The Numerical Stability of Barycentric Lagrange Interpolation

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- Why the need for alternate forms?
- The solution?
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Why the need for alternate forms?

The main reasons:

- **To reduce time complexity**, as the original formulation using the lagrange basis polynomials needs the computation of products of differences. This becomes an issue in bigger datasets as the complexity is $\mathbf{O}(n^2)$

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- **To reduce time complexity**, as the original formulation using the lagrange basis polynomials needs the computation of products of differences. This becomes an issue in bigger datasets as the complexity is $\mathbf{O}(n^2)$
- Adding a new data point is costly as well since the lagrange polynomials need to be computed again.
- Lagrange interpolation is numerically unstable especially if the node points are not chosen apporpriately.

The solution?

- To come up with alternate formulations of the lagrange interpolation methods that reduce the time complexity and are numerically stable.

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- To come up with alternate formulations of the lagrange interpolation methods that reduce the time complexity and are numerically stable.
- Two modified forms have been formulated, both being the **Barycentric forms** of Lagrange interpolation.

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The error analysis has been done for both these forms and compared with the Newton divided difference form for two different functions.

Error Analysis and Stability

Let's define some preliminaries

- Condition Number
- Notions of Stability
- Lebesgue Constant
- Floating Point Arithmetic Model

Condition Number

The condition number of p_n at x with respect to f is defined for $p_n(x) \neq 0$:

$$\text{cond}(x, n, f) = \lim_{\Delta f \to 0} \sup \frac{|p_f(x) - p_{f+\Delta f}(x)|}{|\epsilon p_f(x)|}, \text{ where } |\Delta f| \leq \epsilon |f|. \tag{1}$$

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In the notation, cond(x, n, f), the term 'n' indicates the dependence on the points x_j.

$$\text{cond}(x, n, f) = \sum_{j=0}^{n} \left| \frac{\ell_j(x) f_j}{p_n(x)} \right| \geq 1, \tag{2}$$

and for any Δf with $|\Delta f| \leq \epsilon |f|$ we have

$$\frac{|p_f(x) - p_{f+\Delta f}(x)|}{|p_f(x)|} \leq \text{cond}(x, n, f) \cdot \epsilon \tag{3}$$

Notions of Stability: Forward Stable

A numerical method is **forward stable** if the error between **computed solution** and actual solution is small:

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Forward Error Bound:

The forward error can be bounded as:

$$\frac{|\hat{f}(x) - f(x)|}{|f(x)|} \leq \text{cond}(x, n, f) \cdot u$$

where u is the **unit roundoff** (machine precision) and $\text{cond}(x, n, f)$ is the **condition number**.

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Backward Stability Condition:

A method is backward stable if:

$$|\delta x| \leq u \cdot |x|$$

meaning the computed result corresponds to a nearby problem.

Lebesgue Constant

We assume that the nodes, x_j lie in $[-1, 1]$ and express the bound in terms of Λ_n, the Lebesgue constant associated with the points x_j:

$$\Lambda_n = \sup_{f \in C([-1,1])} \frac{||P_n f||_\infty}{||f||_\infty} \tag{4}$$

where P_n is the operator mapping f to its interpolating polynomial at the points x_j,

$$||f|| = \max_{x \in [-1,1]} |f(x)|$$

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It can be shown that:

$$\Lambda_n = \sup_{x \in [-1,1]} \sum_{j=0}^{n} |\ell_j(x)| \tag{5}$$

Floating-Point Arithmetic Model

Standard Model of Floating-Point Arithmetic (Higham, 2002):

$$\mathrm{fl}(x \,\mathrm{op}\, y) = (x \,\mathrm{op}\, y)(1+\delta)^{\pm 1}, \quad |\delta| \le u$$

where:

- $\mathrm{fl}(x \,\mathrm{op}\, y)$: result in floating-point arithmetic
- u: unit roundoff
- δ: rounding error, $|\delta| \le u$

This formula models how floating-point arithmetic introduces errors in operations like $+, -, *, /$.

Relative Error Accumulation

Relative Error Counter:

$$\langle k \rangle = \prod_{i=1}^{k} (1+\delta_i)^{\rho_i}, \quad \rho_i = \pm 1, \quad |\delta_i| \le u$$

This captures how errors accumulate over multiple operations.

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Bound on Relative Error:

$$|\langle k \rangle - 1| \le \gamma_k = \frac{ku}{1-ku}$$

where k is the number of operations and u is the unit roundoff. The error grows as a function of ku, controlling the relative error in the worst case.

1st Form: Definition

Recall traditional Lagrange representation of an interpolating polynomial is written as:

$$p_n(x) = \sum_{j=0}^{n} \mathcal{L}_j(x) f_j \tag{6}$$

where $\mathcal{L}_j(x) = \prod_{k=0, k\neq j}^{n} \frac{x - x_k}{x_j - x_k}$

We can rewrite this expression as follows:

$$p_n(x) = \underbrace{\left(\prod_{k=0}^{n}(x - x_k)\right)}_{\ell(x)} \sum_{j=0}^{n} \left(\frac{1}{x - x_j}\right) \underbrace{\left(\prod_{k=0,k\neq j}^{n} \frac{1}{x_j - x_k}\right)}_{w_j} f_j \quad (7)$$

$$\boxed{p_n(x) = \ell(x) \sum_{j=0}^{n} \left(\frac{w_j}{x - x_j}\right) f_j} \quad (8)$$

The form presented in Eq.8 is called the **First Form of the Barycentric Interpolation Formula**.

Error Analysis on First Form:

- The computed weights w_j satisfy:

$$\hat{w}_j = w_j \langle 2n \rangle_j, \quad j = 0 : n \tag{9}$$

- The computed $\hat{l}(x)$ satisfies:

$$\hat{l}(x) = l(x) \langle 2n + 1 \rangle_j \tag{10}$$

- The computed interpolating polynomial $\hat{p}_n(x)$ satisfies:

$$\boxed{\hat{p}_n(x) = l(x) \sum_{j=0}^{n} \frac{w_j}{x - x_j} f_j \langle 5n + 5 \rangle_j} \tag{11}$$

Error Analysis on the First Form:

- Eq.11 can be interpreted as the value of $\hat{p}_n(x)$ computed in presence of perturbations in f_i. Therefore, it is backward stable.
- We can also bound a forward error as follows:

$$\frac{\|p_n(x) - \hat{p}_n(x)\|}{\|p_n(x)\|} \leq (\gamma_{5N+5})\text{cond}(x, n, f) \tag{12}$$

- The First Form representation is shown to be **Backward Stable and Forward Stable**.

2nd Form: Definition

If the function values f_j are 1, they will be interpolated by $p_n(x) = 1$. From eq. 8, this gives,

$$\ell(x) = \frac{1}{\sum_{j=0}^{n} \left(\frac{w_j}{x - x_j} \right)} \tag{13}$$

Substituting the expression for $\ell(x)$ in eq. 8, we obtain:

$$\boxed{p_n(x) = \frac{\sum_{j=0}^{n} \left(\frac{w_j}{x - x_j} \right) f_j}{\sum_{j=0}^{n} \left(\frac{w_j}{x - x_j} \right)}} \tag{14}$$

The form presented in Eq.14 is called the **Second (proper) form of the barycentric formula**.

Error and Stability Analysis of the Second Form

- **Different stability properties expected**: This formula is obtained by using a mathematical identity that does not necessarily hold in floating point arithmetic.
- The computed interpolating polynomial is:

$$\hat{p}_n(x) = \frac{\sum_{j=0}^{n} \left(\frac{w_j}{x-x_j}\right) f_j \langle 3n+4 \rangle_j}{\sum_{j=0}^{n} \left(\frac{w_j}{x-x_j}\right) \langle 3n+2 \rangle_j} \tag{15}$$

- Now, from eq. 2,

$$\mathrm{cond}(x, n, f) = \frac{\sum_{j=0}^{n} \left|\frac{w_j f_j}{x-x_j}\right|}{\left|\sum_{j=0}^{n} \frac{w_j f_j}{x-x_j}\right|} \tag{16}$$

Error and Stability Analysis of the Second Form

- Similarly,

$$\text{cond}(x, n, 1) = \frac{\sum_{j=0}^{n} \left| \frac{w_j}{x - x_j} \right|}{\left| \sum_{j=0}^{n} \frac{w_j}{x - x_j} \right|} \tag{17}$$

- Theorem: The computed $\hat{p}_n(x)$ satisfies,

$$\frac{|p_n(x) - \hat{p}_n(x)|}{|p_n(x)|} \leq (3n+4)u \frac{\sum_{j=0}^{n} \left| \frac{w_j f_j}{x - x_j} \right|}{\left| \sum_{j=0}^{n} \frac{w_j f_j}{x - x_j} \right|} + (3n+2)u \frac{\sum_{j=0}^{n} \left| \frac{w_j}{x - x_j} \right|}{\left| \sum_{j=0}^{n} \frac{w_j}{x - x_j} \right|} + O(u^2)$$

$$= (3n+4)u\,\text{cond}(x, n, f) + (3n+2)u\,\text{cond}(x, n, 1) + O(u^2) \tag{18}$$

Error and Stability Analysis of the Second Form

- It can be observed that for suitable choice of the x_j the second term in eq. 18 can be made arbitrarily large.
- For further analysis, we see that the Lebesgue constant may be written as:

$$\Lambda_n = \sup_{x \in [-1,1]} \sum_{j=0}^{n} |\ell_j(x)| \tag{19}$$

- Therefore, the forward error bound is given by:

$$\frac{|p_n(x) - \hat{p}_n(x)|}{|p_n(x)|} \leq -(3n+4)u\mathrm{cond}(x, n, f) + (3n+2)u\Lambda_n + O(u^2) \tag{20}$$

Chebyshev Points

For the Chebyshev points of the first kind (the zeros of the degree $(n+1)$ Chebyshev polynomial) and the Chebyshev points of the second kind (the extreme points of the degree n Chebyshev polynomial):

$$\Lambda_n \leq \frac{2}{\pi} \log(n+1) + 1.$$

For other 'good' sets of points, Λ_n is also slowly growing.

Comparison with Equally Spaced Points

For equally spaced points, Λ_n grows exponentially at a rate proportional to:

$$\frac{2^n}{n \log n}.$$

Stability Variation with Node Distributions

- We conclude that while the barycentric formula is not forward stable in general, it can be significantly less accurate than the modified Lagrange formula only for a poor choice of interpolating points and special functions f.
- More specifically, for both sets of Chebyshev points, the barycentric formula is guaranteed to be forward stable - that is, it produces relative errors bounded by:

$$g(n)u \cdot \mathrm{cond}(x, n, f), \quad \text{with } g \text{ a slowly growing function of } n.$$

- The computational advantage of the barycentric formula over the modified Lagrange formula: Since the w_j appear linearly in both the numerator and the denominator, they can be rescaled ($w_j \leftarrow \alpha w_j$) to avoid overflow and underflow.

Numerical Experiments

- Computations performed in MATLAB with $u \approx 10^{-16}$.
- 30 equally spaced points x_j on $[-1, 1]$ (thus $n = 29$).
- Evaluated interpolant at 100 points on $[-1 + 10^3\epsilon, 1 - 10^3\epsilon]$.
- 'Exact' values obtained using 50-digit arithmetic with MATLAB's Symbolic Math Toolbox.

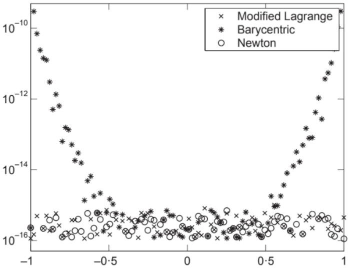
Function values for first case: $f_j = 0$ for $j = 0 : n-1$ and $f_n = 1$, $\mathrm{cond}(x, n, f) = 1$ and that $\Lambda_n = 3 \times 10^6$

Runge function for second case and $\mathrm{cond}(x, n, f) = 7.5$

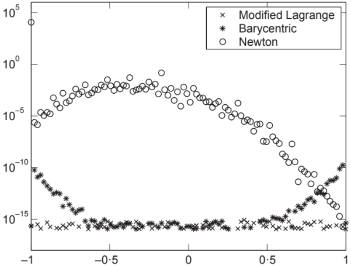
Observations from Numerical Experiments

- Modified Lagrange formula performs stably; barycentric formula performs unstably.
- Newton divided difference formula performs stably in increasing order but unstably in decreasing order.

N. J. HIGHAM



(a) x_i in increasing order.

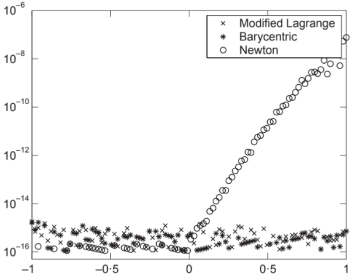


(b) x_i in decreasing order.

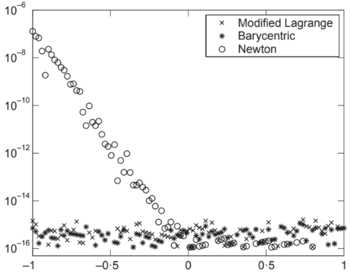
Figure 1: Comparison of Relative errors in computed $p_n(x)$ for 30 equally spaced points x_i

Further Experiments and Conclusions

- A second typical example with the Runge function $f(x) = \frac{1}{1+25x^2}$ and Chebyshev points.
- Modified Lagrange and barycentric formulas behave stably; Newton divided difference becomes unstable.



(a) Increasing order of Chebyshev points.



(b) Decreasing order of Chebyshev points.

Figure 2: Relative errors in computed $p_n(x)$ for 30 Chebyshev points of the first kind.

Conclusions

- It is evident that in all cases the the modified lagrange formula (first form of barycentric lagrange) is stable which implies unconditional stability.

[1] Higham, Nicholas J. / The numerical stability of barycentric Lagrange interpolation. In: IMA Journal of Numerical Analysis. 2004 ; Vol. 24, No. 4. pp. 547-556.

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- It is evident that in all cases the the modified lagrange formula (first form of barycentric lagrange) is stable which implies unconditional stability.
- For the second form we can see see unstability for the dataset having a high Lebesgue constant. The requirement of a low Lebesgue constant for stability implies conditional stability.

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- For the second form we can see see unstability for the dataset having a high Lebesgue constant. The requirement of a low Lebesgue constant for stability implies conditional stability.
- Both barycentric forms perfom significantly better than the Newton divided difference form.

Reference [1]

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Contributions

All team members contributed equally to the completion of this assignment and making of presentation, with active participation in research, analysis, and execution of tasks.