Decision Trees & Time Complexity

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ID3 Algorithm for Decision Tree Building

ID3 (Examples, Target Attribute, Attributes)

- Create a root node for tree
- If all examples are +/-, return root with label = +/-
- If attributes = empty, return root with most common value of Target Attribute in Examples
- Begin
  - A $\leftarrow$ attribute from Attributes which best classifies Examples
  - Root $\leftarrow$ A
  - For each value (v) of A
    - Add new tree branch : A = v
    - Examples_v: subset of examples that A = v
    - If Examples_v is empty: add leaf with label = most common value of Target Attribute
    - Else: ID3 (Examples_v, Target attribute, Attributes - A)

Training Data

Day	Outlook	Temp	Humidity	Windy	Play
D1	Sunny	Hot	High	Weak	No
D2	Sunny	Hot	High	Strong	No
D3	Overcast	Hot	High	Weak	Yes
D4	Rain	Mild	High	Weak	Yes
D5	Rain	Cool	Normal	Weak	Yes
D6	Rain	Cool	Normal	Strong	No
D7	Overcast	Cool	Normal	Strong	Yes
D8	Sunny	Mild	High	Weak	No
D9	Sunny	Cool	Normal	Weak	Yes
D10	Rain	Mild	Normal	Weak	Yes
D11	Sunny	Mild	Normal	Strong	Yes
D12	Overcast	Mild	High	Strong	Yes
D13	Overcast	Hot	Normal	Weak	Yes
D14	Rain	Mild	High	Strong	No

Entropy calculated

We have 14 examples in S: 5 No, 9 Yes

$$\text{Entropy}(S) = -p_{\text{No}} \log_2 p_{\text{No}} - p_{\text{Yes}} \log_2 p_{\text{Yes}}$$
$$= -\frac{5}{14} \log_2 \left(\frac{5}{14}\right) - \frac{9}{14} \log_2 \left(\frac{9}{14}\right) = 0.940$$

Information Gain for Outlook

Outlook	Play
Sunny	No
Sunny	No
Overcast	Yes
Rain	Yes
Rain	Yes
Rain	No
Overcast	Yes
Sunny	No
Sunny	Yes
Rain	Yes
Sunny	Yes
Overcast	Yes
Overcast	Yes
Rain	No

Information Gain for Outlook

Outlook	Play
Sunny	No
Sunny	No
Sunny	No
Sunny	Yes
Sunny	Yes

We have 2 Yes, 3 No Entropy $= -\frac{3}{5}\log_2\left(\frac{3}{5}\right) - \frac{2}{5}\log_2\left(\frac{2}{5}\right) = 0.971$

Outlook	Play
Overcast	Yes
Overcast	Yes
Overcast	Yes
Overcast	Yes

We have 4 Yes, 0 No Entropy $= 0$ (pure subset)

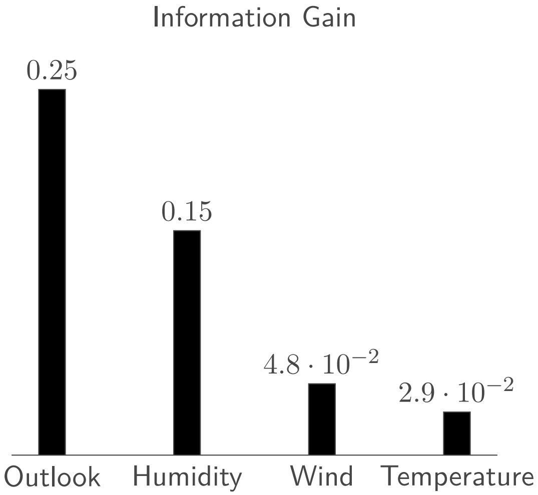
Outlook	Play
Rain	Yes
Rain	Yes
Rain	No
Rain	Yes
Rain	No

We have 3 Yes, 2 No Entropy $= -\frac{3}{5}\log_2\left(\frac{3}{5}\right) - \frac{2}{5}\log_2\left(\frac{2}{5}\right) = 0.971$

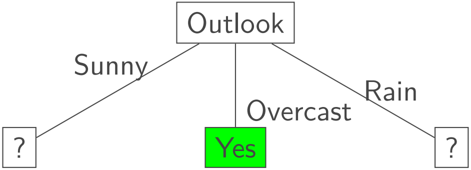
Information Gain

$$
\begin{aligned}
\mathrm{Gain}(S, \mathsf{Outlook}) &= \mathsf{Entropy}(S) - \\
&\sum_{v \in \{\mathsf{Rain,\ Sunny,\ Overcast}\}} \frac{|S_v|}{|S|} \mathsf{Entropy}(S_v) \\
&= 0.940 - \frac{5}{14} \times 0.971 - \frac{4}{14} \times 0 - \frac{5}{14} \times 0.971 \\
&= 0.940 - 0.347 - 0 - 0.347 \\
&= 0.246
\end{aligned}
$$

Information Gain



Learnt Decision Tree

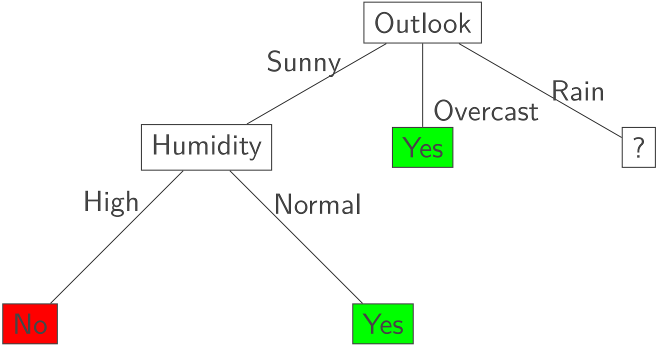


Calling ID3 on Outlook=Sunny

Day	**Temp**	**Humidity**	**Windy**	**Play**
D1	Hot	High	Weak	No
D2	Hot	High	Strong	No
D8	Mild	High	Weak	No
D9	Cool	Normal	Weak	Yes
D11	Mild	Normal	Strong	Yes

- Gain($S_{\text{Outlook=Sunny}}$, Temp) = Entropy(2 Yes, 3 No) - (2/5)*Entropy(0 Yes, 2 No) -(2/5)*Entropy(1 Yes, 1 No) - (1/5)*Entropy(1 Yes, 0 No)
- Gain($S_{\text{Outlook=Sunny}}$, Humidity) = Entropy(2 Yes, 3 No) - (2/5)*Entropy(2 Yes, 0 No) -(3/5)*Entropy(0 Yes, 3 No) $\Longrightarrow$ **maximum possible for the set**
- Gain($S_{\text{Outlook=Sunny}}$, Windy) = Entropy(2 Yes, 3 No) - (3/5)*Entropy(1 Yes, 2 No) -(2/5)*Entropy(1 Yes, 1 No)

Learnt Decision Tree

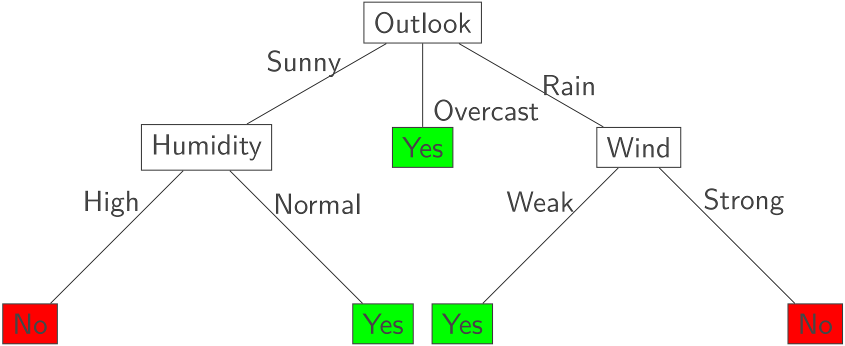


Calling ID3 on (Outlook=Rain)

Day	Temp	Humidity	Windy	Play
D4	Mild	High	Weak	Yes
D5	Cool	Normal	Weak	Yes
D6	Cool	Normal	Strong	No
D10	Mild	Normal	Weak	Yes
D14	Mild	High	Strong	No

- The attribute Windy gives the highest information gain

Learnt Decision Tree



DT Complexity Analysis

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  - $S =$ samples at a given node

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  - If sort here: $O(S \log S)$ sort $+$ $O(S)$ scan $\Rightarrow O(S \log S)$.
- Over all M features: $O(MS)$ (presorted) vs. $O(MS \log S)$ (naïve).

Training Data (Real Features)

Day	Temperature	Humidity	Wind Speed	Play
D1	34.2	85	6.2	No
D2	33.5	88	14.3	No
D3	31.8	90	5.7	Yes
D4	27.6	80	7.5	Yes
D5	23.9	65	9.1	Yes
D6	22.5	70	15.0	No
D7	24.1	60	13.5	Yes
D8	29.8	82	6.8	No
D9	25.3	72	7.0	Yes
D10	26.7	68	8.2	Yes
D11	28.9	75	12.0	Yes
D12	30.5	89	11.5	Yes
D13	32.0	70	5.9	Yes
D14	27.2	85	13.2	No

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Result (both tasks): Train $= O(MN \log N)$.

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- Sum over $\ell = 0$ to $h - 1$ ($h \approx \log N$):

$$\sum MN(\log N - \ell) = MN \cdot \frac{(\log N)(\log N + 1)}{2}.$$

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Result: Train $= O(MN(\log N)^2)$ if you re-sort.

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- Worst case (unbalanced): $h = \Theta(N) \Rightarrow O(N_{\text{test}} N)$.

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- Difference is only in constants:
  - Entropy slower than Gini (logs).
  - MSE similar to Gini.
- Binary features: no sorting needed, but per-level scan still linear. What about fixed discrete features?

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Takeaway: Both Gini/Entropy and MSE need linear sweeps.