CS 225A: Experimental Robotics
Homework 0

Name: Abhyudit Singh Manhas
SUID: 06645995

Problem 1

(a) We have:

$$^2P_4 = \begin{bmatrix} L_4 \cos(90^\circ + \theta_3) \\ 0 \\ L_4 \sin(90^\circ + \theta_3) \end{bmatrix} = \begin{bmatrix} -L_4 \sin\theta_3 \\ 0 \\ L_4 \cos(\theta_3) \end{bmatrix} \tag{1}$$

Also:

$$^1P_4 = {}^1P_2 + {}^2P_4$$

$$\implies {}^1P_4 = \begin{bmatrix} 0 \\ 0 \\ d_2 \end{bmatrix} + \begin{bmatrix} -L_4 \sin\theta_3 \\ 0 \\ L_4 \cos(\theta_3) \end{bmatrix} = \begin{bmatrix} -L_4 \sin\theta_3 \\ 0 \\ d_2 + L_4 \cos(\theta_3) \end{bmatrix} \tag{2}$$

Finally, we get:

$$^0P_4 = {}^0_1R\,{}^1P_4$$

$$\implies {}^0P_4 = \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 & 0 \\ \sin\theta_1 & \cos\theta_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -L_4 \sin\theta_3 \\ 0 \\ d_2 + L_4 \cos(\theta_3) \end{bmatrix}$$

$$\implies {}^0P_4 = \begin{bmatrix} -L_4 \cos\theta_1 \sin\theta_3 \\ -L_4 \sin\theta_1 \sin\theta_3 \\ d_2 + L_4 \cos(\theta_3) \end{bmatrix} \tag{3}$$

(b) The matrix ${}^2_4R$ is given by:

$${}^2_4R = \begin{bmatrix} X_4 \cdot X_2 & Y_4 \cdot X_2 & Z_4 \cdot X_2 \\ X_4 \cdot Y_2 & Y_4 \cdot Y_2 & Z_4 \cdot Y_2 \\ X_4 \cdot Z_2 & Y_4 \cdot Z_2 & Z_4 \cdot Z_2 \end{bmatrix} = \begin{bmatrix} \cos\theta_3 & -\sin\theta_3 & 0 \\ 0 & 0 & -1 \\ \sin\theta_3 & \cos\theta_3 & 0 \end{bmatrix} \tag{4}$$

Noting that ${}^{1}_{2}R = I_{3\times 3}$, we have:

$$ {}^{0}_{4}R = \left({}^{0}_{1}R\right)\left({}^{1}_{2}R\right)\left({}^{2}_{4}R\right) $$

$$ \implies {}^{0}_{4}R = \left({}^{0}_{1}R\right)\left({}^{2}_{4}R\right) = \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 & 0 \\ \sin\theta_1 & \cos\theta_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta_3 & -\sin\theta_3 & 0 \\ 0 & 0 & -1 \\ \sin\theta_3 & \cos\theta_3 & 0 \end{bmatrix} $$

$$ \implies {}^{0}_{4}R = \begin{bmatrix} \cos\theta_1\cos\theta_3 & -\cos\theta_1\sin\theta_3 & \sin\theta_1 \\ \sin\theta_1\cos\theta_3 & -\sin\theta_1\sin\theta_3 & -\cos\theta_1 \\ \sin\theta_3 & \cos\theta_3 & 0 \end{bmatrix} \tag{5} $$

(c) From 3, we have

$$ \begin{aligned} x &= -L_4\cos\theta_1\sin\theta_3 \\ y &= -L_4\sin\theta_1\sin\theta_3 \\ z &= d_2 + L_4\cos(\theta_3) \end{aligned} \tag{6} $$

The linear Jacobian of the end-effector is then given by:

$$ {}^{0}J_v = \begin{bmatrix} \frac{\partial x}{\partial\theta_1} & \frac{\partial x}{\partial d_2} & \frac{\partial x}{\partial\theta_3} \\ \frac{\partial y}{\partial\theta_1} & \frac{\partial y}{\partial d_2} & \frac{\partial y}{\partial\theta_3} \\ \frac{\partial z}{\partial\theta_1} & \frac{\partial z}{\partial d_2} & \frac{\partial z}{\partial\theta_3} \end{bmatrix} $$

$$ \implies {}^{0}J_v = \begin{bmatrix} L_4\sin\theta_1\sin\theta_3 & 0 & -L_4\cos\theta_1\cos\theta_3 \\ -L_4\cos\theta_1\sin\theta_3 & 0 & -L_4\sin\theta_1\cos\theta_3 \\ 0 & 1 & -L_4\sin\theta_3 \end{bmatrix} \tag{7} $$

(d) The angular Jacobian of the end-effector is given by:

$$ {}^{0}J_\omega = \begin{bmatrix} \bar{\epsilon}_1\,{}^{0}Z_1 & \bar{\epsilon}_2\,{}^{0}Z_2 & \bar{\epsilon}_3\,{}^{0}Z_3 \end{bmatrix} $$

$$ \implies {}^{0}J_\omega = \begin{bmatrix} {}^{0}Z_1 & 0 & {}^{0}Z_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & \sin\theta_1 \\ 0 & 0 & -\cos\theta_1 \\ 1 & 0 & 0 \end{bmatrix} \tag{8} $$

where ${}^{0}Z_3 = {}^{0}Z_4$ is the last column of ${}^{0}_{4}R$ given by 5.

(e) The linear singularities of the robot are at $\theta_3 = -90°$, $0°$, and $90°$.
The robot for $\theta_3 = -90°$ is shown below. The singular direction is Y_4.

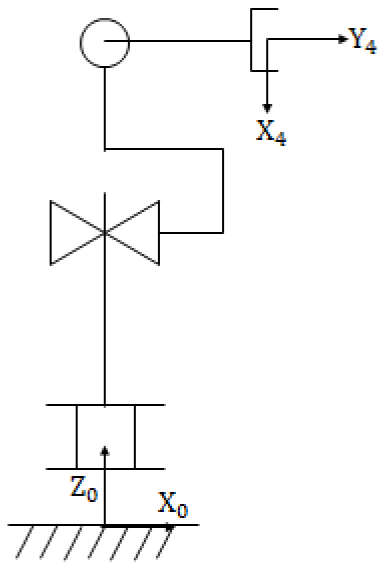


Figure 1: Singular configuration corresponding to $\theta_3 = -90°$

The robot for $\theta_3 = 0°$ is shown below. The singular direction is Z_4.

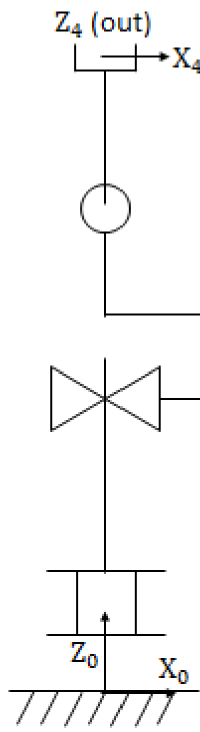


Figure 2: Singular configuration corresponding to $\theta_3 = 0°$

The robot for $\theta_3 = 90°$ is shown below. The singular direction is Y_4.

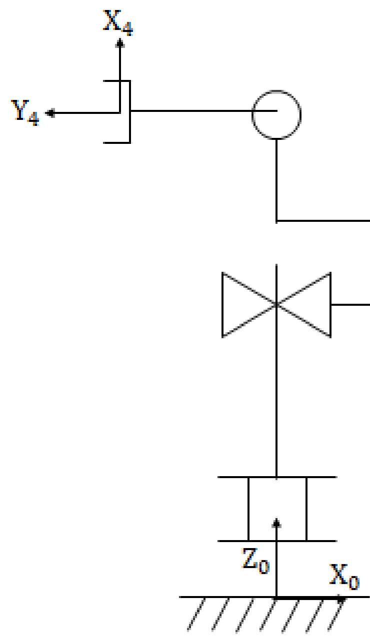


Figure 3: Singular configuration corresponding to $\theta_3 = 90°$

(f) For $\theta_3 = -90°$, we have:

$$^0J_v(\theta_3 = -90°) = \begin{bmatrix} -L_4 \sin\theta_1 & 0 & 0 \\ L_4 \cos\theta_1 & 0 & 0 \\ 0 & 1 & L_4 \end{bmatrix} \tag{9}$$

This is a singularity because the end-effector cannot move along the Y_4 direction. For $\theta_1 = 0°$, $^0Y_4 = X_0 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T$, and for $\theta_1 = 90°$, $^0Y_4 = Y_0 = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}^T$.
For $\theta_3 = 0°$, we have:

$$^0J_v(\theta_3 = 0°) = \begin{bmatrix} 0 & 0 & -L_4 \cos\theta_1 \\ 0 & 0 & -L_4 \sin\theta_1 \\ 0 & 1 & 0 \end{bmatrix} \tag{10}$$

This is a singularity because the end-effector cannot move along the Z_4 direction. For $\theta_1 = 0°$, $^0Z_4 = Y_0 = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}^T$, and for $\theta_1 = 90°$, $^0Z_4 = X_0 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T$.
For $\theta_3 = 90°$, we have:

$$^0J_v(\theta_3 = 90°) = \begin{bmatrix} L_4 \sin\theta_1 & 0 & 0 \\ -L_4 \cos\theta_1 & 0 & 0 \\ 0 & 1 & -L_4 \end{bmatrix} \tag{11}$$

This is a singularity because the end-effector cannot move along the Y_4 direction. For $\theta_1 = 0°$, $^0Y_4 = X_0 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T$, and for $\theta_1 = 90°$, $^0Y_4 = Y_0 = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}^T$.

(g) Since the robot is massless except for the end-effector, the mass matrix is given by:

$$M = m_4 J_v^T J_v$$

$$\implies M = m_4 \begin{bmatrix} L_4 \sin\theta_1 \sin\theta_3 & -L_4 \cos\theta_1 \sin\theta_3 & 0 \\ 0 & 0 & 1 \\ -L_4 \cos\theta_1 \cos\theta_3 & -L_4 \sin\theta_1 \cos\theta_3 & -L_4 \sin\theta_3 \end{bmatrix} \begin{bmatrix} L_4 \sin\theta_1 \sin\theta_3 & 0 & -L_4 \cos\theta_1 \cos\theta_3 \\ -L_4 \cos\theta_1 \sin\theta_3 & 0 & -L_4 \sin\theta_1 \cos\theta_3 \\ 0 & 1 & -L_4 \sin\theta_3 \end{bmatrix}$$

$$\implies M = \begin{bmatrix} m_4 L_4^2 \sin^2\theta_3 & 0 & 0 \\ 0 & m_4 & -m_4 L_4 \sin\theta_3 \\ 0 & -m_4 L_4 \sin\theta_3 & m_4 L_4^2 \end{bmatrix} \tag{12}$$

(h) The gravity vector is given by:

$$G = -m_4 J_v^T g = -m_4 \begin{bmatrix} L_4 \sin\theta_1 \sin\theta_3 & -L_4 \cos\theta_1 \sin\theta_3 & 0 \\ 0 & 0 & 1 \\ -L_4 \cos\theta_1 \cos\theta_3 & -L_4 \sin\theta_1 \cos\theta_3 & -L_4 \sin\theta_3 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ -g \end{bmatrix}$$

$$\implies G = \begin{bmatrix} 0 \\ m_4 g \\ -m_4 g L_4 \sin\theta_3 \end{bmatrix} \tag{13}$$

Problem 2

(a) The schematic alongwith the frames that correspond to the model described by the urdf file is shown below.

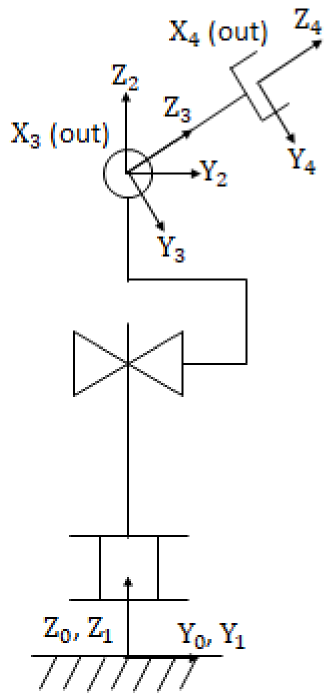


Figure 4: Frames as per the model described by the urdf

(b) The position of the end-effector in frame {3} is:

$$^{3}P_4 = \begin{bmatrix} 0 \\ 0 \\ L_4 \end{bmatrix} \tag{14}$$

(c) The results from SAI are:

```
========================================= Q2-c-i

End effector position for configuration i
  0 2.5 1.5

========================================= Q2-c-ii

End effector position for configuration ii
       -2.5 1.53081e-16         1.5
```

(i) $\theta_1 = 0°$, $d_2 = 1.5$ m, $\theta_3 = -90°$
For this, from 1 (a) (equation 3), we get

$$^{0}P_4 = \begin{bmatrix} -L_4 \cos\theta_1 \sin\theta_3 \\ -L_4 \sin\theta_1 \sin\theta_3 \\ d_2 + L_4 \cos(\theta_3) \end{bmatrix} = \begin{bmatrix} L_4 \\ 0 \\ d_2 \end{bmatrix} = \begin{bmatrix} 2.5 \\ 0 \\ 1.5 \end{bmatrix} \tag{15}$$

We observe that the result from SAI is not consistent with our expression from 1 (a). This is because frame {0} in question 1 is rotated with respect to frame {0} in figure 4 by 90° about Z_0. Keeping this in mind, the result from SAI can then be obtained by

$$^0P_4 = R_Z(\theta = 90^\circ)\begin{bmatrix} 2.5 \\ 0 \\ 1.5 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 2.5 \\ 0 \\ 1.5 \end{bmatrix} = \begin{bmatrix} 0 \\ 2.5 \\ 1.5 \end{bmatrix} \tag{16}$$

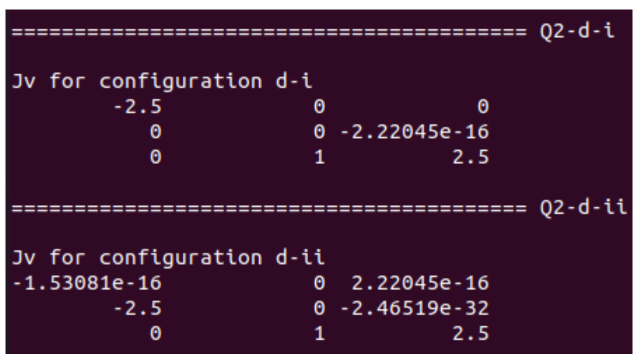
(ii) $\theta_1 = 90^\circ$, $d_2 = 1.5$ m, $\theta_3 = -90^\circ$
For this, from 1 (a) (equation 3), we get

$$^0P_4 = \begin{bmatrix} -L_4\cos\theta_1\sin\theta_3 \\ -L_4\sin\theta_1\sin\theta_3 \\ d_2 + L_4\cos(\theta_3) \end{bmatrix} = \begin{bmatrix} 0 \\ L_4 \\ d_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 2.5 \\ 1.5 \end{bmatrix} \tag{17}$$

We observe that the result from SAI is again not consistent with our expression from 1 (a). This is again explained by the fact that frame {0} in question 1 is rotated with respect to frame {0} in figure 4 by 90° about Z_0. The result from SAI can then be obtained by

$$^0P_4 = R_Z(\theta = 90^\circ)\begin{bmatrix} 0 \\ 2.5 \\ 1.5 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 2.5 \\ 1.5 \end{bmatrix} = \begin{bmatrix} -2.5 \\ 0 \\ 1.5 \end{bmatrix} \tag{18}$$

(d) The results from SAI are:



```
========================================== Q2-d-i

Jv for configuration d-i
        -2.5             0             0
           0             0  -2.22045e-16
           0             1           2.5

========================================== Q2-d-ii

Jv for configuration d-ii
-1.53081e-16             0   2.22045e-16
        -2.5             0  -2.46519e-32
           0             1           2.5
```

(i) $\theta_1 = 0^\circ$, $d_2 = 1.5$ m, $\theta_3 = -90^\circ$
For this, from 1 (c) (equation 7), we get

$$^0J_v = \begin{bmatrix} L_4\sin\theta_1\sin\theta_3 & 0 & -L_4\cos\theta_1\cos\theta_3 \\ -L_4\cos\theta_1\sin\theta_3 & 0 & -L_4\sin\theta_1\cos\theta_3 \\ 0 & 1 & -L_4\sin\theta_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ L_4 & 0 & 0 \\ 0 & 1 & L_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 2.5 & 0 & 0 \\ 0 & 1 & 2.5 \end{bmatrix} \tag{19}$$

We observe that the result from SAI is not consistent with our expression from 1 (c). This is because frame {0} in question 1 is rotated with respect to frame {0} in figure 4 by 90° about Z_0. Keeping this in mind, the result from SAI can then be obtained by the transformation:

$$^0J_v = R_Z(90^\circ)\begin{bmatrix} 0 & 0 & 0 \\ 2.5 & 0 & 0 \\ 0 & 1 & 2.5 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 \\ 2.5 & 0 & 0 \\ 0 & 1 & 2.5 \end{bmatrix} = \begin{bmatrix} -2.5 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 2.5 \end{bmatrix} \tag{20}$$

(ii) $\theta_1 = 90°$, $d_2 = 1.5$ m, $\theta_3 = -90°$
For this, from 1 (c) (equation 7), we get

$$ {}^0J_v = \begin{bmatrix} L_4\sin\theta_1\sin\theta_3 & 0 & -L_4\cos\theta_1\cos\theta_3 \\ -L_4\cos\theta_1\sin\theta_3 & 0 & -L_4\sin\theta_1\cos\theta_3 \\ 0 & 1 & -L_4\sin\theta_3 \end{bmatrix} = \begin{bmatrix} -L_4 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & L_4 \end{bmatrix} = \begin{bmatrix} -2.5 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 2.5 \end{bmatrix} \tag{21} $$

We observe that the result from SAI is again not consistent with our expression from 1 (c). This is again explained by the fact that frame {0} in question 1 is rotated with respect to frame {0} in figure 4 by 90° about Z_0. The result from SAI can then be obtained by

$$ {}^0J_v = R_Z(90°)\begin{bmatrix} -2.5 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 2.5 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} -2.5 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 2.5 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ -2.5 & 0 & 0 \\ 0 & 1 & 2.5 \end{bmatrix} \tag{22} $$

(e) We have from 1 (g) (equation 12)

$$ m_{11} = m_4 L_4^2 \sin^2\theta_3, \quad m_{22} = m_4, \quad m_{33} = m_4 L_4^2 \tag{23} $$

(i) $\theta_1 = 0°$, $d_2 = 1.5$ m, and θ_3 varies from $-90°$ to $90°$.
For this case, the plot of m_{11}, m_{22} and m_{33} versus θ_3 is shown below.

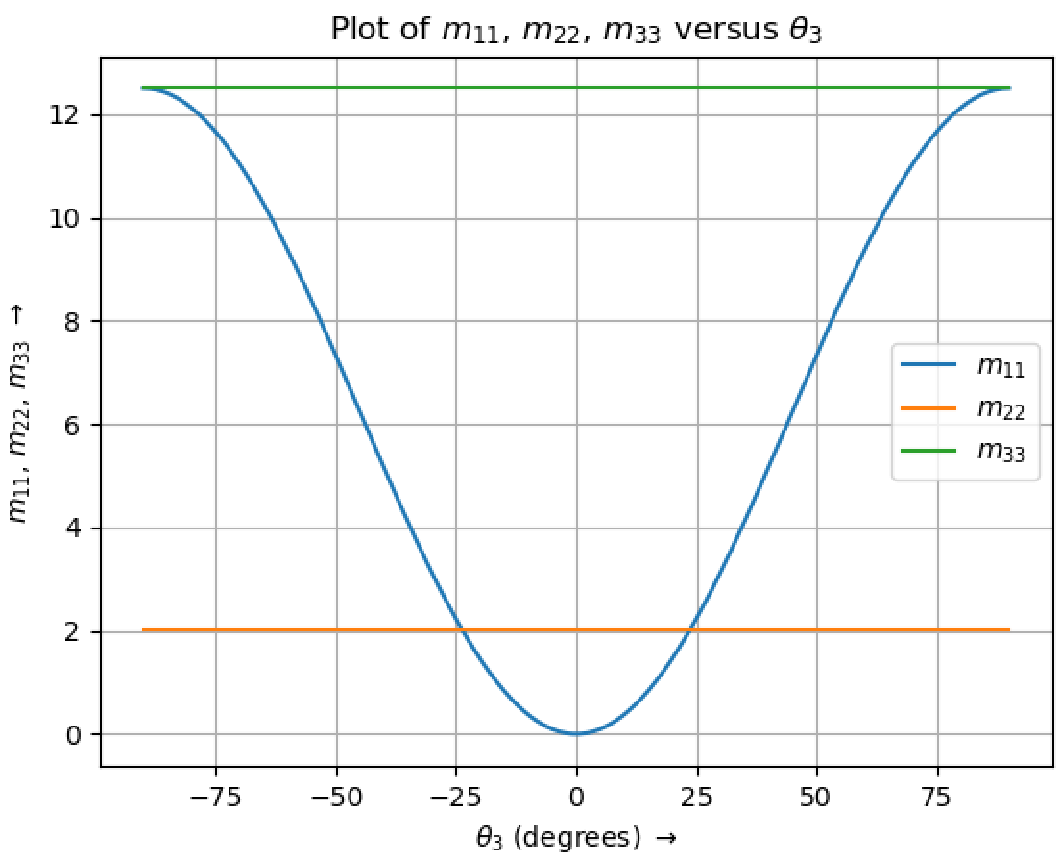


Figure 5: Plot of m_{11}, m_{22} and m_{33} versus θ_3

We observe that the plots agree with 23. That is, m_{11} varies sinusoidally (starts at $m_4 L_4^2 = 12.5$, decreases to 0, and then increases back to $m_4 L_4^2 = 12.5$), m_{22} remains constant and equal to 2 ($= m_4$), and m_{33} remains constant and equal to 12.5 ($= m_4 L_4^2$).

(ii) $\theta_1 = 0°$, $\theta_3 = 0°$, and d_2 varies from 0 to 2 m.
For this case, the plot of m_{11}, m_{22} and m_{33} versus d_2 is shown below.

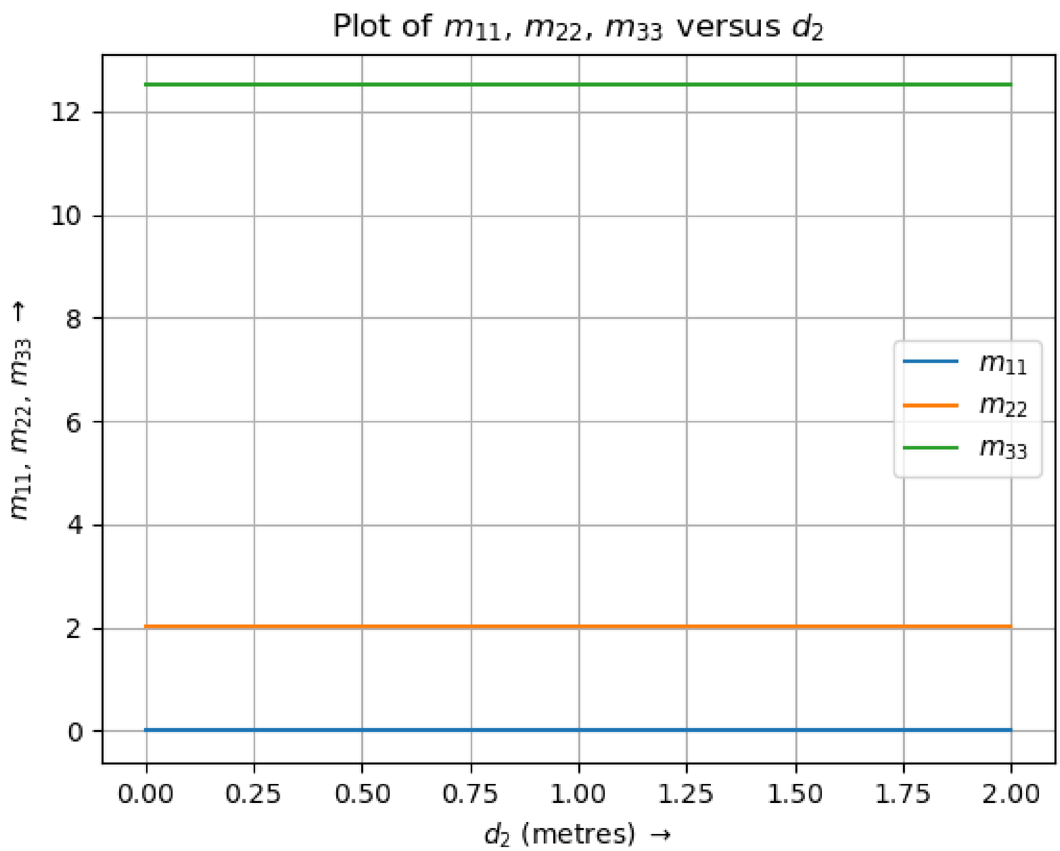


Figure 6: Plot of m_{11}, m_{22} and m_{33} versus d_2

We again observe that the plots agree with 23. Since m_{11} does not depend on d_2, it remains constant and equal to 0 (since $\theta_3 = 0°$). Furthermore, m_{22} remains constant and equal to 2, and m_{33} remains constant and equal to 12.5.

(f) We have from 1 (h) (equation 13)

$$G_1 = 0, \quad G_2 = m_4 g, \quad G_3 = -m_4 g L_4 \sin\theta_3 \tag{24}$$

(i) $\theta_1 = 0°$, $d_2 = 1.5$ m, and θ_3 varies from $-90°$ to $90°$.
For this case, the plot of G_1, G_2 and G_3 versus θ_3 is shown below.

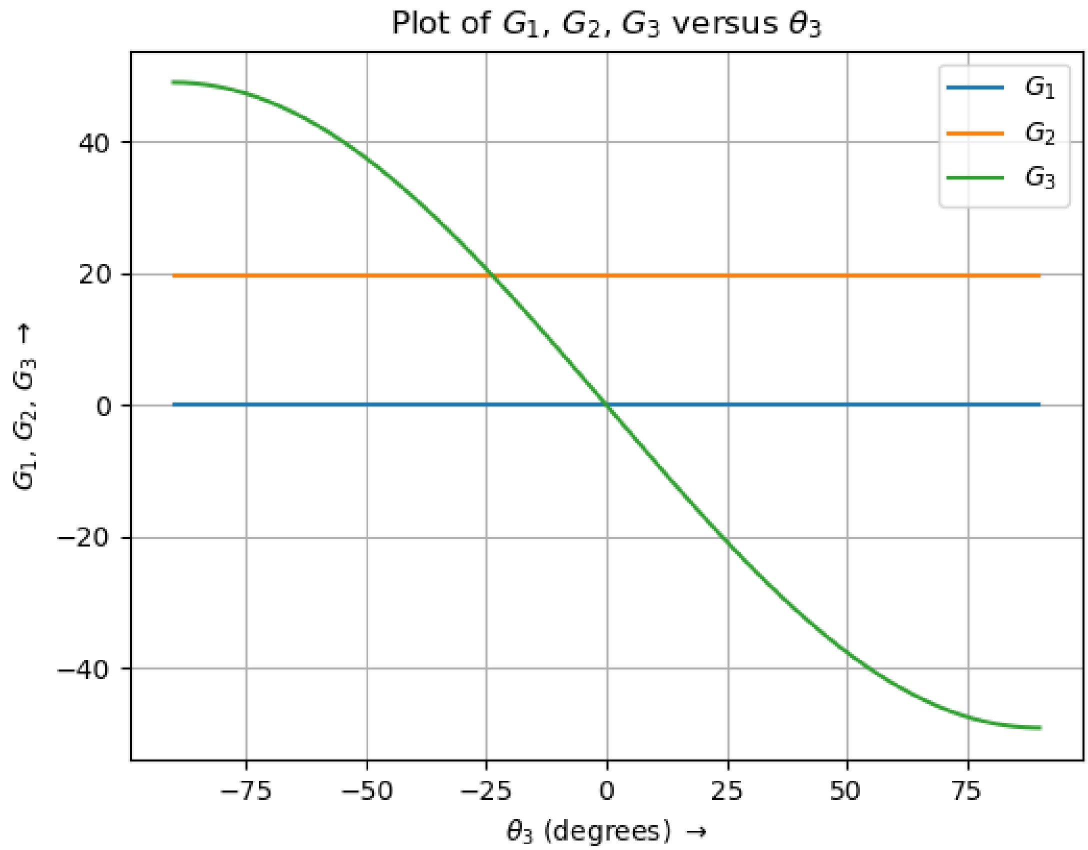


Figure 7: Plot of G_1, G_2 and G_3 versus θ_3

We observe that the plots agree with 24. That is, G_3 varies sinusoidally from $m_4 g L_4 = 49$ to -49, G_1 remains constant and equal to 0, and G_2 remains constant and equal to 19.6 ($= m_4 g$).

(ii) $\theta_1 = 0°$, $\theta_3 = 0°$, and d_2 varies from 0 to 2 m.
For this case, the plot of G_1, G_2 and G_3 versus d_2 is shown below.

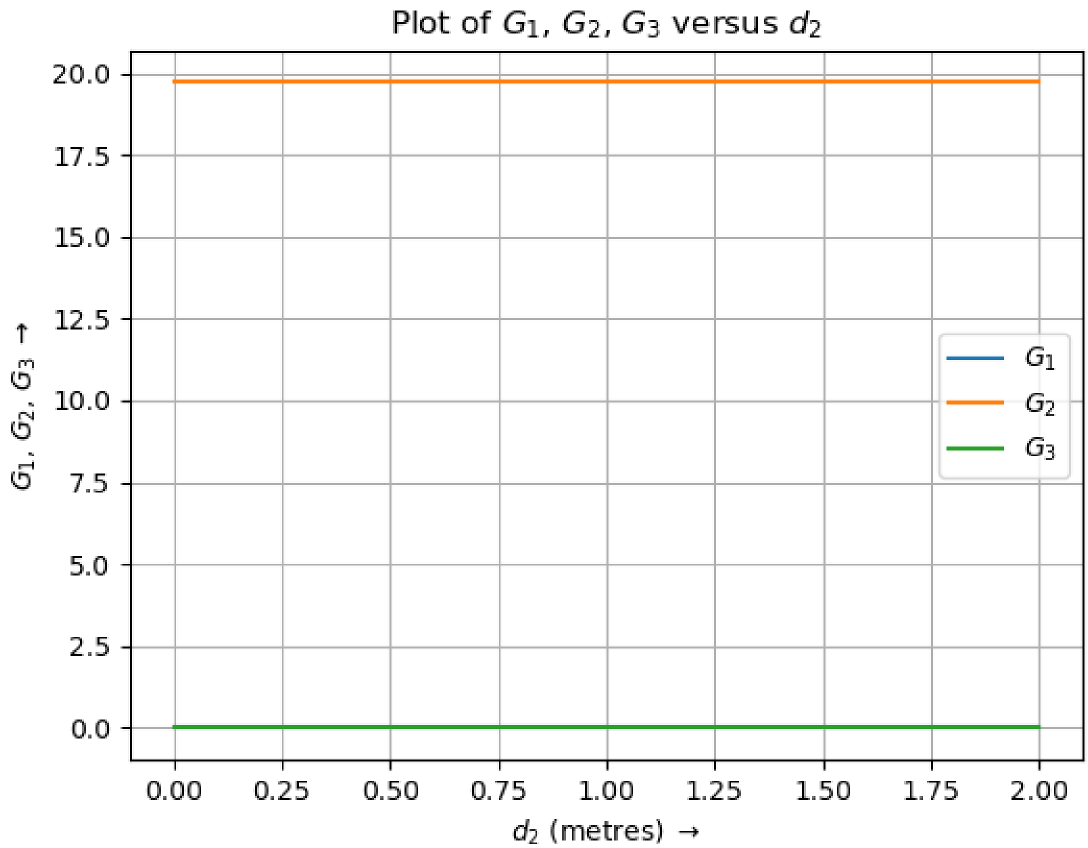


Figure 8: Plot of G_1, G_2 and G_3 versus d_2

We again observe that the plots agree with 24. Since G_3 does not depend on d_2, it remains constant and equal to 0 (since $\theta_3 = 0°$). Furthermore, G_1 remains constant and equal to 0, and G_2 remains constant and equal to 19.6.

(g) The gravity vector for the RPRP robot becomes:

$$G_1 = 0, \quad G_2 = m_4 g, \quad G_3 = -m_4 g d_4 \sin\theta_3, \quad G_4 = m_4 g \cos\theta_3 \tag{25}$$

We now look at the case where $\theta_1 = 0°$, $d_2 = 1.5$ m, $\theta_3 = 45°$ and d_4 varies from 0 to 2 m. For this, the plot of G_1, G_2, G_3 and G_4 versus d_4 is shown below.

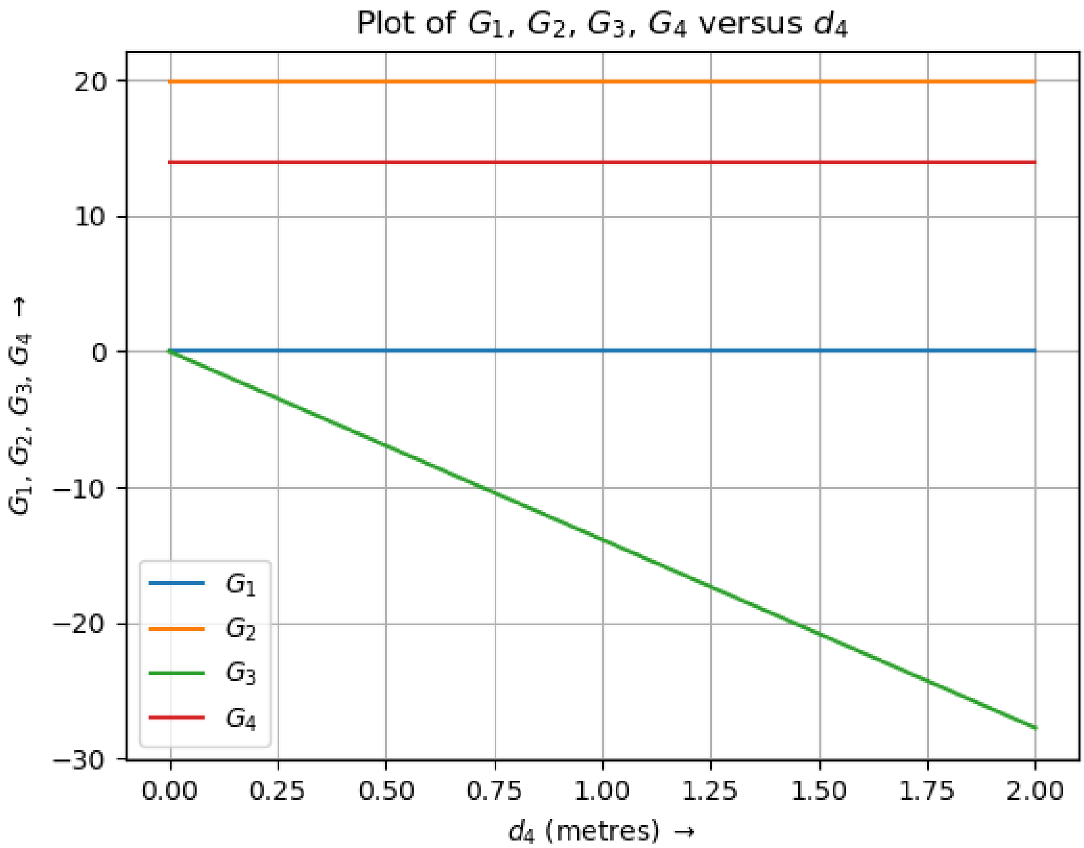


Figure 9: Plot of G_1, G_2, G_3 and G_4 versus d_4

We observe that the plots agree with 25. G_1 and G_2 remain constant and equal to 0 and 19.6 respectively. G_3 decreases linearly (as d_4 is varied) from 0 to -27.72 ($= -m_4 g(2 \sin 45°)$). Since G_4 does not depend on d_4, it stays constant and equal to 13.86 ($= m_4 g \cos 45°$).