AS5850 Project

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1 Problem Statement

Write a 2D FEM code, to solve a plane stress mechanical equilibrium equation over a domain as shown in the figure below for the linear elastic case, with C^0- quadratic isoparametric quadrilateral elements. The flux smoothing and averaging has to be implemented while computing the stress components. Here edges $\overline{AB}$ and $\overline{CD}$ are traction-free ($\sigma_{yy} = \tau_{xy} = 0$), edges $\overline{DE}$ and $\overline{AF}$ are traction-free ($\sigma_{xx} = \tau_{xy} = 0$), where horizontal and vertical directions are considered as x and y axes, respectively. Edge $\overline{BC}$ has roller support, i.e., $v = 0$ and $\tau_{xy} = 0$, and edge $\overline{EF}$ has $\sigma_{yy} = \sigma^0$ and $\tau_{xy} = 0$. Consider suitable material properties.

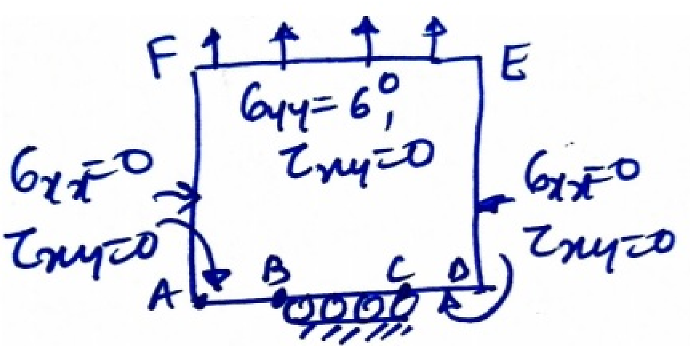

Figure 1: Axially loaded 2D domain

2 Weak form and development of element equations

For the case of plane stress, we have

$$\sigma_{zz} = \tau_{xz} = \tau_{yz} = \gamma_{xz} = \gamma_{yz} = 0$$

with the rest of the stress and strain components being non-zero. Assuming no body forces i.e. $f_x = f_y = 0$, the two governing equations reduce to

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = 0$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} = 0$$

We then write the residuals

$$R_x = \frac{\partial \widetilde{\sigma}_{xx}}{\partial x} + \frac{\partial \widetilde{\tau}_{xy}}{\partial y}$$

$$R_y = \frac{\partial \widetilde{\tau}_{xy}}{\partial x} + \frac{\partial \widetilde{\sigma}_{yy}}{\partial y}$$

The weighted integral statements thus become

$$\iint R_x \phi_i^{(e)} t dx dy = 0$$

$$\iint R_y \phi_i^{(e)} t dx dy = 0$$

where $\phi_i^{(e)}$ are the trial functions of the element, i ranges from 1 to n, the number of nodes in an element, and t is the thickness of the specimen. For convenience, let $\phi_i^{(e)}(x) = \phi_i(x)$. Assuming a constant thickness, we have

$$\iint \left(\frac{\partial \widetilde{\sigma}_{xx}}{\partial x} + \frac{\partial \widetilde{\tau}_{xy}}{\partial y} \right) \phi_i^{(e)} dx dy = 0$$

$$\iint \left(\frac{\partial \widetilde{\tau}_{xy}}{\partial x} + \frac{\partial \widetilde{\sigma}_{yy}}{\partial y} \right) \phi_i^{(e)} dx dy = 0$$

Using the chain rule

$$\frac{\partial}{\partial x}(\widetilde{\sigma}_{xx}\phi_i) = \phi_i \frac{\partial \widetilde{\sigma}_{xx}}{\partial x} + \widetilde{\sigma}_{xx} \frac{\partial \phi_i}{\partial x}$$

and similarly for other terms, we get

$$\iint \left(\widetilde{\sigma}_{xx} \frac{\partial \phi_i}{\partial x} + \widetilde{\tau}_{xy} \frac{\partial \phi_i}{\partial y} \right) dx dy = \iint \left(\frac{\partial}{\partial x}(\widetilde{\sigma}_{xx}\phi_i) + \frac{\partial}{\partial y}(\widetilde{\tau}_{xy}\phi_i) \right) dx dy$$

$$\iint \left(\widetilde{\tau}_{xy} \frac{\partial \phi_i}{\partial x} + \widetilde{\sigma}_{yy} \frac{\partial \phi_i}{\partial y} \right) dx dy = \iint \left(\frac{\partial}{\partial x}(\widetilde{\tau}_{xy}\phi_i) + \frac{\partial}{\partial y}(\widetilde{\sigma}_{yy}\phi_i) \right) dx dy$$

Using the divergence theorem

$$\iint \left(\frac{\partial G_x}{\partial x} + \frac{\partial G_y}{\partial y} \right) dxdy = \oint (n_x G_x + n_y G_y) ds$$

the equations further reduce to

$$\iint \left(\tilde{\sigma}_{xx} \frac{\partial \phi_i}{\partial x} + \tilde{\tau}_{xy} \frac{\partial \phi_i}{\partial y} \right) dxdy = \oint (\tilde{\sigma}_{xx} n_x + \tilde{\tau}_{xy} n_y) \phi_i ds$$

$$\iint \left(\tilde{\tau}_{xy} \frac{\partial \phi_i}{\partial x} + \tilde{\sigma}_{yy} \frac{\partial \phi_i}{\partial y} \right) dxdy = \oint (\tilde{\tau}_{xy} n_x + \tilde{\sigma}_{yy} n_y) \phi_i ds$$

Since

$$t_x^{(\hat{n})} = \tilde{\sigma}_{xx} n_x + \tilde{\tau}_{xy} n_y$$
$$t_y^{(\hat{n})} = \tilde{\tau}_{xy} n_x + \tilde{\sigma}_{yy} n_y$$

the equations finally reduce to

$$\iint \left(\tilde{\sigma}_{xx} \frac{\partial \phi_i}{\partial x} + \tilde{\tau}_{xy} \frac{\partial \phi_i}{\partial y} \right) dxdy = \oint t_x^{(\hat{n})} \phi_i ds$$

$$\iint \left(\tilde{\tau}_{xy} \frac{\partial \phi_i}{\partial x} + \tilde{\sigma}_{yy} \frac{\partial \phi_i}{\partial y} \right) dxdy = \oint t_y^{(\hat{n})} \phi_i ds$$

Letting

$$\tilde{u} = \sum_{j=1}^{n} u_j \phi_j$$

$$\tilde{v} = \sum_{j=1}^{n} v_j \phi_j$$

we have

$$\tilde{\epsilon}_x = \frac{\partial \tilde{u}}{\partial x} = \sum_{j=1}^{n} u_j \frac{\partial \phi_j}{\partial x}$$

$$\tilde{\epsilon}_y = \frac{\partial \tilde{v}}{\partial y} = \sum_{j=1}^{n} v_j \frac{\partial \phi_j}{\partial y}$$

$$\tilde{\gamma}_{xy} = \frac{\partial \tilde{u}}{\partial y} + \frac{\partial \tilde{v}}{\partial x} = \sum_{j=1}^{n} \left(u_j \frac{\partial \phi_j}{\partial y} + v_j \frac{\partial \phi_j}{\partial x} \right)$$

Let

$$[B]^{(e)} = \begin{bmatrix} \frac{\partial \phi_1}{\partial x} & 0 & \frac{\partial \phi_2}{\partial x} & 0 & \dots & \frac{\partial \phi_n}{\partial x} & 0 \\ 0 & \frac{\partial \phi_1}{\partial y} & 0 & \frac{\partial \phi_2}{\partial y} & \dots & 0 & \frac{\partial \phi_n}{\partial y} \\ \frac{\partial \phi_1}{\partial y} & \frac{\partial \phi_1}{\partial x} & \frac{\partial \phi_2}{\partial y} & \frac{\partial \phi_2}{\partial x} & \dots & \frac{\partial \phi_n}{\partial y} & \frac{\partial \phi_n}{\partial x} \end{bmatrix}$$

and

$$\{a\} = \begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ \vdots \\ u_n \\ v_n \end{bmatrix}$$

Hence, we can write

$$\{\tilde{\epsilon}\}^{(e)} = \begin{bmatrix} \tilde{\epsilon}_x \\ \tilde{\epsilon}_y \\ \tilde{\gamma}_{xy} \end{bmatrix} = [B]^{(e)} \{a\}$$

Letting

$$[\phi]^{(e)} = \begin{bmatrix} \phi_1 & 0 & \phi_2 & 0 & \dots & \phi_n & 0 \\ 0 & \phi_1 & 0 & \phi_2 & \dots & 0 & \phi_n \end{bmatrix}$$
$$\implies \{\tilde{u}\}^{(e)} = [\phi]^{(e)} \{a\}$$

The constitutive relation is

$$\{\tilde{\sigma}\}^{(e)} = [C] \{\tilde{\epsilon}\}^{(e)}$$
$$\implies \{\tilde{\sigma}\}^{(e)} = [C][B]^{(e)} \{a\}$$

Finally, letting

$$\{\tilde{t}\} = \begin{bmatrix} t_x^{(\hat{n})} \\ t_y^{(\hat{n})} \end{bmatrix}$$

the weighted integral equation can be rewritten as

$$\iint [B]^T \{\tilde{\sigma}\} dxdy = \oint [\phi]^T \{\tilde{t}\} ds$$
$$\implies \left(\iint [B]^T [C] [B] dxdy \right) \{a\} = \oint [\phi]^T \{\tilde{t}\} ds$$
$$\implies [K]^{(e)} \{a\}^{(e)} = \{F\}^{(e)}$$

where

$$[K]^{(e)} = \iint^{(e)} [B]^T [C] [B] dxdy$$

$$\{F\}^{(e)} = \oint^{(e)} [\phi]^T \{\tilde{t}\} ds$$

We use a C^0-quadratic isoparametric quadrilateral element with 8 nodes, 4 of them being on the edges, and 4 of them being on the centre of the edges. The trial functions are given by

$$\phi_1(\xi,\eta) = -\frac{1}{4}(1-\xi)(1-\eta)(1+\xi+\eta)$$

$$\phi_2(\xi,\eta) = -\frac{1}{4}(1+\xi)(1-\eta)(1-\xi+\eta)$$

$$\phi_3(\xi,\eta) = -\frac{1}{4}(1+\xi)(1+\eta)(1-\xi-\eta)$$

$$\phi_4(\xi,\eta) = -\frac{1}{4}(1-\xi)(1+\eta)(1+\xi-\eta)$$

$$\phi_5(\xi,\eta) = \frac{1}{2}(1-\xi^2)(1-\eta)$$

$$\phi_6(\xi,\eta) = \frac{1}{2}(1+\xi)(1-\eta^2)$$

$$\phi_7(\xi,\eta) = \frac{1}{2}(1-\xi^2)(1+\eta)$$

$$\phi_8(\xi,\eta) = \frac{1}{2}(1-\xi)(1-\eta^2)$$

The mapping is given by

$$x^{(e)} = \sum_{j=1}^{8} x_j^{(e)}\phi_j(\xi,\eta), \qquad y^{(e)} = \sum_{j=1}^{8} y_j^{(e)}\phi_j(\xi,\eta)$$

Since all 4 interior domain nodes are placed at the center of the respective edges i.e $x_5^{(e)} = \dfrac{x_1^{(e)} + x_2^{(e)}}{2}$ and so on, we get

$$x = \frac{1}{4}[x_1(\xi-1)(\eta-1) + x_2(\xi+1)(1-\eta) + x_3(\xi+1)(\eta+1) + x_4(\eta+1)(1-\xi)]$$

$$\implies \frac{\partial x}{\partial \xi} = \frac{1}{4}[(x_2-x_1)(1-\eta) + (x_3-x_4)(1+\eta)]$$

$$\frac{\partial x}{\partial \eta} = \frac{1}{4}[(x_4-x_1)(1-\xi) + (x_3-x_2)(1+\xi)]$$

Similarly, we have

$$\frac{\partial y}{\partial \xi} = \frac{1}{4}[(y_2-y_1)(1-\eta) + (y_3-y_4)(1+\eta)]$$

$$\frac{\partial y}{\partial \eta} = \frac{1}{4}[(y_4-y_1)(1-\xi) + (y_3-y_2)(1+\xi)]$$

Defining the Jacobian $J^{(e)}(\xi,\eta)$ as

$$J^{(e)}(\xi,\eta) = \begin{bmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial y}{\partial \xi} \\ \dfrac{\partial x}{\partial \eta} & \dfrac{\partial y}{\partial \eta} \end{bmatrix}$$

we have

$$\begin{bmatrix} \dfrac{\partial \phi_i}{\partial \xi} \\ \dfrac{\partial \phi_i}{\partial \eta} \end{bmatrix} = J^{(e)}(\xi,\eta) \begin{bmatrix} \dfrac{\partial \phi_i}{\partial x} \\ \dfrac{\partial \phi_i}{\partial y} \end{bmatrix}$$

$$\implies \begin{bmatrix} \dfrac{\partial \phi_i}{\partial x} \\ \dfrac{\partial \phi_i}{\partial y} \end{bmatrix} = J^{-1} \begin{bmatrix} \dfrac{\partial \phi_i}{\partial \xi} \\ \dfrac{\partial \phi_i}{\partial \eta} \end{bmatrix}$$

This can be used to evaluate the $[B]$ matrix defined earlier. The expression for $[K]$ becomes

$$[K]^{(e)} = \int_{-1}^{1}\int_{-1}^{1} [B]^T[C][B]\det(J)d\xi d\eta$$

Using a Gauss-Quadrature rule, this can we written as

$$[K]^{(e)} = \sum_{k=1}^{n}\sum_{l=1}^{n} w_{nk}w_{nl}\left([B]^T[C][B]\det(J)\right)\Big|_{(\xi_{nl},\eta_{nl})}$$

Similarly for the forcing vector we have

$$\{F\}^{(e)} = \int_{-1}^{1} [\phi]^T\{\tilde{t}\}J_\Gamma^{(e)}\Big|_{(\xi,1)} d\xi$$

where $J_\Gamma^{(e)}$ is the boundary Jacobian given by

$$J_\Gamma^{(e)} = \sqrt{\left(\frac{\partial x}{\partial \xi}\right)^2 + \left(\frac{\partial y}{\partial \xi}\right)^2}$$

Hence, using Gauss-quadrature again we get

$$\{F\}^{(e)} = \sum_{l=1}^{n} w_{nl}\left([\phi]^T\{\tilde{t}\}J_\Gamma^{(e)}\right)\Big|_{(\xi_{nl},1)}$$

For this case

$$p(\text{order of trial function polynomial}) = 3$$

$$m(\text{highest derivative of trial function in weak form}) = 1$$

$$\implies p - m + 1 = 3 - 1 + 1 = 3$$

Hence, we use a 3-point rule when using the quadrature. After this, the flux/stress is evaluated at the Gauss points (ξ_i,η_i) (optimum flux sampling points) using

$$\{\tilde{\sigma}\}_{opt} = [C][B]\{a\}\Big|_{(\xi_i,\eta_i)}$$

We then use least square approximation to compute flux at the nodes using

$$\{\tilde{\sigma}\}_{nodes} = [TR]\{\tilde{\sigma}\}_{opt}$$

where $[TR]$ is the transformation matrix given by

$$[TR] = [E][S]$$

The matrices $[E]$ and $[S]$ are

$$[E] = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & -1 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 0 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}$$

$$[S] = \frac{1}{567}\begin{bmatrix} 63 & 63 & 63 & 63 & 63 & 63 & 63 & 63 & 63 \\ -122 & -122 & -122 & 0 & 0 & 0 & 122 & 122 & 122 \\ -122 & 0 & 122 & -122 & 0 & 122 & -122 & 0 & 122 \end{bmatrix}$$

After computing the nodal flux values, flux averaging is performed. For a node shared by N elements, the average flux at the node is

$$\{\widetilde{\sigma}\}_{node} = \frac{1}{N}\sum_{(e)}\{\widetilde{\sigma}\}_{node}^{(e)}$$

Also, the compliance matrix $[C]$ for plane stress is

$$[C] = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & 1-\nu \end{bmatrix}$$

where E is the Young's modulus, and ν is the Poisson's ratio of the material. Here the material chosen is steel, with $E = 200$ GPa, and $\nu = 0.3$.

3 Explanation of Code and Results

All these are implemented in a MATLAB program. We first find the elemental stiffness matrices using the 3-point rule, which are then assembled into a global stiffness matrix $[K]$, which turns out to be a banded matrix. A similar thing is done for the forcing vector $\{F\}$. After this, the displacement boundary conditions are enforced, which are $v = 0$ over edge $\overline{BC}$, and $u = 0$ at the mid-point of $\overline{BC}$. From this, we get a reduced system of equations, which are then solved to get the vector of unknown displacements $\{a\}$. After this flux computation i.e. flux smoothing and averaging is performed to evaluate flux at the nodes. Finally, we evaluate the flux at some test points for convergence analysis.
The dimensions of the plate are taken to be $L_x = L_y = 1$ m. Also B and C are assumed to be located symmetrically with respect to the centerline of the plate such that $\overline{BC} = \frac{\overline{AD}}{2}$. We first apply a force per unit length of 50000 MPa m, so that the displacements u and v are of appreciable magnitude. The two figures below represent the plate before and after the application of stress.

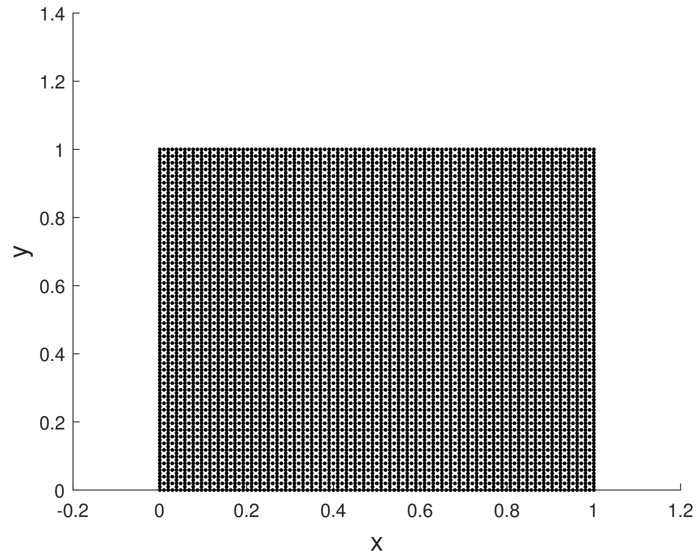

Figure 2: Plate before stress

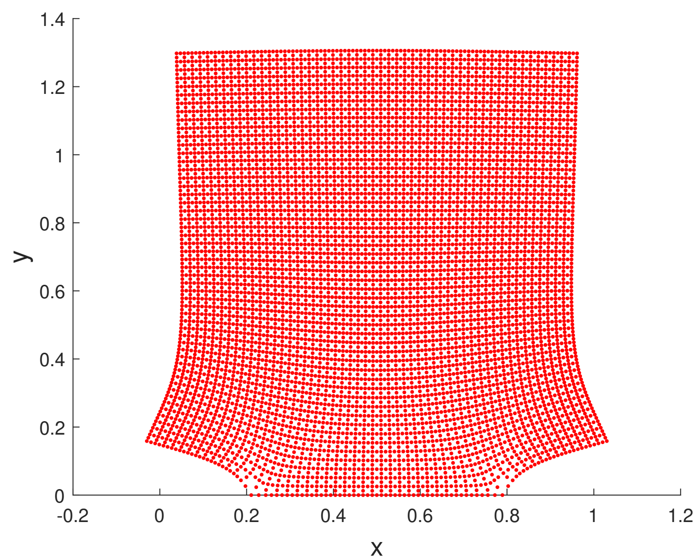

Figure 3: Plate after stress

The plate clearly displaces the way it should, which validates our FEM analysis. The analysis hereafter is carried out for a force per unit length of 16.8 MPa m. For convergence analysis, we evaluate the three stress components (flux) at 49 predefined test points in the domain, whose coordinates are given by

$$(x, y) \in \left\{\frac{1}{10}, \frac{7}{30}, \frac{11}{30}, \frac{1}{2}, \frac{19}{30}, \frac{23}{30}, \frac{9}{10}\right\}$$

We start with $N = 28$ elements along edge $\overline{AF}$, where the total number of nodes are given by $3N^2 - 2N$. The three stress components (σ_x, σ_y and τ_{xy}) are evaluated at the 49 test points and stored in three 7×7 matrices $[\sigma_x]$, $[\sigma_y]$ and $[\tau_{xy}]$ respectively. N is increased by 4 everytime until

$$e_1 = \frac{||[\sigma_x]_{i+1} - [\sigma_x]_i||}{||[\sigma_x]_{i+1}||} < \epsilon$$

$$e_2 = \frac{||[\sigma_y]_{i+1} - [\sigma_y]_i||}{||[\sigma_y]_{i+1}||} < \epsilon$$

$$\text{and } e_3 = \frac{||[\tau_{xy}]_{i+1} - [\tau_{xy}]_i||}{||[\tau_{xy}]_{i+1}||} < \epsilon$$

where ϵ is a pre-specified tolerance. We take $\epsilon = 0.018$. Below is the table of convergence representing this.

N	e_1	e_2	e_3
28	1	1	1
32	0.0775	0.0110	0.0649
36	0.0398	0.0069	0.0522
40	0.0241	0.0055	0.0291
44	0.0217	0.0049	0.0182
48	0.0193	0.0034	0.0150
52	0.0176	0.0029	0.0085

We also plot the relative norm versus N, as shown below.

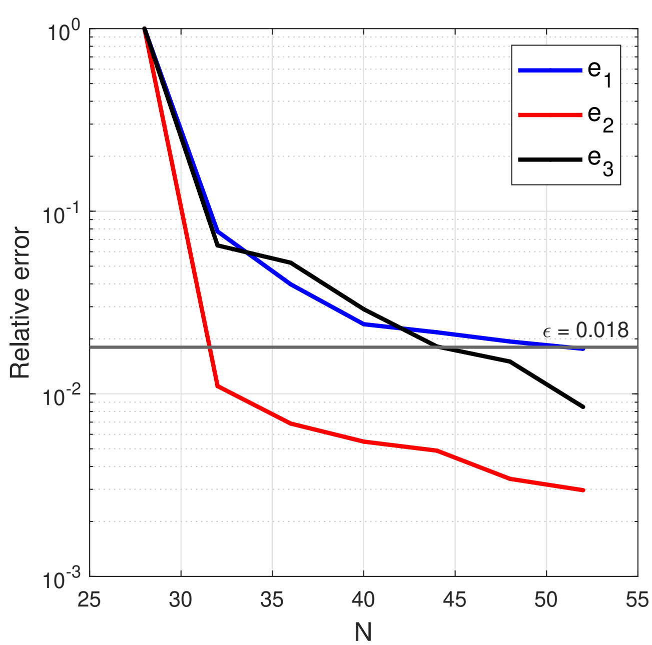

Figure 4: Plot of relative norm versus N

We observe that the relative norm keeps decreasing as we increase N. Hence, h-convergence is demonstrated. The iteration stops at $N = 52$ (8008 nodes). The 3D scatter plots of the displacements u and v at the nodes for this case are shown below.

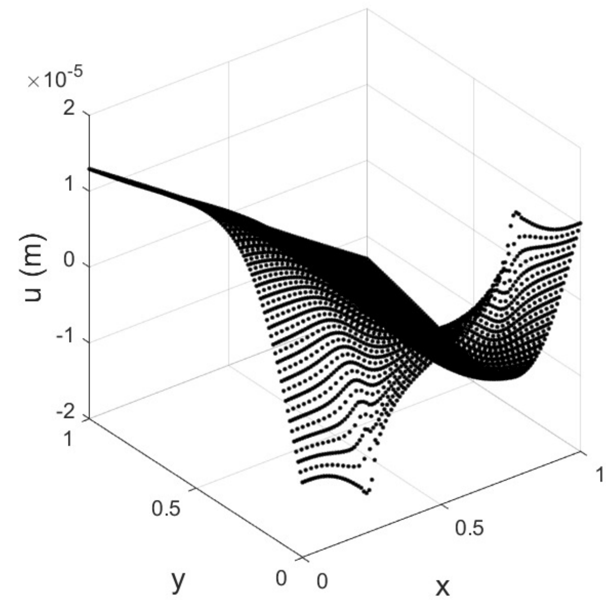

Figure 5: 3D scatter plot of u at the nodes

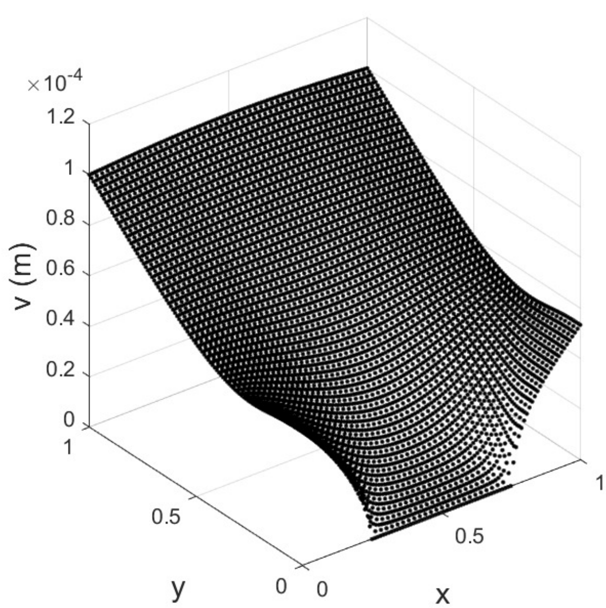

Figure 6: 3D scatter plot of v at the nodes

As expected, the plot of u is asymmetric, and the plot of v is symmetric about the centreline. Next, we plot the flux. The heatmap of the three stress components (σ_x, σ_y and τ_{xy}) at the nodes are shown below.

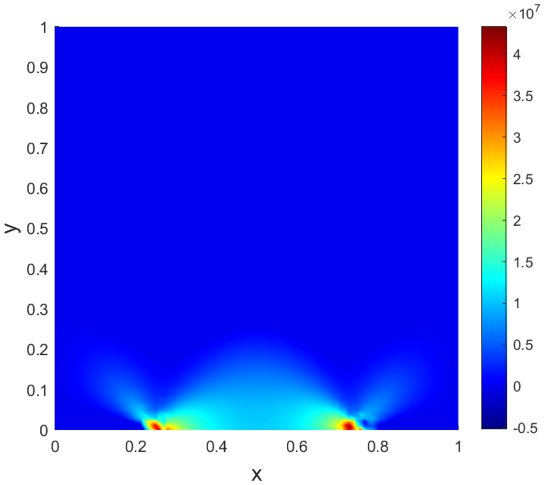

Figure 7: Heatmap of σ_x at the nodes

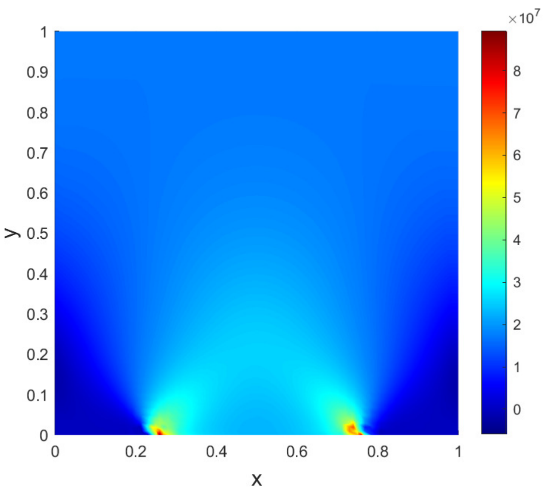

Figure 8: Heatmap of σ_y at the nodes

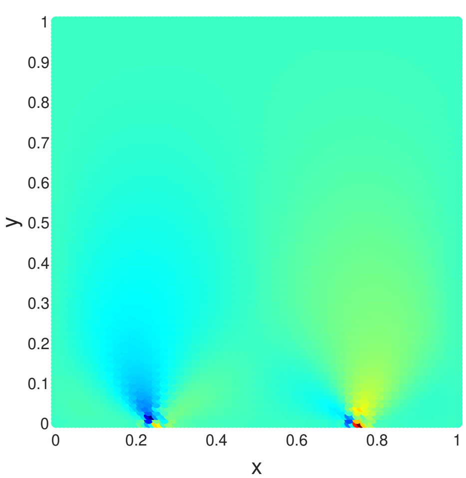

Figure 9: Heatmap of τ_{xy} at the nodes

4 Conclusion

We have solved a 2D problem by FEM using C^0-quadratic isoparametric quadrilateral elements. We first developed the weak form and the element equations for the stiffness matrix and forcing vector. We then incorporated all this in a MATLAB code. For flux computations, we also performed flux smoothing (using least squares approximation) and averaging. For convergence analysis, we evaluated the flux at some test points and observed that the relative norm kept decreasing as we increased the number of elements. Thus h-convergence was also demonstrated. Finally, we plotted the displacements and the stresses/flux at all the nodes.

5 References

1) Ottosen, N. S., and Petersson, H., Introduction to the Finite Element Method, Prentice Hall, 1992.
2) `https://caendkoelsch.wordpress.com/2017/12/03/how-are-stiffness-matrices-assembled-in-fem/`

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