Lasso Regression

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Outline

Introduction and Motivation

What is Lasso Regression?

Definition: LASSO

Least **A**bsolute **S**hrinkage and **S**election **O**perator

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Definition: LASSO

Least **A**bsolute **S**hrinkage and **S**election **O**perator

Key Points: Key Properties

- Uses L1 penalty (absolute values) instead of L2 penalty
- Leads to **sparse solutions** (many coefficients become exactly zero)
- Performs automatic feature selection
- Popular for high-dimensional problems

Mathematical Formulation

Problem: Why Not Just Use Ridge?

Important: Limitation of Ridge Regression

Ridge regression shrinks coefficients but **never makes them exactly zero**

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Example: High-Dimensional Problem

- 1000 features, only 50 are truly relevant
- Ridge gives tiny but non-zero coefficients for irrelevant features
- Model is not interpretable
- Need automatic feature selection!

Lasso Objective Function: Constrained Form

Definition: Constrained Optimization

Find $\boldsymbol{\theta}_{\text{opt}}$ such that:

$$\boldsymbol{\theta}_{\text{opt}} = \arg\min_{\boldsymbol{\theta}} \|(\mathbf{y} - \mathbf{X}\boldsymbol{\theta})\|_2^2 \text{ subject to } \|\boldsymbol{\theta}\|_1 \leq s$$

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L1 Norm (Manhattan Distance)

$$\|\boldsymbol{\theta}\|_1 = |\theta_1| + |\theta_2| + \cdots + |\theta_d| = \sum_{j=1}^{d} |\theta_j|$$

Lasso Objective Function: Penalized Form

Theorem: Using Lagrangian Duality (KKT Conditions)

Constrained form is equivalent to:

$$\boldsymbol{\theta}_{\text{opt}} = \arg\min_{\boldsymbol{\theta}} \underbrace{\|(\mathbf{y} - \mathbf{X}\boldsymbol{\theta})\|_2^2 + \lambda\|\boldsymbol{\theta}\|_1}_{\text{Lasso Objective}}$$

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Key Points: Key Components

- $\|(\mathbf{y} - \mathbf{X}\boldsymbol{\theta})\|_2^2$: **Data fitting term** (minimize prediction error)
- $\lambda\|\boldsymbol{\theta}\|_1$: **L1 penalty** (encourage sparsity)
- $\lambda \geq 0$: **Regularization parameter** (controls sparsity)

The Challenge: Non-Differentiability

Important: Problem

The L1 norm $\|\boldsymbol{\theta}\|_1 = \sum_j |\theta_j|$ is **not differentiable** at $\theta_j = 0$

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Cannot Use Standard Calculus

$$\frac{\partial}{\partial \boldsymbol{\theta}} \left[\|(\mathbf{y} - \mathbf{X}\boldsymbol{\theta})\|_2^2 + \lambda \|\boldsymbol{\theta}\|_1 \right] = 0$$

This fails because $\frac{\partial |\theta_j|}{\partial \theta_j}$ is undefined at $\theta_j = 0$

Solution Approaches

Key Points: Three Main Approaches

- **Coordinate Descent**: Optimize one coefficient at a time
- **Subgradient Methods**: Generalize derivatives to non-smooth functions
- **Proximal Methods**: Use soft-thresholding operators

Example: Focus

We'll concentrate on coordinate descent - most popular for Lasso

Why Lasso Gives Sparsity

Sparsity: The Key Question

Important: Central Question

Why does Lasso produce sparse solutions while Ridge doesn't?

Sparsity: The Key Question

Important: Central Question

Why does Lasso produce sparse solutions while Ridge doesn't?

Key Points: Two Perspectives

- **Geometric**: Shape of constraint regions
- **Algorithmic**: Behavior of optimization algorithms

Sparsity: The Key Question

Important: Central Question

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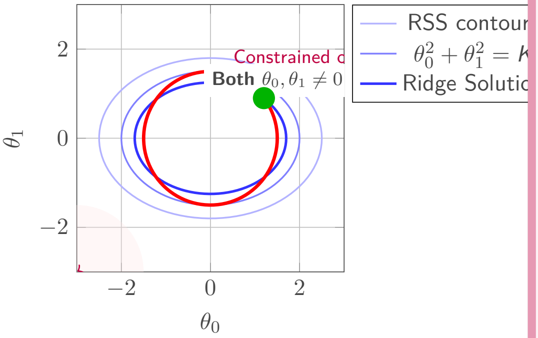
Key Points: Two Perspectives

- **Geometric**: Shape of constraint regions
- **Algorithmic**: Behavior of optimization algorithms

Example: Preview

We'll see why L_p norms with $p < 2$ promote sparsity

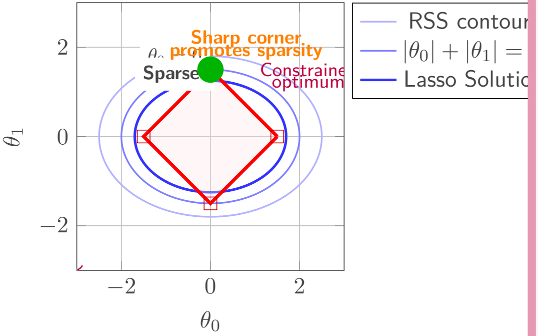
L2 Norm: Ridge Regression



Key Points: L2 Constraint Properties

- **Shape:** Perfect circle
- **Boundary:** Smooth everywhere
- **Intersection:** Rarely on axes
- **Result:** No sparsity
- **Effect:** Shrinks coefficients proportionally

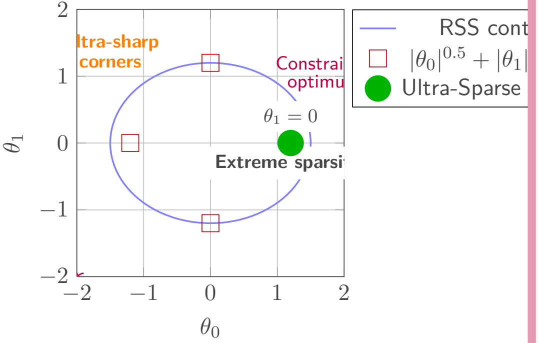
L1 Norm: Lasso Regression



Key Points: L1 Constraint Properties

- **Shape:** Diamond/rhombus
- **Corners:** Sharp at axes
- **Intersection:** High probability on axes
- **Result:** Automatic sparsity!
- **Effect:** Sets coefficients to

L_p Norm: $0 < p < 1$ (Example: $p = 0.5$)



Key Points: L_p Properties $(p < 1)$

- **Shape:** Highly concave
- **Corners:** Ultra-sharp at axes
- **Sparsity:** Extremely high probability
- **Optimization:** Non-convex, much harder
- **Trade-off:**

Sparsity Trend: $L_2 \to L_1 \to L_p$

Theorem: Key Insight

As p decreases from 2 to 1 to $p < 1$:

- Constraint regions become more **pointed** at axes
- Probability of intersection at axes **increases**
- Sparsity **increases**
- Optimization difficulty **increases**

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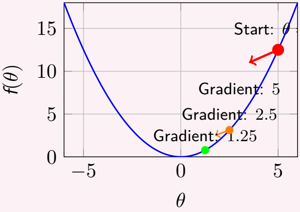
- Optimization difficulty **increases**

Important: Why $p = 1$ is Special

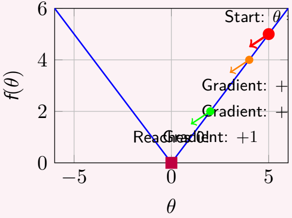
- Still promotes sparsity (sharp corners)
- Remains convex (unlike $p < 1$)
- Computationally tractable

L2 vs L1: Gradient Descent Behavior

Key Points: L2 Penalty: $f(\theta) = \frac{1}{2}\theta^2$



Key Points: L1 Penalty: $f(\theta) = |\theta|$



Example: Key Difference

L2: Gradient $\propto \theta$ (decreases). L1: Constant gradient $= \pm 1$

Gradient Descent: Step 1

L2 Update

- $\frac{df}{d\theta} = \theta = 5$
- $\theta_{new} = 5 - 0.5 \times 5 = 2.5$
- $f(2.5) = 3.125$

L1 Update

- $\frac{df}{d\theta} = \text{sign}(\theta) = +1$
- $\theta_{new} = 5 - 0.5 \times 1 = 4.5$
- $f(4.5) = 4.5$

Important: Key Difference

L2 gradient depends on θ value, L1 gradient is constant ± 1

Gradient Descent: Multiple Steps

Key Points: L2 Sequence

- $\theta_0 = 5.0$
- $\theta_1 = 2.5$
- $\theta_2 = 1.25$
- $\theta_3 = 0.625$
- $\theta_4 = 0.3125$
- $\vdots$ (never exactly 0)

Key Points: L1 Sequence

- $\theta_0 = 5.0$
- $\theta_1 = 4.5$
- $\theta_2 = 4.0$
- $\theta_3 = 3.5$
- $\vdots$
- $\theta_{10} = 0.0$ (exactly!)

Theorem: Sparsity Mechanism

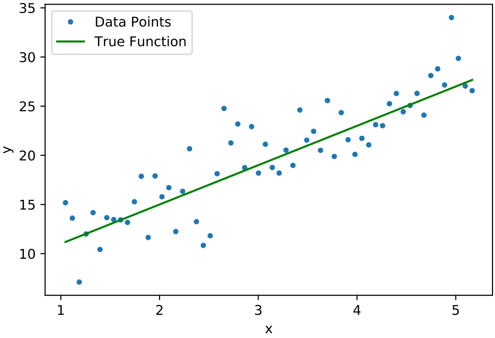
L1 penalty creates **constant gradient** that drives parameters to exactly zero in finite steps!

Geometric Interpretation

Sample Dataset for Demonstration

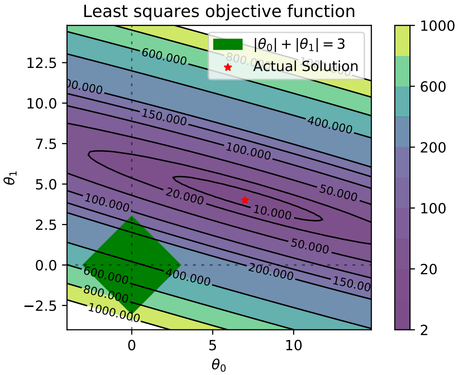
Example: True Function

We'll demonstrate Lasso on a simple linear relationship: $y = 4x + 7$



Sample data from $y = 4x + 7$ with noise

Geometric Interpretation: L1 vs L2 Constraints



L1 vs L2 constraint regions

Key Points: Key Insight

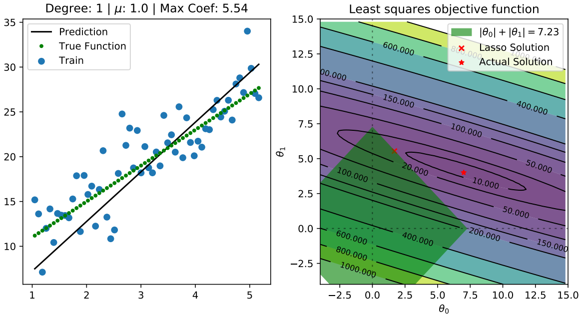
Diamond corners ⇒ exact zeros! Circle ⇒ no sparsity.

Regularization Effects

Effect of λ on Solution Path

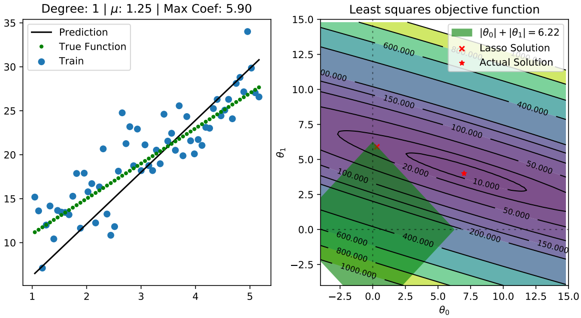
Important: Regularization Parameter

λ controls fit vs sparsity trade-off



$\lambda = 1.0$ - Moderate regularization

Increasing Regularization: $\lambda = 1.25$

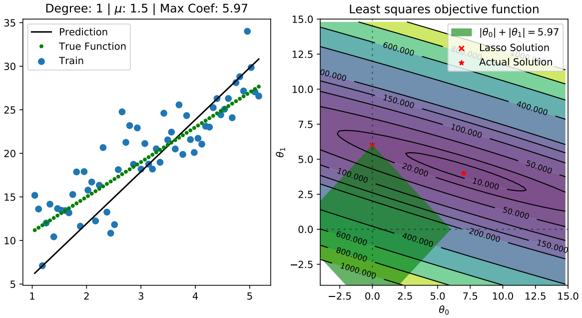


$\lambda = 1.25$ - Higher regularization

Key Points: Observation

As λ increases $\rightarrow$ solution becomes sparser

Further Regularization: $\lambda = 1.5$

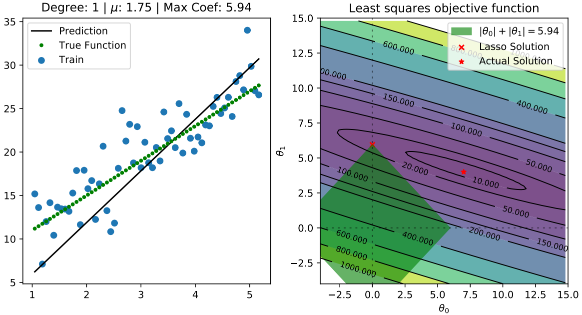


$\lambda = 1.5$ - Even higher regularization

Important: Sparsity Effect

More coefficients $\rightarrow$ exactly zero

High Regularization: $\lambda = 1.75$

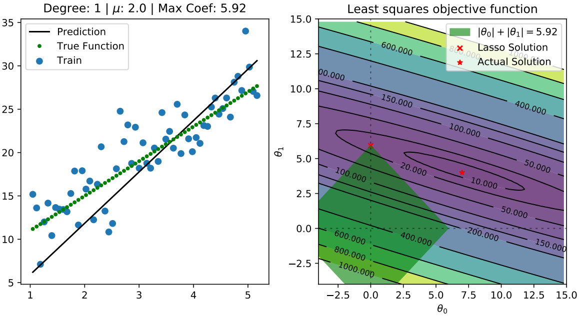


$\lambda = 1.75$ - Strong regularization

Feature Selection

Automatic selection of most important features

Maximum Regularization: $\lambda = 2.0$

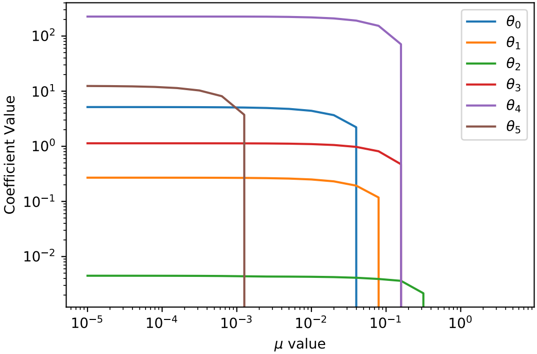


$\lambda = 2.0$ - Very strong regularization

Important: Extreme Sparsity

Only most crucial features remain non-zero

Lasso Regularization Path



Coefficient values vs λ

Key Points: Key Observations

- Coefficients shrink to zero as λ increases
- Different coefficients zero at different λ

Feature Selection Properties

Lasso for Automatic Feature Selection

Definition: Automatic Feature Selection

Lasso performs regression and feature selection simultaneously by setting irrelevant coefficients to exactly zero

Key Points: Key Advantages

- **Sparsity**: Many coefficients $\rightarrow$ exactly zero
- **Interpretability**: Understand which features matter
- **Efficiency**: Fewer parameters, faster prediction

Real-World Feature Selection

Example: Genomics Example

Start with 20,000 genes $\rightarrow$ Lasso selects 50 relevant ones

Key Points: Other Applications

- Text mining: 100k+ words $\rightarrow$ select key terms
- Finance: 1000+ indicators $\rightarrow$ find predictive signals
- Image processing: millions of pixels $\rightarrow$ identify features

Subgradient Methods

What is a Subgradient?

Definition: Subgradient

A subgradient generalizes the concept of gradient to convex but non-differentiable functions

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A subgradient generalizes the concept of gradient to convex but non-differentiable functions

Example: Classic Example

For $f(x) = |x|$:

- $f'(x) = 1$ when $x > 0$
- $f'(x) = -1$ when $x < 0$
- $f'(0)$ is undefined, but subgradient $\in [-1, 1]$

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A subgradient generalizes the concept of gradient to convex but non-differentiable functions

Example: Classic Example

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- $f'(x) = 1$ when $x > 0$
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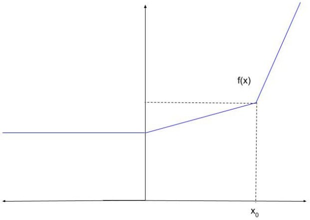
Important: Why Important for Lasso?

The L1 penalty $|\theta_j|$ is non-differentiable at $\theta_j = 0$

Subgradient: Visual Intuition

Important: Task

Find the "derivative" of $f(x)$ at the non-differentiable point $x = x_0$



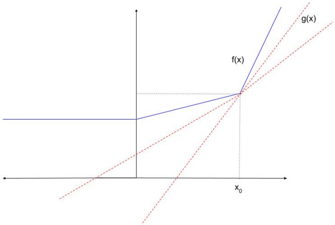
Non-differentiable function at x_0

Subgradient Construction

Construction Method

Find a differentiable function $g(x)$ such that:

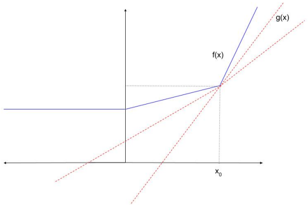
- $g(x_0) = f(x_0)$ (intersects at the point)
- $g(x) \leq f(x)$ for all x (lies below or on f)



Computing the Subgradient

Theorem: Subgradient Definition

Slope of $g(x)$ at $x = x_0$ gives subgradient of f at x_0



Slope $\rightarrow$ subgradient

Subgradient Sets

Key Points: Key Insight

Multiple supporting lines $\Rightarrow$ set of valid subgradients

Example: For $f(x) = |x|$

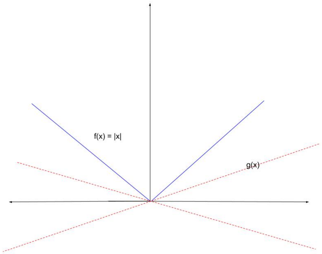
At $x = 0$: subgradient set is $[-1, +1]$

Example: Subgradient of $f(x) = |x|$

Subgradient Set

For $f(x) = |x|$ at $x = 0$:

$$\partial f(0) = [-1, 1]$$



Lines with slope in $[-1, 1]$ support $|x|$ at $x = 0$

Connection to Lasso

Important: Lasso Connection

This subgradient concept is exactly what we need for the L1 penalty term!

Key Points: Next Step

We'll use subgradients to derive coordinate descent for Lasso

Coordinate Descent Algorithm

Introduction to Coordinate Descent

Definition: Coordinate Descent

Optimization method: minimize one coordinate at a time

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Definition: Coordinate Descent

Optimization method: minimize one coordinate at a time

Key Points: Key Idea

- Hard: optimize all coordinates together
- Easy: optimize one coordinate at a time
- Perfect for non-differentiable Lasso!

Coordinate Descent Algorithm

Algorithm Overview

$$\min_{\boldsymbol{\theta}} f(\boldsymbol{\theta}) \text{ becomes } \min_{\theta_j} f(\theta_1, \ldots, \theta_{j-1}, \theta_j, \theta_{j+1}, \ldots, \theta_d)$$

Important: Process

Cycle through coordinates, optimizing one at a time

Coordinate Descent Properties

Key Points: Advantages

- **No step-size**: Exact 1D minimization
- **Convergence**: Guaranteed for convex Lasso
- **Efficient**: Closed-form updates

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Selection Strategies

Cyclic, Random, or Greedy coordinate selection

Worked Example

Coordinate Descent Example Setup

Example: Problem

Learn $y = \theta_0 + \theta_1 x$ using coordinate descent on the dataset below

x	y
1	1
2	2
3	3

Initial Conditions

- Initial parameters: $(\theta_0, \theta_1) = (2, 3)$
- We'll run for 2 iterations
- Using standard least squares (no regularization for simplicity)

Coordinate Descent : Example

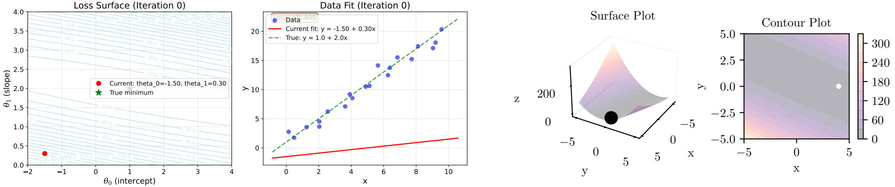
Our predictor, $\hat{y} = \theta_0 + \theta_1 x$

Error for i^{th} datapoint, $\epsilon_i = y_i - \hat{y}_i$
$\epsilon_1 = 1 - \theta_0 - \theta_1$
$\epsilon_2 = 2 - \theta_0 - 2\theta_1$
$\epsilon_3 = 3 - \theta_0 - 3\theta_1$

$\mathrm{MSE} = \frac{\epsilon_1^2+\epsilon_2^2+\epsilon_3^2}{3} = \frac{14+3\theta_0^2+14\theta_1^2-12\theta_0-28\theta_1+12\theta_0\theta_1}{3}$

Iteration 0

$$\text{MSE} = \frac{1}{3}(14 + 3\theta_0^2 + 14\theta_1^2 - 12\theta_0 - 28\theta_1 + 12\theta_0\theta_1)$$



Coordinate Descent : Example

Iteration 1

INIT: $\theta_0 = 2$ and $\theta_1 = 3$

$\theta_1 = 3$ optimize for θ_0

Coordinate Descent : Example

Iteration 1

INIT: $\theta_0 = 2$ and $\theta_1 = 3$

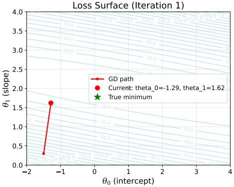
$\theta_1 = 3$ optimize for θ_0

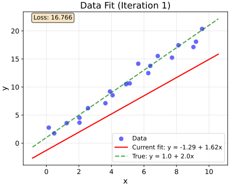
$\frac{\partial \, \mathrm{MSE}}{\partial \theta_0} = 6\theta_0 + 24 = 0$

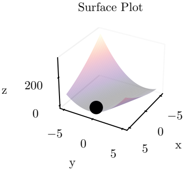
$\theta_0 = -4$

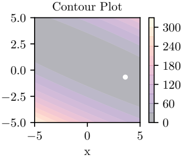
Iteration 1

$$\text{MSE} = \frac{1}{3}(14 + 3\theta_0^2 + 14\theta_1^2 - 12\theta_0 - 28\theta_1 + 12\theta_0\theta_1)$$









Coordinate Descent : Example

Iteration 2

INIT: $\theta_0 = -4$ and $\theta_1 = 3$

$\theta_0 = -4$ optimize for θ_1

Coordinate Descent : Example

Iteration 2

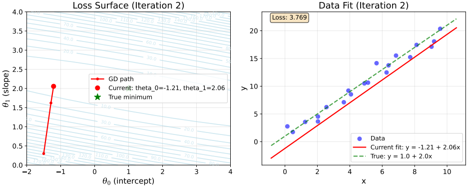
INIT: $\theta_0 = -4$ and $\theta_1 = 3$

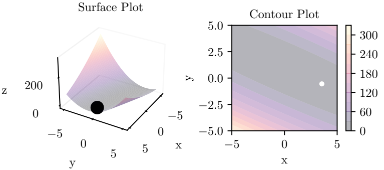
$\theta_0 = -4$ optimize for θ_1

$\theta_1 = 2.7$

Iteration 2

$$\text{MSE} = \frac{1}{3}(14 + 3\theta_0^2 + 14\theta_1^2 - 12\theta_0 - 28\theta_1 + 12\theta_0\theta_1)$$





Coordinate Descent : Example

Iteration 3

INIT: $\theta_0 = -4$ and $\theta_1 = 2.7$

$\theta_1 = 2.7$ optimize for θ_0

Coordinate Descent : Example

Iteration 3

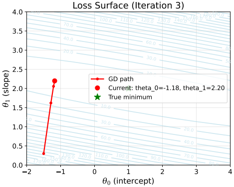
INIT: $\theta_0 = -4$ and $\theta_1 = 2.7$

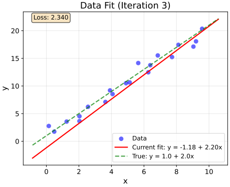
$\theta_1 = 2.7$ optimize for θ_0

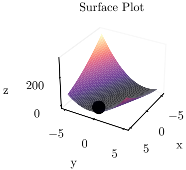
$\theta_0 = -3.4$

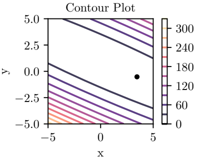
Iteration 3

$$\text{MSE} = \frac{1}{3}(14 + 3\theta_0^2 + 14\theta_1^2 - 12\theta_0 - 28\theta_1 + 12\theta_0\theta_1)$$







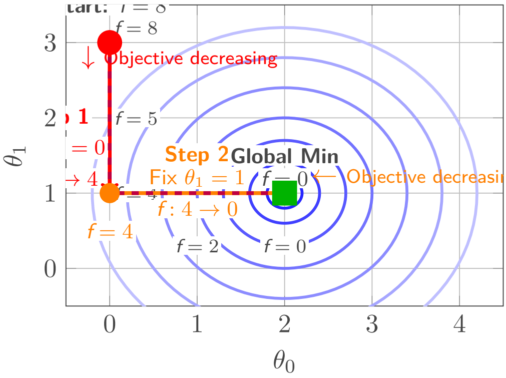


Visual Coordinate Descent

Coordinate Descent: Visual Algorithm

Example: Setup

Minimize $f(\theta_0, \theta_1) = (\theta_0 - 2)^2 + (\theta_1 - 1)^2$ starting from $(0, 3)$



Coordinate Descent: Step-by-Step

Step 1: Fix $\theta_0 = 0$

Minimize: $f(0, \theta_1) = 4 + (\theta_1 - 1)^2$

$$\frac{\partial f}{\partial \theta_1} = 2(\theta_1 - 1) = 0$$

$$\theta_1^* = 1$$

Step 2: Fix $\theta_1 = 1$

Minimize: $f(\theta_0, 1) = (\theta_0 - 2)^2$

$$\frac{\partial f}{\partial \theta_0} = 2(\theta_0 - 2) = 0$$

$$\theta_0^* = 2$$

Key Points: Key Observations

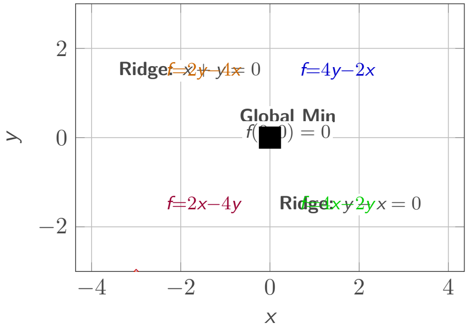
- Each step moves parallel to coordinate axes
- Each step finds exact 1D minimum
- Converges to global minimum in 2 steps (for this quadratic)

When Coordinate Descent Fails

Function: $f(x, y) = |x + y| + 3|y - x|$

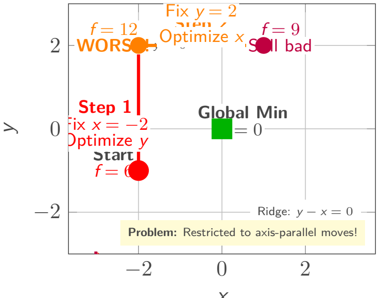
Example: Problematic Function

Let's visualize: $f(x, y) = |x + y| + 3|y - x|$ - a non-smooth function with ridges



Important: Non-Smooth Surface with Ridges

Coordinate Descent Failure: Step by Step



Function Values

- Start $(-2, -1)$: $f = 6$
- After Step 1 $(-2, 2)$: $f = 12 \uparrow$ WORSE
- After Step 2 $(1, 2)$: $f = 9$ Still bad
- Global Min $(0, 0)$: $f = 0$ ✓ Optimal

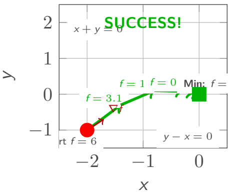
Important: Root Cause

1D Optimization Prob-

Gradient Descent vs Coordinate Descent

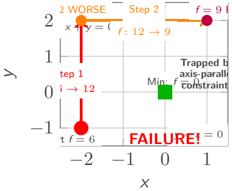
Key Points: Gradient Descent

Strategy: Move in direction of steepest descent - can move diagonally



Important: Coordinate Descent

Constraint: Only axis-parallel moves - gets trapped!



Why Coordinate Descent Fails Here

Important: Problem

Function $f(x, y) = |x + y| + 3|y - x|$ is non-separable

Analysis

- Start at $(-2, -1)$: $f(-2, -1) = |-3| + 3|-1| = 6$
- Fix $x = -2$, optimize y: optimal $y = 2$
- New point $(-2, 2)$: $f(-2, 2) = 12$ (worse!)

When Coordinate Descent Fails

Theorem: Failure Conditions

- Non-separable functions
- Strong coupling between variables
- Need simultaneous movement in multiple directions

Key Points: Fortunately

Lasso objective IS separable, so coordinate descent works well!

Coordinate Descent for Unregularised Regression

- Express error as a difference of y_i and $\hat{y}_i$

$$\hat{y}_i = \sum_{j=0}^{d} \theta_j x_i^j = \theta_0 x_i^0 + \theta_1 x_i^1 + \theta_2 x_i^2 + \ldots + \theta_d x_i^d \qquad (1)$$

$$\epsilon_i = y_i - \hat{y}_i = y_i - \theta_0 x_i^0 - \theta_1 x_i^1 - \ldots - \theta_d x_i^d = y_i - \sum_{j=0}^{d} \theta_j x_i^j \quad (2)$$

Coordinate Descent for Unregularised regression

$$\sum_{i=1}^{n} \epsilon^2 = \mathrm{RSS} = \sum_{i=1}^{n} \left(y_i - \left(\theta_0 x_i^0 + \ldots + \theta_j x_i^j + \theta_d x_i^d \right) \right)^2$$

Coordinate Descent for Unregularised regression

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$$\frac{\partial \, \mathrm{RSS} \left(\theta_j \right)}{\partial \theta_j} = 2 \sum_{i=1}^{n} \left(y_i - \left(\theta_0 x_i^0 + \ldots + \theta_j x_i^j + \ldots \right) \right) \left(-x_i^j \right)$$

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$$= 2 \sum_{i=1}^{n} \left(y_i - \left(\theta_0 x_i^0 + \ldots + \theta_d x_i^d \right) \right) \left(-x_i^j \right) + 2 \sum_{i=1}^{n} \theta_j (x_i^j)^2$$

Coordinate Descent for Unregularised regression

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where:

$$\hat{y}_i^{(-j)} = \theta_0 x_i^0 + \ldots + \theta_d x_i^d$$

is $\hat{y}_i$ without θ_j

Coordinate Descent for Unregularised regression

$$\text{Set } \frac{\partial \, \mathrm{RSS}\,(\theta_j)}{\partial \theta_j} = 0$$

$$\theta_j = \sum_{i=1}^{n} \frac{\left(y_i - \left(\theta_0 x_i^0 + \ldots + \theta_d x_i^d\right)\right)\left(x_i^j\right)}{\left(x_i^j\right)^2} = \frac{\rho_j}{z_j}$$

$$\rho_j = \sum_{i=1}^{n} x_i^j \left(y_i - \hat{y}_i^{(-j)}\right) \quad \text{and} \quad z_j = \sum_{i=1}^{n} \left(x_i^j\right)^2$$

z_j is the squared of ℓ_2 norm of the j^{th} feature

Mathematical Derivation

Lasso Coordinate Descent: Setup

Lasso Objective

$$\text{Minimize } \underbrace{\sum_{i=1}^{n}(y_i - \hat{y}_i)^2 + \lambda \sum_{j=0}^{d} |\theta_j|}_{\text{Lasso Objective}}$$

Lasso Coordinate Descent: Setup

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Key Points: Key Definitions

- $\rho_j = \sum_{i=1}^{n} x_{ij}(y_i - \hat{y}_i^{(-j)})$ (partial residual correlation)
- $z_j = \sum_{i=1}^{n} x_{ij}^2$ (feature norm squared)
- $\hat{y}_i^{(-j)}$ = prediction without j-th feature

Coordinate Descent: Subgradient Analysis

Subgradient of Lasso Objective w.r.t. θ_j

$$\frac{\partial}{\partial \theta_j}(\text{Lasso Objective}) = -2\rho_j + 2\theta_j z_j + \lambda \frac{\partial}{\partial \theta_j}|\theta_j|$$

Coordinate Descent: Subgradient Analysis

Subgradient of Lasso Objective w.r.t. θ_j

$$\frac{\partial}{\partial \theta_j}(\text{Lasso Objective}) = -2\rho_j + 2\theta_j z_j + \lambda \frac{\partial}{\partial \theta_j}|\theta_j|$$

Theorem: Subgradient of $|\theta_j|$

$$\frac{\partial}{\partial \theta_j}|\theta_j| = \begin{cases} +1 & \text{if } \theta_j > 0 \\ [-1, +1] & \text{if } \theta_j = 0 \\ -1 & \text{if } \theta_j < 0 \end{cases}$$

Case Analysis: $\theta_j > 0$

Case 1: $\theta_j > 0$

Subgradient is $+1$, so optimality condition:

$$-2\rho_j + 2\theta_j z_j + \lambda = 0$$

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Theorem: Solution

$$\theta_j = \frac{\rho_j - \lambda/2}{z_j}$$

This is valid when $\rho_j > \lambda/2$ (ensures $\theta_j > 0$)

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$$\theta_j = \frac{\rho_j - \lambda/2}{z_j}$$

This is valid when $\rho_j > \lambda/2$ (ensures $\theta_j > 0$)

Important: Soft Thresholding

Notice the $-\lambda/2$ term: this "shrinks" the coefficient!

Case Analysis: $\theta_j < 0$

Case 2: $\theta_j < 0$

Subgradient is -1, so optimality condition:

$$-2\rho_j + 2\theta_j z_j - \lambda = 0$$

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$$\theta_j = \frac{\rho_j + \lambda/2}{z_j}$$

This is valid when $\rho_j < -\lambda/2$ (ensures $\theta_j < 0$)

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Important: Symmetric Shrinkage

Same shrinkage effect, but in the opposite direction!

Case Analysis: $\theta_j = 0$

Case 3: $\theta_j = 0$

Subgradient $\in [-1, +1]$, so optimality requires:

$$0 \in [-2\rho_j - \lambda, -2\rho_j + \lambda]$$

Case Analysis: $\theta_j = 0$

Case 3: $\theta_j = 0$

Subgradient $\in [-1, +1]$, so optimality requires:

$$0 \in [-2\rho_j - \lambda, -2\rho_j + \lambda]$$

Theorem: Zero Condition

This happens when:

$$-2\rho_j - \lambda \leq 0 \text{ and } -2\rho_j + \lambda \geq 0$$

$$\Rightarrow -\frac{\lambda}{2} \leq \rho_j \leq \frac{\lambda}{2}$$

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Important: Sparsity Mechanism

If correlation $|\rho_j|$ is small ($< \lambda/2$), set $\theta_j = 0$!

Soft-Thresholding Operator

Definition: Complete Lasso Update Rule

$$\theta_j = \begin{cases} \frac{\rho_j + \lambda/2}{z_j} & \text{if } \rho_j < -\lambda/2 \\ 0 & \text{if } |\rho_j| \leq \lambda/2 \\ \frac{\rho_j - \lambda/2}{z_j} & \text{if } \rho_j > \lambda/2 \end{cases}$$

Soft-Thresholding Properties

Key Points: Key Properties

- **Shrinkage**: Coefficients pulled toward zero
- **Selection**: Small coefficients $\rightarrow$ exactly zero
- **Soft-thresholding**: Smooth shrinkage + selection

Example: Intuition

Weak correlation $|\rho_j| \leq \lambda/2 \Rightarrow$ eliminate feature!

Lasso vs Ridge Comparison

Lasso vs Ridge: Key Differences

Property	Ridge (L2)	Lasso (L1)
Penalty	$\sum \theta_j^2$	$\sum \lvert\theta_j\rvert$
Sparsity	Never exactly zero	Can be exactly zero
Feature Selection	No	Yes
Differentiable	Yes	No (at $\theta_j = 0$)
Solution Method	Closed form	Coordinate descent
Constraint Shape	Circle	Diamond
Best for	Multicollinearity	Feature selection

When to Use Lasso vs Ridge

Key Points: Use Lasso When:

- High-dimensional data ($p >> n$)
- Need interpretable model
- Expect only few features are truly relevant
- Want automatic feature selection

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Key Points: Use Ridge When:

- All features might be somewhat relevant
- Multicollinearity is the main problem
- Want to keep all features with reduced impact
- Need stable solution with correlated features

Summary and Applications

Lasso Regression: Summary

Definition: Lasso in a Nutshell

Lasso = Linear regression + L1 penalty for automatic feature selection

Key Points: Key Advantages

- Regression + feature selection simultaneously
- Sparse, interpretable models
- Handles high-dimensional data well

Lasso: Limitations and Applications

Key Points: Limitations

- Arbitrary selection among correlated features
- May underperform when all features are relevant

Example: Applications

Genomics, text mining, signal processing, finance, marketing analytics