Linearization of Quadrotor Equations of Motion

Quadrotor Equations of Motion

Linear momentum balance:

$$m\ddot{\mathbf{r}} = \begin{bmatrix} 0 \\ 0 \\ -mg \end{bmatrix} + R \begin{bmatrix} 0 \\ 0 \\ F_1 + F_2 + F_3 + F_4 \end{bmatrix}$$

Angular momentum balance:

$$I \begin{bmatrix} \dot{p} \\ \dot{q} \\ \dot{r} \end{bmatrix} = \begin{bmatrix} L(F_2 - F_4) \\ L(F_3 - F_1) \\ M_1 - M_2 + M_3 - M_4 \end{bmatrix} - \begin{bmatrix} p \\ q \\ r \end{bmatrix} \times I \begin{bmatrix} p \\ q \\ r \end{bmatrix}$$

Quadrotor Equations of Motion

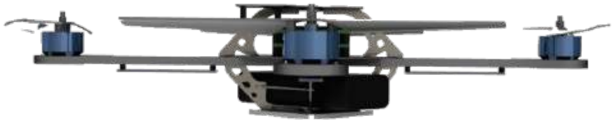
Linear momentum balance:

$$m\ddot{\mathbf{r}} = \begin{bmatrix} 0 \\ 0 \\ -mg \end{bmatrix} + R \begin{bmatrix} 0 \\ 0 \\ u_1 \end{bmatrix}$$

Angular momentum balance:

$$I \begin{bmatrix} \dot{p} \\ \dot{q} \\ \dot{r} \end{bmatrix} = \begin{bmatrix} u_{2x} \\ u_{2y} \\ u_{2z} \end{bmatrix} - \begin{bmatrix} p \\ q \\ r \end{bmatrix} \times I \begin{bmatrix} p \\ q \\ r \end{bmatrix}$$

Equilibrium Hover Configuration



$$\mathbf{r} = \mathbf{r}_0, \theta = \phi = 0, \psi = \psi_0$$

$$\dot{\mathbf{r}} = 0, \dot{\theta} = \dot{\phi} = \dot{\psi} = 0$$

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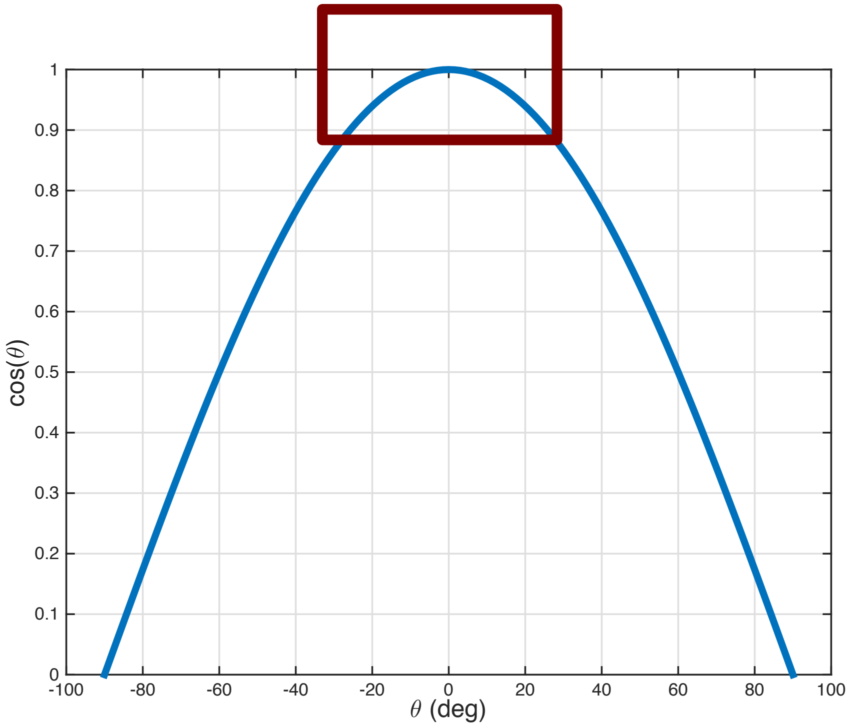
Linearization of Trigonometric Functions

What is the value of $\cos(\theta)$ near $\theta = 0$?



Can be approximated with the Taylor Series:

$$\cos(\theta)$$

$$\approx \cos(\theta)|_{\theta=0} + \frac{d\cos(\theta)}{d\theta}|_{\theta=0}\theta$$

$$+ \text{higher order terms}$$

$$\approx 1 - \sin(\theta)|_{\theta=0}\theta$$

$$\approx 1$$

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Linearization of Trigonometric Functions

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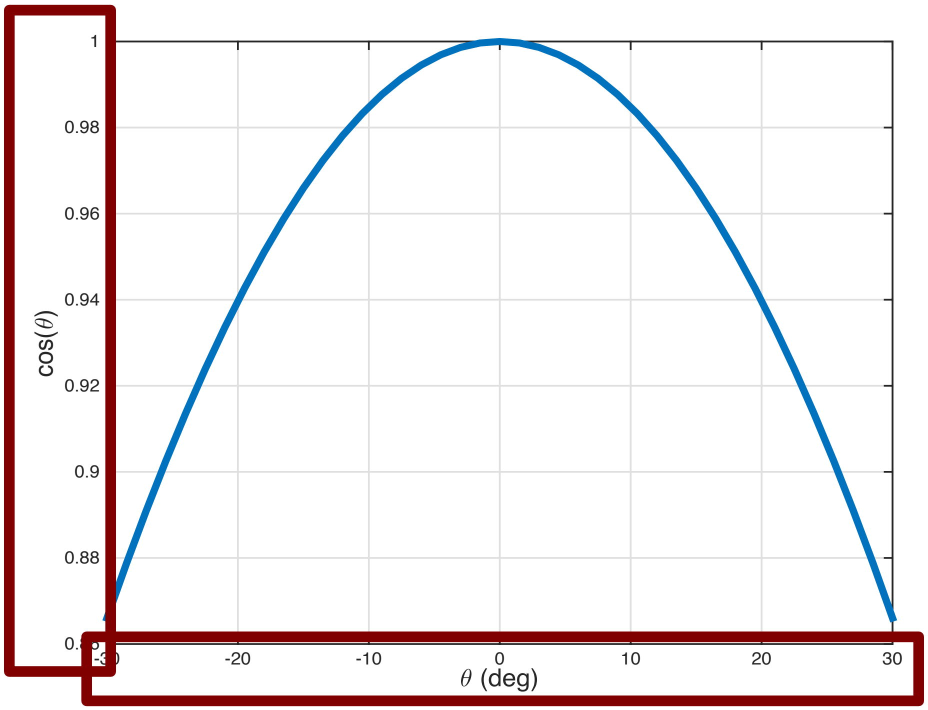

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Linearization of Trigonometric Functions

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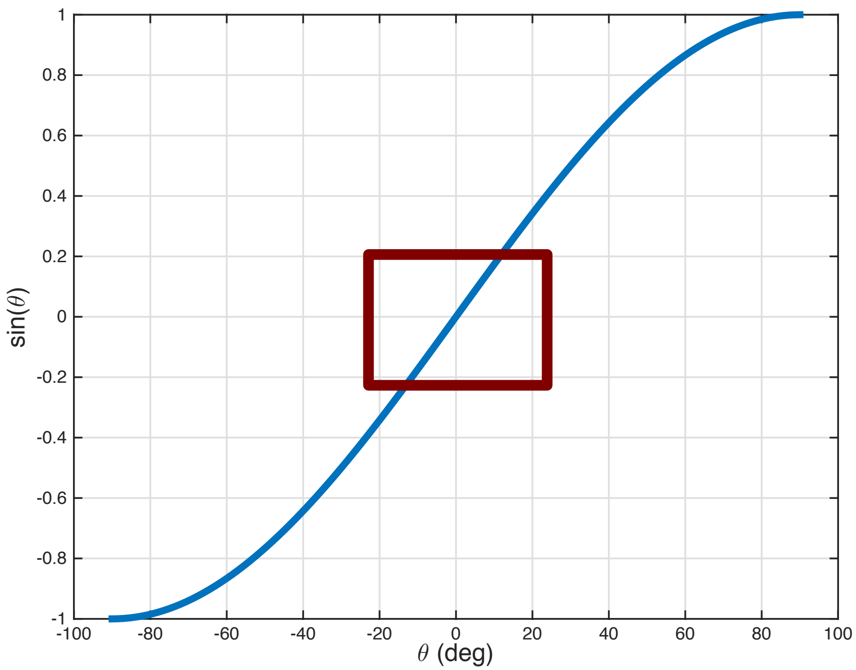

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$$+ \text{higher order terms}$$

$$\approx 0 + \cos(\theta)|_{\theta=0}\theta$$

$$\approx \theta$$

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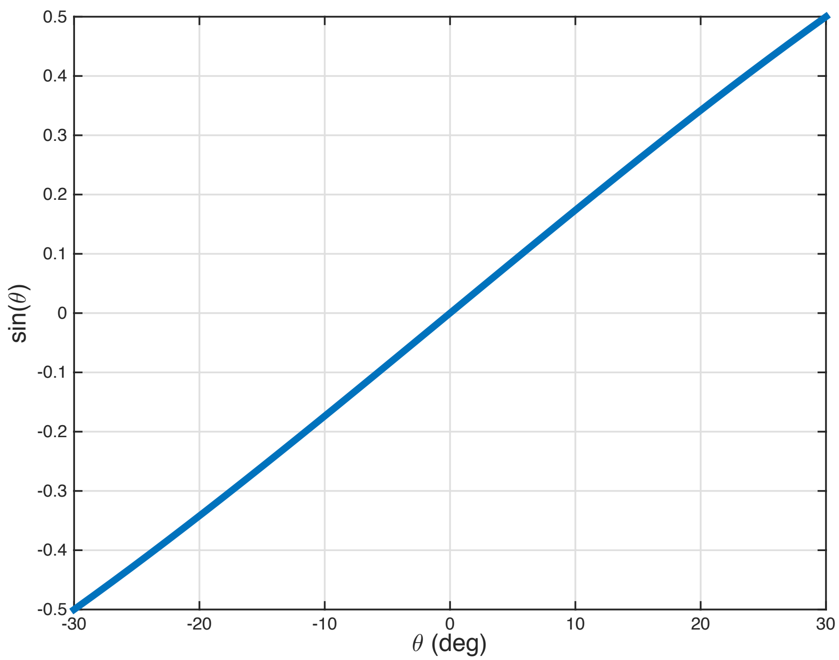
Linearization of Trigonometric Functions

What is the value of $\sin(\theta)$ near $\theta = 0$?



sine function looks linear around $\theta = 0$

Can be approximated with the Taylor Series:

$$\sin(\theta)$$

$$\approx \sin(\theta)|_{\theta=0} + \frac{d\sin(\theta)}{d\theta}|_{\theta=0}\theta$$

$$+ \text{ higher order terms}$$

$$\approx 0 + \cos(\theta)|_{\theta=0}\theta$$

$$\approx \theta$$

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Linearized Equations of Motion

What are the equations of motion of the quadrotor when it is near the equilibrium hover configuration?

$$\mathbf{r} \approx \mathbf{r}_0, \theta \approx \phi \approx 0, \psi \approx \psi_0$$

$$\dot{\mathbf{r}} \approx 0, \dot{\theta} \approx \dot{\phi} \approx \dot{\psi} \approx 0$$

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Linear Momentum Equation Near Hover

$$m\ddot{\mathbf{r}} = \begin{bmatrix} 0 \\ 0 \\ -mg \end{bmatrix} + R \begin{bmatrix} 0 \\ 0 \\ u_1 \end{bmatrix}$$

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Linear Momentum Equation Near Hover

$$m\ddot{\mathbf{r}} = \begin{bmatrix} 0 \\ 0 \\ -mg \end{bmatrix} + \begin{bmatrix} c\psi c\theta - s\phi s\psi s\theta & -c\phi s\psi & c\psi s\theta + c\theta s\phi s\psi \\ c\theta s\psi + c\psi s\theta s\phi & c\phi c\psi & s\psi s\theta - c\psi c\theta s\phi \\ -c\phi s\theta & s\phi & c\phi c\theta \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ u_1 \end{bmatrix}$$

Linear Momentum Equation Near Hover

$$m\ddot{x} = (c\psi s\theta + c\theta s\phi s\psi)\, u_1$$
$$m\ddot{y} = (s\psi s\theta - c\psi c\theta s\phi)\, u_1$$
$$m\ddot{z} = -mg + (c\phi c\theta)\, u_1$$

Substituting in the approximation:

$$\sin(\theta) \approx \theta, \sin(\phi) \approx \phi, \cos(\theta) \approx \cos(\phi) \approx 1$$

Linear Momentum Equation Near Hover

$$m\ddot{x} = (\theta c\psi + \phi s\psi)\, u_1$$
$$m\ddot{y} = (\theta s\psi - \phi c\psi)\, u_1$$
$$m\ddot{z} = -mg + u_1$$

The second derivative of position is proportional to u_1 !

Angular Rates Near Hover

$$\begin{bmatrix} p \\ q \\ r \end{bmatrix} = \begin{bmatrix} c\theta & 0 & -c\phi s\theta \\ 0 & 1 & s\phi \\ s\theta & 0 & c\phi c\theta \end{bmatrix} \begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix}$$

Angular Rates Near Hover

$$p = \dot{\phi} c\theta - \dot{\psi} c\phi s\theta$$

$$q = \dot{\theta} + \dot{\psi} s\phi$$

$$r = \dot{\phi} s\theta + \dot{\psi} c\phi c\theta$$

Substituting in the approximation:

$$\sin(\theta) \approx \theta, \sin(\phi) \approx \phi, \cos(\theta) \approx \cos(\phi) \approx 1$$

Angular Rates Near Hover

$$p = \dot{\phi} - \dot{\psi}\theta$$

$$q = \dot{\theta} + \dot{\psi}\phi$$

$$r = \dot{\phi}\theta + \dot{\psi}$$

Substituting in the approximation:

$$\dot{\psi}\theta \approx \dot{\psi}\phi \approx \dot{\phi}\theta \approx 0$$

Higher order terms: Product of two terms around 0 is approximately 0.

Angular Rates Near Hover

$$p = \dot{\phi}$$

$$q = \dot{\theta}$$

$$r = \dot{\psi}$$

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Angular Momentum Equation Near Hover

$$I \begin{bmatrix} \dot{p} \\ \dot{q} \\ \dot{r} \end{bmatrix} = \begin{bmatrix} u_{2x} \\ u_{2y} \\ u_{2z} \end{bmatrix} - \begin{bmatrix} p \\ q \\ r \end{bmatrix} \times I \begin{bmatrix} p \\ q \\ r \end{bmatrix}$$

Substituting in the approximation:

$$I_{xy} \approx I_{yx} \approx I_{xz} \approx I_{zx} \approx I_{yz} \approx I_{zy} \approx 0$$

Angular Momentum Equation Near Hover

$$\begin{bmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{bmatrix} \begin{bmatrix} \dot{p} \\ \dot{q} \\ \dot{r} \end{bmatrix} = \begin{bmatrix} u_{2x} \\ u_{2y} \\ u_{2z} \end{bmatrix} - \begin{bmatrix} 0 & r & -q \\ -r & 0 & p \\ q & -p & 0 \end{bmatrix} \begin{bmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{bmatrix} \begin{bmatrix} p \\ q \\ r \end{bmatrix}$$

Angular Momentum Equation Near Hover

$$I_{xx}\dot{p} = u_{2x} - I_{yy}qr + I_{zz}qr$$
$$I_{yy}\dot{q} = u_{2y} + I_{xx}pr - I_{zz}pr$$
$$I_{zz}\dot{r} = u_{2z} - I_{xx}pq + I_{yy}pq$$

Substituting in the approximation:

$$qr \approx pr \approx pq$$
$$\approx \dot{\theta}\dot{\psi} \approx \dot{\phi}\dot{\psi} \approx \dot{\phi}\dot{\theta} \approx 0$$

Higher order terms: Product of two terms around 0 is approximately 0.

Angular Momentum Equation Near Hover

$$I_{xx}\dot{p} = u_{2x}$$

$$I_{yy}\dot{q} = u_{2y}$$

$$I_{zz}\dot{r} = u_{2z}$$

Substituting in the approximation:

$$p \approx \dot{\phi}$$

$$q \approx \dot{\theta}$$

$$r \approx \dot{\psi}$$

Angular Momentum Equation Near Hover

$$\ddot{\phi} = \frac{u_{2x}}{I_{xx}}$$

$$\ddot{\theta} = \frac{u_{2y}}{I_{yy}}$$

$$\ddot{\psi} = \frac{u_{2z}}{I_{zz}}$$

Equations of Motion

Recall the linearized linear momentum equation:

$$m\ddot{x} = (\theta c\psi + \phi s\psi)\, u_1$$

Differentiating the equation:

$$m\dddot{x} = (\theta c\psi + \phi s\psi)\, \dot{u}_1 + \left(\dot{\theta} c\psi - \theta s\psi\dot{\psi} + \dot{\phi} s\psi + \phi c\psi\dot{\psi}\right) u_1$$

Differentiating again:

$$m\ddddot{x} = (\theta c\psi + \phi s\psi)\, \ddot{u}_1 + 2\left(\dot{\theta} c\psi - \theta s\psi\dot{\psi} + \dot{\phi} s\psi + \phi c\psi\dot{\psi}\right) \dot{u}_1 +$$
$$\left(\ddot{\theta} c\psi - \dot{\theta} s\psi\dot{\psi} - \theta s\psi\ddot{\psi} - \theta c\psi\dot{\psi}^2 + \ddot{\phi} s\psi + \dot{\phi} c\psi\dot{\psi} + \phi c\psi\ddot{\psi} - \phi c\psi\dot{\psi}^2\right) u_1$$

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Equations of Motion

Substituting in the approximation:

$$\ddot{\phi} = \frac{u_{2x}}{I_{xx}}, \ddot{\theta} = \frac{u_{2y}}{I_{yy}}, \ddot{\psi} = \frac{u_{2z}}{I_{zz}}$$

The linear momentum equation becomes:

$$m\ddot{\ddot{x}} = \ldots + \left(\frac{u_{2y}}{I_{yy}}c\psi + \frac{u_{2z}}{I_{zz}}\theta(c\psi - s\psi) + \frac{u_{2x}}{I_{xx}}s\psi\right)u_1$$

The fourth derivative of position is proportional to u_2 !