AS5580 - Optional Assignment

Abhigyan Roy AE21B002
Agni Ravi Deepa AE21B101

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Question 1

Problem 1.1.3 Show that the fundamental polynomial $L_i(x)$ can be expressed in the form

$$L_i(x) = \frac{w(x)}{(x - x_i)w'(x_i)},$$

where $w(x) = (x - x_0)(x - x_1)\cdots(x - x_n)$. By differentiating the above expression for $L_i(x)$ and using L'Hospital's rule, show further that

$$L_i'(x_i) = \frac{1}{2}\frac{w''(x_i)}{w'(x_i)}.$$

$$L_j(x) = \prod_{\substack{k=0 \\ k \neq j}}^{n} \frac{(x - x_k)}{(x_j - x_k)}$$

$$L_j(x) = \frac{(x - x_1)}{(x_j - x_1)} \times \prod_{\substack{k=0 \\ k \neq j}}^{n} \frac{(x - x_k)}{(x_j - x_k)}$$

$$L_{ij}(x) = \frac{\omega(x)}{(x - x_i)} \times \prod_{\substack{k=0 \\ k \neq i}}^{n} \frac{(x_i - x_k)}{(x_i - x_k)}$$

$$\omega(x) = (x - x_0)(x - x_1)\cdots(x - x_n)$$

$$\text{we know that } \omega(x_i) = 0$$

$$\therefore \omega(x) - \omega(x_i) = (x - x_1)(x - x_2)\cdots(x - x_n)$$

$$\frac{\omega(x) - \omega(x_i)}{x - x_i} = \prod_{\substack{k=0 \\ k \neq i}}^{n} (x - x_k)$$

Taking the limit as $x \to x_i$ on both sides

$$\omega'(x_i) = \prod_{\substack{k=0 \\ k\neq i}}^{n} (x_i - x_k)$$

substituting in the above relation

$$L_{ij}(x) = \frac{\omega(x)}{(x - x_i)\omega'(x_i)}$$

$$L_{ij}(x) = \frac{1}{\omega'(x_i)} \left(\frac{(x - x_i)\omega'(x_i) - \omega(x)}{(x - x_i)^2} \right)$$

Using L'Hopital's rule

$$= \frac{1}{\omega'(x_i)} \left(\frac{(x - x_i)\omega''(x_i) + \omega'(x_i) - \omega'(x)}{2(x - x_i)} \right)$$

$$L_{ij}(x_i) = \frac{\omega''(x_i)}{2\omega'(x_i)}$$

Question 2

Problem 1.1.4 Show that

$$\det \mathbf{V} = \det \mathbf{M},$$

where $\mathbf{V}$ and $\mathbf{M}$ are defined by (1.7) and (1.15), respectively.

$$V = \begin{bmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^n \\ 1 & x_1 & x_1^2 & \cdots & x_1^n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^n \end{bmatrix} \quad \text{Vandermonde matrix}$$

$$M = \begin{bmatrix} \Pi_0(x_0) & 0 & \cdots & 0 \\ \Pi_0(x_1) & \Pi_1(x_1) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \Pi_0(x_n) & \Pi_1(x_n) & \cdots & \Pi_n(x_n) \end{bmatrix} \text{where } \Pi_i(x) = \prod_{\substack{j=0 \\ j \neq i}}^{n} (x - x_j) \quad 1 \leq i \leq n$$

$$\det(V) = \det(V[x_0, x_1, \ldots, x_n]) = \prod_{j=0}^{n-1} \prod_{\substack{i=0 \\ i \neq j}}^{n} (x_n - x_j)$$

$$= \det(V[x_0, x_1, \ldots, x_{n-1}]) \prod_{\substack{j=0 \\ j \neq i}}^{n} (x_n - x_j)$$

$$= 1 \cdot (x_1 - x_0) \prod_{i=1}^{n-1} (x_{n-1} - x_j) \prod_{i=0}^{n-1} (x_n - x_i)$$

$$= \Pi(x_0)\Pi(x_1) \cdots \Pi(x_n)$$

$$= \det(M)$$

Question 3

Problem 1.5.2 Show that for any fixed integer r such that $0 \leq r \leq k$,

$$\prod_{j \neq r}([r] - [j]) = (-1)^{k-r} q^{r(2k-r-1)/2} [r]![k-r]!,$$

where the product is taken over all j from 0 to k, but excluding r. Hint: Split the product into two factors, one corresponding to the values of j such that $0 \leq j < r$, and the other to the values of j such that $r < j \leq k$.

Step 1: Understanding the Expression

The notation $[r]$ typically refers to the q-analog of an integer r, defined as:

$$[r] = \frac{1 - q^r}{1 - q}. \tag{1}$$

Thus, the product on the left-hand side can be expanded as:

$$\prod_{j \neq r} ([r] - [j]) = \prod_{j \neq r} \left(\frac{1 - q^r}{1 - q} - \frac{1 - q^j}{1 - q} \right).$$

This simplifies to:

$$\prod_{j \neq r} ([r] - [j]) = \prod_{j \neq r} \frac{q^j - q^r}{1 - q}$$

Step 2: Simplifying the Product

$$\prod_{j \neq r} \frac{(q^j - q^r)}{1 - q} = \prod_{j \neq r} q^j \frac{(1 - q^{r-j})}{1 - q}$$

Thus, using 1 the product becomes:

$$\prod_{j \neq r} q^j [r - j]$$

$$= q^{(0+1+....+r-1)+(r+1+.....+k)} \left([r-0][r-1] \cdots [r-(r-1)]\right) \left([r-(r+1)][r-(r+2)] \cdots [r-k]\right)$$

$$= q^{\left(\sum_{j=0}^{r-1} j\right)+\left(\sum_{j=r+1}^{k} j\right)} \left(\prod_{j=0}^{r-1}[r-j]\right)\left(\prod_{j=r+1}^{k}[r-j]\right)$$

The summation for the power of q by adding and subtracting r would be $q^{\frac{k(k+1)}{2}-r}$,

$$= q^{\frac{k^2+k-2r}{2}} \left(\prod_{j=0}^{r-1}[r-j]\right)\left(\prod_{j=r+1}^{k}[r-j]\right)$$

and the result for the next product is $[r]!$

$$= q^{\frac{k^2+k-2r}{2}}[r]!\left(\prod_{j=r+1}^{k}[r-j]\right) \tag{2}$$

where,

$$\left(\prod_{j=r+1}^{k}[r-j]\right) = ([-1][-2]\cdots[-(k-r)]) = \left(\prod_{j=1}^{k-r}[-j]\right)$$

Here, we need to make a substitution for

$$[-j] = \frac{1-q^-j}{1-q} = -[j]q^{-j}$$

So,

$$\left(\prod_{j=r+1}^{k}[r-j]\right) = \left(\prod_{j=1}^{k-r}-[j]q^{-j}\right)$$

We put this back in product 2

$$= q^{\frac{k^2+k-2r}{2}}[r]!\left(\prod_{j=1}^{k-r}-[j]q^{-j}\right) = q^{\frac{k^2+k-2r}{2}}[r]!\left((-1)^{k-r}[k-r]!q^{-\Sigma_{j=1}^{j=k-r} j}\right)$$

$$= q^{\frac{k^2+k-2r}{2}}[r]!\left((-1)^{k-r}[k-r]!q^{-\frac{(k-r+1)(k-r)}{2}}\right)$$

Simplifying this we get the final answer.

Step 3: Final Form of the Product

$$\prod_{j\neq r}([r]-[j]) = (-1)^{k-r}q^{r(2k-r-1)/2}[r]![k-r]!$$