AS5580 : Quiz 2

(1a)

$P_n(x) \longrightarrow n^{th}$ order monic polynomial

Assume $|P_n(x)| < \frac{1}{2^{n-1}}$

$q_n(x) = P_n(x) - \underset{\downarrow \text{ monic Chebyshev polynomial}}{T_n(x)}$ $\quad x_j \in \{-1, x_1, \ldots x_{n-1}, 1\} \rightarrow (n+1)$ node points

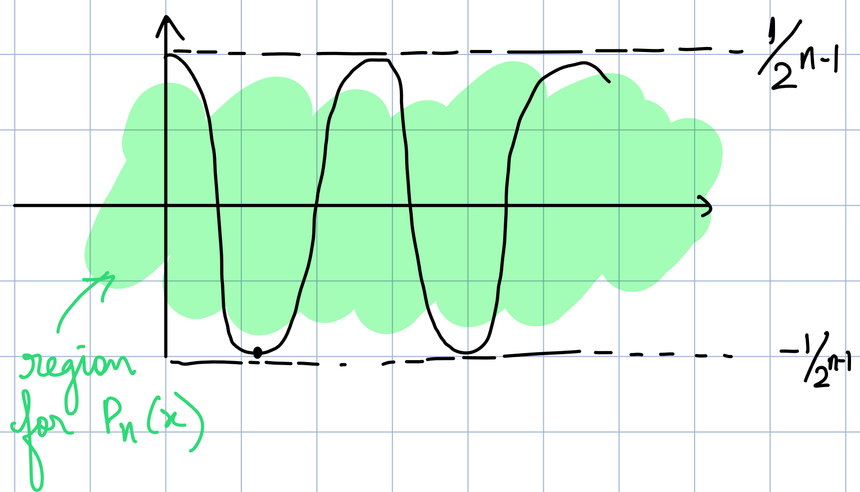
$x_1 \ldots x_{n-1} \rightarrow T_n'(x) = 0$

$\Downarrow$

$T_n(x)$ is an extrema

$\Rightarrow q_n$ is $(n-1)^{th}$ order

at extrema, $P_n(x_j) - T_n(x_j)$ keep flipping signs based on our assumption



$\Rightarrow q_n$ changes sign n times.

$\therefore$ q_n is $(n-1)^{th}$ order

but contradicts

$\therefore |P_n(x)| \geq \frac{1}{2^{n-1}}$

(1b)

$$P_n(x) = \sum_{k=0}^{n} c_k \phi_k \xrightarrow{\text{approximates}} x^{n+1}$$

$$P_n(x) - x^{n+1} = \prod_{k=0}^{n} \frac{(x - x_k)}{\cancel{(n+1)!}} \cdot \cancel{(n+1)!} = \sum_{k=0}^{n+1} a_k \phi_k$$

$$J = \int (P_n(x) - x^{n+1})^2 \, dx$$

$$\frac{\partial J}{\partial c_j} = \int 2 \, (\underbrace{P_n(x) - x^{n+1}}_{\sum_{k=0}^{n+1} a_k \phi_k}) \, (\phi_j) \, dx = 0$$

$$j = 0, \; - - - - , \; n$$

$$\Rightarrow \quad 2 \int \sum_{k=0}^{n+1} a_k \phi_k \phi_j \, dx = 0$$

$j = 0$

$$\int a_0 \phi_0^2 + \cancelto{0}{a_1 \phi_1 \phi_0 + - - - + a_{n+1} \phi_{n+1} \phi_0} = 0$$

$$\Rightarrow a_0 = 0$$

$\vdots$

$j = n$

$$\int \cancelto{0}{a_0 \phi_0 \phi_n + a_1 \phi_1 \phi_n + - - } + a_n \phi_n^2 + \cancelto{0}{a_{n+1} \phi_{n+1} \phi_n} = 0$$

$$\Rightarrow a_n = 0$$

$$\Rightarrow \quad a_k = 0 \quad \forall \, k = 0, \, - - - - , \, n$$

$$a_{n+1} \neq 0$$

(2) x_k , $k=0,1,2,\ldots,n$ roots of $T_{n+1}(x)=0$

α_j — roots of $T_n(x)$

$$\sum_{k=0}^{n}\prod_{j=0}^{n-1}(x_k-\alpha_j)$$

for

$$=\sum_{k=0}^{n}T_n(x_k)\,\frac{w_j}{w_j}=\frac{1}{w_j}\sum_{k=0}^{n}T_n(x_k)\,w_j$$

$$=\frac{1}{\underset{\pi/(n+1)}{\cancel{w_j}}}\int_{-1}^{1}\frac{T_n(x)\cdot\overset{T_0(x)}{1}\,dx}{\sqrt{1-x^2}}\quad,\ n\geq 2$$

$$=\frac{n+1}{\cancel{\pi}}\cdot\frac{\cancel{\pi}}{2}$$

$$=\frac{n+1}{2}$$

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(3) $$T_{n+1}(x)\,T_{n-1}(x)=\frac{1}{2}\left[T_{2n}(x)+T_2(x)\right]\qquad n>0$$

$$\cos\big((n+1)\overset{\theta}{\cos^{-1}x}\big)\,\cos\big((n-1)\overset{\theta}{\cos^{-1}x}\big)$$

sum of cos

$$=\frac{1}{2}\Big(\cos\big((n+1)\theta+(n-1)\theta\big)+\cos\big((n+1)\theta-(n-1)\theta\big)\Big)$$

$$=\frac{1}{2}\big(\cos 2n\theta+\cos 2\theta\big)=\frac{1}{2}\big(\cos(2n\cos^{-1}x)+\cos(2\cos^{-1}x)\big)$$

$$=\frac{1}{2}\big(T_{2n}(x)+T_2(x)\big)$$

(4) $x_j \in \{-1, 1\}$ $\quad j \in \{0, 1, \ldots, n-1\} \rightarrow$ roots of $T_n(x)$

x_n — Gauss Lobatto node $\rightarrow T_n'(x_n) = 0 \Rightarrow T_n(x_n) = \pm 1$

$$\int_{-1}^{1} L_n^2(x) \frac{dx}{\sqrt{1-x^2}}$$

$$L_k(x) = \prod_{\substack{k=0 \\ j \neq k}}^{n} \frac{(x - x_k)}{(x_j - x_k)}$$

$\downarrow$ using Chebyshev nodes

$$= \int_{-1}^{1} T_n^2(x) \frac{dx}{\sqrt{1-x^2}}$$

$$L_n(x) = \prod_{\substack{k=0 \\ j \neq k}}^{n} \frac{(x - x_k)}{\cancel{(x_n - x_k)}} \rightarrow \cancel{T_n(x_n)} \rightarrow \pm 1$$

$$= \boxed{\frac{\pi}{2}}$$

$$L_n(x) = \pm \prod_{\substack{k=0 \\ j \neq k}}^{n} (x - x_k) = \pm T_n(x)$$

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(5) $\displaystyle\sum_{k=0}^{n} x_k^4 \times \frac{w_k}{w_k}$ $\quad x_k, \ k = 0, 1, \ldots, n \quad , n \geq 2$

$$= \frac{1}{w_k} \sum_{k=0}^{n} \underbrace{x_k^4}_{f(x_k)} w_k = \frac{1}{w_k} \sum_{k=0}^{n} f(x_k) \int_{-1}^{1} L_k(x_k) dx \approx \int f(x) w(x) dx$$

$$\Rightarrow w_k = \pi / n+1$$

$$= \frac{n+1}{\pi} \int_{-1}^{1} \frac{x^4}{\sqrt{1-x^2}} dx$$

$x = \sin\theta$

$dx = \cos\theta \, d\theta$

$$= \frac{n+1}{\pi} \int_{-\pi/2}^{\pi/2} \frac{\sin^4\theta}{\cancel{\cos\theta}} \cancel{\cos\theta} \, d\theta$$

$$= 2\frac{(n+1)}{\pi} \int_{0}^{\pi/2} \sin^4\theta \, d\theta$$

$$\int_{0}^{\pi/2} \sin^{2n} dx = \frac{(2n)!}{(2^n n!)^2} \frac{\pi}{2} = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdots (2n)} \frac{\pi}{2}$$

$$\int_{0}^{\pi/2} \sin^{2n+1} dx = \frac{(2^n n!)^2}{(2n+1)!} = \frac{2 \cdot 4 \cdots 2n}{3 \cdot 5 \cdot 7 \cdots (2n+1)}$$

$$= \cancel{2} \frac{(n+1)}{\cancel{\pi}} \cdot \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{\cancel{\pi}}{\cancel{2}}$$

$$= \boxed{\frac{3}{8}(n+1)}$$