Lasso Regression

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Introduction and Motivation

What is Lasso Regression?

Definition: LASSO

Least **A**bsolute **S**hrinkage and **S**election **O**perator

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Definition: LASSO

Least **A**bsolute **S**hrinkage and **S**election **O**perator

Key Points: Key Properties

- Uses L1 penalty (absolute values) instead of L2 penalty
- Leads to **sparse solutions** (many coefficients become exactly zero)
- Performs automatic feature selection
- Popular for high-dimensional problems

Mathematical Formulation

Problem: Why Not Just Use Ridge?

Important: Limitation of Ridge Regression

Ridge regression shrinks coefficients but **never makes them exactly zero**

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Example: High-Dimensional Problem

- 1000 features, only 50 are truly relevant
- Ridge gives tiny but non-zero coefficients for irrelevant features
- Model is not interpretable
- Need automatic feature selection!

Lasso Objective Function

Definition: Constrained Form

$$\boldsymbol{\theta}_{\text{opt}} = \arg\min_{\boldsymbol{\theta}} \|(\mathbf{y} - \mathbf{X}\boldsymbol{\theta})\|_2^2 \text{ subject to } \|\boldsymbol{\theta}\|_1 \leq s$$

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Theorem: Penalized Form (Using Lagrangian Duality)

Constrained form is equivalent to:

$$\boldsymbol{\theta}_{\text{opt}} = \arg\min_{\boldsymbol{\theta}} \underbrace{\|(\mathbf{y} - \mathbf{X}\boldsymbol{\theta})\|_2^2 + \lambda\|\boldsymbol{\theta}\|_1}_{\text{Lasso Objective}}$$

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L1 Norm (Manhattan Distance)

$$\|\boldsymbol{\theta}\|_1 = |\theta_1| + |\theta_2| + \cdots + |\theta_d| = \sum_{j=1}^{d} |\theta_j|$$

The Challenge: Non-Differentiability

Important: Problem

The L1 norm $\|\boldsymbol{\theta}\|_1 = \sum_j |\theta_j|$ is **not differentiable** at $\theta_j = 0$

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Cannot Use Standard Calculus

$$\frac{\partial}{\partial \boldsymbol{\theta}} \left[\|(\mathbf{y} - \mathbf{X}\boldsymbol{\theta})\|_2^2 + \lambda \|\boldsymbol{\theta}\|_1 \right] = 0$$

This fails because $\frac{\partial |\theta_j|}{\partial \theta_j}$ is undefined at $\theta_j = 0$

Key Points: Solution Approaches

- **Coordinate Descent**: Optimize one coefficient at a time
- **Subgradient Methods**: Generalize derivatives to non-smooth functions

Why Lasso Gives Sparsity

Sparsity: The Key Question

Important: Central Question

Why does Lasso produce sparse solutions while Ridge doesn't?

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Key Points: Two Perspectives

- **Geometric**: Shape of constraint regions
- **Algorithmic**: Behavior of optimization algorithms

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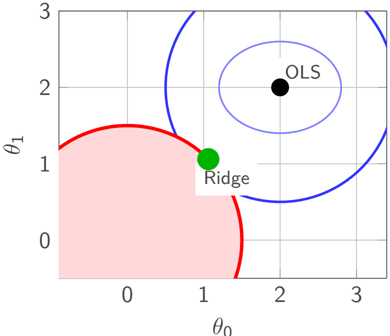
- **Algorithmic**: Behavior of optimization algorithms

Example: Preview

We'll see why L_p norms with $p < 2$ promote sparsity

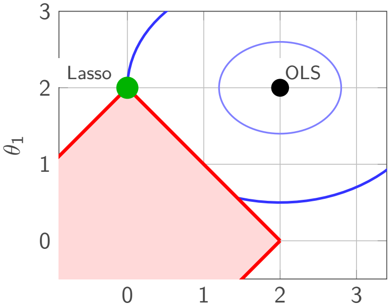
L2 Norm: Ridge Constraint



Key Points: L2 Properties

- **Shape**: Perfect circle
- **Constraint**: $\theta_0^2 + \theta_1^2 \leq c$
- **Boundary**: Smooth everywhere
- **Intersection**: Rarely on axes
- **Result**: No sparsity

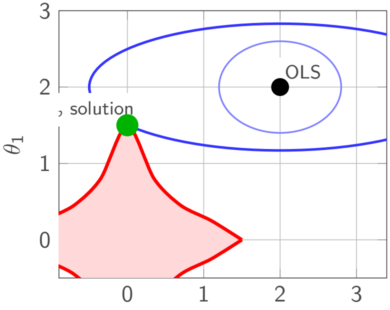
L1 Norm: Lasso Constraint



Key Points: L1 Properties

- **Shape**: Diamond/rhombus
- **Constraint**: $|\theta_0| + |\theta_1| \leq c$
- **Corners**: Sharp at axes
- **Intersection**: High probability on axes
- **Result**: Automatic sparsity!

L_p Norm: Even More Sparsity ($p < 1$)



Key Points: L_p Properties ($p < 1$)

- **Shape**: Highly concave
- **Constraint**: $(|\theta_0|^p + |\theta_1|^p)^{1/p} \leq c$
- **Corners**: Ultra-sharp at axes
- **Sparsity**: Extremely high
- **Problem**: Non-convex!

Sparsity Progression: $L_2 \to L_1 \to L_p$

Theorem: Key Insight

As p decreases from 2 to 1 to $p < 1$:

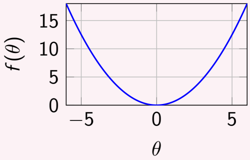
- Constraint regions become more **pointed** at axes
- Probability of intersection at axes **increases**
- Sparsity **increases**
- Optimization difficulty **increases**

Example: Why $p = 1$ is Special

- Still promotes sparsity (sharp corners)
- Remains convex (unlike $p < 1$) and Computationally tractable
- Perfect balance of sparsity and solvability

L2 vs L1: Gradient Behavior

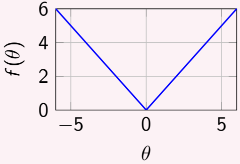
Key Points: L2 Penalty: $f(\theta) = \frac{1}{2}\theta^2$



Gradient: $\frac{df}{d\theta} = \theta$

Shrinks proportionally to current value

Key Points: L1 Penalty: $f(\theta) = |\theta|$



Subgradient: $\text{sign}(\theta) = \pm 1$

Constant push toward zero

L2 vs L1: Gradient Behavior

Example: Example: Start at $\theta = 5$

L2: $5 \to 2.5 \to 1.25 \to 0.625 \to \dots$ (never exactly zero)

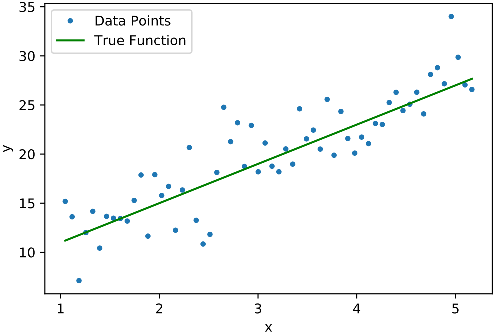
L1: $5 \to 4.5 \to 4.0 \to 3.5 \to \dots \to 0$ (reaches zero in finite steps)

Geometric Interpretation

Sample Dataset for Demonstration

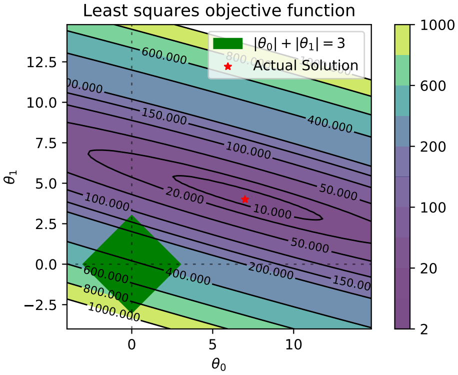
Example: True Function

We'll demonstrate Lasso on a simple linear relationship: $y = 4x + 7$



Sample data from $y = 4x + 7$ with noise

Geometric Interpretation: L1 vs L2 Constraints



L1 vs L2 constraint regions

Key Points: Key Insight

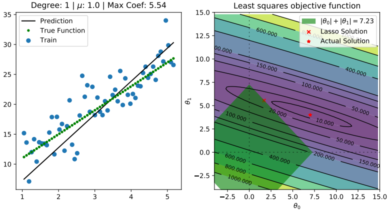
Diamond corners ⇒ exact zeros! Circle ⇒ no sparsity.

Regularization Effects

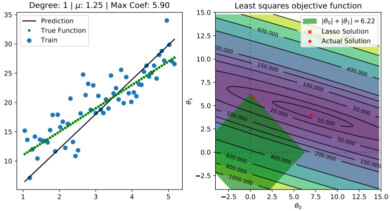
Effect of λ on Solution Path

Important: Regularization Parameter

λ controls fit vs sparsity trade-off

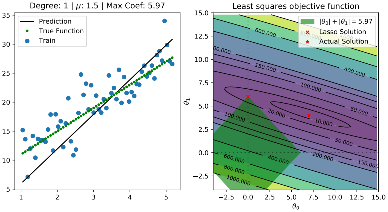


$\lambda = 1.0$ - Moderate

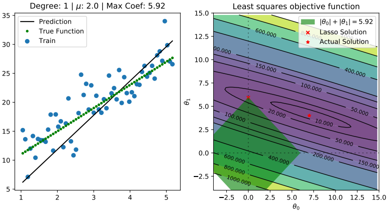


$\lambda = 1.25$ - Higher

Increasing Regularization Strength



$\lambda = 1.5$ - Strong

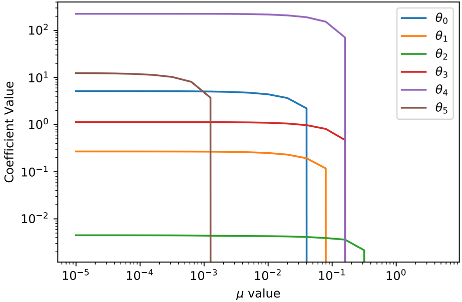


$\lambda = 2.0$ - Very strong

Key Points: Observation

As λ increases $\rightarrow$ more coefficients become exactly zero (automatic feature selection)

Lasso Regularization Path



Coefficient values vs λ

Key Points: Key Observations

- Coefficients shrink to zero as λ increases
- Natural feature selection ordering

Feature Selection Properties

Lasso for Automatic Feature Selection

Definition: Automatic Feature Selection

Lasso performs regression and feature selection simultaneously by setting irrelevant coefficients to exactly zero

Key Points: Key Advantages

- **Sparsity**: Many coefficients $\rightarrow$ exactly zero
- **Interpretability**: Understand which features matter
- **Efficiency**: Fewer parameters, faster prediction

Subgradient Methods

What is a Subgradient?

A subgradient generalizes the concept of gradient to convex but non-differentiable functions

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Example: Classic Example

For $f(x) = |x|$:

- $f'(x) = 1$ when $x > 0$
- $f'(x) = -1$ when $x < 0$
- $f'(0)$ is undefined, but subgradient $\in [-1, 1]$

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A subgradient generalizes the concept of gradient to convex but non-differentiable functions

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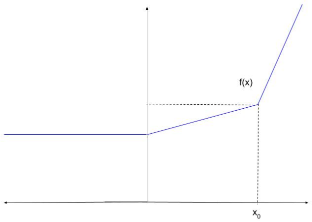
Important: Why Important for Lasso?

The L1 penalty $|\theta_j|$ is non-differentiable at $\theta_j = 0$

Subgradient: Visual Intuition



Non-differentiable function at x_0

Important: Task

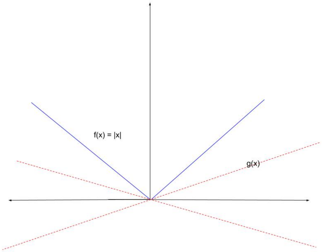
Find the "derivative" of $f(x)$ at the non-differentiable point $x = x_0$

Construction

Find differentiable $g(x)$ such that:

- $g(x_0) = f(x_0)$
- $g(x) \leq f(x)$ for all x

Subgradient of $|x|$ at $x = 0$



Supporting lines with slopes in $[-1, 1]$

Subgradient Set

For $f(x) = |x|$ at $x = 0$:

$$\partial f(0) = [-1, 1]$$

Key Points: Key Insight

Multiple supporting lines $\Rightarrow$ set of valid subgradients

Important: Lasso Connection

This subgradient concept is exactly what we need for the L1 penalty term!

Coordinate Descent Algorithm

Introduction to Coordinate Descent

Definition: Coordinate Descent

Optimization method: minimize one coordinate at a time

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Definition: Coordinate Descent

Optimization method: minimize one coordinate at a time

Key Points: Key Idea

- Hard: optimize all coordinates together
- Easy: optimize one coordinate at a time
- Perfect for non-differentiable Lasso!

Algorithm Overview

$$\min_{\boldsymbol{\theta}} f(\boldsymbol{\theta}) \text{ becomes } \min_{\theta_j} f(\theta_1, \ldots, \theta_{j-1}, \theta_j, \theta_{j+1}, \ldots, \theta_d)$$

Coordinate Descent Properties

Key Points: Advantages

- **No step-size**: Exact 1D minimization
- **Convergence**: Guaranteed for convex Lasso
- **Efficient**: Closed-form updates

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Selection Strategies

Cyclic, Random, or Greedy coordinate selection

Important: Process

Cycle through coordinates, optimizing one at a time until convergence

Worked Example

Coordinate Descent Example Setup

Learn $y = \theta_0 + \theta_1 x$ using coordinate descent on the dataset below

x	**y**
1	1
2	2
3	3

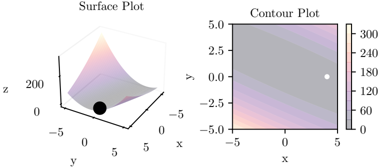
Setup

- Initial parameters: $(\theta_0, \theta_1) = (2, 3)$
- MSE $= \frac{14+3\theta_0^2+14\theta_1^2-12\theta_0-28\theta_1+12\theta_0\theta_1}{3}$
- Using standard least squares (no regularization for simplicity)

Coordinate Descent Iterations

Iteration 1:
INIT: $\theta_0 = 2$ and $\theta_1 = 3$

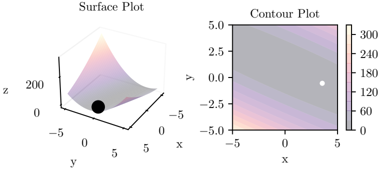
Fix $\theta_1 = 3$, optimize θ_0:
$\frac{\partial \text{MSE}}{\partial \theta_0} = 6\theta_0 + 24 = 0$
$\theta_0 = -4$



Starting point

Iteration 2:
INIT: $\theta_0 = -4$ and $\theta_1 = 3$

Fix $\theta_0 = -4$, optimize θ_1:
$\theta_1 = 2.7$



After 2 iterations

Visual Coordinate Descent

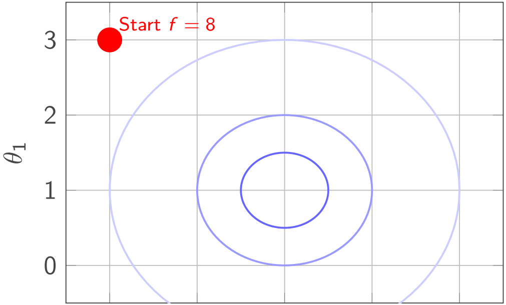
Coordinate Descent: Setup

Example: Problem

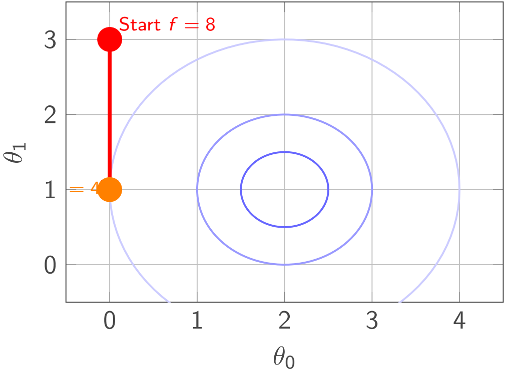
Minimize

$$f(\theta_0, \theta_1) = (\theta_0 - 2)^2 + (\theta_1 - 1)^2$$

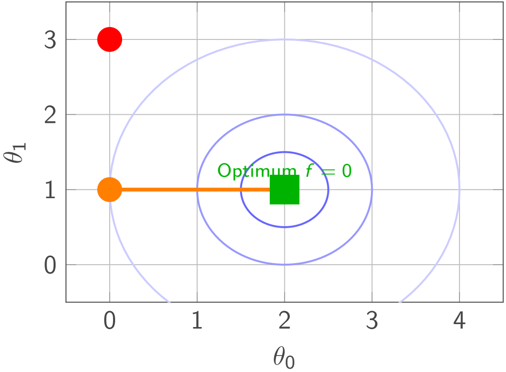
starting from $(0, 3)$



Coordinate Descent: Step 1



Coordinate Descent: Step 2



Failure of Coordinate Descent

Mathematical Derivation

Lasso Coordinate Descent: Setup

Lasso Objective

$$\text{Minimize } \sum_{i=1}^{n}(y_i - \hat{y}_i)^2 + \lambda \sum_{j=0}^{d} |\theta_j|$$

Lasso Coordinate Descent: Setup

Lasso Objective

$$\text{Minimize} \sum_{i=1}^{n}(y_i - \hat{y}_i)^2 + \lambda \sum_{j=0}^{d} |\theta_j|$$

Key Points: Key Definitions

- $\rho_j = \sum_{i=1}^{n} x_{ij}(y_i - \hat{y}_i^{(-j)})$ (partial residual correlation)
- $z_j = \sum_{i=1}^{n} x_{ij}^2$ (feature norm squared)
- $\hat{y}_i^{(-j)}$ = prediction without j-th feature

Coordinate Update Rule

Fix all θ_k for $k \neq j$, minimize w.r.t. θ_j:

Subgradient Analysis

Subgradient of Lasso Objective w.r.t. θ_j

$$\frac{\partial}{\partial \theta_j}(\text{Lasso}) = -2\rho_j + 2\theta_j z_j + \lambda \frac{\partial}{\partial \theta_j}|\theta_j|$$

Subgradient Analysis

Subgradient of Lasso Objective w.r.t. θ_j

$$\frac{\partial}{\partial \theta_j}(\text{Lasso}) = -2\rho_j + 2\theta_j z_j + \lambda \frac{\partial}{\partial \theta_j}|\theta_j|$$

Theorem: Subgradient of $|\theta_j|$

$$\frac{\partial}{\partial \theta_j}|\theta_j| = \begin{cases} +1 & \text{if } \theta_j > 0 \\ [-1, +1] & \text{if } \theta_j = 0 \\ -1 & \text{if } \theta_j < 0 \end{cases}$$

Soft-Thresholding Solution

Theorem: Complete Lasso Update Rule

$$\theta_j = \begin{cases} \frac{\rho_j + \lambda/2}{z_j} & \text{if } \rho_j < -\lambda/2 \\ 0 & \text{if } |\rho_j| \leq \lambda/2 \\ \frac{\rho_j - \lambda/2}{z_j} & \text{if } \rho_j > \lambda/2 \end{cases}$$

Important: Sparsity Mechanism

If correlation $|\rho_j| \leq \lambda/2$ is weak, set $\theta_j = 0$!

Key Points: Soft-Thresholding Properties

- **Shrinkage**: Coefficients pulled toward zero
- **Selection**: Small coefficients $\rightarrow$ exactly zero

Lasso vs Ridge Comparison

Lasso vs Ridge: Key Differences

Property	Ridge (L2)	Lasso (L1)
Penalty	$\sum \theta_j^2$	$\sum \|\theta_j\|$
Sparsity	Never exactly zero	Can be exactly zero
Feature Selection	No	Yes
Differentiable	Yes	No (at $\theta_j = 0$)
Solution Method	Closed form	Coordinate descent
Constraint Shape	Circle	Diamond
Best for	Multicollinearity	Feature selection

Key Points: When to Use Each

Lasso: High-dimensional data, need interpretable model, expect few relevant features
Ridge: All features somewhat relevant, multicollinearity issues, want stable solution

Summary and Applications

Lasso Regression: Summary

Theorem: Three-Part Understanding

Visual: L1 diamond constraint $\rightarrow$ sparsity at sharp corners
Algorithmic: Coordinate descent + soft-thresholding $\rightarrow$ exact zeros
Mathematical: Subgradients handle non-differentiability elegantly

Key Points: Key Advantages

- Regression + feature selection simultaneously
- Sparse, interpretable models
- Handles high-dimensional data well

Key Points: Limitations

Applications and Extensions

Example: Real-World Applications

- **Genomics**: 20,000+ genes $\rightarrow$ identify disease markers
- **Text Mining**: 100k+ words $\rightarrow$ sentiment analysis features
- **Signal Processing**: Sparse signal reconstruction
- **Finance**: Risk factor selection from hundreds of indicators
- **Marketing**: Customer segmentation with key attributes

Key Points: Extensions

- **Elastic Net**: Combines L1 + L2 penalties
- **Group Lasso**: Selects groups of related features
- **Fused Lasso**: Enforces smoothness in ordered features