}_1 = \alpha \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

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