AS5580: Pseudospectral Methods for Optimal Control

Tutorial (27/09/24) Solutions

Question 1

Let $L_j(x)$ be Lagrange interpolating basis polynomials based on roots of $T_{n+1}(x)$. For weight function $w(x) = \dfrac{1}{\sqrt{1-x^2}}$, show that $L_j(x)$ is orthogonal to $L_k(x)$ when $k \neq j$. Also determine

$$\int_{-1}^{1} L_k^2(x)\frac{dx}{\sqrt{1-x^2}}$$

For showing orthogonality, we have to show that the integral

$$I = \int_{-1}^{1} L_k(x)L_j(x)\frac{dx}{\sqrt{1-x^2}}$$

is zero for $k \neq j$ and non-zero for $k = j$. Using the Gauss-Jacobi formula, we have:

$$I = \sum_{i=0}^{n} L_k(x_i)L_j(x_i)w_i$$

where, x_i are the roots of $T_{n+1}(x)$. Now, from the properties of Lagrange interpolating polynomials, we know that:

$$L_k(x_i) = \delta_{ik} = \begin{cases} 1 & i = k \\ 0 & i \neq k \end{cases}$$

Similarly, $L_j(x_i) = \delta_{ij}$. Thus, the integral is non-zero only when $i = j = k$. Thus:

$$I = \begin{cases} 0 & k \neq j \\ w_k & k = j \end{cases}$$

Hence, we proved the required orthogonality condition. Now, it was shown in class that $w_k = \dfrac{\pi}{n+1}$ for Chebyshev nodes. Thus, the answer to second part is this value.

$$\boxed{I = \begin{cases} 0 & k \neq j \\ \dfrac{\pi}{n+1} & k = j \end{cases}}$$

Question 2

Let x_k, $k = 0, 1, \ldots, n$ be roots of Chebyshev polynomial $T_{n+1}(x) = 0$. Show that

$$\sum_{k=0}^{n} \frac{1}{1-x_k} = (n+1)^2$$

The Chebyshev polynomial can be defined as:

$$T_{n+1}(x) = \frac{1}{2^n} \prod_{k=0}^{n} (x - x_k)$$

where, x_k are the Chebyshev nodes. Taking logarithm on both sides:

$$\log\left(T_{n+1}(x)\right) = \sum_{k=0}^{n} \log(x - x_k) - \log(2^n)$$

Taking derivative w.r.t. x, we have:

$$\frac{T'_{n+1}(x)}{T_{n+1}(x)} = \sum_{k=0}^{n} \frac{1}{x - x_k}$$

Thus, if we substitute $x = 1$, we get the required LHS. Thus, we have to prove that:

$$\frac{T'_{n+1}(1)}{T_{n+1}(1)} = (n+1)^2$$

Using the cosine definition of T_{n+1}, we have:

$$\begin{aligned}
T_{n+1}(x) &= \cos\left((n+1)\cos^{-1} x\right) \\
\implies T_{n+1}(\cos\theta) &= \cos\left((n+1)\theta\right) \\
x = 1 &\implies \theta = 0 \\
\implies T_{n+1}(1) &= 1
\end{aligned}$$

Taking derivative w.r.t. θ:

$$\begin{aligned}
T'_{n+1}(\cos\theta) &= (n+1)\ \frac{\sin(n+1)\theta}{\sin\theta} \\
\implies T'_{n+1}(1) &= (n+1)\ \lim_{\theta\to 0} \frac{\sin(n+1)\theta}{\sin\theta}
\end{aligned}$$

Using L'Hôpital's rule:

$$\implies T'_{n+1}(1) = (n+1)^2$$

Hence proved that:

$$\boxed{\sum_{k=0}^{n} \frac{1}{1-x_k} = (n+1)^2}$$

Question 3

The function x^{n+1} is interpolated by a polynomial $P_n(x)$ using nodes x_k, $k = 0, 1, 2, \ldots, n$ which are roots of $T_{n+1}(x)$. Evaluate the integral

$$\int_{-1}^{1} \left(x^{n+1} - P_n(x)\right)^2 \frac{dx}{\sqrt{1-x^2}}$$

Cauchy's Remainder Theorem gives us:

$$f(x) - P_n(x) = \frac{\prod_{k=0}^{n}(x - x_k)}{(n+1)!} f^{(n+1)}(\zeta)$$

Here, $f(x) = x^{n+1}$. Thus, we have:

$$\begin{aligned} f(x) - P_n(x) &= \prod_{k=0}^{n}(x - x_k) \\ &= \frac{T_{n+1}(x)}{2^n} \end{aligned}$$

Thus, the integral becomes:

$$\begin{aligned} I &= \int_{-1}^{1} \left(x^{n+1} - P_n(x)\right)^2 \frac{dx}{\sqrt{1-x^2}} \\ &= \int_{-1}^{1} \left(\frac{T_{n+1}(x)}{2^n}\right)^2 \frac{dx}{\sqrt{1-x^2}} \\ &= \frac{1}{2^{2n}} \int_{-1}^{1} T_{n+1}^2(x) \frac{dx}{\sqrt{1-x^2}} \\ &= \frac{1}{2^{2n}} \frac{\pi}{2} \\ &\boxed{I = \frac{\pi}{2^{2n+1}}} \end{aligned}$$

Question 4

Determine the monic quadratic polynomial $P_2(x)$ which is an element of an orthogonal basis (monic polynomials) associated with interval $[-1,\ 1]$ and weight function $w(x) = 1$. Given the interval and the weight function, it understood that we have to find the Legendre polynomials. Since the polynomials are monic, $P_0(x) = 1$. Then, $P_1(x) = x + a$. Then, using orthogonality, we have:

$$\int_{-1}^{1} P_0(x)P_1(x)w(x)dx = 0$$

$$\int_{-1}^{1} 1(x+a)1dx = 0$$

$$\left(\frac{x^2}{2} + ax\right)\Bigg|_{x=-1}^{x=1} = 0$$

$$\implies a = 0$$

$$\implies P_1(x) = x$$

Similarly, $P_2(x) = x^2 + bx + c$. Using orthogonality, we get:

$$\int_{-1}^{1} P_0(x)P_2(x)w(x)dx = 0$$

$$\int_{-1}^{1} 1(x^2 + bx + c)1dx = 0$$

$$\left(\frac{x^3}{3} + \frac{bx^2}{2} + cx\right)\Bigg|_{x=-1}^{x=1} = 0$$

$$\implies c = -\frac{1}{3}$$

and:

$$\int_{-1}^{1} P_1(x)P_2(x)w(x)dx = 0$$

$$\int_{-1}^{1} x(x^2 + bx + c)1dx = 0$$

$$\left(\frac{x^4}{4} + \frac{bx^3}{3} + \frac{cx^2}{2}\right)\Bigg|_{x=-1}^{x=1} = 0$$

$$\implies b = 0$$

$$\implies \boxed{P_2(x) = x^2 - \frac{1}{3}}$$