Proportional Navigation with a Maneuvering Target

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Abstract

A qualitative analysis of the trajectories of a missile pursuing a maneuvering target according to the proportional navigation law is presented. Conditions for a missile to reach the target from any initial state are determined. The existence of a boundary for the required missile acceleration is demonstrated.

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*See "From-the-Editors," this issue, page 257.

Introduction

One of the most frequently utilized methods of navigation for controlling the trajectory of an interceptor missile is proportional navigation, a homing guidance technique in which the missile turn rate is directly proportional to the turn rate in space of the line of sight.

When the target maneuvers with normal acceleration, the trajectories of the missile are defined by a nonlinear time-varying system of differential equations which cannot be solved in analytic terms.

In classical theory [1], [2], strong assumptions are made in order to reduce this system of differential equations to a unique linear time-varying differential equation of the first order solvable in analytic terms.

The purpose of this paper is to show that qualitative methods can be applied to obtain the general solution when planar pursuit is considered. The methods applied here are an extension of such methods utilized in [3] for the case of a nonmaneuvering target.

System Equations

Planar pursuit with a maneuvering target is depicted in Fig. 1. Target T and missile M are considered as geometric points with constant velocities V_T and V_M and normal accelerations a_T and a_M, respectively. The pursuit will be described in a relative system of coordinates centered at T and axis T_x parallel to V_{T0}, the initial direction of V_T.

Letting a dot denote differentiation with respect to time, we have the equations of motion

$$V_\theta = r\dot{\theta} = V_M \sin\alpha + V_T \sin(\theta - \beta) \qquad (1)$$

$$V_r = \dot{r} = V_M \cos\alpha - V_T \cos(\theta - \beta) \qquad (2)$$

$$\dot{\beta} = a_T/V_T \qquad (3)$$

$$\dot{\gamma} = a_M/V_M. \qquad (4)$$

When M follows a proportional navigation law, a_M, the normal missile acceleration, is defined by

$$a_M = NV_M\dot{\theta} \qquad (5)$$

where N is the navigation constant.

From Fig. 1,

$$\alpha = \gamma - \theta + 2\pi. \qquad (6)$$

Substituting (5) into (4), integrating with initial conditions θ_0, γ_0, and substituting the resulting γ into (6),

$$\alpha = k\theta - \varphi_0 \qquad (7)$$

where

$$\varphi_0 = k\theta_0 - \alpha_0 \qquad (8)$$

$$k = N - 1. \qquad (9)$$

Assuming now a_T constant and integrating (3) with respect to t,

$$\beta = a_{vT}t \qquad (10)$$

where $a_{vT} = a_T/V_T$. Finally, substituting α from (7) and β

from (10), respectively, into (1) and (2),

$$r\dot{\theta} = V_\theta(\theta, t) \tag{11}$$

$$\dot{r} = V_r(\theta, t) \tag{12}$$

where

$$V_\theta(\theta, t) = V_M \sin(k\theta - \varphi_0) + V_T \sin(\theta - a_{vT}t) \tag{13}$$

$$V_r(\theta, t) = V_M \cos(k\theta - \varphi_0) - V_T \cos(\theta - a_{vT}t). \tag{14}$$

The second-order nonlinear time-varying system of differential equations formed by (11) and (12) completely defines the pursuit. Its solutions will provide the trajectories of the missile in the relative system of coordinates previously defined.

An analysis of the solutions of this system of differential equations will be performed in the following sections.

Qualitative Study

The solutions of the nonlinear time-varying system of differential equations (11) and (12) are not available in closed form. For this reason the qualitative methods developed in [3] for the case of a missile pursuing a nonmaneuvering target according to proportional navigation will be used here to extend the previous results to the more general case of a maneuvering target.

The values of θ which make V_θ and V_r equal to zero are defined, respectively, by the following two implicit functions of time:

$$V_\theta/V_T = \nu \sin(k\theta_\theta - \varphi_0) + \sin(\theta_\theta - a_{vT}t) = 0 \tag{15}$$

$$V_r/V_T = \nu \cos(k\theta_r - \varphi_0) - \cos(\theta_r - a_{vT}t) = 0 \tag{16}$$

where $\nu = V_M/V_T$.

Assuming for the moment t constant, the following can be demonstrated [3].

Lemma 1: If $\nu > 1$, $k\nu > 1$, then the roots of (15) and (16) alternate along the θ axis.

Lemma 2: If $\nu > 1, k\nu > 1$, then

$$V_r(\theta_\theta, t)\,\frac{\partial V_\theta}{\partial \theta}\,(\theta_\theta, t) > 0$$

where θ_θ is a root of $V_\theta = 0$.

From Lemmas 1 and 2 it results that V_θ and V_r can be depicted in all generality for a constant t, as is shown in Fig. 2.

Separating now t in (15) and (16),

$$t = \Phi_\theta(\theta_\theta) = \begin{cases} \Phi_{1\theta}(\theta_\theta) = \dfrac{1}{a_{vT}}\left\{\theta_\theta - \arcsin\left[-\nu \sin(k\theta - \varphi_0)\right]\right\} \\ \Phi_{2\theta}(\theta_\theta) = \dfrac{1}{a_{vT}}\left\{\theta_\theta + \arcsin\left[-\nu \sin(k\theta - \varphi_0)\right] - \pi\right\} \end{cases} \tag{17}$$

$$t = \Phi_r(\theta_r) = \begin{cases} \Phi_{1r}(\theta_r) = \dfrac{1}{a_{vT}}\left\{\theta_r - \arccos\left[\nu \cos(k\theta_r - \varphi_0)\right]\right\} \\ \Phi_{2r}(\theta_r) = \dfrac{1}{a_{vT}}\left\{\theta_r + \arccos\left[\nu \cos(k\theta_r - \varphi_0)\right]\right\}. \end{cases} \tag{18}$$

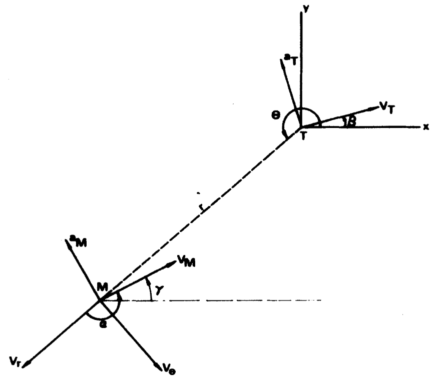
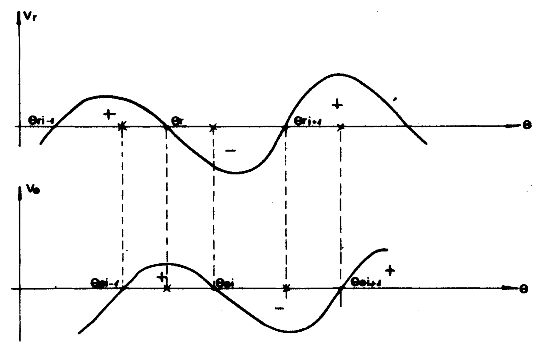


Fig. 1. Plane pursuit.

Fig. 2. V_r and V_θ versus θ (t = constant).



It is readily seen that functions Φ_θ and Φ_r exist in the real field if and only if

$$|\sin(k\theta_\theta - \varphi_0)| \leqslant 1/\nu \tag{19}$$

and

$$|\cos(k\theta_r - \varphi_0)| \leqslant 1/\nu, \tag{20}$$

respectively.

Conditions (19) and (20) can be rewritten in the form

$$|k\theta_\theta - \varphi_0 + n\pi| \leqslant \arcsin(1/\nu) \tag{21}$$

$$|k\theta_r - \varphi_0 + n\pi + \pi/2| \leqslant \arcsin(1/\nu). \tag{22}$$

Replacing φ_0 by its value given by (8) and rearranging,

$$\theta_{no} - (1/k)\arcsin(1/\nu) \leqslant \theta_\theta \leqslant \theta_{no} + (1/k)\arcsin(1/\nu) \tag{23}$$

$$\theta_{n0} + \pi/2k - (1/k) \arcsin (1/\nu) \leqslant \theta_r \leqslant \theta_{n0} + \pi/2k + (1/k) \arcsin (1/\nu) \qquad (24)$$

where

$$\theta_{n0} = \theta_0 - \alpha_0/k - n\pi/k. \qquad (25)$$

Conditions (23) and (24) are depicted in Fig. 3 in the circle of radius unity. Equations (23) and (24) determine two classes of sectors in the polar plane which will be denoted by S_θ and S_r, respectively. These sectors have the following properties:

1) S_θ and S_r are independent of the target acceleration.
2) They do not intersect if $V_M > \sqrt{2}V_T$ (i.e., $\nu > \sqrt{2}$).
3) The angle ω they span is

$$\omega = (2/k) \arcsin (1/\nu).$$

4) The separation between two successive S_r, S_θ sector bisectrices is $\delta = \pi/2k$.

Lemma 3: Given any real $t = t_1$, if $\nu > 1$ and $k\nu > 1$, there exists one and only one value of $\theta = \theta_\theta$ (θ_r) in each sector $S_\theta(S_r)$ such that $V_\theta(\theta_\theta, t_1) = 0$ $(V_r(\theta_r, t) = 0)$.

The proof can be found in Appendix I.

Let S_θ and S_r be disjoint, i.e., $V_M > \sqrt{2}V_T$.

In Fig. 4, where V_r and V_θ are depicted once again for a constant t, eight different classes of sectors can be distinguished:

$$S_\theta^+ = \{\theta\text{: given any real } t,\ V_\theta(\theta, t) = 0,\ V_r(\theta, t) > 0\}$$
$$S_\theta^- = \{\theta\text{: given any real } t,\ V_\theta(\theta, t) = 0,\ V_r(\theta, t) < 0\}$$
$$S_r^+ = \{\theta\text{: given any real } t,\ V_r(\theta, t) = 0,\ V_\theta(\theta, t) > 0\}$$
$$S_r^- = \{\theta\text{: given any real } t,\ V_r(\theta, t) = 0,\ V_\theta(\theta, t) < 0\}$$

$$\sigma_\theta^+ = \{\theta\text{: } V_\theta(\theta, t) > 0, \text{ for all real } t\}$$
$$\sigma_\theta^- = \{\theta\text{: } V_\theta(\theta, t) < 0, \text{ for all real } t\}$$
$$\sigma_r^+ = \{\theta\text{: } V_r(\theta, t) > 0, \text{ for all real } t\}$$
$$\sigma_r^- = \{\theta\text{: } V_r(\theta, t) < 0, \text{ for all real } t\}.$$

We now have all the necessary elements to perform an analysis of the trajectories of the missile, as presented in the following sections.

Polar Plane

In this section conditions under which the missile can reach the target will be determined.

In Fig. 5 the plane of the relative pursuit with the sectors defined in the previous section is depicted. The following theorem will now be proved.

Theorem 1: A missile M pursuing a maneuvering target T according to the proportional navigation law, with N and V_M such that

1) $V_M > \sqrt{2}V_T$
2) $N > 1 + V_T/V_M$

reaches the target from any initial state $M_0(r_0, \theta_0)$ exterior

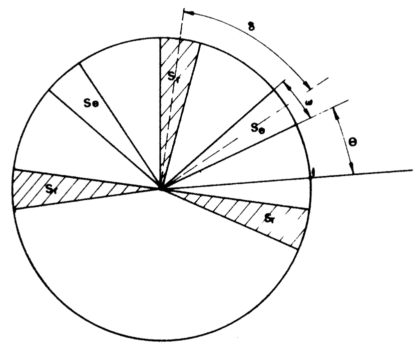
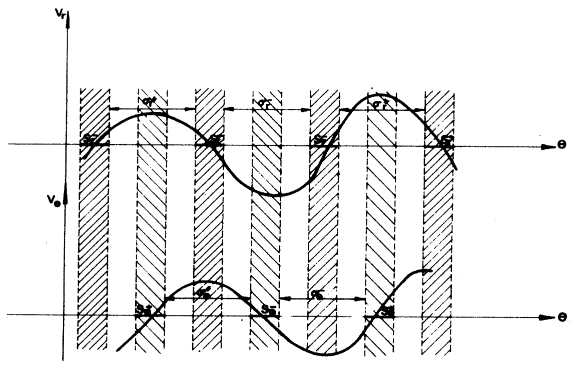


Fig. 3. S_θ and S_r sectors.

Fig. 4. S and σ classes of sectors.



to S_θ^+. Moreover, M arrives at T in the interior of a sector S_θ^-.

Proof: Let $M_0 \in \sigma_\theta^+ \cap \sigma_r^+$, where

$$V_{\theta_0} > 0, \quad V_{r_0} > 0.$$

M starts its course going away from the target. V_θ and V_r satisfy, respectively,

1) $V_\theta(\theta, t) > 0$, for $\theta_0 \leqslant \theta < \theta_{\theta_1}$ and all t
2) $V_r(\theta, t) \leqslant V_M + V_T$, for all θ and t.

$V_\theta(\theta, t)$ is a continuous function. There consequently exists $V_{\theta m}$ such that $0 < V_{\theta m} \leqslant V_\theta$, for $\theta_0 \leqslant \theta < \theta_{\theta_1}$ and all t.

Integrating 2),

$$\int_0^t \dot{r}\ dt \leqslant \int_0^t (V_M + V_T)\ dt.$$

Hence,

$$r \leqslant r_0 + (V_M + V_T)t.$$

It follows then

$$V_{\theta m} \leqslant V_\theta = r\dot{\theta} \leqslant [r_0 + (V_M + V_T)t]\,\dot{\theta}$$

for $\theta_0 \leqslant \theta < \theta_{\theta_1}$ and all t.

Rearranging,

$$\dot{\theta} \geqslant \frac{V_{\theta m}}{r_0 + (V_M + V_T)t}.$$

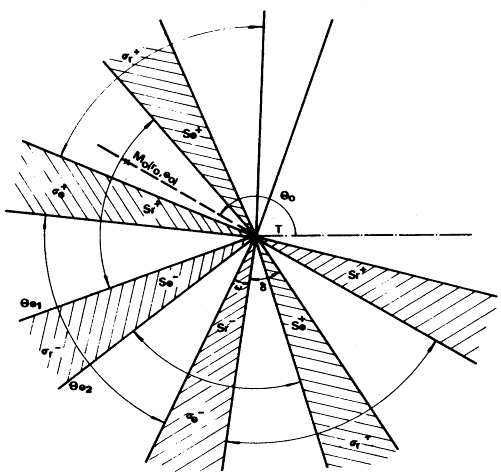


Fig. 5. Polar plane.

Integrating,

$$\int_0^t \dot{\theta}\ dt \geqslant \int_0^t \frac{V_{\theta m}}{r_0 + (V_M + V_T)t}\ dt,$$

whence

$$\theta \geqslant \theta_0 + \frac{V_{\theta m}}{V_M + V_T} \log \frac{r_0 + (V_M + V_T)t}{r_0}.$$

From here it results that $\theta(t)$ will be greater than any $\theta_1 < \theta_{\theta 1}$ for a finite t. In other words, θ increases uniformly for increasing t, and M will approach S_θ^- after entering σ_r^-.

In an equivalent form it can be shown that with initial conditions r_0, θ_0 such that

$$M_0 \in \sigma_\theta^- \cap \sigma_r^+$$

(i.e., M starts its course going away from the target, as in the previous case, but now $V_\theta(\theta, t) < 0$), θ decreases uniformly for increasing t, and M approaches S_θ^- after entering σ_r^-.

For increasing t, a trajectory entering $\sigma_r^-\ [V_r(\theta, t) < 0]$ will tend to the origin. Given the fact that the trajectory can arrive at the origin only along one of the directions $\theta = \theta_\theta\ [V_\theta(\theta_\theta, t) = 0]$, the relative trajectory of M enters S_θ^-. Once in S_θ^-, recalling the previous steps, M remains in it until T is reached.

Finally we remark that if M starts its course at $M_0 \in S_\theta^-$, the entire trajectory will be interior to S_θ^-.

Remark 1: The condition

$$V_M > \sqrt{2} V_T$$

is required in order to have S_r and S_θ disjoint, which in turn assures that M enters σ_r^-.

Remark 2: The condition

$$M_0 \notin S_\theta^+$$

must be added to assure $V_\theta \neq 0$ for $V_r > 0$.

Remark 3: It is worthwhile to rewrite the conditions defining S_θ in terms of α. Substituting α from (7) into (23) where θ_{n0} was substituted from (25),

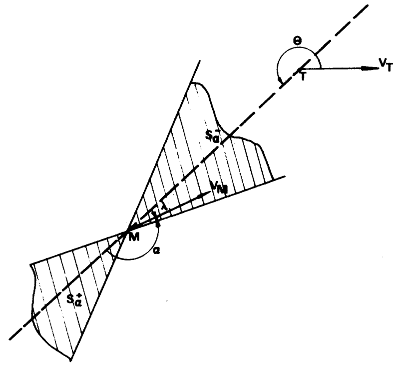


Fig. 6. S_α^+ and S_α^- sectors.

$$-\arcsin(1/\nu) - n\pi \leqslant \alpha_\theta \leqslant \arcsin(1/\nu) - n\pi. \qquad (26)$$

Condition (26) defines two sectors,

$$S_\alpha^+ = \{\alpha_\theta : -\arcsin(1/\nu) \leqslant \alpha_\theta \leqslant \arcsin(1/\nu)\}$$

$$S_\alpha^- = \{\alpha_\theta : -\arcsin(1/r) - \pi \leqslant \alpha_\theta \leqslant \arcsin(1/\nu) - \pi\}.$$

S_α^- and S_α^+ are depicted in Fig. 6. S_α^- corresponds to V_M pointing to the semiplane containing the target T, and S_α^+ corresponds to V_M in the opposite direction.

The essential differences between S_θ and S_α are the following.

1) The sectors S_α are independent of the initial conditions.
2) As opposed to S_θ, which is absolutely referred, S_α permanently rotates in keeping with the line of sight.

The final part of Theorem 1 assures us, now in terms of α, that if $V_{M_0} \in S_\alpha^-$, $V_M \in S_\alpha^-$ along the entire trajectory.

This result is of practical interest. $\lambda = \pi - \alpha$ is the angle formed by the missile and tracking system axes. If the angle of attack is neglected, V_M and the missile axis are aligned. The maximum lead angle which may be required by the missile appears to be

$$\lambda_M = \arcsin(1/\nu).$$

This result establishes a relation, not previously known, between the ratio of velocities ν and the maximum value of the lead angle λ.

Remark 4: We recall once again that S_θ and S_r are geometric elements depending only on two parameters, $\nu = V_M/V_T$, the ratio of velocities, and N, the navigation constant. The value of the target acceleration plays no role in the development of Theorem 1.

Let us consider now the case $M_0 \in S_\theta^+$. For this purpose let us differentiate (11), where V_θ was replaced from (13), with respect to time t:

$$r\ddot{\theta} + \dot{r}\dot{\theta} = kV_M \cos(k\theta - \varphi_0) + V_T \cos(\theta - a_{\nu T} t)(\dot{\theta} - a_{\nu T}). \qquad (27)$$

Substituting $\dot{r}$ from (12) into (27) and rearranging,

$$r\ddot{\theta} = V_S \dot{\theta} - a_T \cos(\theta - a_{\nu T} t) \qquad (28)$$

where

$$V_S = \frac{\partial V_\theta}{\partial \theta} - V_r = (k-1)V_M \cos(k\theta - \varphi_0) + 2V_T \cos(\theta - a_{vT}t). \quad (29)$$

Lemma 4: If $\theta \in S_\theta^+$ (S_θ^-), then

$$V_S \geqslant V_e \quad (V_S \leqslant V_e)$$

where

$$V_e = (N-2)\sqrt{V_M^2 - V_T^2} - 2V_T.$$

The proof is given in Appendix II.

Lemma 5: Let $\theta = \theta(t)$ be a solution of system (11), (12) such that

$$\dot{\theta}(t) > \dot{\theta}_1(t) \quad (\dot{\theta}(t) < \dot{\theta}_1(t))$$

where $\dot{\theta}_1(t)$ is defined by

$$V_S\dot{\theta}_1 - a_T \cos(\theta - a_{vT}t) = 0. \quad (30)$$

Then, if

1) $V_S > 0$, $\dot{\theta}$ is an increasing (decreasing) function of time
2) $V_S < 0$, $\dot{\theta}$ is a decreasing (increasing) function of time.

The proof can be found in Appendix III.

We have now all the elements to prove the following theorem.

Theorem 2: If $M_0 \in S_\theta^+$ and

1) $V_M > \sqrt{2}V_T$
2) $N > 2 + (2V_T/\sqrt{V_M^2 - V_T^2}) > 1 + V_T/V_M$
3) $|\dot{\theta}_0| > |a_T|/[(N-2)\sqrt{V_M^2 - V_T^2} - 2V_T]$,

the missile M reaches the target T.

Proof: The proof reduces to show that when conditions 1)-3) are fulfilled, $V_\theta \neq 0$ for $M \in S_\theta^+$. This in turn implies that Theorem 1 can be applied to prove the capture of T.

From Lemma 4 with $M_0 \in S_\theta^+$,

$$V_S \geqslant V_e = (N-2)\sqrt{V_M^2 - V_T^2} - 2V_T > 0,$$

the last inequality being derived from condition 2) above.

Let us consider

$$\dot{\theta}_1(t) = \frac{a_T \cos(\theta - a_{vT}t)}{V_S}.$$

With $V_S > 0$,

$$-\frac{|a_T|}{V_e} \leqslant -\frac{|a_T|}{V_S} \leqslant \dot{\theta}_1(t) \leqslant \frac{|a_T|}{V_S} \leqslant \frac{|a_T|}{V_e}.$$

From Lemma 5 it follows that if $|\dot{\theta}_0| > |a_T|/V_e$,

$$|\dot{\theta}| > \frac{|a_T|}{V_e}$$

for $M \in S_\theta^+$. This implies that M goes out of S_θ^+ and the proof follows as in Theorem 1.

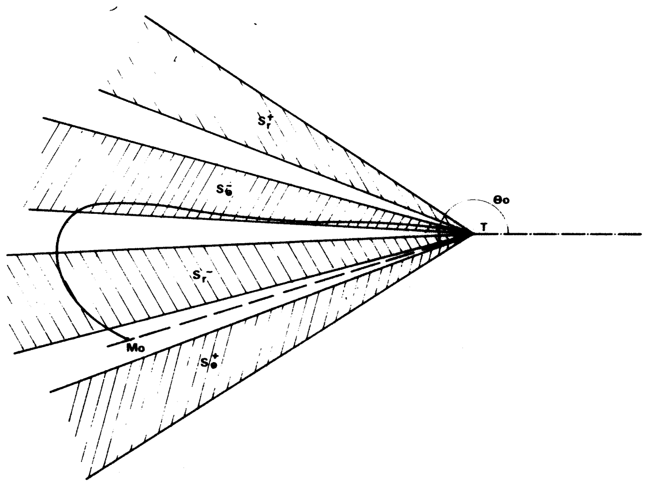


Fig. 7. Relative trajectory.

In Fig. 7 a typical trajectory obtained by numerical integration is depicted.

Missile Acceleration

The strongest limitation in air-to-air missiles is due to the missile acceleration capability.

In [4] it was proved that a missile M with constant velocity V_M, pursuing a maneuvering target T with constant velocity V_T and normal acceleration a_T, can capture T from any state if (and only if) $a_M > a_T$.

Theoretically this result can be implemented as it is done in the proof of this theorem. In practice this result has not, at the present time, a practical value.

The reason which made proportional navigation the most commonly utilized technique in air-to-air missiles is precisely its ease of implementation. But even though it is widely utilized, the determination in a rigorous form of the maximum required missile acceleration remained an open field.

The following theorem represents a partial result in this area.

Theorem 3: The normal acceleration of an ideal missile pursuing a maneuvering target T, according to the proportional navigation law, and starting at $M_0 \in S_\theta^-$ with

1) $V_M > \sqrt{2}V_T$
2) $N > 2 + (2V_T/\sqrt{V_M^2 - V_T^2}) > 2 + 2(V_T/V_M)$,

is such that if:

a) $|a_{M_0}| > a_{M_1}$, where

$$a_{M_1} = \frac{N}{(N-2)\sqrt{(V_M/V_T)^2 - 1} - 2} \frac{V_M}{V_T} |a_T| \quad (31)$$

then a_M will decrease until $|a_M| < a_{M_1}$;

b) if $|a_{M_0}| < a_{M_1}$, then $|a_M| < a_{M_1}$.

Proof: From Theorem 1, if M_0 starts its course at S_θ^-, the entire trajectory belongs to S_θ^-.

Now, if $M \in S_\theta^-$, it follows from Lemma 4 that

$$V_S \leqslant V_e < 0,$$

the last inequality being derived from condition 2) above.

Let us consider

$$\dot{\theta}_1 = \frac{a_T \cos(\theta - a_{vT} t)}{V_S}. \tag{32}$$

With $V_S < 0$,

$$\dot{\theta}_1(t) \leqslant -\frac{|a_T|}{V_S} \leqslant -\frac{|a_T|}{V_e}$$

and

$$\dot{\theta}_1(t) \geqslant \frac{|a_T|}{V_S} \geqslant \frac{|a_T|}{V_e}.$$

From Lemma 5 it follows straightforwardly that if

$$|\dot{\theta}_0| > -\frac{|a_T|}{V_e},$$

$|\dot{\theta}|$ is a decreasing function of time, and this at least until

$$|\dot{\theta}| < -\frac{|a_T|}{V_e}.$$

In terms of missile acceleration, if

$$|a_{M_0}| = NV_M|\dot{\theta}_0| > -\frac{NV_M|a_T|}{V_e} = a_{M_1},$$

then $|a_M|$ will decrease until

$$|a_M| < a_{M_1}.$$

Part b) follows directly from the proof of part a).

Remark 5: For a nonmaneuvering target $a_{M_1} = 0$, and the result is simply that a_M decreases uniformly to zero. Recall that in [3] a more restricted result was found involving only the behavior of a_M in the neighborhood of the collision course. Now, with stronger conditions, a more general result is found applying to a_M in all the S_θ^- sector.

Remark 6: For N and (V_M/V_T) going to infinity,

$$\lim a_{M_1} = |a_T|.$$

From Fig. 1, for a rear attack,

$$\pi/2 < \theta - \beta < 3\pi/2$$

which in turn implies

$$\cos(\theta - \beta) < 0. \tag{33}$$

For this case the following result can be proved.

Theorem 4: The normal acceleration of a missile M pursuing a maneuvering target T, according to the proportional navigation law, starting at $M_0 \in S_\theta^-$ and with

1) $V_M > \sqrt{2} V_T$
2) $N > 1 + V_M/V_T$,

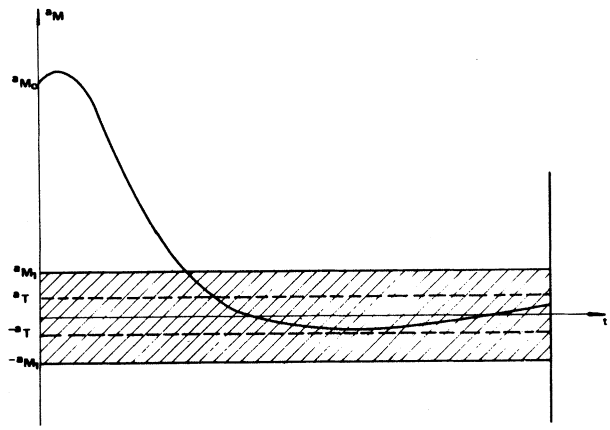


Fig. 8. Missile acceleration.

is such that if the entire pursuit is restricted to the rear of the target

$$|a_M| < (V_M/V_T)|a_T| \tag{34}$$

if $|a_{M_0}| < (V_M/V_T)\,|a_T|$. If $|a_{M_0}| > (V_M/V_T)|a_T|$, $|a_T|$ will decrease until (34) is fulfilled.

Proof: For $M_0 \in S_\theta^- \Rightarrow M \in S_\theta^-$ thus

$$V_r < 0.$$

From (14),

$$V_M \cos(k\theta - \varphi_0) < V_T \cos(\theta - a_{vT} t).$$

Applying this to V_S,

$$V_S = (k-1)V_M \cos(k\theta - \varphi_0) + 2V_T \cos(\theta - a_{vT} t)$$
$$< (k+1)V_T \cos(\theta - a_{vT} t). \tag{34'}$$

With $V_S < 0$,

$$\frac{\cos(\theta - a_{vT} t)}{V_S} < \frac{1}{NV_T}.$$

For a rear attack, recalling (33), $\cos(\theta - a_{vT} t) < 0$, where β was replaced from (10), and from (34') $V_S < 0$. Then, for $a_T > 0$,

$$0 < \dot{\theta}_1 = \frac{a_T \cos(\theta - a_{vT} t)}{V_S} < \frac{a_T}{NV_T}.$$

For $a_T < 0$,

$$\frac{a_T}{NV_T} < \dot{\theta}_1 < 0.$$

Applying Lemma 5 now, the conclusion follows straightforwardly.

Fig. 8 shows the missile acceleration a_M as a function of time for the trajectory depicted in Fig. 7.

Summary and Conclusions

In this paper an analysis of the trajectories of a missile pursuing a maneuvering target according to the proportional navigation law was performed.

The qualitative methods applied here are revealed to be powerful enough to overcome the difficulties encountered with a nonlinear time-varying system of differential equations.

The division of the relative plane of the pursuit into classes of sectors provided a clear insight into the missile behavior and enables one to find conditions under which the missile can reach the target from any initial state.

The difficult task of determining boundaries for the missile acceleration was undertaken and results were obtained for the case where the missile lies in preassigned regions of the relative plane of the pursuit.

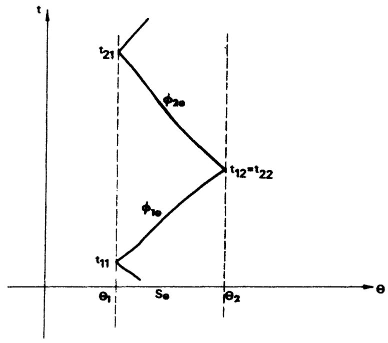


Fig. 9. Φ_θ versus θ.

Appendix I

Proof of Lemma 3

Let us determine

$$\frac{d\Phi_{1\theta}}{d\theta} = \frac{1}{a_{\nu T}}\left[1 + \frac{k\nu \cos(k\theta - \varphi_0)}{\sqrt{1 - [\nu \sin(k\theta - \varphi_0)]^2}}\right]. \qquad (35)$$

Setting $d\Phi_{1\theta}/d\theta$ to zero,

$$\frac{k\nu \cos(k\theta - \varphi_0)}{\sqrt{1 - [\nu \sin(k\theta - \varphi_0)]^2}} = -1. \qquad (36)$$

Squaring and rearranging,

$$k^2\nu^2 \cos^2(k\theta - \varphi_0) = 1 - \nu^2 \sin^2(k\theta - \varphi_0). \qquad (37)$$

Replacing $\sin^2(k\theta - \varphi_0)$ by $1 - \cos^2(k\theta - \varphi_0)$ and rearranging once again,

$$\cos^2(k\theta - \varphi_0) = \frac{1 - \nu^2}{\nu^2(k^2 - 1)}. \qquad (38)$$

For $\nu > 1$ and $k\nu > 1$, (38) has no roots in the real field. Consequently, $d\Phi_{1\theta}/d\theta$ is sign definite and the extrema of $\Phi_{1\theta}$ are located at the S_θ boundaries.

In an equivalent form it can be shown that $d\Phi_{2\theta}/d\theta$ also has no roots in the real field, being in consequence sign definite. The extrema of $\Phi_{2\theta}$ are thus also located at the S_θ boundaries.

The S_θ boundaries are defined by

$$\nu \sin(k\theta - \varphi_0) = \pm 1. \qquad (39)$$

Hence,

$$\theta_1^+ = [\varphi_0 + \arcsin(1/\nu) + 2n\pi]/k \qquad (40)$$

$$\theta_2^+ = [\varphi_0 - \arcsin(1/\nu) + 2n\pi]/k \qquad (41)$$

or

$$\theta_1^- = \pi/k - \theta_1^+ \qquad (42)$$

$$\theta_2^- = \pi/k - \theta_2^+ \qquad (43)$$

are in terms of θ the S_θ boundaries. Substituting the first set of θ_1, θ_2 values into $\Phi_{1\theta}$ and $\Phi_{2\theta}$,

$$t_{11} = \Phi_{1\theta}(\theta_1^+) = \frac{1}{a_{\nu T}}\left\{\frac{1}{k}[\varphi_0 + \arcsin(1/\nu)] - \frac{3}{2}\pi + 2n\pi\right\} \qquad (44)$$

$$t_{12} = \Phi_{1\theta}(\theta_2^+) = \frac{1}{a_{\nu T}}\left\{\frac{1}{k}[\varphi_0 - \arcsin(1/\nu)] - \frac{\pi}{2} + 2n\pi\right\} \qquad (45)$$

$$t_{21} = \Phi_{2\theta}(\theta_1^+) = \frac{1}{a_{\nu T}}\left\{\frac{1}{k}[\varphi_0 + \arcsin(1/\nu)] + \frac{\pi}{2} + 2n\pi\right\} \qquad (46)$$

$$t_{22} = \Phi_{2\theta}(\theta_2^+) = \frac{1}{a_{\nu T}}\left\{\frac{1}{k}[\varphi_0 - \arcsin(1/\nu)] - \frac{\pi}{2} + 2n\pi\right\}. \qquad (47)$$

Representing $\Phi_{1\theta}$ and $\Phi_{2\theta}$ in Fig. 9, the result follows straightforwardly.

For the second set of θ_1, θ_2 values, the proof follows in an equivalent form. For the case of V_r similar steps can be followed to arrive at the proof.

Appendix II

Proof of Lemma 4

For θ belonging to S_θ^+ (S_θ^-)

1) $|\sin(k\theta - \varphi_0)| \leqslant 1/\nu$
2) $S_\theta^+ \in \sigma_\theta^+$ ($S_\theta^- \in \sigma_\theta^-$) $\Rightarrow V_r(\theta, t) > 0$ ($V_r < 0$), for all t.

From 1) it follows that

$$|\cos(k\theta - \varphi_0)| \geqslant \cos[\arcsin(1/\nu)] = \frac{\sqrt{\nu^2 - 1}}{\nu}.$$

From 2),

$$\cos(k\theta - \varphi_0) > 0 \quad (\cos(k\theta - \varphi_0) < 0).$$

In consequence,

$$\cos(k\theta - \varphi_0) \geqslant \frac{\sqrt{\nu^2 - 1}}{\nu} \left(\cos(k\theta - \varphi_0) \leqslant -\frac{\sqrt{\nu^2 - 1}}{\nu}\right).$$

With

$$\cos(\theta - a_{vT}t) \geqslant -1 \quad (\cos(\theta - a_{vT}t) \leqslant 1)$$

it follows that

$$V_S \geqslant (k-1)V_M \frac{\sqrt{v^2-1}}{v} - 2V_T$$

$$\left(V_S \leqslant -(k-1)V_M \frac{\sqrt{v^2-1}}{v} + 2V_T.\right)$$

Appendix III

Proof of Lemma 5

$\dot{\theta} > \dot{\theta}_1$ $(\dot{\theta} < \dot{\theta}_1)$ can be rewritten as

$$\dot{\theta} = \dot{\theta}_1 + \dot{\theta}_S \quad (\dot{\theta} = \dot{\theta}_1 - \dot{\theta}_S) \tag{48}$$

where $\dot{\theta}_S = \dot{\theta}_S(t) > 0$. Substituting $\dot{\theta}$ from (48) into (27),

$$r\ddot{\theta} = V_S\dot{\theta}_S \quad (r\ddot{\theta} = -V_S\dot{\theta}_S).$$

Hence,

1) if $V_S > 0 \Rightarrow \ddot{\theta} > 0$ $(\ddot{\theta} < 0)$, then $\dot{\theta}$ is an increasing (decreasing) function of time;
2) if $V_S < 0 \Rightarrow \ddot{\theta} < 0$ $(\ddot{\theta} > 0)$, then $\dot{\theta}$ is a decreasing (increasing) function of time.

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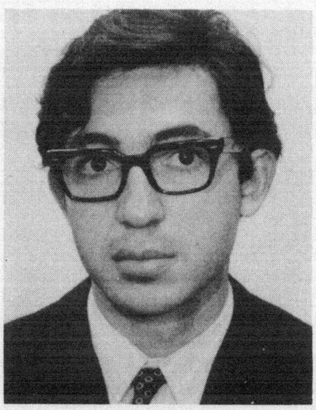
References

[1] J.J. Jerger, *Systems Preliminary Design*. Princeton, N.J.: Van Nostrand, 1960.

[2] S.A. Murtaugh and H.E. Criel, "Fundamentals of proportional navigation," *IEEE Spectrum*, vol. 3, pp. 75-85, December 1966.

[3] M. Guelman, "A qualitative study of proportional navigation," *IEEE Trans. Aerospace and Electronic Systems*, vol. AES-7, pp. 637-643, July 1971.

[4] E. Cockayne, "Plane pursuit with curvature constraints," *SIAM J. Appl. Math.*, vol. 15, pp. 1511-1516, November 1967.



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