Chapter 7

Chebyshev Polynomials

7.1 Definition of Chebyshev Polynomials

The Chebyshev polynomials $T_n(x)$ are defined recursively and can be expressed explicitly as:

$$T_n(x) = \cos(n \cos^{-}(x)), \quad \text{for } x \in [-1, 1].$$

7.2 Examples of the First Few Chebyshev Polynomials

$$\begin{aligned}
T_0(x) &= 1, \\
T_1(x) &= x, \\
T_2(x) &= 2x^2 - 1, \\
T_3(x) &= 4x^3 - 3x, \\
T_4(x) &= 8x^4 - 8x^2 + 1.
\end{aligned}$$

7.3 Orthogonality Property

The Chebyshev polynomials $T_n(x)$ satisfy the orthogonality condition:

$$\int_{-1}^{1} \frac{T_m(x)T_n(x)}{\sqrt{1-x^2}}\, dx = \begin{cases} 0, & \text{if } m \neq n, \\ \pi, & \text{if } m = n = 0, \\ \frac{\pi}{2}, & \text{if } m = n \neq 0. \end{cases}$$

7.4 Nodes of Chebyshev Polynomials

The nodes x_k (roots) of the Chebyshev polynomial $T_n(x)$ are given by:

$$x_k = \cos\left(\frac{2k+1}{2n}\pi\right), \quad k = 0, 1, 2, \ldots, n.$$

7.5 Derivative of Chebyshev Polynomials

The derivative $T'_{n+1}(x)$ can be computed as:

$$T'_{n+1}(x) = nU_n(x),$$

where $U_n(x)$ is the Chebyshev polynomial of the second kind.

7.6 Weights for Numerical Integration

The weights for numerical integration using Chebyshev polynomials are:

$$w_k = \frac{\pi}{n+1}, \quad \text{for } k = 0, 1, 2, \ldots, n.$$

where k are the node points

7.7 Recurrence Relation

The Chebyshev polynomials $T_n(x)$ can be generated using the recurrence relation:

$$T_{n+1}(x) + T_{n-1}(x) = 2xT_n(x) \quad \text{for } n \geq 2$$

with initial conditions:

$$T_0(x) = 1, \quad T_1(x) = x.$$

7.8 Chebyshev-Gauss-Lobatto Nodes

Definition: Chebyshev-Gauss-Lobatto nodes are specific collocation points used for interpolation and numerical methods involving Chebyshev polynomials. These nodes are particularly effective in minimizing the Runge phenomenon and ensuring numerical stability in polynomial interpolations.

7.8.1 Mathematical Formulation

The Chebyshev-Gauss-Lobatto nodes are defined in the interval $[-1, 1]$ as:

$$x_k = \cos\left(\frac{k\pi}{N}\right), \quad k = 0, 1, 2, \ldots, N,$$

where N is the total number of nodes.

7.8.2 Properties

- **Boundary Inclusion**: The nodes include the endpoints $x_0 = -1$ and $x_N = 1$, which is particularly useful for enforcing boundary conditions in numerical methods.
- **Clustering at Endpoints**: The distribution of nodes clusters more densely near the endpoints of the interval, which helps in reducing errors in polynomial approximations.

7.8.3 Applications

- **Spectral and Pseudospectral Methods**: Chebyshev-Gauss-Lobatto nodes are commonly used for constructing differentiation matrices in pseudospectral methods.
- **Interpolation and Approximation**: These nodes are employed in polynomial interpolation to achieve better accuracy and stability.

7.8.4 Example

For $N = 4$, the Chebyshev-Gauss-Lobatto nodes are:

$$x_0 = \cos\left(\frac{0\pi}{4}\right) = 1, \quad x_1 = \cos\left(\frac{1\pi}{4}\right) = \frac{\sqrt{2}}{2}, \quad x_2 = \cos\left(\frac{2\pi}{4}\right) = 0,$$

$$x_3 = \cos\left(\frac{3\pi}{4}\right) = -\frac{\sqrt{2}}{2}, \quad x_4 = \cos\left(\frac{4\pi}{4}\right) = -1.$$

7.8.5 Differentiation Matrix

In pseudospectral methods, the differentiation matrix D is constructed using Chebyshev-Gauss-Lobatto nodes. The matrix D approximates the derivative of a function sampled at these nodes and is defined by:

$$D_{kj} = \begin{cases} \frac{2N^2+1}{6}, & \text{for } k = j = 0, \\ \frac{-c_k}{c_j} \frac{(-1)^{k+j}}{x_k - x_j}, & \text{for } k \neq j, \\ -\frac{x_k}{2(1-x_k^2)}, & \text{for } 1 \leq k = j \leq N-1, \\ -\frac{2N^2+1}{6}, & \text{for } k = j = N, \end{cases}$$

where:

$$c_k = \begin{cases} 2, & \text{for } k = 0 \text{ or } k = N, \\ 1, & \text{otherwise.} \end{cases}$$

7.8.6 Orthogonality and Weights

Chebyshev polynomials of the first kind $T_n(x)$ are orthogonal with respect to the weight function $w(x) = \frac{1}{\sqrt{1-x^2}}$ over $[-1, 1]$:

$$\int_{-1}^{1} T_m(x) T_n(x) w(x)\, dx = \begin{cases} 0, & \text{for } m \neq n, \\ \pi, & \text{for } m = n = 0, \\ \frac{\pi}{2}, & \text{for } m = n \neq 0. \end{cases}$$

The quadrature weights w_k associated with Chebyshev-Gauss-Lobatto nodes are given by:

$$w_k = \frac{\pi}{N}\left(1 - x_k^2\right).$$

7.9 Conclusion

Chebyshev-Gauss-Lobatto nodes play a vital role in numerical approximation methods due to their favorable distribution and properties that facilitate accurate polynomial interpolation and stable numerical differentiation.