Chapter 5

Weistrass Theorem

The Weistress approximation theorem states that any continuous function defined on a closed interval can be uniformly approximated by polynomial functions to any degree of accuracy.

Weierstrass Theorem (Bernstein polynomial)

$$\exists\ P_n(x) \text{ such that } |f(x) - P_n(x)| < \epsilon$$

$$f(x),\ \ x \in [0,1]$$

$$|f(x) - f(x_0)| < \epsilon$$

$$x \in [x_0 - \delta, x_0 + \delta] \quad \text{closed interval (can go to } \infty \text{ at endpoints)}$$

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$$

$$(a + (1-x))^n = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} = 1$$

Bernstein polynomial terms: $B_k(x)$

$$B_n(f,x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) b_k(x)$$

$$\Rightarrow B_n(f) \to f$$

$$|B_n(f,x) - f(x)| = \left| \sum_{k=0}^{n} \left(f\left(\frac{k}{n}\right) - f(x) \right) b_k(x) \right|$$

Trying to find upper bound for this

$$|x - \frac{k}{n}| \leq \delta$$

$$f(x) \leq M \quad \text{bounded and continuous in interval taken}$$

$$|f(x) - f(x_0)| \leq 2M$$

Further Analysis on Bernstein Polynomials

$$\left|\sum_{k=0}^{n}\left(f\left(\frac{k}{n}\right)-f(x)\right)b_k(x)\right| = \sum_{k=0}^{n}\left|f\left(\frac{k}{n}\right)-f(x)\right|b_k(x)$$

$$|B_n(f,x)-f(x)| \le \sum_{|x-\frac{k}{n}|\le\delta}\left|f\left(\frac{k}{n}\right)-f(x)\right|b_k(x) + \sum_{|x-\frac{k}{n}|>\delta}\left|f\left(\frac{k}{n}\right)-f(x)\right|b_k(x)$$

$$\le \sum_{|x-\frac{k}{n}|\le\delta}\frac{\epsilon}{2}b_k(x) + \sum_{|x-\frac{k}{n}|>\delta}\frac{2M}{n\delta^2}b_k(x)$$

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Using bounds and conditions, we proceed as follows:

$$= \sum_{|x-\frac{k}{n}|\le\delta}\frac{\epsilon}{2}b_k(x) + \frac{2M}{n\delta^2}\sum_{|x-\frac{k}{n}|>\delta}b_k(x)$$

Expanding on this:

$$= \frac{\epsilon}{2}\sum b_k(x) + \frac{2M}{n\delta^2}\sum\left(\frac{k}{n}-x\right)^2 b_k(x)$$

Define parameters for probability and expectation:

$$p = nx, \quad \sigma = \sqrt{nx(1-x)}, \quad E[(k-\mu)^2] = \sigma^2 = E[k^2]-E[k]^2$$

$$E[k] = \sum k\cdot p(k), \quad \sigma^2 = E[(k-\mu)^2]$$

Where:

$$E[(k-\mu)^2] = E[k^2]-E[k]^2 \quad \text{with} \quad \mu = nx$$

$$B_n(f,x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k}x^k(1-x)^{n-k}$$

$$|B_n(f,x)-f(x)| = \left|\sum_{k=0}^{n}\left(f\left(\frac{k}{n}\right)-f(x)\right)\binom{n}{k}x^k(1-x)^{n-k}\right|$$

$$\le \sum\left|f\left(\frac{k}{n}\right)-f(x)\right|\binom{n}{k}x^k(1-x)^{n-k}$$

$$= \sum_{|x-\frac{k}{n}|\le\delta}\left|f\left(\frac{k}{n}\right)-f(x)\right|\binom{n}{k}x^k(1-x)^{n-k} + \sum_{|x-\frac{k}{n}|>\delta}\left|f\left(\frac{k}{n}\right)-f(x)\right|\binom{n}{k}x^k(1-x)^{n-k}$$

Define these two regions as:

$$A = \sum_{|x-\frac{k}{n}|\leq\delta} \left| f\left(\frac{k}{n}\right) - f(x) \right| \binom{n}{k} x^k (1-x)^{n-k} \leq \frac{\epsilon}{2}$$

$$B = \sum_{|x-\frac{k}{n}|>\delta} \left| f\left(\frac{k}{n}\right) - f(x) \right| \binom{n}{k} x^k (1-x)^{n-k}$$

For B:

$$\leq \sum_{k\in B} \frac{2M}{n^2\delta^2} \binom{n}{k} x^k (1-x)^{n-k}$$

Using the probability statistics:

$$\leq \frac{\epsilon}{2} + \frac{2M}{n\delta^2}$$

We adjust so that:

$$\frac{M}{2n\delta} = \frac{\epsilon}{2} \Rightarrow \epsilon$$

Thus:

$$|B_n(f,x) - f(x)| \leq \epsilon$$

Hence, this justifies that interpolation is valid.

$$\int_0^1 h(t)\,dt$$

And constraints:

$$\mathbf{D}\mathbf{x} = A$$

$$\int_0^T h(t)\,dt \approx \int_0^T \sum_{k=0}^n h(t_k) L_k(t)\,dt = \sum_{k=0}^n h(t_k) \underbrace{\int_0^T L_k(t)\,dt}_{w_k}$$

$$h \in P_n$$

can this also be reversed order to make it exact, not approximating the function.

we're approximating the integral to a higher

5.1 Bernstein Polynomials and Their Applications

Definition: Bernstein polynomials are a particular set of polynomials that are used to approximate continuous functions on a closed interval $[0, 1]$. They are fundamental in approximation theory and are used to prove the Weierstrass approximation theorem, which states that any continuous function can be uniformly approximated by polynomials.

5.1.1 Mathematical Definition

Given a function $f(x)$ defined on the interval $[0,1]$, the Bernstein polynomial $B_n(f,x)$ of degree n is defined as:

$$B_n(f,x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k}x^k(1-x)^{n-k},$$

where $\binom{n}{k}$ is the binomial coefficient.

5.1.2 Properties

- **Pointwise Convergence**: The sequence $B_n(f,x)$ converges to $f(x)$ as $n \to \infty$ for any continuous function $f(x)$ on $[0,1]$.
- **Preservation of Monotonicity**: If $f(x)$ is a monotonically increasing or decreasing function, then so is $B_n(f,x)$.
- **Preservation of Convexity**: If $f(x)$ is convex, then $B_n(f,x)$ is also convex for sufficiently large n.

5.1.3 Weierstrass Approximation Theorem

The Weierstrass approximation theorem states that for any continuous function $f(x)$ defined on a closed interval $[0,1]$ and for any $\epsilon > 0$, there exists a polynomial $P_n(x)$ such that:

$$|f(x) - P_n(x)| < \epsilon \quad \text{for all } x \in [0,1].$$

Using Bernstein polynomials, this theorem can be demonstrated by showing that:

$$\lim_{n\to\infty} B_n(f,x) = f(x).$$

5.1.4 Derivation and Analysis

To show the convergence:

$$|B_n(f,x) - f(x)| = \left|\sum_{k=0}^{n}\left(f\left(\frac{k}{n}\right) - f(x)\right)\binom{n}{k}x^k(1-x)^{n-k}\right|.$$

For small $\delta > 0$, we can split the summation into two parts:

$$\sum_{|x-\frac{k}{n}|\le\delta}\left|f\left(\frac{k}{n}\right) - f(x)\right|\binom{n}{k}x^k(1-x)^{n-k} + \sum_{|x-\frac{k}{n}|>\delta}\left|f\left(\frac{k}{n}\right) - f(x)\right|\binom{n}{k}x^k(1-x)^{n-k}.$$

Upper Bound Analysis:

$$|B_n(f,x) - f(x)| \le \sum_{|x-\frac{k}{n}|\le\delta}\frac{\epsilon}{2}\binom{n}{k}x^k(1-x)^{n-k} + \sum_{|x-\frac{k}{n}|>\delta}\frac{2M}{n\delta^2}\binom{n}{k}x^k(1-x)^{n-k}.$$

5.1.5 Example: Bernstein Polynomials for $f(x) = x^2$

For $f(x) = x^2$, the Bernstein polynomial of degree n is:

$$B_n(x^2, x) = \sum_{k=0}^{n} \left(\frac{k}{n}\right)^2 \binom{n}{k} x^k (1-x)^{n-k}.$$

5.2 Applications of Bernstein Polynomials

- **Approximation Theory**: Bernstein polynomials are used to approximate continuous functions due to their convergence properties.
- **Computer Graphics**: They are employed in Bézier curves, which are essential in vector graphics and computer-aided geometric design.
- **Numerical Analysis**: Bernstein polynomials help in numerical integration and solving differential equations.