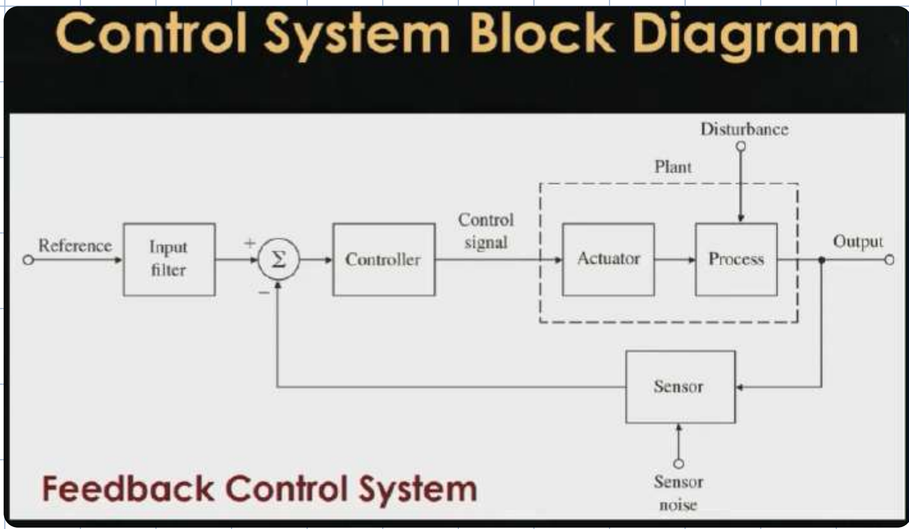
Control Systems Analysis : Modelling of Dynamical Systems [COURSERA]



* control systems can be of any field like machines, biological, monetary, behavioural, etc
* control systems → need actuators designed based on "model" of "plant" ↳ have sensors which give feedback to controller

V 1.1.2

Modelling the Plant

① our focus – analytically derived models (anything else possible?)

- → based on fundamental laws (eg. Newton's laws, etc)
- → used for simpler systems
- → yields DE as a function of system parameters
- → more general, gives better insights

for eg, pendulum system run by a motor → non linear time invariant

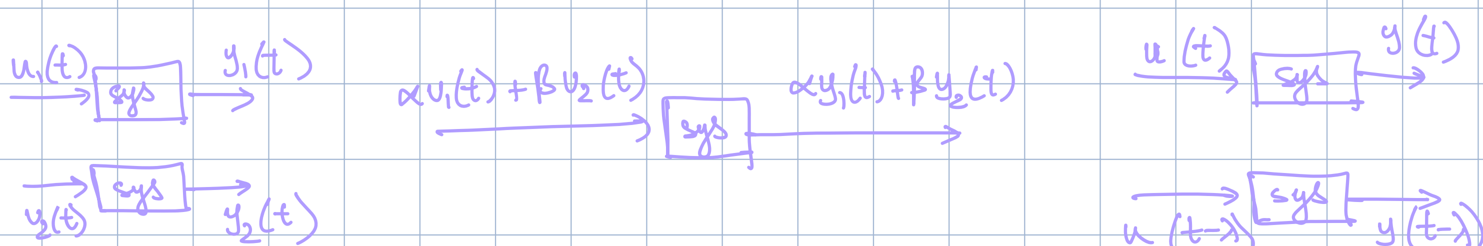
② experimentally derived models (system identification)

- → time domain, frequency domain, and other methods
- → used for more complex systems
- → yields numerical DE or transfer functions for particular systems

Linear Time Invariant (LTI) — LTI controllers for LTI models of plants

linear — system follows principle of superposition

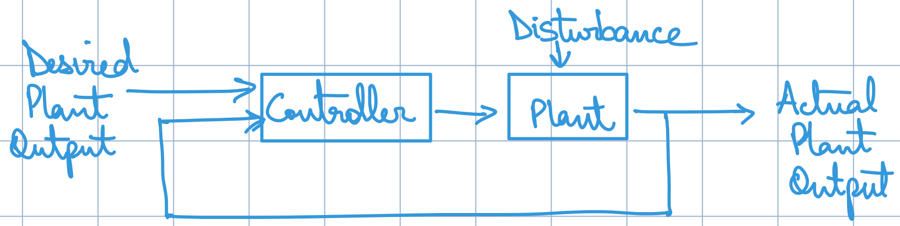
time invariant —



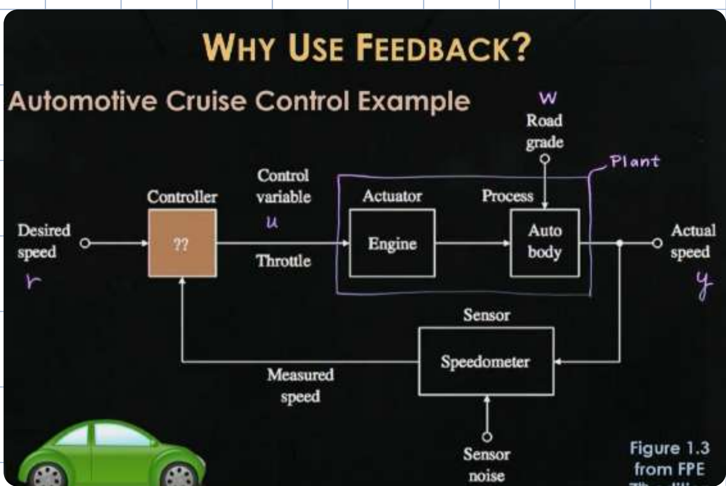
* Continuous Time vs Digital Control

most modern control systems, the plant is continuous time but the controller is often implemented on a microprocessor which results in digital control

(when sampling speeds are fast (relative to dynamics of system), the digital controller can be approximated as a continuous time controller)



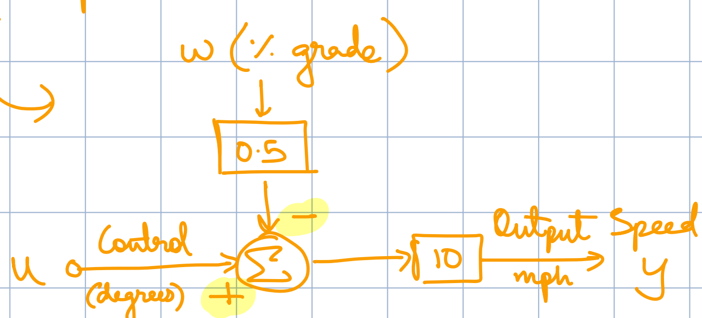
V1.1.3



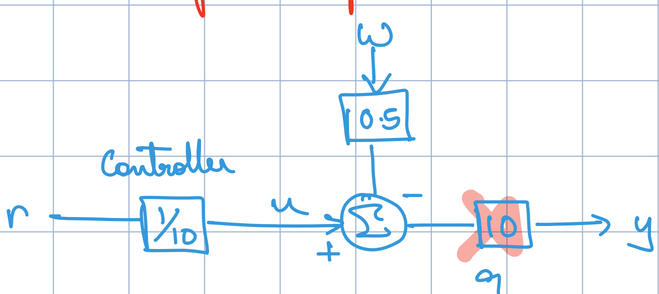
Assumptions

- ignore dynamic response and focus on steady-state behaviour
- assume linear system behaviour over range of speeds increased

from Experimental Data



open-loop



without modelling error

$$y = 10(u - 0.5w)$$
$$= 10\left(\frac{r}{10} - 0.5w\right)$$

$$y = r - 5w \Rightarrow y = r \text{ if } w = 0$$

depicts sensitivity to disturbance

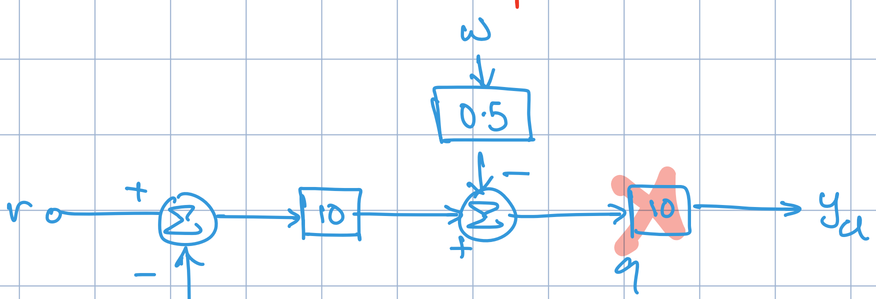
with modelling errors

$$y = 9(u - 0.5w) = 9\left(\frac{r}{10} - 0.5w\right)$$

$$y = 0.9r - 0.45w$$

10% tracking error if $w = 0$

Closed-loop Control



without modelling error

$$y_a = 10(u - 0.5w)$$
$$y = 10(10(r - y_a) - 0.5w)$$
$$y_a = 100r - 100y_a - 5w$$
$$101 y_a = 100r - 5w$$
$$\Rightarrow \quad y_a = \frac{100}{101} r - \frac{5}{101} w$$

with modelling error

$$y_a = 9(u - 0.5w)$$
$$= 9(10(r - y_a) - 0.5w)$$
$$y_a = 90r - 90y_a - 4.5w$$
$$y_a = \frac{90}{91} r - \frac{4.5}{90} w \quad \approx \text{similar}$$

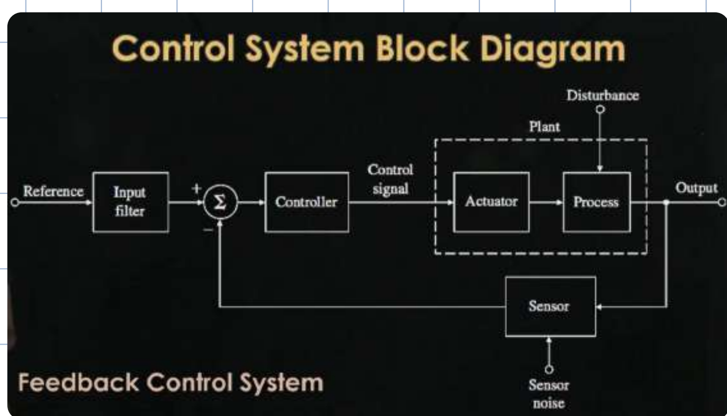
less than 5% sensitivity due to controller

if $w = 0 \rightarrow y \sim 1\%$ of tracking speed (input)

→ Is this a "good" control system?

→ What happens as controller gain increases or decreases

input filter if inverse of multiplier of r to get perfect tracking.



→ Open loop systems – don't use measurements of system behaviour so are insensitive to modelling errors or disturbances

→ Closed loop control systems – feedback measurements of system behaviour leads to more robustness and is sensitive to deviations what is Laplace Transform?

Laplace Transform Review

$$F(s) = \int_{-\infty}^{\infty} f(t) e^{-st} dt \qquad \mathbb{1}(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}$$

region of convergence

$$F(s) = \int_{-\infty}^{\infty} a\mathbb{1}(t) e^{-st} dt = \frac{a}{s} \text{ assuming } Re\{s\} > 0$$

Usually dealing with causal signal, so generally assumed $Re\{s\} > 0$

$F(s)$	$f(t),\ t \geq 0$		
1	$\delta(t)$	$\frac{a}{a^2+s^2}$	$\sin at$
$1/s$	$\mathbb{1}(t)$	$\frac{s}{s^2+a^2}$	$\cos at \rightarrow \frac{e^{-jat}+e^{jat}}{2}$
$1/s^2$	$t \cdot \mathbb{1}(t)$	$\frac{s+a}{(s+a)^2+b^2}$	$e^{-at}\cos bt$
$2!/s^3$	$t^2 \cdot \mathbb{1}(t)$	$\frac{b}{(s+a)^2+b^2}$	$e^{-at}\sin bt$
$1/(s+a)$	$e^{-at} \cdot \mathbb{1}(t)$	$\frac{a^2+b^2}{s[(s+a)^2+b^2]}$	$1-e^{-at}\left(\cos bt + \frac{a}{b}\sin bt\right)$

one sided Laplace transform, $\mathcal{L}\{f(t)\} = F(s) = \int_{0^-}^{\infty} f(t) e^{-st} dt$

impulsive input at $t=0$

Q:

$$\frac{a}{s(s+4)} = \frac{a}{4s} - \frac{a}{4(s+4)}$$

$$\frac{10}{(-4+j3)(j3)} = \frac{10}{-9-12j} = \frac{10}{-15^2}(-9+12j) = \frac{12}{-9} = -\frac{4}{3}$$

Practice Assgn

① $F(s) = \frac{s+5}{s+1} \Rightarrow s = -5-j4 \Rightarrow \frac{-j4}{-4-j4} = \frac{j}{1+j} \Rightarrow \frac{1}{\sqrt{2}} = 0.707$ $|F(s)|$

② $90 - 45 = 45°$

③ $s = -3-j3 \Rightarrow F(s) = +\frac{2-j3}{-2-j3} \Rightarrow |F(s)| = 1$

④ $\tan^{-1}\left(\frac{-3}{2}\right) - \tan^{-1}\left(\frac{-3}{-2}\right) = -112.62°$

$-56.31 \quad - \quad -123.69$

Superposition Property.

$$\mathcal{L}\{\alpha f_1(t) + \beta f_2(t)\} = \alpha F_1(s) + \beta F_2(s)$$ (prove using definition)

Properties of Laplace Transforms (selected)

Laplace Transform	Time Function	Comment
$F(s)$	$f(t)$	Transform pair
$\alpha F_1(s) + \beta F_2(s)$	$\alpha f_1(t) + \beta f_2(t)$	Superposition
$s^m F(s) - s^{m-1} f(0) - s^{m-2} \dot{f}(0) - \cdots - f^{(m-1)}(0)$	$f^{(m)}(t)$	Differentiation
$\frac{1}{s} F(s)$	$\int_0^t f(\zeta)\, d\zeta$	Integration
$F_1(s) F_2(s)$	$f_1(t) * f_2(t)$	Convolution
$\lim_{s\to\infty} sF(s)$	$f(0^+)$	Initial Value Theorem
$\lim_{s\to 0} sF(s)$	$\lim_{t\to\infty} f(t)$	Final Value Theorem

(From FPE edition 7th)

** derivation

Differentiation

Assuming causal signal & one-sided Laplace transform

$$\mathcal{L}\left\{\frac{df}{dt}\right\} = sF(s) - f(0)$$

$$\mathcal{L}\left\{\frac{df}{dt}\right\} = sF(s) \quad \text{or} \quad \frac{df}{dt} \overset{\mathcal{L}}{\longleftrightarrow} sF(s)$$

Integration: $\int_0^t f(\xi)\, d\xi \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s} F(s)$

Convolution

$$f_1(t) * f_2(t) \overset{\mathcal{L}}{\longleftrightarrow} F_1(s) F_2(s)$$

$$\int_{-\infty}^{\infty} f_1(\tau) f_2(t-\tau)\, d\tau$$

$$H(s) \times U(s) = Y(s)$$

$u(t) \rightarrow \boxed{h(t)} \rightarrow y(t) = h(t) * u(t)$ $\quad \hookrightarrow y(t) = \mathcal{L}^{-1}(Y(s))$

Impulse Response for LTI system

Inverse Laplace Transform

$$f(t) = \frac{1}{2\pi j} \int_{\sigma_c - j\infty}^{\sigma_c + j\infty} F(s) e^{st} ds$$

limits – vertical line in S plane in the region of convergence (ROC)

$\sigma_c \in$ ROC

usually find causal $f(t)$ that corresponds to it.

Partial Fraction Expansion (distinct poles)

For real systems, $m \le n$

leading gain

zeros

$$F(s) = \frac{b_1 s^m + b_2 s^{m-1} + \cdots + b_{m+1}}{s^n + a_1 s^{n-1} + \cdots + a_n} = \underset{\underset{b_1}{\shortparallel}}{K} \frac{\prod_{i=1}^{m} (s - z_i)}{\prod_{i=1}^{n} (s - p_i)}$$

poles

$$= \underbrace{\frac{c_1}{s - p_1}}_{} + \frac{c_2}{s - p_2} + \cdots + \frac{c_n}{s - p_n}$$

inverse is $c_1 e^{p_1 t} \cdot 1(t)$

$$\Rightarrow f(t) = \sum_{i=1}^{n} c_i e^{p_i t} \, 1(t)$$

$$(s - p_1) F(s) = c_1 + \frac{(s - p_1)}{(s - p_2)} c_2 + \cdots + \frac{(s - p_1)}{(s - p_n)} c_n$$

@ $s = p_1 \Rightarrow c_1 = (s - p_1) F(s) \big|_{s = p_1}$

$$\Rightarrow c_i = (s - p_i) F(s) \big|_{s = p_i}$$

suppose p_1 is repeated 3 times

$$F(s) = \frac{c_1}{s - p_1} + \frac{c_2}{(s - p_1)^2} + \frac{c_3}{(s - p_1)^3} + \frac{c_4}{(s - p_4)} + \cdots + \frac{c_n}{s - p_n}$$

$$(s - p_1)^3 F(s) = (s - p_1)^2 c_1 + (s - p_2) c_2 + c_3 + \frac{c_4}{(s - p_4)}(s - p_1)^3 + \cdots + \frac{c_n}{(s - p_n)}(s - p_1)^3$$

@ $s = p_1 \Rightarrow c_3 = (s - p_1)^3 F(s) \big|_{s = p_1}$

derivative

$$\frac{d}{ds}\left((s-p_1)^3 F(s)\right) = 2(s-p_1)c_1 + c_2 + 0 + c_4 (s-p_1)^2(\ldots) + \ldots c_n (s-p_1)^2(.)$$

$$@\ s=p_1 \Rightarrow \quad c_2 = \frac{d}{ds}\left[(s-p_1)^3 F(s)\right]\Big|_{s=p_1}$$

derivative

$$\frac{d^2}{ds^2}\left((s-p_1)^3 F(s)\right) = 2c_1 + 0 + 0 + \ldots + c_4 (s-p_1)(\ldots) + \ldots + c_n (s-p_1)(..)$$

$$@\ s=p_1 \Rightarrow \quad c_1 = \frac{1}{2}\frac{d^2}{ds^2}\left((s-p_1)^3 F(s)\right)\Big|_{s=p_1}$$

pole p having multiciplity k

$$c_{k-i} = \frac{1}{i!}\left\{\frac{d^i}{ds^i}\left[(s-p)^k F(s)\right]\right\}\Big|_{s=p} \quad \text{for } i=0,\ldots,k-1$$

are the coefficients associated with pole p

* for complex conjugate poles, alternate method (in PFE supplement)

Time domain

ODE relating $y(t)\ u(t)$

$$\begin{array}{ccc} h(t),\ u(t) & \xrightarrow{\ \mathcal{L}\ } & \text{Frequency Domain} \quad H(s)\ U(s) \\ \downarrow \text{Integration} & & \downarrow \text{Algebraic Manipulation} \\ y(t) = h(t) * u(t) & \xleftarrow{\ \mathcal{L}^{-1}\ } & Y(s) = H(s)\,U(s) \end{array}$$

Practice Assgn

$$\frac{dy(t)}{dt} + 4y(t) = 5u_{in}(t) \qquad y(0)=2,\ u_{in}(t) = 1(t)$$

Solve using Laplace.

Week 1 Assgn

① ②
$$F(s) = \frac{s+5}{s+1} \qquad s = -1 + j3$$

$$\frac{4+j3}{j3} = -\frac{4}{3}j + 3 \qquad |F(s)| = \frac{5}{3}, \quad \tan^{-1}\left(\frac{9}{-4}\right)$$

③
$$2/s^3 + \frac{3}{(s+2)^2 + 3^2}$$

④
$$\mathcal{L}\{t \sin(at)\} = \frac{2as}{(s^2+a^2)^2}$$

V. 2.1.1

Modelling Mechanical Systems

Assumptions: (ignore mass & rotational effects of wheel, total friction from surface as $-b\dot{d}$)

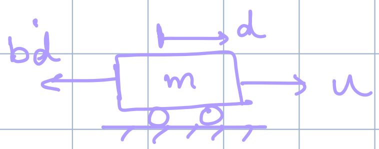
d: displacement

NLM 2nd

$$\sum F = ma$$
$$\sum M = I\alpha$$

→ FBD & define coordinates
↓
EOM

Case 1: Cart Model

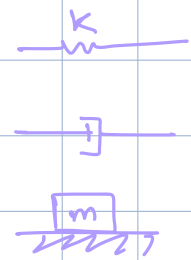


$$u - b\dot{d} = m\ddot{d}$$

spring K $\quad F_{spring} = -k(d - d_{rest})$

damper $\quad F_{damper} = -b\dot{d}$

friction $\quad F_{friction} = -b\dot{d}$



Cart Motion Example $\quad d(t) = ?$ if $u(t) = 1(t)$ assume $d(0) = 0$, $\dot{d}(0) = 0$, $m = b = 1$ $\qquad$ V 2.1.2

→ solve ODE

→ use Laplace transform ✓✓

(used superposition too)

$$u - b\dot{d} = m\ddot{d} \xrightarrow[\text{(from table)}]{\mathcal{L}} U(s) - b\,sD(s) = m\,s^2 D(s)$$

$$U(s) = (ms^2 + bs)D(s)$$

* Transfer function is the dynamic ratio of output to input.

$$H(s) = \frac{D(s)}{U(s)} = \frac{1}{ms^2 + bs}$$

$u(t) = \mathbb{1}(t) \rightarrow U(s) = 1/s$

$$D(s) = \frac{c_1}{s} + \frac{c_2}{s^2} + \frac{c_3}{s+1}$$

$c_3 = (s+1)\,D(s)\big|_{s=-1}$, - - - -
$= 1$

$$\Rightarrow D(s) = \frac{1/s}{(ms+b)\,s} = \frac{1}{s^2(ms+b)} = \frac{1}{s^2(s+1)} = \frac{1}{s+1} - \frac{1}{s} + \frac{1}{s^2}$$

$$\Rightarrow d(t) = \mathcal{L}^{-1}\{D(s)\} = \left(e^{-t} - 1 + t\right)\cdot \underbrace{\mathbb{1}(t)}_{}$$

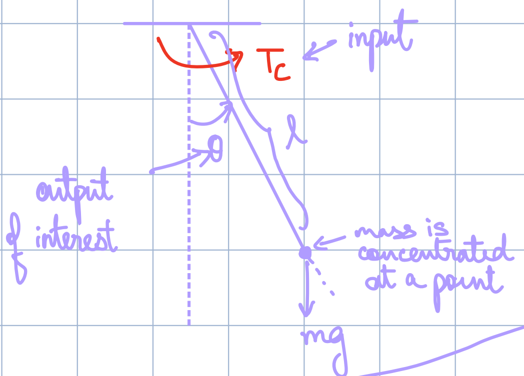
↳ assumed as causal signal

$$\boxed{d(t) = \left(e^{-t} - 1 + t\right)\cdot \mathbb{1}(t)}$$

3rd term will dominate in time @ steady state
⇒ answer makes sense

v 2.1.3

Pendulum Model & Linearisation



$$ml^2\ddot{\theta} = T_c - mg\,l\sin\theta$$

$$\ddot{\theta} + \frac{g}{l}\sin\theta = \frac{T_c}{ml^2}$$
↓
non linear EOM

does not follow superposition

$\theta \rightarrow \boxed{\sin(\ )} \rightarrow \sin\theta$

$\pi/3 \qquad \sqrt{3}/2$
$+\ \pi/6 \qquad 1/2$
$\pi/2 \longrightarrow 1 \neq \frac{\sqrt{3}+1}{2}$

* Taylor series expansion of a function $f(x)$

$$f(x) = \sum_{i=0}^{\infty} \frac{f^{i}(a)}{i!}(x-a)^i = f(a) + \frac{df(x)}{dx}\bigg|_{x=a}(x-a) + \text{- - - - -}$$

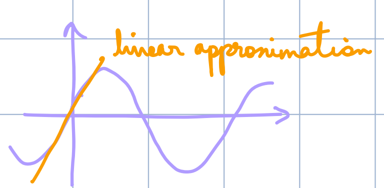
expansion done at point a ; assume function is infinitely differentiable

⇒ keeping only first 2 terms yields a linear function approximation of $f(x)$ near point a (approximation becomes more accurate closer to a) → $(x-a)$ terms become smaller when closer

$\theta_0 = 0$

$$\sin\theta = f(\theta) \approx f(\theta_0) + \frac{df(\theta)}{d\theta}\bigg|_{\theta=\theta_0}(\theta-\theta_0) = \cancel{\sin\theta_0}^{0} + \cancel{\cos\theta_0}^{1}(\theta - \cancel{\theta_0}^{0}) = \theta \Rightarrow f(\theta) = \sin\theta \approx \theta$$

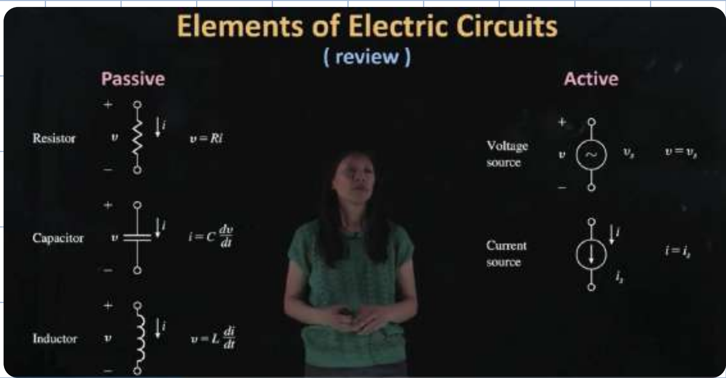
↓
good for approx $(-25^\circ, 25^\circ)$

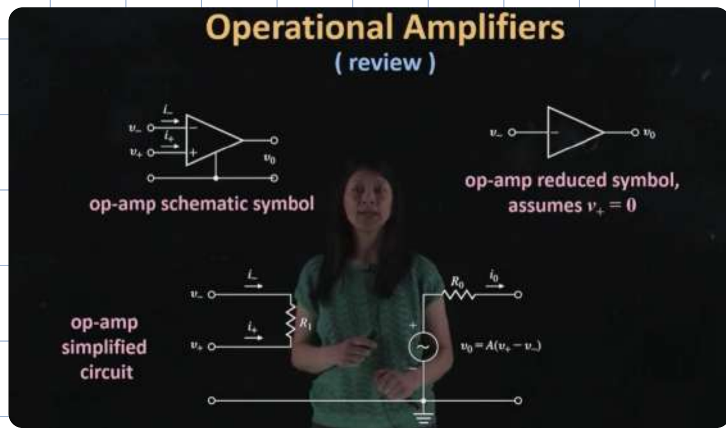


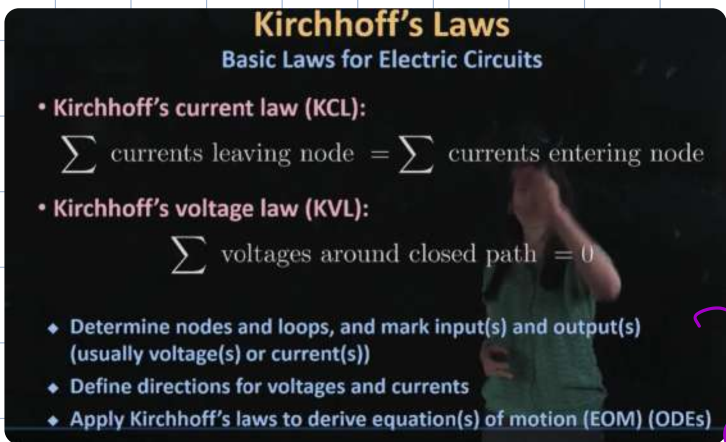
Q $T = 2\pi\sqrt{\frac{I}{mg\frac{l}{2}}}$, $I = \frac{ml^2}{3}$ (linear approx) $\Rightarrow 1 = 2\pi\sqrt{\frac{2l}{3g}} \Rightarrow l = \frac{3g}{8\pi^2} = 0.3727\,m$

→ most systems are non-linear but can be represented fairly well with linear models

v2.2.1

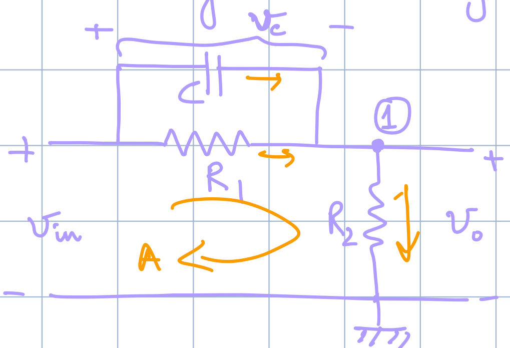






v2.2.2

Electric System Modelling : Lead Network Example.



KCL @ 1 $\quad C\frac{dv_c}{dt} + \frac{v_c}{R_1} = \frac{v_o}{R_2}$

KVL around loop A : $\quad v_{in} - v_c - v_o = 0$

(assume zero IC's)

$C(sV_c(s) - \underbrace{V_c(0)}_{0}) + \frac{V_c(s)}{R_1} = \frac{V_o(s)}{R_2}$

$V_{in}(S) - V_c(S) - V_o(S) = 0$

take Laplace transform → find transfer function from v_{in} to v_{out} to better understand dynamics of system.

$$V_c(s) = V_{in}(s) - V_o(s) \longrightarrow Cs\left(V_{in}(s) - V_o(s)\right) + \frac{\left(V_{in}(s) - V_o(s)\right)}{R_1} = \frac{V_o(s)}{R_2}$$

$$Cs\left(1 - \frac{V_o(s)}{V_{in}(s)}\right) + \frac{1}{R_1}\left(1 - \frac{V_o(s)}{V_{in}(s)}\right) = \frac{V_o(s)}{R_2 V_{in}(s)}$$

Why is this called a "lead network"?

$$\frac{V_o(s)}{V_{in}(s)}\left(-Cs - \frac{1}{R_1} - \frac{1}{R_2}\right) = -Cs - \frac{1}{R_1}$$

$\downarrow$

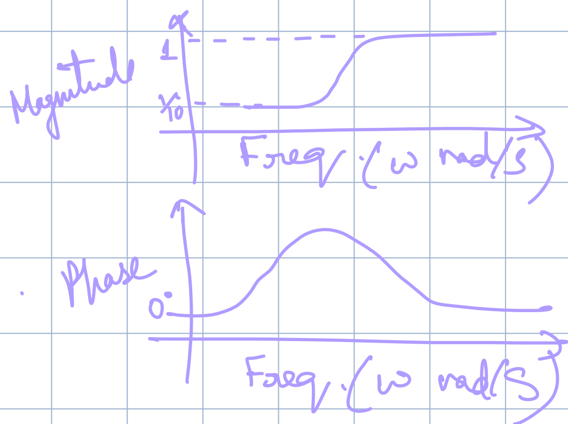
look at Bode plots for this network

$$H(s) = \frac{V_o(s)}{V_{in}(s)} = \frac{s + \frac{1}{R_1 C}}{s + \frac{1}{R_1 C} + \frac{1}{R_2 C}} = \frac{s+b}{s+a}$$

$\underline{a > b}$

eg

$$\sin(\omega t) \longrightarrow \boxed{\frac{s+1}{s+10}} \longrightarrow A\sin(\omega t + \phi)$$



at steady state, same frequency output for sinusoidal input!

$$A = |H(j\omega)| \quad , \quad \phi = \angle H(j\omega)$$

$$H(j\omega) = \frac{1 + j\omega}{10 + j\omega} \qquad \omega \ll 1 \rightarrow |H(j\omega)| = 1/10, \ \phi = 0^\circ$$

$$\omega \gg 1 \rightarrow |H(j\omega)| = 1, \ \phi = 0^\circ$$

$$\angle H(j\omega) = \tan^{-1}\left(\frac{\omega}{1}\right) - \tan^{-1}\left(\frac{\omega}{10}\right) \longrightarrow \text{always positive}$$

when $H(j\omega) > 1/10$, increases slowly till a peak, then drops.

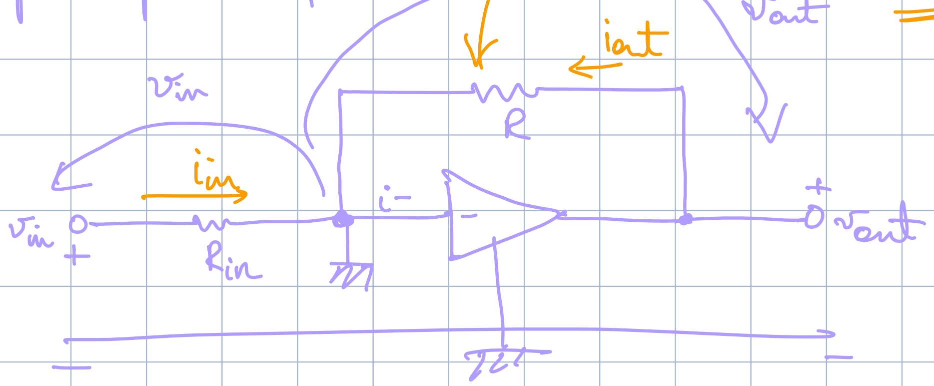
$\downarrow$

if we look at output response wrt input signal, some initial transient state but reaches steady state but looks like it's ahead, due to phase

Op-Amp Example

v2.2.3

|| → $v_{out} = -\frac{1}{CR_{in}} v_{in}$



$$\frac{v_{in}}{R_{in}} + \frac{v_{out}}{R} = 0$$

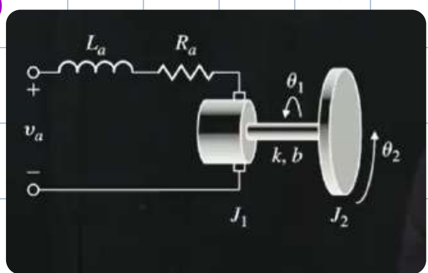
$$v_{out} = -\left(\frac{R}{R_{in}}\right) v_{in}$$

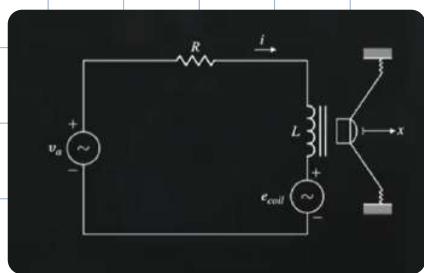
some negative gain

✱ if I want positive gain, cascade 2 of these circuits

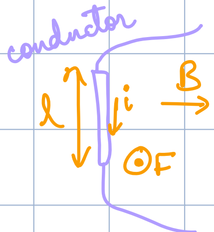
Modelling of ElectroMechanical Systems

v2.3.1





- Motor law $F = Bli$

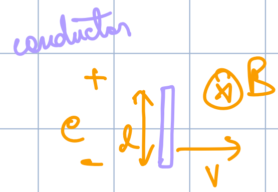


for particular system or geometry:

$F = K_m i$

$T = K_m i$

- Generator law $e = Blv$



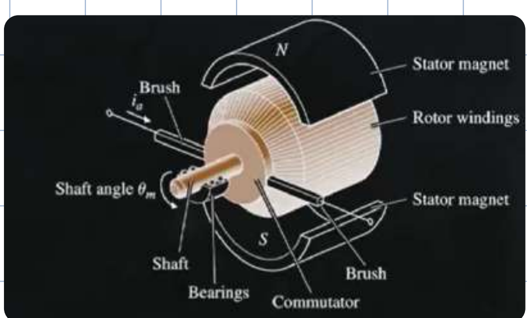
for particular system

$e = K_g v$

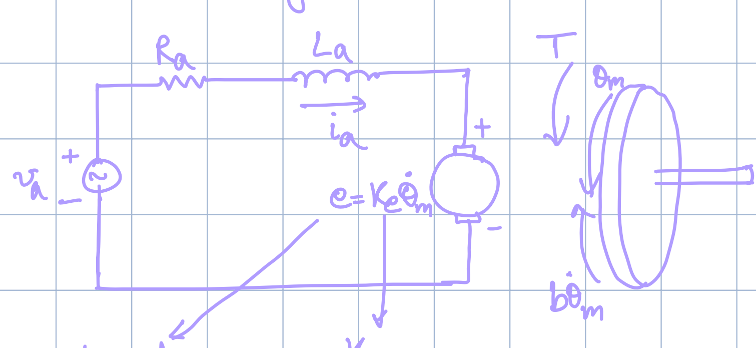
$e = K_g \theta$

combine these eqⁿ with those of electric & mechanical components

v2.3.2



Control v_a to yield a desired θ_m. Assume $T = K_t i_a$



EoM: $v_a - K_e\dot{\theta}_m = R_a i_a + L_a \frac{di_a}{dt}$

$T = K_t i_a$

$T - b\dot{\theta}_m = J_m\ddot{\theta}_m$

$$\ddot{\theta}_m + \frac{b}{J_m}\dot{\theta}_m = \frac{K_t}{J_m} i_a$$

Transfer function from v_a to θ_m (eliminate i_a variable, assume zero IC's)

$$\Theta_m(s)\left(s^2 + \frac{b}{J_m}s\right) = \frac{K_t}{J_m} I_a(s)$$

$$V_a(s) - K_e s\Theta_m(s) = (R_a + L_a s) I_a(s)$$

$$V_a(s) = \left(K_e s + \left(\frac{J_m s^2}{K_t} + \frac{b}{K_t}s\right)(R_a + L_a s)\right)\Theta_m(s)$$

$$V_a(s) = \left[\frac{J_m L_a}{K_t}s^3 + \left(\frac{R_a J_m}{K_t} + \frac{L_a b}{K_t}\right)s^2 + \left(\frac{R_a b}{K_t} + K_e\right)s\right]\Theta_m(s)$$

$$\frac{\Theta_m(s)}{V_a(s)} = \frac{1}{\cancel{\frac{J_m L_a}{K_t}}s^3 + \left(\frac{R_a J_m}{K_t} + \cancel{\frac{L_a b}{K_t}}\right)s^2 + \left(\frac{R_a b}{K_t} + K_e\right)s}$$

* often L_a is negligible relative to other factors

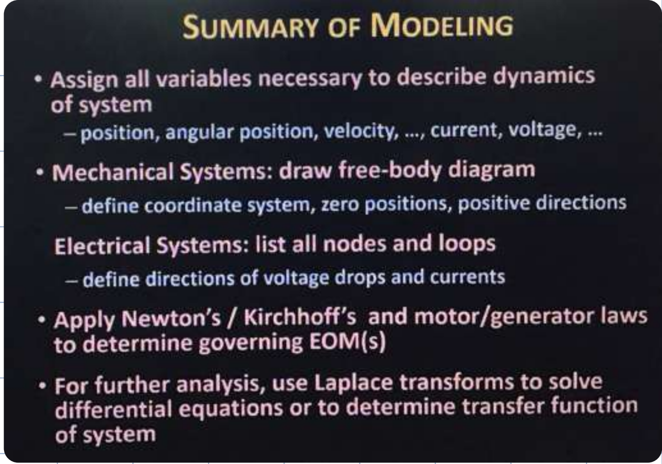
$$\frac{\Theta_m(s)}{V_a(s)} = \frac{K_t/R_a}{s\left(J_m s + \frac{bR_a + K_e K_t}{R_a}\right)} \cdot \frac{\left(\frac{R_a}{bR_a + K_e K_t}\right)}{\left(\frac{R_a}{bR_a + K_e K_t}\right)}$$

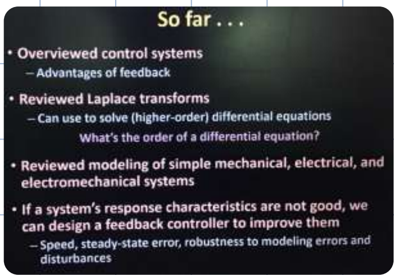
$$= \frac{K}{s(\tau s + 1)}$$ $\rightarrow$ a pole at origin and a real pole

usually want rotor speed of motor

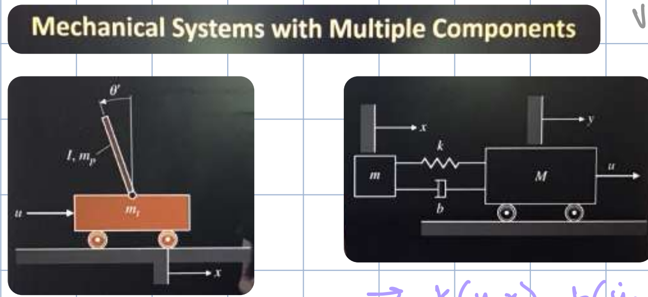
$$w = \dot{\theta}_m \rightarrow \Omega(s) = s\Theta_m(s) \quad \text{(zero IC's)}$$

$$\Rightarrow \frac{\Omega(s)}{V_a(s)} = \frac{K}{\tau s + 1}$$



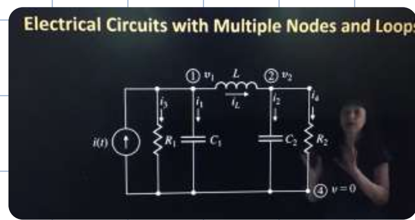


v2.3.3

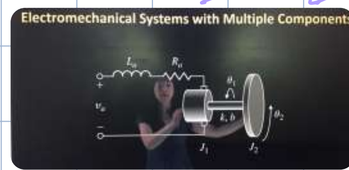


– draw FBD of both mech. sys.
– coupled by reaction forces

→ $k(y-x)$ $b(\dot{y}-\dot{x})$
→ similarly.



ext. version



(Week 2 Assgn. pdf stored)

v3.1.1

v3.1.2

→ transient response — how does the system respond initially

→ steady-state response — how does the system behave in the long run?

$$f(t) \overset{\mathcal{L}}{\longleftrightarrow} F(s)$$

Initial Value Theorem $\quad \lim_{s\to\infty} sF(s) = f(0^+)$

Final Value Theorem (conditional)

If all poles of $sF(s)$ are in the left half of the s-plane, then

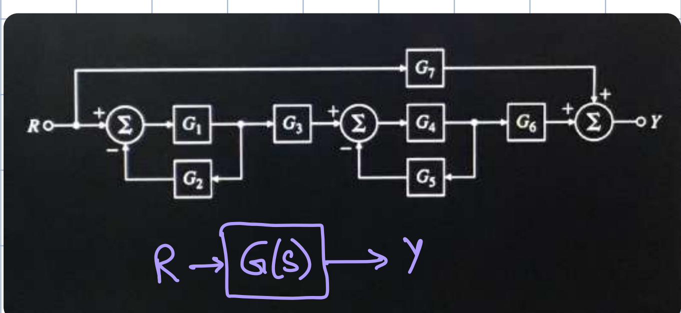
$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$$

→ checking stability (if unstable, no final steady value)

v3.1.2

Block Diagram Analysis

(can do this or go back to governing eqn's to eliminate the intermediatery variables)



Block Diagram Simplification Rules.

① Series / Product

$$U_1(s) \to \boxed{G_1} \to \boxed{G_2} \to Y_2(s)$$

$$Y_2 = G_2 G_1 U_1 \;\to\; \frac{Y_2(s)}{U_1(s)} = G_2 G_1$$

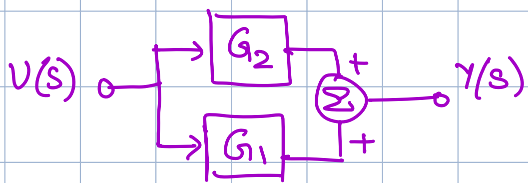
$$U_1 \to \boxed{G_1 G_2} \to Y_2$$

if multiple inputs, i.e vector, the block (eg G_1) are matrices

* commutative if scalar

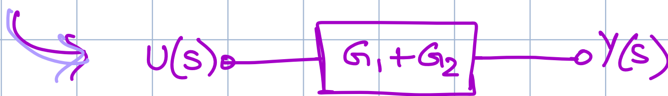
meaning single input

② Parallel / Sum

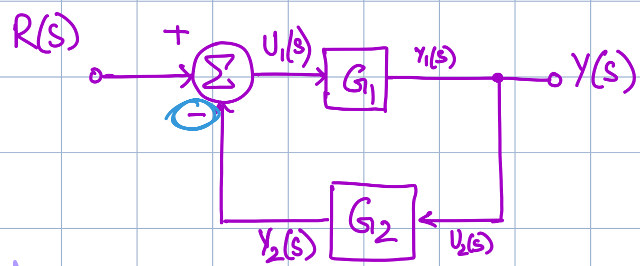


$Y = G_1 U + G_2 U = (G_1 + G_2) U$

$\frac{Y(S)}{U(S)} = G_1 + G_2$



③ Feedback Loop Rule



* one loop system
  - forward gain $= G_1$
  - loop gain $= G_1 G_2$
* negative feedback

IMPORTANT!

$$\boxed{\frac{Y(S)}{R(S)} = \frac{\text{forward gain}}{1 + \text{loop gain}}}$$

this sign would reverse if positive feedback

top part

$Y = G_1 U_1 = G_1 (R - Y_2)$

$Y = G_1 (R - G_2 Y)$

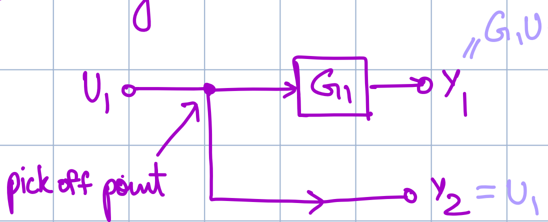
bottom part of loop

$Y(1 + G_2 G_1) = G_1 R$

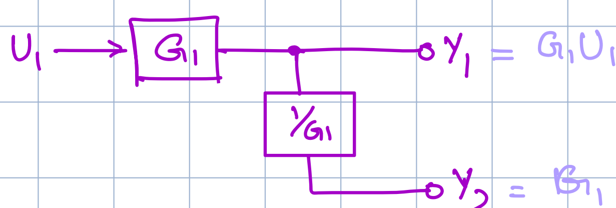
$\frac{Y(S)}{R(S)} = \frac{G_1}{1 + G_2 G_1}$

Block Diagram Manipulation

① Moving a node

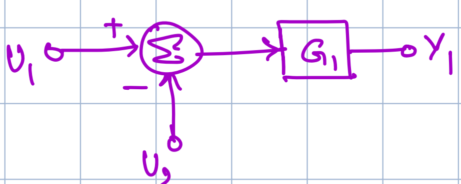


$\equiv$



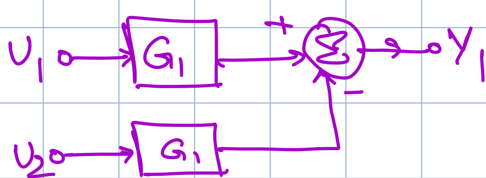
$Y_2 = G_1 \frac{1}{G_1} U_1 = U_1$

② Moving a summer



$Y_1 = G_1 (U_1 - U_2)$

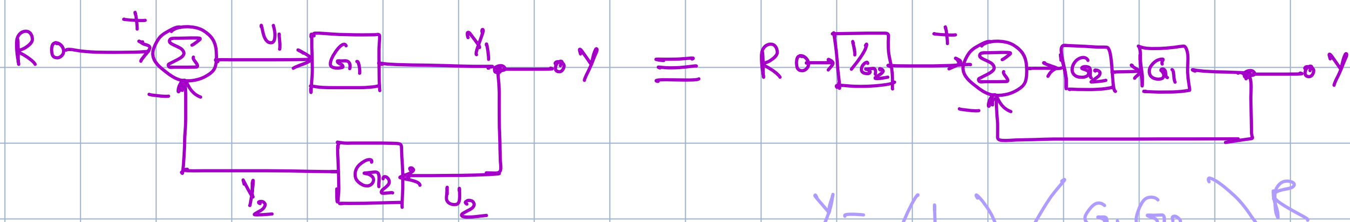
$\equiv$



$Y_1 = G_1 U_1 - G_1 U_2$

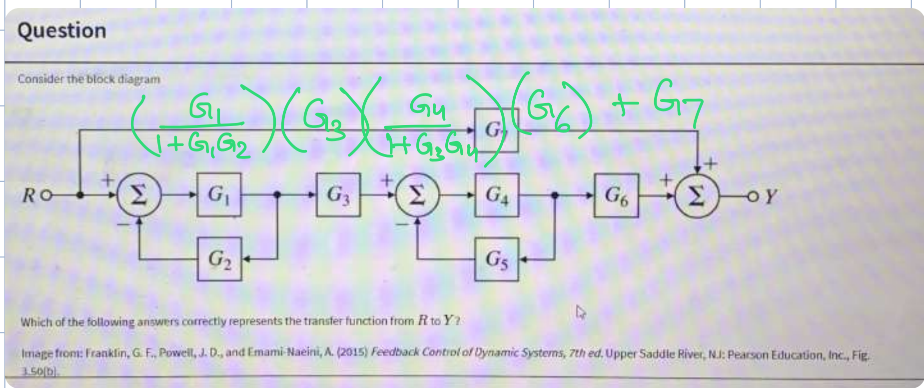
$= G_1 (U_1 - U_2)$

③ Moving a block from feedback path to forward path.



$$Y = \frac{G_1}{1+G_1G_2}R$$

$$Y = \left(\frac{1}{G_2}\right)\left(\frac{G_1G_2}{1+G_1G_2}\right)R$$



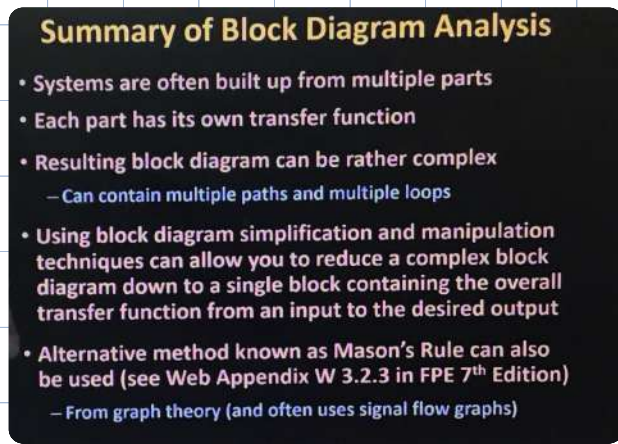
Question

Consider the block diagram

$$\left(\frac{G_1}{1+G_1G_2}\right)(G_3)\left(\frac{G_4}{1+G_3G_4}\right)(G_6)+G_7$$

Which of the following answers correctly represents the transfer function from R to Y?

Image from: Franklin, G. F., Powell, J. D., and Emami-Naeini, A. (2015) *Feedback Control of Dynamic Systems, 7th ed.* Upper Saddle River, NJ: Pearson Education, Inc., Fig. 3.50(b).

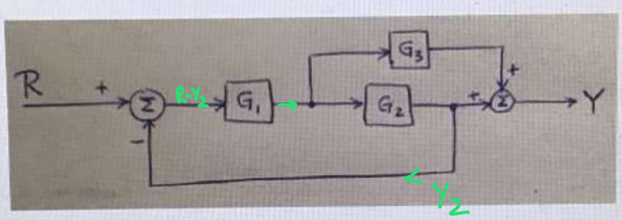


Summary of Block Diagram Analysis

- Systems are often built up from multiple parts
- Each part has its own transfer function
- Resulting block diagram can be rather complex
  - Can contain multiple paths and multiple loops
- Using block diagram simplification and manipulation techniques can allow you to reduce a complex block diagram down to a single block containing the overall transfer function from an input to the desired output
- Alternative method known as Mason's Rule can also be used (see Web Appendix W 3.2.3 in FPE 7th Edition)
  - From graph theory (and often uses signal flow graphs)

Week 3 Assgn Prac.

1. Consider the system depicted in the following block diagram:



What is the overall transfer function from R to Y?

forward path

$$G_2G_1(R-Y_2)+G_3(R-Y_2)G_1 = Y$$

$$Y_2 = G_2G_1(R-Y_2)$$

$$Y_2 = \frac{G_2G_1R}{1+G_1G_2}$$

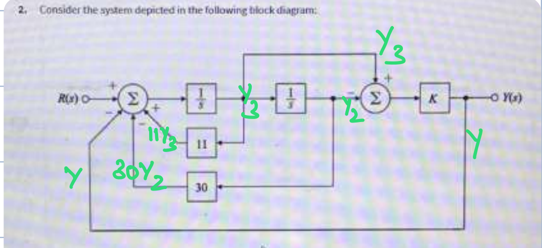
$$(R-Y_2)G_1(G_2+G_3) = Y$$

$$R\left(1-\frac{G_1G_2}{1+G_1G_2}\right)G_1(G_2+G_3) = Y$$

$$R\left(\frac{1}{1+G_1G_2}\right)G_1(G_2+G_3) = Y$$

$$\frac{Y}{R} = \frac{G_1(G_2+G_3)}{1+G_1G_2}$$

2. Consider the system depicted in the following block diagram:



What is the overall transfer function from R to Y?

Instructions for entering mathematical expressions can be found here. Express your answer in terms of "K" and the Laplace variable "s".

$$\left(-(R+11Y_3-30Y_2-Y)\frac{1}{s}\cdot\frac{1}{s}+Y_3\right)K = Y$$

$$Y_3 = (R+11Y_3-30Y_2-Y)\,1/s$$

$$Y_2 = Y_3/s$$

$$Y_3/s$$

$$K(-\cancel{Y_2}+Y_3) = Y \Rightarrow Y_3 = \frac{Y}{K(1-\frac{1}{s})}$$

$$Y_2 = \frac{Y}{K(s-1)}$$

$$(s-11)\frac{Y}{K(1-\frac{1}{s})} - R + \frac{30Y}{K(s-1)} + Y = 0$$

$$Y\left(\frac{s(s-11)}{K(s-1)} + \frac{30}{K(s-1)} + 1\right) = R$$

$$Y\left(\frac{s^2-11s+30+K(s-1)}{K(s-1)}\right) = R \rightarrow \frac{Y}{R} = \frac{K(s-1)}{(s^2+(K-11)s+30-K)}$$

Introduction to Dynamic Response.

V3.2.1

Transfer Function (TFs)

$$U_{in}(s) \longrightarrow \boxed{H(s)} \longrightarrow Y(s)$$

$$H(s) = \frac{Y(s)}{U_{in}(s)} = \frac{b_1 s^m + b_2 s^{m-1} + \cdots + b_{m+1}}{s^n + a_1 s^{n-1} + \cdots + a_n} = \frac{b(s)}{a(s)} = b_i \frac{\prod_{i=1}^{m}(s-z_i)}{\prod_{i=1}^{n}(s-p_i)}$$

zeros

poles

- for real systems, $m \leq n$
- n – order of system = order of corresponding DE relating $y(t)$ to $u_{in}(t)$

focus on 1st & 2nd order system

$$= \underbrace{\frac{C_1}{s-p_1} + \cdots}_{\text{1st order terms}} + \underbrace{\frac{C_i s + C_{i+1}}{s^2+2\sigma_i s+(\sigma_i^2+\omega_{d_i}^2)}}_{\text{2nd order terms}} + \cdots$$

usually left half plane we take

(real poles $s=-\sigma$) $\sigma>0$

(complex conjugate poles $s=\sigma \pm j\omega_d$)

overall system is a sum of 1st & 2nd order components; usually one or two of these components will be dominant

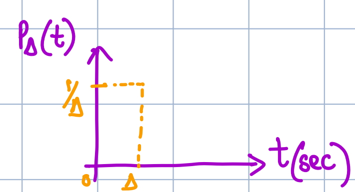
V3.2.2

1st Order System

$$U_{in}(s) \longrightarrow \boxed{H(s)} \longrightarrow Y(s)$$

$$H(s) = \frac{1}{s+\sigma} \quad \text{pole @ } s=-\sigma$$

Impulse Input $\quad u_{in}(t) = \delta(t) \quad \delta(t) = \lim_{\Delta\to 0} P_\Delta(t)$



$$\mathcal{L}\{\delta(t)\} = \int_{0^-}^{\infty} \delta(t) e^{-st}\,dt = \int_{0^-}^{0^+} \delta(t) e^{-st}\,dt = 1 \qquad \text{(sifting property)}$$

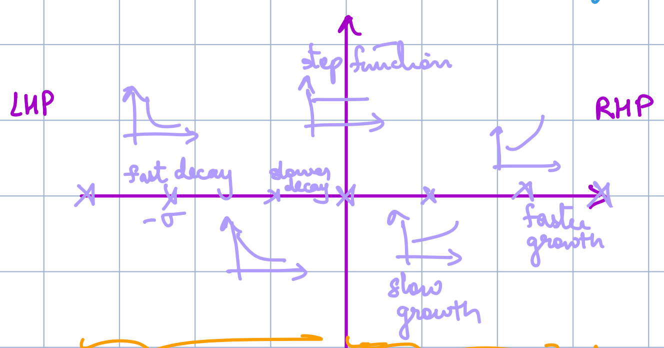
* Impulse Response of System is just the inverse Laplace transform of the Transfer function of the system $(Y(s) = H(s))$

$$\Rightarrow y(t) = h(t) = \mathcal{L}^{-1}\{H(s)\}$$

Impulse Response $h(t) = e^{-\sigma t} 1(t)$

$$\int_{-\infty}^{\infty} \delta(t)\, g(t)\, dt = g(0) \quad \text{[Sifting property]} \quad , \int_{-\infty}^{\infty} \delta(t-1000)\,(t^2+3)\,dt = 1000^2+3$$

1st Order System Impulse Response as a function of pole location $\quad H(s) = \dfrac{1}{s+\sigma} \quad h(t) = e^{-\sigma t} 1(t)$



stable poles. unstable poles.

If a system transfer function is n^{th} order with only real poles, the pole furthest to the right in the s-plane is dominant.

→ so best approximation of the system would be the term with right most pole as a first-order system

eg. $H(s) = \dfrac{C_1}{s+\sigma_1} + \dfrac{C_2}{s+\sigma_2} + \dfrac{C_3}{s+\sigma_3} + \dfrac{C_4}{s} + \dfrac{C_5}{s+\sigma_5} + \dfrac{C_6}{s+\sigma_6} + \dfrac{C_7}{s+\sigma_7}$

eg. if all poles considered $\dfrac{C_7}{s+\sigma_7}$
but if only stable poles then $\dfrac{C_3}{s+\sigma_3}$

$\delta(t) \rightarrow \boxed{H(s)} \rightarrow y(t)$

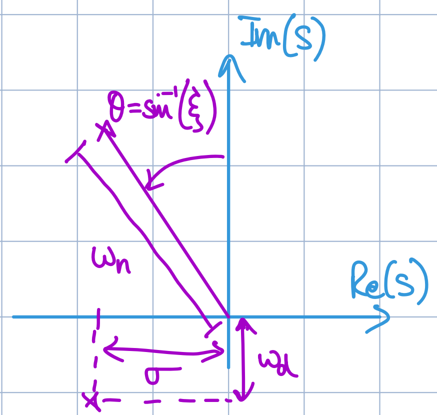
accuracy of approximation is more accurate if spaced-out well and not clustered.

v3.2.3

2nd Order System $\quad H(s) = \dfrac{b_1}{s^2 + a_1 s + a_2} \qquad U_{in}(s) \rightarrow \boxed{H(s)} \rightarrow Y(s)$

$$b_1 = a_2 = \omega_n^2, \quad a_1 = 2\xi\omega_n \longrightarrow H(s) = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

↳ with this definition we'll have DC gain as 1.



$$s = \underbrace{-\xi\omega_n}_{\sigma} \pm j\,\underbrace{\omega_n\sqrt{1-\xi^2}}_{\omega_d}$$

damping

undamped natural frequency

distance of pole from imaginary axis

ω_d ↓ distance of pole from real axis

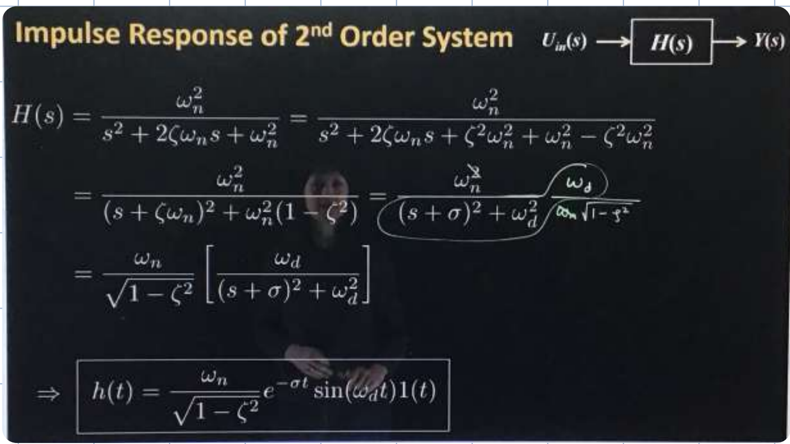
$\omega_n = \sqrt{(\xi\omega_n)^2 + \omega_d^2}$
$= \sqrt{\sigma^2 + \omega_d^2}$

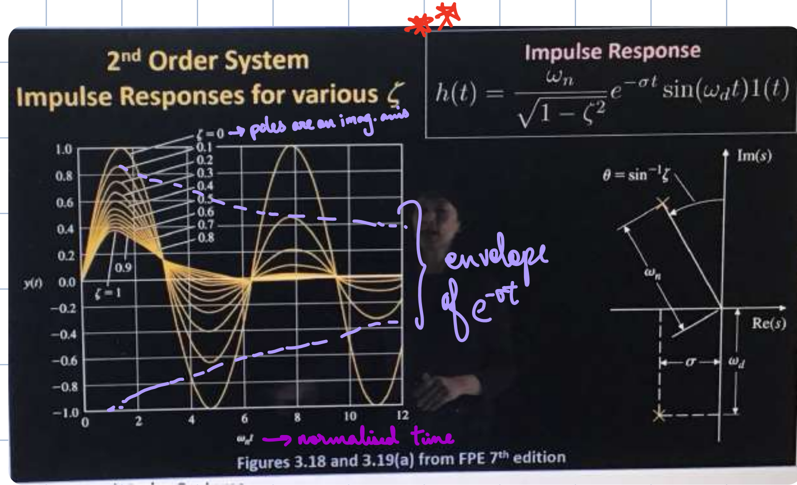
$\underline{\omega_n > 0}$.

actual frequency/ damped natural frequency

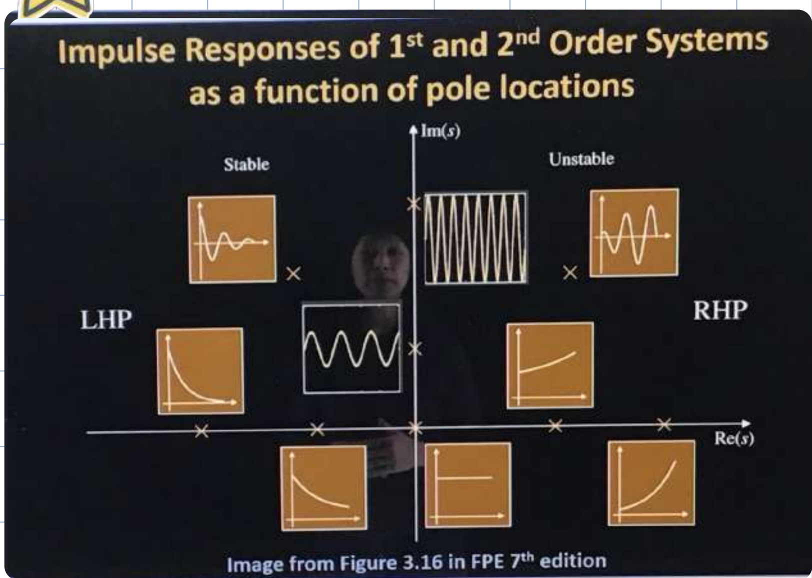
$0 < \xi < 1 \rightarrow$ for stable roots/poles.

(when $\xi > 1$, 2 real poles;
$\xi < 1$, left plane, unstable system)



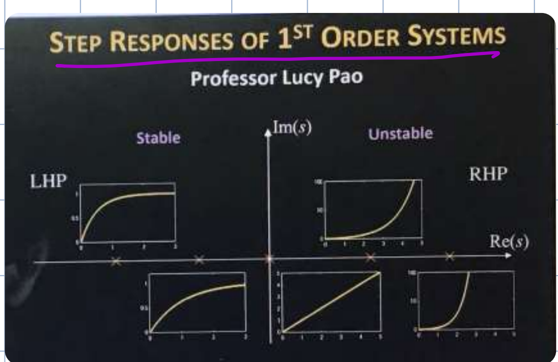


for $\xi > 1$, envelope increases, $(\sigma < 0)$.
poles on the right half plane



(assumed complex conjugate plots below the axis.)

v 3.2.4



$H(s) = \frac{b}{s+\sigma}$ pole @ $s = -\sigma$

U_{in}

t

$U_{in}(s) \rightarrow \boxed{H(s)} \rightarrow Y(s)$

$U_{in}(t) = 1(t)$

$\hookrightarrow U_{in}(s) = 1/s$

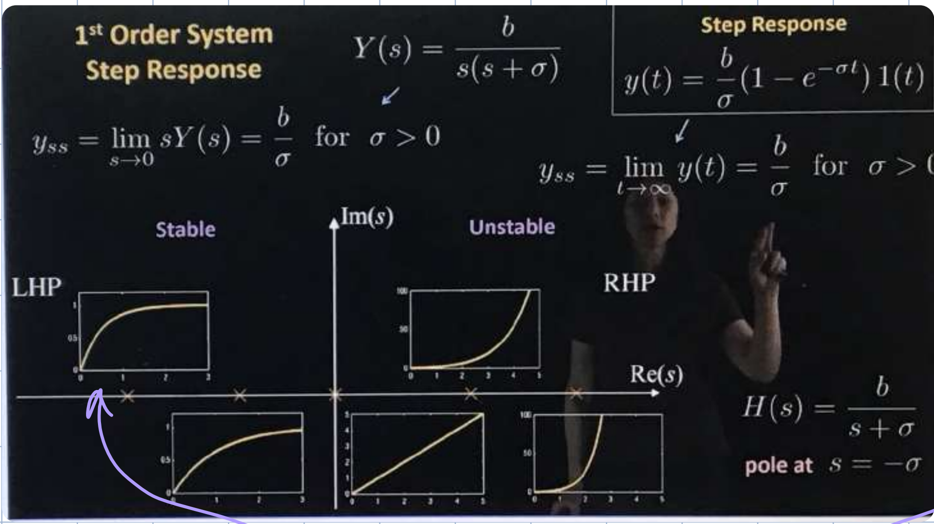
$Y(s) = H(s)\,U_{in}(s)$

$= \frac{b}{s+\sigma} \cdot \frac{1}{s}$

$C_1 = sY(s)\big|_{s=0} = b/\sigma \longleftarrow Y(s) = \frac{C_1}{s} + \frac{C_2}{s+\sigma}$

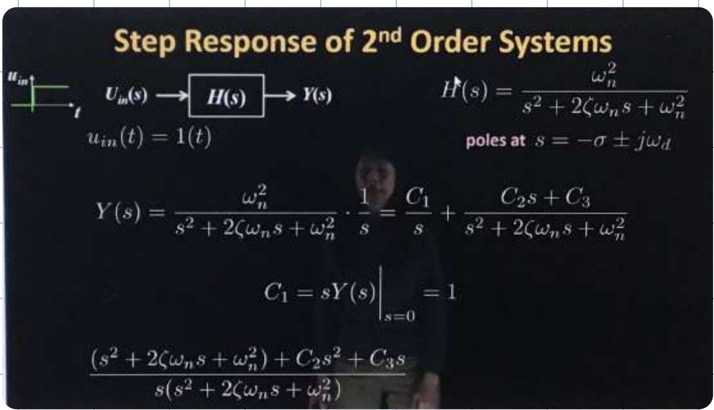
$C_2 = (s+\sigma)\,Y(s)\big|_{s=-\sigma} = -\frac{b}{\sigma}$

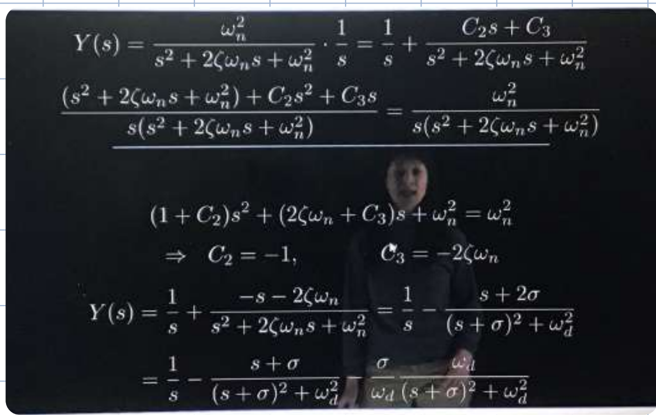
$\Rightarrow y(t) = \frac{b}{\sigma}1(t) + \left(-\frac{b}{\sigma}\right)e^{-\sigma t}1(t) \Rightarrow \boxed{y(t) = \frac{b}{\sigma}\left(1-e^{-\sigma t}\right)1(t)}$ [causal system]



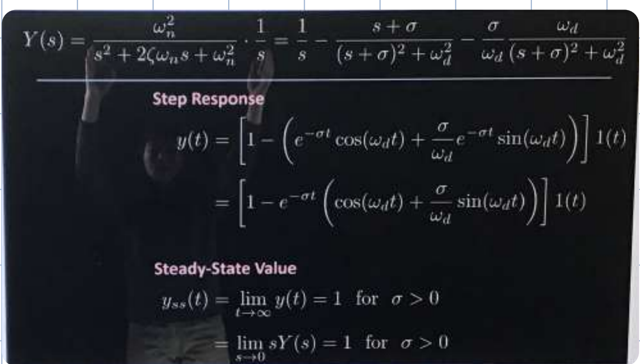
further left means faster convergence / decay

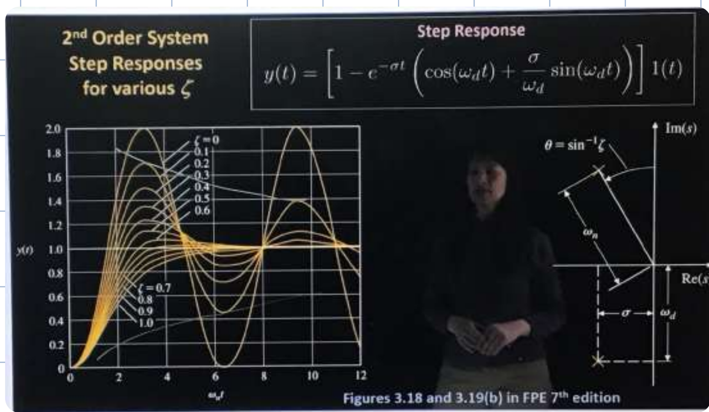
V3.2.5





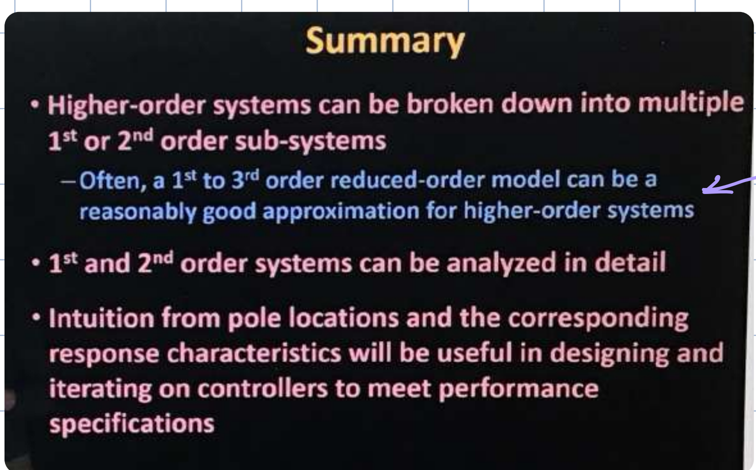
$\frac{s+a}{(s+a)^2+b^2} \xleftrightarrow{\mathcal{L}} e^{-at}\cos bt$ $\quad$ $\frac{b}{(s+a)^2+b^2} \xleftrightarrow{\mathcal{L}} e^{-at}\sin bt$





(similar to what we saw for impulse response)

V3.2.6



if any pole/poles is much farther to the right of the s-plane are dominant, Week 3 Graded Assgn

IVT: $\lim_{s\to\infty} sF(s) = f(0^+)$ $\qquad$ $\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$

① $G(s) \quad \dfrac{-2(s-1)}{(s+1)(s+2)}$

$$Y(s)=G(s)U(s) = \frac{-2\ (s-1)}{s(s+1)(s+2)} \qquad y f(0^+) = \frac{-2\left(1-\frac{1}{s}\right)\frac{1}{s}}{\left(1+\frac{1}{s}\right)\left(1+\frac{2}{s}\right)} = 0$$

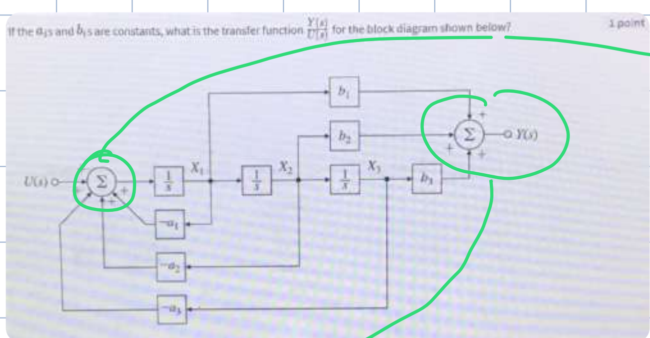
② $y(t) \xleftrightarrow{\mathcal{L}} Y(s)$

$$\lim_{t\to\infty} y_1(t) = \lim_{s\to 0} Y_1(s) = \lim_{s\to 0} \frac{-2\ (s-1)}{(s+1)(s+2)} = \underline{\underline{1}}$$

③ $$\dot{y}_1(0^+) = \lim_{s\to\infty} s^2 Y_1(s)$$

$$= \lim_{s\to\infty} \frac{-2\ s(s-1)}{(s+1)(s+2)} = \underline{\underline{-2}}$$

④ $$\lim_{t\to\infty} \dot{y}_1(t) = \lim_{s\to 0} s^2 Y_1(s) = -\frac{2(s-1)s}{(s+1)(s+2)} = \underline{\underline{0}}$$

⑤ 

$$X_1 = \frac{1}{s}\left(U - a_1X_1 - a_2X_2 - a_3X_3\right)$$

$$X_2 = \frac{X_1}{s}, \quad X_3 = \frac{X_2}{s}$$

$$(s+a_1)X_1 = U - a_2\left(\frac{X_1}{s}\right) - a_3\left(\frac{X_1}{s^2}\right)$$

$$X_1 = \frac{s^2}{s^2+a_1s^2+a_2s+a_3}\,U$$

$\to X_2, X_3$

$\to Y(s) = b_1X_1 + b_2X_2 + b_3X_3$

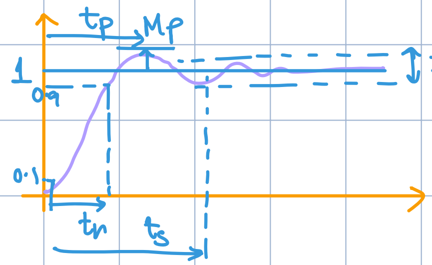
$$\frac{Y(s)}{U(s)} = \frac{b_1s^2+b_2s+b_3}{s^2+a_1s^2+a_2s+a_3}$$

⑥ $v_{in}(t) = \operatorname{sgn}\{h(-t)\}$

$$h(t) = \begin{cases} -1 & h(-t)<0 \\ 0 & h(-t)=0 \\ 1 & h(-t)>0 \end{cases}$$

$$y(0) = \int_{-\infty}^{\infty} h(\tau)\, v_{in}(0-\tau)\, d\tau = \int_{-\infty}^{\infty} h(\tau) \operatorname{sgn}\{h(\tau)\}\, d\tau = \int_{-\infty}^{\infty} |h(\tau)|\, d\tau$$

Transient Step Response Performance Specifications



step responses are common because we tend to change systems from one fixed state/setpoint to another.

t_r : rise time — 0 or 0.1 to 0.9 level time duration

t_p : peak time — time till maximum overshoot occurs

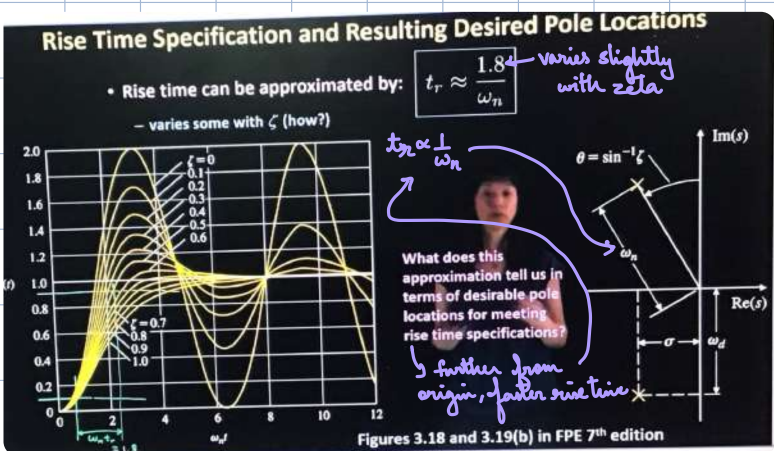
M_p : maximum overshoot

t_s : settling time — Time after which response will stay within some settle boundary

we will develop some rules of thumb based on 2nd order system response and desired pole location:

$$y(t) = \left[1 - e^{-\sigma t}\left(\cos(\omega_d t) + \frac{\sigma}{\omega_d}\sin(\omega_d t)\right)\right] 1(t)$$

Rise Time & Settling Time Specification and Desired Pole Location



* for 1% settling time,

$$e^{-\sigma t_s} = 0.01$$

$$t_s = \frac{2\ln 10}{\sigma} \approx \frac{4.6}{\sigma} = \frac{4.6}{\xi\omega_n}$$

further poles implies faster settling time → more into left half plane.

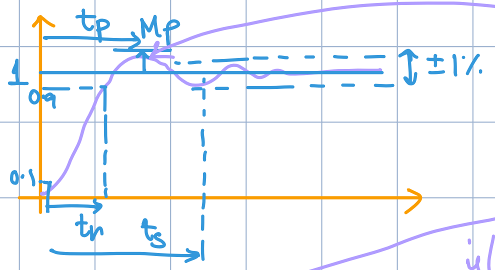
Practice Assgn :- 5% settling time

$$e^{-\sigma t_s} = 0.05 \Rightarrow t_s = \frac{\ln 20}{\sigma} = \frac{2.99}{\sigma}$$

Peak Time & Overshoot Specifications and Desired Pole locations

$$U_{in}(s) \longrightarrow \boxed{H(s)} \longrightarrow Y(s) \qquad H(s) = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

* step response of second-order system (with no finite zeros)



$y(t)$ is flat, $\dot{y}(t) = 0$

first time when this happens

$$y(t) = 1 - e^{-\sigma t}\left(\cos(\omega_d t) + \frac{\sigma}{\omega_d}\sin(\omega_d t)\right)$$

$$\dot{y}(t) = \sigma e^{-\sigma t}\left(\cancel{\cos(\omega_d t)} + \frac{\sigma}{\omega_d}\sin(\omega_d t)\right) - e^{-\sigma t}\left(-\omega_d \sin(\omega_d t) + \cancel{\sigma\cos(\omega_d t)}\right)$$

$$\dot{y}(t) = 0 \Rightarrow e^{-\sigma t}\underbrace{\left(\frac{\sigma^2}{\omega_d} + \omega_d\right)}_{>0}\sin(\omega_d t) = 0.$$

$$\Downarrow$$

$$\omega_d t = n\pi \quad (n=1).$$

$$\boxed{t_p = \pi/\omega_d}$$

→ can use as general rule of thumb for higher order systems too of damped frequency of dominant mode

$$y(t_p) = 1 - e^{-\frac{\sigma\pi}{\omega_d}}\left(\cancelto{-1}{\cos\pi} + \frac{\sigma}{\omega_d}\cancelto{0}{\sin\pi}\right)$$

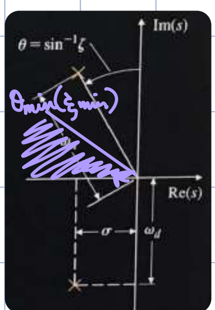
$$\boxed{y(t_p) = 1 + e^{-\sigma\pi/\omega_d}} = 1 + M_p \Rightarrow M_p = \frac{\sigma\pi}{\omega_d} = \frac{\xi\omega_n\pi}{\omega_n\sqrt{1-\xi^2}}$$

$$\boxed{M_p = e^{\frac{-\xi\pi}{\sqrt{1-\xi^2}}}, \quad 0 < \xi < 1}$$

$\xi < 0$ → grows

$\xi = 0$ → no steady state, only oscillatory

$\xi \geq 1$ → real axis overdamped

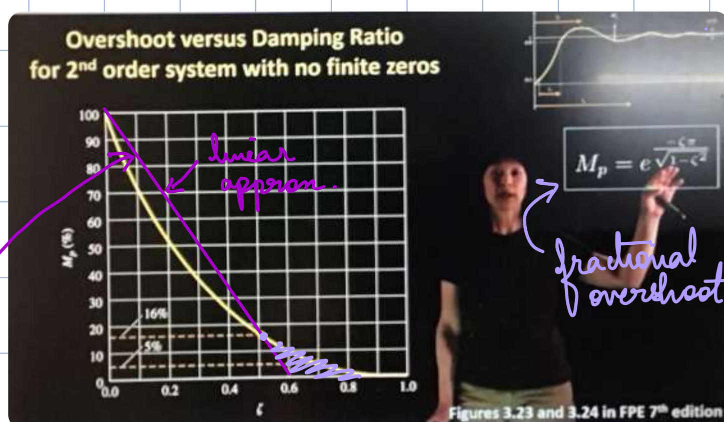


as $\xi\uparrow$, $\theta\uparrow$ → $M_p\downarrow$

region till which over shoot is fine,

$\theta > \theta_{min}$ for

$M_p < M_{p_{max}}$ ← constraint



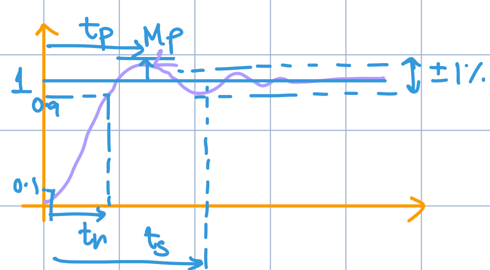
this does not work for very low overshoot requirements

$$M_p = 100 - \frac{100}{0.6}\xi$$

$$0 < \xi < 0.6 \leftarrow M_p = 100\left(1 - \frac{\xi}{0.6}\right)$$

v4.1.5

Step Response Specifications : Analysis vs Design.

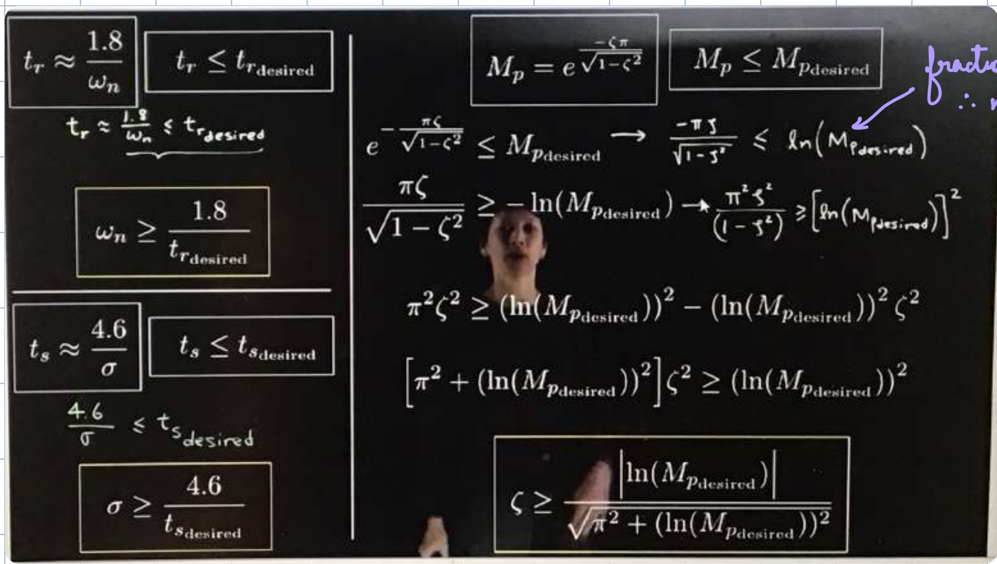


- Analysis - given your system and estimates of ξ and ω_n (hence, σ and ω_d)

$$t_r \approx \frac{1.8}{\omega_n}, \quad t_s = \frac{4.6}{\sigma}, \quad t_p = \frac{\pi}{\omega_d}, \quad M_p = e^{\frac{-\xi\pi}{\sqrt{1-\xi^2}}}$$

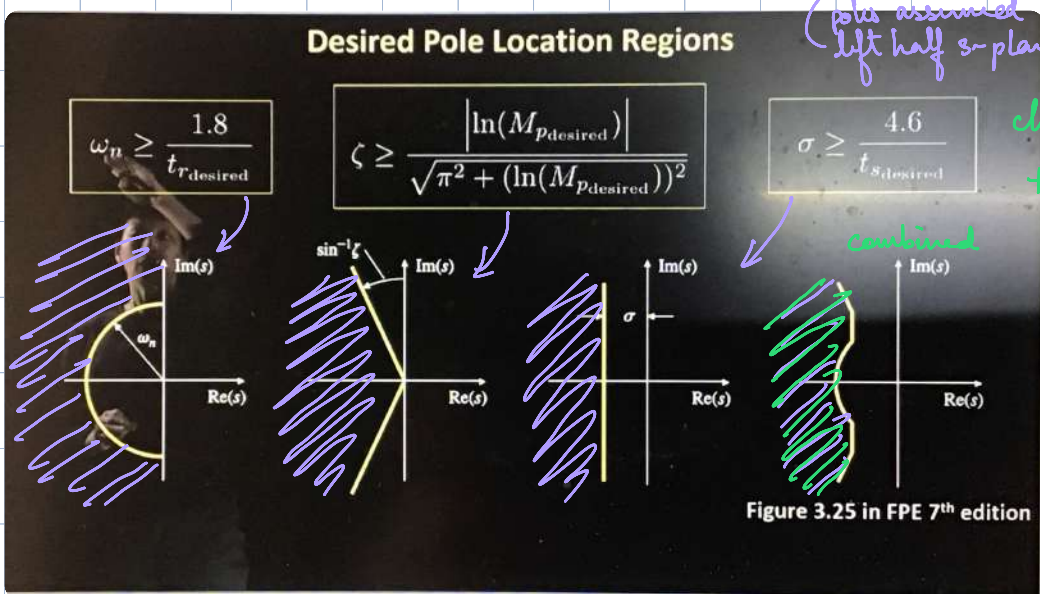
• Design – given desired specifications for t_r, t_s and M_p, the above relations can be used to determine the desired location of the system poles.

$t_r < t_{r_{desired}}$; $t_s < t_{s_{desired}}$; $M_p \leq M_{p_{desired}}$



fractional, b/w 0 & 1, ∴ negative.

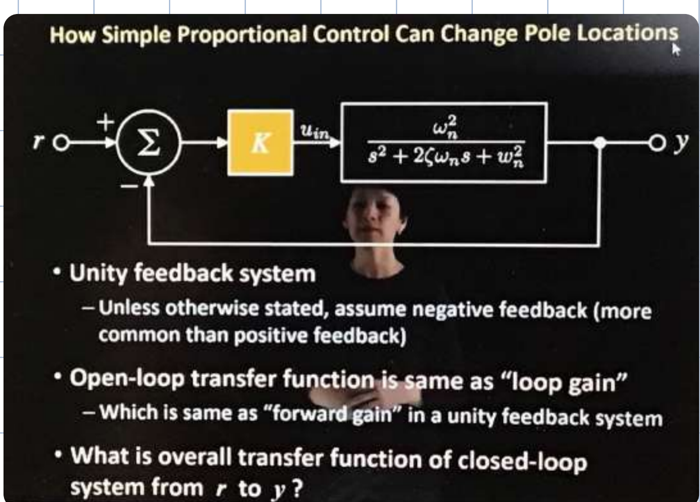
(poles assumed in left half s-plane)

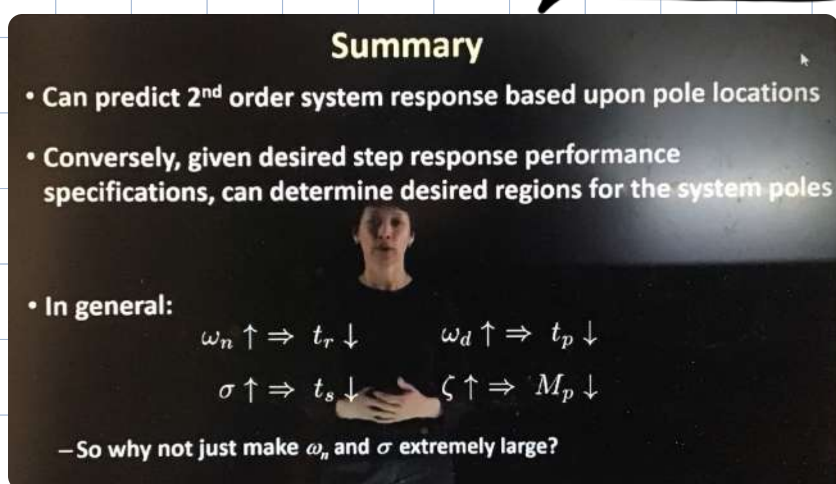


closed loop poles → basically the modified open loop original system with a controller

these are used as approx. for higher order systems too.

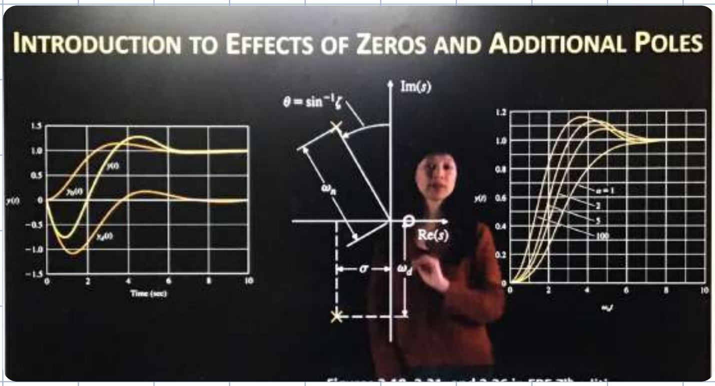
IMPORTANT!

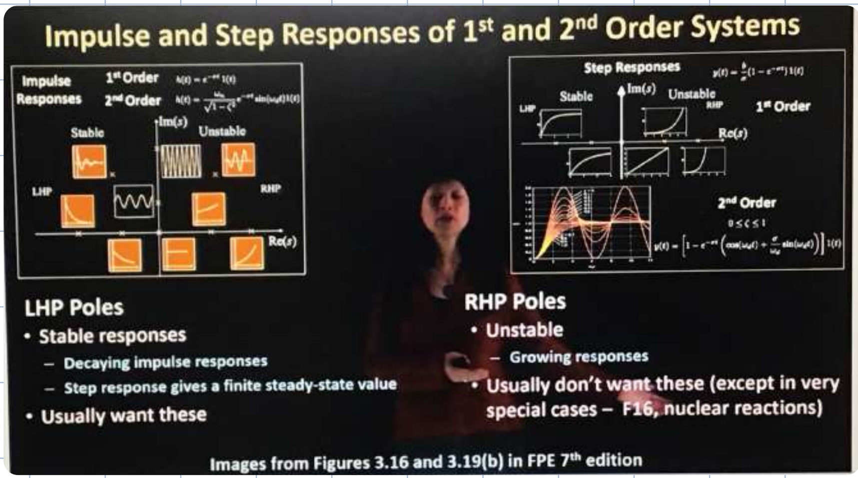




Effects of Zeros and Additional Poles

v4.2.1





DC Gain of a System

v4.2.2

$u_{in} = 1(t)$

$U_{in}(s) \longrightarrow \boxed{H(s)} \longrightarrow Y(s) \longrightarrow Y(s) = H(s) \cdot \frac{1}{s} \Rightarrow y_{ss} = \lim_{s\to 0} sY(s) = \lim_{s\to 0} H(s)$

- DC gain of a system is the ratio of the output of a system to a constant input.

when stable system, $\text{DC gain} = \lim_{s\to 0} H(s)$

same as applying unit-step input to system $H(s)$ and computing the final value of the system response using the Final Value Theorem.

2^{nd} order system considered had $\underbrace{\text{unit DC gain}}$, $H(s) = \dfrac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$

often desired but not always the case.

Effect of Zeros.

v4.2.3

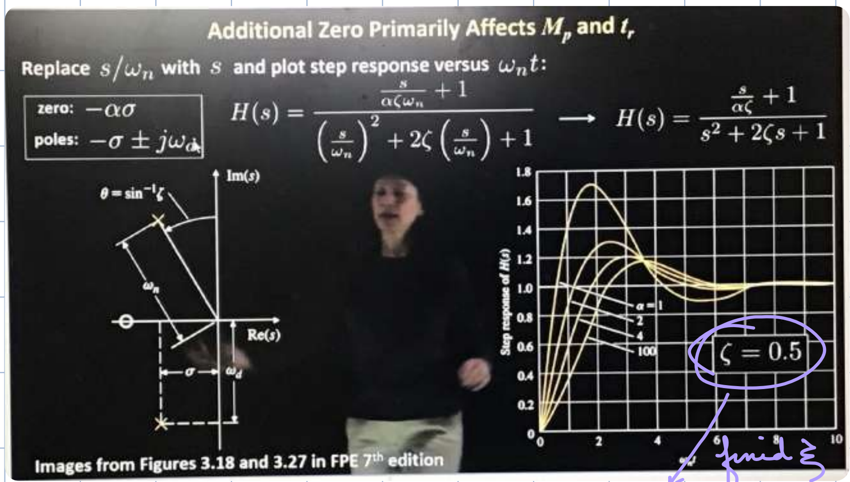
- how does an additional zero in the 2^{nd} order system change the response?

Consider transform with "normalised" time and zero locations

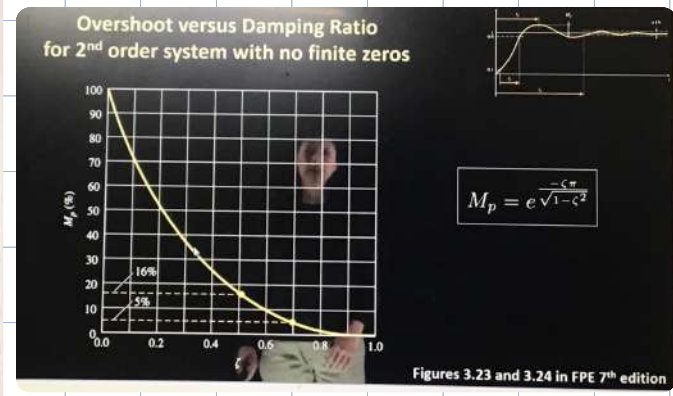
$$H(s) = \frac{\frac{s}{\alpha\xi\omega_n} + 1}{(s/\omega_n)^2 + 2\xi(s/\omega_n) + 1}$$

$s = -\xi\omega_n \pm j\omega_n\sqrt{1-\xi^2} = \sigma + j\omega_d$ — poles are unchanged

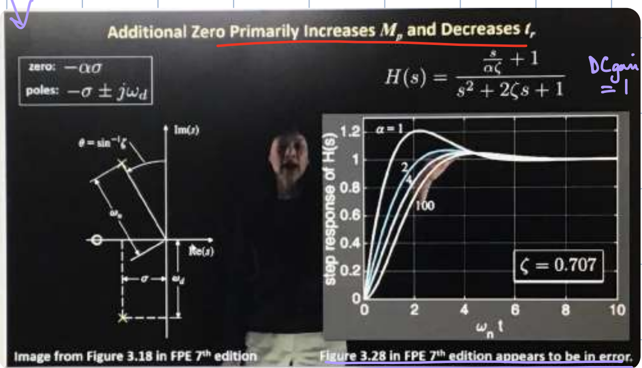
zero at $s = -\alpha\xi\omega_n = -\alpha\sigma$ ($\alpha > 0$, zero in left half plane)

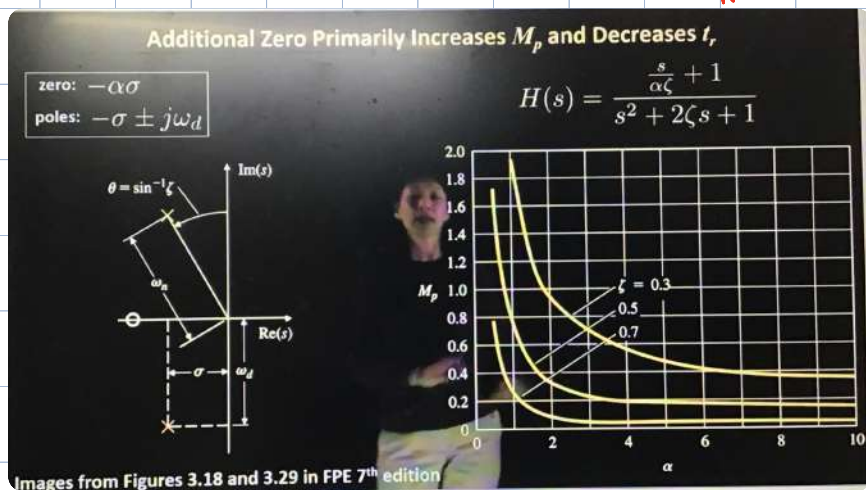


fixed ξ



as $\alpha\uparrow$, the overshoot will converge to the case where there is no zero (~18%) @ almost greater than 4x from $j\omega$ as the poles.



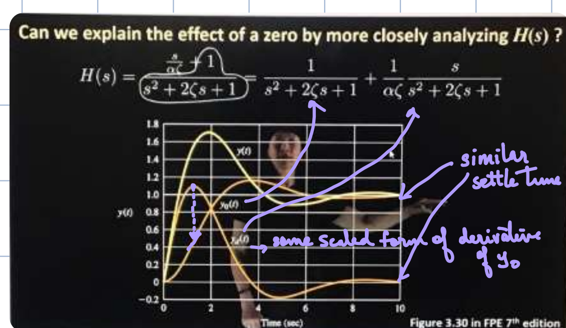


v4.2.4

Effect of Zeros: Analytical Explanation

$$H(s) = \frac{\frac{s}{\alpha\xi}+1}{s^2+2\xi s+1} = \frac{1}{s^2+2\xi s+1} + \frac{1}{\alpha\xi}\cdot\frac{s}{s^2+2\xi s+1}$$

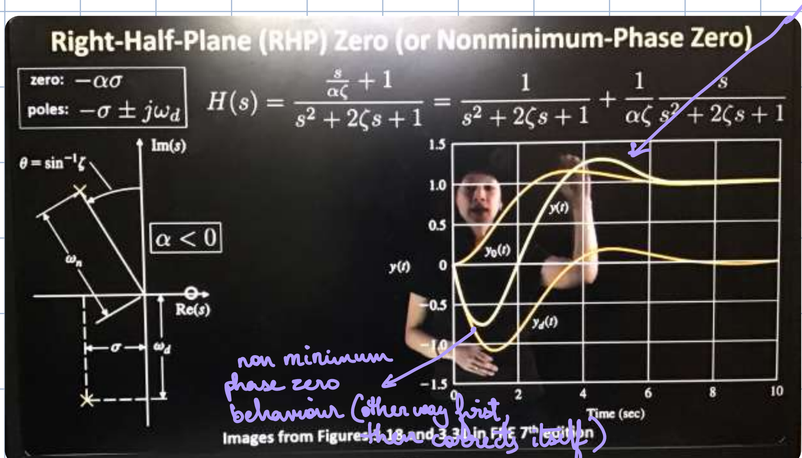
as $\alpha\uparrow$, lesser effect of 2nd term



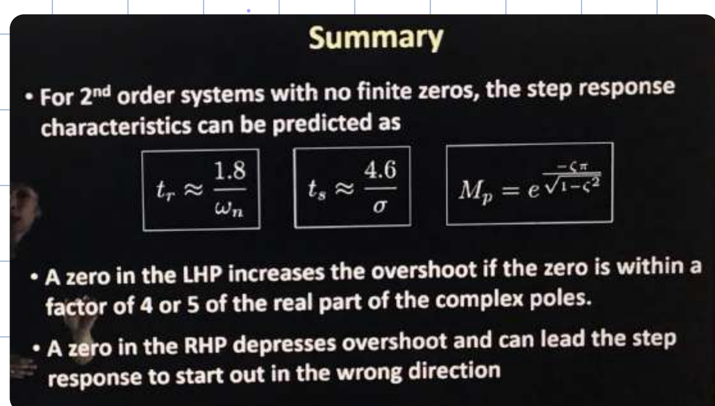
similar settle time

some scaled form of derivative of y_0

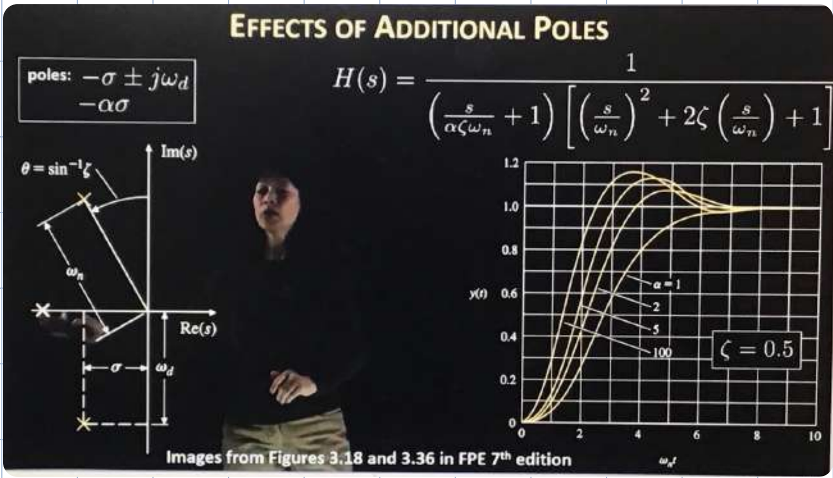
suppressed overshoot

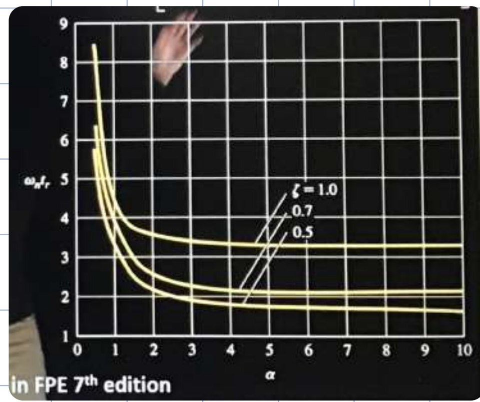


non minimum phase zero behaviour (other way first then corrects itself)



v4.2.5





$\alpha > 0 \rightarrow$ stable system; similarly as $\alpha\uparrow$, additional pole no effect.

$\alpha\downarrow \rightarrow$ rise time increase, decrease overshoot
$\downarrow$
effect more intense

Near Pole - Zero Cancellations

v4.2.6

$$H_1(s) = \frac{2}{s+2} \qquad H_2(s) = \frac{2(s+1+\epsilon)}{(1+\epsilon)(s+1)(s+2)} \qquad H_3(s) = \frac{2(s-1+\epsilon)}{(1-\epsilon)(s-1)(s+2)}$$

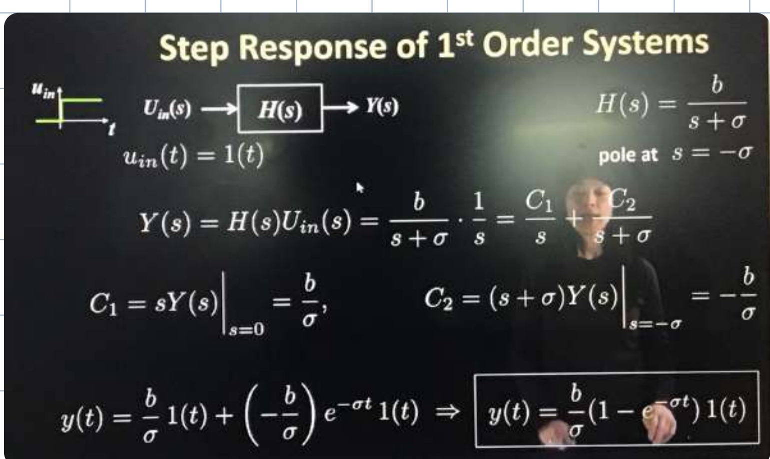
to keep same DC gain

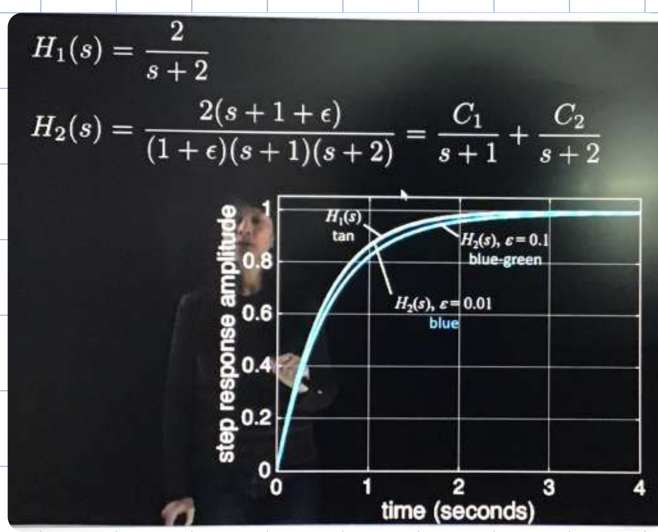
Is the behaviour of $H_2(s)$ and $H_3(s)$, both similar to $H_1(s)$? Why or why not?

$$= \frac{C_1}{s+1} + \frac{C_2}{s+2} \qquad = \frac{C_1}{s-1} + \frac{C_2}{s+2}$$

$$C_1 = (s-p)\,H(s)\big|_{s=p}$$

$\Rightarrow$ If $H(s)$ has a zero near pole @ $s=p$, then C_1 is small.

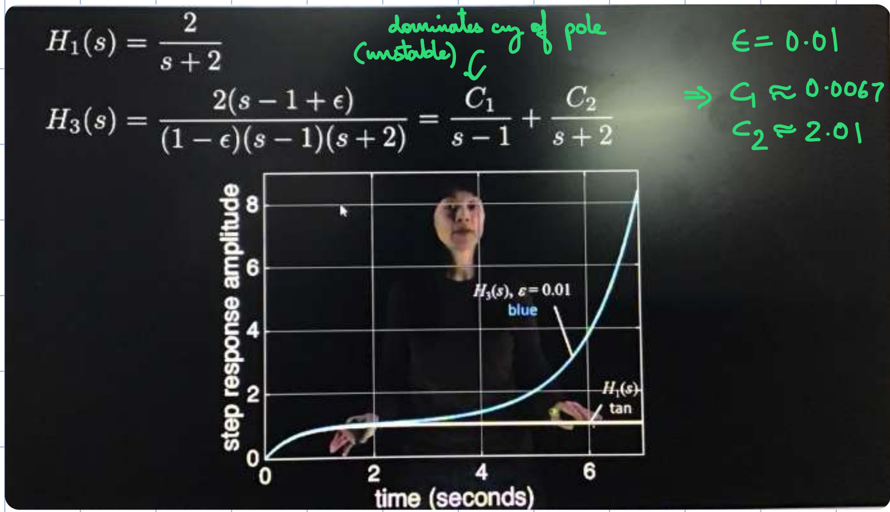


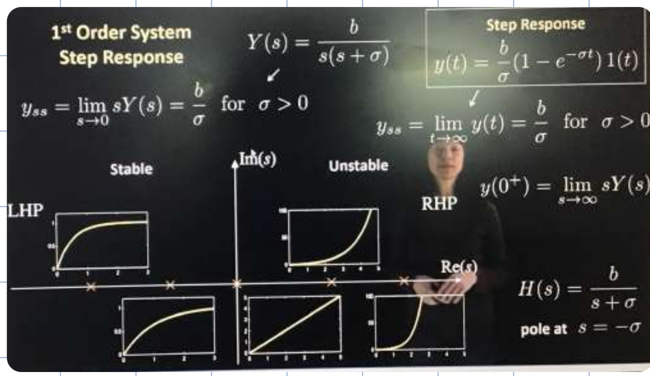


$\epsilon = 0.01$
$\Rightarrow C_1 \approx 0.02$
$C_2 \approx 1.96$

$\epsilon = 0.1$
$\Rightarrow C_1 \approx 0.18$
$C_2 \approx 1.64$

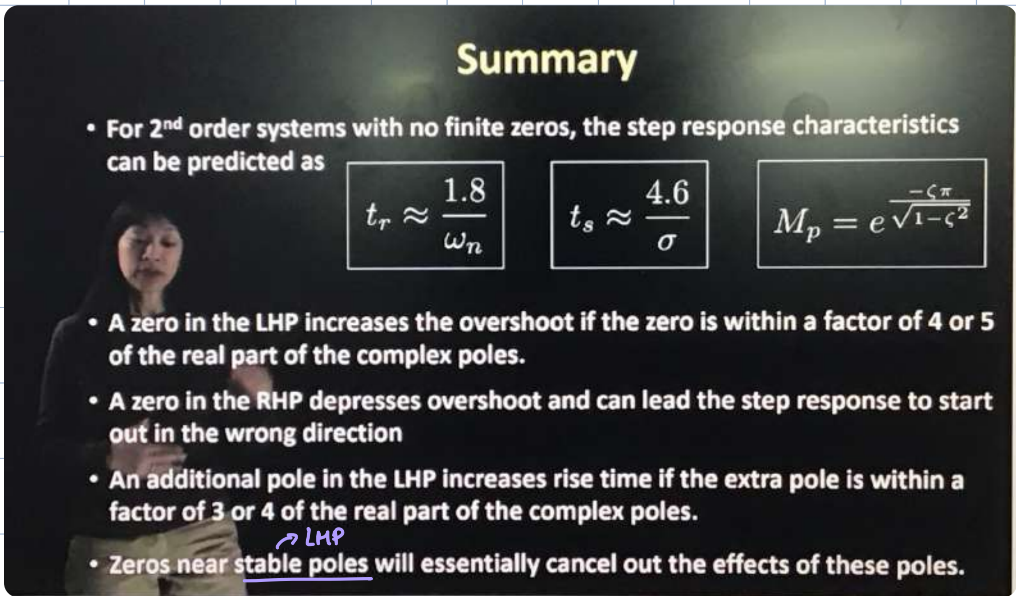
$\Rightarrow$ stable pole-zero cancellations have similar response to OG.





$\Rightarrow$ unstable pole-zero cancellations will have different response

$\checkmark$ 2.4.7

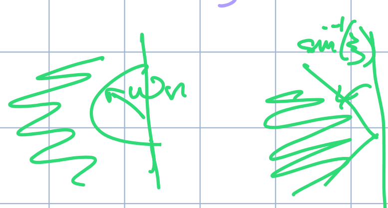


Week 4 Assessment

(1)

$$1 + G(s) = 1 + \frac{K}{s(s+2)} =$$

$$\frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$$



$$t_r \leq 0.6$$

$$\frac{1.8}{\omega_n} \leq 0.6$$

$$\omega_n \geq 3$$

&

$$\sqrt{K} \geq 3 \Rightarrow \underline{K = 9}$$

$$0.05 \geq e^{-\frac{\xi\pi}{\sqrt{1-\xi^2}}}$$

$$\frac{-\xi\pi}{\sqrt{1-\xi^2}} \geq \ln 0.05$$

$$\frac{\xi^2}{1-\xi^2} \geq \left(\frac{\ln 0.05}{\pi}\right)^2 \Rightarrow \xi \geq \sqrt{\frac{\left(\frac{\ln 0.05}{\pi}\right)^2}{1 + \left(\frac{\ln 0.05}{\pi}\right)^2}}$$

$$\xi \geq 0.7$$

(2) $s^2+2s+K=0$

$s = -1 \pm \sqrt{1-K}$ $\qquad \omega$-

$\xi = \frac{1}{\cancel{\omega_n}\sqrt{K}} = 0.7 \Rightarrow K = 2.04$, $\quad \underline{\underline{K \geq 2.04}}$

(3) $t_r = \frac{1.8}{\omega_n} = \frac{1.8}{\sqrt{K}}$ if $K=3$, $\Rightarrow t_r = 1.039$

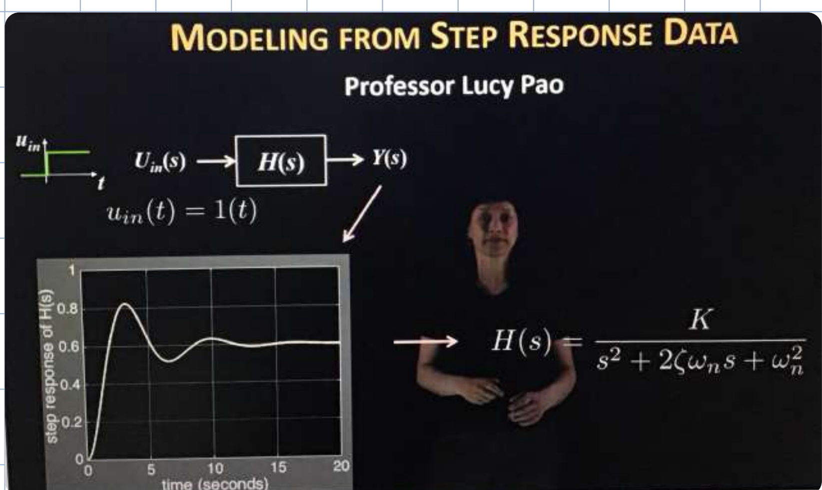
(4) $s = -1+j = -\xi\omega_n \pm \sqrt{1-\xi^2}\,\omega_n$

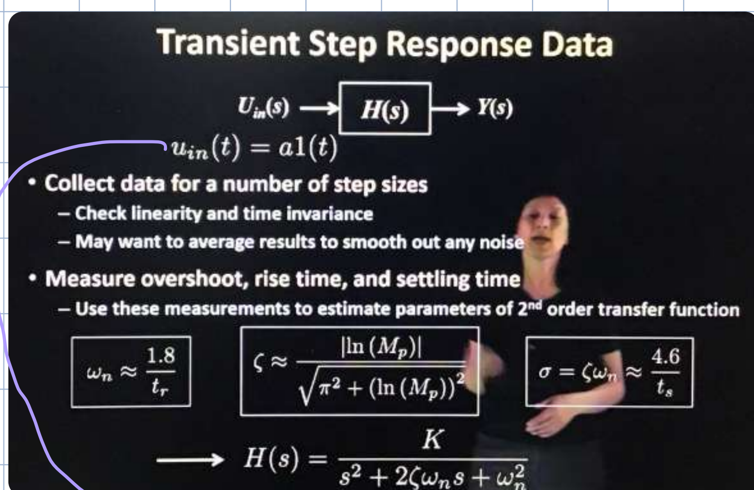
$\omega_n = \sqrt{2}$, $\quad \xi = 1/\sqrt{2}$

$t_r = \frac{1.8}{\cancel{\omega_n}\,1.414} \approx 1.2$ $\qquad M_p = e^{-\frac{\xi\pi}{\sqrt{1-\xi^2}}} = e^{-\frac{\cancel{1/\sqrt{2}}\,\pi}{\cancel{1/\sqrt{2}}}} = e^{-\pi} \approx \frac{1}{9} \approx 0.1$

v5.1.2

Modelling from Transient Step Response Data.

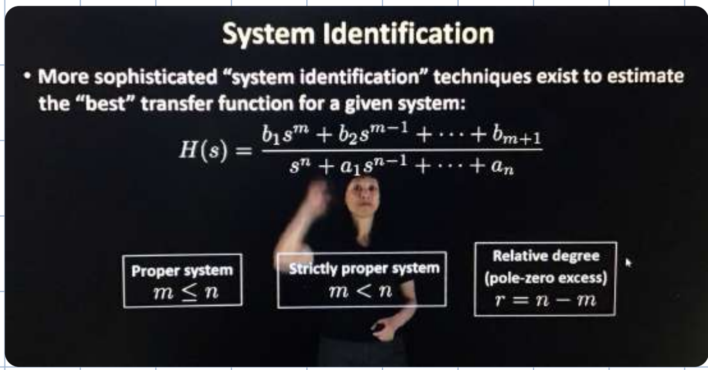




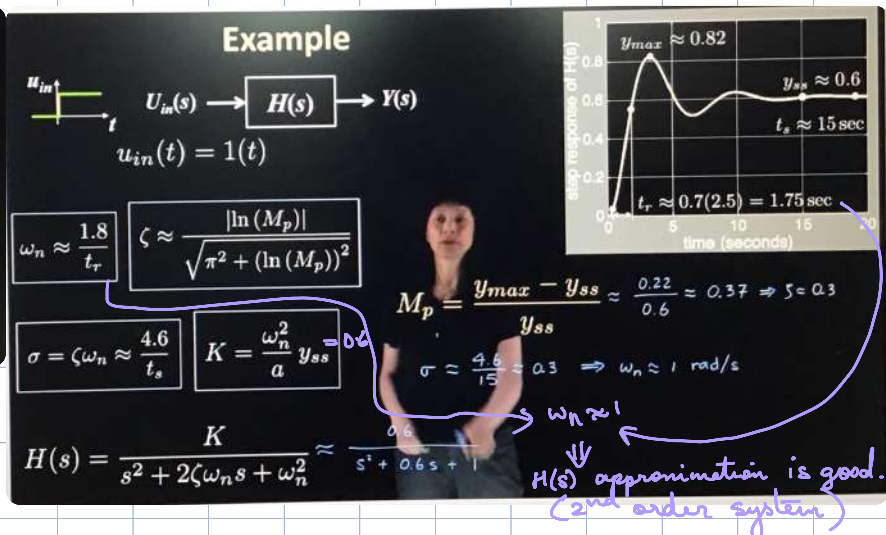
$y_{ss} = \lim_{t\to\infty} y(t) = \lim_{s\to 0} sY(s) = \lim_{s\to 0} s \cdot \frac{K}{s^2+2\xi\omega_n s+\omega_n^2} \cdot \frac{a}{s} = \frac{Ka}{\omega_n^2}$

$\Rightarrow K = \frac{\omega_n^2\, y_{ss}}{a}$

the modelling method outlined here works well if system is predominantly 2nd order.



exactly proper system $\rightarrow m = n$



$\Rightarrow$ If a system is predominantly 2^{nd} order, the damping ξ, natural frequency ω_n and steady state (or DC) gain can be reasonably estimated from step response data.

V5.1.3

Example and Introduction to Proportional Control

$$u_{in}(t) \longrightarrow \boxed{\frac{1}{s^2+2s+4}} \longrightarrow y(t)$$

$u_{in}(t) = 1(t)$

→ Reasonably sketch the step response $y(t)$

→ Can proportional control be used to decrease t_r?

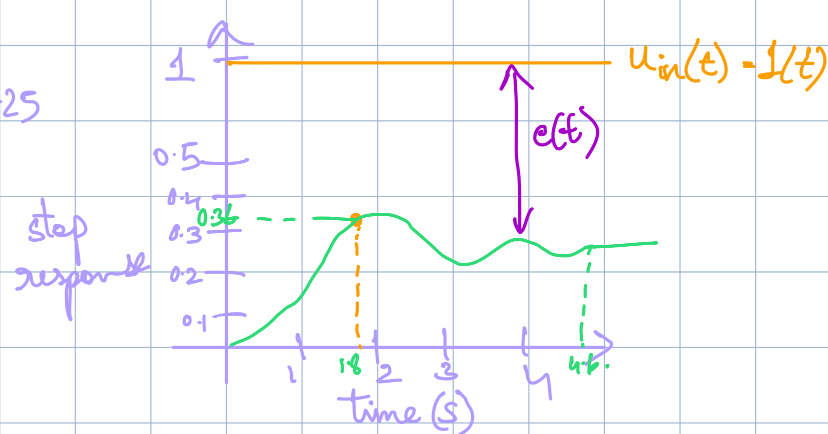
$$\frac{1}{s^2+2\xi\omega_n s+\omega_n^2} \Rightarrow \omega_n = 2 \text{ rad/s}, \quad \xi = \tfrac{1}{2}, \quad \sigma = \xi\omega_n$$

$$\Rightarrow t_r \approx \frac{1.8}{\omega_n} = \frac{1.8}{2} = \underline{\underline{0.9\,s}} \qquad t_s \approx \frac{4.6}{\sigma} = \underline{\underline{4.6\,s}} \qquad M_p = e^{\frac{-\xi\pi}{\sqrt{1-\xi^2}}} \approx \underline{\underline{0.16}}$$

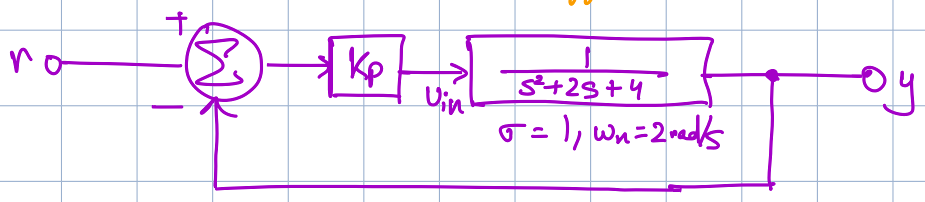
$$t_p = \frac{\pi}{\omega_d = \omega_n\sqrt{1-\xi^2}} \approx \frac{\pi}{2\sqrt{3/4}} \approx \frac{\pi}{\sqrt{3}} \approx \underline{\underline{1.8\,s}}$$

$$\text{DC gain} = \lim_{s\to 0} H(s) = \frac{1}{4} = 0.25$$

$$e(t) = u_{in}(t) - y(t) \approx 0.75$$



How does Proportional Control affect Closed-Loop Poles?



$$\frac{Y(s)}{R(s)} = \frac{\text{forward gain}}{1 + \text{loop gain}}$$

$$H_{cl}(s) = \frac{Y(s)}{R(s)} = \frac{\frac{K_p}{s^2+2s+4}}{1 + \frac{K_p}{s^2+2s+4}} = \frac{K_p}{s^2+2s+4+K_p} = \frac{K_{cl}}{s^2 + 2\xi_{cl}\omega_{n_{cl}} s + \omega_{n_{cl}}^2}$$

$\sigma_{cl} = 1$; $t_{s_{cl}} = t_s$

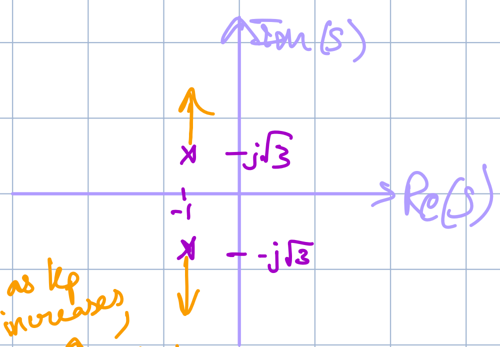
$\omega_{n_{cl}} = \sqrt{4+K_p} > 2$. (assumed $K_p > 0$)

$\Rightarrow \omega_{n_{cl}} > \omega_{cl}$.

$\xi_{cl}\,\omega_{n_{cl}} = 1 \Rightarrow \xi_{cl} = \frac{1}{\sqrt{4+K_p}}$

$t_{r_{cl}} \approx \frac{1.8}{\omega_{n_{cl}}} \approx \frac{1.8}{\sqrt{4+K_p}} \leq 0.8$ (if)

$\Rightarrow K_p \geq 1.0625$



as K_p increases, $\omega_n \uparrow$, $\xi \downarrow$ → more overshoot.

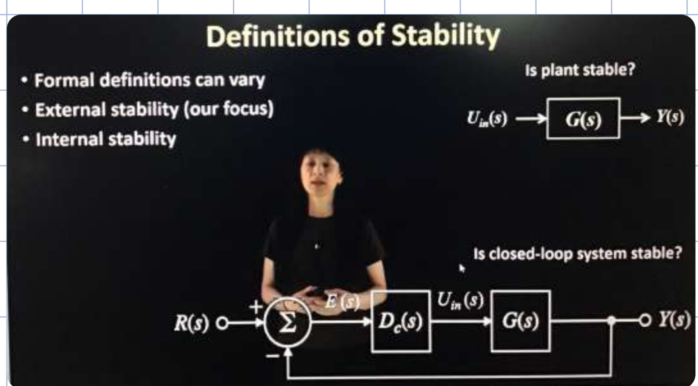
DC gain $= \lim_{s\to 0} sH_{cl}(s) = \frac{K_p}{4+K_p}$

$K_p \uparrow$ → better tracking of steady state input ⇒ lower error.

→ Feedback control can change the pole locations of the closed-loop system (without feedback is open loop)

→ design feedback controllers to improve closed-loop performance of systems

v 5.2.1



Both $G(s)$ & $H(s)$ (or $P(s)$) will be often used to denote plant transfer functions

Both $G_{cl}(s)$ & $H_{cl}(s)$ will be often used to denote closed-loop transfer functions.

Both $D(s)$ and $D_c(s)$ (or $C(s)$) are often used to denote the controller or compensators

BIBO Stability

- A system is Bounded-Input Bounded-Output (BIBO) stable if every bounded input leads to a bounded output.

$$U_{in}(s) \longrightarrow \boxed{H(s)} \longrightarrow Y(s)$$

- What condition must a system satisfy for BIBO stability?
  - Given an LTI system with input $u_{in}(t)$, output $y(t)$, and impulse response $h(t)$, then

$$y(t) = \int_{-\infty}^{\infty} h(\tau)u_{in}(t-\tau)d\tau$$

  - If $u_{in}(t)$ is bounded, then there is a positive constant M such that

$$|u_{in}(t)| \le M < \infty \text{ for all } t$$

- The output must be bounded by

$$|y(t)| = \left|\int_{-\infty}^{\infty} h(\tau)u_{in}(t-\tau)d\tau\right| \le \int_{-\infty}^{\infty} |h(\tau)||u_{in}(t-\tau)|d\tau \le M\int_{-\infty}^{\infty} |h(\tau)|d\tau$$

- The output $y(t)$ is bounded if

Time-domain condition for BIBO stability $\boxed{\int_{-\infty}^{\infty} |h(\tau)|d\tau < \infty}$

- This is a sufficient condition for BIBO stability.
- It can be shown that this is also a necessary condition for BIBO stability.

constructed a bounded input with unbounded output

v5.2.2

Time-domain condition for BIBO stability $\boxed{\int_{-\infty}^{\infty} |h(\tau)|d\tau < \infty}$

- If for every time the above sufficient condition is not satisfied, we can construct a bounded input $u_{in}(t)$ such that $y(t)$ is not bounded, then the above condition is also a necessary condition for BIBO stability.
  - Consider the input

$$u_{in}(t) = \text{sgn}\{h(-t)\} = \begin{cases} -1, & h(-t) < 0 \\ 0, & h(-t) = 0 \\ 1, & h(-t) > 0 \end{cases}$$

  - Clearly, $|u_{in}(t)| \le 1$

$$u_{in}(t) = \text{sgn}\{h(-t)\} = \begin{cases} -1, & h(-t) < 0 \\ 0, & h(-t) = 0 \\ 1, & h(-t) > 0 \end{cases} \qquad |u_{in}(t)| \le 1$$

$$y(t) = \int_{-\infty}^{\infty} h(\tau)u_{in}(t-\tau)d\tau = \int_{-\infty}^{\infty} h(\tau)\text{sgn}\{h(\tau-t)\}d\tau$$

$$y(0) = \int_{-\infty}^{\infty} h(\tau)\text{sgn}\{h(\tau)\}d\tau = \int_{-\infty}^{\infty} |h(\tau)|d\tau$$

v5.2.3

What are the requirements on $H(s)$ for BIBO stability?

$$H(s) = \frac{b_1 s^m + b_2 s^{m-1} + \cdots + b_m}{s^n + a_1 s^{n-1} + \cdots + a_n} = \frac{C_1}{s-p_1} + \frac{C_2}{s-p_2} + \cdots + \frac{C_n}{s-p_n}$$

(assuming distinct poles)

$$\Rightarrow h(t) = \left[\sum_{i=1}^{n} C_i e^{p_i t}\right] 1(t)$$

When are we guaranteed that $\int_{-\infty}^{\infty} |h(\tau)|d\tau < \infty$ is bounded?

$\text{Re}(p_i) < 0$

What are the requirements on $H(s)$ for BIBO stability?

$$H(s) = \frac{b_1 s^m + b_2 s^{m-1} + \cdots + b_m}{s^n + a_1 s^{n-1} + \cdots + a_n}$$

What if there are repeated poles?

If there is a pole p_i that is repeated twice, the corresponding term in $h(t)$ is of the form: $\underline{C_i t e^{p_i t} 1(t)}$

When are we guaranteed that $\int_{-\infty}^{\infty} |h(\tau)|d\tau < \infty$ is bounded?

same here (checks out with limits).

What are the requirements on $H(s)$ for BIBO stability?

$$H(s) = \frac{b_1 s^m + b_2 s^{m-1} + \cdots + b_m}{s^n + a_1 s^{n-1} + \cdots + a_n}$$

What if there are repeated poles?

If there is a pole p_i that is repeated m times, the corresponding term in $h(t)$ is of the form: $C_i \frac{1}{(m-1)!} t^{m-1} e^{p_i t} 1(t)$

When are we guaranteed that $\int_{-\infty}^{\infty} |h(\tau)|d\tau < \infty$ is bounded?

IMPORTANT!

Summary of BIBO Stability

- A system is Bounded-Input Bounded-Output (BIBO) stable if every bounded input leads to a bounded output.
  - Time-domain condition: $\boxed{\int_{-\infty}^{\infty} |h(\tau)|d\tau < \infty}$
  - Frequency-domain condition: $\boxed{\text{All poles of } H(s) \text{ must be inside the LHP.}}$

v5.2.4

Routh's Stability Criterion

- Procedure to determine whether all roots of characteristic equation

$$s^n + a_1 s^{n-1} + \cdots + a_n = 0$$

are inside of the LHP

- Given transfer function

$$H(s) = \frac{b_1 s^m + b_2 s^{m-1} + \cdots + b_m}{s^n + a_1 s^{n-1} + \cdots + a_n}$$

  - Difficult to solve for poles for $n > 2$
  - Routh's stability criterion extremely useful when
    - Only need to know stability (are all poles in LHP?)
    - If want to know conditions on some parameters of the system such that the system is stable

Necessary (but not Sufficient) Condition for Stability

- All the coefficients of the characteristic polynomial must be positive

$$s^n + a_1 s^{n-1} + \cdots + a_n = (s+p_1)(s+p_2)(s+p_3)\cdots(s+p_n)$$

For stability, $\mathrm{Re}(-p_i) < 0 \Leftrightarrow \mathrm{Re}(p_i) > 0 \Rightarrow \begin{cases} a_1 = \sum_{i=1}^{n} p_i > 0 \\ a_2 = \sum \prod p_i > 0 \\ \vdots \\ a_n = \prod_{i=1}^{n} p_i > 0 \end{cases}$

$\Rightarrow a_i > 0$ for $i = 1, 2, \ldots, n$

v5.2.5

Necessary and Sufficient Condition for Stability

$$s^n + a_1 s^{n-1} + \cdots + a_n = 0$$

- All the elements in the first column of the Routh array are positive

Routh array:

(This ends up being an approximately triangular matrix)

$$\begin{bmatrix} s^n & | & 1 & a_2 & a_4 & \cdots \\ s^{n-1} & | & a_1 & a_3 & a_5 & \cdots \\ s^{n-2} & | & b_1 & b_2 & b_3 & \cdots \\ s^{n-3} & | & c_1 & c_2 & c_3 & \cdots \\ \vdots & | & & \vdots & & \\ s^1 & | & * & & & \\ s^0 & | & * & & & \end{bmatrix}$$