AS5580 : QUIZ 1

① $L_j(x)$ — Lagrange interpolating basis polynomials on roots of $T_{n+1}(x)$

$$\downarrow$$

$$x_K = \cos \frac{(2k-1)}{2}\frac{\pi}{n+1}$$

$$w(x) = \frac{1}{\sqrt{1-x^2}}$$

$$L_j(x_k) = \delta_{jk} = \begin{cases} 1 & j = k \\ 0 & j \neq k \end{cases}$$

$$\int_{-1}^{1} \frac{L_j(x)\, L_k(x)}{\sqrt{1-x^2}} = \sum_{i=1}^{n+1} w_i L_j(x_i)\, L_k(x_i)$$ — Gaussian quadrature

$$L_j(x_i)\, L_k(x_i) = \delta_{ij}\delta_{ik}$$

$$\downarrow$$

non zero when $i=j=k$

but $j \neq k$

$$= 0$$

$\therefore$ $L_j(x)$ & $L_k(x)$ are orthogonal

$$\int_{-1}^{1} \frac{L_k^2(x)}{\sqrt{1-x^2}} = \sum_{i=1}^{n+1} w_j\, L_k^2(x_j)$$

$$x_i = \frac{(2i-1)}{2}\frac{\pi}{n+1}$$

$$w_j = \frac{\pi}{n+1} \quad \forall j \qquad w_j = \int_{-1}^{1} L_j(x)\,dx \qquad L_j(x) = \prod_{\substack{i \neq j \\ i=0}} \frac{x-x_i}{x_j-x_i}$$

$$= \sum_{j=1}^{n+1} \frac{\pi}{n+1} \cdot \delta_{jk}$$ → for one specific k

$$= \frac{\pi}{n+1}$$

② $\sum_{k=0}^{n} \frac{1}{1-x_k}$

$$C(x) = \sum_{k=0}^{n} \log(x - x_k) = \log \prod_{k=0}^{n} (x - x_k)$$

$$C(x) = \log(T_{n+1}(x))$$

$$C'(x) = \sum_{k=0}^{n} \frac{1}{x - x_k}$$

$$T'_{n+1}(x) = \cancel{+}(n+1)\sin((n+1)\theta) \cancel{+} \frac{1}{\sin\theta}$$

$$C'(1) = \sum_{k=0}^{n} \frac{1}{1 - x_k}$$

$$T'_{n+1}(x) = \cancel{+}(n+1)\frac{\sin((n+1)\cos^{-1}x)}{\sin(\cos^{-1}x)}$$

$$\sum_{k=0}^{n} \frac{1}{1-x_k} = C'(1)$$

$$\lim_{\theta \to 0} = (n+1)^2$$

$$= T'_{n+1}(1)$$

$$= (n+1)^2$$

$$C'(x) = \frac{T'_{n+1}(x)}{T_{n+1}(x)}$$

$$C'(1) = \frac{T'_{n+1}(1)}{\cancel{T_{n+1}(1)} \to 1} = T'_{n+1}(1)$$

③ $\displaystyle\int_{-1}^{1} \left[x^{n+1} - P_n(x)\right]^2 \frac{dx}{\sqrt{1-x^2}}$ $\qquad P_n(x) = \sum_{k=0}^{n} c_k T_k(x)$

Cauchy's Remainder Theorem,

$$f(x) - P_n(x) = \frac{\prod_{k=0}^{n} (x - x_k)}{(n+1)!} f^{n+1}(\xi)$$

$$f(x) = x^{n+1} \quad f(x) - P_n(x) = \prod_{k=0}^{n} (x - x_k) = \frac{1}{2^n} T_{n+1}(x)$$

$$I = \int_{-1}^{1} \left(\frac{T_{n+1}(x)}{2^n}\right)^2 \frac{dx}{\sqrt{1-x^2}} = \frac{1}{2^{2n}} \int_{-1}^{1} T_{n+1}^2(x) \frac{dx}{\sqrt{1-x^2}} = \frac{\pi}{2^{2n+1}}$$

④ monic quadratic $P_2(x)$ — orthogonal basis $[-1, 1]$ $\quad w(x) = 1$

$$P_2(x) = x^2 + bx + c$$

$$\int_{-1}^{1} \cancel{P_0(x)}^{\,1} P_2(x) = 0 \qquad \int_{-1}^{1} \cancel{P_1(x)}^{\,x} P_2(x) = 0$$

$$\int_{-1}^{1} (x^2 + bx + c)\, dx = 0 \qquad \int_{-1}^{1} (x^3 + bx^2 + cx)\, dx = 0$$

$$\left[\frac{x^3}{3} + b\frac{x^2}{2} + cx\right]_{-1}^{1} = 0 \qquad \left[\frac{x^4}{4} + \frac{bx^3}{3} + \frac{cx^2}{2}\right]_{-1}^{1} = 0$$

$$\frac{2}{3} + 2c = 0 \Rightarrow c = -1/3 \qquad 0 + \frac{2b}{3} + 0 = 0$$

$$b = 0$$

$$\Rightarrow P_2(x) = x^2 - 1/3$$