Multivariate Normal Distribution II

Nipun Batra and teaching staff

IIT Gandhinagar

August 30, 2025

Detour: Inverse of partioned symmetric matrix [1]

Consider an $n \times n$ symmetric matrix $\mathbf{A}$ and divide it into four blocks
$\mathbf{A} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{12}^\top & \mathbf{A}_{22} \end{bmatrix}$
For example, let $n = 3$, we have

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 6 \\ 3 & 6 & 8 \end{bmatrix}$$

We could for example have
$\mathbf{A}_{11} = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}$ and $\mathbf{A}_{12} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$ and $\mathbf{A}_{22} = \begin{bmatrix} 8 \end{bmatrix}$

[1] Courtesy: http://fourier.eng.hmc.edu/e161/lectures/gaussianprocess/node6.html

Detour: Inverse of partioned symmetric matrix

Question: Write $\mathbf{B} = \mathbf{A}^{-1}$ in terms of the four blocks

$$\mathbf{B} = \begin{bmatrix} \mathbf{B}_{11} & \mathbf{B}_{12} \\ \mathbf{B}_{12}^\top & \mathbf{B}_{22} \end{bmatrix} = \mathbf{A}^{-1}$$

$\mathbf{A}_{11}$ and $\mathbf{B}_{11} \in \mathbb{R}^{p \times p}$
$\mathbf{A}_{22}$ and $\mathbf{B}_{22} \in \mathbb{R}^{q \times q}$
$\mathbf{A}_{12} = \mathbf{A}_{21}^\top$ and $\mathbf{B}_{12} = \mathbf{B}_{21}^\top \in \mathbb{R}^{p \times q}$
and, $p + q = n$

Detour: Inverse of partioned symmetric matrix

$$\mathbf{I}_n = \mathbf{A}\mathbf{A}^{-1} = \mathbf{A}\mathbf{B}$$
$$= \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{12}^\top & \mathbf{A}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{B}_{11} & \mathbf{B}_{12} \\ \mathbf{B}_{12}^\top & \mathbf{B}_{22} \end{bmatrix} =$$
$$\begin{bmatrix} \mathbf{A}_{11}\mathbf{B}_{11} + \mathbf{A}_{12}\mathbf{B}_{12}^\top & \mathbf{A}_{11}\mathbf{B}_{12} + \mathbf{A}_{12}\mathbf{B}_{22} \\ \mathbf{A}_{12}^\top\mathbf{B}_{11} + \mathbf{A}_{22}\mathbf{B}_{12}^\top & \mathbf{A}_{12}^\top\mathbf{B}_{12} + \mathbf{A}_{22}\mathbf{B}_{22} \end{bmatrix} = \begin{bmatrix} \mathbf{I}_p & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_q \end{bmatrix}$$

Thus, we have

$$\mathbf{A}_{11}\mathbf{B}_{11} + \mathbf{A}_{12}\mathbf{B}_{12}^\top = \mathbf{I}_p$$
$$\mathbf{A}_{11}\mathbf{B}_{12} + \mathbf{A}_{12}\mathbf{B}_{22} = \mathbf{0}^{p\times q}$$
$$\mathbf{A}_{12}^\top\mathbf{B}_{11} + \mathbf{A}_{22}\mathbf{B}_{12}^\top = \mathbf{0}^{q\times p}$$
$$\mathbf{A}_{12}^\top\mathbf{B}_{12} + \mathbf{A}_{22}\mathbf{B}_{22} = \mathbf{I}_q$$

Detour: Inverse of partioned symmetric matrix

Moving the expressions around we get the following results.

$$
\begin{aligned}
\mathbf{B}_{11} &= (\mathbf{A}_{11} - \mathbf{A}_{12}\mathbf{A}_{22}^{-1}\mathbf{A}_{12}^\top)^{-1} \\
&= \mathbf{A}_{11}^{-1} + \mathbf{A}_{11}^{-1}\mathbf{A}_{12}(\mathbf{A}_{22} - \mathbf{A}_{12}^\top\mathbf{A}_{11}^{-1}\mathbf{A}_{12})^{-1}\mathbf{A}_{12}^\top\mathbf{A}_{11}^{-1} \\
\mathbf{B}_{22} &= (\mathbf{A}_{22} - \mathbf{A}_{12}^\top\mathbf{A}_{11}^{-1}\mathbf{A}_{12})^{-1} \\
&= \mathbf{A}_{22}^{-1} + \mathbf{A}_{22}^{-1}\mathbf{A}_{12}^\top(\mathbf{A}_{11} - \mathbf{A}_{12}\mathbf{A}_{22}^{-1}\mathbf{A}_{12}^\top)^{-1}\mathbf{A}_{12}\mathbf{A}_{22}^{-1} \\
\mathbf{B}_{12}^\top &= -\mathbf{A}_{22}^{-1}\mathbf{A}_{12}^\top(\mathbf{A}_{11} - \mathbf{A}_{12}\mathbf{A}_{22}^{-1}\mathbf{A}_{12}^\top)^{-1} \\
\mathbf{B}_{12}^\top &= -\mathbf{A}_{11}^{-1}\mathbf{A}_{12}^\top(\mathbf{A}_{22} - \mathbf{A}_{12}^\top\mathbf{A}_{11}^{-1}\mathbf{A}_{12})^{-1}
\end{aligned}
$$

Determinant of Partitioned Symmetric Matrix

Theorem: Determinant of a partitioned symmetric matrix can be written as follows

$$\begin{aligned}\det(\mathbf{A}) = \det\begin{pmatrix}\mathbf{A}_{11} & \mathbf{A}_{12}\\ \mathbf{A}_{21} & \mathbf{A}_{22}\end{pmatrix}\\ = \det(\mathbf{A}_{11})\det(\mathbf{A}_{22} - \mathbf{A}_{12}^{\top}\mathbf{A}_{11}^{-1}\mathbf{A}_{12})\\ = \det(\mathbf{A}_{22})\det(\mathbf{A}_{11} - \mathbf{A}_{12}\mathbf{A}_{22}^{-1}\mathbf{A}_{12}^{\top})\end{aligned}$$

Determinant of Partitioned Symmetric Matrix

Proof: Note that

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{0} \\ \mathbf{A}_{12}^\top & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{I} & \mathbf{A}_{11}^{-1}\mathbf{A}_{12} \\ \mathbf{0} & \mathbf{A}_{22} - \mathbf{A}_{12}^\top \mathbf{A}_{11}^{-1} \mathbf{A}_{12} \end{bmatrix}$$
$$= \begin{bmatrix} \mathbf{I} & \mathbf{A}_{12} \\ \mathbf{0} & \mathbf{A}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{A}_{11} - \mathbf{A}_{12}\mathbf{A}_{22}^{-1}\mathbf{A}_{12}^\top & \mathbf{0} \\ \mathbf{A}_{22}^{-1}\mathbf{A}_{21} & \mathbf{I} \end{bmatrix}$$

The theorem is proved as we also know that

$$\det(\mathbf{AB}) = \det(\mathbf{A})\det(\mathbf{B})$$

and

$$\det\begin{pmatrix} \mathbf{B} & \mathbf{0} \\ \mathbf{C} & \mathbf{D} \end{pmatrix} = \det\begin{pmatrix} \mathbf{B} & \mathbf{C} \\ \mathbf{0} & \mathbf{D} \end{pmatrix} = \det(\mathbf{B})\det(\mathbf{D})$$

Marginalisation and Conditional of multivariate normal[2]

Assume an n-dimensional random vector

$$\mathbf{x} = \begin{bmatrix} \mathbf{x}_1 & \mathbf{x}_2 \end{bmatrix}$$

has a normal distribution $\mathcal{N}(\boldsymbol{\mu}, \ \boldsymbol{\Sigma})$ with

$$\boldsymbol{\mu} = \begin{bmatrix} \boldsymbol{\mu}_1 \\ \boldsymbol{\mu}_2 \end{bmatrix} \text{ and } \boldsymbol{\Sigma} = \begin{bmatrix} \boldsymbol{\Sigma}_{11} & \boldsymbol{\Sigma}_{12} \\ \boldsymbol{\Sigma}_{21} & \boldsymbol{\Sigma}_{22} \end{bmatrix}$$

where $\mathbf{x}_1$ and $\mathbf{x}_2$ are two subvectors of respective dimensions p and q with $p + q = n$. Note that $\boldsymbol{\Sigma} = \boldsymbol{\Sigma}^\top$, and $\boldsymbol{\Sigma}_{21} = \boldsymbol{\Sigma}_{21}^\top$.

[2]Courtesy: http://fourier.eng.hmc.edu/e161/lectures/gaussianprocess/node7.html.

Marginalisation and Conditional of multivariate normal

Theorem:
part a: The marginal distributions of $\mathbf{x}_1$ and $\mathbf{x}_2$ are also normal with mean vector $\boldsymbol{\mu}_i$ and covariance matrix $\boldsymbol{\Sigma}_{ii}$ $(i = 1, 2)$, respectively.
part b: The conditional distribution of $\mathbf{x}_i$ given $\mathbf{x}_j$ is also normal with mean vector

$$\boldsymbol{\mu}_{i|j} = \boldsymbol{\mu}_i + \boldsymbol{\Sigma}_{ij}\boldsymbol{\Sigma}_{jj}^{-1}(\mathbf{x}_j - \boldsymbol{\mu}_j)$$

Marginalisation and Conditional of multivariate normal

Proof:
The joint density of $\mathbf{x}$ is:

$$f(\mathbf{x}) = f(\mathbf{x}_1, \mathbf{x}_2) = \frac{1}{(2\pi)^{n/2}\det(\boldsymbol{\Sigma})^{1/2}} exp\left[-\frac{1}{2}Q(\mathbf{x}_1, \mathbf{x}_2)\right]$$

where Q is defined as

$$Q(\mathbf{x}_1, \mathbf{x}_2) = (\mathbf{x} - \boldsymbol{\mu})^\top \boldsymbol{\Sigma}^{-1}(\mathbf{x} - \boldsymbol{\mu})$$

$$= [(\mathbf{x}_1 - \boldsymbol{\mu}_1)^\top, (\mathbf{x}_2 - \boldsymbol{\mu}_2)^\top] \begin{bmatrix} \boldsymbol{\Sigma}^{11} & \boldsymbol{\Sigma}^{12} \\ \boldsymbol{\Sigma}^{21} & \boldsymbol{\Sigma}^{22} \end{bmatrix} \begin{bmatrix} \mathbf{x}_1 - \boldsymbol{\mu}_1 \\ \mathbf{x}_2 - \boldsymbol{\mu}_2 \end{bmatrix}$$

$$= (\mathbf{x}_1 - \boldsymbol{\mu}_1)^\top \boldsymbol{\Sigma}^{11}(\mathbf{x}_1 - \boldsymbol{\mu}_1) + 2(\mathbf{x}_1 - \boldsymbol{\mu}_1)^\top \boldsymbol{\Sigma}^{12}(\mathbf{x}_2 - \boldsymbol{\mu}_2) + (\mathbf{x}_2 - \boldsymbol{\mu}_2)^\top \cdots$$

$$\cdots \boldsymbol{\Sigma}^{22}(\mathbf{x}_2 - \boldsymbol{\mu}_2)$$

Marginalisation and Conditional of multivariate normal

Here we have assumed

$$\boldsymbol{\Sigma}^{-1} = \begin{bmatrix} \boldsymbol{\Sigma}_{11} & \boldsymbol{\Sigma}_{12} \\ \boldsymbol{\Sigma}_{21} & \boldsymbol{\Sigma}_{22} \end{bmatrix}^{-1} = \begin{bmatrix} \boldsymbol{\Sigma}^{11} & \boldsymbol{\Sigma}^{12} \\ \boldsymbol{\Sigma}^{21} & \boldsymbol{\Sigma}^{22} \end{bmatrix}$$

According to inverse of a partitioned symmetric matrix we have,

$$\begin{aligned}
\boldsymbol{\Sigma}^{11} &= \left(\boldsymbol{\Sigma}_{11} - \boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22}^{-1}\boldsymbol{\Sigma}_{12}^{\top}\right)^{-1} \\
&= \boldsymbol{\Sigma}_{11}^{-1} + \boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12}\left(\boldsymbol{\Sigma}_{22} - \mathbf{A}_{12}^{\top}\boldsymbol{\Sigma}_{11}^{\top}\boldsymbol{\Sigma}_{12}\right)^{-1}\boldsymbol{\Sigma}_{12}^{\top}\boldsymbol{\Sigma}_{11}^{-1} \\
\boldsymbol{\Sigma}^{22} &= \left(\boldsymbol{\Sigma}_{22} - \boldsymbol{\Sigma}_{12}^{\top}\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12}\right)^{-1} \\
&= \boldsymbol{\Sigma}_{22}^{-1} + \boldsymbol{\Sigma}_{22}^{-1}\boldsymbol{\Sigma}_{12}^{\top}\left(\boldsymbol{\Sigma}_{11} - \boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22}^{-1}\boldsymbol{\Sigma}_{12}^{\top}\right)^{-1}\boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22}^{-1} \\
\boldsymbol{\Sigma}^{12} &= -\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12}\left(\boldsymbol{\Sigma}_{22} - \boldsymbol{\Sigma}_{12}^{\top}\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12}\right)^{-1} = \left(\boldsymbol{\Sigma}^{21}\right)^{\top}
\end{aligned}$$

Marginalisation and Conditional of multivariate normal

Substituting the second expression for $\boldsymbol{\Sigma}^{11}$, first expression for $\boldsymbol{\Sigma}^{22}$, and $\boldsymbol{\Sigma}^{12}$ into $Q(\mathbf{x}_1, \mathbf{x}_2)$ to get:

$$
\begin{aligned}
&Q\left(\mathbf{x}_1, \mathbf{x}_2\right) = \\
&\left(\mathbf{x}_1 - \boldsymbol{\mu}_1\right)^\top \left[\boldsymbol{\Sigma}_{11}^{-1} + \boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12}\left(\boldsymbol{\Sigma}_{22} - \mathbf{A}_{12}^\top\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12}\right)^{-1}\boldsymbol{\Sigma}_{12}^\top\boldsymbol{\Sigma}_{11}^{-1}\right]\left(\mathbf{x}_1 - \boldsymbol{\mu}_1\right) \\
&- 2\left(\mathbf{x}_1 - \boldsymbol{\mu}_1\right)^\top \left[\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12}\left(\boldsymbol{\Sigma}_{22} - \boldsymbol{\Sigma}_{12}^\top\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12}\right)^{-1}\right]\left(\mathbf{x}_2 - \boldsymbol{\mu}_2\right) \\
&+ \left(\mathbf{x}_2 - \boldsymbol{\mu}_2\right)^\top \left[\left(\boldsymbol{\Sigma}_{22} - \boldsymbol{\Sigma}_{12}^\top\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12}\right)^{-1}\right]\left(\mathbf{x}_2 - \boldsymbol{\mu}_2\right) \\
= &\left(\mathbf{x}_1 - \boldsymbol{\mu}_1\right)^\top \boldsymbol{\Sigma}_{11}^{-1}\left(\mathbf{x}_1 - \boldsymbol{\mu}_1\right) \\
&+ \left(\mathbf{x}_1 - \boldsymbol{\mu}_1\right)^\top \boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12}\left(\boldsymbol{\Sigma}_{22} - \mathbf{A}_{12}^\top\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12}\right)^{-1}\boldsymbol{\Sigma}_{12}^\top\boldsymbol{\Sigma}_{11}^{-1}\Big]\left(\mathbf{x}_1 - \boldsymbol{\mu}_1\right) \\
&- 2\left(\mathbf{x}_1 - \boldsymbol{\mu}_1\right)^\top \left[\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12}\left(\boldsymbol{\Sigma}_{22} - \boldsymbol{\Sigma}_{12}^\top\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12}\right)^{-1}\right]\left(\mathbf{x}_2 - \boldsymbol{\mu}_2\right) \\
&+ \left(\mathbf{x}_2 - \boldsymbol{\mu}_2\right)^\top \left[\left(\boldsymbol{\Sigma}_{22} - \boldsymbol{\Sigma}_{12}^\top\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12}\right)^{-1}\right]\left(\mathbf{x}_2 - \boldsymbol{\mu}_2\right)
\end{aligned}
$$

Marginalisation and Conditional of multivariate normal

$$
\begin{aligned}
&= (\mathbf{x}_1 - \boldsymbol{\mu}_1)^\top \boldsymbol{\Sigma}_{11}^{-1} \\
&\quad + \left[(\mathbf{x}_2 - \boldsymbol{\mu}_2) - \boldsymbol{\Sigma}_{12}^\top \boldsymbol{\Sigma}_{11}^{-1} (\mathbf{x}_1 - \boldsymbol{\mu}_1)\right]^\top \cdot \left(\boldsymbol{\Sigma}_{22} - \boldsymbol{\Sigma}_{12}^\top \boldsymbol{\Sigma}_{11}^{-1} \boldsymbol{\Sigma}_{12}\right)^{-1} \\
&\quad \cdot \left[(\mathbf{x}_2 - \boldsymbol{\mu}_2) - \boldsymbol{\Sigma}_{12}^\top \boldsymbol{\Sigma}_{11}^{-1} (\mathbf{x}_1 - \boldsymbol{\mu}_1)\right]
\end{aligned}
$$

The last equal sign is due to the following equations for any vectors $\mathbf{u}$ and $\mathbf{v}$ and a symmetric matrix $\mathbf{A} = \mathbf{A}^\top$:

$$
\begin{aligned}
&\mathbf{u}^\top \mathbf{A} \mathbf{u} - 2\mathbf{u}^\top \mathbf{A} \mathbf{v} + \mathbf{v}^\top \mathbf{A} \mathbf{v} = \mathbf{u}^\top \mathbf{A} \mathbf{u} - \mathbf{u}^\top \mathbf{A} \mathbf{v} - \mathbf{u}^\top \mathbf{A} \mathbf{v} + \mathbf{v}^\top \mathbf{A} \mathbf{v} \\
=&\mathbf{u}^\top \mathbf{A}(\mathbf{u} - \mathbf{v}) - (\mathbf{u} - \mathbf{v})^\top \mathbf{A} \mathbf{v} = \mathbf{u}^\top \mathbf{A}(\mathbf{u} - \mathbf{v}) - \mathbf{v}^\top \mathbf{A}(\mathbf{u} - \mathbf{v}) \\
=&(\mathbf{u} - \mathbf{v})^\top \mathbf{A}(\mathbf{u} - \mathbf{v}) = (\mathbf{v} - \mathbf{u})^\top \mathbf{A}(\mathbf{v} - \mathbf{u})
\end{aligned}
$$

Marginalisation and Conditional of multivariate normal

We define
$\mathbf{b} \triangleq \boldsymbol{\mu}_2 + \boldsymbol{\Sigma}_{12}^{\top}\boldsymbol{\Sigma}_{11}^{-1}\left(\mathbf{x}_1 - \boldsymbol{\mu}_1\right)$

$$\mathbf{A} \triangleq \boldsymbol{\Sigma}_{22} - \boldsymbol{\Sigma}_{12}^{\top}\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12}$$

and

$$\begin{cases} Q_1\left(\mathbf{x}_1\right) \quad \triangleq \left(\mathbf{x}_1 - \boldsymbol{\mu}_1\right)^{\top}\boldsymbol{\Sigma}_1^{-1}\left(\mathbf{x}_1 - \boldsymbol{\mu}_1\right) \\ \mathbf{N} = \left[\left(\mathbf{x}_2 - \boldsymbol{\mu}_2\right) - \boldsymbol{\Sigma}_{12}^{\top}\boldsymbol{\Sigma}_{11}^{-1}\left(\mathbf{x}_1 - \boldsymbol{\mu}_1\right)\right] \\ Q_2\left(\mathbf{x}_1, \mathbf{x}_2\right) \triangleq \mathbf{N}^{\top}\left(\boldsymbol{\Sigma}_{22} - \boldsymbol{\Sigma}_{12}^{\top}\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12}\right)^{-1}\mathbf{N} \\ = \left(\mathbf{x}_2 - \mathbf{b}\right)^{\top}\mathbf{A}^{-1}\left(\mathbf{x}_2 - \mathbf{b}\right) \end{cases}$$

and get

$$Q\left(\mathbf{x}_1, \mathbf{x}_2\right) = Q_1\left(\mathbf{x}_1\right) + Q_2\left(\mathbf{x}_1, \mathbf{x}_2\right)$$

Marginalisation and Conditional of multivariate normal

Now the joint distribution can be written as:

$$
\begin{aligned}
f(\mathbf{x}) = f(\mathbf{x}_1, \mathbf{x}_2) &= \frac{1}{(2\pi)^{n/2} \det(\boldsymbol{\Sigma})^{1/2}} \exp\left[-\frac{1}{2} Q(\mathbf{x}_1, \mathbf{x}_2)\right] \\
&= \frac{1}{(2\pi)^{n/2} \det(\boldsymbol{\Sigma}_{11})^{1/2} \det(\boldsymbol{\Sigma}_{22} - \boldsymbol{\Sigma}_{12}^\top \boldsymbol{\Sigma}_{11}^{-1} \boldsymbol{\Sigma}_{12})^{1/2}} \exp\left[-\frac{1}{2} Q(\mathbf{x}_1, \mathbf{x}_2)\right] \\
&= \frac{1}{(2\pi)^{p/2} \det(\boldsymbol{\Sigma}_{11})^{1/2}} \exp\left[-\frac{1}{2} (\mathbf{x}_1 - \boldsymbol{\mu}_1)^\top \boldsymbol{\Sigma}_{11}^{-1} (\mathbf{x}_1 - \boldsymbol{\mu}_1)\right] \\
&\quad \times \frac{1}{(2\pi)^{q/2} \det(\mathbf{A})^{1/2}} \exp\left[-\frac{1}{2} (\mathbf{x}_2 - \mathbf{b})^\top \mathbf{A}^{-1} (\mathbf{x}_2 - \mathbf{b})\right] \\
&= \mathcal{N}(\boldsymbol{\mu}_1,\ \boldsymbol{\Sigma}_{11})_{\mathbf{x}_1}\ \mathcal{N}(\mathbf{b},\ \mathbf{A})_{\mathbf{x}_2}
\end{aligned}
$$

The third equal sign is due to Determinant of a partitioned symmetric matrix:

$$
\det(\boldsymbol{\Sigma}) = \det(\boldsymbol{\Sigma}_{11}) \det(\boldsymbol{\Sigma}_{22} - \boldsymbol{\Sigma}_{12}^\top \boldsymbol{\Sigma}_{11}^{-1} \boldsymbol{\Sigma}_{12})
$$

Marginalisation and Conditional of multivariate normal

The marginal distribution of $\mathbf{x}_1$ is

$$
\begin{aligned}
f_1(\mathbf{x}_1) &= \int f(\mathbf{x}_1, \mathbf{x}_2)\, d\mathbf{x}_2 \\
&= \frac{1}{(2\pi)^{p/2} \det(\boldsymbol{\Sigma}_{11})^{1/2}} \exp\left[-\frac{1}{2}(\mathbf{x}_1 - \boldsymbol{\mu}_1)^\top \boldsymbol{\Sigma}_{11}^{-1} (\mathbf{x}_1 - \boldsymbol{\mu}_1)\right]
\end{aligned}
$$

and the conditional distribution of $\mathbf{x}_2$ given $\mathbf{x}_1$ is

$$
\begin{aligned}
f_{2|1}(\mathbf{x}_2|\mathbf{x}_1) &= \frac{f(\mathbf{x}_1, \mathbf{x}_2)}{f(\mathbf{x}_1)} \\
&= \frac{1}{(2\pi)^{q/2} \det(\mathbf{A})^{1/2}} \exp\left[-\frac{1}{2}(\mathbf{x}_2 - \mathbf{b})^\top \mathbf{A}^{-1} (\mathbf{x}_2 - \mathbf{b})\right]
\end{aligned}
$$

Marginalisation and Conditional of multivariate normal

with

$$\mathbf{b} = \boldsymbol{\mu}_2 + \boldsymbol{\Sigma}_{12}^{\top}\boldsymbol{\Sigma}_{11}^{-1}(\mathbf{x}_1 - \boldsymbol{\mu}_1)$$

$$\mathbf{A} = \boldsymbol{\Sigma}_{22} - \boldsymbol{\Sigma}_{12}^{\top}\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12}$$