Chapter 4

Complex Analysis

4.1 Brief Theory Introduction

4.1.1 Limits

A complex function $f(z)$ is said to have a limit L as $z \to z_0$ if:

$$\lim_{z \to z_0} f(z) = L,$$

where L is a complex number. This implies that for every $\epsilon > 0$, there exists a $\delta > 0$ such that $|f(z) - L| < \epsilon$ whenever $0 < |z - z_0| < \delta$.

4.1.2 Derivatives

The derivative of $f(z)$ at $z = z_0$ is given by:

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0},$$

provided the limit exists. A function is holomorphic if it is differentiable everywhere in its domain.

4.1.3 Analytic Functions and Cauchy-Riemann Equations

A function $f(z) = u(x, y) + iv(x, y)$, where $u(x, y)$ and $v(x, y)$ are real-valued functions, is said to be analytic in a domain D if it is differentiable at every point in D. Differentiability in the complex sense implies that the limit:

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z}$$

exists and is independent of the direction from which $\Delta z \to 0$.

For $f(z)$ to be analytic, it is necessary and sufficient that:

1. $u(x, y)$ and $v(x, y)$ have continuous first-order partial derivatives, and
2. The Cauchy-Riemann equations are satisfied:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

If these conditions hold, $f(z)$ is both holomorphic (complex differentiable) and harmonic (satisfying Laplace's equation):

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad \Delta v = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0.$$

Functions that are analytic throughout the entire complex plane are called *entire* functions. Examples include e^z, $\sin z$, and $\cos z$.

4.1.4 Complex Integration

For a curve Γ parameterized by $z(t) = x(t) + iy(t)$, the integral of $f(z)$ along Γ is:

$$\int_\Gamma f(z)\,dz = \int_a^b f(z(t))z'(t)\,dt.$$

4.1.5 ML Inequality

If $f(z)$ is continuous on a contour Γ, then:

$$\left|\int_\Gamma f(z)\,dz\right| \leq M \cdot L,$$

where M is the maximum of $|f(z)|$ on Γ and L is the length of Γ.

4.1.6 Cauchy Integral Theorem

Simply Connected Domains

The Cauchy Integral Theorem states that if $f(z)$ is analytic in a simply connected domain D and Γ is any closed contour within D, then:

$$\int_\Gamma f(z)\,dz = 0.$$

A domain is ***simply connected*** if any closed contour in the domain can be continuously shrunk to a point without leaving the domain (i.e., the domain has no "holes").

Multiply Connected Domains

For a multiply connected domain, which contains holes or excluded regions, the theorem requires special handling of singularities. Let Γ be a positively oriented closed contour, and let there be n disjoint regions within Γ, each bounded by negatively oriented closed contours $\Gamma_1, \Gamma_2, \ldots, \Gamma_n$. If $f(z)$ is analytic in the multiply connected domain but may have singularities inside Γ, the integral around Γ is related to the sum of integrals around the excluded regions:

$$\int_\Gamma f(z)\,dz = \int_{\Gamma_1} f(z)\,dz + \int_{\Gamma_2} f(z)\,dz + \cdots + \int_{\Gamma_n} f(z)\,dz.$$

If the function $f(z)$ is analytic everywhere except at isolated singularities inside Γ, the integral can be evaluated using the Residue Theorem:

$$\int_\Gamma f(z)\,dz = 2\pi i \sum \text{Residues of } f(z) \text{ inside } \Gamma.$$

This approach allows us to handle more complex domains by effectively "subtracting out" the effects of the excluded regions or singularities.

Practical Example for Multiply Connected Domains

Consider $f(z) = \frac{1}{z}$ in the annular region $1 < |z| < 2$. Let Γ_1 be the circle $|z| = 2$ (positively oriented) and Γ_2 be the circle $|z| = 1$ (negatively oriented). Then:

$$\int_{\Gamma_1} \frac{1}{z}\,dz + \int_{\Gamma_2} \frac{1}{z}\,dz = 2\pi i - 2\pi i = 0.$$

Here, the contribution from each contour cancels, demonstrating the nature of integrals in multiply connected domains.

4.1.7 Cauchy Integral Formula

If $f(z)$ is analytic inside and on a simple closed contour Γ, and z_0 is inside Γ, then:

$$f(z_0) = \frac{1}{2\pi i} \int_\Gamma \frac{f(z)}{z - z_0}\,dz.$$

4.1.8 Taylor and Laurent Series

Taylor Series

If $f(z)$ is analytic in a disk $|z - z_0| < R$, it can be expressed as a Taylor series centered at z_0:

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n,$$

where the coefficients a_n are given by:

$$a_n = \frac{1}{n!} \frac{d^n f}{dz^n}(z_0).$$

1. **Radius of Convergence:** The series converges for $|z - z_0| < R$, where R is the distance from z_0 to the nearest singularity.

2. **Application:** Taylor series are commonly used to approximate functions locally around z_0.

Laurent Series

If $f(z)$ is analytic in an annular region $R_1 < |z - z_0| < R_2$, it can be expressed as a Laurent series:

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n,$$

where the coefficients a_n are given by:

$$a_n = \frac{1}{2\pi i} \int_{\Gamma} \frac{f(w)}{(w - z_0)^{n+1}} \, dw,$$

and Γ is a positively oriented contour within the annular region.

1. **Principal Part:** The terms $\sum_{n=-\infty}^{-1} a_n (z - z_0)^n$ describe the behavior near singularities.

2. **Regular Part:** The terms $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ represent the analytic behavior.

Comparison of Taylor and Laurent Series

- Taylor series are valid in simply connected domains without singularities.
- Laurent series are used in regions containing isolated singularities, combining both regular and singular behavior.

Example: Expansion Around Singularities

For $f(z) = \frac{1}{z(z-1)}$:

1. **At $z_0 = 0$:** The Laurent series in $|z| < 1$ is:

$$f(z) = \frac{1}{z} - 1 - z - z^2 - \cdots .$$

2. **At $z_0 = 1$:** The Laurent series in $|z - 1| < 1$ is:

$$f(z) = \frac{1}{z-1} + \frac{1}{z-1} \sum_{n=1}^{\infty} z^{-n}.$$

4.1.9 Residue Theorem

The Residue Theorem provides an efficient way to evaluate complex integrals. If $f(z)$ is analytic in a domain D except for isolated singularities $z_1, z_2, \ldots, z_n$ inside a closed contour Γ, then:

$$\int_{\Gamma} f(z) \, dz = 2\pi i \sum_{k=1}^{n} \operatorname{Res}(f, z_k),$$

where $\operatorname{Res}(f, z_k)$ is the residue at z_k.

Residue at Isolated Singularities

1. **Simple Pole at z_0:** For a simple pole at z_0,

$$\mathrm{Res}(f, z_0) = \lim_{z \to z_0} (z - z_0) f(z).$$

2. **Pole of Order m at z_0:** For a pole of order m,

$$\mathrm{Res}(f, z_0) = \frac{1}{(m-1)!} \lim_{z \to z_0} \frac{d^{m-1}}{dz^{m-1}} \left[(z - z_0)^m f(z) \right].$$

Applications of Residue Theorem

1. **Computing Contour Integrals:** Evaluate $\int_\Gamma \frac{e^z}{z(z-1)}\, dz$, where Γ encloses $z = 0$ and $z = 1$:

$$\mathrm{Res}\left(\frac{e^z}{z(z-1)}, z = 0 \right) = 1,$$
$$\mathrm{Res}\left(\frac{e^z}{z(z-1)}, z = 1 \right) = e.$$

   Then:

$$\int_\Gamma \frac{e^z}{z(z-1)}\, dz = 2\pi i \left(1 + e \right).$$

2. **Evaluating Real Integrals:** The integral $\int_{-\infty}^{\infty} \frac{\cos(x)}{x^2+1}\, dx$ can be converted to a contour integral using:

$$f(z) = \frac{e^{iz}}{z^2 + 1}.$$

   Using residues at $z = \pm i$, we find the value of the integral.

4.2 Usage in PseudoSpecteal Methods

$$f(x) - P_n(x) = \frac{1}{2\pi i} \oint_C \phi(x, z) dz$$

Assume that the poles of ϕ are x and x_k, where k = 0, 1, ... , n