Chapter 9

Fourier Series

The Fourier series of a general function, f(x) can be written as

$$S_n(x) = a_0 + \sum_{k=1}^{\infty} \left(a_k \cos\left(\frac{2\pi kx}{T}\right) + b_k \sin\left(\frac{2\pi kx}{T}\right)\right) \tag{9.1}$$

where, the coefficients are

$$a_0 = \frac{1}{T}\int_0^T f(x)\,dx \tag{9.2}$$

$$a_k = \frac{2}{T}\int_0^T f(x)\cos\left(\frac{2\pi kx}{T}\right)dx \quad \text{for } k \geq 1 \tag{9.3}$$

$$b_k = \frac{2}{T}\int_0^T f(x)\sin\left(\frac{2\pi kx}{T}\right)dx \quad \text{for } k \geq 1 \tag{9.4}$$

It can also be written in complex form as

$$f(x) \sum_{n=-\infty}^{\infty} c_n e^{ikx} \tag{9.5}$$

where

$$c_n = \frac{2}{T}\int_0^T f(x)e^{-ikx}\,dx \tag{9.6}$$

but we are considering a periodic function f(x) with x ∈ [0, T]. So we can essentially take a part of a general function and consider that to be the period, hence we can do this analysis on any general function.

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9.1 Fourier Series Convergence

- What happens to $\lim_{k\to\infty} a_k$ and $\lim_{k\to\infty} b_k$? Do they both tend to zero?
- Does $\lim_{n\to\infty} S_n(x)$ converge to $f(x)$?

9.1.1 Numerical Proof

We can first try to answer these questions numerically using two methods for comparison, Fast Fourier Transform (FFT) and Numerical integration with the trapezoidal rule (trapz).

1. **Initialization:**
   - Define the number of sample points $N = 20$ and the time vector x spanning $[0, 2\pi]$.
2. **Performing FFT:**
   - Compute $F = \text{fft}(\text{func})$, where $\text{func} = \sin(x)$.
   - Extract magnitudes $|F|$ and phases $\angle F$ to find the Fourier coefficients:

$$a_k = \frac{2 \cdot \text{Re}(F)}{N}, \quad k = 1, 2, \ldots, \frac{N}{2}$$
$$b_k = -\frac{2 \cdot \text{Im}(F)}{N}$$

3. **Numerical Integration (Trapz):**
   - Calculate a_k and b_k using:

$$a_k = \frac{2}{T} \int_0^T f(x) \cos(2kx)\, dx$$
$$b_k = \frac{2}{T} \int_0^T f(x) \sin(kx)\, dx$$

   - The integrals are approximated with MATLAB's `trapz` function.
4. **Reconstruction:**
   - Reconstruct $f(x)$ from the Fourier series coefficients using:9

$$f(x) \approx \frac{a_0}{2} + \sum_{k=1}^{n} [a_k \cos(kx) + b_k \sin(kx)]$$

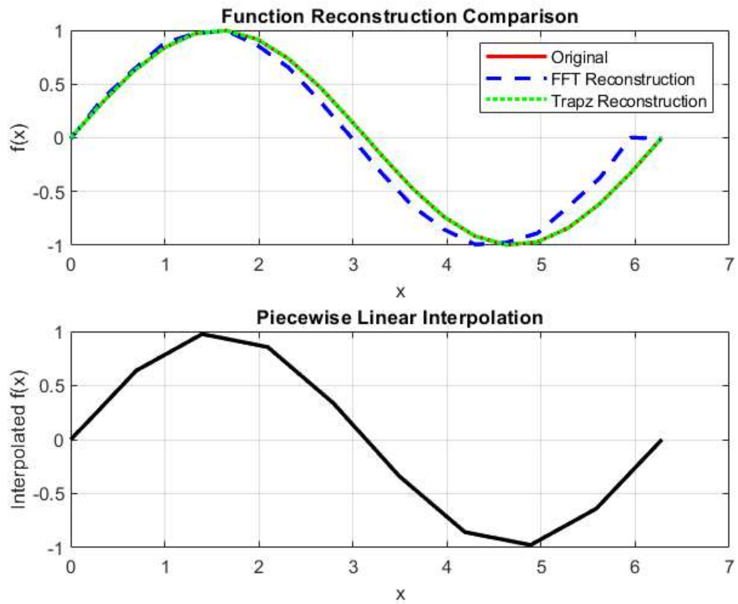
5. **Interpolation:**
   - Use `interp1` for piecewise linear interpolation to create a denser set of points for visualization.

Visualization

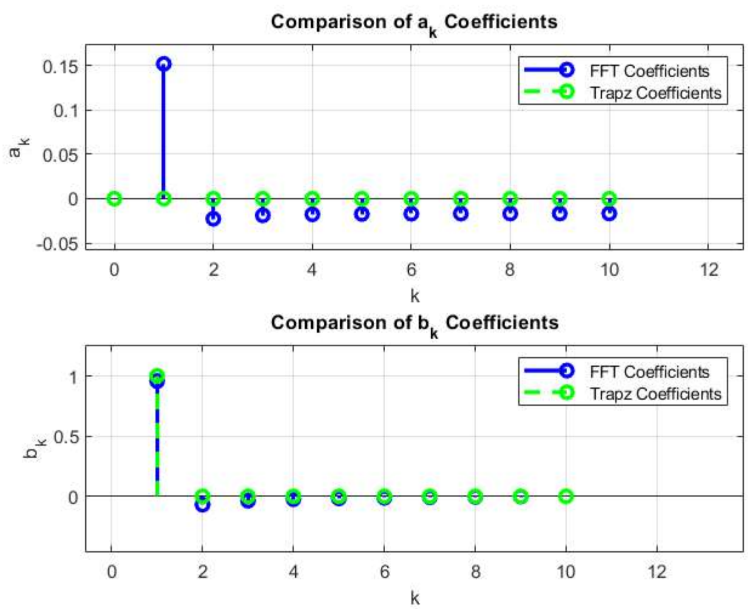
The code generates plots to:

- Compare the original function with its FFT-based and numerical integration-based reconstructions.

- Show a piecewise linear interpolation of $f(x)$.



- Display the Fourier coefficients a_k and b_k obtained from both methods.



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A similar piecewise linear interpolation can be done manually but it is a very tedious process. For example, take the function sin^2x with 3 points including ends (done in my notes). We have the coefficients $b_k = 0$ and

$$a_k = \begin{cases} \frac{-4}{\pi}k^2 & \text{k is odd} \\ 0 & \text{k is even} \end{cases}$$

So, here $S_n(x) = \Sigma_{k=0}^{n} a_k \; cos(kx)$ with calculated a_k as shown.

$|f(x) - sin^2(x)| = \frac{1}{(n+1)^2} + \frac{1}{(n+1)^2} + \dots$

(there's some proof about how the summation is is bounded or something, idk; then its extended to piecewise linear interpolation for general function, just writing out in therms of variabes basically)

9.1.2 Analytical Proof

The sine and cosine functions are orthogonal over the interval $[0, 2\pi]$. This property is fundamental in Fourier series analysis. The orthogonality conditions are defined as follows:

- **Orthogonality of Sine Functions:**

$$\int_0^{2\pi} \sin(mx)\sin(nx)\,dx = \begin{cases} 0, & \text{for } m \neq n \\ \pi, & \text{for } m = n \neq 0 \end{cases}$$

- **Orthogonality of Cosine Functions:**

$$\int_0^{2\pi} \cos(mx)\cos(nx)\,dx = \begin{cases} 0, & \text{for } m \neq n \\ \pi, & \text{for } m = n \neq 0 \\ 2\pi, & \text{for } m = n = 0 \end{cases}$$

- **Orthogonality Between Sine and Cosine:**

$$\int_0^{2\pi} \sin(mx)\cos(nx)\,dx = 0, \quad \text{for any } m, n.$$

These properties mean that when integrating the product of two different sine or cosine functions over a full period $[0, 2\pi]$, the result is zero. This orthogonality is crucial for expressing periodic functions as a sum of sine and cosine terms in a Fourier series.

proof that Sn(x) goes to zero if n tends to infinity, Sn(x) tends to f(x),

9.2 Collocation using Fourier Series as Basis

$S_n(x)$ is the approximation for the function f(x) using its Fourier series representation. The collocation points are x_k. The Fourier series decomposes a periodic function into a sum of sine and cosine terms as

$$S_n(x) = \alpha_0 + \sum_{m=1}^{\infty} \left(\alpha_m \cos\left(\frac{2\pi mx}{T}\right) + \beta_m \sin\left(\frac{2\pi mx}{T}\right) \right)$$

and

$$S_n(x_k) = f(x_k)$$

with $x_k = \frac{Tk}{2n}$ for k = 0, 1, 2, ... 2n+1

Using relations 9 and 9, where dx = $\frac{T}{2n+1}$

$$\alpha_m = \frac{2}{2n+1} \sum_{k=1}^{2n+1} f(x_k) \cos\left(\frac{2\pi m x}{T}\right)$$

$$\beta_m = \frac{2}{2n+1} \sum_{k=1}^{2n+1} f(x_k) \sin\left(\frac{2\pi m x}{T}\right)$$

Example: Fourier Series Approximation of (Numerical)

We approximate a given piecewise function $f(x)$ using its Fourier series representation. A periodic function $f(x)$ defined over the interval $[0, 2\pi]$ can be approximated by a Fourier series.

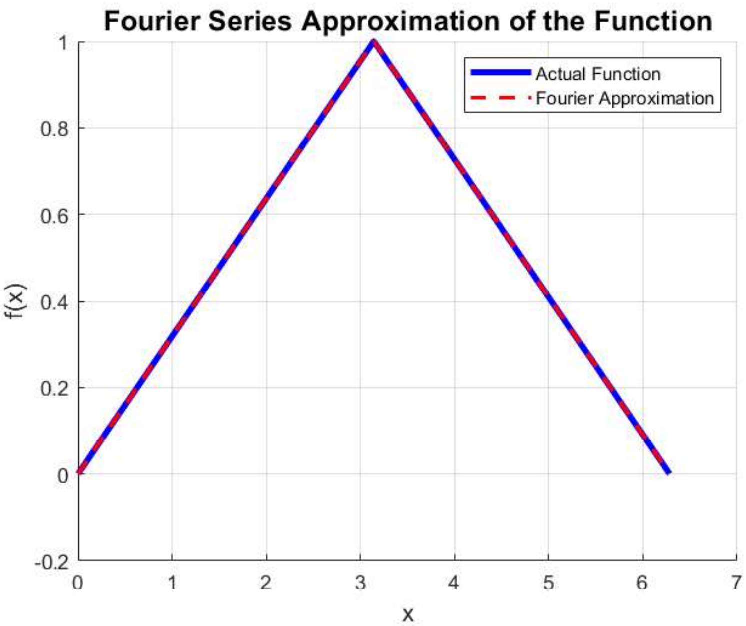
- **Initialization and Sample Points:**
  - The number of terms in the series n is set.
  - The vector x represents the points sampled from the interval $[0, 2\pi]$, specifically $2n + 1$ points for better accuracy.
- **Definition of the Function:**
  - The piecewise function is defined as:

$$f(x) = \begin{cases} \frac{x}{\pi}, & \text{for } 0 \leq x < \pi, \\ 2 - \frac{x}{\pi}, & \text{for } \pi \leq x \leq 2\pi. \end{cases}$$

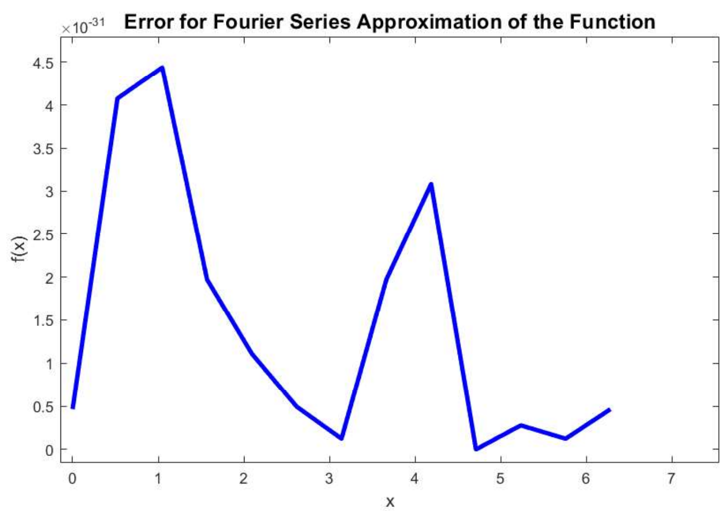
  - This is represented as two segments, `fx1` for $[0, \pi]$ and `fx2` for $[\pi, 2\pi]$.
- **Calculation of Coefficients:** The Fourier coefficients α_m and β_m using numerical integration (via the `trapz` function) as shown above for a specific piecewise function defined over the interval $[0, 2\pi]$.
- **Function Reconstruction:**
  - The function $S_n(x)$ is reconstructed using the calculated α_m and β_m coefficients.
  - The approximation $S_n(x)$ is then compared to the original piecewise function $f(x)$ by calculating the squared error $e(x)$.

Plotting and Analysis

- The first plot displays the original function $f(x)$ and its Fourier series approximation $S_n(x)$.



- The second plot visualizes the squared error $e(x)$ across the interval, which helps to assess the quality of the approximation.



This numerical approach highlights the convergence behavior of the Fourier series in approximating piecewise continuous functions.