Contour Plots & Gradients

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Definition: What is a Contour Plot?

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Circular Contours

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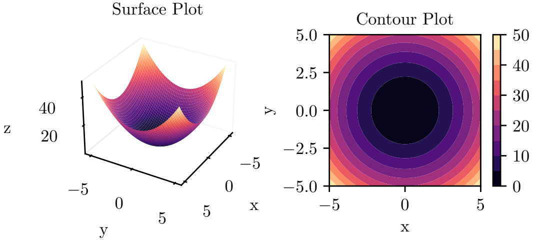
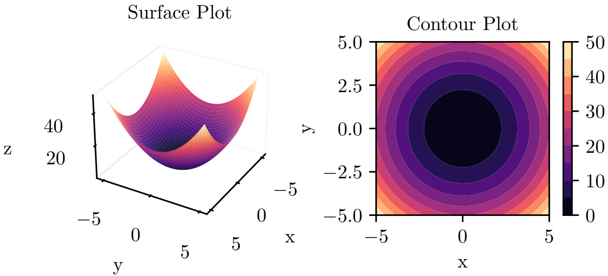


Introduction to Contour Plots



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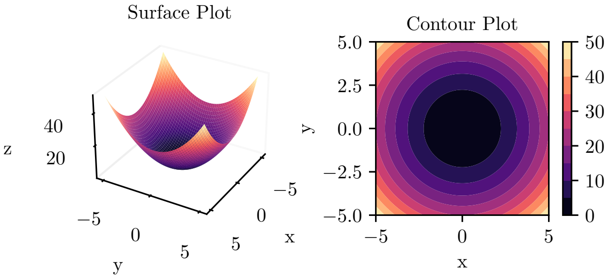

Key Points:

Key Insight: Each contour line represents all points (x, y) where $f(x, y) = K$ for a specific constant K

Contour Example: Parabolic Function

Example: Function: $z = f(x, y) = x^2$

Note: This function depends only on x, not on y!

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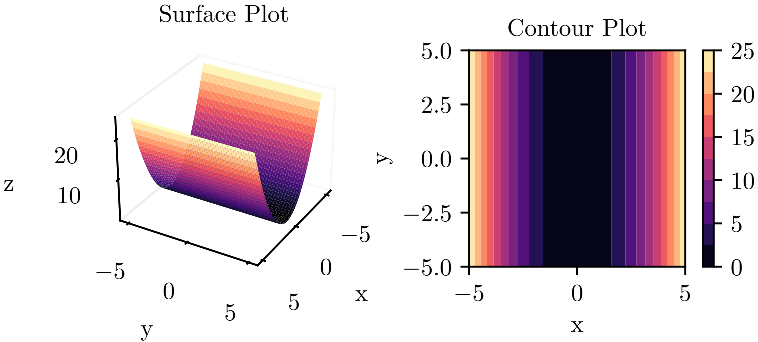

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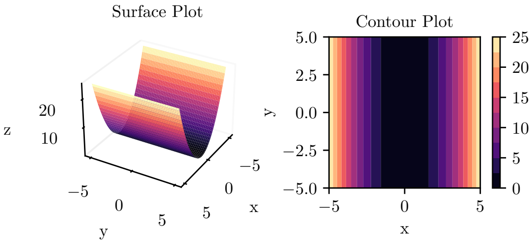

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Observation: Contour lines are vertical because $f(x, y) = x^2$ is constant for all y values when x is fixed

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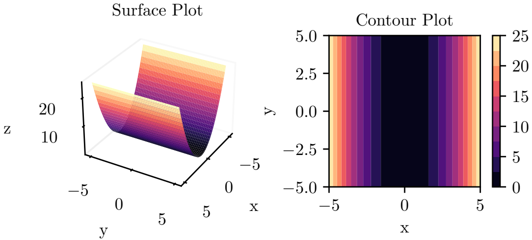

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Observation: Contour lines are vertical because $f(x, y) = x^2$ is constant for all y values when x is fixed

Important: ML Connection

This represents: A loss function that doesn't depend on one of the parameters!

Contour Example: Manhattan Distance

Example: Function: $z = f(x, y) = |x| + |y|$

Also known as: Manhattan distance or L1 norm

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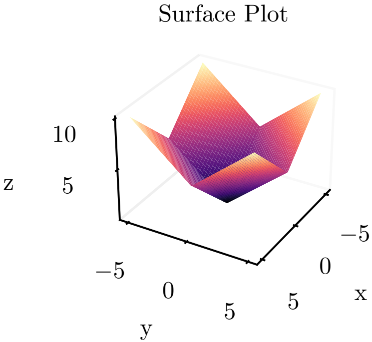

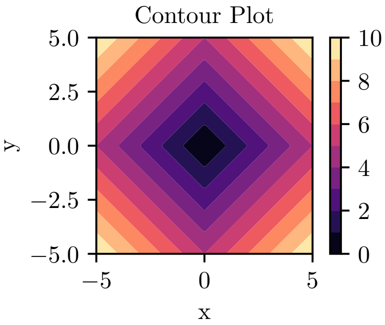

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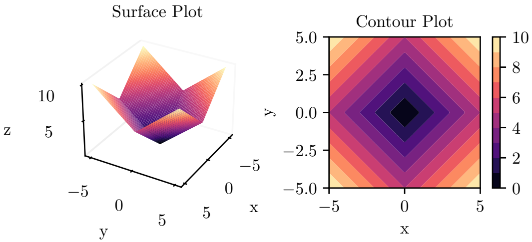

Key Points:

Shape: Diamond-shaped contours due to absolute value functions

Important: ML Connection

This represents: L1 regularization in machine learning (promotes sparsity!)

Contour Example: Polynomial Function

Example: Function: $z = f(x, y) = x^2 \cdot y$

Type: Mixed polynomial (quadratic in x, linear in y)

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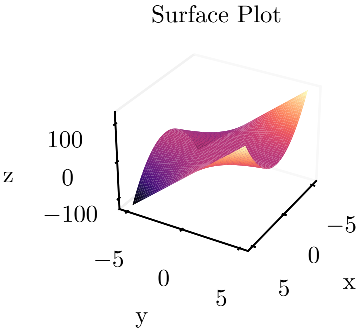

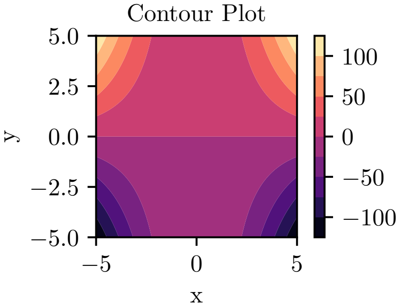

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Key Features:

- Asymmetric contours

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This represents: Complex loss surfaces with variable interactions

Contour Example: Hyperbolic Function

Example: Function: $z = f(x, y) = xy$

Type: Bilinear function (linear in each variable separately)

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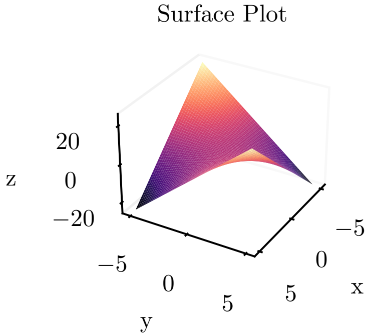

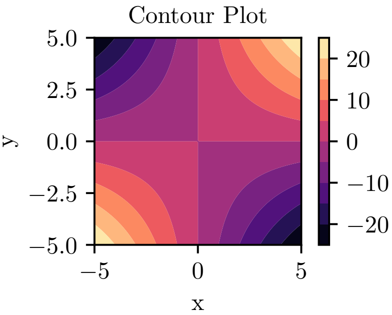

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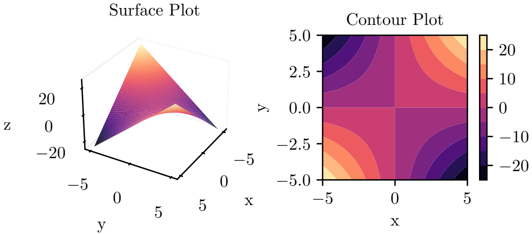

Key Points:

Shape: Hyperbolic contours with saddle point at the origin

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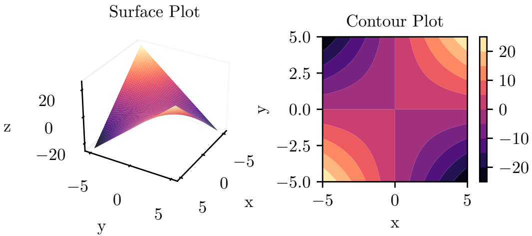

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Important: ML Significance

Saddle points: Common in neural network optimization - neither minimum nor maximum!

Gradients and Contour Plots

Understanding Gradients

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Gradient ∇f: Vector pointing in the direction of steepest increase of function f

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- **Magnitude:** Rate of steepest change
- **Contour relationship:** Always perpendicular to contour lines

Understanding Gradients

Example: Fundamental Insight

All points on the same contour have identical $f(x, y)$ values

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Moving along a contour: No change in function value
Moving perpendicular to contour: Maximum change in function value

Important: ML Application

Gradient descent: Move opposite to gradient direction to minimize loss!

Gradients Visualized: Circular Contours

Example: Function: $z = f(x, y) = x^2 + y^2$

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Gradient: $\nabla f = \begin{bmatrix} 2x \\ 2y \end{bmatrix}$

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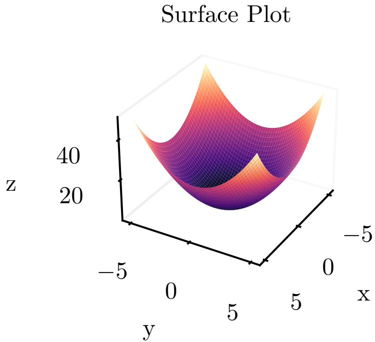

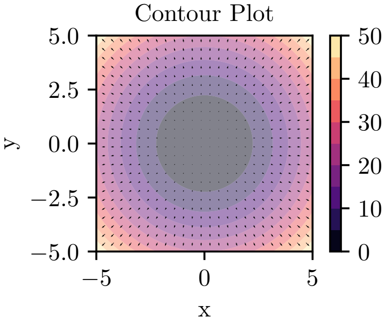

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Key Points:

Observations:

- Gradient arrows point radially outward

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Important: Perfect for Optimization

This is an ideal optimization landscape: Single global minimum at origin!

Gradient Properties: Key Insights

Important: Direction Interpretation

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Steepest Ascent: Gradient ∇f points toward maximum increase in $f(x, y)$

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- **Same contour:** All points have identical $f(x, y)$ values

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Key Points: Contour Relationship

- **Same contour:** All points have identical $f(x, y)$ values
- **Gradient direction:** Always perpendicular to contour lines
- **Zero gradient:** Occurs at critical points (minima, maxima, saddle points)

Gradient Properties: Key Insights

Definition: Machine Learning Connection

Optimization algorithms use gradients to:

- Find minimum loss (gradient descent: $\theta_{new} = \theta_{old} - \alpha \nabla L$)
- Navigate complex parameter spaces
- Escape saddle points
- Converge to optimal solutions

Summary: Contours and Gradients in ML

Key Points: What We Learned

- **Contour plots:** Visualize function behavior in 2D

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- **Different shapes:** Circular, diamond, hyperbolic, asymmetric
- **Gradients:** Point toward steepest function increase
- **Perpendicular relationship:** Gradients $\perp$ contours

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Important: ML Applications

- **Loss landscapes:** Understanding optimization challenges

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These concepts enable understanding of:

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These concepts enable understanding of:

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- Learning rate selection

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- Convergence analysis