Department Of Aerospace Engineering
Indian Institute Of Technology Madras

AS5580
Pseudo-Spectral Methods for Optimal Control

Lecture Note No. 11

Complex Analysis Review

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August 21, 2024

Contents

1 Introduction

In this report, the notes for the lecture conducted for the course **Pseudo-Spectral Methods for Optimal Control (AS5580)** conducted by Prof. Ranjith Mohan. The topic presented in this report is the ***Complex Analysis Review***.

2 Complex Differentiation & Integration

Limit and Continuity:

A function $f(z) : \mathbb{C} \to \mathbb{C}$ is said to have the limit l when z approaches z_0, written as $\lim_{z \to z_0} f(z) = l$, if f is defined in a neighbourhood of z_0 (except maybe at z_0 exactly) and for every $\epsilon \in \mathbb{R}^+$, we can find a $\delta \in \mathbb{R}^+$ such that for all $z \neq z_0$ in the disk $|z - z_0| < \delta$, we have

$$|f(z) - l| < \epsilon$$

This formal definition is same as that we use in a real number context, but the primary difference is in how this translates in a geometric sense. In the real case, x can approach x_0 from only the real line (left and right sides). In complex plane, the point can be approached from all directions and thus, the value should approach l from all directions.

The function is defined to be *continuous* at $z = z_0$ if

$$\lim_{z \to z_0} f(z) = f(z_0)$$

Derivative:

The derivative is defined in the same manner as in real case. The *derivative* of the function is denoted and defined as

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \tag{1}$$

provided that this limit exists. The function is then said to be *differentiable* at z_0. Here, again, the limit should exist and approach the same value $f'(z_0)$ from all directions to be termed as differentiable. The above equation can be alternatively written using $z = z_0 + \Delta z$ as

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

Analytic Function:

A function is said to be *analytic* in a domain D if $f(z)$ is defined and differentiable at all points of D. It is said to be analytic at a point z_0 *if* it is analytic in a neighbourhood around z_0.

To find if a function is analytic, we can use the ***Cauchy-Riemann Equations***, which together with certain continuity and differentiability criteria, form a necessary and sufficient condition for a function to be analytic. They can be found as follows:

First, we define the function f as summation of two real functions $u(x, y)$ and $v(x, y)$ as

$$f(z) = f(x, y) = u(x, y) + iv(x, y)$$

where, we define $z = x + iy$. First, we assume the function to be differentiable. Now, according to definition of limits in complex plane, z can approach z_0 from anywhere in the limit in Eqn. 1. Thus, we can use two different paths as shown in Fig. 1. On comparing the derivative for these two cases, we will get the CR Equations.

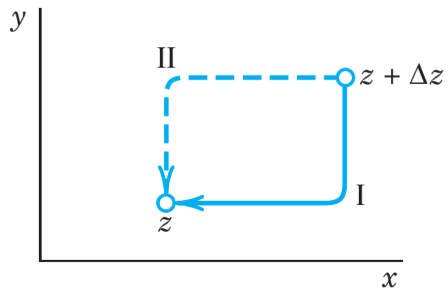

Figure 1: The two paths taken for Proof of CR Equations

First, write $\Delta z = \Delta x + i\Delta v$. Thus, we have the derivative equation as

$$f'(z) = \lim_{\Delta z \to 0} \frac{[u(x+\Delta x, y+\Delta y) + iv(x+\Delta x, y+\Delta y)] - [u(x,y) + iv(x,y)]}{\Delta x + i\Delta y}$$

For the first path, we make $\Delta y \to 0$ first and then $\Delta x \to 0$. Thus, we have

$$f'(z) = \lim_{\Delta x \to 0} \frac{u(x+\Delta x, y) - u(x,y)}{\Delta x} + i \lim_{\Delta x \to 0} \frac{v(x+\Delta x, y) - v(x,y)}{\Delta x}$$

where, the two limits on right exist because $f'(z)$ exists and they correspond to partial derivative of u and v with respect to x (by definition). Thus, we have

$$f'(z) = u_x + iv_x \tag{2}$$

For the second path, we have $\Delta x \to 0$ first then $\Delta y \to 0$. Through the same methodology, we get

$$f'(z) = \lim_{\Delta y \to 0} \frac{u(x, y+\Delta y) - u(x,y)}{i\Delta y} + i \lim_{\Delta y \to 0} \frac{v(x, y+\Delta y) - v(x,y)}{i\Delta y}$$

which we get as

$$f'(z) = -iu_y + v_y \tag{3}$$

Equating the real and imaginary parts of both we get

$$\begin{aligned} u_x &= v_y \\ v_x &= -u_y \end{aligned} \tag{4}$$

These are the ***Cauchy-Riemann Equations***.

Complex Integration:

In complex analysis, the indefinite integral $\int f(z)\,dz$ is a function $F(z)$ whose derivative is the given analytic function $f(z)$. The definite integral is called as a complex line integral and is written as $\int_C f(z)\,dz$. Here, the integrand is integrated over the curve C, called as the path of integration. For analytic functions, using the indefinite integral, we can write it as

$$\int_C f(z)dz = F(b) - F(a) \tag{5}$$

where, the curve C joins the two points a and b.

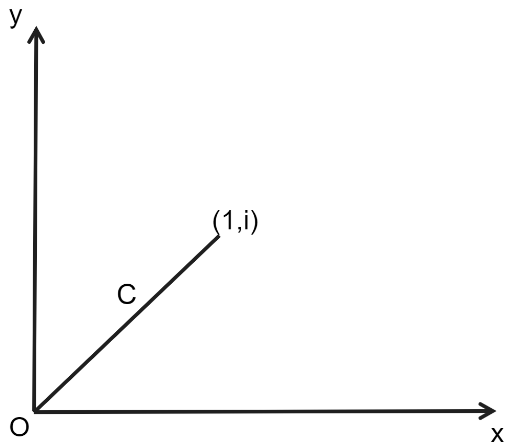

Figure 2: Path of integration

For example, $f(z) = z$ can be integrated on the path shown in Fig. 2 as

$$\begin{aligned}\int\limits_C f(z)\,dz &= \int_0^{1+i} zdz \\ &= \frac{(1+i)^2}{2} - \frac{0^2}{2} \\ &= i\end{aligned}$$

It can be parameterized and computed as well, which is given as

$$\int\limits_C f(z)dz = \int_a^b f[z(t)]\dot{z}(t) \tag{6}$$

where, C is denoted with $z(t)$ and $a \leq t \leq b$. In the above example, the path C can be parameterized as $z = t(1+i)$, where $0 \leq t \leq 1$. Thus, $dz = (1+i)dt$. Thus,

$$\begin{aligned}\int\limits_C f(z)\,dz &= \int_0^1 (1+i)t(1+i)dt \\ &= (1+i)^2 \int_0^1 tdt \\ &= \frac{(1+i)^2}{2} \\ &= i\end{aligned}$$

Expanding the function in terms of its real and imaginary parts, we get

$$\begin{aligned}\int\limits_C f(z)dz &= \int\limits_C (u+iv)d(x+iy) \\ &= \int\limits_C udx - vdy + i\int\limits_C udy + vdx\end{aligned} \tag{7}$$

In the above example, $u = x$ and $v = y$. The path C is given by $x = y$, which implies $u = x = y = v$ on the path C. So, we get

$$\begin{aligned}\int_C f(z)dz &= \int_0^1 xdx - \int_0^1 ydy + i\int_0^1 ydy + i\int_0^1 xdx \\ &= \frac{1}{2} - \frac{1}{2} + i\frac{1}{2} + i\frac{1}{2} \\ &= i\end{aligned}$$

In general, the complex integral depends on both the end points as well as the path of integration.

ML Inequality:

For a function f, we have

$$\left|\int_C f(z)dz\right| \leq ML \tag{8}$$

where, the function satisfies $|f(z)| \leq M$ everywhere on the curve C and L is the length of the curve C.

Proof: Consider $n+1$ randomly spaced points $z_0, z_1, \ldots, z_n$ on curve C, which divides it into n sections. We can then approximate the integral as a sum as

$$\int_C f(z)dz = \sum_{m=1}^{n} f(\zeta_m)\Delta z_m$$

where, ζ_m is any arbitrary point in the section z_{m-1} to z_m and $\Delta z_m = z_m - z_{m-1}$. As n increases, the sum eventually converges to the integral.

Now, using triangle inequality, we can write the sum

$$\int_C f(z)dz \leq \sum_{m=1}^{n} |f(\zeta_m)||\Delta z_m|$$

Since it is given that f is bounded by M on the curve, we can write it as

$$\begin{aligned}\int_C f(z)dz &\leq M\sum_{m=1}^{n} |\Delta z_m| \\ &= ML\end{aligned}$$

Hence, the inequality has been proved.

Cauchy's Integral Theorem:

For any analytic function $f(z)$ in a simply connected domain D, for every closed path C in D, we have

$$\oint_C f(z)dz = 0 \tag{9}$$

Proof: Since the function f is analytic in domain D, its derivative $f'(z)$ exists. With the *additional assumption* that $f'(z)$ is continuous (which can be proved to be true for analytic functions), we have that u and v have continuous partial derivatives.

Now, consider the first integral from Eqn. 7. Using Green's Theorem, we have

$$\oint_C u dx - v dy = \iint_R \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx dy$$

where, R is the region bounded by curve C. By the second CR equation (Eqn. 4), the RHS is zero. Similarly, the second integral of Eqn. 7 will be zero using the Green's Theorem and first CR equation. Thus, Cauchy's Integral Theorem has been proved.

Using this theorem, we can prove that: ***For a function f analytic in the domain D, integration of $f(z)$ is independent of the path.***

Cauchy's Integral Theorem for Multiply Connected Domain:

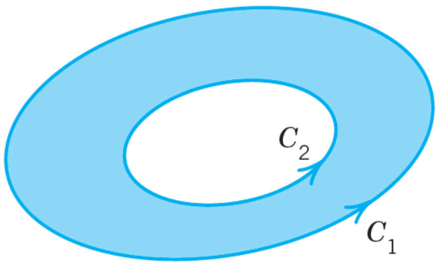

Figure 3: Multiply Connected Domain D

Let D be a doubly connected domain, bounded by outer curve C_1 and inner curve C_2 (shown as blue region in Fig. 3). If a function $f(z)$ is analytic in any domain $D*$ which contains D, C_1 and C_2, then

$$\oint_{C_1} f(z)dz = \oint_{C_2} f(z)dz \tag{10}$$

Proof: Make a cut C_3 in the domain, which has its two endpoints as curves C_1 and C_2. Now, the new domain is a simply connected one, bounded by C_1, C_3, C_2 (in reverse direction) and again C_3 (in reverse direction). In this domain, using Cauchy's Integral Theorem (CIT), we have

$$\oint_C f(z)dz = \oint_{C_1} f(z)dz + \oint_{C_3} f(z)dz - \oint_{C_3} f(z)dz - \oint_{C_3} f(z)dz = 0$$

$$\implies \oint_{C_1} f(z)dz = \oint_{C_2} f(z)dz$$

Cauchy's Integral Formula:

Let function f be an analytic function in a simply connected domain D. Then for any point z_0 in domain D,

$$f(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z - z_0} dz \tag{11}$$

where, C is any closed curve in domain D which encloses z_0.
Proof: First, we will prove a useful result, which will be used to prove the formula.

Define $g(z) = \dfrac{f(z) - f(z_0)}{z - z_0}$. We can see that $g(z)$ is analytic everywhere except at z_0, but is bounded there, since by definition

$$g(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = f'(z_0)$$

Now, using the CIT for doubly connected domain, we can choose a circle C_1, which is completely enclosed by C and we will have

$$\oint_C g(z)dz = \oint_{C_1} g(z)dz$$

We can use the ML inequality on C_1, we have

$$\oint_C g(z)dz \leq f_{max}(2\pi r)$$

Now, since g is not analytic only at z_0, we can shrink the size of C_1 to as small as we want. But, as $r \to 0$, the integral goes to 0. Thus, we have

$$\oint_C g(z)dz = 0$$

Expanding this using definition of $g(z)$, we get

$$\begin{aligned} &\oint_C \frac{f(z) - f(z_0)}{z - z_0} dz = 0 \\ \implies &\oint_C \frac{f(z)}{z - z_0} dz = \oint_C \frac{f(z_0)}{z - z_0} dz \end{aligned}$$

Again consider a circle C_1, centered at z_0, completely enclosed by C. We can thus write $z - z_0 = re^{i\theta}$. This gives us $dz = re^{i\theta} i d\theta$. So, the RHS becomes

$$\begin{aligned} \oint_{C_1} \frac{f(z_0)}{z - z_0} dz &= f(z_0) \oint_{C_1} \frac{dz}{z - z_0} \\ &= f(z_0) \int_0^{2\pi} \frac{re^{i\theta} i d\theta}{re^{i\theta}} \\ &= f(z_0) \int_0^{2\pi} i d\theta \\ &= f(z_0)\, 2\pi i \end{aligned}$$

Thus, we have

$$f(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z - z_0} dz$$

Thus, we have derived the Cauchy's Integral Formula.

3 Taylor & Laurent Series Expansions

Taylor Series:

The Taylor series of a function $f(z)$, the complex analog of the real Taylor series is:

$$f(z) = \sum_{n=1}^{\infty} a_n (z - z_0)^n \tag{12}$$

Where,

$$\begin{aligned} a_n &= \frac{1}{n!} f^{(n)}(z_0) \\ &= \frac{1}{2\pi i} \oint_C \frac{f(z')}{(z' - z_0)^{n+1}} dz' \end{aligned} \tag{13}$$

Here, we integrate counterclockwise around a simple closed path C that contains z_0 in its interior and is such that $f(z)$ is analytic in a domain containing C and every point inside C.

Proof:

From **Cauchy's Integral Formula**, if a complex function $f(z')$ is analytic within and on a closed contour C inside a simply-connected domain, and if z is any point in the middle of C, then we can write:

$$f(z) = \frac{1}{2\pi i} \oint_C \frac{f(z')}{z' - z} dz' \tag{14}$$

We take a circle with center z_0 and interior in C. Let us now take $1/(z' - z)$ and perform the following algebraic manipulation:

$$\frac{1}{z' - z} = \frac{1}{z' - z_0 - (z - z_0)} = \frac{1}{(z' - z_0)\left(1 - \dfrac{z - z_0}{z' - z_0}\right)} \tag{15}$$

Making use of the fact that z' is on C while z is inside C, we can write:

$$\left| \frac{z - z_0}{z' - z_0} \right| < 1$$

Given this inequality, we notice that $1/\left(1 - \dfrac{z - z_0}{z' - z_0}\right)$ is the infinte sum of a geometric series and can be rewritten as:

$$\frac{1}{\left(1 - \dfrac{z - z_0}{z' - z_0}\right)} = \sum_{n=0}^{\infty} \left(\frac{z - z_0}{z' - z_0}\right)^n \tag{16}$$

Therefore we can rewrite 15 as:

$$\frac{1}{z' - z} = \frac{1}{z' - z_0} \left(\sum_{n=0}^{\infty} \left(\frac{z - z_0}{z' - z_0}\right)^n \right) \tag{17}$$

Substituting 17 in 14 we get:

$$\begin{aligned} f(z) &= \frac{1}{2\pi i} \oint_C \frac{f(z')}{z' - z_0} \left(\sum_{n=0}^{\infty} \left(\frac{z - z_0}{z' - z_0}\right)^n \right) dz' \\ &= \sum_{n=0}^{\infty} \left(\frac{1}{2\pi i} \oint_C \frac{f(z')}{(z' - z_0)^{n+1}} dz' \right) (z - z_0)^n \end{aligned} \tag{18}$$

From the above expression and the definition of the Taylor Series given in 12, we can observe that:

$$a_n = \frac{1}{2\pi i} \oint_C \frac{f(z')}{(z' - z_0)^{n+1}} dz' \tag{19}$$

Differentiating $f(z_0)$ from equation 14 with respect to z_0, we get:

$$f^{(1)}(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z')}{(z' - z_0)^2} dz \tag{20}$$

Similarly, we get the second derivative as:

$$f^{(2)}(z_0) = \frac{2}{2\pi i} \oint_C \frac{f(z')}{(z' - z_0)^3} dz \tag{21}$$

By observing the pattern and generalising for the n^{th} derivative, we get:

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_C \frac{f(z')}{(z' - z_0)^{n+1}} dz \tag{22}$$

Therefore, we can write:

$$a_n = \frac{1}{n!} f^{(n)}(z_0) \tag{23}$$

Therefore, we get the final form of the Taylor Series as:

$$\begin{aligned} f(z) &= \sum_{n=0}^{\infty} a_n (z - z_0)^n \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(z_0)(z - z_0)^n \\ &= \sum_{n=0}^{\infty} \left(\frac{1}{2\pi i} \oint_C \frac{f(z')}{(z' - z_0)^{n+1}} dz' \right) (z - z_0)^n \end{aligned} \tag{24}$$

Laurent Series:

Laurent series generalize Taylor series. If, in an application, we want to develop a function $f(z)$ in powers of $z - z_0$ when $f(z)$ is singular at z_0, we cannot use a Taylor series. Instead we can use a new kind of series, called Laurent series. The Laurent series consists of positive integer powers of $z - z_0$ (and a constant) as well as ***negative integer powers*** of $z - z_0$.

The Laurent Series of a function $f(z)$ is:

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n} \tag{25}$$

Where,

$$\begin{aligned} a_n &= \frac{1}{2\pi i} \oint_C \frac{f(z')}{(z' - z_0)^{n+1}} dz' \\ b_n &= \frac{1}{2\pi i} \oint_C (z' - z_0)^{n-1} f(z') dz' \end{aligned} \tag{26}$$

Here, we integrate counterclockwise around a simple closed path C that lies in the annulus and encircles the inner circle, as in figure 4. Here, the function in analytic in the shaded region and singular at z_0. This series converges and represents $f(z)$ in the enlarged open annulus obtained from the given annulus by continuously increasing the outer circle C_1 and decreasing C_2 until each of the two circles reaches a point where $f(z)$ is singular.

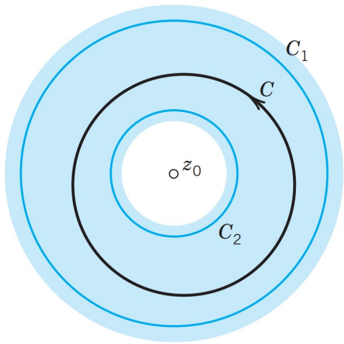

Figure 4: Laurent's Theorem

Proof:

Let us take z such that it lies in the analytic region, that is, in between the contours C_1 and C_2. Then by **Cauchy's Integral Formula**, we can write:

$$f(z) = \frac{1}{2\pi i}\oint_{C_1 - C_2} \frac{f(z')}{z' - z} dz' \tag{27}$$

We can split the above integral into two integrals I_1 and I_2 as follows:

$$\begin{aligned} I_1 &= \frac{1}{2\pi i}\oint_{C_1} \frac{f(z')}{z' - z} dz' \\ I_2 &= \frac{1}{2\pi i}\oint_{C_2} \frac{f(z')}{z' - z} dz' \end{aligned} \tag{28}$$

First, we take I_2. We can perform algebraic manipulation on I_2 and rewrite it as:

$$I_2 = \frac{1}{2\pi i}\oint_{C_2} \frac{-f(z')}{(z - z_0)\left(1 - \dfrac{z' - z_0}{z - z_0}\right)} dz' \tag{29}$$

Using the fact that z lies in between C_1 and C_2, z' lies on C_2 and z_0 lies at the center of C_2, we have:

$$\left|\frac{z' - z_0}{z - z_0}\right| < 1$$

Therefore, we observe that $1/\left(1 - \dfrac{z' - z_0}{z - z_0}\right)$ can be rewritten as the infinite sum of a geometric series. Therefore we can rewrite I_2 as:

$$I_2 = \frac{1}{2\pi i}\oint_{C_2} \frac{-f(z')}{(z - z_0)} \sum_{n=0}^{\infty} \left(\frac{z' - z_0}{z - z_0}\right)^n dz' \tag{30}$$

We know that $f(z)$ is analytic in $C_1 - C_2$. We also know that $(z' - z_0)^n$ is analytic. Therefore we can say that their product is analytic in $C_1 - C_2$. Therefore we have:

$$\oint_{C_1 - C_2} f(z')(z' - z_0)^n dz' = 0 \tag{31}$$

Therefore, we can say that:

$$\oint_{C_2} f(z')(z' - z_0)^n dz' = \oint_{C_1} f(z')(z' - z_0)^n dz' \tag{32}$$

Using this, we can rewrite I_2 as:

$$\begin{aligned} I_2 &= -\sum_{n=0}^{\infty}\left(\frac{1}{2\pi i}\oint_{C_1} f(z')(z'-z_0)^n dz'\right)\frac{1}{(z-z_0)^{n+1}} \\ &= -\sum_{n=1}^{\infty}\left(\frac{1}{2\pi i}\oint_{C_1} f(z')(z'-z_0)^{n-1} dz'\right)\frac{1}{(z-z_0)^{n}} \end{aligned} \tag{33}$$

Similarly, we perform the following algebraic manipulations on I_1:

$$\begin{aligned} I_1 &= \frac{1}{2\pi i}\frac{f(z')}{z'-z}dz' \\ &= \frac{1}{2\pi i}\frac{f(z')}{(z'-z_0)\left(1-\dfrac{z-z_0}{z'-z_0}\right)}dz' \end{aligned} \tag{34}$$

Using the fact that z lies in between C_1 and C_2, z' lies on C_1 and z_0 lies at the center of C_2 and C_1, we have:

$$\left|\frac{z-z_0}{z'-z_0}\right| < 1$$

Therefore, we observe that $1/\left(1-\dfrac{z-z_0}{z'-z_0}\right)$ can be rewritten as the infinite sum of a geometric series. Therefore we can rewrite I_1 as:

$$I_1 = \sum_{n=0}^{\infty}\left(\frac{1}{2\pi i}\oint_{C_1}\frac{f(z')}{(z'-z_0)^{n+1}}dz'\right)(z-z_0)^n \tag{35}$$

Now, we can write 27 as:

$$\begin{aligned} f(z) &= \frac{1}{2\pi i}\oint_{C_1-C_2}\frac{f(z')}{z'-z}dz' \\ &= \frac{1}{2\pi i}\oint_{C_1}\frac{f(z')}{z'-z}dz' - \frac{1}{2\pi i}\oint_{C_2}\frac{f(z')}{z'-z}dz' \\ &= I_1 - I_2 \end{aligned} \tag{36}$$

Using the expressions for I_1 and I_2 derived in 35 and 33 respectively, we can write:

$$\begin{aligned} f(z) &= I_1 - I_2 \\ &= \sum_{n=0}^{\infty}\left(\frac{1}{2\pi i}\oint_{C_1}\frac{f(z')}{(z'-z_0)^{n+1}}dz'\right)(z-z_0)^n \\ &+ \sum_{n=1}^{\infty}\left(\frac{1}{2\pi i}\oint_{C_1} f(z')(z'-z_0)^{n-1} dz'\right)\frac{1}{(z-z_0)^{n}} \end{aligned} \tag{37}$$

Comparing the above expression with the definition of the Laurent Series given in 25, we get:

$$a_n = \frac{1}{2\pi i}\oint_{C}\frac{f(z')}{(z'-z_0)^{n+1}}dz' \tag{38}$$

$$b_n = \frac{1}{2\pi i}\oint_{C}(z'-z_0)^{n-1}f(z')dz' \tag{39}$$

Therefore we get the final form of the Laurent Series as:

$$
\begin{aligned}
f(z) &= \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n} \\
&= \sum_{n=0}^{\infty} \left(\frac{1}{2\pi i} \oint_C \frac{f(z')}{(z'-z_0)^{n+1}} dz' \right) (z-z_0)^n \\
&+ \sum_{n=1}^{\infty} \left(\frac{1}{2\pi i} \oint_C (z'-z_0)^{n-1} f(z') dz' \right) \frac{1}{(z-z_0)^n}
\end{aligned}
\tag{40}
$$

4 Residue Theorem

4.1 Singularities and Zeroes

Singularities

We say that a function $f(Z)$ **is singular** or **has a singularity** at a point $z = z_0$ if $f(z)$ is not analytic (or not even defined) at $z = z_0$, but every neighborhood of $z = z_0$ contains points at which $f(z)$ is analytic. We can also say that $z = z_0$ is a **singular point** of $f(z)$.

We call $z = z_0$ an isolated singularity of $f(Z)$ if $z = z_0$ has a neighborhood without further singularities of $f(Z)$. Isolated singularities of $f(Z)$ at $z = z_0$ can be classified by the Laurent series (as given in Eq. 40) valid **in the immediate neighborhood** of the singular point $z = z_0$ except at z_0 itself, that is, in a region of the form

$$0 < |z - z_0| < R$$

The sum of the first series is analytic at $z = z_0$ as we know from the Sec. 3. The second series, containing the negative powers, is called the *principal part* of Eq. 40. If the second part has only finitely many terms, it is of the form

$$\frac{b_1}{(z-z_0)} + \cdots + \frac{b_m}{(z-z_0)^m}, \qquad (b_m \neq 0) \tag{41}$$

Then the singularity of $f(Z)$ at $z = z_0$ is called a **pole**, and m is called its **order**. Poles of the first order are also known as **simple poles**. If the principal part of Eq. 40 has infinitely many terms, we say that $f(Z)$ has at $z = z_0$ an **isolated essential singularity**.

The classification of singularities into poles and essential singularities is not merely a formal matter, because the behavior of an analytic function in a neighborhood of an essential singularity is entirely different from that in the neighborhood of a pole.

Formally, a **Pole** is defined as

> *If $f(z)$ is analytic and has a pole at $z = z_0$ then $|f(z)| \to \infty$ as $z \to z_0$ in any manner.*

Zeroes of an Analytic Function

A **zero** of an analytic function $f(Z)$ in a domain D is a $z = z_0$ in D such that $f(z_0) = 0$. A zero has **order** n if not only f but also the derivatives $f', f'', \cdots, f^{(n-1)}$ are all 0 at $z = z_0$ but $f(z_0) \neq 0$.

4.2 Residual Integration

Recall that complex integrals can be directly solved by using Cauchy's integral formula (Sec. 2). The purpose of Cauchy's residue integration method is the evaluation of integrals

$$\oint_C f(z)dz$$

taken around a simple closed path C. The idea is as follows.

If $f(z)$ is analytic everywhere on C and inside C, such an integral is *zero* by Cauchy's integral theorem (Sec. 2), and we are done. The situation changes if $f(z)$ has a **singularity** at a point $z = z_0$ inside C but is otherwise analytic on C and inside C as before. Then $f(z)$ has a Laurent series

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \frac{b_1}{(z - z_0)} + \frac{b_2}{(z - z_0)^2} + \cdots$$

that converges for all points near $z = z_0$ (except at $z = z_0$ itself), in some domain of the form $0 < |z - z_0| < R$. Now, the coefficient of the first negative power $1/(z - z_0)$ of this Laurent series is given by the integral formula in Eq. 39 in Sec. 3 with $n = 1$,

$$b_1 = \frac{1}{2\pi i} \oint_C f(z)dz$$

So, if we can find b_1 by a suitable methods and then use the above formula for for evaluating the integral, that is,

$$\oint_C f(z)dz = 2\pi i b_1 \tag{42}$$

Here we integrate counterclockwise around a simple closed path C that contains $z = z_0$ in its interior (but no other singular points of $f(z)$ on or inside C).

The coefficient b_1 is called the **residue** of $f(z)$ at $z = z_0$ and we denote it by

$$b_1 = \operatorname*{Res}_{z=z_0} f(z) \tag{43}$$

Proof:

We want to prove the relation in Eq. 42. Let's consider the Laurent Series as given in Eq. 40 as our starting point.

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n}$$

In the above representation, we have the Laurent series written as sum of two series, one in terms of a_n and the other in terms of b_n. Calling them respective **Part A** and **Part B**, we will now perform term-by-term integration on both parts.

Starting with **Part A**, let $f(z) = a_n(z - z_0)^n$, where we define $(z - z_0) = re^{i\theta}$, as a circle C of radius r centered at z_0. Now, we integrate counterclockwise around this simple closed path C that contains $z = z_0$ in its interior as follows.

$$\begin{aligned}\oint_C f(z)dz = \oint_C a_n (z - z_0)^n dz &= \int_0^{2\pi} a_n r e^{in\theta} r e^{i\theta} i d\theta \\ &= \int_0^{2\pi} a_n i r^2 e^{i(n+1)\theta} d\theta \\ &= a_n i r^2 \int_0^{2\pi} e^{i(n+1)\theta} d\theta \\ &= a_n r^2 \frac{[e^{i(n+1)2\pi} - e^{i(n+1)0}]}{(n+1)} = 0\end{aligned}$$

Here, we are using the fact that $e^{i(n+1)2\pi} = \cos(2(n+1)\pi) + i\sin(2(n+1)\pi) = 1$ and $e^{i(n+1)0} = \cos((n+1)0) + i\sin((n+1)0) = 1$. So, we can write it as

$$\oint_C f(z)dz = \oint_C a_n(z-z_0)^n dz = 0 \qquad \forall n \in \mathbb{N} \cup \{0\} \tag{44}$$

Next, repeating the same fot **Part B**, let $f(z) = b_n/(z-z_0)^n$, where we define $(z-z_0) = re^{i\theta}$, as a circle C of radius r centered at z_0. Now, we integrate counterclockwise around this simple closed path C that contains $z = z_0$ in its interior as follows.

$$\begin{aligned}\oint_C f(z)dz = \oint_C \frac{b_n}{(z-z_0)^n}dz &= \int_0^{2\pi} \frac{b_n}{r}e^{-in\theta}re^{i\theta}id\theta \\ &= \int_0^{2\pi} b_n ie^{-i(n-1)\theta}d\theta\end{aligned}$$

Here, the value of above integral is determined by the value taken by $n \in \mathbb{N}$. We have two distinguishing cases with $n = 1$ and $n \neq 1$. So, we have

$$\int_0^{2\pi} b_n ie^{-i(n-1)\theta}d\theta = \left\{ \begin{array}{ll} \int_0^{2\pi} b_1 id\theta, & \text{for } \; n = 1 \\ \int_0^{2\pi} b_n ie^{-i(n-1)\theta}d\theta, & \text{for } \; n \neq 1 \end{array} \right\}$$

For $n \neq 1$, similar to **Part A**, using the fact that $e^{-i(n-1)2\pi} = \cos(2(n-1)\pi) - i\sin(2(n-1)\pi) = 1$ and $e^{-i(n-1)0} = \cos((n-1)0) - i\sin((n-1)0) = 1$, we get

$$\oint_C f(z)dz = \oint_C \frac{b_n}{(z-z_0)^n}dz = 0 \qquad \forall n \in \mathbb{N}\,,\; n \neq 1 \tag{45}$$

And for $n = 1$, by simple integration, we get

$$\oint_C f(z)dz = \oint_C \frac{b_n}{(z-z_0)^n}dz = 2\pi i b_1 \tag{46}$$

where b_1 is called the **residue** of $f(z)$ at $z = z_0$.

4.3 Formulas for Residues

To calculate a residue at a pole, we need not produce a whole Laurent series, but we can derive formulas for residues as follows.

Simple Pole at z_0: First formula for the residue at a simple pole is

$$\operatorname*{Res}_{z=z_0} f(z) = b_1 = \lim_{z\to z_0}(z-z_0)f(z) \tag{47}$$

The second formula for the residue at a simple pole is

$$\operatorname*{Res}_{z=z_0} f(z) = \operatorname*{Res}_{z=z_0} \frac{p(z)}{q(z)} = \frac{p(z_0)}{q'(z_0)} \tag{48}$$

In Eq. 48 we assume that $f(z) = p(z)/q(z)$ with $p(z_0) \neq 0$ and $q(z)$ has a simple zero at z_0, so that $f(z)$ has a simple pole at z_0.

4.4 Several Singularities Inside the Contour (Residue Theorem)

Residue integration can be extended from the case of a single singularity to the case of several singularities within the contour C. This is the purpose of the residue theorem.

Residue Theorem:

Let $f(z)$ be analytic inside a simple closed path C and on C, except for finitely many singular points $z_1, z_2, \cdots, z_k$ inside C. Then the integral of $f(z)$ taken counterclockwise around C equals $2\pi i$ times the sum of the residues of $f(z)$ at $z_1, z_2, \cdots, z_k$

$$\oint_C f(z)dz = 2\pi i \sum_{j=1}^{k} \operatorname*{Res}_{z=z_j} f(z) \tag{49}$$

Proof:

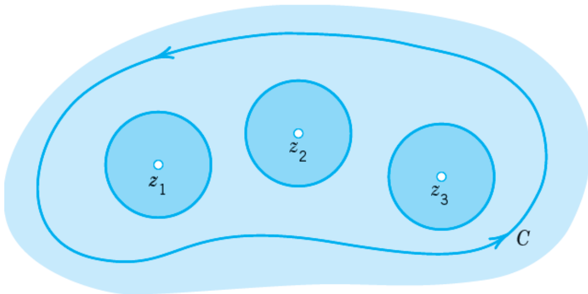

Figure 5: Residue Theorem

We enclose each of the singular points z_j in a circle C_j with radius small enough that those k circles and C are all separated. This is shown in Fig. 5 where $k = 3$ is assumed. Then $f(z)$ is analytic in the multiply connected domain D bounded by C and $C_1, C_2, \cdots, C_k$ and on the entire boundary of D. From Cauchy's integral theorem we thus have

$$\oint_C f(z)dz + \oint_{C_1} f(z)dz + \oint_{C_2} f(z)dz + \cdots + \oint_{C_k} f(z)dz = 0$$

the integral along C being taken *counterclockwise* and the other integrals *clockwise*. We take the integrals over $C_1, C_2, \cdots, C_k$ to the right and compensate the resulting minus sign by reversing the sense of integration. Thus, we have

$$\oint_C f(z)dz = \oint_{C_1} f(z)dz + \oint_{C_2} f(z)dz + \cdots + \oint_{C_k} f(z)dz \tag{50}$$

where all the integrals are now taken counterclockwise. By using Eqs. 42 and 43, we get

$$\oint_{C_j} f(z)dz = 2\pi i \operatorname*{Res}_{z=z_j} f(z) \quad \text{where } j = 1, 2, \cdots, k \tag{51}$$

So, when Eq. 51 is substituted in the RHS of Eq. 50, we obtain the result stated in the residue theorem (Eq. 49).

5 References

1. Kreyszig, E., Kreyszig, H.,, Norminton, E. J. (2011). *Advanced Engineering Mathematics*. Hoboken, NJ: Wiley. ISBN: 0470458364