Fault-Tolerant All-Pairs Mincuts

Undergraduate Project (UGP)

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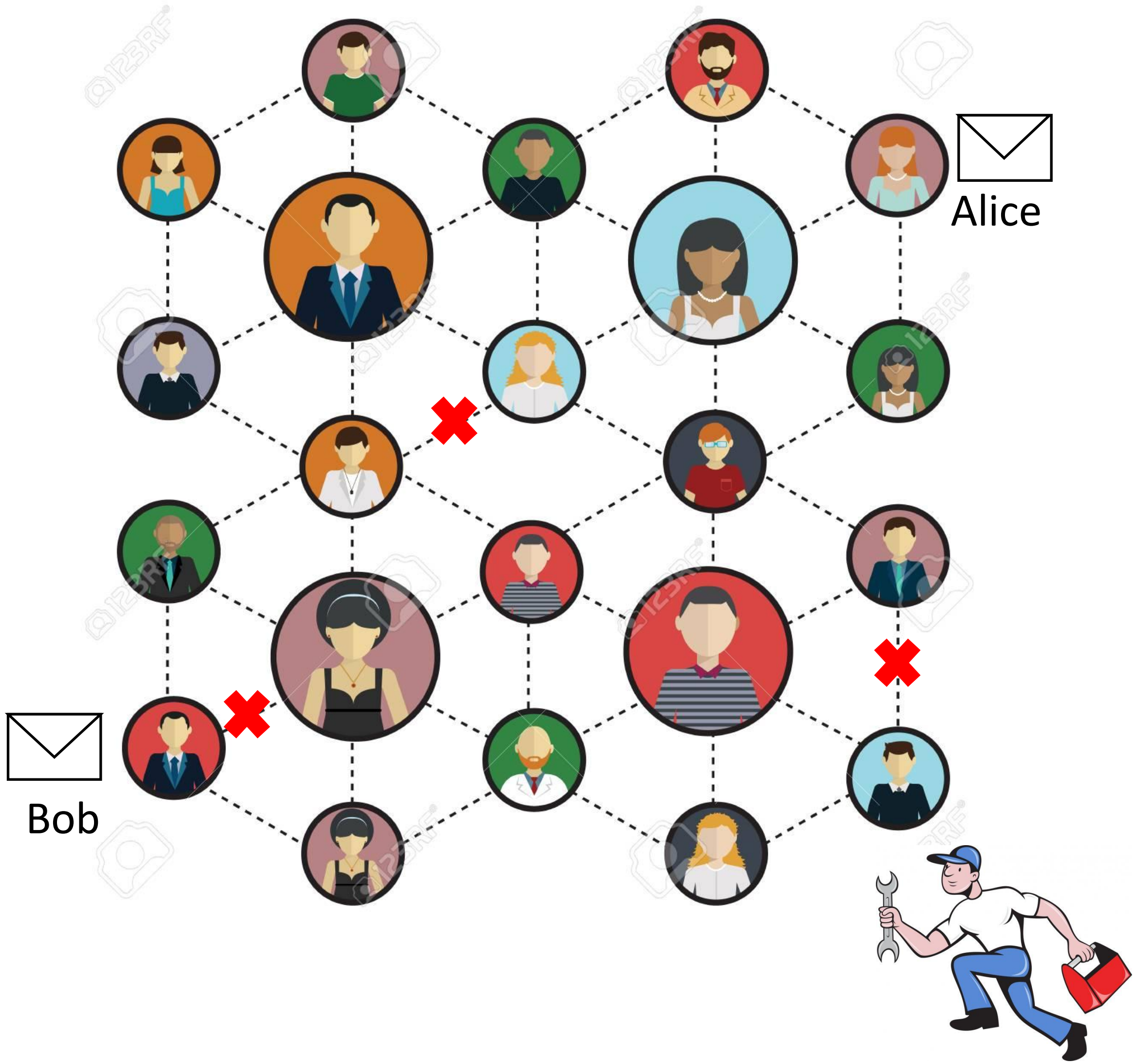
Fault-tolerant Model



- Given k nodes or edge failures. Report a specified query (shortest path, connectivity etc.) in presence of failures.
- Aim
  - Data structure should be very compact.
  - Query time should be small.
  - Often a space-time trade-off.

Some classic fault-tolerance results

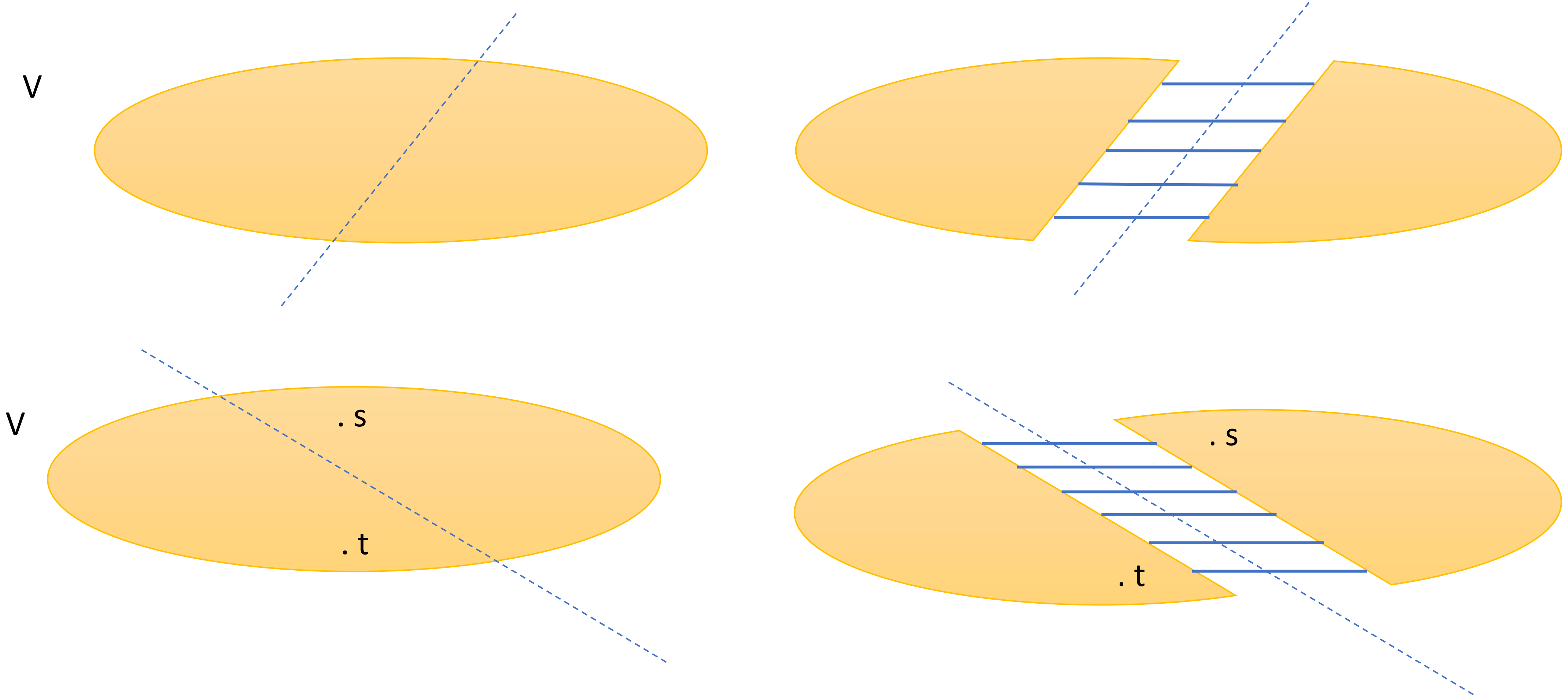
- Shortest Path with single vertex/edge failure (directed weighted graph)
  - Demetrescu et al. [SIAMJ 2008] $O(n^2 \log n)$ space $O(1)$ query time.
  - Bernstein and Karger [STOC 2009] $O(n^2 \log n)$ space $O(1)$ query time.
- Fault Tolerant Spanners for General Graphs
  - Chechik et al. [SIAMJ 2010]
- Our results
- All-pairs Mincuts with single edge failure (undirected unweighted graph)
  - $O(n^2)$ space and $O(1)$ query time.*
  - $O(m)$ space and $O(\min(m, nc_{s,t}))$ query time. *

*Surender Baswana, Abhyuday Pandey
Fault-Tolerant All-Pairs Mincuts. CoRR abs/2011.03291 (2020) submitted to STOC'21

Difficulty of the problem

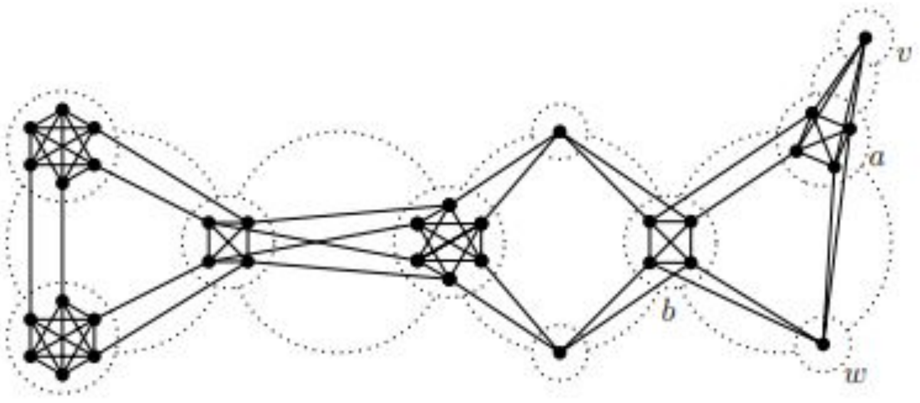
- Consider an undirected unweighted graph G=(V,E)
- **Fact:** Failure of an edge affects the value of Mincut/Maxflow iff the edge lies in some mincut.
- How many Mincuts are possible between a designated source and sink vertex?
  - Exponential [Picard and Quaryenne ‘80]
- And we wish to deal with all possible pair of vertices.

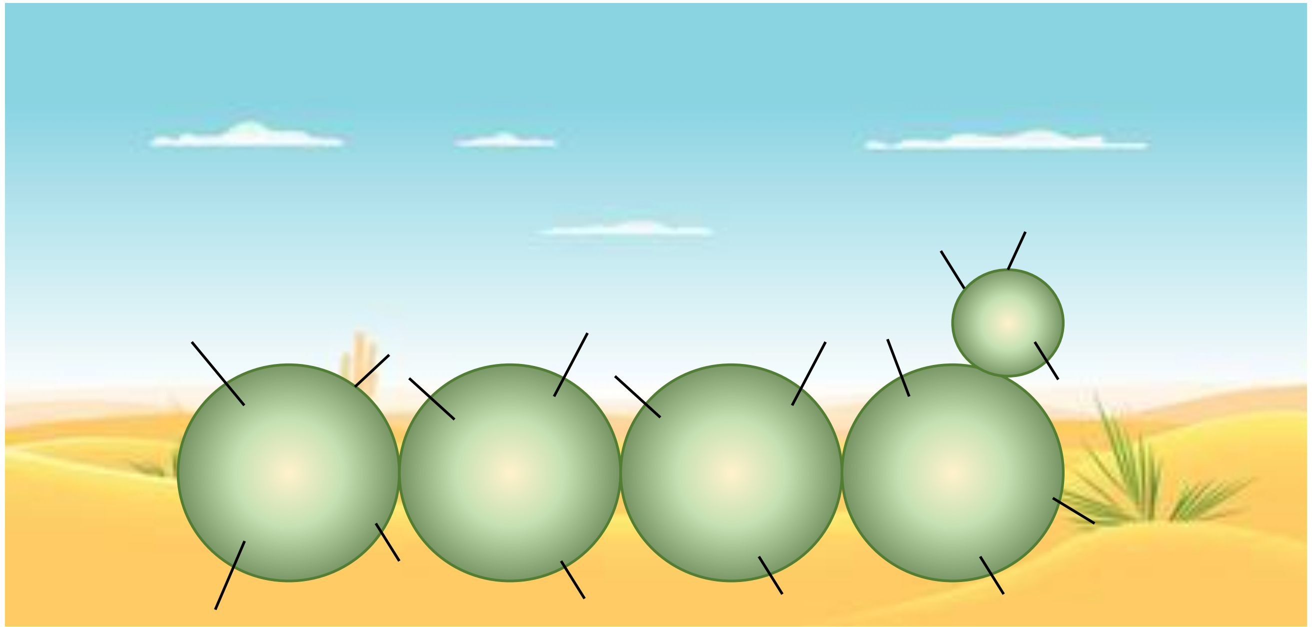
Global Mincuts and (s,t)-Mincuts



Dinitz, Karzanov and Lomonosov [1976]

- All global Mincuts form a nice tree like structure called ‘cactus’.

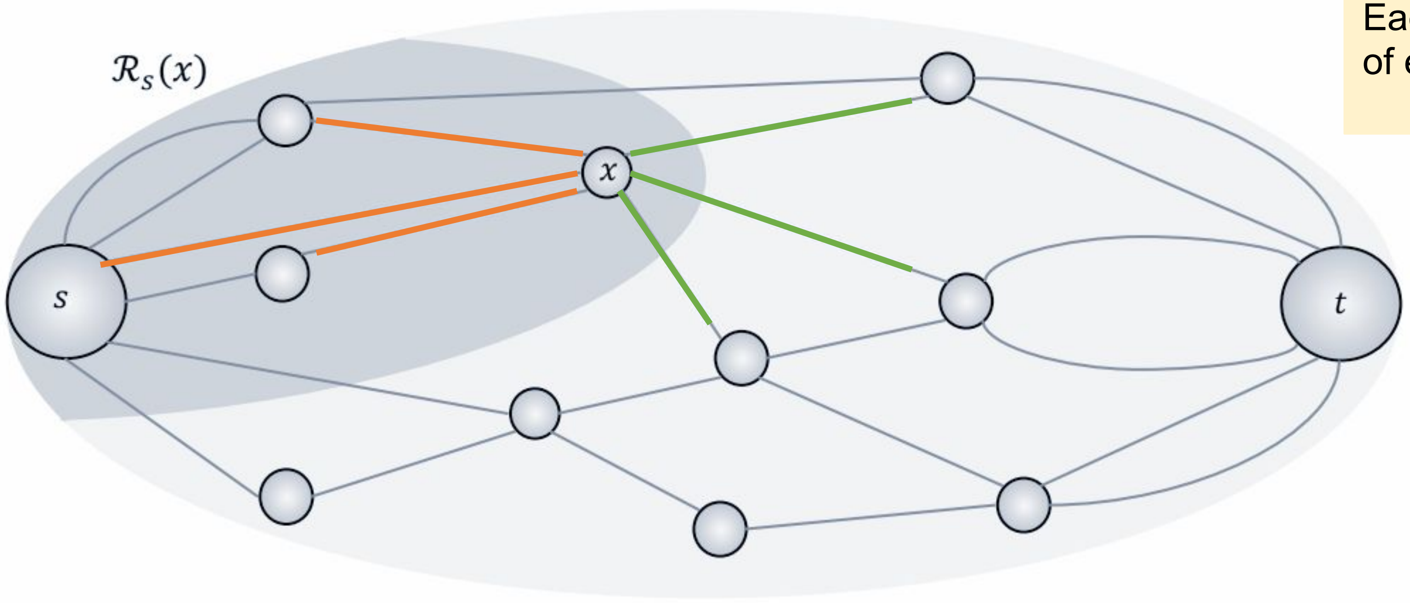




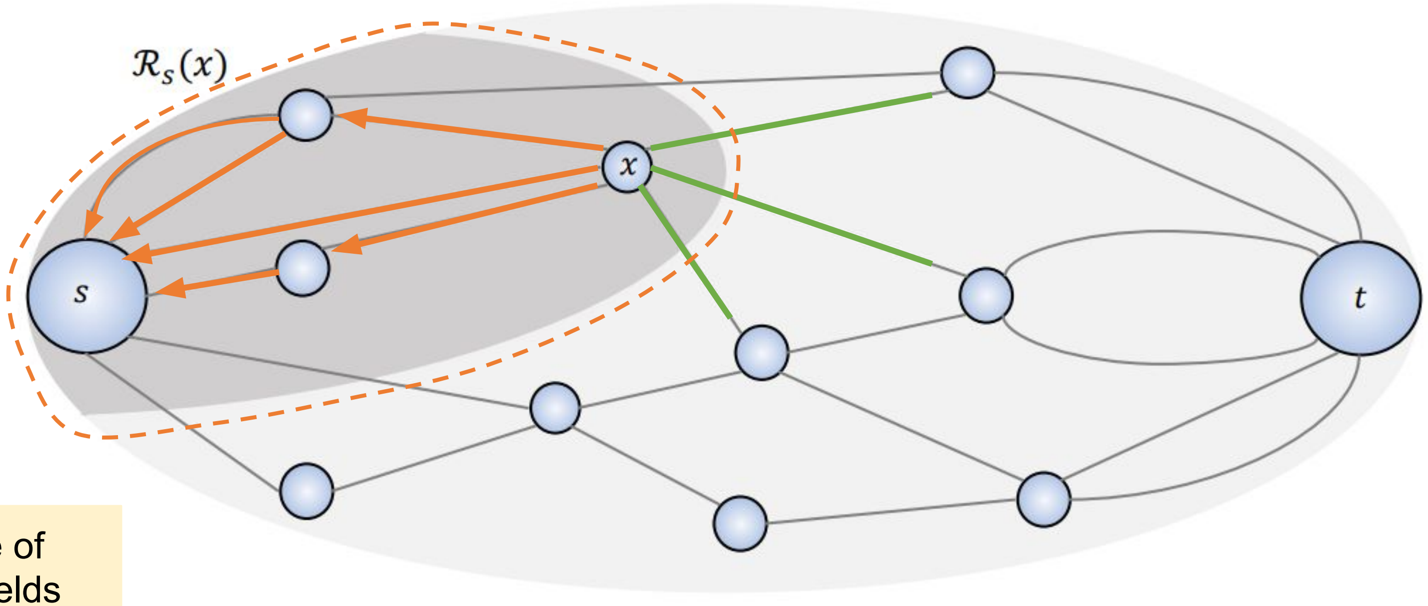
A 2D cactus (two cycles intersect at at most one vertex)

Picard and Quaryenne [1980]
Dinitz and Vainshtein [1994]

- All (s,t)-Mincuts can be stored in a strip (dag-like structure).



Each vertex has a 2 partition of edges incident on it.



"Expanding out" in one of the partition (side-s) yields **reachability cone** in side-s represented as $R_s(x)$

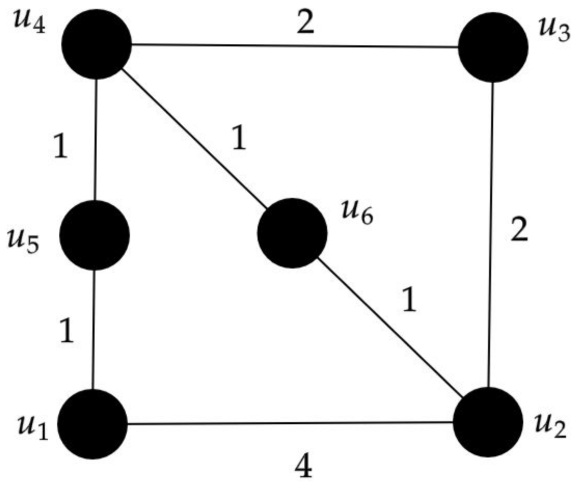
- Each (s,t)-mincut can be represented by union of reachability cones on side-s.
- Any union of reachability cones (on side-s) forms a (s,t)-mincut.

Dinitz and Vainshtein [1994]

- Generalized the cactus and strip.
- Steiner Mincuts
  - Minimum set of edges that divide a set $S \subseteq V$ often called "Steiner Set".
  - All Steiner Mincuts can be stored in O(min(m,nc_S)) space in a data structure called *Connectivity Carcass*.
  - S=V : Steiner Mincuts = Global Mincuts
  - S={s,t} : Steiner Mincuts = (s,t)-Mincuts

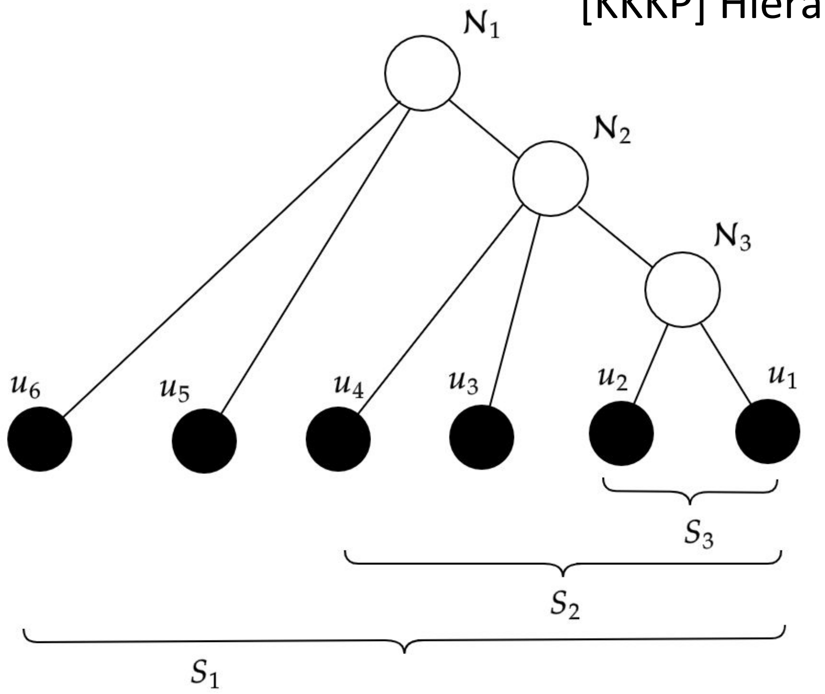
Katz Katz Korman Peleg [2004]

- Hierarchical Structure of all connectivity classes.



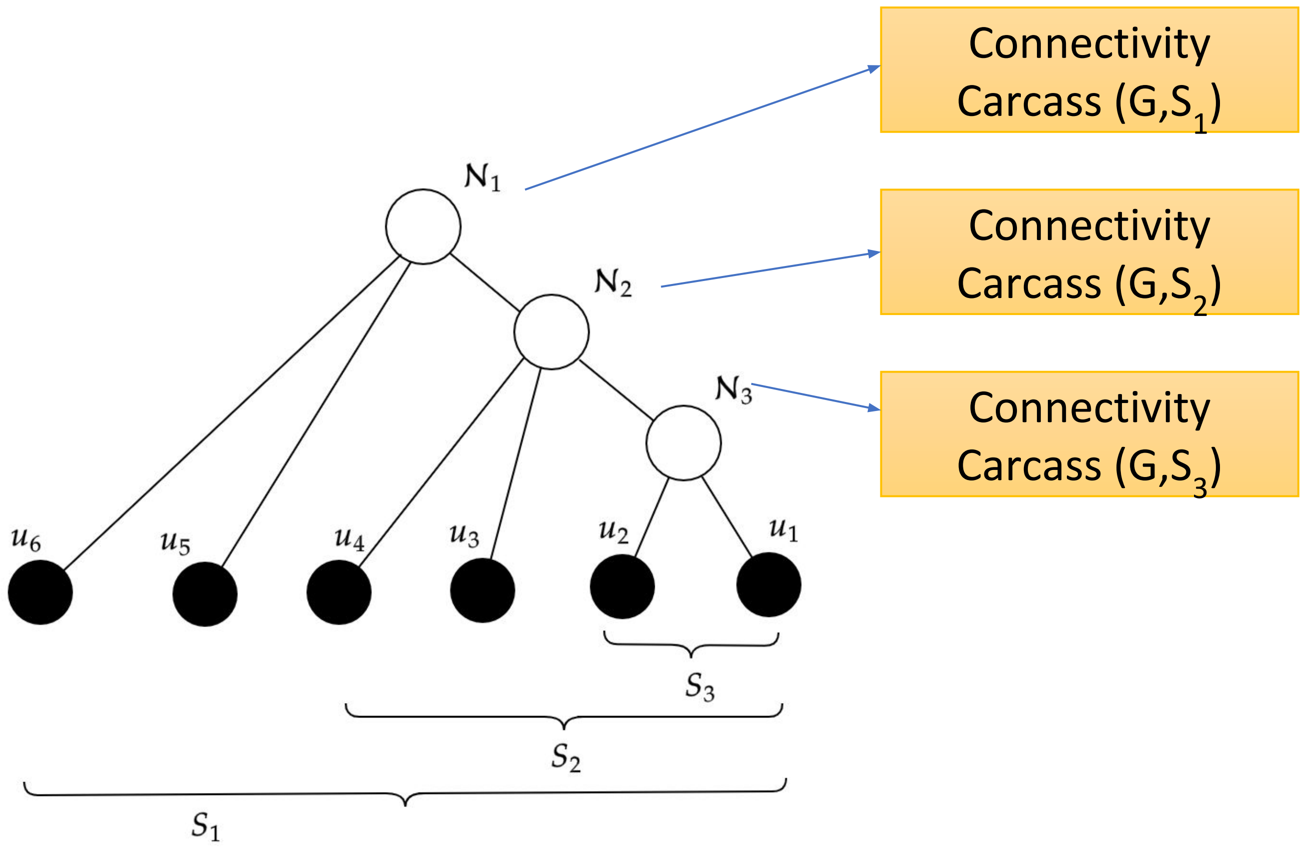
Toy Graph G

[KKKP] Hierarchical Tree



Hierarchical Structure
of vertices

An O(mn) size DS for ft-all-pairs-mincuts

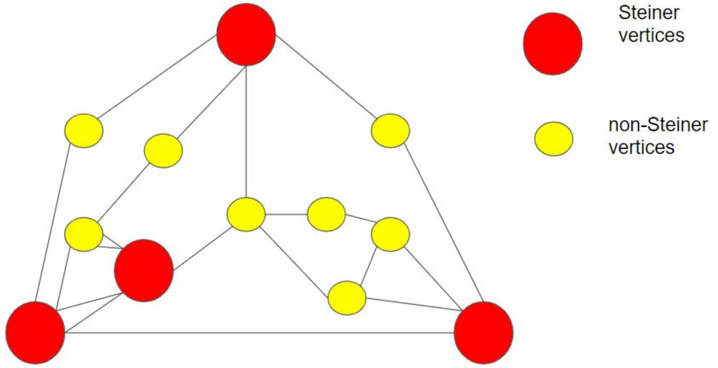


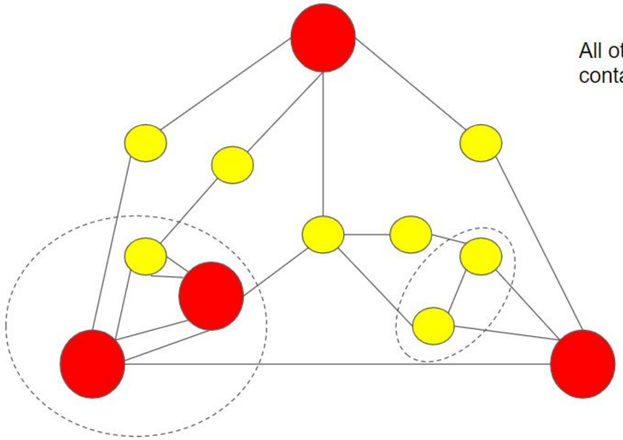
[KKKP] Hierarchical Tree

Total Space Taken
$\leq$ n x m
= O(mn)

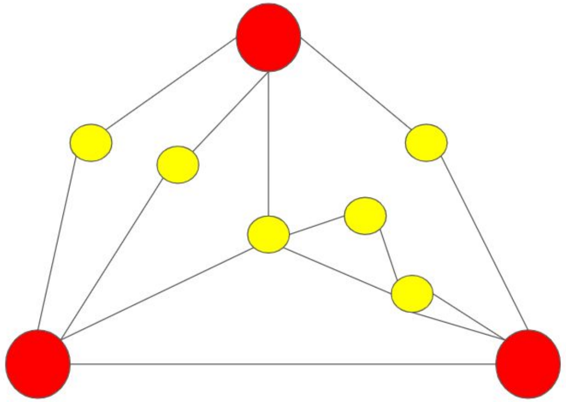
A closer look into the connectivity carcass …

G:=



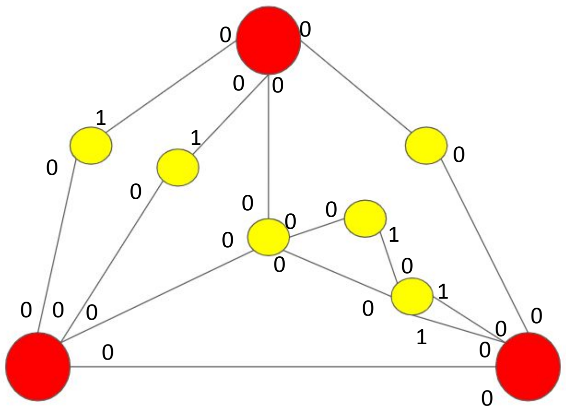


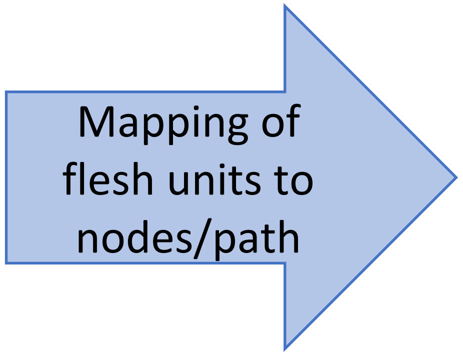
All other equivalent sets contain exactly one vertex.



A closer look …

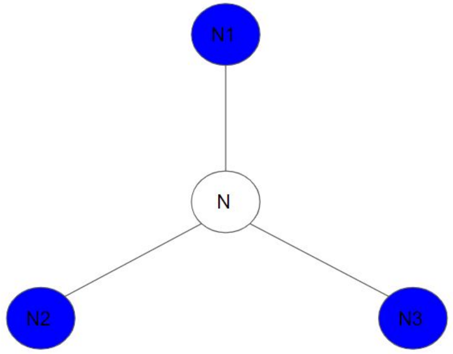
- Flesh graph

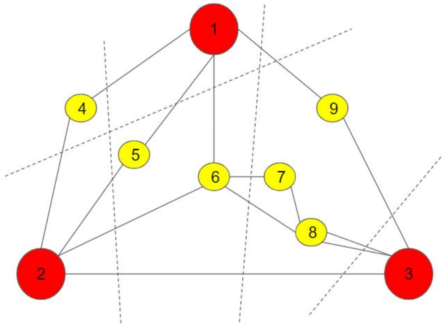




- Skeleton (cactus graph)

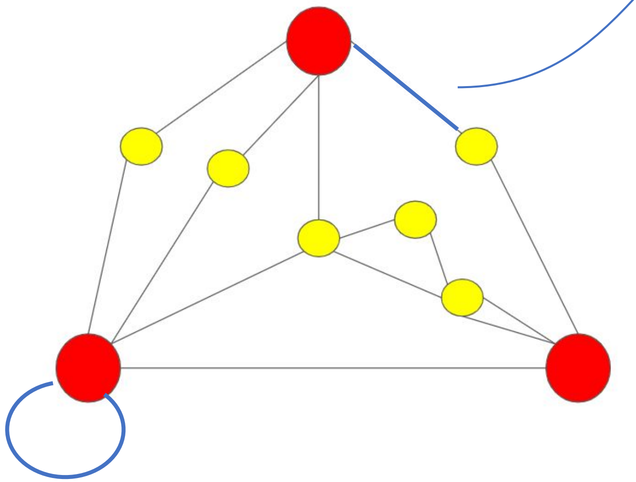
Skeleton :=

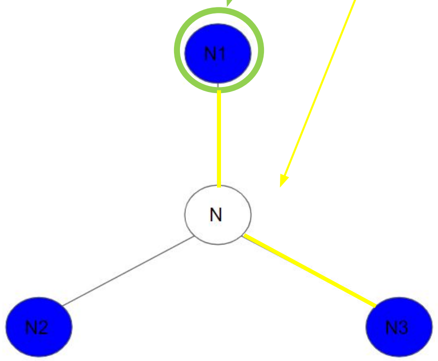




Extending the notion to mapping of an edge

$$P(e) = P(v_1).P(v_2)$$



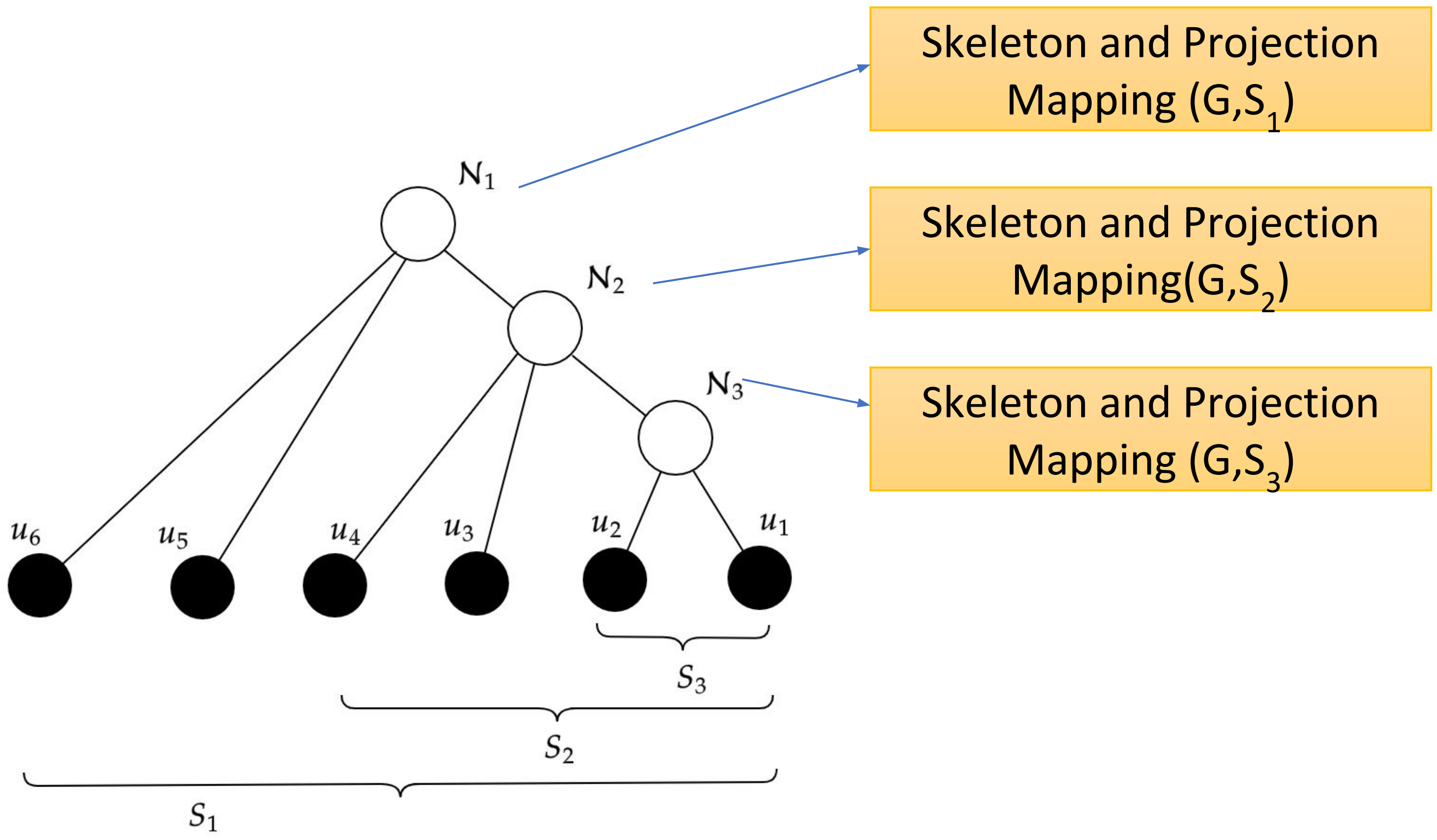


$$P(e) = \emptyset$$

An edge (u,v) lies in a (s,t)-mincut where s and t are separated by steiner mincut iff

$$P(u,v) \cap P(s,t) \neq \emptyset$$

An $O(n^2)$ size DS for ft-all-pairs-Mincuts

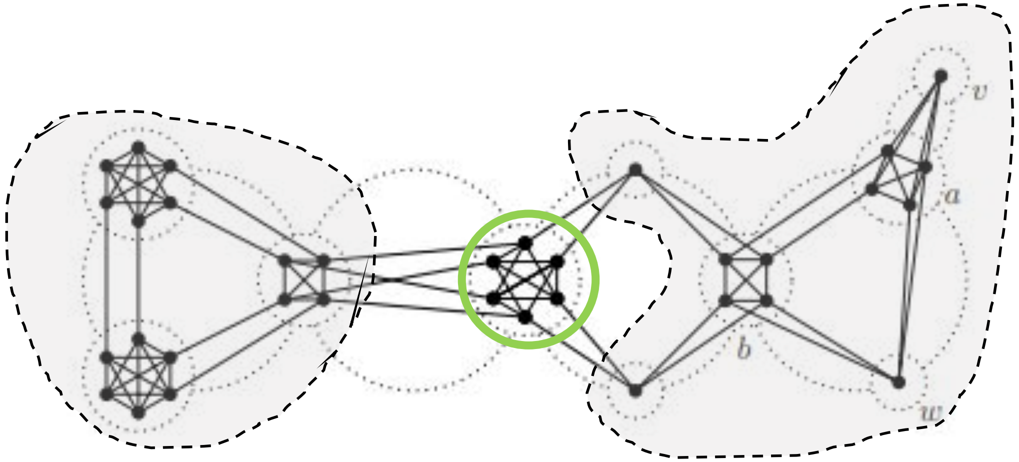


[KKKP] Hierarchical Tree

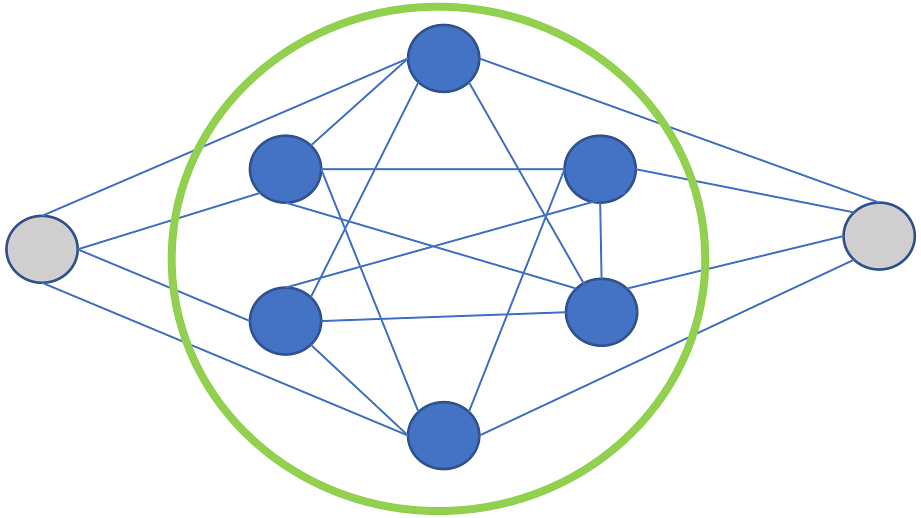
Total Space Taken
$\leq c \times n \times n$
$= O(n^2)$

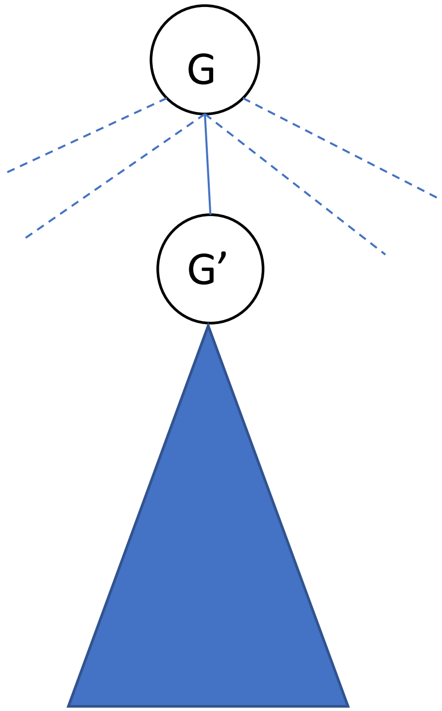
Insights into O(m) size DS

G:=



G’:=





[KKKP] Hierarchical Tree

Working graph can be made smaller with graph contractions.

Central Ideas

- Graph Contraction can be generalized for all levels.
- Information about edges lost in contraction can be retrieved efficiently.
- $O(m)$ space and $O(\min(m, nc_S))$ query time.

Future Work

- Turning the ft-DS into a sensitivity oracle.
  - Handling edge insertions as well in same time bounds.
- Efficient construction of Data Structure.
  - A trivial construction will require m maxflow/mincut computations.
  - O(mn) algorithm may be possible.
- Fully-Dynamic exact all-pairs Mincut.

Thanks for listening

Questions?