Minimum Jerk Trajectory

Design a trajectory $x(t)$ such that $x(0) = a$, $x(T) = b$

$$x^\star(t) = \underset{x(t)}{\operatorname{argmin}} \int_0^T \mathcal{L}\left(\dddot{x}, \ddot{x}, \dot{x}, x, t\right) dt$$

$$\mathcal{L} = \left(\dddot{x}\right)^2$$

Euler-Lagrange:

$$\underbrace{\boxed{\frac{\partial \mathcal{L}}{\partial x} - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) + \frac{d^2}{dt^2}\left(\frac{\partial \mathcal{L}}{\partial \ddot{x}}\right)}}_{0} \boxed{- \frac{d^3}{dt^3}\left(\frac{\partial \mathcal{L}}{\partial x^{(3)}}\right)} = 0$$

$$x^{(6)} = 0$$

$$x = c_5 t^5 + c_4 t^4 + c_3 t^3 + c_2 t^2 + c_1 t + c_0$$

Solving for Coefficients

$$x = c_5 t^5 + c_4 t^4 + c_3 t^3 + c_2 t^2 + c_1 t + c_0$$

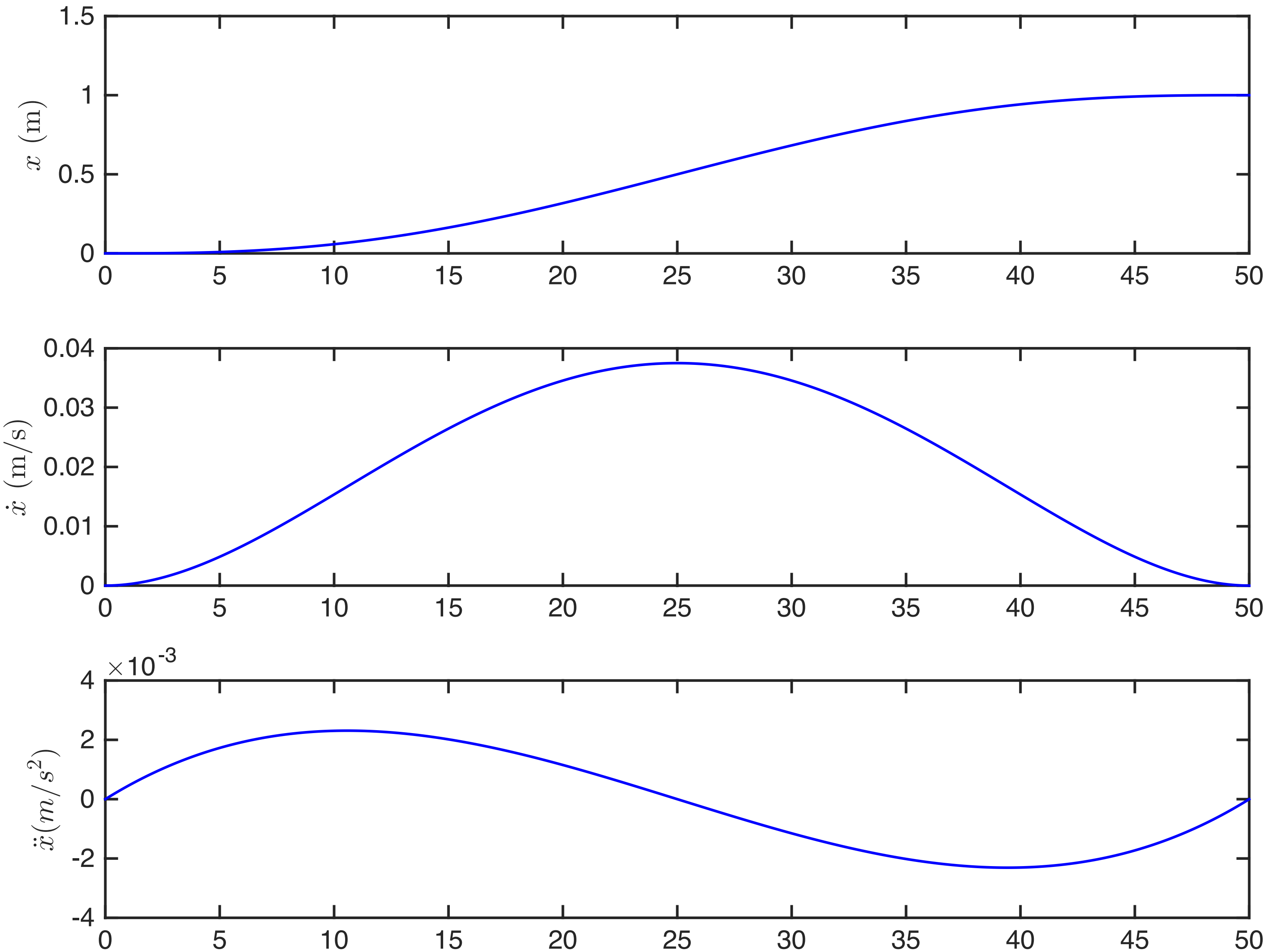
Boundary conditions:

	Position	Velocity	Acceleration
$t = 0$	a	0	0
$t = T$	b	0	0

Solve:

$$\begin{bmatrix} a \\ b \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ T^5 & T^4 & T^3 & T^2 & T & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 5T^4 & 4T^3 & 3T^2 & 2T & 1 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 20T^3 & 12T^2 & 6T & 2 & 0 & 0 \end{bmatrix} \begin{bmatrix} c_5 \\ c_4 \\ c_3 \\ c_2 \\ c_1 \\ c_0 \end{bmatrix}$$

Minimum Jerk Trajectory



a=0, b=1, T=50

Extensions to multiple dimensions

$$(x^\star(t), y^\star(t)) = \arg \min_{x(t), y(t)} \int_0^T \mathcal{L}\left(\dot{x}, \dot{y}, x, y, t\right) dt$$

Euler Lagrange Equation

Necessary condition satisfied by the "optimal" function

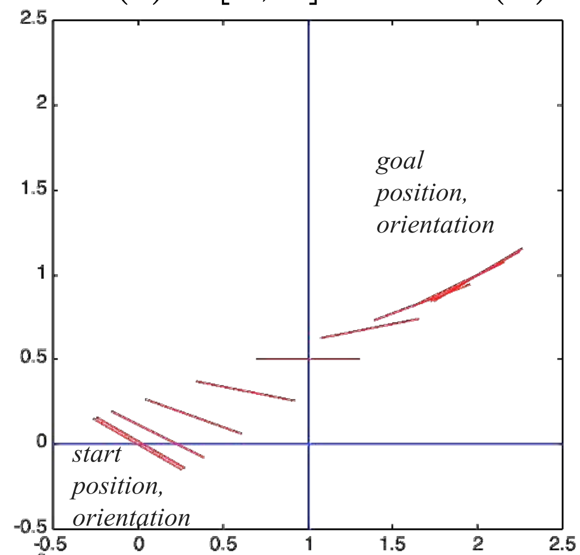
$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) - \frac{\partial \mathcal{L}}{\partial x} = 0$$
$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{y}}\right) - \frac{\partial \mathcal{L}}{\partial y} = 0$$

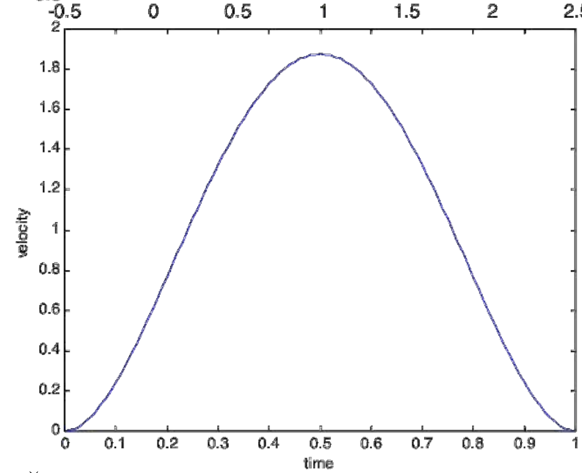
Minimum Jerk for Planar Motions

Minimum-jerk trajectory in (x, y, θ)

$$\min_{x(t),y(t),\theta(t)} \int_0^1 \left(\dddot{x}^2 + \dddot{y}^2 + \dddot{\theta}^2\right) dt$$

$$\mathbf{x}^\star(t) : [0,1] \rightarrow SE(2)$$





Human two-handed manipulation tasks

- Noise in the neural control signal increases with size of the control signal
- Rate of change of muscle fiber lengths is critical in relaxed, voluntary motions

G.J. Garvin, M. Žefran, E.A. Henis, V. Kumar, Two-arm trajectory planning in a manipulation task, *Biological Cybernetics,* January 1997, Volume 76, Issue 1, pp 53-62

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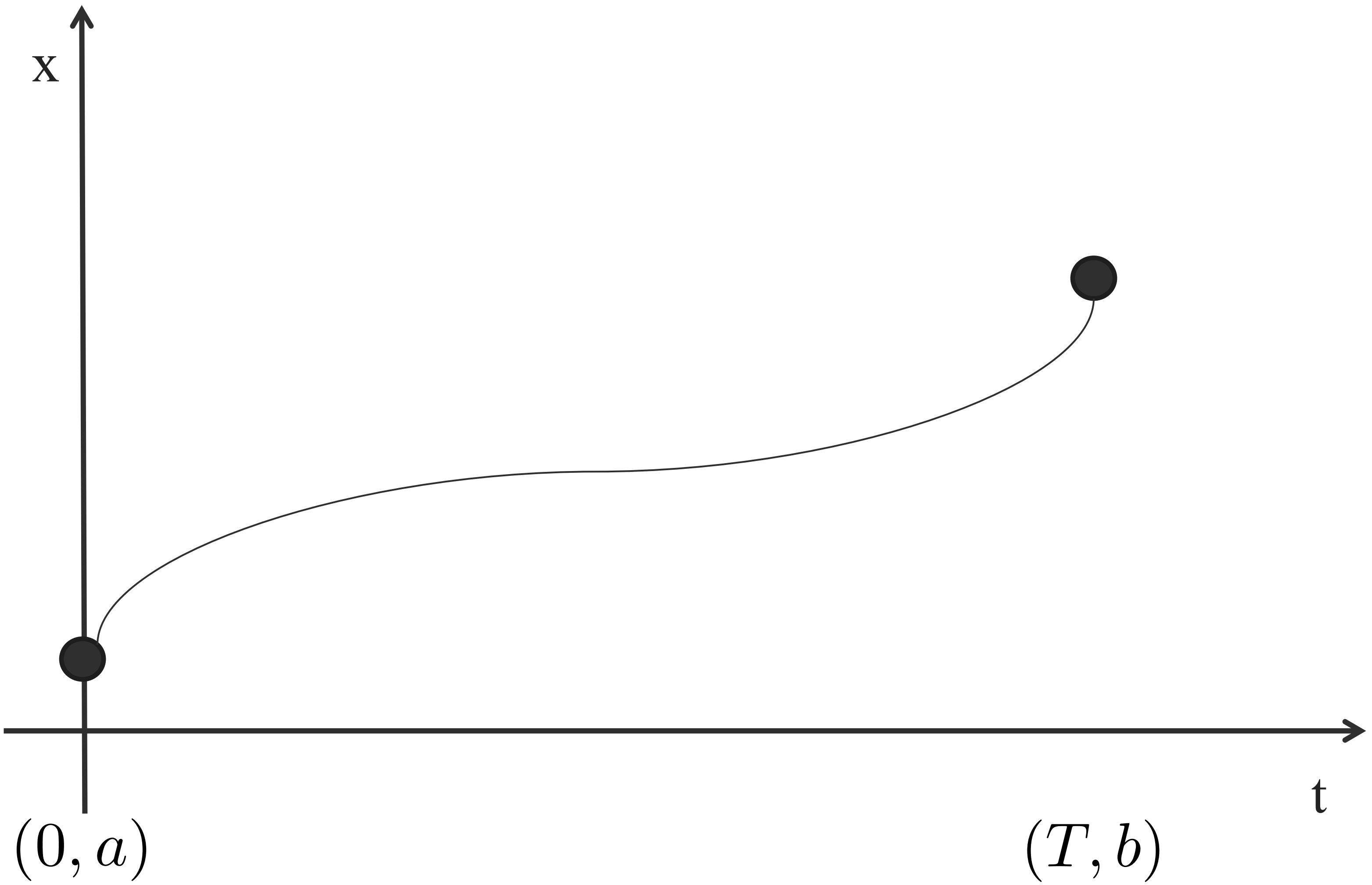
Waypoint Navigation

Smooth 1D Trajectories

Design a trajectory $x(t)$ such that $x(0) = a$, $x(T) = b$

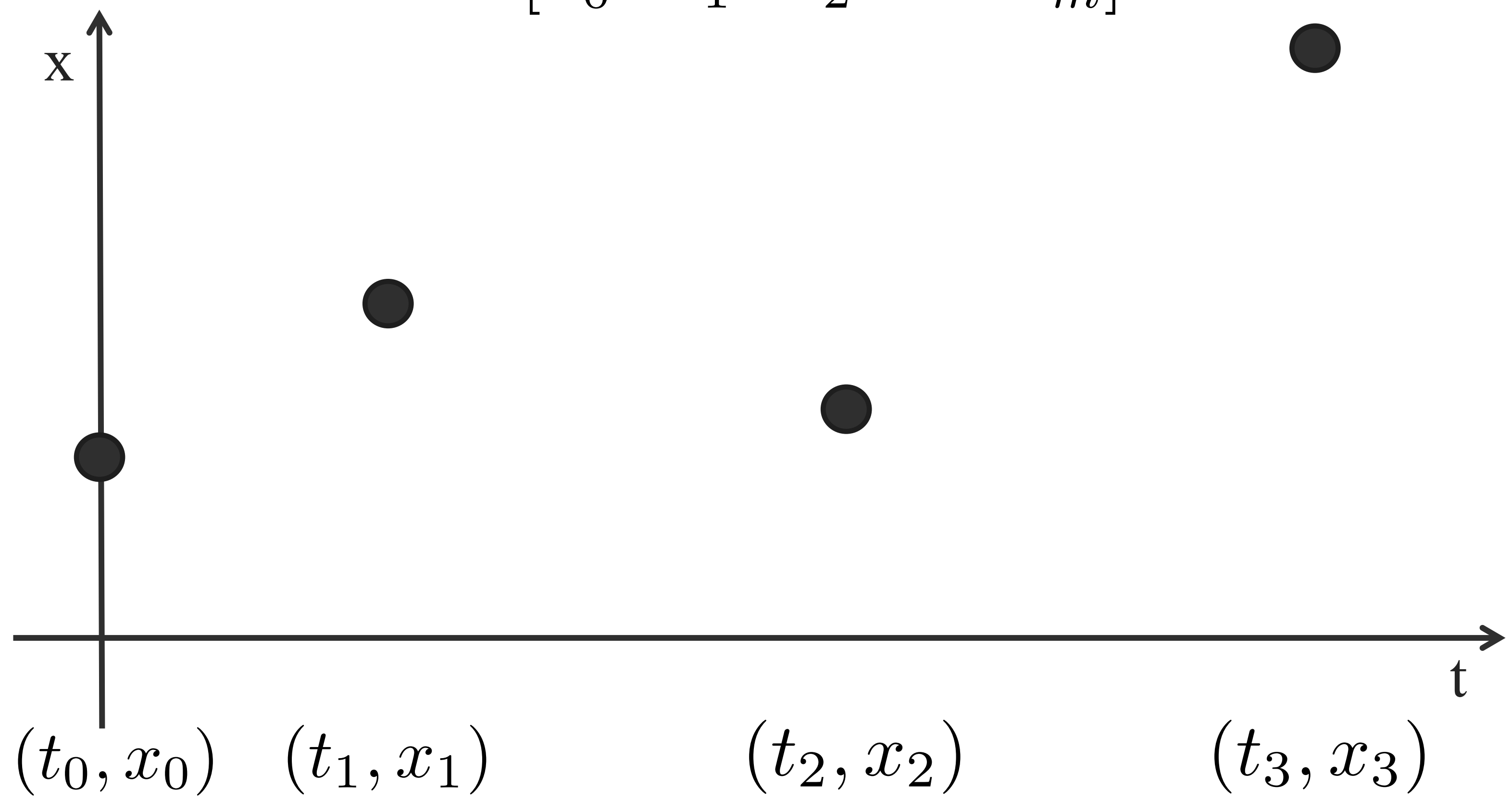


Multi-Segment 1D Trajectories

Design a trajectory x(t) such that:

$$t = [t_0 \quad t_1 \quad t_2 \quad \dots \quad t_m]^T$$

$$x = [x_0 \quad x_1 \quad x_2 \quad \dots \quad x_m]^T$$



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Multi-Segment 1D Trajectories

Design a trajectory x(t) such that:

$$t = [t_0 \quad t_1 \quad t_2 \quad \ldots \quad t_m]^T$$

$$x = [x_0 \quad x_1 \quad x_2 \quad \ldots \quad x_m]^T$$

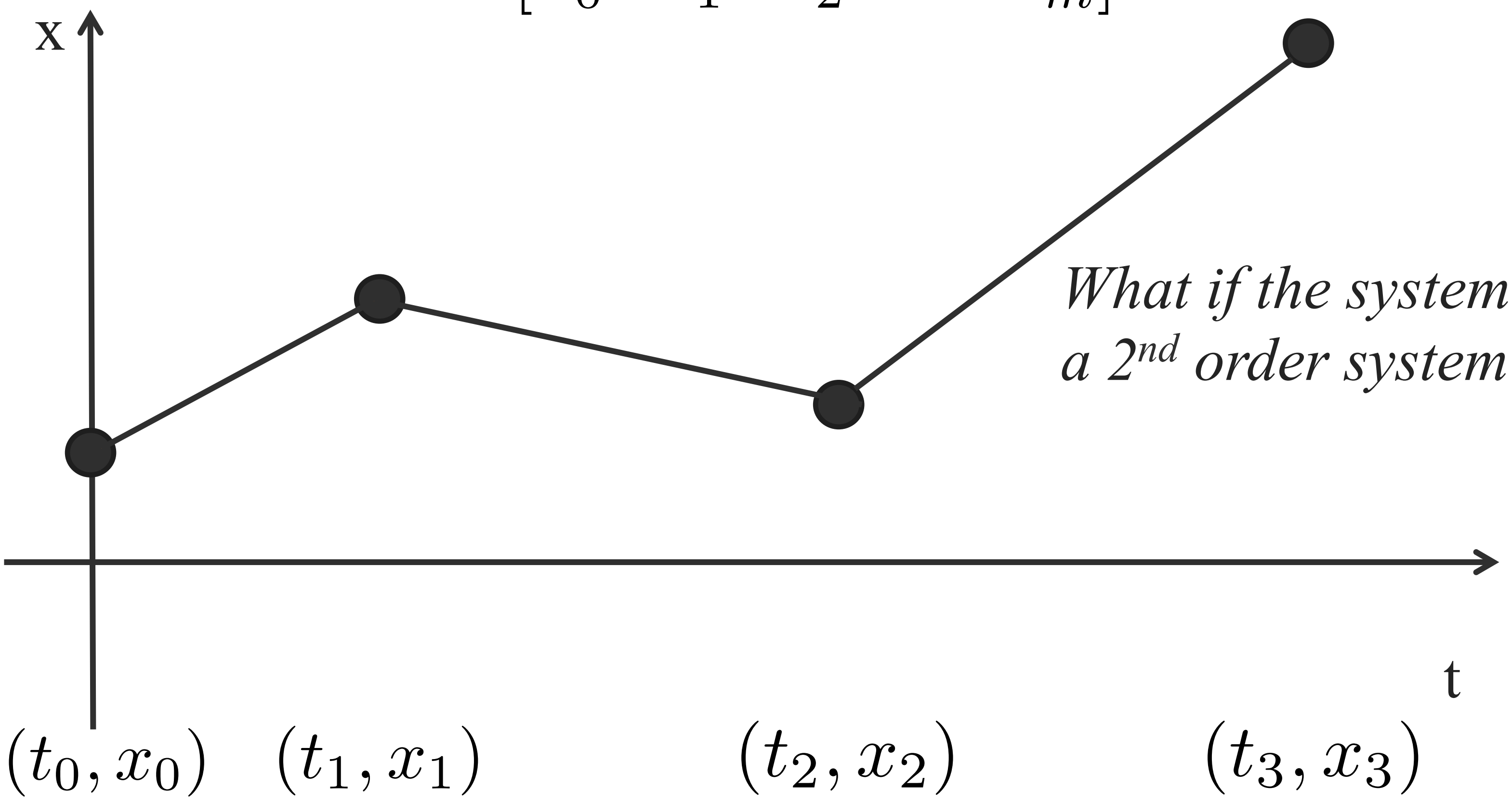
Define piecewise continuous trajectory:

$$x(t) = \begin{cases} x_1(t), & t_0 \leq t < t_1 \\ x_2(t), & t_1 \leq t < t_2 \\ \ldots & \\ x_m(t), & t_{m-1} \leq t < t_m \end{cases}$$

Continuous but not Differentiable

Design a trajectory $x(t)$ such that:

$$t = [t_0 \quad t_1 \quad t_2 \quad \ldots \quad t_m]^T$$
$$x = [x_0 \quad x_1 \quad x_2 \quad \ldots \quad x_m]^T$$



What if the system is a 2nd order system?

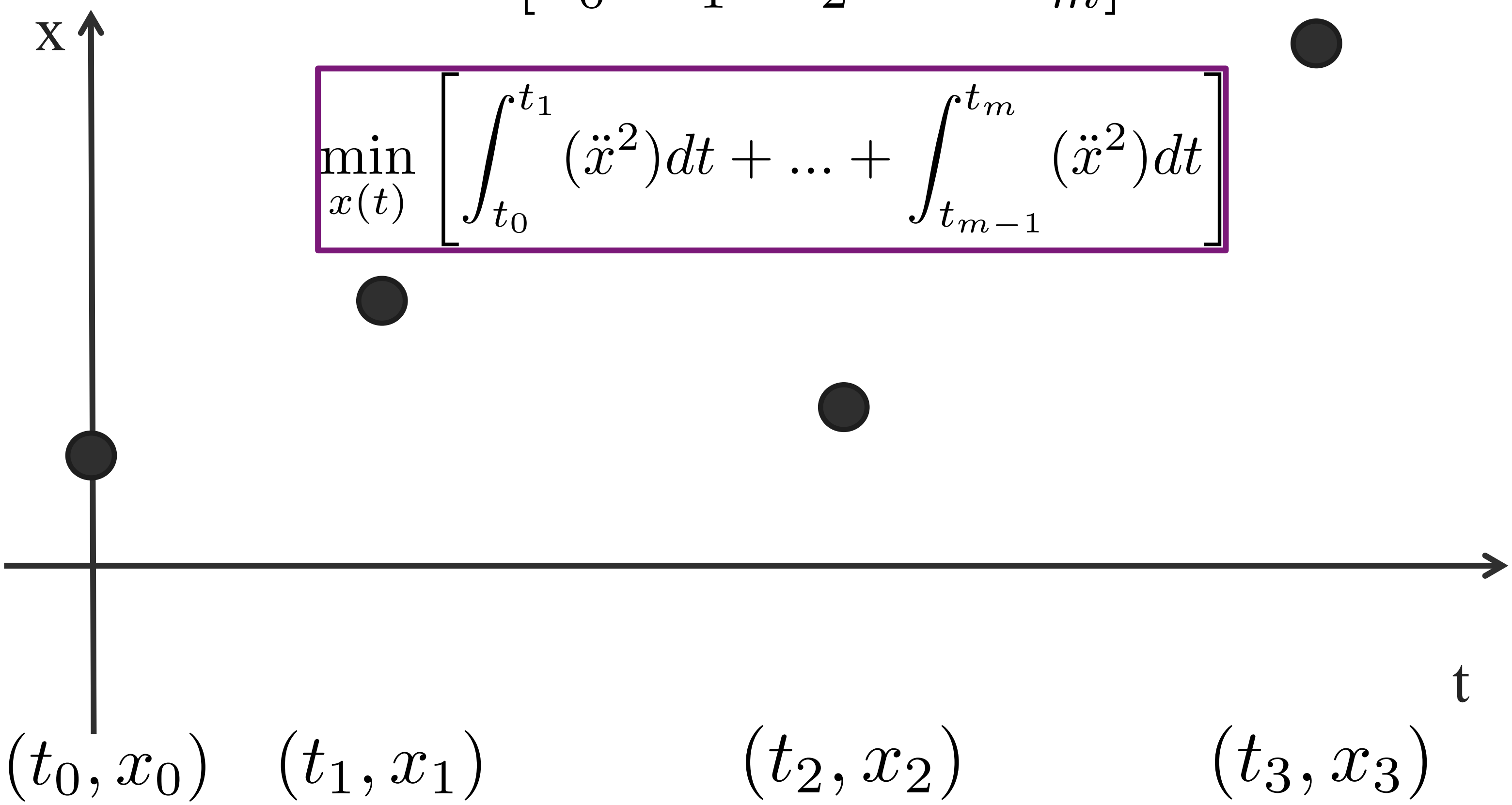
Minimum Acceleration Curve for 2nd Order Systems

Design a trajectory $x(t)$ such that:

$$t = [t_0 \quad t_1 \quad t_2 \quad \dots \quad t_m]^T$$

$$x = [x_0 \quad x_1 \quad x_2 \quad \dots \quad x_m]^T$$

$$\boxed{\min_{x(t)} \left[\int_{t_0}^{t_1} (\ddot{x}^2) dt + \dots + \int_{t_{m-1}}^{t_m} (\ddot{x}^2) dt \right]}$$



Cubic Spline

Design a trajectory x(t) such that:

$$t = [t_0 \quad t_1 \quad t_2 \quad \dots \quad t_m]^T$$

$$x = [x_0 \quad x_1 \quad x_2 \quad \dots \quad x_m]^T$$

$$\boxed{\min_{x(t)} \left[\int_{t_0}^{t_1} (\ddot{x}^2) dt + \dots + \int_{t_{m-1}}^{t_m} (\ddot{x}^2) dt \right]}$$

$$x(t) = \begin{cases} x_1(t) = c_{1,3}t^3 + c_{1,2}t^2 + c_{1,1}t + c_{1,0}, & t_0 \leq t < t_1 \\ x_2(t) = c_{2,3}t^3 + c_{2,2}t^2 + c_{2,1}t + c_{2,0}, & t_1 \leq t < t_2 \\ \dots \\ x_m(t) = c_{m,3}t^3 + c_{m,2}t^2 + c_{m,1}t + c_{m,0}, & t_{m-1} \leq t < t_m \end{cases}$$

$4m$ degrees of freedom

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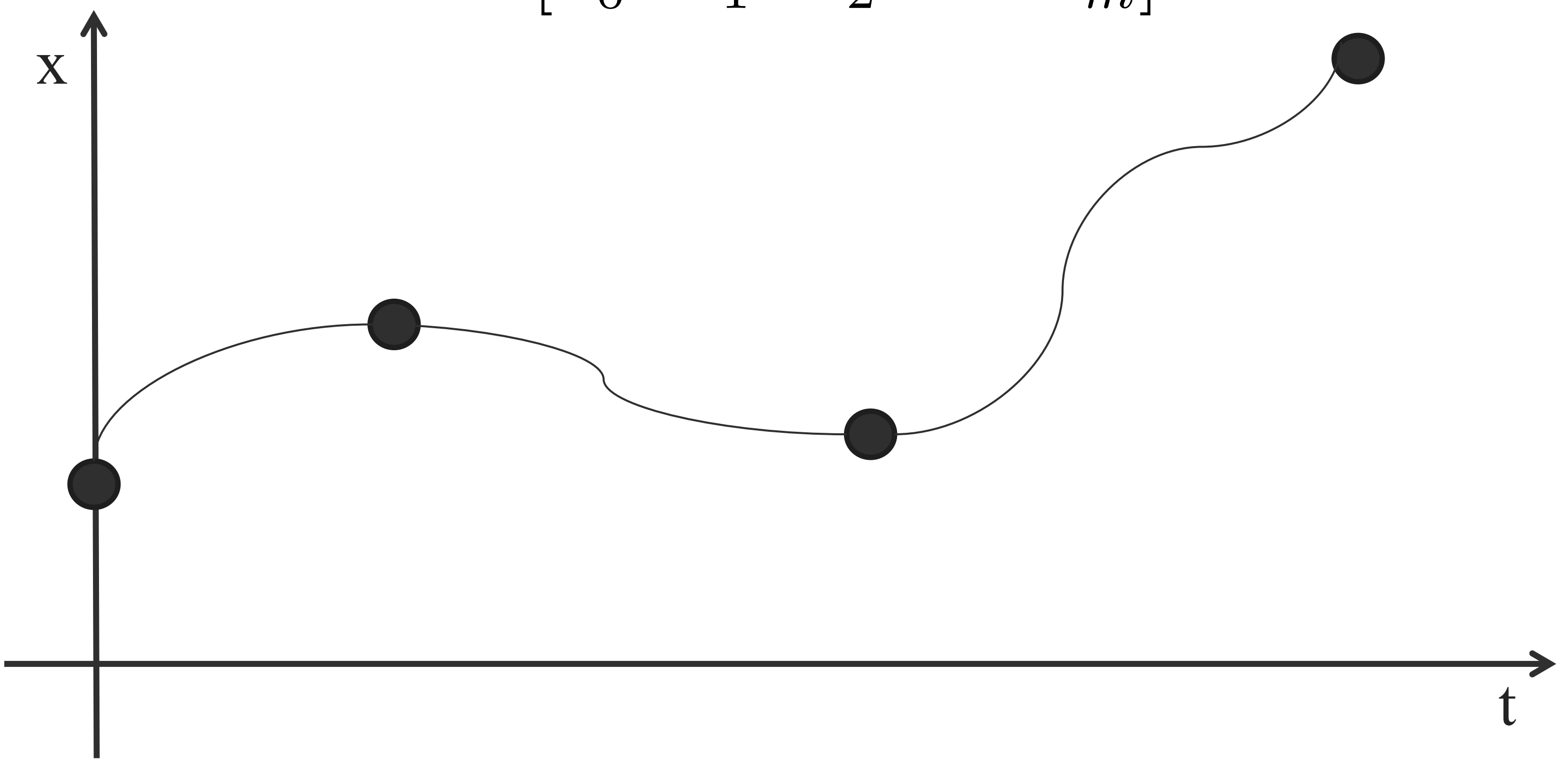
Cubic Spline

Design a trajectory x(t) such that:

$$t = [t_0 \quad t_1 \quad t_2 \quad \ldots \quad t_m]^T$$

$$x = [x_0 \quad x_1 \quad x_2 \quad \ldots \quad x_m]^T$$

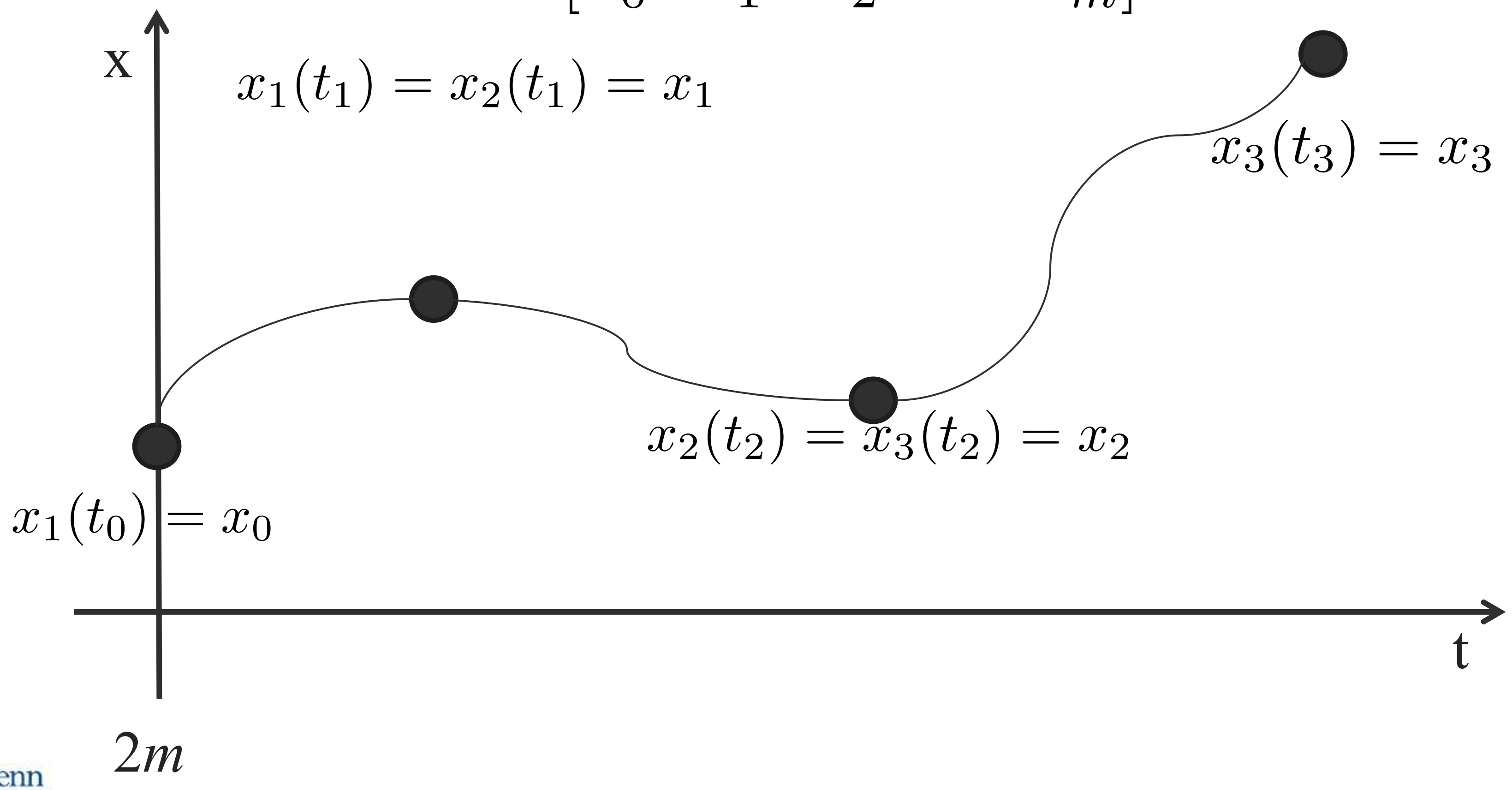


Cubic Spline

Design a trajectory x(t) such that:

$$t = [t_0 \quad t_1 \quad t_2 \quad \dots \quad t_m]^T$$

$$x = [x_0 \quad x_1 \quad x_2 \quad \dots \quad x_m]^T$$

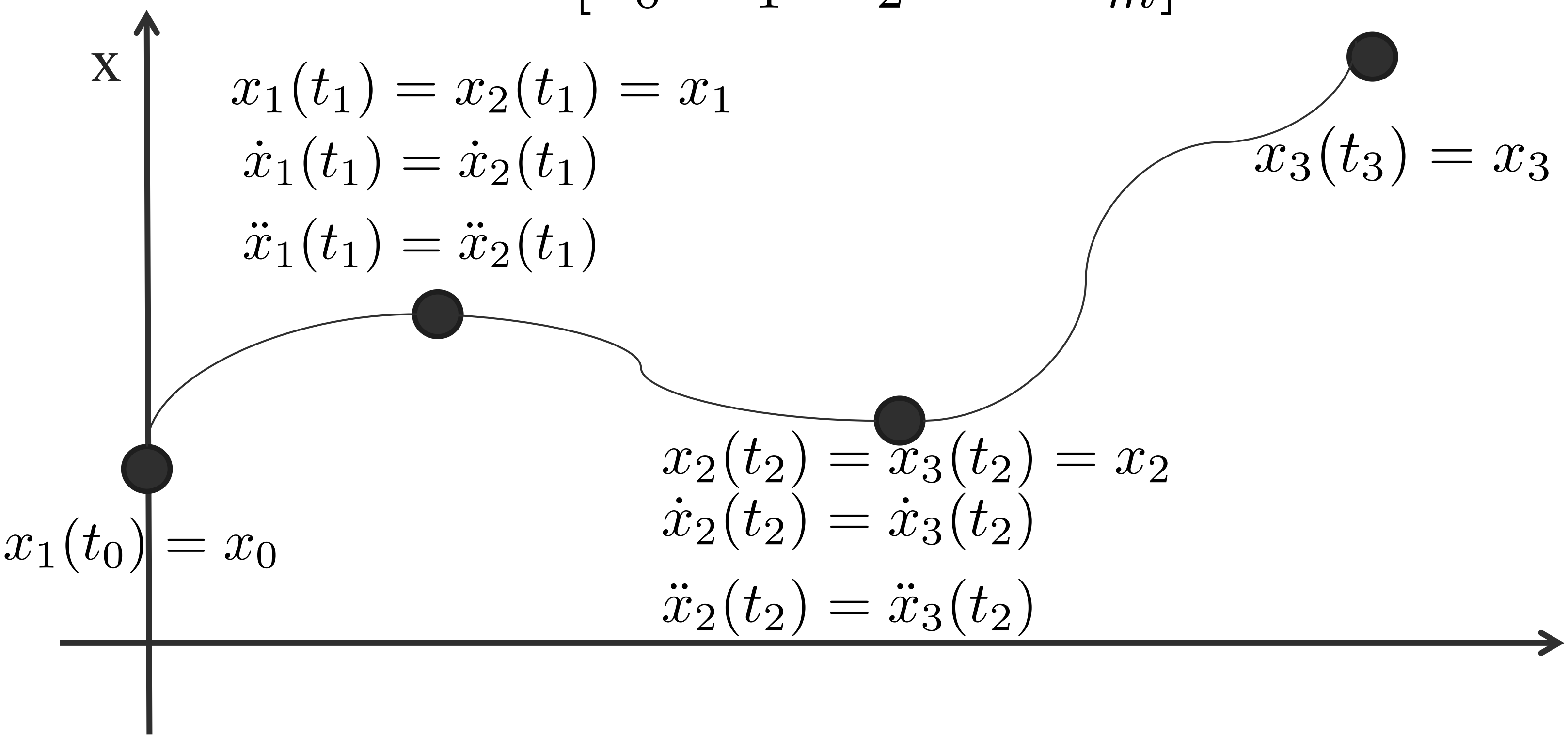


Cubic Spline

Design a trajectory x(t) such that:

$$t = [t_0 \quad t_1 \quad t_2 \quad \dots \quad t_m]^T$$

$$x = [x_0 \quad x_1 \quad x_2 \quad \dots \quad x_m]^T$$



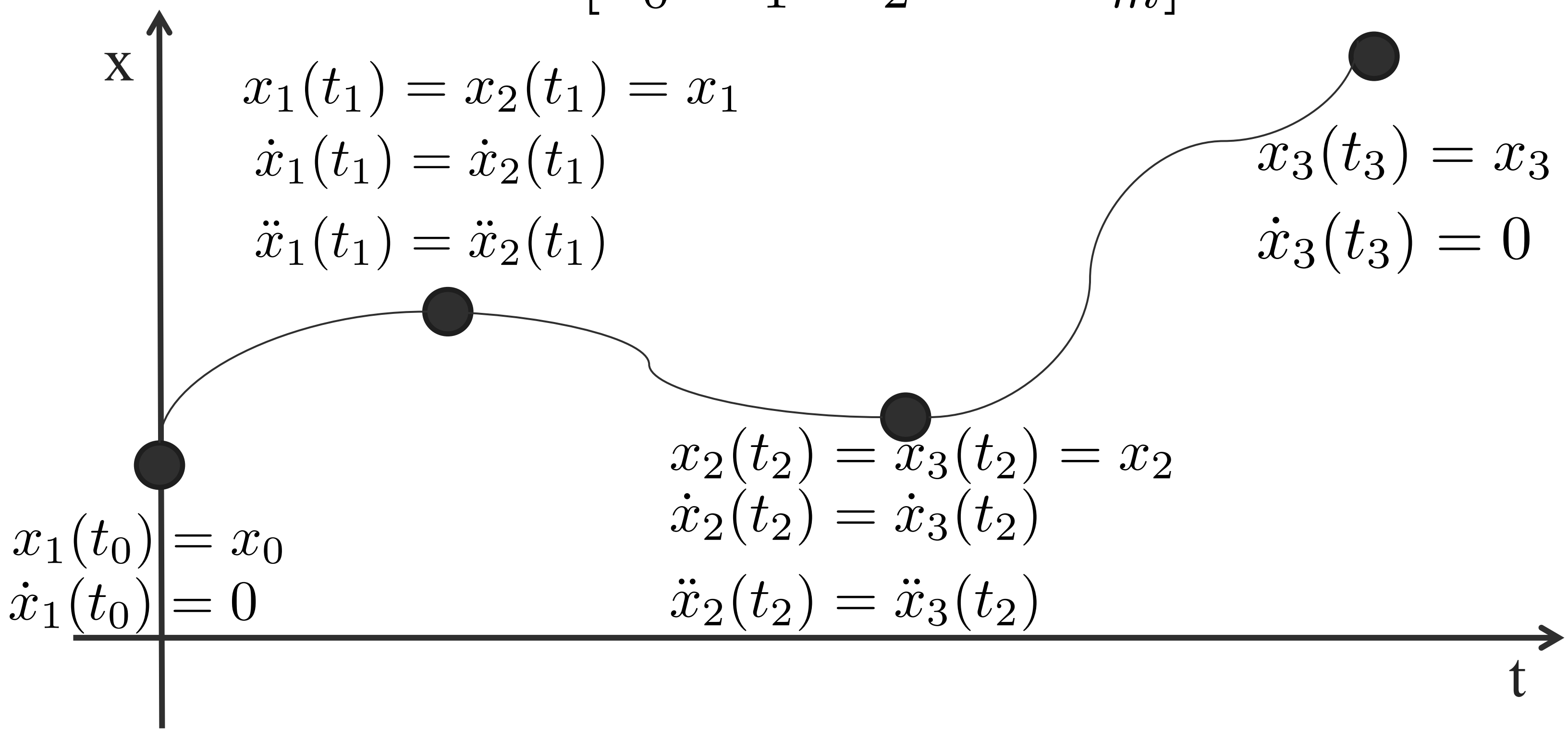
$2m+2(m-1)$

Cubic Spline

Design a trajectory x(t) such that:

$$t = [t_0 \quad t_1 \quad t_2 \quad \ldots \quad t_m]^T$$

$$x = [x_0 \quad x_1 \quad x_2 \quad \ldots \quad x_m]^T$$



$2m+2(m-1)+2 = 4m$ constraints

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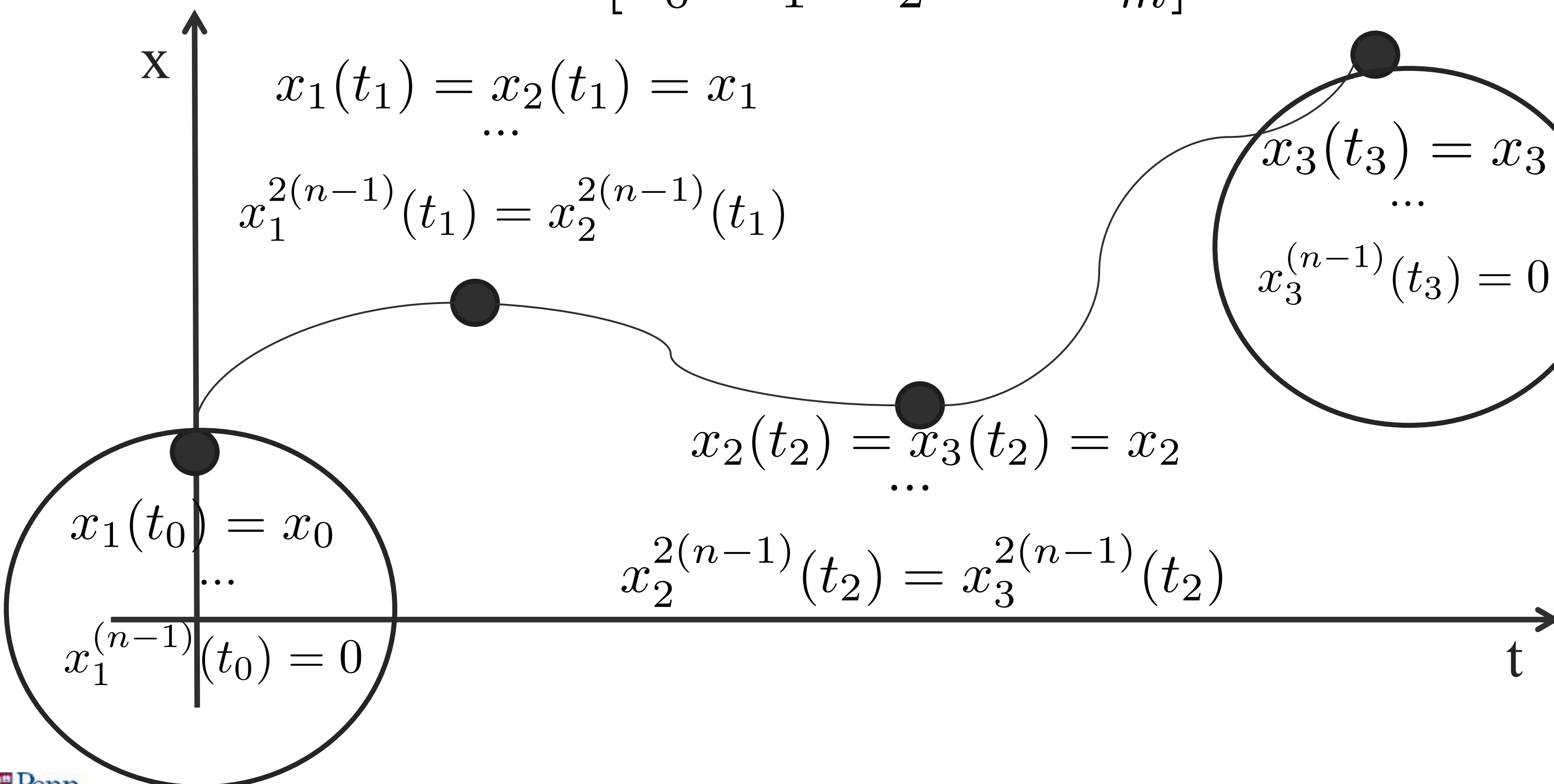
Spline for nth order system

Design a trajectory x(t) such that:

$$t = [t_0 \quad t_1 \quad t_2 \quad \dots \quad t_m]^T$$

$$x = [x_0 \quad x_1 \quad x_2 \quad \dots \quad x_m]^T$$

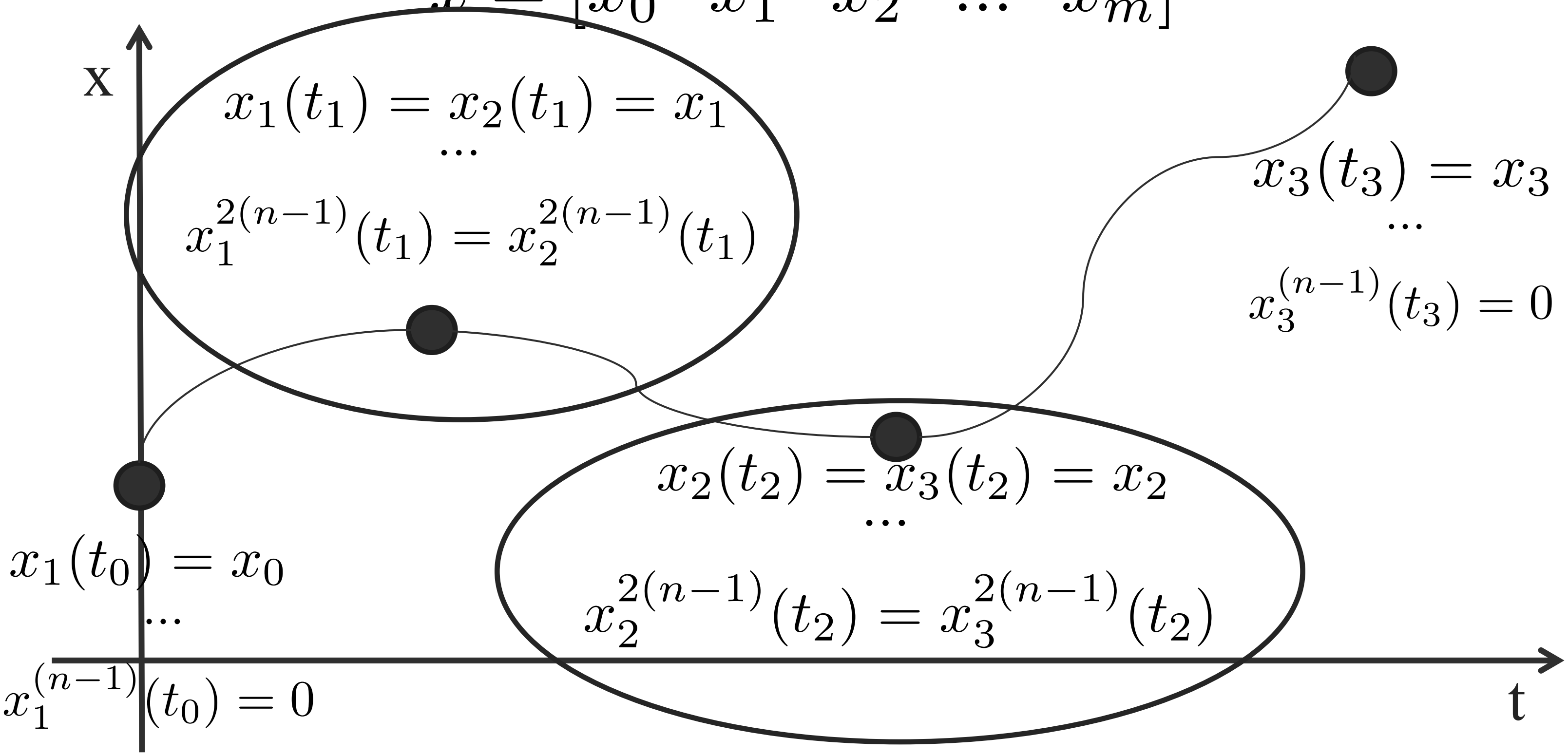


Spline for nth order system

Design a trajectory x(t) such that:

$$t = [t_0 \quad t_1 \quad t_2 \quad \dots \quad t_m]^T$$

$$x = [x_0 \quad x_1 \quad x_2 \quad \dots \quad x_m]^T$$



Motion Planning of Quadrotors