K-Nearest Neighbors Approximation

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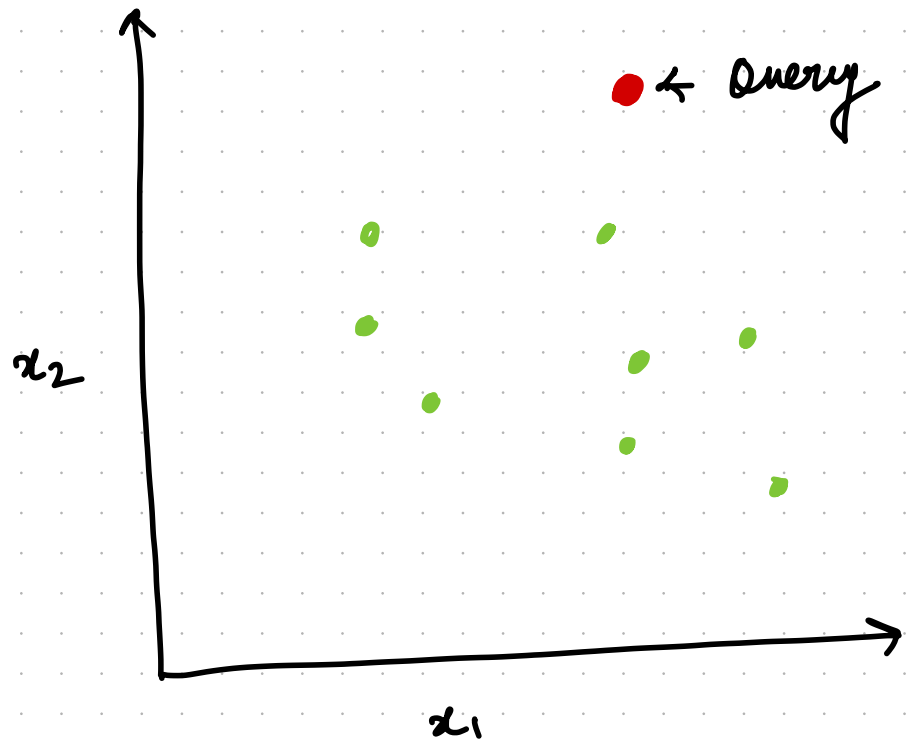
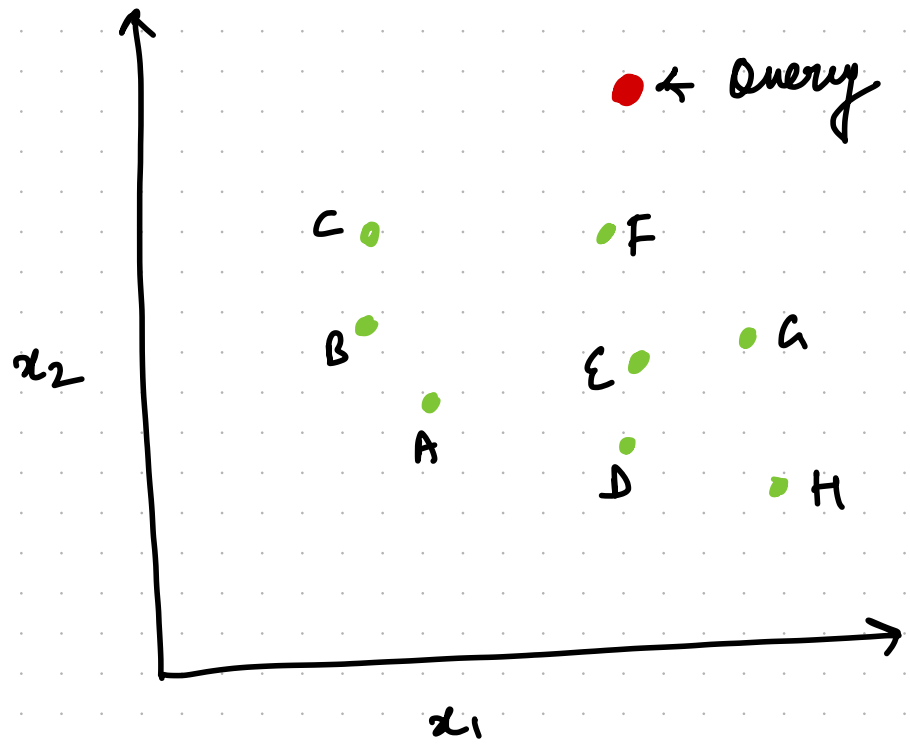
July 29, 2025

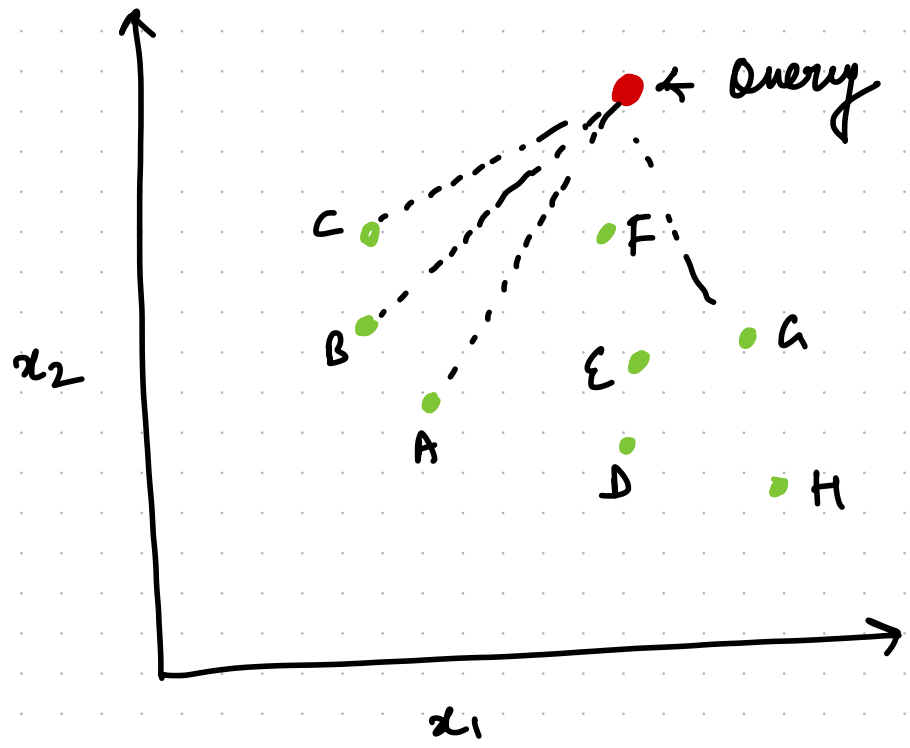


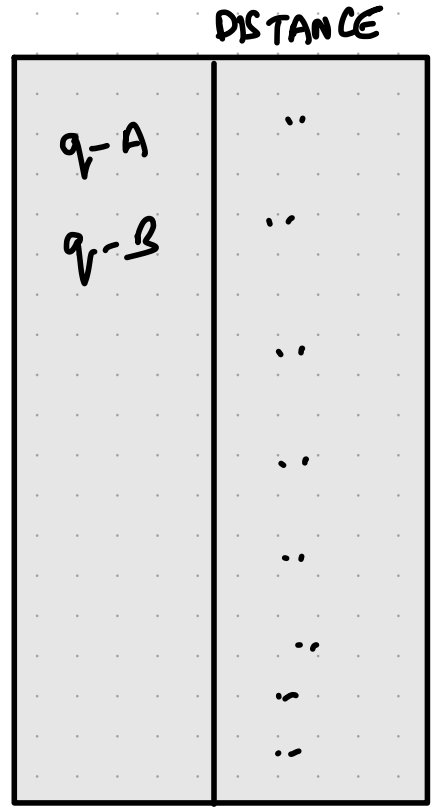
Find 1-NN for query point $'q' \in R^D$



Find 1-NN for query point "q" $\in R^D$

TRAIN SET is $X \in R^{N \times D}$





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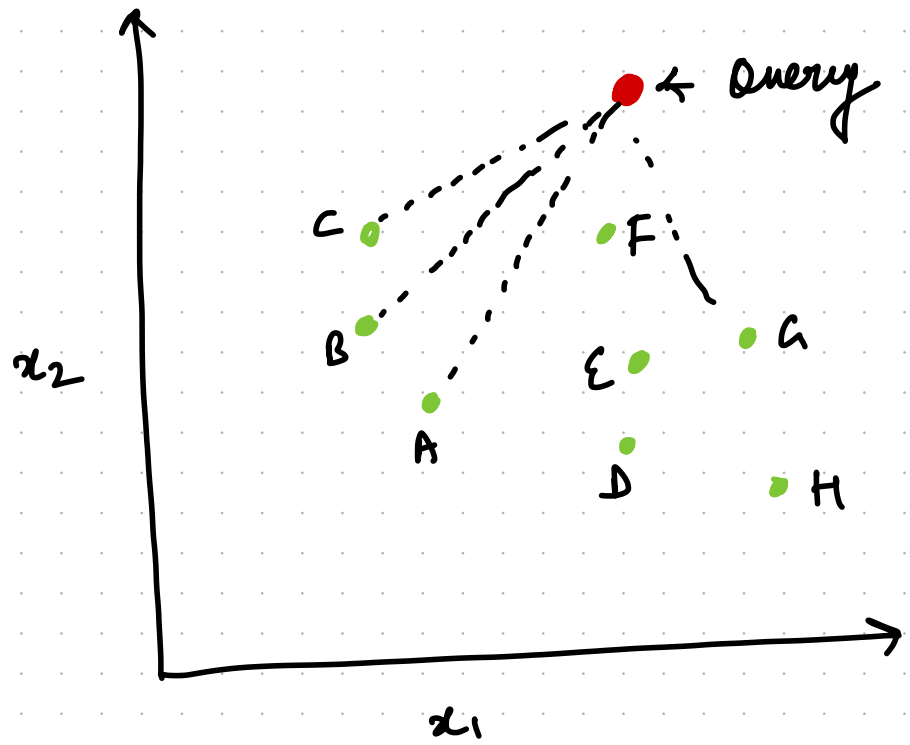
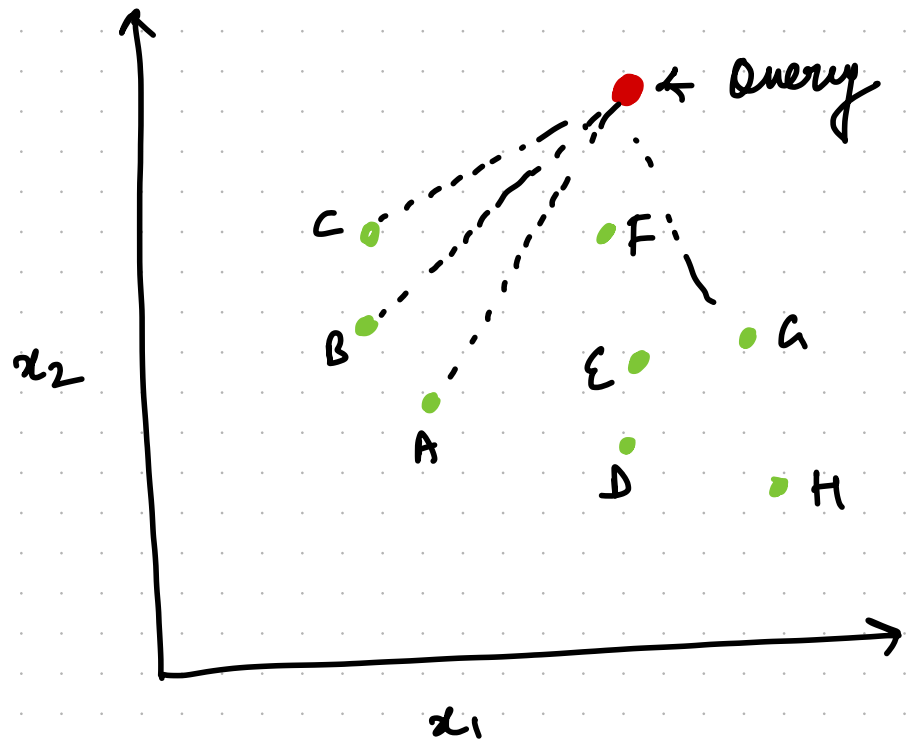


	DISTANCE
q-A	..
q-B	..
	..
	..
	..
	..
	..
	..

STEPS FOR FINDING $D(q, A)$



	DISTANCE
q-A	..
q-B	..
	..
	..
	..
	..
	..
	..

STEPS FOR FINDING $D(q, A)$

$$D(q,A) = \sqrt{(q_1 - A_1)^2 + \dots (q_D - A_D)^2}$$

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SUBTRACTIONS

MULTIPLICATIONS

ADDITIONS

SQRT

STEPS FOR FINDING $D(q, A)$

$$D(q, A) = \sqrt{(q_1 - A_1)^2 + \dots (q_D - A_D)^2}$$

\# SUBTRACTIONS D

\# MULTIPLICATIONS D

\# ADDITIONS D

\# SQRT 1

STEPS FOR FINDING $D(q, A)$

$$D(q,A) = \sqrt{(q_1 - A_1)^2 + \dots (q_D - A_D)^2}$$

\# SUBTRACTIONS D

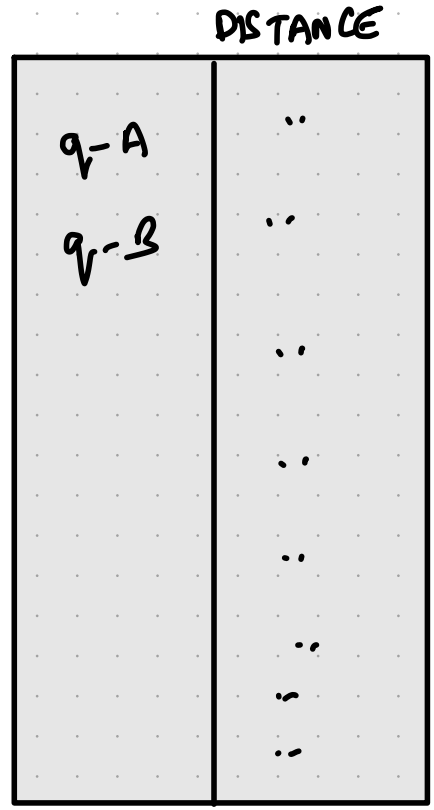
\# MULTIPLICATIONS D

\# ADDITIONS D

\# SQRT 1

Total time for $D(q,A)$ or DISTANCE BTW 1 PAIR $= O(D)$

Q) Time required for creating table



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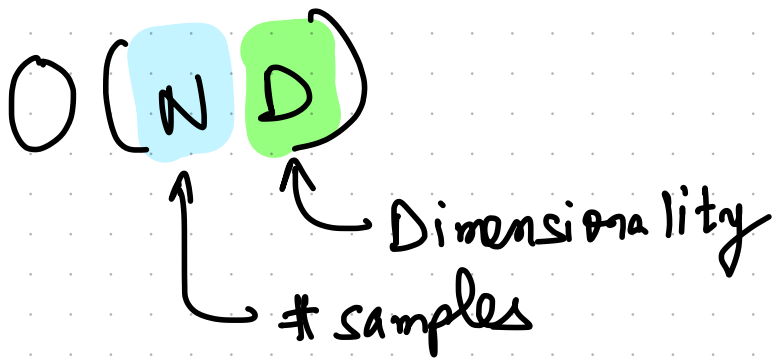
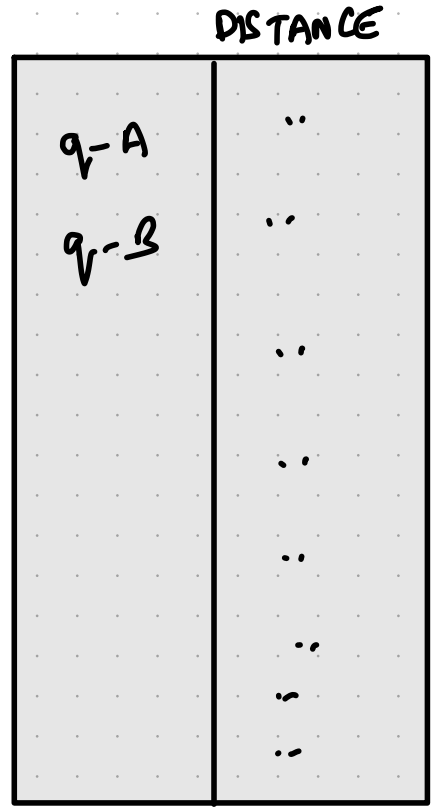




Q) Time required for finding 1-NN?

	DISTANCE
q-A	20
q-B	30
.	2
.	8
.	9
.	..
	34

Q) Time required for finding 1-NN?

$O(N)$: Linear search

	DISTANCE
q-A	20
q-B	30
.	2
.	8
.	9
.	..
	34

Q) Overall time complexity

$O(ND)$

↑ Linear in # samples (sometimes millions of samples)

	DISTANCE
q-A	20
q-B	30
.	2
.	8
.	9
.	..
	34

Goal:

Reduce $O(ND)$

Target

How?

Goal:

Reduce $O(ND)$

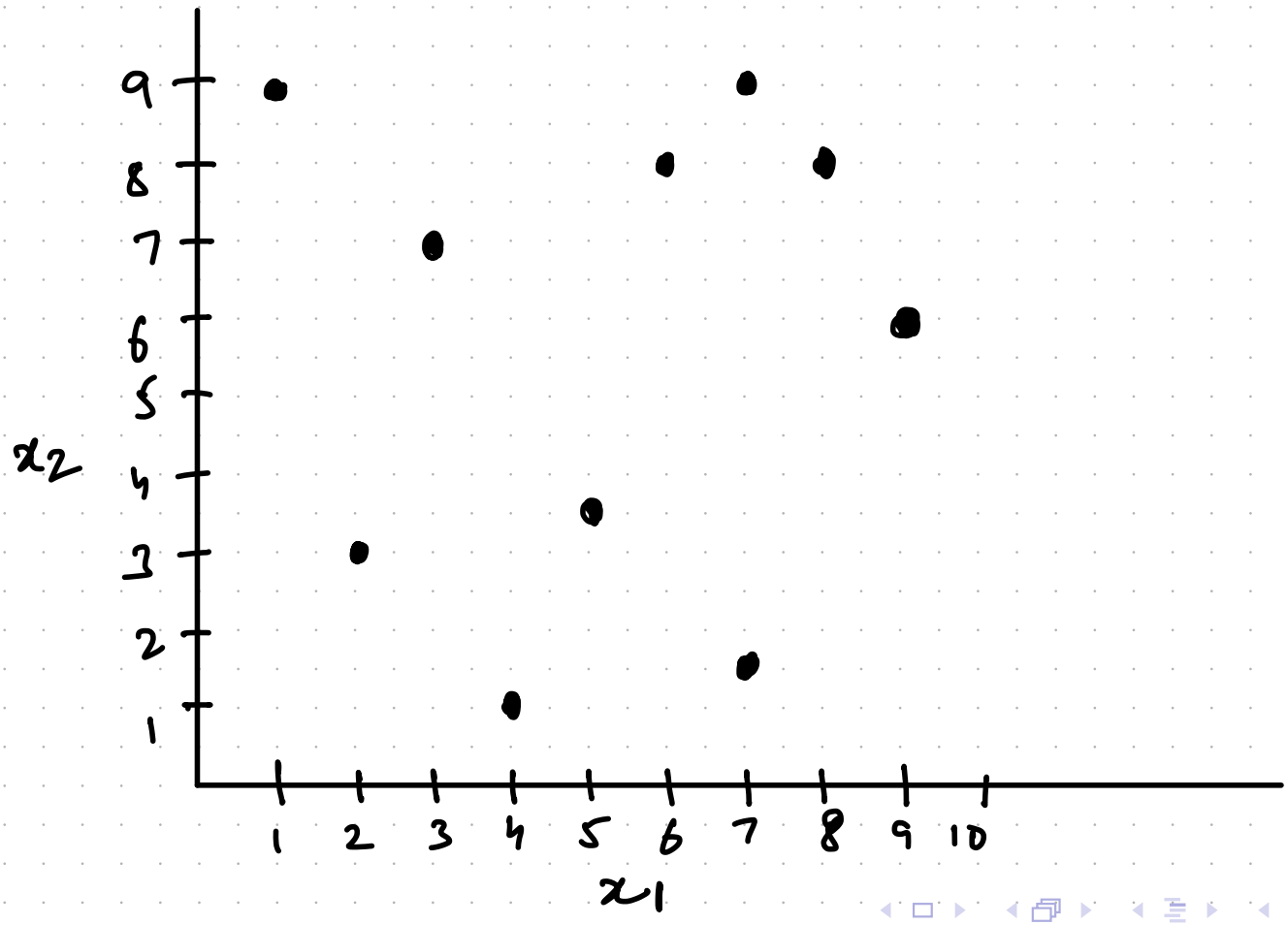
↑ Target

How? (Hints)

- Decision trees
- Search for subset of examples
- Current algorithm does nothing at training time.

K-D trees (Victor Lavarenta slides)

- $X = \{(1,9), (2,3), (4,1), (3,7), (5,4), (6,8), (7,2), (8,8), (7,9), (9,6)\}$



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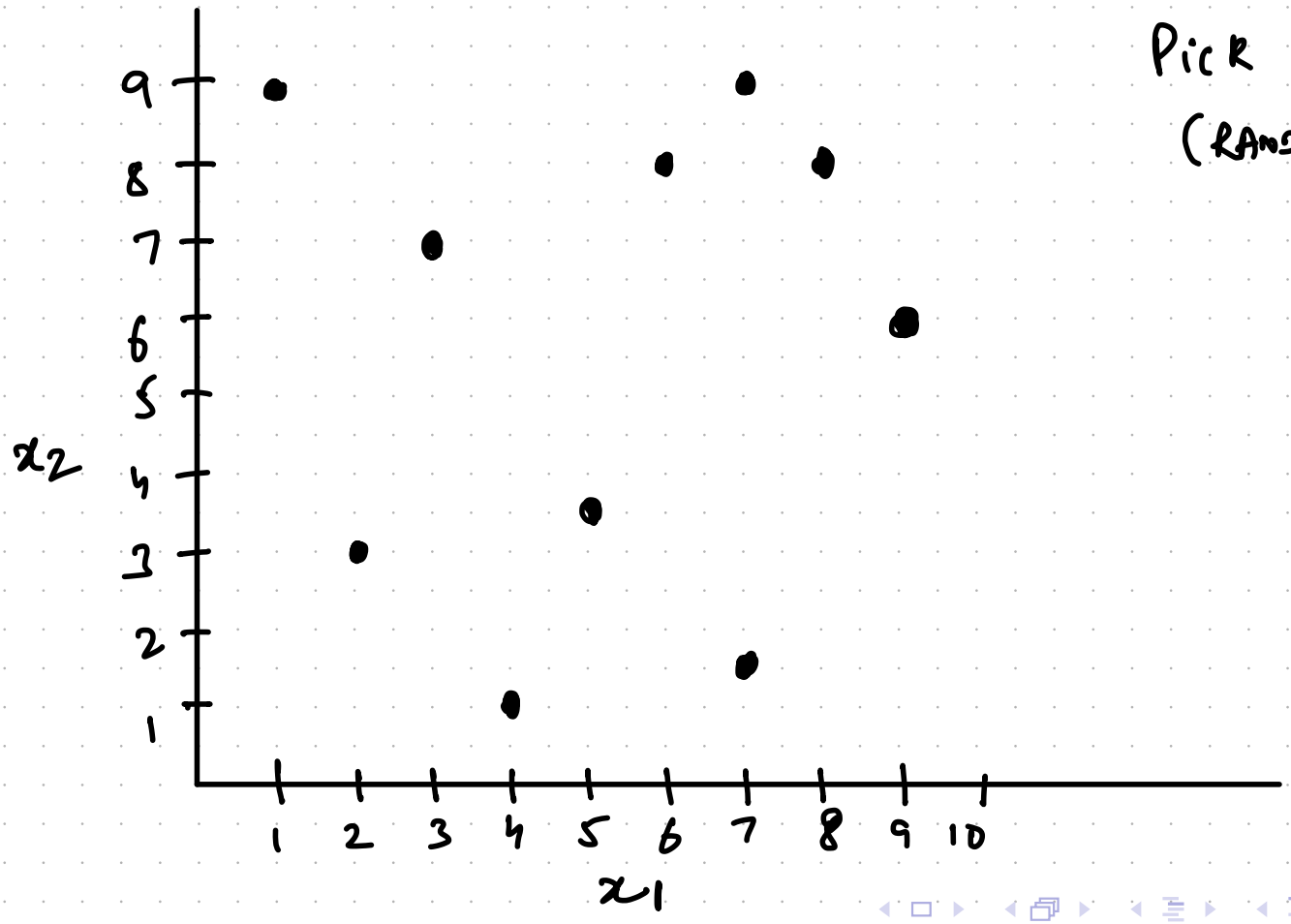

Pick dim x_1 (RANDOM{x_1, x_2})

K-D trees (Victor Lavarenko slides)

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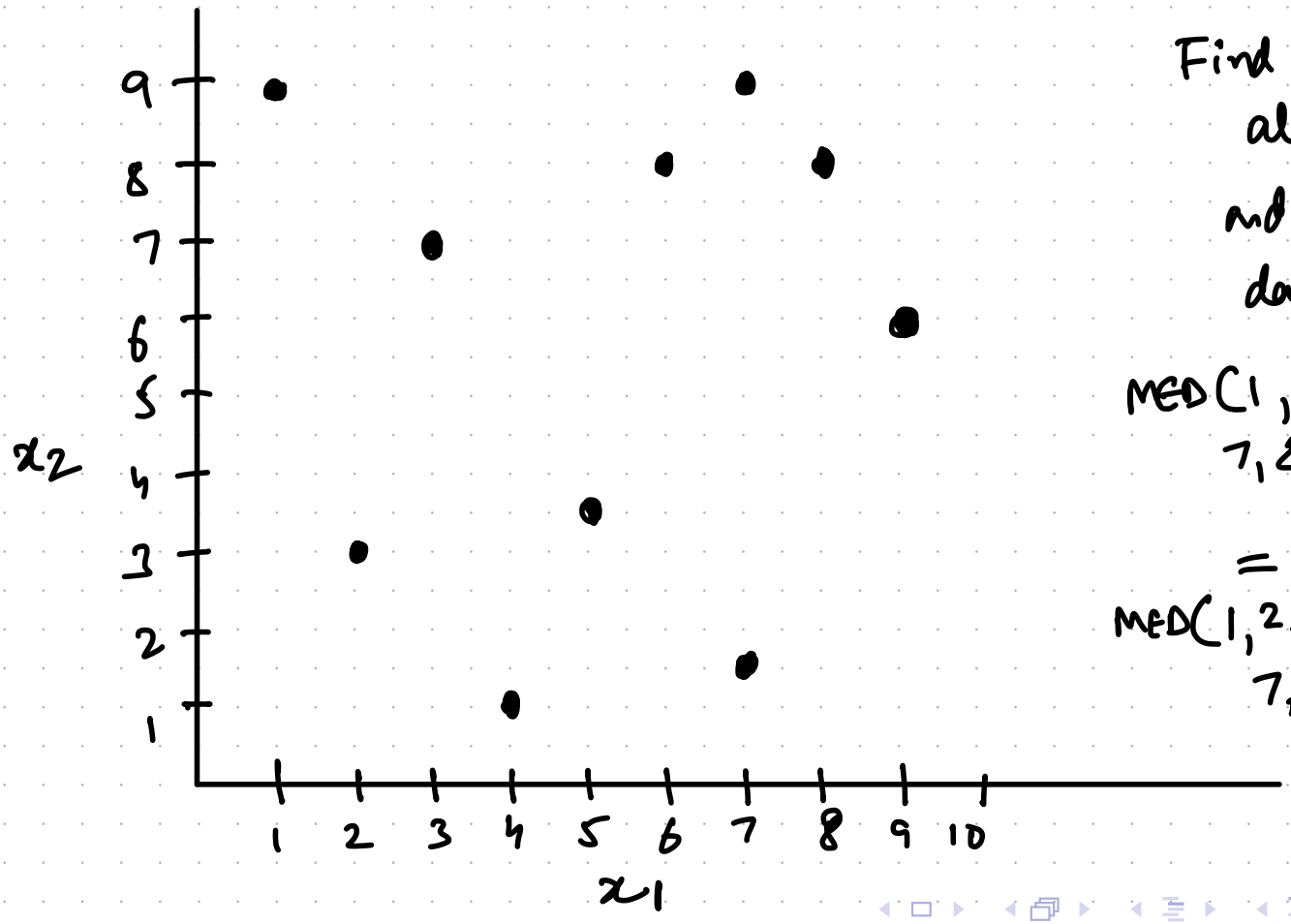

Find median along x_1 and split data

MED(1, 2, 4, 3, 5, 6, 7, 8, 7, 9)

=

MED(1, 2, 3, 4, 5, 6, 7, 7, 8, 9)

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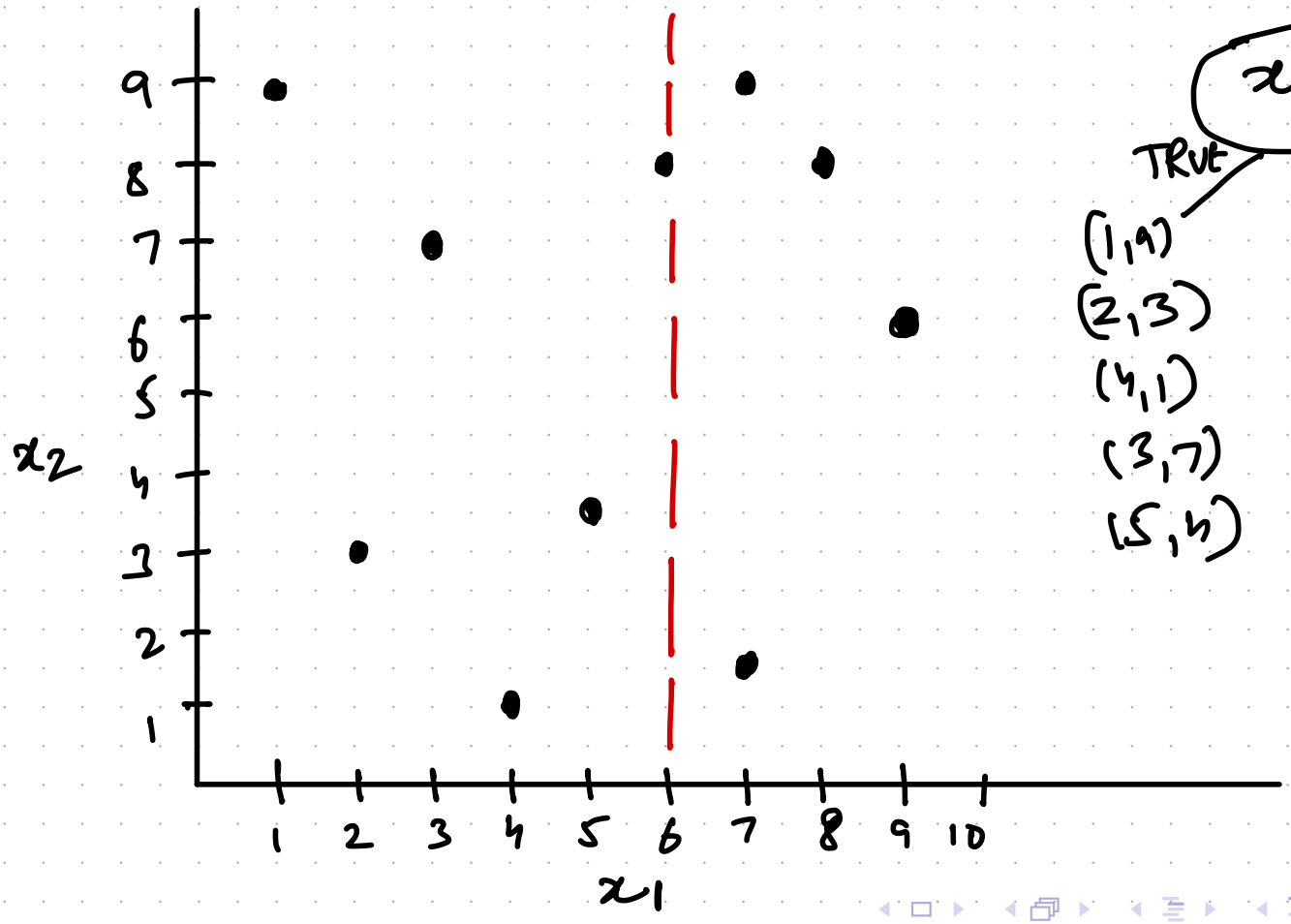

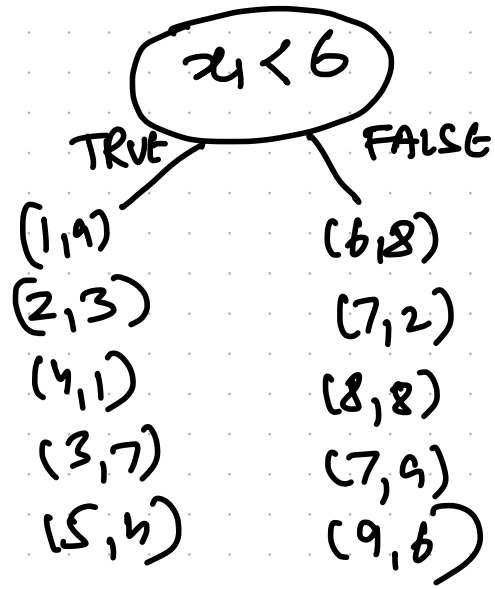

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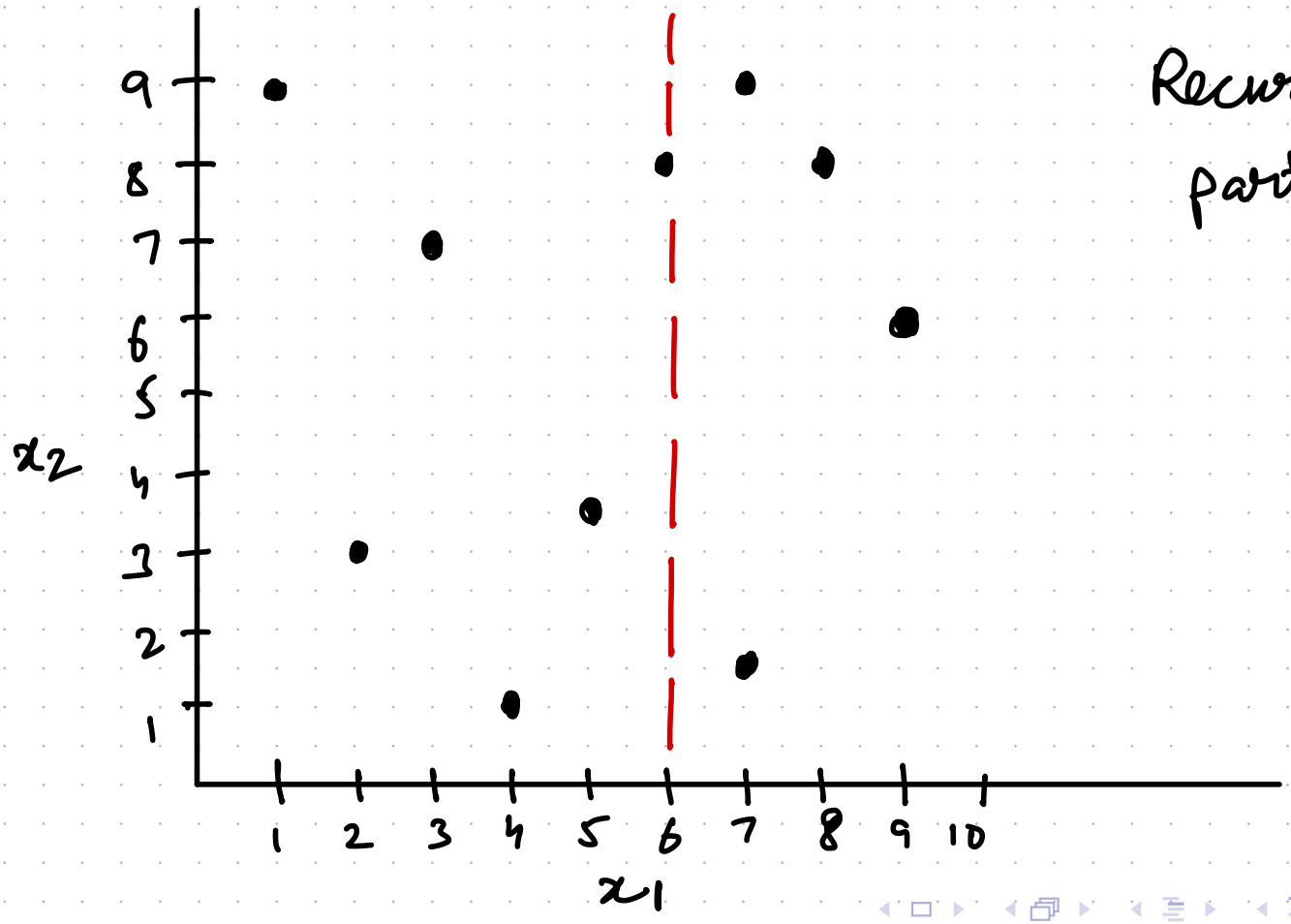

Recursively partition

K-D trees (Victor Lavarenko slides)

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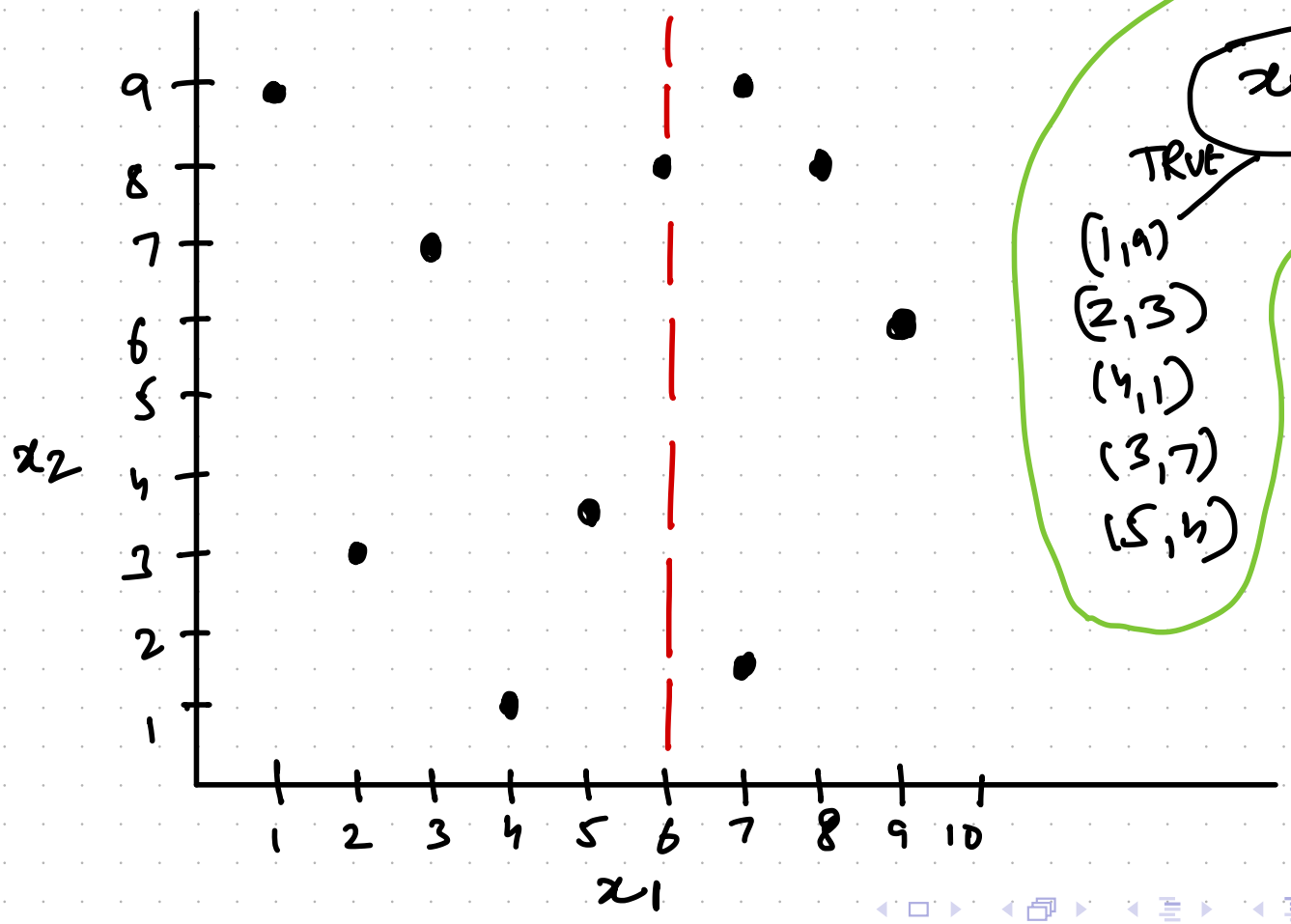

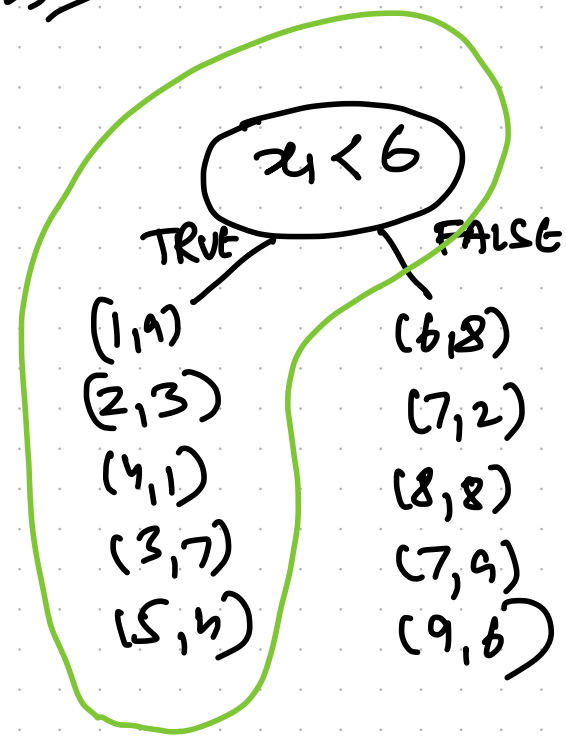

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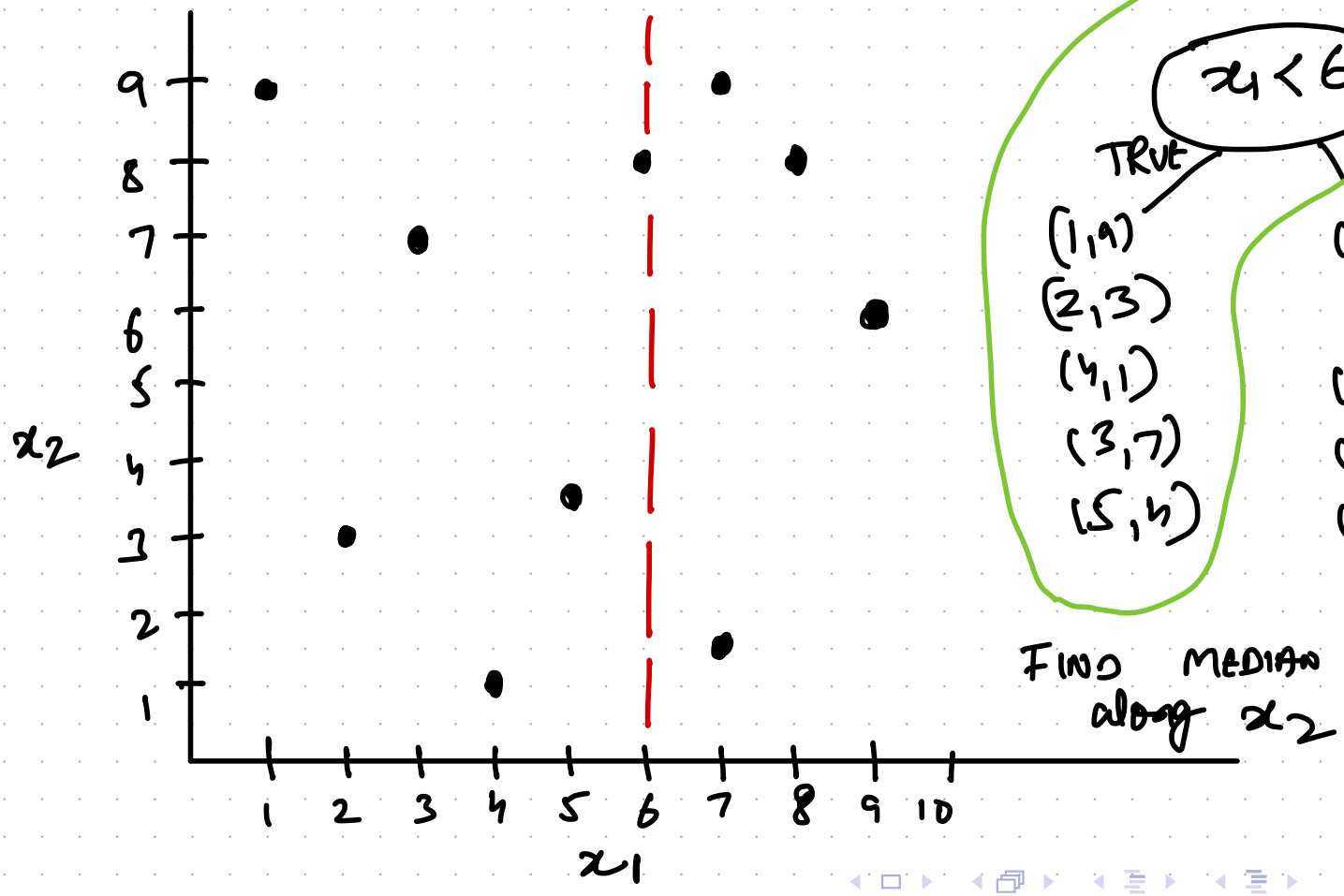

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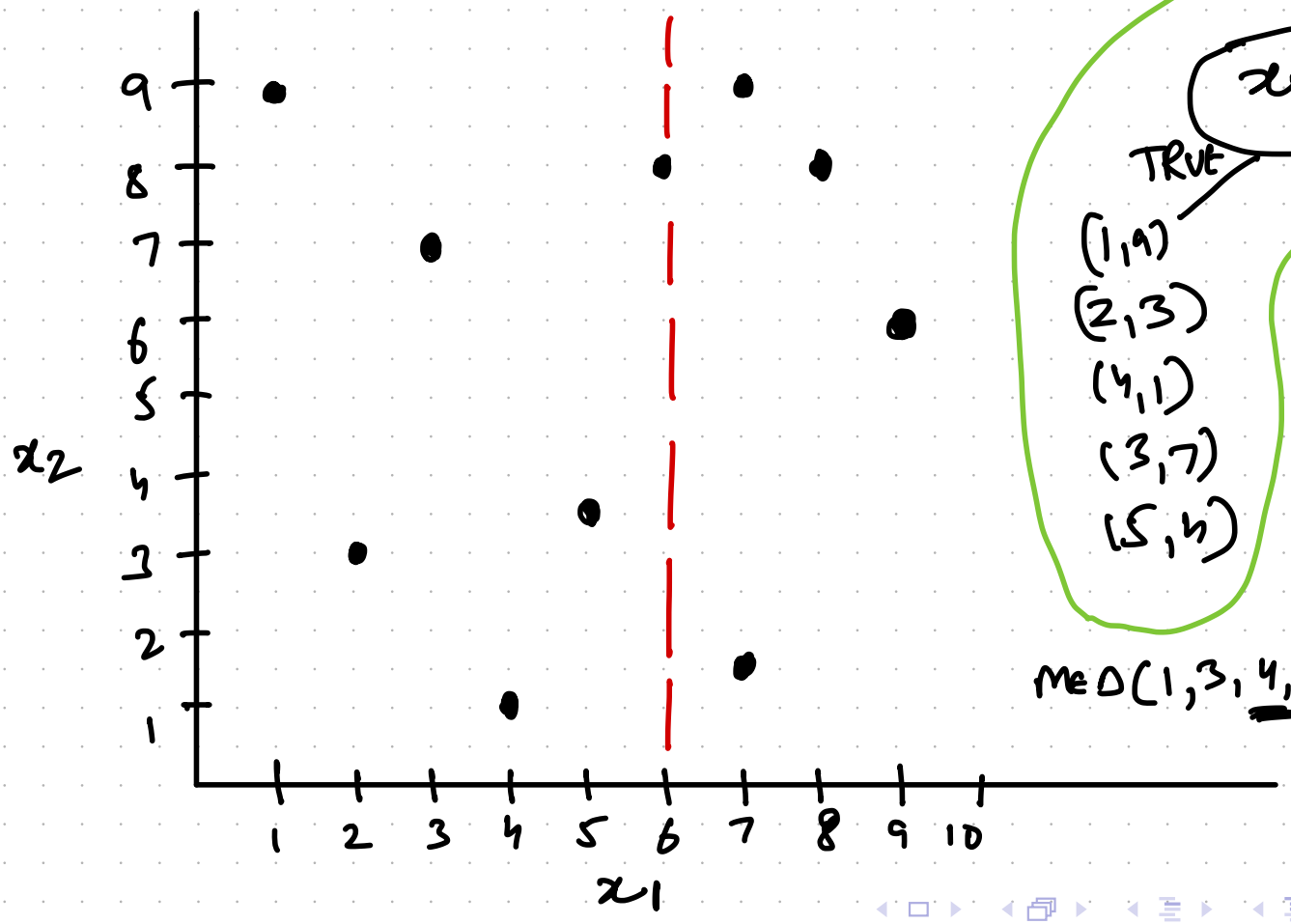

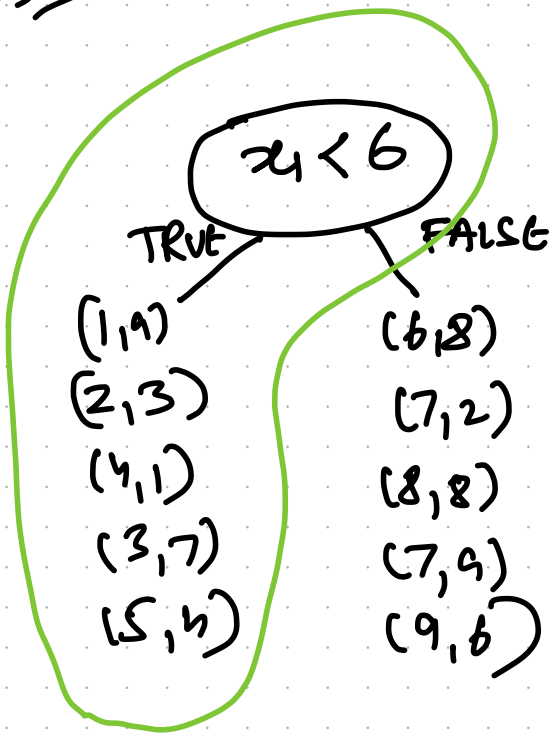

MED(1,3,4,7,9)

K-D trees (Victor Lavarenko slides)

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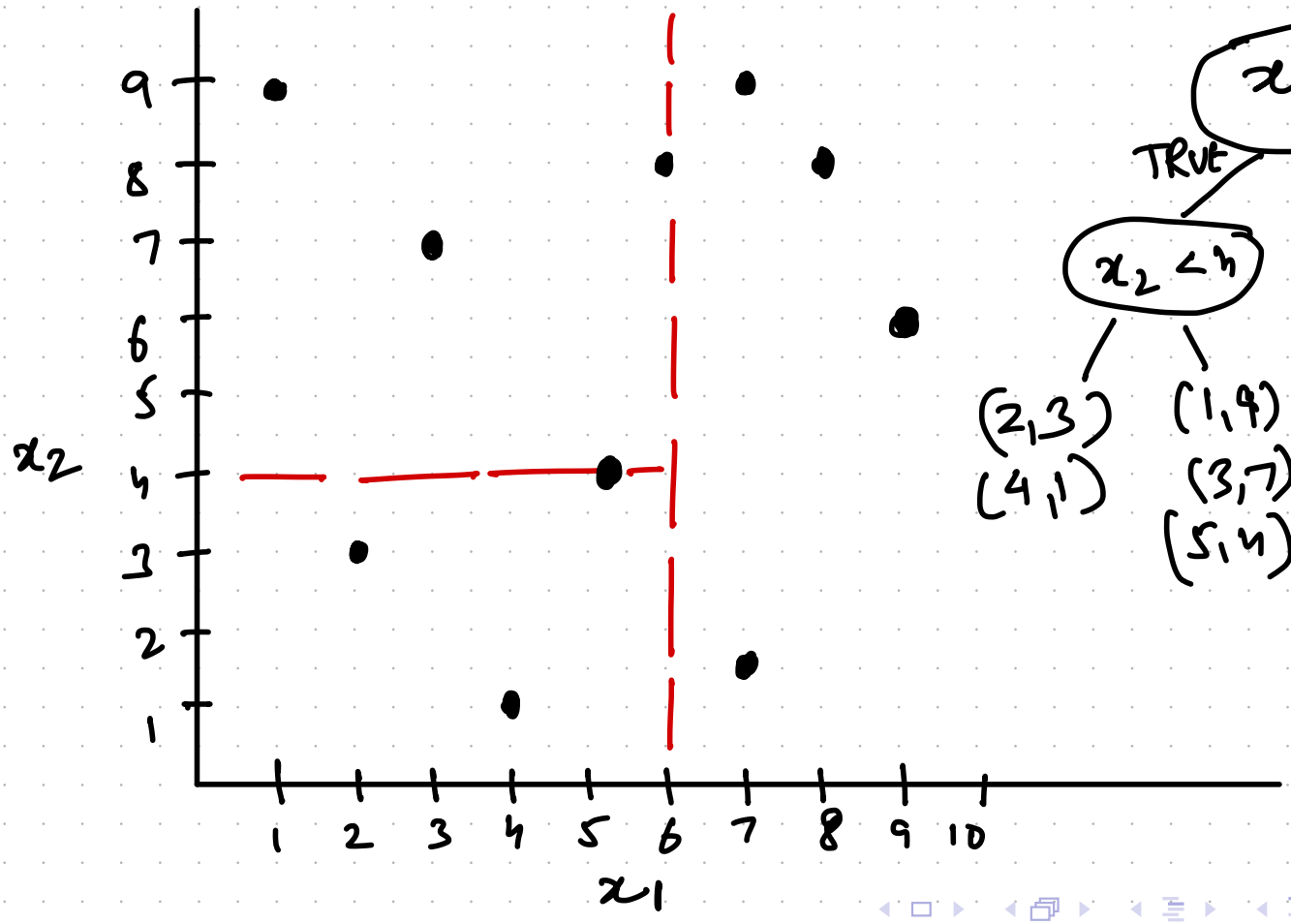

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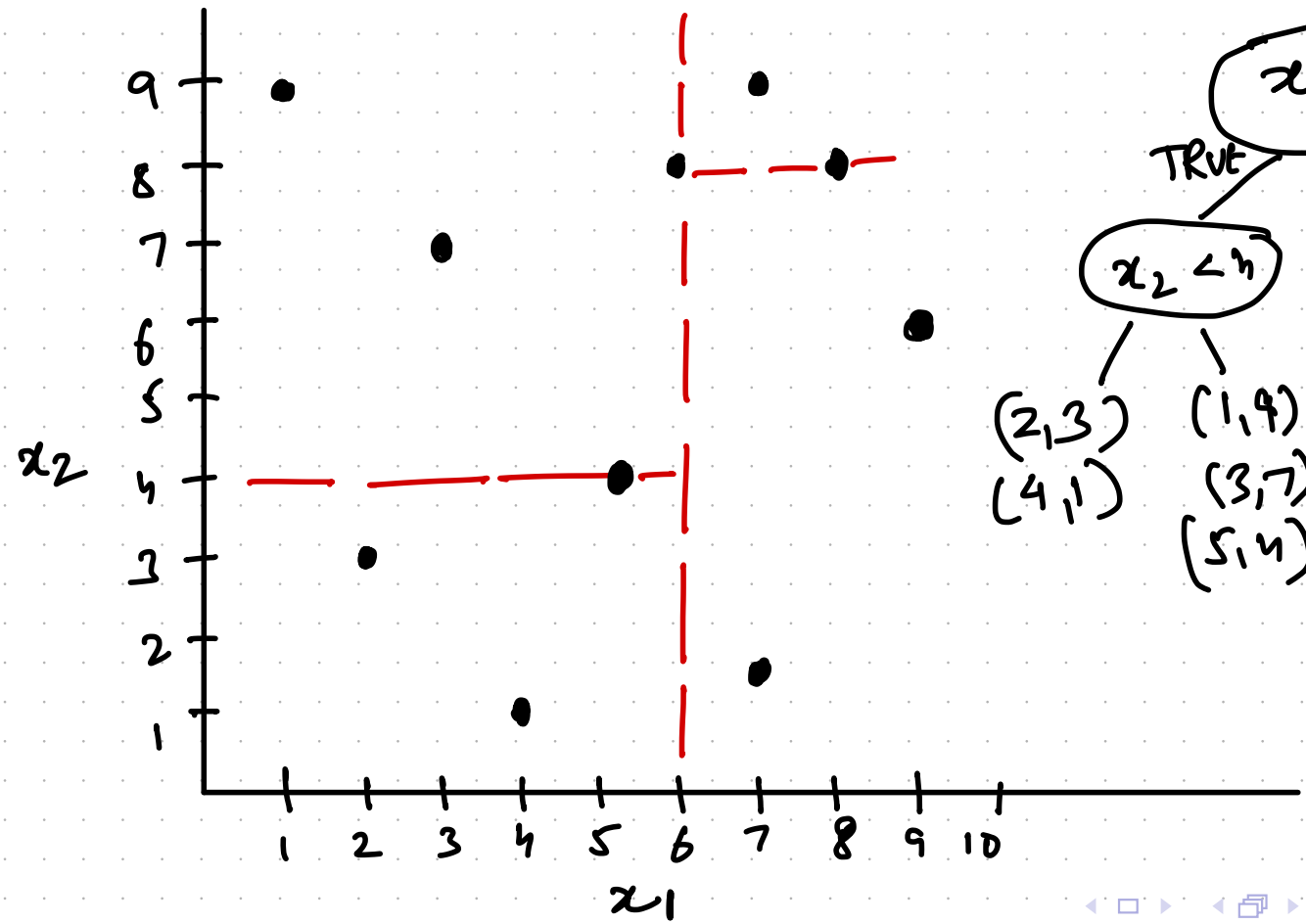
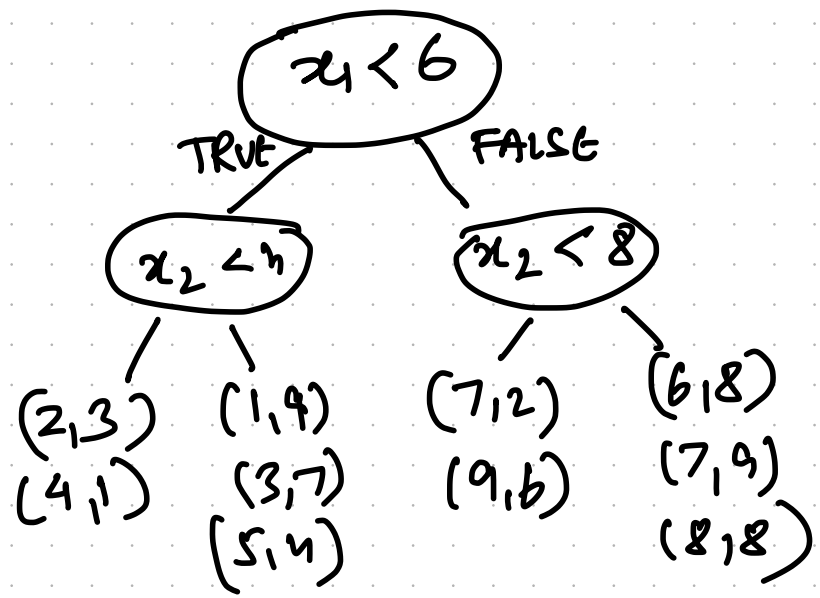
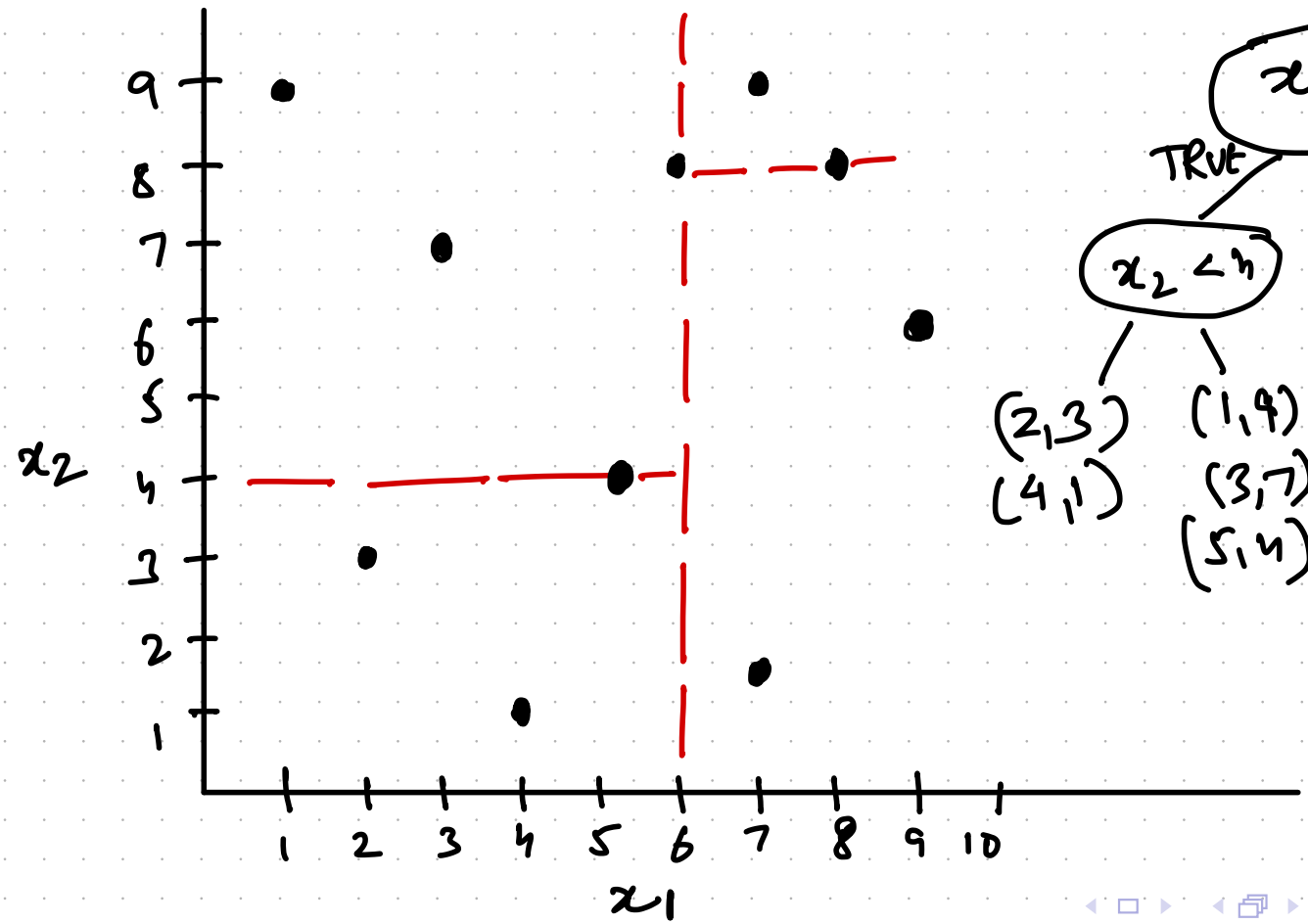
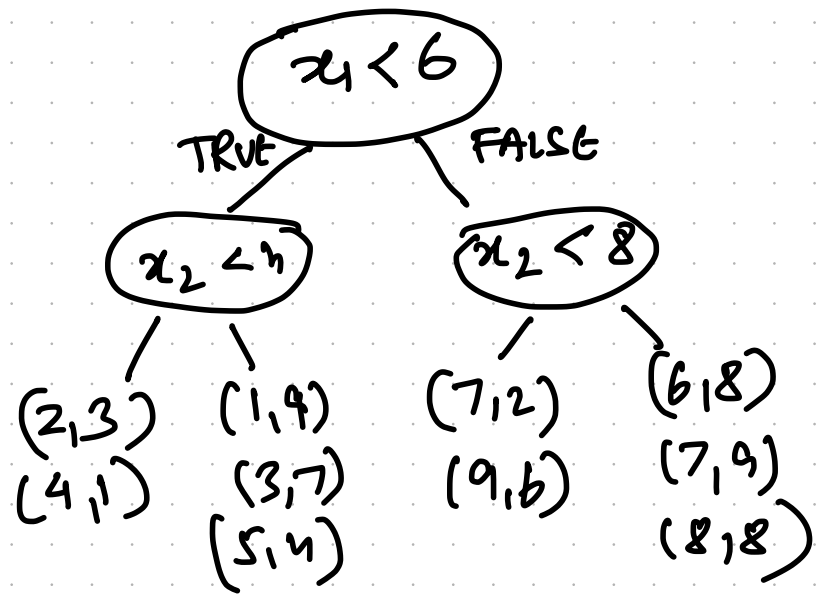




K-D trees (Victor Lavarenko slides)

- query pt (7, 4)





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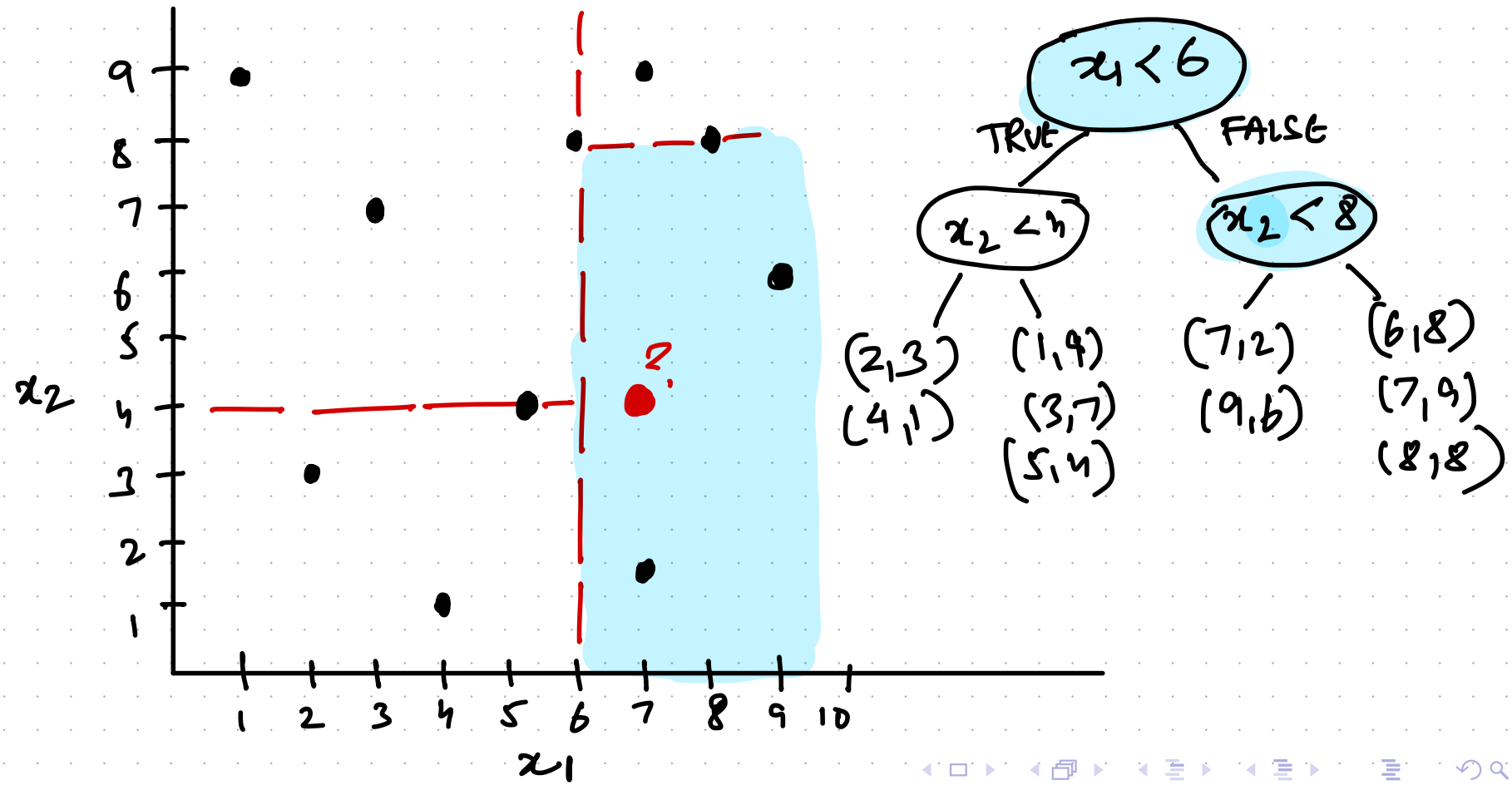
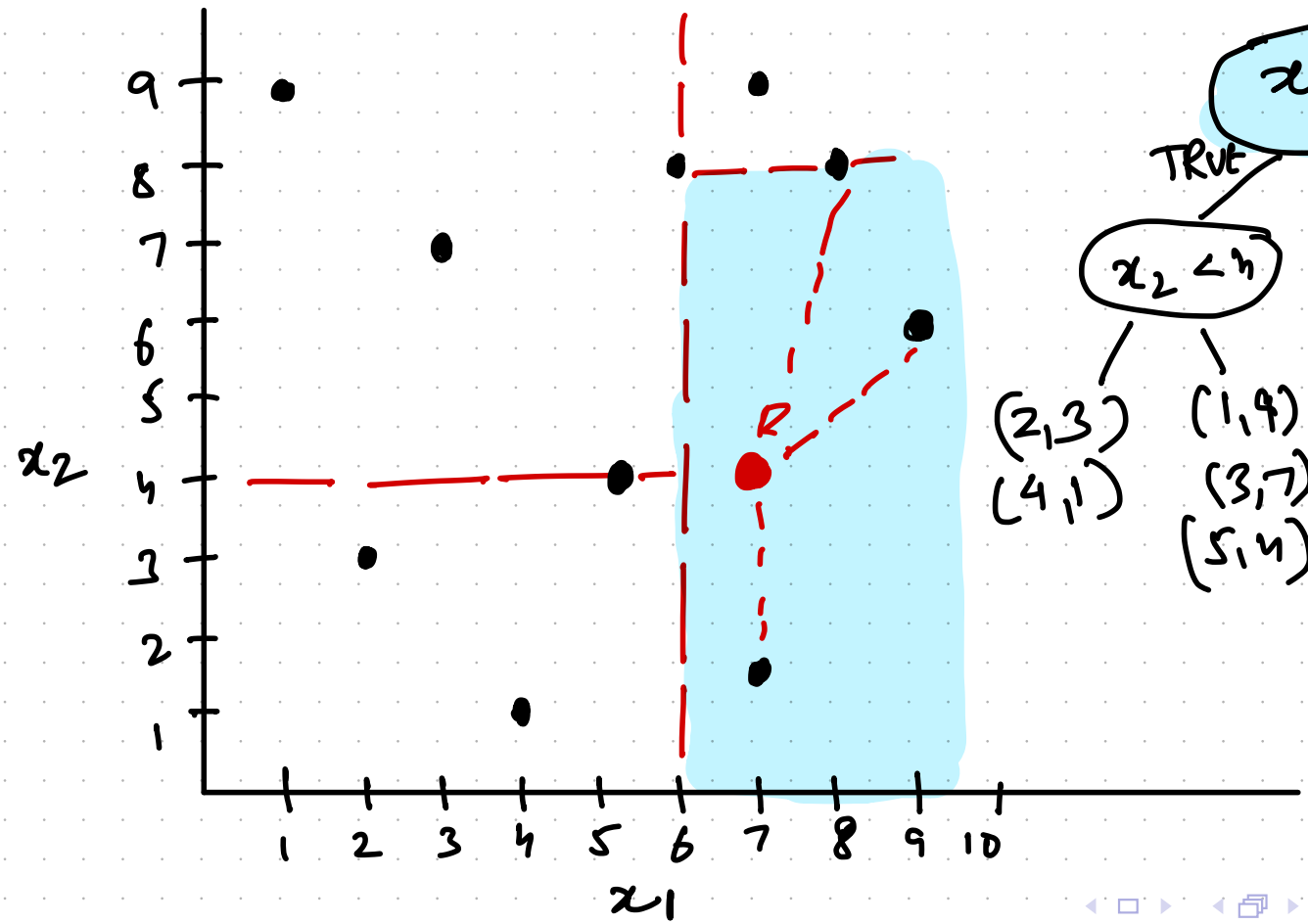
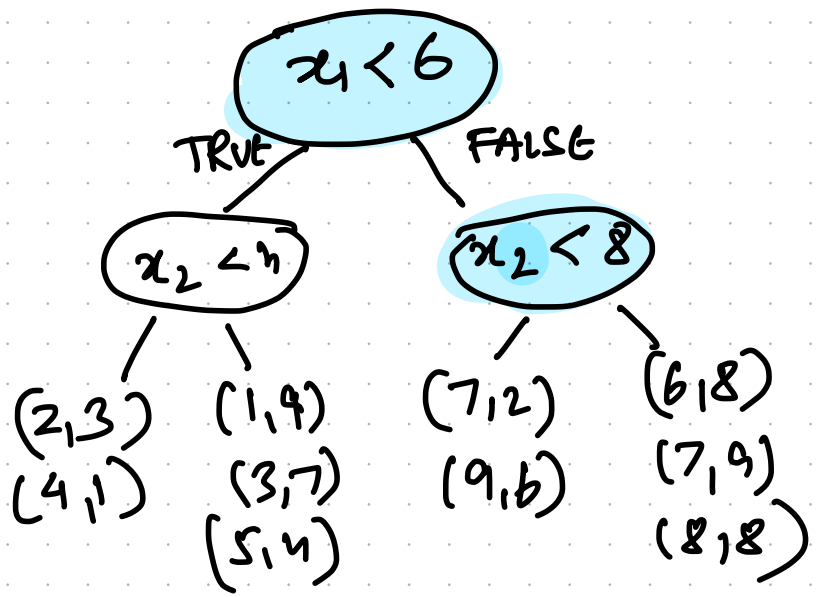


K-D trees (Victor Lavarenko slides)

- query pt (7, 4)

FOR finding NNs, look in subspace



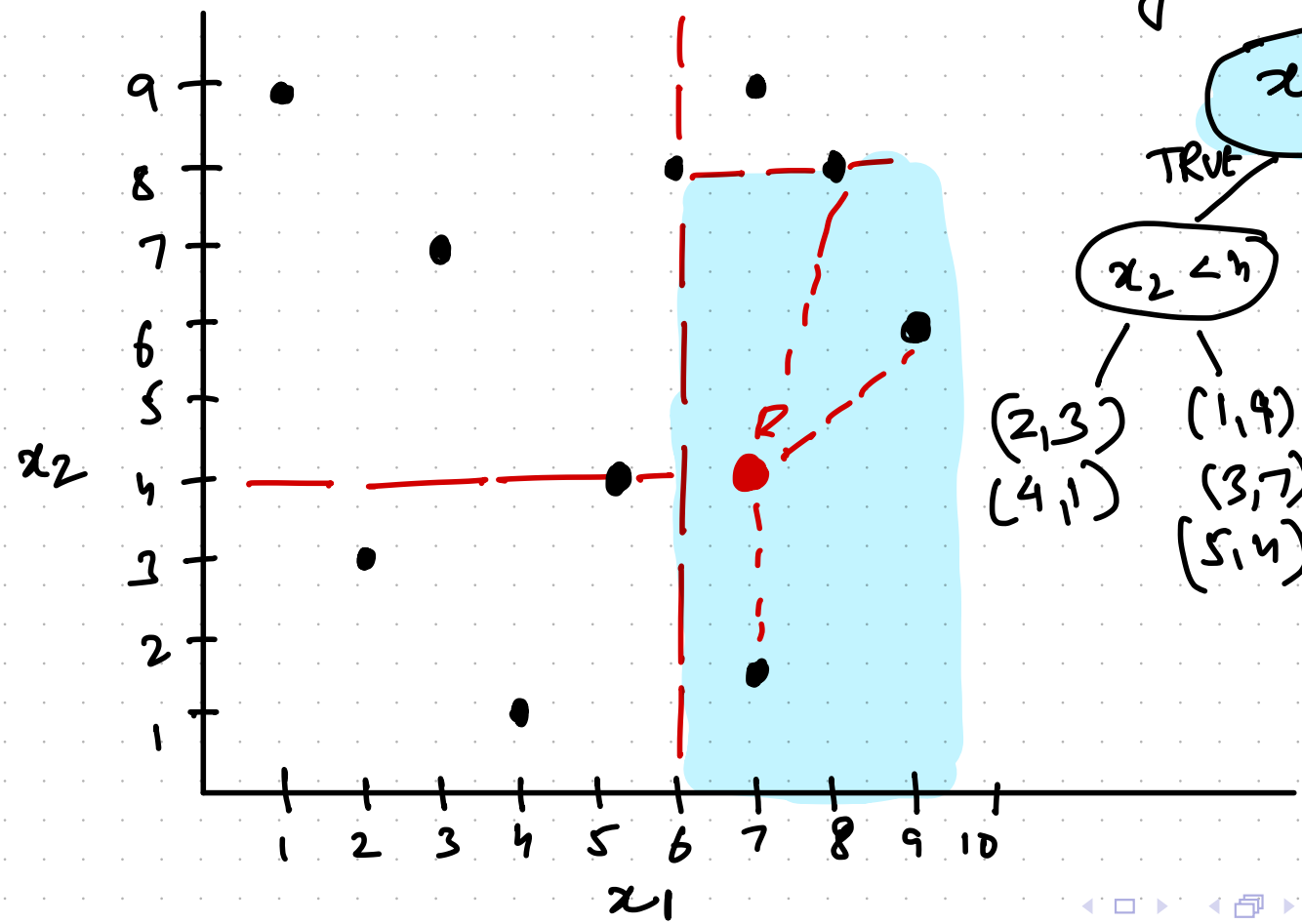


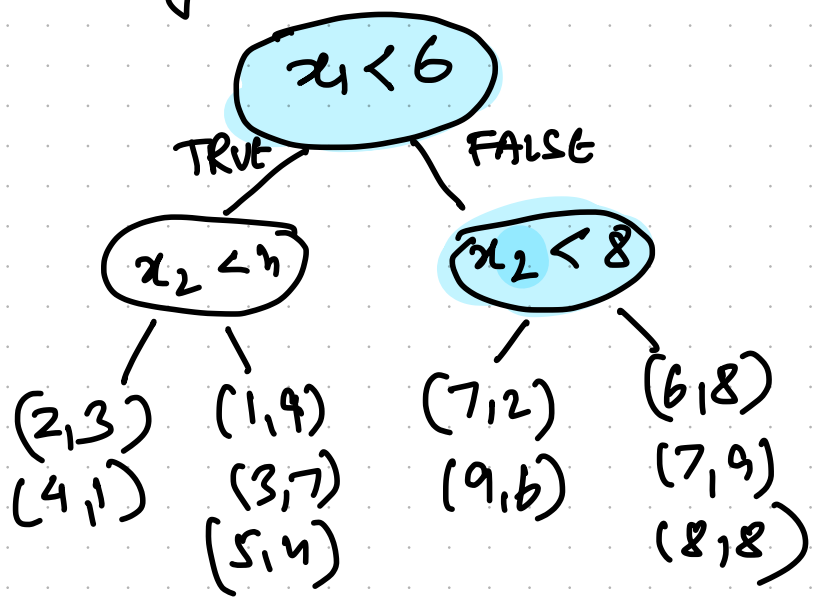
K-D trees (Victor Lavrenko slides)

- Query pt (7, 4)

FOR finding NNs, look in subspace

✔ Way lesser comparisons





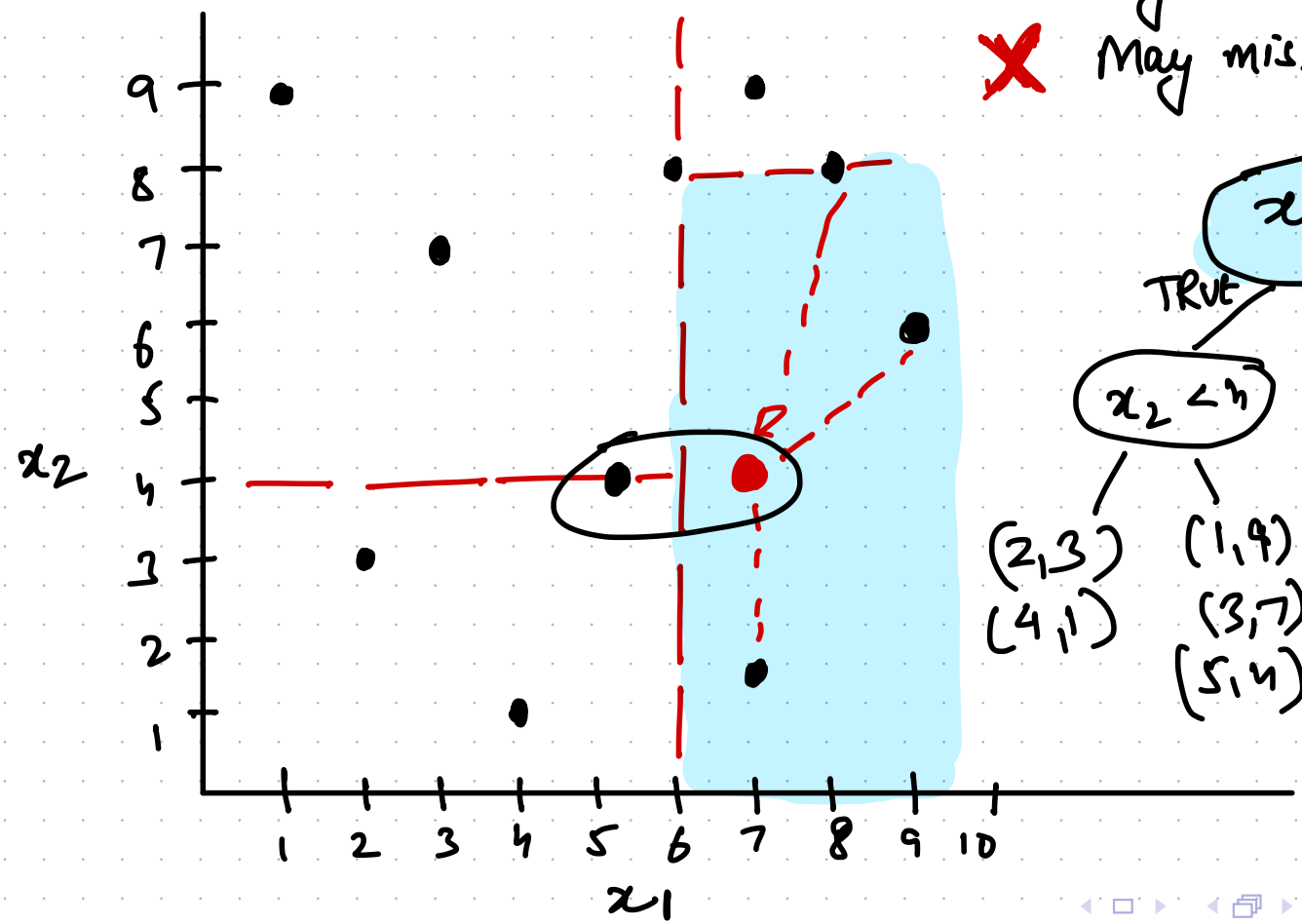
K-D trees (Victor Lavrenko slides)

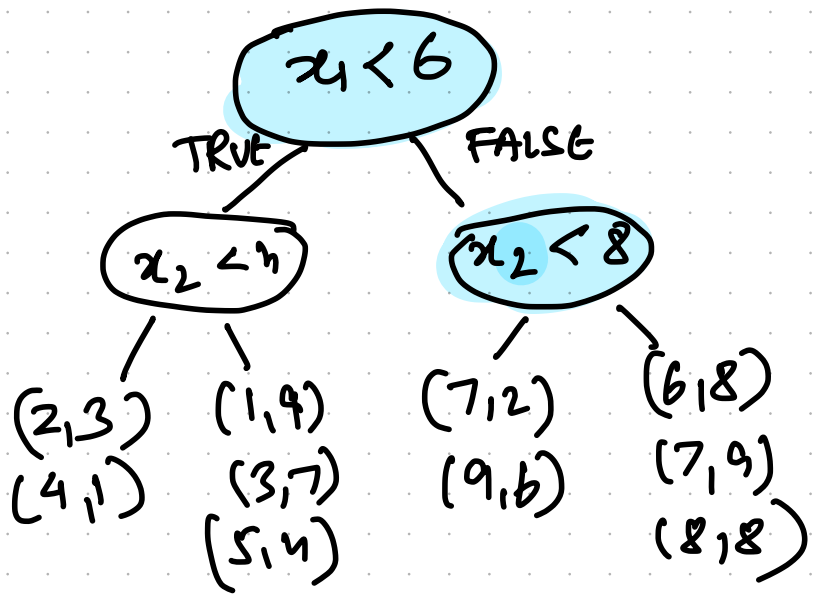
- Query pt (7, 4)

FOR finding NNs, look in subspace

✔ Way lesser comparisons

✘ May miss closest neighbors





KD trees (time complexity)

TRAINING TIME (Ref: Wikipedia KD trees)

KD trees (time complexity)

TRAINING Time

samples in subtrees

KD trees (time complexity)

TRAINING Time

samples in subtrees

Depth 0

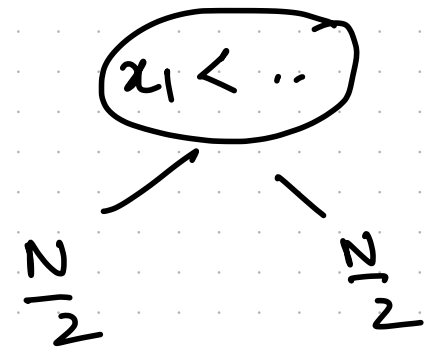
|

N

KD trees (time complexity)

TRAINING Time

samples in subtrees

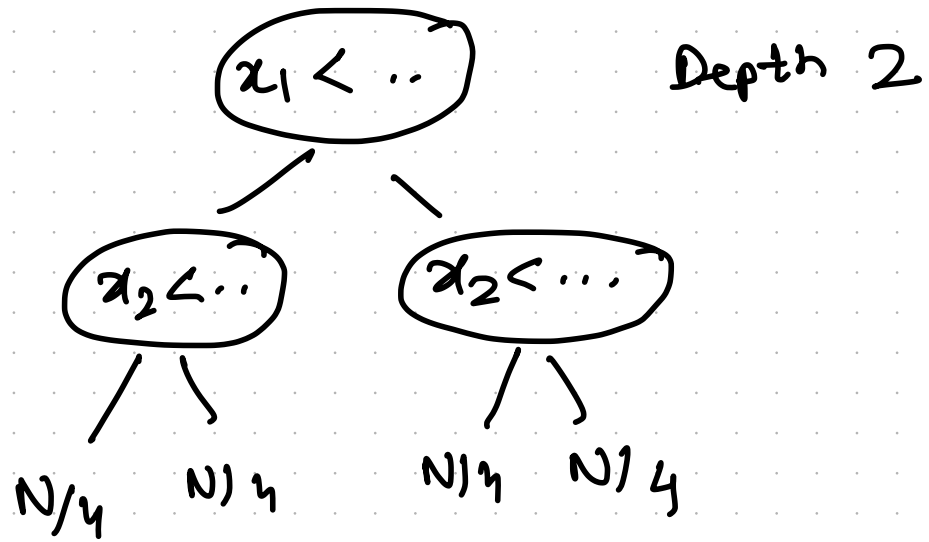


Depth ?

KD trees (time complexity)

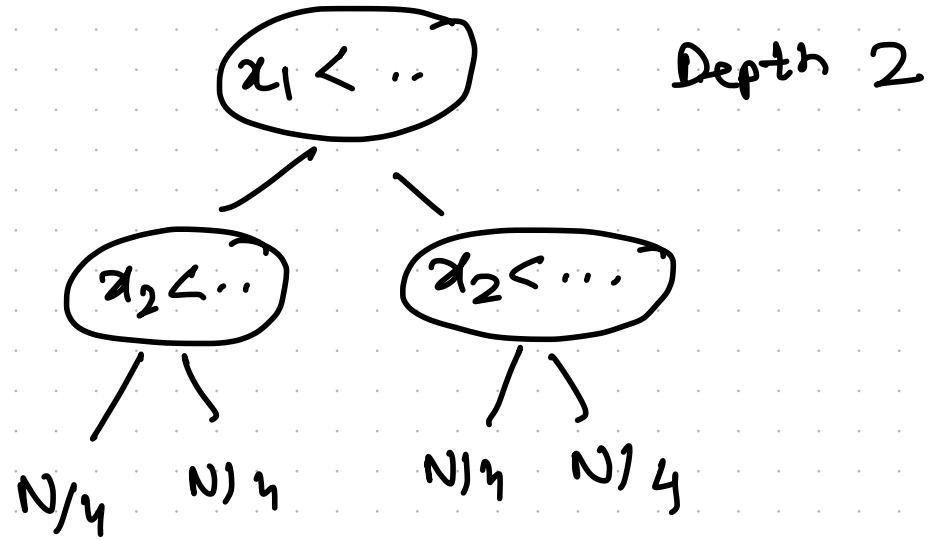
TRAINING Time

samples in subtrees



KD trees (time complexity)

TRAINING Time



Depth $O(\log_2 N)$

KD trees (time complexity)

TRAINING TIME

- We have $O(\log_2 N)$ levels.
- For each level,
  - Sort "N" examples to find median.
    - $O(N \log_2 N)$ time
- Total time to build data structure $O(N \log^2 N)$

KD trees (time complexity)

* let's assume 'S' (≈40) is leaf size
  (# samples at which we stop partitioning)

* what is time complexity of query?

KD trees (time complexity)

* let's assume 'S' (≈40) is leaf size
  (# samples at which we stop partitioning)

* what is time complexity of query?
  — How many levels to read of?

KD trees (time complexity)

* let's assume 'S' (≈40) is leaf size
  (# samples at which we stop partitioning)

* what is time complexity of query?
  - How many levels to reach leaf?
    - $O(\log N)$
  - How many computat°s per level?

KD trees (time complexity)

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  - How many levels to reach leaf?
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  - How many comptat's per level?
    - One 1-D comparison (e.g. $x_1 < \theta$)
  - How many comptat's at leaf?
    - $O(D * S)$

KD trees (time complexity)

* what is time complexity of query?

  - How many levels to reach leaf?
    - $O(\log N)$
  - How many comptat's per level?
    - One 1-D comparison (e.g. $x_1 < \delta$)
  - How many comptat's at leaf?
    - $O(D * S)$

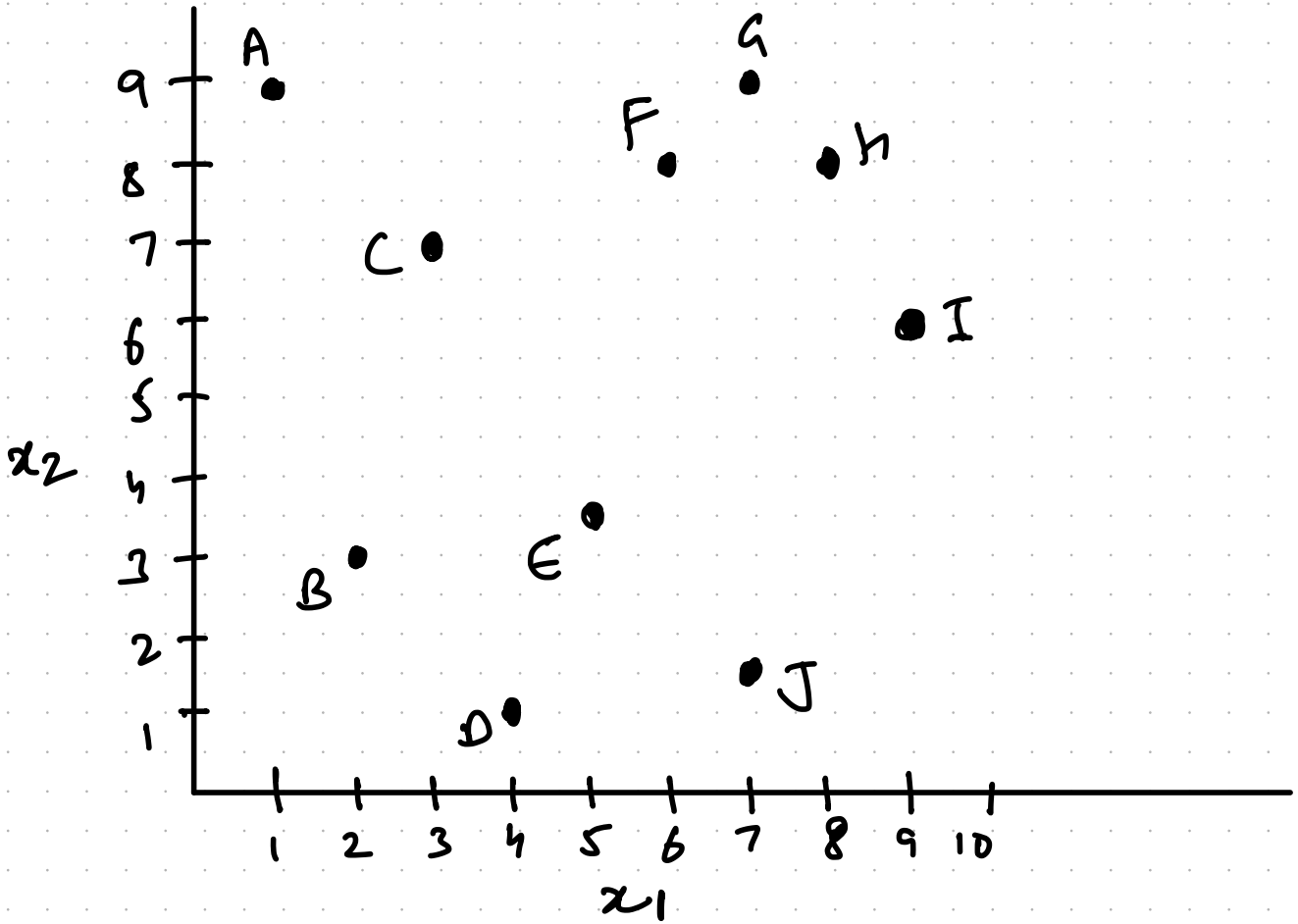
TOTAL QUERY $O(\log N + D*S)$

if $D << N$ and S is small

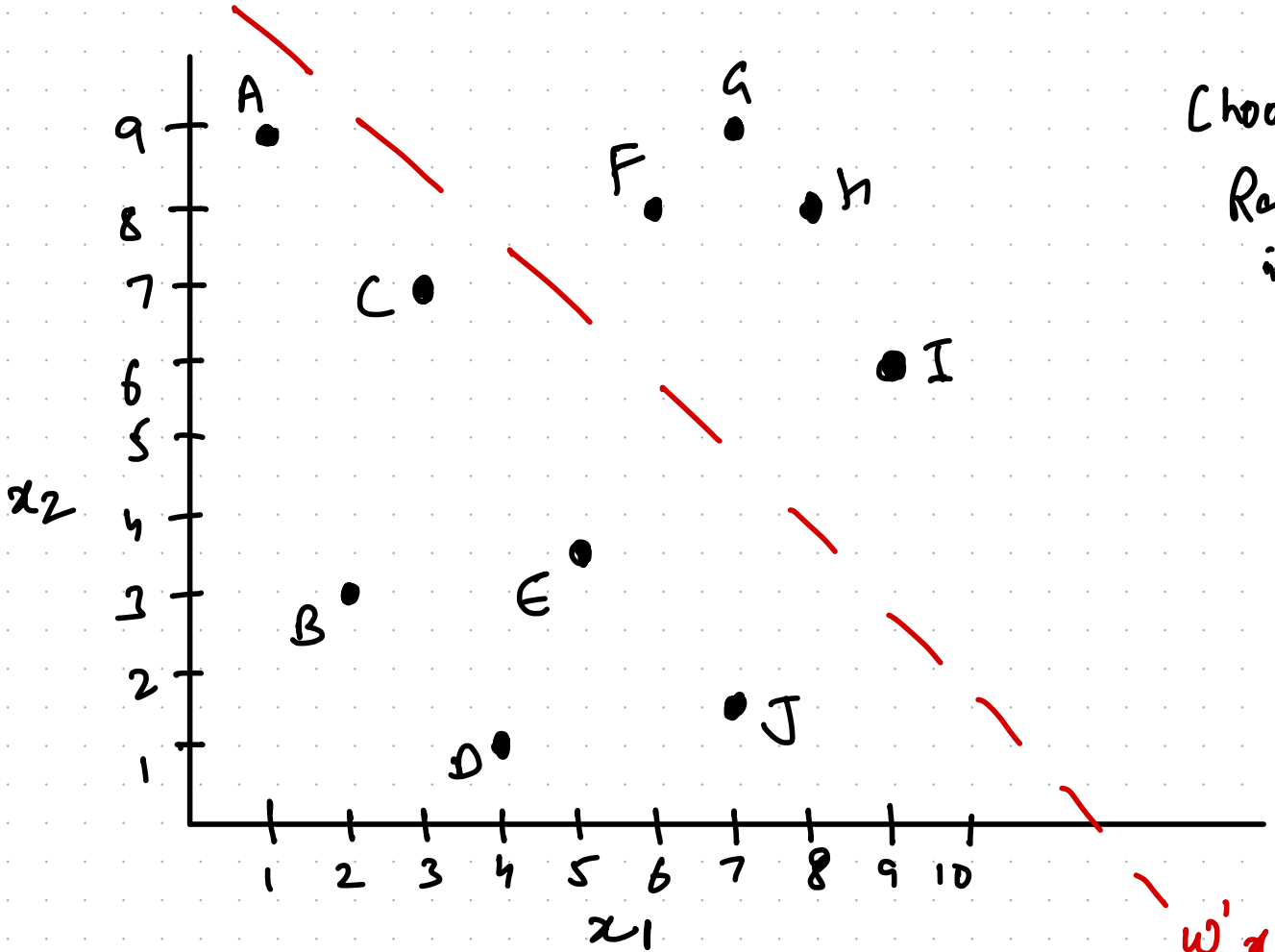
$O(\log N)$

Locality Sensitive Hashing (LSH) w/ Random Projection

(Machine Learning Interview.com)



Locality Sensitive Hashing (LSH) w/ Random Projection

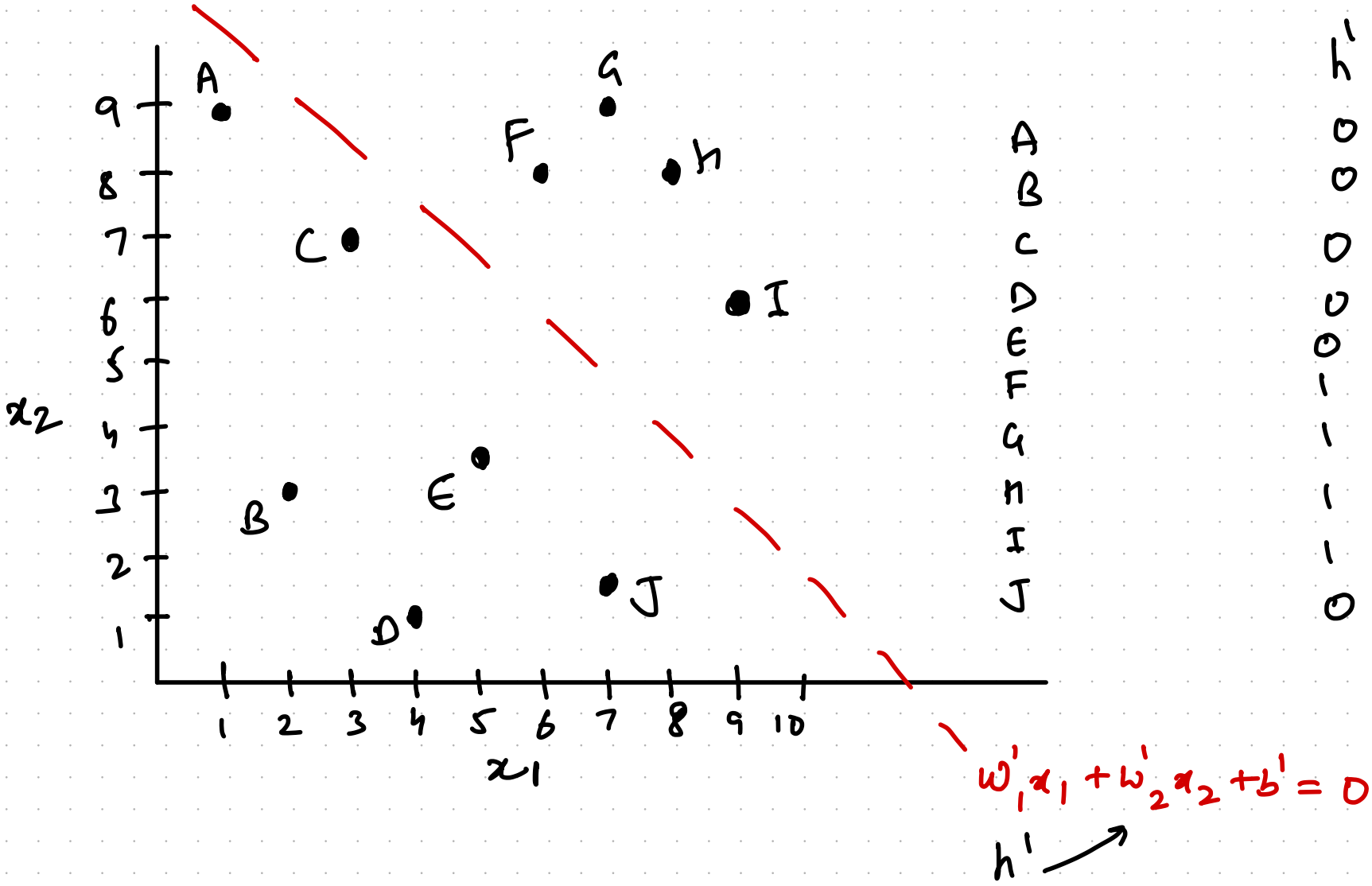


Choose h':
Random hyperplane in 'D' dim

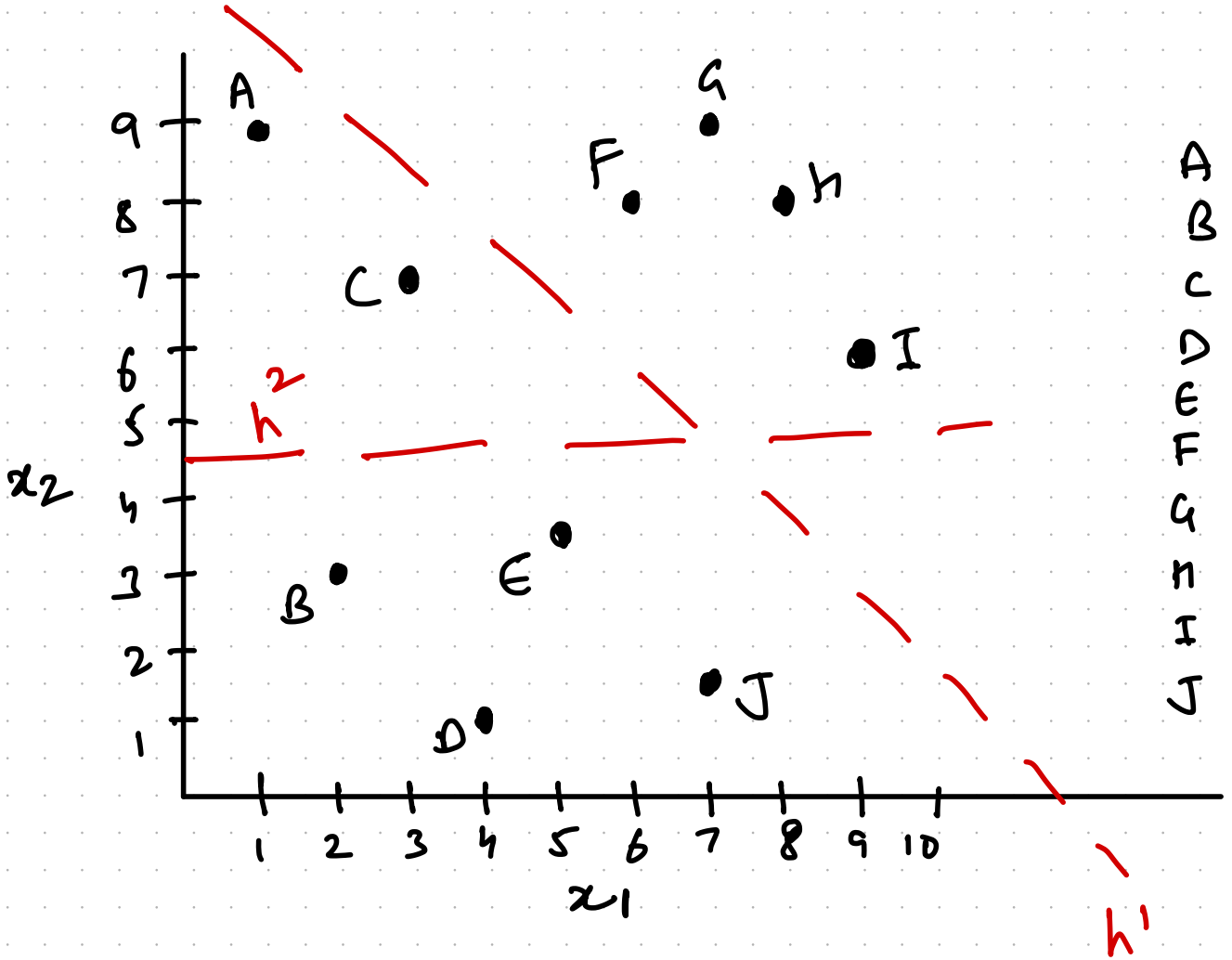
$w'_1 x_1 + w'_2 x_2 + b' = 0$

h'

Locality Sensitive Hashing (LSH) w/ Random Projection

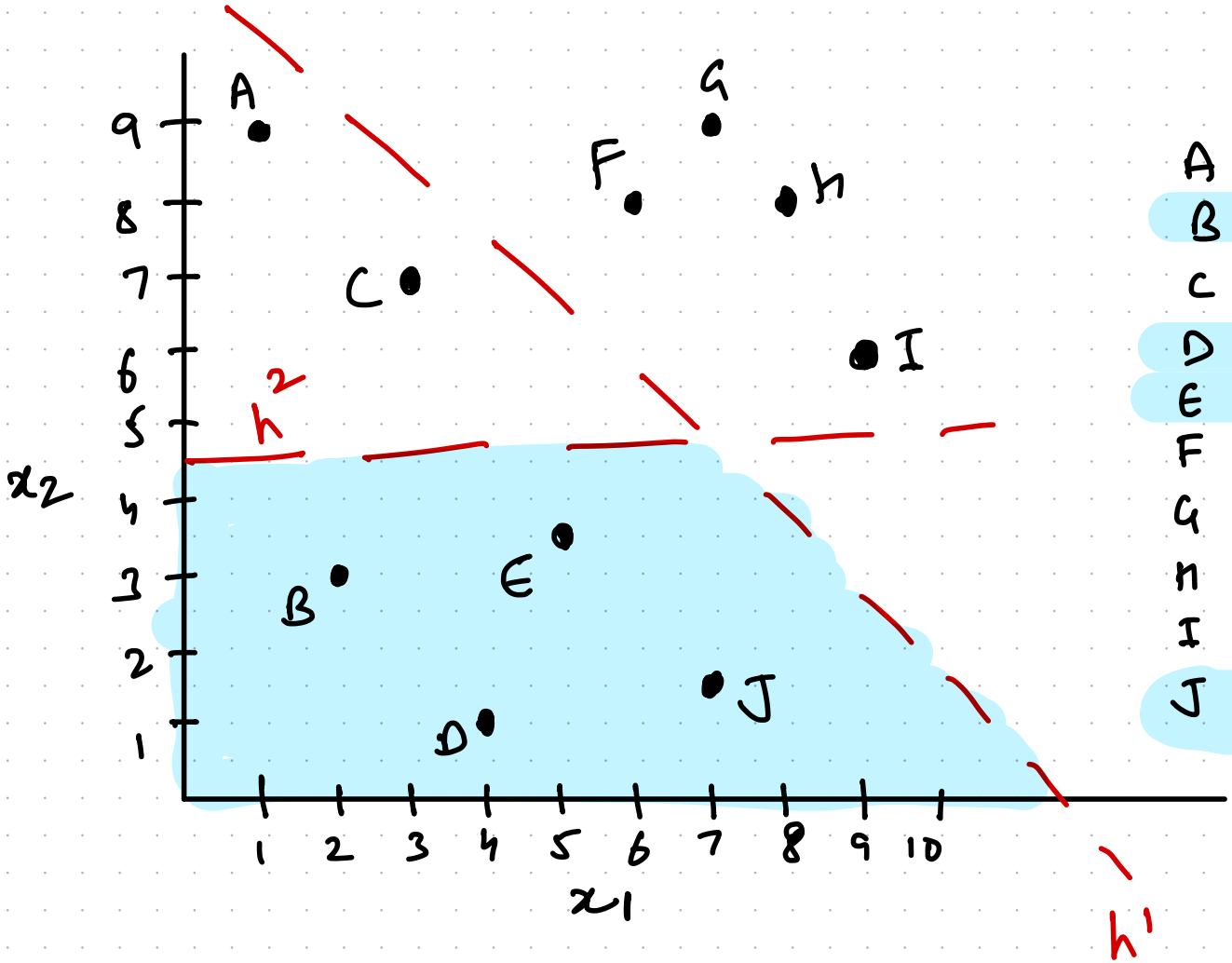


Locality Sensitive Hashing (LSH) w/ Random Projection



	h^2	h^1
A	1	0
B	0	0
C	1	0
D	0	0
E	0	0
F	1	1
G	1	1
H	1	1
I	1	1
J	1	0
	0	0

Locality Sensitive Hashing (LSH) w/ Random Projection



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D	0	0
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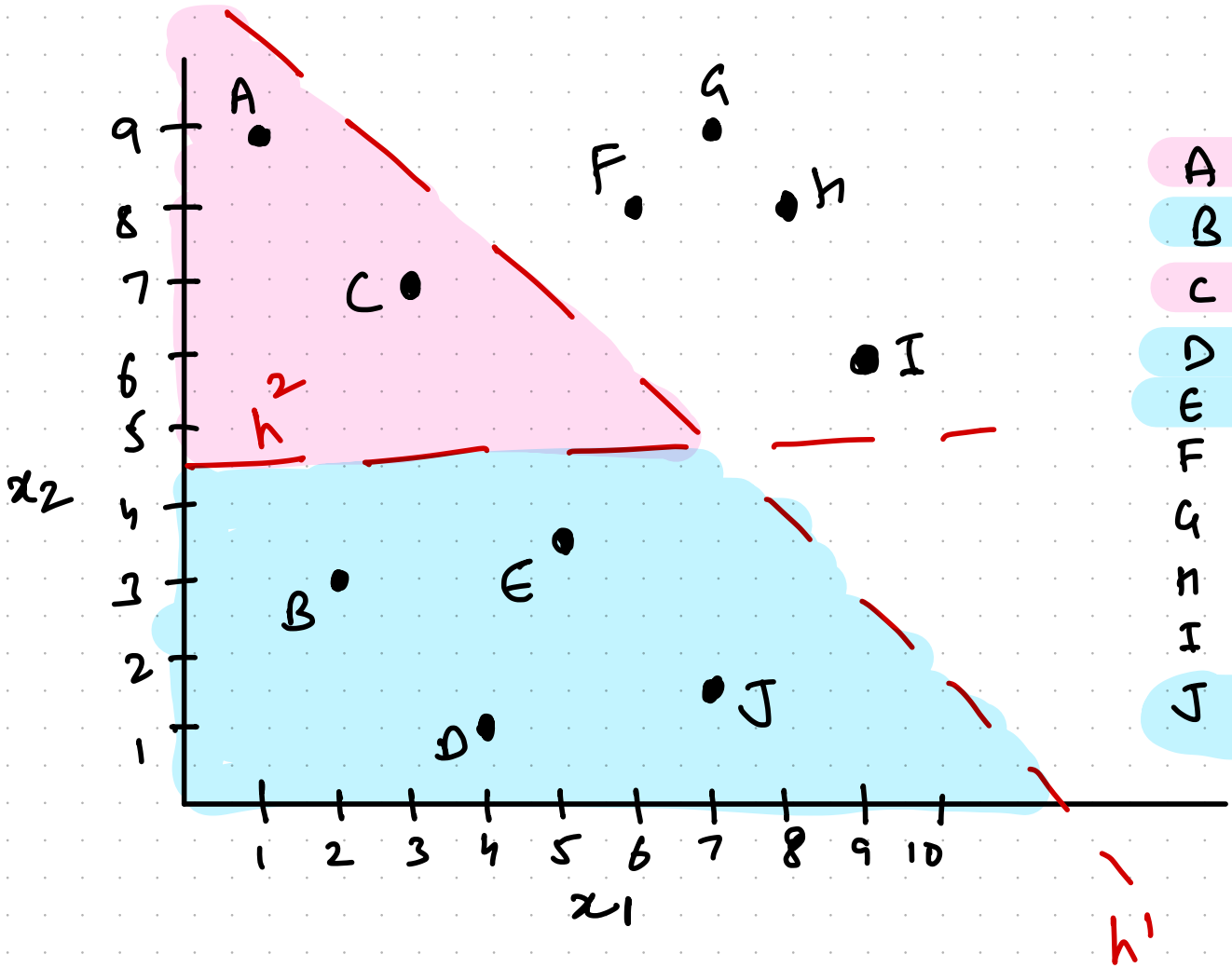
F	1	1
G	1	1
H	1	1
I	1	1
J	0	0

Locality Sensitive Hashing (LSH) w/ Random Projection



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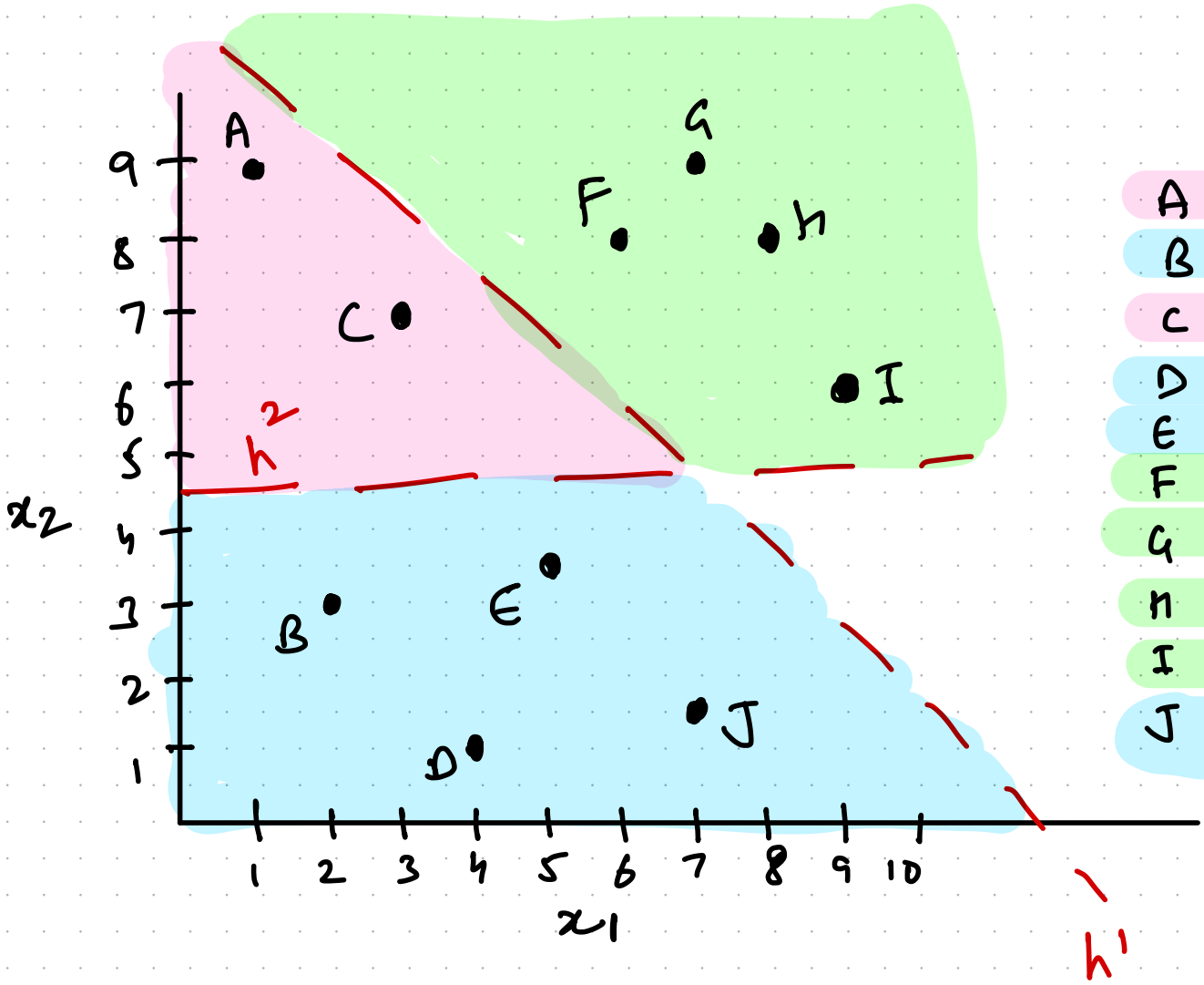
Locality Sensitive Hashing (LSH) w/ Random Projection



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D	0	0
E	0	0
F	1	1
G	1	1
H	1	1
I	1	1
J	0	0

Practical implementat^{n}

* How to sample hyperplane?

$i = 1 \dots D$: — sampled from

$w_i \sim N(\dots, \dots)$

$b \sim N(\dots, \dots)$

Practical implementat^{n}

* How to get $h^{i}(x) = 0$ or 1

$$h^{i}(x) = \text{SIGN}\left(b + w^{i}_{1} x_1 + w^{i}_{2} x_2 + \cdots\right)$$

Practical implementat^{n}

* How to vectorize finding Hash Table

$X = \begin{bmatrix} & \\ & \\ & \end{bmatrix}$

Practical implementat^{n}

* How to vectorize finding Hash Table

$$X = \begin{bmatrix} & \\ & \\ & \end{bmatrix}_{N \times D}$$

$$x' = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \begin{bmatrix} & \\ & \\ & \end{bmatrix}_{N \times D+1}$$

Practical implementat^n

* How to vectorize finding Hash Table

$$x' = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}_{N \times D+1}$$

let # planes be 'p'

$$w = \begin{bmatrix} \cdots & \cdots \\ N(\cdots, \cdots) & \\ \cdots & \cdots \end{bmatrix}_{D+1 \times p}$$

Practical implementation

* How to vectorize finding Hash Table

$$x' = \begin{bmatrix} 1 & & \\ 1 & & \\ \vdots & & \\ 1 & & \end{bmatrix}_{N \times D+1}$$

$$H = \underset{N \times P}{SGN}\left(x' W\right)$$

let # planes be 'P'

$$W = \begin{bmatrix} & N(\cdots,\cdots) & \\ & & \\ & & \end{bmatrix}_{D+1 \times P}$$

Time complexity

- Assuming one (1) set of hash functions

$\rightarrow \quad H_{N \times P} = SGN(X_{N \times D} \cdot R_{D \times P})$

Time $= O(N \times D \times P)$

Time to generate $R \simeq O(D \times P)$

Thus, at training time : $O(N \times D \times P)$

At testing time,

→ Computing hash on $q_{1 \times D}$ takes $O(DP)$

→ Assuming 'T' points in bucket:

$O(T * D)$ to find closest neighbour

At testing time,

→ Computing hash on $q_{1 \times D}$ takes $O(DP)$

→ Assuming 'T' points in bucket:

$O(T * D)$ to find closest neighbour

→ What is T in terms of N and P (on average)

At testing time,

→ Computing hash on $q_{1 \times D}$ takes $O(DP)$

→ Assuming 'T' points in bucket:

$O(T * D)$ to find closest neighbour

→ What is T in terms of N and P (on average)

$$T \approx \frac{N}{2^P}$$

At testing time,

→ Computing hash on query takes $O(DP)$

→ Assuming 'T' points in bucket:

$O(T * D)$ to find closest neighbour

→ What is T in terms of N and P (on average)

$$T \approx \frac{N}{2^P}$$

→ Test Time $= O\left(DP + \frac{DN}{2^P}\right)$