MA365

Advanced Mathematics for Engineers/Scientists

MA365 ADVANCED MATHEMATICS FOR ENGINEERS/SCIENTISTS

ABET eCOURSEBOOK AY2020

TABLE OF CONTENTS

SECTION I. COURSE ASSESSMENTS

TAB A .... AY2020 Course Assessment

TAB B .... AY2019 Course Assessment

TAB C .... AY2018 Course Assessment

TAB D .... AY2017 Course Assessment

NOTE: THE COURSE WAS NOT OFFERED PRIOR TO AY2017

SECTION II. EXAMINATIONS

TAB E .... WPR #1

TAB F .... WPR #2

TAB G .... Term End Examination

TAB H .... Final Course Grades

SECTION III. ASSIGNMENTS

TAB I .... Problem Sets

TAB J .... Article Critique

SECTION I

COURSE ASSESSMENTS

TAB A AY2020 COURSE ASSESSMENT

MADN-MATH 11 January 2020

MEMORANDUM THRU Service Program Director, LTC(P) Michael Scioletti, Department of Mathematical Sciences, United States Military Academy, West Point, New York 10996

FOR COL Tina Hartley, Head, Department of Mathematical Sciences, United States Military Academy, West Point, New York 10996

SUBJECT: MA365, Advanced Mathematics for Engineers and Scientists, Course End Report, AY20-1

1. Purpose. The intent of this memorandum is to review and assess MA365, Advanced Mathematics for Engineers & Scientists, as it was conducted in AY20-1. The following course summary and assessment is based on instructor and cadet feedback.

2. Background. MA365 is a course designed for the advanced mathematics student that has completed courses in differential equations and vector calculus (those that have completed MA153 and MA255) and will study ME, CE, EE, NE, Physics or Space Science. MA365 was created due to the approximately 75% overlap in the MA364 and MA153/MA255 curricula (topics in differential equations and vector calculus); MA365 begins where the advanced core mathematics program ends. The advanced engineering course offering includes topics in linear algebra, complex variables, Fourier series, partial differential equations, and computational mathematics. There is approximately a 25% overlap in the MA364 and MA365 curricula. Common topics in both courses include complex variables, Fourier series, and partial differential equations.

3. Course Enrollment. AY20-1 was the fifth semester the Math Department has offered MA365 as its own course since inception in AY17. In AY20-1, there were five sections of MA365 (71 cadets). MA365 is conducted as a 75-minute, 30 lesson course. LtCol Kris Ahlers taught two sections, while Dr. Ensela Mema taught the remaining three sections. The table below shows the number of cadets enrolled by major.

Major	Number of Cadets
Mechanical Engineering	37
Electrical Engineering	1
Chemical Engineering	11
Physics	8
Nuclear Engineering	12
Plebe (undeclared)	2

4. Assessment and Recommendations.

a. Bottom Line up Front: The overall average for the course was an 89.5%. There were no course failures, one C-, and one A+ grades. The course average for the TEE was 89.1%.

b. Course Performance. The average grades in AY20-1 were very similar to those earned by cadets in previous semesters. In fact, the overall course average and TEE average have all been within one percentage point of one another.

c. Develop a Fundamental Set of Mathematical Skills: MA365 is designed as a set of three "micro-courses" that develop a critical set of mathematical skills which will enable students to better understand the language of their respective Engineering field. Each of the three blocks focuses on the foundational material of MA363, MA371, and MA484.

d. Theory Application Days: Included this year was 5 lessons which we used to explore course topics more in depth. These lessons were distributed throughout the course and served as a method to review, study, and critique scholarly articles as well as tackle linear algebra and PDEs across real world applications.

(1) Block 1: Linear Algebra and Complex Variables. The cadets began the course with a 11 lesson block that focused on the use of linear algebra as a means to efficiently solve system of linear equations. Additionally, the students were introduced to eigenvectors, eigenvalues and diagonalization of matrices, which were reinforced later in the course in solving real world application problems during the theory application days.

	Problem Set 1	Problem Set 2	WPR 1
Mean	88.54	87.14	267.18 (89.1%)
Standard Deviation	10.65	16.95	25.58
Median	90.5	92	271.5
Mode	94	100	270

Cadets did reasonably well on the problem sets in Block 1. The problems were designed to be more computationally challenging than those they encountered in the daily WebAssign homework. The first WPR had several conceptual questions presented in T/F or multiple choice format. As in previous semesters, some cadets bristled against questions of this type, despite being exposed to the concepts in class on multiple occasions.

(2) Block 2: Partial Differential Equations (PDEs). This 12-lesson block introduced students to solution techniques for PDEs, focusing primarily on Fourier series expansion methods. The block began with one lesson on ODE review to refamiliarize stdents with topics encountered in MA153. Topics then included orthogonal functions, Fourier series, boundary value problems, the Sturm Liouville problem, introduction to separable PDEs and the three classical PDEs: the heat equation, the wave equation, and the Laplace equation.

	Problem Set 3	Problem Set 4	WPR 2
Mean	86.24	90.12	272.58 (90.1%)
Standard Deviation	14.06	11.78	18.92
Median	91.5	93	275.5
Mode	100	93	289

Performance improved on the problem sets, and performance on WPR 2 when compared to previous semesters. As noted last semester, I again observed that cadets have largely failed to retain knowledge on the theory and techniques for solving second order linear differential equations that they learned in MA153. I suspect this may be because all students now taking MA365 took MA153 and MA255 in its current sequence—differential equations first, followed by multivariable calculus. Perhaps one of the disadvantages of placing differential equations first in the advanced math program is less retention of the material in follow-on courses. In addition, many of the students taking MA365 may be 4 semesters removed from when they previously took MA153. This semester we implemented a one day review of the techniques for solving ordinary differential equations, in order to try and improve the student experience during this block.

(3) Block 3: Numerical Solutions of Differential Equations. This block focused on the algortihms used to approximated solutions to ODEs and PDEs. During Block 3, we covered the following topics:

- review of Euler methods, Runge-Kutta (RK)
- RK solutions to higher order equations and systems
- 2nd order BVPs.

This block was largely hands-on as the cadets had to work with different software packages to apply the numerical techniques. The software packages were selected by the student as the course is now technology agnostic. I showed and provided examples of how to build the RK methods and Euler forms in Excel. The cadets utilized a range of technology solutions to answer the course problems in this sections. The most dominate programs seen were: Mathematica, Excel, MatLab, Python, R, and Julia.

(4) The Term End Exam: The TEE was a 12 question survey of the entire course. All three block were reprented across the 600 point exam. Students were authorized the TI-30 and TI-36 (FE approved) calculator and up to two 8.5"x11" note sheets. Most students finished the exam between 2 hours and 45 minutes and 3 hours.

	TEE Overall
Mean	534.57 (89.1%)
Standard Deviation	4.71
Median	527
Mode	527

d. Course Sustains for the Upcoming Semester (AY20-2):

(1) Topical make up of the course. For two academic years now, the topics covered in this course seem appropriate and are meeting the needs of the client departments. Anecdotal feedback from Mechanical Engineering students in particular that have taken the course suggest that the PDE section is very important to study within their discipline.

(2) Use of Out of Class Problem Sets. Continue to assign relevant, yet challenging, problem sets. Problem sets extend the topics learned in class into relevant applications and allow for the assignment of more complicated problems than those tested on exams. The use of problem sets in this manner also encourages students to interact outside of class. The inclusion of a traffic network on Problem Set 1 continues to be well-received.

(3) Theory Application Day. Students initially struggled with the idea of article reviews or of applying mathematical theory to real world problems without the confines of root calculation. However, it was interesting to see their development in how they were able to communicate in mathematics without resulting to calculations. I think their confidence in there own understanding of the material was greatly impacted by this and turned the mathematics into something which could be more readily applied to situations the students may encounter outside of the confines of academia.

(4) Survey Results: Overall the end of course survey results for the class were quite positive. Students noted an appreciation for the ODE review day as well as for the opportunity to explore the math utilized a little more deeply in their own self selected applications. There was also some frustration with the lack of depth in the topics as many noted that the course seemed to speed through subject matter. Another element of frustration to some was the technical article/review assignment, as some wondered why there were doing that work in a math class. However, as CD, I think there is significant value to that assignment and will continue to include it as part of any course I instruct.

e. Recommended Improvements for the Upcoming Semester (AY 20-2): Based on my observations and student feedback, I offer the following recommendations for the next iteration of MA365

(1) Continue to assess the scheduling of course lessons within the 75-minute paradigm. The thought going into this semester was that 75-minute lessons should give more time during class for student engagement and practice. Having proportionally less "overhead" at the beginning of class and the opportunity to merge some lessons should also gain some efficiency over the current 55-minute model. This was true to some extent but not universally true. The effect of the schedule adjustment on in-class time for students was largely neutral for Blocks 1 and 2. In essence, the same amount of material still needed to be presented and it was distributed the same proportionally across the 75-minute class periods as it was in 55-minute class periods. However, 75-minute class periods did allow students longer blocks of uninterrupted time for hands-on work in Block 3.

(2) Become better integrated with the curriculum of the Advanced Math Program. All students that take MA365 are graduates of the advanced math program. There are opportunities in the advanced math program to highlight concepts that students will see again in MA365.

5. Conclusions. AY20-1 was a great success and has provided a lot of insights in the transition to the 75-minute model. As instructors, we did note that any efficiencies we thought we might gain in having less overhead periods at the start of class were neutralized by longer periods elapsing in between classes. Because we met less frequently, the material was sometimes not as fresh in the students' minds from the previous lesson, so we did have to take some time to refresh before we were able to move on. Again, my most distressing takeaway from the course, though, is that the retention of material learned in MA153 is sorely lacking. What is encouraging is that the advanced math program has been deliberate about re-visiting differential equations from time to time in MA255 projects and exercises. This should help going forward.

KRISTOPHER H.O. AHLERS
LtCol, USAF
MA365 Course Director

TAB B AY2019 COURSE ASSESSMENT

DEPARTMENT OF THE ARMY
DEPARTMENT OF MATHEMATICAL SCIENCES
UNITED STATES MILITARY ACADEMY
WEST POINT, NEW YORK 10996-1704

MADN-MATH 15 July 2019

MEMORANDUM THRU Service Program Director, COL Paul Goethals, Department of Mathematical Sciences, United States Military Academy, West Point, New York 10996

FOR COL Tina Hartley, Head, Department of Mathematical Sciences, United States Military Academy, West Point, New York 10996

SUBJECT: MA365, Advanced Mathematics for Engineers and Scientists, Course End Report, AY19-2

1. Purpose. The intent of this memorandum is to review and assess MA365, Advanced Mathematics for Engineers & Scientists, as it was conducted in AY19-2. The following course summary and assessment is based on instructor and cadet feedback.

2. Background. MA365 is a course designed for the advanced mathematics student that has completed courses in differential equations and vector calculus (those that have completed MA153 and MA255) and will study ME, CvE, EE, NE, Physics or Space Science. MA365 was created due to the approximately 75% overlap in the MA364 and MA153/MA255 curricula (topics in differential equations and vector calculus); MA365 begins where the advanced mathematics program ends. The advanced engineering course offering includes topics in linear algebra, complex variables, Fourier series, partial differential equations, and computational mathematics. There is approximately a 25% overlap in the MA364 and MA365 curricula. The topics in both courses include complex variables, Fourier series, and partial differential equations.

3. Course Enrollment. AY19-2 was the fourth semester the Math Department offered MA365 as its own course following the successful pilot of MA364X in AY17. In AY19-2, there were two sections of MA365 (37 cadets). MA365 continued its transition to the 75-minute, 30 lesson delivery model developed in AY19-1. LtCol Kris Ahlers taught both sections and served as the course director. The table below shows the number of cadets enrolled by major.

Major	Number of Cadets
Mechanical Engineering	15
Electrical Engineering	7
Chemical Engineering	6
Physics	5
Nuclear Engineering	4

4. Assessment and Recommendations.

a. Bottom Line up Front: The overall average for the course was an 91.3%. There were no course failures, one D, and three A+ grades. The course average for the TEE was 90.1%.

b. Course Performance. The average grades in AY19-2 were very similar to those earned by cadets in previous semesters. In fact, the overall course average and TEE average have all been within one percentage point of one another.

c. Develop a Fundamental Set of Mathematical Skills: MA365 is designed as a set of three "micro-courses" that develop a critical set of mathematical skills which will enable students to better understand the language of their respective Engineering field. Each of the three blocks has the foundational material of a stand-alone course.

(1) Block 1: Linear Algebra and Complex Variables. The cadets began the course with a 11 lesson block that focused on the use of linear algebra as a means to efficiently solve system of linear equations. Additionally, the students were introduced to eigenvectors, eigenvalues and diagonalization of matrices.

	Problem Set 1	Problem Set 2	WPR 1
Mean	90.8	96.6	264.5
Standard Deviation	17.5	4.8	29.6
Median	97	100	273
Mode	100	100	280

Cadets did reasonably well on the problem sets in Block 1. The problems were designed to be more of a stretch than those they encountered in the daily WebAssign homework. The first WPR had several conceptual questions presented in T/F or multiple choice format. As in previous semesters, some cadets bristled against questions of this type, despite being exposed to the concepts in class on multiple occasions.

(2) Block 2: Partial Differential Equations (PDEs). This 12-lesson block introduced students to solution techniques for PDEs, focusing primarily on Fourier series expansion methods. The block began with one lesson on vector spaces and the inner product, one half lesson on orthogonal functions, then one and one half lessons on Fourier series, and then two lessons on boundary value problems, the Sturm Liouville problem, and an introduction to separable PDEs. Finally, two lessons each were devoted to classical PDEs: the heat equation, the wave equation, and the Laplace equation.

	Problem Set 3	Problem Set 4	WPR 2
Mean	91.4	89.4	275
Standard Deviation	13.9	13.4	13.5
Median	98	96	276
Mode	100	94	286

Performance for the seconde WPR fell in comparison to AY19-1 and in general the students struggled with the material in Block 2. As noted last semester, I again observed that cadets have largely failed to retain knowledge on the theory and techniques for solving second order linear differential equations that they learned in MA153. I suspect this may be because all students now taking MA365 took MA153 and MA255 in its current sequence—differential equations first, followed by multivariable calculus. Perhaps one of the disadvantages of placing differential equations first in the advanced math program is less retention of the material in follow-on courses.

(3) Block 3: Numerical Solutions of Differential Equations. This block focused on the algortihms used to approximated solutions to ODEs and PDEs. During Block 3, we covered the following topics:

- review of Euler methods, Runge-Kutta (RK)
- RK solutions to higher order equations and systems
- 2nd order BVPs.
- Finite Difference Method approximations of the three classical PDEs

This block was largely hands-on as the cadets had to work with software to apply the numerical techniques. I showed and provided examples of how to build the RK methods and most cadets adopted the same form. I did provide a template for the finite-difference method techniques, but made them adapt them to specific problems and boundary conditions. All of the numerical methods for the problems we encountered could be implemented in Microsoft Excel.

Besides the WebAssign problems assigned for each lesson, the only other graded event in Block 3 was a technology lab. The tech lab was designed for students to explore one problem and then apply solution techniques they had learned from both Blocks 2 and 3. Up until now, the numerical methods and the technology lab have all been implemented in Excel. One area of confusion that manifest itself with the use of Excel is that students sometimes equate the cells of the spreadsheet with a physical element of area, instead of as a point or node as they should. This can lead to improper discretizations of the area being studied. Perhaps the transition to Matlab will alleviate the corrupting influence of Excel's visual cues.

(4) The Term End Exam: The TEE was split into two components: a non-tech portion (worth 75%) and a technology portion (worth 25%). Students took the non-technology portion first and had to submit it before being issued the technology-portion of the exam. Distinct colors were used for the two portions to aid the management of the transition. It was up to the individual students as to how much of the 3.5 hour TEE session they allocated for each of the TEE portions. Only after receiving the technology portion could students use their computers. They were allowed to use Mathematica, Excel, and/or Matlab and any pre-existing files they had created. They were specifically instructed to not use connectivity or any external resources through the computer. For the entire TEE they were allowed to use up to two 8.5"x11" note sheets.

	Non-Tech	Tech	TEE Overall
Mean	399	140	539.1
Standard Deviation	27.2	11.1	34
Median	403	148	544
Mode	392	150	544

d. Course Sustains for the Upcoming Semester (AY20-1):

(1) Topical make up of the course. For two academic years now, the topics covered in this course seem appropriate and are meeting the needs of the client departments. Anecdotal feedback from Mechanical Engineering students in particular that have taken the course suggest that the PDE section is very important to study within their discipline.

(2) Use of Out of Class Problem Sets. Continue to assign relevant, yet challenging, problem sets. Problem sets extend the topics learned in class into relevant applications and allow for the assignment of

more complicated problems than those tested on exams. The use of problem sets in this manner also encourages students to interact outside of class. The inclusion of a traffic network on Problem Set 1 continues to be well-received.

(3) Two portions for TEE. Despite concerns about the possibility of technical glitches during the TEE (cadet computer crashes, battery charge runs out, etc.), the benefit of assessing cadets on numerical solution techniques outweighed the risks. To mitigate the risk, I announced the format and procedures for the TEE well in advance, as well as providing extra electrical power sources in the TEE classroom. No cadet experienced any problems with their laptop during the TEE and the procedures for exchanging the two portions of the exam worked flawlessly for the fourth sesmester in a row.

(4) Revision of the Tech Lab. The tech lab was re-designed to be a more interesting application of numerically approximating a solution to Laplace's Equation. Students were asked to model a real-world scenario of a griddle on two different burners that had warmed to equilibrium. Students were engaged and thoughtful on their approach to this problem.

e. Recommended Improvements for the Upcoming Semester (AY 20-1): Based on my observations and student feedback, I offer the following recommendations for the next iteration of MA365

(1) Re-think tech portion. The tech portion of the course has continued to frustrate students with the applications back to the math which was taught in class. I want the class to be a confidence builder for students to tackle the applications in their major courses and currently I am unsure if the tech portion of the course is a useful tool to that end.

(2) Continue to assess the scheduling of course lessons within the 75-minute paradigm. The thought going into this semester was that 75-minute lessons should give more time during class for student engagement and practice. Having proportionally less "overhead" at the beginning of class and the opportunity to merge some lessons should also gain some efficiency over the current 55-minute model. This was true to some extent but not universally true. The effect of the schedule adjustment on in-class time for students was largely neutral for Blocks 1 and 2. In essence, the same amount of material still needed to be presented and it was distributed the same proportionally across the 75-minute class periods as it was in 55-minute class periods. However, 75-minute class periods did allow students longer blocks of uninterrupted time for hands-on work in Block 3. I recommend continuing to refine the structure of lessons to best take advantage of the 75-minute class periods.

(3) Become better integrated with the curriculum of the Advanced Math Program. All students that take MA365 are graduates of the advanced math program. There are opportunities in the advanced math program to highlight concepts that students will see again in MA365. I recommend that the MA365 course director and AMP program director coordinate the instruction of some of these concepts in a longitudinal sense.

5. Conclusions. AY19-2 was a well conducted semester and has provided a lot of insights in the transition to the 75-minute model. I did note that any efficiencies we thought we might gain in having less overhead periods at the start of class were neutralized by longer periods elapsing in between classes. Because we met less frequently, the material was sometimes not as fresh in the students' minds from the previous lesson, so we did have to take some time to refresh before we were able to move on. Again, my most distressing takeaway from the course, though, is that the retention of material learned in MA153 is sorely lacking. What is encouraging is that the advanced math program has been deliberate about re-

visiting differential equations from time to time in MA255 projects and exercises. This should help going forward.

Encl
1. Assessment of Student Outcomes

KRISTOPHER H.O. AHLERS
LtCol, USAF
MA365 Course Director

Department of Mathematical Sciences Assessment Program
(Embedded Course Indicator Descriptions)

Course: MA365
Term: AY 19-2
Course Director: LtCol Kris Ahlers

Student Outcome 4: Communicate mathematics, both orally and in writing.

This outcome could be observed through cadet performance in class during board presentations and on all of the Problem Sets and the Tech Lab. As is typical, cadets are given several opportunities throughout the semester to synthesize what they've learned to communicate orally to their instructors and classmates. Immediate feedback is given to validate their presentations or to correct as appropriate.

Discussion questions are included in the problem sets for students to comment on their results or to express their understanding of concepts more deeply. The technology lab had several discussion questions that made up the bulk of their submission. Students did very well communicating their work in this way and a few students were especially excellent in the thoroughness and depth of their communications.

In addition, this semester featured a writing critique exercise. In this exercise the students were asked to select a peer reviewed journal article in which to summarize and then critique. This was a multi-page writing assignment in which the cadets would then present their work.

Student Outcome 7: Understand the role of mathematical sciences (in our world) by analyzing applied problems through disciplinary, multidisciplinary, and interdisciplinary approaches.

This outcome was reinforced throughout the course. All problem sets contained application-motivated problems. The technology lab was designed to have students explore the modeling of an everyday phenonmenon using concepts and techniques they learned in Blocks 2 and 3. I would like to integrate even more application problems into the problem sets. Several exam questions included material which applied to problems the students could relate to and appreciate the connection of the underlying theory. Care was taken to introduce a variety of applications that represented multiple disciplines.

TAB C AY2018 COURSE ASSESSMENT

DEPARTMENT OF THE ARMY
DEPARTMENT OF MATHEMATICAL SCIENCES
UNITED STATES MILITARY ACADEMY
WEST POINT, NEW YORK 10996-1704

MADN-MATH 13 June 2018

MEMORANDUM THRU Service Program Director, COL Paul Goethals, Department of Mathematical Sciences, United States Military Academy, West Point, New York 10996

FOR COL Tina Hartley, Head, Department of Mathematical Sciences, United States Military Academy, West Point, New York 10996

SUBJECT: MA365, Advanced Mathematics for Engineers and Scientists, Course End Report, AY18-2

1. Purpose. The intent of this memorandum is to review and assess MA365, Advanced Mathematics for Engineers & Scientists, as it was conducted in AY18-2. The following course summary and assessment is based on instructor and cadet feedback.

2. Background. MA365 is a course designed for the advanced mathematics student that has completed courses in differential equations and vector calculus (those that have completed MA153 and MA255) and will study ME, CvE, EE, NE, Physics or Space Science. MA365 was created due to the approximately 75% overlap in the MA364 and MA153/MA255 curricula (topics in differential equations and vector calculus); MA365 begins where the advanced mathematics program ends. The advanced engineering course offering includes topics in linear algebra, complex variables, Fourier series, partial differential equations, and computational mathematics. There is approximately a 25% overlap in the MA364 and MA365 curricula. The topics in both courses include complex variables, Fourier series, and partial differential equations.

3. Course Enrollment. AY18-2 was the second semester the Math Department offered MA365 as its own course following the successful pilot of MA364X in AY17. In AY18-2, there were four sections of MA365 (67 cadets). LTC Mike Findlay taught two sections and served as the course director. Dr. Kayla Blyman and Dr. Ivan Dungan taught one section each. The table below shows the number of cadets enrolled by major.

Major	Number of Cadets
Mechanical Engineering	32
Civil Engineering	2
Electrical Engineering	10
Chemical Engineering	11
Physics	7
Nuclear Engineering	1
Space Science	3
Environmental Engineering	1

4. Assessment and Recommendations.

a. Bottom Line up Front: The overall average for the course was an 89.4%. There were no course failures and five A+ grades. The course average for the TEE was 89.8%.

b. Course Performance. The average grades in AY18-2 were very similar to those earned by cadets in previous semesters. In fact, the overall course average and TEE average have all been within one percentage point of one another.

c. Develop a Fundamental Set of Mathematical Skills: MA365 is designed as a set of three “micro-courses” that develop a critical set of mathematical skills which will enable students to better understand the language of their respective Engineering field. Each of the three blocks has the foundational material of a stand-alone course.

(1) Block 1: Linear Algebra and Complex Variables. The cadets began the course with a 13 lesson block that focused on the use of linear algebra as a means to efficiently solve system of linear equations. Additionally, the students were introduced to eigenvectors, eigenvalues and diagonalization of matrices.

	Problem Set 1	Problem Set 2	WPR 1
Mean	90.99	92.89	265.43 (88.5%)
Standard Deviation	4.92	5.67	20.77
Median	92	95	272
Mode	93	97	280

Cadets did very well on the problem sets in Block 1. The problems were designed to be more of a stretch than those they encountered in the daily WebAssign homework. The first WPR had several conceptual questions presented in T/F or multiple choice format. As in previous semesters, some cadets bristled against questions of this type, despite being exposed to the concepts in class on multiple occasions.

(2) Block 2: Partial Differential Equations (PDEs). This 17-lesson block introduced students to solution techniques for PDEs, focusing primarily on Fourier series expansion methods. The block began with one lesson on vector spaces and the iner product, one lesson on orthogonal functions, then two lessons on Fourier series, and then three lessons on boundary value problems, the Sturm Liouville problem, and an introduction to separable PDEs. Finally, three lessons each were devoted to classical PDEs: the heat equation, the wave equation, and the Laplace equation.

	Problem Set 3	Problem Set 4	WPR 2
Mean	87.23	89.75	267.76 (89.3%)
Standard Deviation	8.47	6.63	21.14
Median	89	90	270
Mode	90	93	284

Performance slipped a little on the problem sets, although performance on WPR 2 in the aggregate was as good as it has been for any semester. In general, cadets struggled most with finding eigenvalues and the associated eigenfunctions of a given boundary value problem (BVP). This struggle has been a recurring theme, so I’ve had more keen attention on this issue. What has struck me in this semester more so in the others is that cadets have largely failed to retain knowledge on the theory and techniques for solving second order linear differential equations that they learned in MA153.

(3) Block 3: Numerical Solutions of Differential Equations. This block focused on the algortihms used to approximated solutions to ODEs and PDEs. During Block 3, we covered the following topics:

- review of Euler methods, Runge-Kutta (RK)
- RK solutions to higher order equations and systems
- 2nd order BVPs.
- Finite Difference Method approximations of the three classical PDEs

This block was largely hands-on as the cadets had to work with software to apply the numerical techniques. I showed and provided examples of how to build the RK methods and most cadets adopted the same form. I did provide a template for the finite-difference method techniques, but made them adapt them to specific problems and boundary conditions. All of the numerical methods for the problems we encountered could be implemented in Microsoft Excel.

Besides the WebAssign problems assigned for each lesson, the only other graded event in Block 3 was a technology lab. The tech lab was designed for students to explore one problem and then apply solution techniques they had learned from both Blocks 2 and 3.

(4) The Term End Exam: The TEE was split into two components: a non-tech portion (worth 75%) and a technology portion (worth 25%). Students took the non-technology portion first and had to submit it before being issued the technology-portion of the exam. Distinct colors were used for the two portions to aid the management of the transition. It was up to the individual students as to how much of the 3.5 hour TEE session they allocated for each of the TEE portions. Only after receiving the technology portion could students use their computers. They were allowed to use Mathematica, Excel, and/or Matlab and any pre-existing files they had created. They were specifically instructed to not use connectivity or any external resources through the computer. For the entire TEE they were allowed to use up to two 8.5”x11” note sheets.

	Non-Tech	Tech	TEE Overall
Mean	404.37 (89.9%)	134.24 (89.5%)	538.61 (89.8%)
Standard Deviation	23.03	11.01	27.92
Median	407	135	541
Mode	426	144	541

d. Course Sustains for the Upcoming Semester (AY19-1):

(1) Topical make up of the course. For two academic years now, the topics covered in this course seem appropriate and are meeting the needs of the client departments. Anecdotal feedback from Mechanical Engineering students in particular that have taken the course suggest that the PDE section is very important to study within their discipline.

(2) Use of Out of Class Problem Sets. Continue to assign relevant, yet challenging, problem sets. Problem sets extend the topics learned in class into relevant applications and allow for the assignment of more complicated problems than those tested on exams. The use of problem sets in this manner also

encourages students to interact outside of class. The inclusion of a traffic network on Problem Set 1 continues to be well-received.

(3) Authorized References for WPRs and the TEE. Continue to have the student's prepare hand written 8.5" x 11" reference sheets for the WPRs and TEE. This encourages students to review the material and arrive better prepared for the exam. Many students comment that the very act of composing this note sheet is a very useful way to study.

(4) Two portions for TEE. Despite concerns about the possibility of technical glitches during the TEE (cadet computer crashes, battery charge runs out, etc.), the benefit of assessing cadets on numerical solution techniques outweighed the risks. To mitigate the risk, I announced the format and procedures for the TEE well in advance, as well as providing extra electrical power sources in the TEE classroom. No cadet experienced any problems with their laptop during the TEE and the procedures for exchanging the two portions of the exam worked flawlessly for the fourth sesmester in a row.

e. Recommended Improvements for the Upcoming Semester (AY 19-1): Based on my observations and student feedback, I plan to implement the following course adjustments for the next iteration of MA365

(1) Place more emphasis on WebAssign. This semester more than the others, we noticed some diminished participation on WebAssign. Not surprisingly, the students that did the least WebAssign problems also tend to earn the lowest grades on other course events. There also seemed to be an extraordinary amount of students that neglected to complete the Academic Integrity portion of the WebAssign assignments, despite numerous reminders. Assignments do not count if this portion is not done. We will make these items a special point of emphasis in the introduction of subsequent courses.

(2) Adjust the focus of the tech lab. I made more significant adjustments to the tech lab this semester, asking students to explore various solutions of Laplace's Equation and then compare numerical approximations to the exact solutions. This seemed to go over well, but I think it would be better to make this assignment more of a modeling exercise instead of giving the students all of the proposed solutions.

(3) Proof numerical method implementation. This semester was the first time the course was taught by multiple instructors so variety in delivery was expected and encouraged. However, one instructor experimented with implementing the various numerical solutions techniques in Mathematica only to find it was not as accessible as he had hoped. Moreover, the recursive techniques he was employed were inexplicably bogging down the Mathematica engine. Unfortunately, these discoveries were occurring in Block III. In the future, any deviations from proven methods should be approved by the course director in advance.

(4) Introduce more applications. One of the recurring themes in the end of course feedback is a desire for more applications. I do introduce some in the problem sets, and there are a few included in the text, but they tend to be rather canned. It would be good to introduce more applications in the course lesson delivery, especially given the broad range of STEM disciplines serviced by the course.

(5) Gain efficiency with course structure, delivery, and student engagement. This course is transitioning to the 75-minute model next semester. This should give more time during class for student engagement and practice. Having proportionally less "overhead" at the beginning of class and the opportunity to merge some lessons should also gain some efficiency over the current 55-minute model.

5. Conclusions. AY18-2 was a great success and has provided a lot of insights moving forward to the transition to the 75-minute model. Many students commented that the course was hard, but just as many commented that they relished the challenge presented by the course. My most distressing takeaway from the course, though, is that the retention of material learned in MA153 is sorely lacking. I noticed it and the students noted it.

Encl
1. Assessment of Student Outcomes

MICHAEL A. FINDLAY
LTC, FA49/AR
MA365 Course Director

Department of Mathematical Sciences Assessment Program
(Embedded Course Indicator Descriptions)

Course: MA365
Term: AY 18-2
Course Director: LTC Mike Findlay

Student Outcome 1: Demonstrate competence in modeling physical, informational, and social phenomena.

This outcome could be observed through cadet performance on all of the Problem Sets, the Tech Lab, WPR 2, and the TEE.

Cadets did well in constructing appropriate models of a heated plate in problem sets 1 and 2, but did struggle somewhat in identifying the appropriate mathematical representation of boundary conditions given a written description for problems in block 2. Most commonly, they conflated Dirichlet and Neumann conditions. This trend seemed to improve on the relevant TEE question.

There is an encouraging desire from the cadets, though, to be exposed to more applications. As engineering students, they want to know how they can use the math in this course to solve problems in their discipline. A few appreciate the process of identifying assumptions and recognizing the

Student Outcome 3: Achieve mathematical proficiency in breadth and depth

Perhaps the most theoretical of the blocks was Block 1, and students did very well navigating the linear algebra topics and understanding the relevance to solving systems of linear equations.

To strengthen the students' understanding of why we care about orthogonal functions, I placed more emphasis on the connections of geometric vectors to abstract vector spaces and how what we see in generalized vector spaces and the notion of an inner product is actually an idea they've seen before with the dot product.

The superposition principle is something they understand and can recall (perhaps with prompting), but is something I will make the theme of the next semester because it is an important concept for both Blocks 1 and 2.

TAB D AY2017 COURSE ASSESSMENT

DEPARTMENT OF THE ARMY
DEPARTMENT OF MATHEMATICAL SCIENCES
UNITED STATES MILITARY ACADEMY
WEST POINT, NEW YORK 10996-1704

MADN-MATH 5 June 2017

MEMORANDUM THRU Math Electives Program Director, COL Joe Lindquist, Department of Mathematical Sciences, United States Military Academy, West Point, New York 10996

FOR COL Steve Horton, Head, Department of Mathematical Sciences, United States Military Academy, West Point, New York 10996

SUBJECT: MA364X, Advanced Mathematics for Engineers and Scientists, Course End Report, AY17-2

1. Purpose. The intent of this memorandum is to review and assess MA364X, Advanced Mathematics for Engineers & Scientists, as it was conducted in AY17-2. The following course summary and assessment is based on instructor and cadet feedback.

2. Background. MA364X is a temporary pilot course designed for the advanced mathematics student that has completed courses in differential equations and vector calculus (those that have completed MA153 and MA255) and will study ME, CvE, EE, NE, Physics or Interdisciplinary Science. MA364X was created due to the approximately 75% overlap in the MA364 and MA153/MA255 curricula (topics in differential equations and vector calculus); MA364X begins where the advanced mathematics program ends. The advanced engineering course offering includes topics in linear algebra, complex variables, Fourier series, partial differential equations, and computational mathematics. There is a 25% overlap in the MA364 and MA364X curricula. The topics in both courses include complex variables, Fourier series, and partial differential equations.

3. Course Enrollment. AY17-2 was the second semester the Math Department offered MA364X. In AY17-2 there were two sections of MA364X. LTC Mike Findlay taught both sections and served as the course director. Good performance in MA153 and MA255 was used as a screening criteria to place students in MA364X. Of the 39 cadets that were graduates of the advanced math program originally enrolled in MA364, 34 were placed into MA364X; 11 from the class of 2018, and 23 from the class of 2019. The 5 cadets not placed into MA364X were late enrollees in MA364 and schedule conflicts did not allow for them to take MA364X. Their demonstrated performance would otherwise have put them in MA364X. AY17-2 should be the last semester Advanced Math Program (AMP) graduates are not properly enrolled in the correct engineering mathematics course as MA364X has since been formalized as MA365. The table below shows the number of cadets enrolled by major.

Major	Number of Cadets
Mechanical Engineering	15
Civil Engineering	1
Electrical Engineering	11
Chemical Engineering	1
Physics	4
Interdisciplinary Science	2

4. Assessment and Recommendations.

a. Bottom Line up Front: The overall average for the course was an 89.5%. There were no course failures and five A+ grades (after the cut line adjustment). The course average for the TEE was 88.3%.

b. Course Performance. The average grades in AY17-2 were very similar to those in AY17-1. However, the grades were more widely distributed. There were five A+ grades this semester compared to only one last semester and the median scores for most graded events was higher this semester. A few lower-performing students seemed to bring the averages down. One notable difference this semester was the addition of an incentive to practice the Thayer method. There were 100 course points allocated to the cadets' self-assessment of daily class preparation. There seems to be at least anecdotal evidence that the incentive produced positive results. Not surprisingly, the lower-performing cadets were also the ones that did not prepare as much for class.

c. Develop a Fundamental Set of Mathematical Skills: MA364X is designed as a set of three "micro-courses" that develop a critical set of mathematical skills which will enable students to better understand the language of their respective Engineering field. Each of the three blocks has the foundational material of a stand-alone course.

(1) Block 1: Linear Algebra and Complex Variables. The cadets began the course with a 13 lesson block that focused on the use of linear algebra as a means to efficiently solve system of linear equations. Additionally, the students were introduced to eigenvectors, eigenvalues and diagonalization of matrices. In a change from the previous semester, the brief introduction to complex numbers and functions of complex variables was included at the beginning of the block instead of at the end.

	Problem Set 1	Problem Set 2	WPR 1
Mean	91.44	89.41	271.12 (90.4%)
Standard Deviation	10.19	10.24	17.76
Median	94	93.5	272
Mode	97	98	285

Cadets did reasonably well on the problem sets in Block 1, with most cadets receiving an A or A+. The problems were designed to be more of a stretch than those they encountered in the daily WebAssign homework. The first WPR had several conceptual questions presented in T/F or multiple choice format. Anecdotally, it seems they were caught off guard by questions of this type, despite being exposed to the concepts in class on multiple occasions, but they performed much better on these questions than the students did in the previous semester.

(2) <u>Block 2: Partial Differential Equations (PDEs)</u>. This 17-lesson block introduced students to solution techniques for PDES, focusing primarily on Fourier series expansion methods. The block began with one lesson on vector spaces and the iner product, one lesson on orthogonal functions, then two lessons on Fourier series, and then three lessons on boundary value problems, the Sturm Liouville problem, and an introduction to separable PDEs. Finally, three lessons each were devoted to classical PDEs: the heat equation, the wave equation, and the Laplace equation.

	Problem Set 3	Problem Set 4	WPR 2
Mean	88.09	89.09	265.87 (88.6%)
Standard Deviation	9.44	5.71	22.54
Median	90.5	90	270
Mode	96	90	290

Strong performance continued on the problem sets. In general, cadets struggled most with finding eigenvalues and the associated eigenfunctions of a given boundary value problem (BVP).

(3) <u>Block 3: Numerical Solutions of Differential Equations</u>. This block focused on the algortihms used to approximated solutions to ODEs and PDES. During Block 3, we covered the following topics:

- review of Euler methods, Runge-Kutta (RK)
- RK solutions to higher order equations and systems
- 2nd order BVPs.
- Finite Difference Method approximations of the three classical PDEs

This block was largely hands-on as the cadets had to work with software to apply the numerical techniques. I showed and provided examples of how to build the RK methods and most cadets adopted the same form. I did provide a template for the finite-difference method techniques, but made them adapt them to specific problems and boundary conditions. All of the numerical methods for the problems we encountered could be implemented in Microsoft Excel.

Besides the WebAssign problems assigned for each lesson, the only other graded event in Block 3 was a technology lab. The tech lab was designed for students to explore one problem and then apply solution techniques they had learned from both Blocks 2 and 3.

(4) <u>The Term End Exam</u>: The TEE was split into two components: a non-tech portion (worth 75%) and a technology portion (worth 25%). Students took the non-technology portion

first and had to submit it before being issued the technology-portion of the exam. Distinct colors were used for the two portions to aid the management of the transition. It was up to the individual students as to how much of the 3.5 hour TEE session they allocated for each of the TEE portions. Only after receiving the technology portion could students use their computers. They were allowed to use Mathematica, Excel, and/or Matlab and any pre-existing files they had created. They were specifically instructed to not use connectivity or any external resources through the computer. For the entire TEE they were allowed to use up to two 8.5"x11" note sheets.

	Non-Tech	Tech	TEE Overall
Mean	394.71 (87.7%)	135.21 (90.1%)	529.91 (88.3%)
Standard Deviation	32.71	12.63	41.04
Median	402.5	136.5	540.5
Mode	420	150	560

d. Course Sustains for the Upcoming Semester (AY17-2):

(1) Use of WebAssign. The Zill & Wright textbook can be purchased with a WebAssign subscription. There are several representative problems available for each section of the book. WebAssign problems were assigned for each lesson. This compelled cadets to stay engaged with the material. Cadets appreciate the instantaneous feedback they receive, the multiple submission attempts, and other features available to them on the interface. From an instructor standpoint, it's an invaluable enhancement to the course and automates what would otherwise be burdensome grading.

(2) General course structure. The organization of the course into three blocks worked well, seemed to be paced appropriately, and allowed for adequate student assessment.

(3) Use of other textbooks. The Zill & Wright textbook had less problems than I would have preferred for the students to exercise PDE solutions both in and out of class. I augmented in-class board work with problems taken from the MA153 Boyce & DiPrima textbook. This worked well, and I also encouraged cadets to use this text as a supplementary reference. I do recognize this is a fleeting option for cadets though as MA153 has transitioned to the Zill&Wright Differential Equations book, which is largely duplicative of the relevant chapters in our textbook.

(4) Use of Out of Class Problem Sets. Continue to assign relevant, yet challenging, problem sets. Problem sets extend the topics learned in class into relevant applications and allow for the assignment of more complicated problems than those tested on exams. The use of problem sets in this manner also encourages students to interact outside of class.

(5) Authorized References for WPRs and the TEE. Continue to have the student's prepare hand written 8.5" x 11" reference sheets for the WPRs and TEE. This encourages students to review the material and arrive better prepared for the exam.

(6) Two portions for TEE. Despite concerns about the possibility of technical glitches during the TEE (cadet computer crashes, battery charge runs out, etc.), the benefit of assessing cadets on numerical solution techniques outweighed the risks. To mitigate the risk, I announced the format and procedures for the TEE well in advance, as well as providing extra electrical power sources in the TEE classroom. No cadet experienced any problems with their laptop during the TEE and the procedures for exchanging the two portions of the exam worked flawlessly.

e. Recommended Improvements for the Upcoming Semester (AY 18-1): Based on my observations and student feedback, I plan to implement the following course adjustments for the next iteration of MA365 (Fall, AY 2018):

(1) Develop another application problem for PS1. I attempted to include an application of linear systems with a simple truss problem, but it turned out to be too confusing, not everyone had the same base of knowledge, and I found conflicting techniques in the literature—some treated the internal forces as scalars, some as vectors. I ended up dismissing the problem, but only after causing some undue angst.

(2) Adjust the focus of the tech lab. I used largely the same prompt in AY17-1 and AY17-2 for the tech lab. It does connect many of the ideas from Blocks I and II to Block III, but the underlying application is admittedly a bit esoteric for most cadets. I plan to develop a new tech lab that has more practical exercises for cadets to implement the numerical methods they have learned.

(3) Curtail some of the mandatory WebAssign in Block III. There were some time-consuming problems (made up of several individual responses) in the WebAssign associated with Chapter 16. It might be worth looking to see if all that were assigned need to be mandatory.

(4) Better highlight student work. I felt like I got away from board work and student presentations in this semester somewhat, and that should be turned around. For example, there were some students that had created some very ingenious Matlab code to implement numerical techniques that I would have like the opportunity to showcase in class. For the lessons that span 2-3 class meetings especially, there should be ample time set aside for students to present their work.

5. Conclusions. Overall, MA364X was a very successful course pilot. Only minimal changes are planned from the first year to the execution of the formalized course, MA365. Student feedback suggests that they liked the challenge of the course and exposure to new concepts. MA364X more than adequately solves the problem that used to confront certain Advanced Math program (AMP) graduates in fulfilling the requirements for their majors. MA364X and now MA365 can essentially be viewed as an extension of the AMP, giving an enriched experience to certain cadets in select STEM disciplines. Electives program assessments are included in the enclosures.

Encl
1. Assessment of Student Outcomes

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MA364X Course Director

Department of Mathematical Sciences Assessment Program
(Embedded Course Indicator Descriptions)

Course: MA364X
Term: AY 17-2
Course Director: LTC Mike Findlay

Student Outcome 2: Argue and inquire soundly and rigorously; become independent questioners and learners.

This outcome could be observed through cadet performance during board problem briefings and on their performance to open-ended questions posed on out-of-class assignments.

On Problem Set 2, cadets were prompted several times to cite specific theories and terminology in the textbook to justify their answers. Most did this very well. However, the sections of WPR I that seemed to cause the most difficulty were those that were conceptual in nature. So, even though many of the cadets articulated correct arguments in one setting, some could not recall upon that argument in a test setting. Anecdotally, many cadets remarked that they were caught off-guard by the inclusion of conceptual portions in WPR I.

Student Outcome 5: Use technology to model, visualize, and solve complex problems

Technology was used throughout the course. Cadets were asked to use technology on all problem sets to both solve and visualize problems. Little instruction was devoted in class on how to carry out the implementation with software, yet cadets did very well. It was clear they came to the class with a good working knowledge of Mathematica and it was clear that many improved their proficiency throughout the course. There seemed to be less consistency with Microsoft Excel proficiency. Some cadets were very good with it, while others relied heavily on templates I created and struggled somewhat to adapt their spreadsheets to different sized discretizations. Using technology on the TEE again was a success. Course end feedback from the cadets suggests that they really enjoyed the use of technology (it was the most prominent topic in what they liked most about the course) and learning new techniques in both Excel and Mathematica.

Student Outcome 6: Develop attitudes—habits of mind (creative and curious, experimental disposition, critical thinking and reasoning, commitment to life-long learning)

From the course end survey, there was a common comment about liking the wide range of mathematical topics they were exposed to and that they could see the connection with applications in their chosen fields. It's hard to judge if that necessarily means this course has caused them to develop the attitudes and habits of mind we seek.

There was one question on the Tech Lab that perhaps gives the greates insight into their critical thinking and experimental disposition. If they had done their work correctly, the cadets would have produced a numerical solution to a Laplace BVP that resulted in zero errors. They were then asked if they thought if numerical solutions to all Laplace BVPs would behave this way. Almost all guessed that it would, and if they attempted any justification, it was often misapplied or amounted to circular reasoning. A few cadets surmised a higher-order solution may not behave the same, but none actually tested their hypotheses.

Running thoughts on executing MA364X and its curriculum

Lesson 1: It's very ambitious to conduct the course overview and the lesson—as designed—on vector spaces in one class meeting. I anticipated that we would not get through it all, which turned out to be the case, and planned to catch up on Lesson 2. I adjusted the due date for WA Assignment # 1 accordingly. One question that comes to mind: Is the vector space lesson placed well? It is a "mind-blowing" concept on the first day of the semester, and we set aside its concepts to come back to them later in the course. I also wonder if and when we'll come back to them and how they connect exactly to follow-on material in the course. I think it will occur in Block II? This mystery is due in part to my own lack of familiarity in depth with what we're presenting in the course, as I've really only gone deep into Block I at this point. Any value in also including Gram-Schmidt orthogonalization process to the course?
Lesson 2: Good refresher for most students; not intimidated by the material. Unfortunately, we didn't get to the boards in this lesson, because we had to catch up from lesson 1.
Lessons 3/4: It makes sense to spread section 8.2 over two days. My plan was to provide motivation for using matrices to solve systems of linear equations up front by showing a simple system of two linear equations and prompting the students to help me solve it in class using what they've done in their experiences. Most common suggestion was to use one equation to solve one variable in terms of the other, substitute it into the other equation and then back-substitute into the first equation. This worked well to highlight the need for a more systematic way to eliminate variables when working with systems involving more equations and more variables. The remainder of lesson 3 was spent on showing how to transform a system of linear equations into **AX = B**, discussing consistent/inconsistent systems and homogenous systems, and then finally how to solve a system. I chose an example from the book, constructed the augmented matrix **A|B**, then walked through the problem using Gaussian elimination, and then picking up where Gaussian elimination left off with Gauss-Jordan elimination. I then compared the results from the two elimination methods with one another and also the `LinearSolve` command in Mathematica. There wasn't as much time as I forecasted available for board problems: 10 minutes for one section, 6 for another. Plan is to start Lesson 4 off immediately with board work, and then have a discussion of nontrivial solutions of homogenous systems and underdetermined systems, as well as setting up a system through an application of Kirchhoff's laws. This should give the students confidence for a related WebAssign problem as well as for problem 2 on PS1.
I am also thinking about the utility of introducing MatLab in the course for the next go-around. Mathematica is familiar and accessible to the students, and it should work just fine knowing what I know about the course right now, but exposure to and hands-on use of MatLab may have a more lasting benefit for this student population. I'll keep thinking about what the billpayer ought to be for the upfront cost in the students learning this software.
Lesson 4: Worked very well starting out this class with board problems. Students chose from a pool of problems and we ended up with a nice variety of work that allowed us to highlight inconsistent systems and homogenous systems with a trivial solution. Many students are very comfortable with row operations and will even perform several in one iteration. Others struggled somewhat, and more often than not, they neglected to properly denote which row operation they were performing in one step to the next. Concluded a lesson with a Kirchhoff's laws application and was well received. Time available for this lesson was perfect.
Lesson 5: Looking ahead (after lesson 4) to this lesson and I anticipate that section 8.3 does not offer enough material to fill up the class period…Great class after execution. Not a lot of

concepts to "lecture" on, so pace of class benefited. Showing p. 380 from eBook, the summary of everything we learned to that point was a big hit. Ample time available for board problems; cadets worked through all without difficulty. Problem 7 was good to highlight relationship between rank and linearly independent column vectors. Bonus point challenge on discussion questions was also a great idea and well received.

Lesson 6: Great lesson. Some students recalled doing determinants on 2x2 matrices in MA255 (solving systems of ODEs), one thought they had used a determinant to compute a cross product, although none admitted to remembering using a determinant to perform the 2nd derivatives test in multivariable optimization. It was the right call to combine sections 8.4 and 8.5 from the book into one lesson. It would have unnecessarily drawn out the explanation of concepts had it not been done this way. There was a certain amount of momentum maintained being able to show the mechanical execution of cofactor expansion first, and then immediately highlight an easier way to find determinants by leveraging the properties of determinants. The cadets crushed the board problems (7 assigned).

Looking ahead to lesson 7, I'm not sure covering the section on Cramer's rule adds any value to the Linear Algebra Block, let alone the course. Cramer's rule is linked to #10 of the Block 1 learning objectives. I think we look at eliminating this, if it's just presented for sake of theoretical interest.

Lesson 7: Another great lesson. Good flow to the class. The narrative I share with the students is that most of the mechanics involved in linear algebra are tedious and not practical to carry out manually for systems with $n > 3$. Computers are very good at solving these systems, finding determinants, or finding inverses, so that's why we rely on them. But, it's important for engineers to understand the math that's "underneath the hood." Discussion on Cramer's rule only took about 5 minutes and didn't detract from the class. It could probably be omitted with no loss in value to the curriculum. Students' board work was excellent for this lesson.

Lesson 8: Obviously, eigenvalue problem is an important lesson. Students were engaged. I presented both a 2x2 and 3x3 matrix as examples and it worked well. Time it took for presentation of material and examples did not leave time for board problems, but I think this was OK. I may lead off the next class with a couple of problems. I did not get to show them the eigenvalue commands in MMA, but I'll do that in the next lesson. One student asked a very thoughtful question: "What do the eigenvalues tell us about the matrix **A**?" This gave me pause, and I told the class I would think about it over the weekend and give them a good answer at the next class.

Lesson 9: Opened lesson today with answer to question posed at end of lesson 8. Used animated gif on Wikipedia to show how the linear transformation AK will change the orientation of some vectors, but not others. Those whose direction remain the same are eigenvectors of K, and if their magnitude is changed, it is by the value of the corresponding eigenvalue. Also, since we did not do board work during Lesson 8, I did a couple rounds of problems from 8.8 at the beginning of class. This was fruitful, because students needed the practice, and it was good to have discussions about their work. Many students helped each other, which was encouraging. This left only about 15 minutes to cover lesson on orthogonal matrices and I did not get through an example on how to construct an orthogonal matrix from the eigenvectors of a symmetric matrix. I'll take the first five minutes of the next class to complete it. If I incorporate the visual on the eigenvectors into lesson 8 next time, though, this should give some time back to lesson 9.

Lesson 10: Finished example left over from Lesson 9 on constructing an orthogonal matrix from the eigenvectors of a symmetric matrix. Hopefully, we come back to this idea in subsequent

material. Otherwise, it's hard to see why we spend time going over this. Good discussion today on necessary and sufficient conditions and the logic used in mathematical "if, then" statements. Lesson on diagonalization of matrices is a good vehicle to reinforce other ideas, too, like the effect of multiplying a vector by a diagonal matrix. Students were excited to see this in action as we uncoupled a system of linear homogenous differential equations. They claimed to have forgotten what they learned about these in Jedi ODEs, but the rust shook off as we worked through the example. If we're able to buy some time back from other portions of block 1, it might be good to also include section 10.4, solutions to nonhomogenous systems, unless we use the solution technique of diagonalization extensively in the other blocks. Did also introduce the command `Eigensystem` in Mathematica. Some were already aware of it, but I mentioned it's a good way to check their work or to find eigenvalues and eigenvectors in situations where technology is allowed.
Lesson 11: Easy pace to this lesson. Began class with a quick discussion about what i raised to various powers equals, and showed the pattern. Used this as a reference when doing examples on roots of complex numbers—which worked well. All students were comfortable with the idea of the imaginary unit and had familiarity with complex numbers and addition and subtraction. It was good to review how to multiply complex numbers. I deferred discussion on division until after I walked them through complex conjugates. Although I did mention that it wasn't necessary to memorize the general formulae for adding, multiplying, and dividing complex numbers because the same associative, commutative, and distributive laws hold for complex numbers as real numbers, in retrospect it may have been value-added to highlight this specifically by pointing it out in the book or writing it down for their notes. Introduced complex plane and had a leading discussion to get them to buy into polar form of complex numbers. Ran through examples of taking powers and roots of complex numbers, which reinforced finding the modulus and arguments of z and complex multiplication, as well as basic knowledge of the unit circle. Intended to introduce Euler's formula as an aside when speaking about complex numbers represented in polar form, but only remembered to do it for one section.
Lesson 12: Thought going into this lesson is that it will be like drinking from the firehouse for the cadets…we'll see how they do. It's also becoming increasingly apparent that students are not doing the assigned reading before class; I'm strongly considering introducing an incentive for next semester (prep points).
Forecast proved correct. I did not get to the Harmonic conjugate example in the first section and only made it through the first Cauchy-Riemann equation for the example on checking if a complex function is analytic in the second session. I'll catch up with those in Lesson 13. Because of this, I decided to extend the due dates for the Lesson12 WebAssign and PS2. Complex mapping CDF in Mathematica was a big payoff. I think it would be good to teach ahead on 17.4 for Lesson 11 or just generally distribute 17.1-2 and 17.4-17.5 across lessons 11 and 12 in order to better allocate available class time.
Lesson 13: Playing catch up from Lesson 12 ate into the time for Lesson 13; the billpayer was board problems. I'm not convinced yet that we need to keep the dedicated lesson on exponential, logarithmic, trigonometric, and hyperbolic complex functions in the course. It's possible that discussing Euler's formula will be enough. We will end up revisiting Euler's formula again in Section 12.4 when we discuss complex Fourier series. Again, I'm not sure the intricacies and nuances of Sections 13.6-7 are valued added.

Running thoughts on executing MA364X and its curriculum

Lesson 16: A little crunched for time as I spent some time going over the Block I WPR in the beginning. I introduced Chapter 12 by highlighting that it is the first chapter in Part IV, "Partial Differential Equations". Motivation is that we need to learn some math to employ some solution techniques for PDEs. Book keeps going back to the vector analogy to introduce the concepts of inner product, orthogonality, norm, and linear combinations. I did the same, and it seemed to resonate with students. I did put special emphasis on the fact that orthogonality and norms have no geometric significance in the context of functions. Didn't quite get through the general Fourier series expansion in class, but it will be a good leaping off point for Lesson 17.
Lesson 17: Very comfortable pace to this lesson. Even though we had to catch up from Lesson 16 a little bit, there was ample time available to cover the material. I did not derive where the Fourier coefficients came from, left it to the students to explore that on their own time, but instead just gave it to them as a summary, similar to how presented on p. 661. Used 12.2.1 as an example to show how to find coefficients and construct a Fourier series expansion. #1 was a good example as it was a piecewise function and a_n = 0. Used Mathematica to highlight ideas of partial sums, convergence at points of discontinuity, and periodic extensions; this worked very well. Splitting class up into groups to work on the coefficients a_0, a_n, b_n for 12.2.2 was a good use of time. We used results from the three groups to construct another expansion and demonstrate that it was valid for f(x), again in Mathematica. One student asked a good question about cos(-n(Pi)). I had them consider if cos (-x) = cos x. Said we would discuss this in more detail in the next class; very good lead-in for even functions.
Lesson 18: Started this lesson with a good discussion of even and odd functions and their various properties. Easy for students, then, to see how the *b* coefficients dropped out for a cosine series and the *a* coefficients for a sine series; tied this back to why some of our integrals were evaluating to 0 in the previous lesson. Talked briefly about Gibbs phenomenon—not sure this is necessary, although it does satisfy the curiosity of those who might wonder about the spikes seen at points of discontinuity or at the ends of the interval. Spent rest of the time talking about half-range expansions and then building a demonstration of a function expanded on the interval [0,L] with a cosine series, a sine series, and a Fourier series and then comparing the various behaviors of these expansions on the interval [-L,0].
Lesson 19: I know that Complex Fourier Series are important for many applications, but it's curious that we don't highlight constructing complex Fourier series or finding frequency spectra in the course objectives. Maybe we add it or look at how relevant 12.4 is to the rest of the course. Quick review of Euler's formula—we saw this in Block I, so students were quick on the recall. Spent the time to derive the coefficient for a complex Fourier series, rather than just present it as a stand-alone definition. Good to highlight the quote in the text where the author is addressing the students directly on why we choose to have a complex Fourier series expansion in our kit bag. Most students were able to then take what they just saw and find a general c_n for a given function and construct a complex Fourier series. Example problem I used, the general c_n was undefined for n=0, so it was good to have them recognize that in a case like that, they have to compute that coefficient separately. We formed a partial sums function of the complex Fourier series in class in Mathematica, and I asked if they thought the plot of this function would appear on the same plane as the original real-valued function. To their amazement it did, and I showed that evaluating the partial sums function itself (after simplification) yielded only real-valued terms (Euler's formula is embedded within it). Spent only about 3-5 minutes on frequency spectra, but that seemed to be OK, as it seemed students were comfortable with the concept.

Running thoughts on executing MA364X and its curriculum

Lesson 20: Spent some time upfront to follow up on presentations from the CEM Colloquia; may have cost some time on the back end as I didn't get through all of the Sturm-Liouville problem discussion. Otherwise, spent first portion of lesson reviewing and summarizing theory of linear equations. Introduced BVPs by first contrasting them to IVPs. Ran through an example of solving a BVP and that solutions were only present for the eigenvalues.
Lesson 21: Used first part of the lesson to catch up where we had left off with the regular Sturm-Liouville problem. Students were able to easily grasp how examples we showed on Friday were examples of a Sturm-Liouville problem. I did go through the general method of re-casting a 2nd order linear DE into the self-adjoint form, because they have a problem like this on their WebAssign. I do wonder how important being able to re-cast an equation is, though. Is there anything special about recognizing equations (and their side conditions) as Sturm-Liouville problems? I'll wait to see if that comes up again. Introduced PDEs and their terminology and explained what hyperbolic, parabolic, elliptic equations are; cadets were able to quickly determine how to identify each. Ran through example #2 of 13.1 to teach Separation of Variables. Appeared accessible to students; they mostly recalled the technique as applied to ODEs. Referred them back to the table on pages 674-675 for solutions of common ODEs.
Classical PDEs: Looking ahead to the last several lessons of Block II, I think my approach will be to introduce the three classical equations on Lesson 22 and remark on their various applications; introduce terminology for the various side conditions, and then dive deep on the heat equation for the remainder of the lesson and then do practice on the heat equation for lessons 23 and 24. I imagine that I will also introduce some visualizations during the 2nd or 3rd lesson of the heat equation. I plan to follow a similar pattern for the wave equation and Laplace equation.
Lesson 22: A lot of students had questions on a WebAssign problem (separation of variable), so I took the time to work through problem 13.1.12; didn't go all the way to the answer, but it was a strong nudge in the right direction. Rest of lesson was spent introducing the three classical PDEs, classifying them (as a class) as to them being parabolic, hyperbolic, or elliptic. In the first section, I introduced the general boundary conditions first before I started doing a deep dive on the heat equation, but I reversed this sequence in the next section. I think it was better to show the application first and then examples of certain boundary conditions and how they are described generally—Dirichlet, Neumann, or Robin. I also was deliberate about taking time to comment on the assumptions the heat equation makes; seemed to resonate with the students.
Lesson 23: I did not get as far as I thought I would in this class. We spent much of the lesson on boundary conditions, initial conditions and setting up BVPs. Cadets practiced two of these on the boards—with no difficulty. We walked through a more complicated example that allows for lateral heat transfer. Only in the last 10 minutes did I start talking about separation of variables and how it will ultimately lead to a solution that looks like a Fourier series. I think that's OK, because the time we spend on these ideas with the wave equation should pay dividends once we apply them to the wave and Laplace equations. One student asked an interesting question about the long-term behavior of a cooling rod with insulated ends and uniform initial temperature distribution. He hypothesized that the resulting level curves for t will be horizontal. I'll probably use this as an illustration for Lesson 24.
Lesson 24: Easy pace to this lesson and lots of time for practice on the boards. Not many practice problems available from the Zill/Wright text, so I relied heavily on the Boyce/DiPrima ODEs/BVPs text. Students seemed to do well constructing half-range series for given values of k, L, and f(x), when BCs are homogenous. Integration by parts continues to be a nuisance, but I

told cadets that technology is perfectly acceptable method of solving those coefficient integrals; I'm not going to test them on their ability to do integration by parts. There's a stretch problem on WebAssign involving lateral heat loss; I expect that to challenge them somewhat. Looking ahead to the next lesson, I'm starting to see the significance of the regular Sturm-Liouville problem and how we can relate similarities amongst the various BVPs and recognize what the solutions of the separated ODEs are.

Lesson 25: Used the same outline to introduce the wave equation as I did the heat equation, spending time to explain the underlying assumptions and variations of the model when some of those assumptions are relaxed. Devoted a lot of time to setting up BVPs with different BCs, ICs, and extra terms in the equation itself. Students did well with this. With one section, we were able to progress to introducing how we solve the wave equation through separation of variables. Emphasized that the eigenvalues looked the same as they did for the heat equation.

Lesson 26: I've realized the canonical examples of the wave equation are when the BCs are homogenous. In that way, it's straightforward to introduce to students a solution form for u(x,t) that involves an infinite series with two unknown coefficients. The exercise for them, then, becomes solving for the coefficients based on what f(x) and g(x) are. At least half of this class was devoted to students working in groups to solve a problem with a different f(x) (g(x) = 0). They were asked to produce several plots of the solution holding t constant, then holding x constant, and finally producing an animation. Groups will brief their results during Lesson 27.

Lesson 27: Briefings continued today and all but one group across both sections had an opportunity to brief their results. Overall, briefings were quite good. I most impressed by briefs led by two students that are typically quite quiet in class. They demonstrated an unexpected command of the material; one's nuanced discussion of wave interference and reflection was simply excellent. The feedback from students was that the group work and devoted time in class working on these problems was very helpful and increased their level of understanding with the material. I'll plan to do it similarly for the Laplace equation lessons.

Lesson 28: After re-introducing the Laplace equation and then speaking about its applications and the steady-state heated plate application we will use in class, most of the remainder of the lesson was taken up by working through the introductory BVP in 13.5. I used it to summarize the technique of separation of variables, checking for nontrivial solutions for the three eigenvalue cases, and then imposing the boundary conditions to solve for the coefficients. I emphasized that there is no formulaic way to solve Laplace's equation; it depends on the nature of the boundary conditions. This seemed to register with the students. I contrasted what we did in class with the boundary conditions found in the Dirichlet problem, and told them that we will begin the next class period with them solving the Dirichlet problem in groups.

Lesson 29: After taking some time to explain the project (which, incidentally, is more like a Tech lab and should probably be referred as such for the next go-around) at the beginning of class, the rest of the class period was devoted to the students working in groups on solving a Dirichlet problem. I made the work competitive by offering bonus points to the first group to correctly and completely briefing a response. There was good momentum initially and one group in the first section actually asked to brief after only 10-15 minutes of work. Their brief wasn't incorrect; just incomplete. They tried again soon, but were still lacking the case when $\lambda < 0$. Other groups struggled with various aspects of the problem and I prompted them along the way, but no one group was able to pull it all together in one complete brief. I decided to use the first group's boardwork to brief the solution myself at the end of class to hammer the procedure home. By that time, interest was waning anyway. The second section's performance was similar, but one group was able to brief a correct and complete response. Another group claimed to have arrived at the same solution simultaneously. Their boardwork looked sound and having monitored their energy throughout class, I felt no reason to deny them bonus points as well.

Lesson 30: After reviewing the problem set, I spent some time upfront going over the superposition principle and how it can be leveraged to solve a Laplace Equation BVP when there are no opposing homogenous BCs. In retrospect, it might be good to emphasize the superposition principle a little more than I do on solving all of these classical PDEs, because we've been using it over and over when we say $u_n = XY$ and consequently $u(x,y)$ is an infinite Fourier series. Rest of class was spent on the boards with students back in their same groups tackling problem 13.5.3. Process still proves to be time-consuming for students, but it seemed like light bulbs went off for several today. Some recurring things I saw: sin and sinh be treated the same; applying nonhomogenous BC to try and solve for coefficients in one of the ODEs, instead of waiting to do so after constructing the product solution.

Lesson 32: Seemed to be an accessible lesson to the class, as they have seen Euler's Method and the Improved Euler's Method in their ODE course as plebes. Walked them through the background on both methods before implementing in Excel. Took care to use k1 and k2 when describing Improved Euler's Method (weighted average of slopes), so it would be a gentler introduction to RK4. It was clear that I wasn't going to have time to implement RK4 in class, so I introduced the procedure and deferred implementation to the beginning of the next class.

Lesson 33: Lesson a little bogged down today in having the class just replicate what I'm doing in Excel to implement the RK4 method. First, we worked with an example 1st order, then a 2nd order split into a system of two 1st order ODEs. One student brought up that it would be helpful for me to provide the shell of the Excel file, so that they could concentrate on getting the formulae correct. I'll have to do that next time around. Students also struggled with the change in notation the book uses. I thought it useful to explain that solving a 2nd order equation was just a special case of the more general method for solving systems, but book's notation made that transition difficult. 2nd order equations method uses x as the independent variable, and y and u as the dependent variables. General system's method uses t as the independent variable and then x and y as the dependent variables. Compounding the confusion is the mistake in the book on p. 302 that omits many of the "h" factors. I did highlight this and will post the correct equations.

Lesson 34: Spent a little time up front fielding an RK4 question—really just an Excel implementation issue. Then, I derived the finite difference method by showing the central difference approximations for y' and y'' we can obtain from Taylor series expansions. At first, class thought the formula was just a bunch of jibberish, but it became clear once we applied it to a BVP. I was deliberate about drawing out the interval and identifying all of the interior mesh points. For the second class, I actually made a table of x and y and showed how we were just completing the table for the unknown interior y-values. I then ran through an example of how to implement the method (n=4). I conceded to the class that the example I chose was easy to do because P(x) = 0 and Q(x) = -1, but I chose the example for brevity. Class seemed to follow very well, and was able to easily see how we would convert the resulting system of equations into matrix-vector form and then solve with method of choice. Looking ahead, we have two lessons set aside for the Laplace Equation, but the Finite Difference Method applied to the Laplace Equation is pretty straightforward. I think I might reallocate the second day to introducing the Heat Equation. The discussion on stability for both the heat and wave equations is worth spending a little more time on than I think just one lesson per equation would afford.

Running thoughts on executing MA364X and its curriculum

Lesson 35: Lesson was very accessible today, especially since students seen application of the mean-value theorem in PS1, which is essentially what the finite difference approximation of the Laplacian is. That type of problem that we use in PS1 needs to be sustained, because of its value for recall. I showed an example and students were able to easily solve one on their own on the boards (4 equations). I had enough time to show them how to implement Gauss-Seidel iteration in Excel. We don't need two lessons set aside for numerical solutions to the Laplace Equation. I'll use the extra time available now on the heat and wave equations.

Lesson 36: There were no questions at the outset on Gauss-Seidel iteration from Lesson 35, despite it being presented briefly. First, I derived the finite difference approximation we use for the Heat Equation by using difference approximations we've already seen for the Laplace Equation and Euler's method. I adapted the Heat Equation "slider" spreadsheet to use as a tool for showing the explicit finite difference method. I adapted it to also show the value of lambda and what its impact on stability is. To demonstrate how to use the spreadsheet, I used Example 1 from the book, and soon discovered a typo in the boundary conditions (reinforces why we are writing our textbook). We also took time to do 16.2.1 (same as WebAssign #1). 16.2.1 shows concept of instability more readily, and it looked like Example 1 would not become unstable (I thought because of the sinusoidal initial condition), but it does for greater values of t after it appears the system has reached equilibrium (interesting).

Lesson 38: We picked up again on the Heat Equation. I showed them the derivation of the Crank-Nicholson method and my first attempt to implement it into Excel using Gauss-Seidel iteration. We had a good discussion on why it didn't work based on the relationship of the side conditions to the mesh. However, I did have a student in the second section who proposed a different way of implementation. I was intrigued, and I successfully verified his proposal. That's the way I'll introduce Crank-Nicholson from now on. I also showed how one could implement it in Excel by solving an n-1 AX=B system for each j, although everyone agreed that it was a bit more involved to implement. We also had a good discussion on error and stability differences amongst the explicit and implicit techniques. I won't test the students on the Crank-Nicholson method, and I think that will continue in future semesters, however the exposure to it and the discussions it prompts are worth keeping it in the course. I briefly introduced the finite difference method we'll use for the wave equation and highlighted the difficulty we'll have with the j = 1 timeline. The students understood that we're going to have to address the problem of going back in time (before the initial t). We'll pick up there in Lesson 39.

Lesson 39: Took time at the beginning of the class to revisit the Crank-Nicholson method and showed how it could be implemented in Excel with Gauss-Seidel iteration and how it converges on the same values that the solutions to the systems of linear equations produce. Picked up with the "back-in-time" problem and derived the special case expression for j=1 by taking advantage of the initial velocity condition. Showed students how to include g(x_i) on their spreadsheets. Everyone seemed to track on this. Not much anxiety perceived for the computational math block. They all seem comfortable with using the techniques we discussed in class.

SECTION II

EXAMINATIONS

UNITED STATES MILITARY ACADEMY
WEST POINT.

TAB E WPR #1

Instructor Solution

NAME Solutions Section ________ Time Departed __________

WPR 1 – MA365	3 October 2019	300 Points

READ THESE INSTRUCTIONS CAREFULLY BEFORE STARTING WORK:

1. Write your name on every exam page.

2. Authorized references for this exam include one sheet of paper (front and back) of hand-written notes. The issued TI-30 or FEE approved calculator.

3. You will have 75 minutes to complete this exam. Early departure is authorized.

4. Sufficient work is required to indicate clearly your method of reasoning and the operations performed. **SHOW ALL WORK. Clearly indicate your final answer.**

5. Show as much work as necessary to support your answer. A wrong numerical or symbolic solution with no supporting work receives zero credit. It is always best to show intermediate steps to illustrate your problem-solving process.

6. Clearly identify your final answer. You should double underline your answer when appropriate.

7. Use a blank continuation sheet and clearly identify that the problem is continued both on the exam and on the continuation sheet. Use one continuation sheet per problem continued. Be sure, to put your name on each continuation sheet.

Problem	1	2	3	4	5	6	7	8	Total
Possible	42	35	36	32	40	40	30	45	300
Score									

Name ______________________________ Section __________

1. (42 points) Consider the complex function:

$$f(z) = \frac{8+3i}{9-2i}$$

(a) Determine the real, $u(x,y)$, and imaginary part, $v(x,y)$, of $f(z)$.

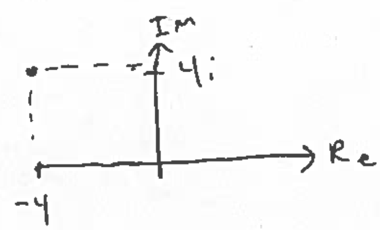
$$f(z) = \frac{8+3i}{9-2i} \cdot \frac{9+2i}{9+2i} = \frac{(8+3i)(9+2i)}{(9-2i)(9+2i)}$$

$$= \frac{72+16i+27i-6}{85} = \frac{66}{85} + \frac{43i}{85}$$

$$Re = \frac{66}{85} \qquad Im = \frac{43}{85}$$

(b) Write the given number in polar form: $-4+4i$

$$||z|| = \sqrt{(-4)^2+(4)^2}$$
$$= \sqrt{16+16} = \sqrt{32} = 16\sqrt{2} = 4\sqrt{2}$$



$$\tan\theta = \frac{y}{x} = \frac{Im}{Re} = \frac{4}{-4} = -1 \qquad \theta_R = \frac{\pi}{4} \qquad \theta = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$$

$$z = r\cos\theta + i\,r\sin\theta$$
$$= 4\sqrt{2}\cos\frac{3\pi}{4} + i\,4\sqrt{2}\sin\frac{3\pi}{4}$$
$$= 4\sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$$

Cadet________________________________ Section ____________

2. (35 points) Mark each statement as True (T) or False (F).

(a) __F__ Any system of n linear equations in n variables always has at least one solution.

(b) __T__ If a system of linear equations has two different solutions, it must have infinitely many solutions.

(c) __T__ If $\mathbf{AB} = \mathbf{C}$ and $\mathbf{C}$ has 1 column, then $\mathbf{B}$ must have 1 column.

(d) __T__ If the system $\mathbf{AX} = \mathbf{B}$, where $\mathbf{B} = \mathbf{0}$, then the system will have at least the trivial solution.

(e) __F__ If $\mathbf{A}$ is a 3×3 matrix and the equation $\mathbf{AX} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ has only one solution, then $\mathbf{A}$ is singular.

(f) __F__ The inverse of any symmetric matrix is equal to its own transpose, i.e. $\mathbf{A}^{-1} = \mathbf{A}^T$.

(g) __T__ All $n \times n$ matrices that have an eigenvalue of 0 are singular.

3. (36 points) Calculate the determinant of $\mathbf{A} = \begin{pmatrix} 0 & -1 & 21 \\ 1-i & 0 & -6 \\ -1 & 0 & 1+i \end{pmatrix}$. Based on your answer, is $\mathbf{A}$ singular? Is $\mathbf{A}$ invertible? Does $\mathbf{A}$ have full rank?

$$\left(\begin{array}{ccc|cc} 0 & -1 & 21 & 0 & -1 \\ 1-i & 0 & -6 & 1-i & 0 \\ -1 & 0 & 1+i & -1 & 0 \end{array}\right)$$

determinat = 4 Thus A is: nonsingular

an Inverse exists and is invertible

A has full rank

Cadet____________________________ Section __________

4. (32 points) The augmented matrices (**A**|**B**) and (**C**|**D**) below are row equivalent. In other words, (**C**|**D**) is the row echelon form obtained after performing row operations on (**A**|**B**). For each of the concepts following, choose (by placing a checkmark) **ONE** mathematical statement that can be made concerning the following properties of the matrix **A** or augmented matrix (**A**|**B**).

$$(\mathbf{A}|\mathbf{B}) = \left(\begin{array}{rrrr|r} 1 & 1 & 2 & 0 & 1 \\ 2 & -1 & 0 & 1 & -2 \\ 1 & -1 & -1 & -2 & 4 \\ 2 & -1 & 2 & -1 & 0 \end{array}\right) \qquad (\mathbf{C}|\mathbf{D}) = \left(\begin{array}{rrrr|r} 1 & 1 & 2 & 0 & 1 \\ 0 & 1 & \frac{3}{2} & 1 & -\frac{3}{2} \\ 0 & 0 & 1 & 8 & -17 \\ 0 & 0 & 0 & 1 & -2 \end{array}\right)$$

(a) For matrix **A**,

i. __X__ the columns are linearly dependent.
ii. _____ the columns are linearly independent.
iii. _____ the columns and rows are both linearly independent.
iv. _____ None of the above.

(b) Matrix **A**

i. _____ has a determinant of 0.
ii. __X__ is nonsingular.
iii. _____ All of the above.
iv. _____ None of the above.

(c) The reduced row echelon form of **A** shows that

i. __X__ **A** is invertible.
ii. _____ the rows of **A** are linearly dependent.
iii. _____ **A** has 0 as one of its eigenvalues.
iv. _____ All of the above.
v. _____ None of the above.

(d) Is the system **AX** = **B** consistent? Why or why not?

i. __X__ Yes, because rank(**A**) = rank(**A**|**B**).
ii. _____ No, because rank(**A**) < rank(**A**|**B**).
iii. _____ No, because the system is nonhomogenous.
iv. _____ Both items ii and iii.
v. _____ None of the above.

Name ______________________ Section __________

5. (40 points) Consider the system of equations:

$$7x_1 - 2x_2 = b_1$$
$$3x_1 - 2x_2 = b_2$$

(a) Write the system in the form **AX=B** and calculate the inverse of **A**

$$\begin{bmatrix} 7 & -2 \\ 3 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

$$A^{-1} = \frac{1}{\det}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{-8}\begin{bmatrix} -2 & 2 \\ -3 & 7 \end{bmatrix} = -\frac{1}{8}\begin{bmatrix} -2 & 2 \\ -3 & 7 \end{bmatrix}$$

$$\det(A) = -14 - (-6) = -8 \qquad \Rightarrow \begin{bmatrix} \frac{1}{4} & -\frac{1}{4} \\ \frac{3}{8} & -\frac{7}{8} \end{bmatrix}$$

(b) Use $X = A^{-1}B$ to solve the system for each matrix **B**.

a. $B = \binom{5}{4}$

$$-\frac{1}{8}\begin{bmatrix} -2 & 2 \\ -3 & 7 \end{bmatrix}\begin{bmatrix} 5 \\ 4 \end{bmatrix} = \frac{1}{-8}\begin{bmatrix} -10 & +10 \\ -15 & +32 \end{bmatrix} = -\frac{1}{8}\begin{bmatrix} 0 \\ 7 \end{bmatrix}$$

b. $B = \binom{10}{50}$

$$-\frac{1}{8}\begin{bmatrix} -2 & 2 \\ -3 & 7 \end{bmatrix}\begin{bmatrix} 10 \\ 50 \end{bmatrix} = -\frac{1}{8}\begin{bmatrix} -20+20 \\ -31+350 \end{bmatrix} = -\frac{1}{8}\begin{bmatrix} 0 \\ 320 \end{bmatrix} = \begin{bmatrix} 0 \\ -40 \end{bmatrix}$$

c. $B = \binom{0}{-20}$

$$-\frac{1}{8}\begin{bmatrix} -2 & 2 \\ -3 & 7 \end{bmatrix}\begin{bmatrix} 0 \\ -20 \end{bmatrix} = -\frac{1}{8}\begin{bmatrix} 0 + -40 \\ 0 + -140 \end{bmatrix} = -\frac{1}{8}\begin{bmatrix} -40 \\ -140 \end{bmatrix}$$

(c) Do the solutions you found in part (b) each represent a unique solution to the system **AX=B**? How can you tell?

Yes, A is not singular and therefore the solution set $A\vec{x} = B$ must be unique

Name ______________________________ Section __________

6. (40 points) Determine the inverse of A

$$A = \begin{bmatrix} -1 & 1 & 2 \\ 3 & -1 & 1 \\ -1 & 3 & 4 \end{bmatrix}$$

Several methods can be used, I used Gauss-Jordan elimination for this problem

$$\left[\begin{array}{ccc|ccc} -1 & 1 & 2 & 1 & 0 & 0 \\ 3 & -1 & 1 & 0 & 1 & 0 \\ -1 & 3 & 4 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} -1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 2 & 7 & 3 & 1 & 0 \\ 0 & 2 & 2 & -1 & 0 & 1 \end{array}\right] \quad \begin{array}{l} \text{Row } 2 + 3 \text{ Row } 1 \\ \text{Row } 3 - \text{Row } 1 \end{array}$$

$$\left[\begin{array}{ccc|ccc} -1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 2 & 7 & 3 & 1 & 0 \\ 0 & 0 & -5 & -4 & -1 & 1 \end{array}\right] \quad \text{Row } 3 - \text{Row } 2$$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & -2 & -1 & 0 & 0 \\ 0 & 1 & 3.5 & 1\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 & \frac{4}{5} & \frac{1}{5} & -\frac{1}{5} \end{array}\right] \quad \begin{array}{l} -\text{Row } 1 \\ \frac{1}{2} \text{ Row } 2 \\ -\frac{1}{5} \text{ Row } 3 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 0 & \frac{3}{5} & \frac{2}{5} & -\frac{2}{5} \\ 0 & 1 & 0 & -1\frac{3}{10} & -\frac{1}{5} & \frac{7}{10} \\ 0 & 0 & 1 & \frac{4}{5} & \frac{1}{5} & -\frac{1}{5} \end{array}\right] \quad \begin{array}{l} \text{Row } 1 + 2 \text{ Row } 3 \\ \text{Row } 2 - 3\frac{1}{2} \text{ Row } 3 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & \\ 0 & 1 & 0 & \boxed{\begin{array}{ccc} -\frac{7}{10} & \frac{1}{5} & \frac{3}{10} \\ -1\frac{3}{10} & -\frac{1}{5} & \frac{7}{10} \\ \frac{4}{5} & \frac{1}{5} & -\frac{1}{5} \end{array}} \\ 0 & 0 & 1 & \end{array}\right]$$

Name ______________________________ Section __________

7. (30 points) Construct an orthogonal matrix from the eigenvalues and eigenvectors of the given symmetric matrix.

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 0 & 1 \\ 0 & 1-\lambda & 0 \\ 1 & 0 & 1-\lambda \end{vmatrix} = (1-\lambda)^3 - 1(1-\lambda)$$

$$= (1-\lambda)\left[(1-\lambda)^2 - 1\right] = (1-\lambda)(\lambda^2 - 2\lambda) = 0 \Rightarrow \lambda = 0, 1, 2$$

Since 0 is an eigenvalue, A is thus singular

$\lambda_1 = 0$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right] \Rightarrow \begin{array}{l} x_1 = -x_3 \\ x_2 = 0 \\ x_1 = -x_3 \end{array} \qquad V_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$\lambda = 1$

$$\left[\begin{array}{ccc|c} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{array}\right] \Rightarrow V_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

V_1, V_2, V_3

$\|V_1\| = \sqrt{2}$

$\lambda = 2$

$$\left[\begin{array}{ccc|c} -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \end{array}\right] \qquad V_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$\|V_2\| = 1$

$\|V_3\| = \sqrt{2}$

Thus an orthogonal set of vectors is

$$\begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ -\frac{1}{\sqrt{2}} \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

orthogonal matrix is then

$$P = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$$

Name ______________________________ Section __________

8. (45 points) Determine whether the given matrix **A** is diagonalizable. If so, find the matrix **P** that diagonalizes **A** and the diagonal matrix **D**, such that $D = P^{-1}AP$

$$A = \begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix}$$

det (A − λI)

$$\begin{vmatrix} -5-\lambda & -3 \\ 5 & 11-\lambda \end{vmatrix} = (-5-\lambda)(11-\lambda) + 15 = 0 \Rightarrow -55 + 5\lambda - 11\lambda + \lambda^2 + 1\text{[illegible]}$$

$$\lambda^2 - 6\lambda - 40 = 0$$

$$(\lambda - 10)(\lambda + 4) = 0$$

eigenvalues are

$\lambda_1 = -4$ $\lambda_2 = 10$

$\frac{1}{5}R_2,\ R_2 + R_1$

$$\left[\begin{array}{cc|c} -5+4 & 3 & 0 \\ 5 & 11+4 & 0 \end{array}\right] = \left[\begin{array}{cc|c} -1 & -3 & 0 \\ 5 & 15 & 0 \end{array}\right] = \left[\begin{array}{cc|c} -1 & -3 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$-K_1 = 3K_2$

therefore $V_1 = \begin{pmatrix} -3 \\ 1 \end{pmatrix}$

$\lambda_2 = 10$

$-\frac{1}{3}R_1,\ R_2 - R_1$

$$\left[\begin{array}{cc|c} -15 & -3 & 0 \\ 5 & 1 & 0 \end{array}\right] = \left[\begin{array}{cc|c} 5 & 1 & 0 \\ 5 & 1 & 0 \end{array}\right] \Rightarrow \left(\begin{array}{cc|c} 5 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

$K_2 = -5K_1$

$V_2 = \begin{pmatrix} -1 \\ 5 \end{pmatrix}$

$$P = (K_1\ K_2) = \begin{pmatrix} -3 & -1 \\ 1 & 5 \end{pmatrix}$$

$$P^{-1} = \frac{1}{-15+1}\begin{pmatrix} 5 & 1 \\ -1 & -3 \end{pmatrix} = \frac{1}{-14}\begin{pmatrix} 5 & 1 \\ -1 & -3 \end{pmatrix} = \begin{pmatrix} \frac{-5}{14} & \frac{-1}{14} \\ \frac{1}{14} & \frac{3}{14} \end{pmatrix}$$

$$P^{-1}AP = \begin{pmatrix} \frac{-5}{14} & \frac{-1}{14} \\ \frac{1}{14} & \frac{3}{14} \end{pmatrix}\begin{pmatrix} -5 & -3 \\ 5 & 11 \end{pmatrix}\begin{pmatrix} -3 & -1 \\ 1 & 5 \end{pmatrix} \Rightarrow \begin{pmatrix} \frac{25}{14} - \frac{5}{14} & \frac{15}{14} - \frac{11}{14} \\ \frac{-5}{14} + \frac{15}{14} & \frac{-3}{14} + \frac{33}{14} \end{pmatrix}\begin{pmatrix} -3 & -1 \\ 1 & 5 \end{pmatrix}$$

$$P^{-1}AP = \begin{pmatrix} \frac{10}{7} & \frac{2}{7} \\ \frac{5}{7} & \frac{15}{7} \end{pmatrix}\begin{pmatrix} -3 & -1 \\ 1 & 5 \end{pmatrix} \Rightarrow \begin{pmatrix} \frac{-30}{7} + \frac{2}{7} & \frac{-10}{7} + \frac{10}{7} \\ \frac{-15}{7} + \frac{15}{7} & \frac{-5}{7} + \frac{75}{7} \end{pmatrix} = \begin{pmatrix} -4 & 0 \\ 0 & 10 \end{pmatrix} = D$$

TAB F WPR #2

Instructor Solution

NAME Solutions Section ________ Time Departed __________

WPR 2 – MA365	14/15 November 2019	300 Points

READ THESE INSTRUCTIONS CAREFULLY BEFORE STARTING WORK:

1. Write your name on every exam page.

2. Authorized references for this exam include one sheet of paper (front and back) of hand-written notes. The issued TI-30 or FEE approved calculator.

3. You will have 75 minutes to complete this exam. Early departure is authorized.

4. Sufficient work is required to indicate clearly your method of reasoning and the operations performed. **SHOW ALL WORK. Clearly indicate your final answer.**

5. Show as much work as necessary to support your answer. A wrong numerical or symbolic solution with no supporting work receives zero credit. It is always best to show intermediate steps to illustrate your problem-solving process.

6. Clearly identify your final answer. You should double underline your answer when appropriate.

7. Use a blank continuation sheet and clearly identify that the problem is continued both on the exam and on the continuation sheet. Use one continuation sheet per problem continued. Be sure, to put your name on each continuation sheet.

Problem	1	2	3	4	5	6	7	Total
Possible	40	35	70	35	60	30	30	300
Score								

Name ______________________________ Section __________

1. (40 pts) Consider the functions: $f_1(x) = x$ *and* $f_2(x) = x^2 + 1$

(a) Is $f_1(x)$ even, odd, or neither? Is $f_2(x)$ even, odd, or neither?

even

(b) Are $f_1(x) = x^3$ *and* $f_2(x) = x^2 + 1$ orthogonal from [-1, 1]? Show your work.

$$\int_{-1}^{1} x^3(x^2+1)\,dx \Rightarrow \int_{-1}^{1} x^5 + x^3\,dx \Rightarrow \left|\frac{x^6}{6} + \frac{x^4}{4}\right|_{-1}^{1}$$

$$\frac{1}{6} + \frac{1}{4} - \left(\frac{1}{6} + \frac{1}{4}\right) = 0$$

Yes, They are orthogonal

(c) Are $f_1(x) = x^3$ *and* $f_2(x) = x^2 + 1$ orthogonal on [0, 1]? Show your work.

$$\int_{0}^{1} x^5 + x^3\,dx = \left|\frac{x^6}{6} + \frac{x^4}{4}\right|_{0}^{1} \quad \frac{1}{6} + \frac{1}{4}$$

No, Not orthogonal

Name ______________________________ Section __________

2. (35 pts) Consider a string of length 4 that coincides with the interval [0, 9] on the x-axis. The left end of the string is secured to the x-axis, but the right end moves in a transverse manner according to $\sin \pi t$. Initially, the string is undisplaced but has the initial velocity $\sin \frac{\pi x}{4}$. Set up the boundary-value problem for the displacement $u(x, t)$. DO NOT SOLVE.

$u(0,t)=0$

$u(4,t)=\sin(\pi t)$

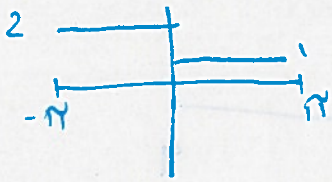
$u(x,0)=0$

$u_t(x,0)=\sin\frac{\pi x}{4}$

$\alpha^2 u_{xx}=u_{tt}$

Name ______________________________ Section __________

3. (70 pts) Expand $f(x) = \begin{cases} 2 & -\pi < x < 0 \\ 1 & 0 \le x < \pi \end{cases}$ in a Fourier series.



$$Sf(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)$$

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,dx = \frac{1}{L}\left[\int_{-\pi}^{0} 2 + \int_{0}^{\pi} 1\right]dx$$

$$= \frac{1}{L}\left[2x\Big|_{-\pi}^{0} + 1x\Big|_{0}^{\pi}\right] = \frac{1}{L}\left[2\pi + \pi\right] = \cancel{2\pi}\ \underline{3}$$

$$a_n = \frac{1}{\pi}\left[\int_{-\pi}^{0} 2\cos nx + \int_{0}^{\pi} \cos nx\right] = \frac{2}{n}\sin nx\Big|_{-\pi}^{0} + \frac{1}{n}\sin nx\Big|_{0}^{\pi}$$

$$-\frac{2}{n}\sin n\pi + \frac{1}{n}\sin n\pi$$

$$a_n = -\frac{1}{n}\sin n\pi$$

$$f(x) = \frac{3}{2} + \sum_{n=1}^{\infty} \frac{1}{n\pi}\left(-1 + (-1)^n\right)\sin nx$$

$$b_n = \frac{1}{\pi}\left[\int_{-\pi}^{0} 2\sin nx + \int_{0}^{\pi} \sin nx\right.$$

$$= -\frac{2}{n}\cos nx + -\frac{1}{n}\cos nx$$

$$= -\frac{2}{n} + \frac{2}{n}(-1)^n + -\frac{1}{n}(-1)^n - \frac{1}{n}$$

$$= -\frac{3}{n} + \frac{1}{n}(-1)^n$$

$$\frac{3}{2} + \sum_{n=1}^{\infty}\left(-\frac{1}{n}\sin n\pi\right)\cos\frac{n\pi x}{L} + \left(-\frac{3}{n} + \frac{1}{n}\right)(-1)^n \sin\frac{n\pi x}{L}$$

$$\frac{1}{n\pi}\left(-1 + (-1)^n\right)$$

Name ______________________________ Section __________

4. (35 pts) For each multiple choice problem, circle the one answer that best answers the question.

(a) (5 pts) Which of the following is NOT an assumption supporting the formulation of the 1-D heat equation?

i. The rod is homogeneous; that is, its mass per unit volume ρ is a constant
ii. The flow of heat takes place only in the x and y-direction
iii. The lateral, or curved, surface of the rod is insulated
iv. No heat is being generated within the rod

(b) (5 pts) The function $f(x) = x^2 + 3$ is:

i. An even function
ii. An odd function
iii. Neither an odd nor even function
iv. Both an odd and an even function

(c) (5 pts) Let $f(x)$ represent an odd function and $g(x)$ represent an even function. Which of the following is NOT always true?

i. $\int_0^a f(x)g(x)dx = 0$
ii. The product $f(x)g(x)$ is an odd function
iii. $2\int_0^a g(x)dx = \int_{-a}^a g(x)dx$
iv. $\int_{-a}^a f(x)g(x)dx = 0$

(d) (4 pts) True or False. In general, two functions are orthogonal on an interval if their graphs are perpendicular to each other at each point of intersection on the interval.

false

Name ______________________________ Section __________

(e) (6 pts) Classify each following PDE as hyperbolic, elliptic, parabolic, or neither:

i. $7\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial t} = 0$ parabolic

ii. $2\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial t^2} = 0$ hyperbolic

iii. $5\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ elliptic

(f) (5 pts) For the one-dimensional heat equation where $0 < x < L$, which of the following boundary conditions indicates that the left end of the rod or wire is held at a constant temperature of 20?

i. $u(x, 0) = 20$
ii. $\left.\frac{\partial u}{\partial x}\right|_{x=0} = 20$
iii. $u(20,\ t) = 0$
iv. $u(0,\ t) = 20$

(g) (5 pts) True or False. Every partial differential equation can be solved by the method of separation of variables.

False

Name ______________________ Section __________

5. (60 pts) Solve the following boundary-value problem, subject to the given conditions. To receive full credit, you must show all of the steps (separation of variables, finding all possible cases for the separation constant, applying the principle of superposition, applying the initial condition, and solving for the coefficients of a Fourier series) in generating the solution.

$$2\frac{\partial^2 u}{\partial x^2}=\frac{\partial u}{\partial t},\ 0<x<2,\ t>0$$
$$U_x(0,t)=0, U_x(2,t)=0$$
$$u(x,0)=x,\ 0<x<2$$

$2u_{xx}=u_t$

$2X''T=T'X$

$\frac{X''}{X}=\frac{T'}{2T}=\gamma$ $\quad X''-\gamma X=0$

$T'=2\gamma T$

$T=ce^{2\gamma t}$ $\quad T_n=\sum_{n=1}^{\infty} C_n e^{-2\left(\frac{n\pi}{2}\right)^2}$

$u=XT$

Case $\gamma=0$

$Ax+B$ $\quad B=?$

$u_x(0,t)=0$ $\quad A=0$

Cas $\gamma>0$

$Ae^{\sqrt{\gamma}x}+Be^{-\sqrt{\gamma}x}=0$

u_x $\cancel{Ae^{\sqrt{\gamma}(0)}+Be^{(0)}=A+B}$

$A\sqrt{\gamma}e^{\sqrt{\gamma}x}-B\sqrt{\gamma}e^{-\sqrt{\gamma}x}=0$

$\sqrt{\gamma}(A-B)=0$ $\quad A=B$

$A\sqrt{\gamma}(e^{2\sqrt{\gamma}}-e^{-2\sqrt{\gamma}})=0$

$A=0$ Then $B=0$

$\gamma<0$

$A\cos\sqrt{-\gamma}x+B\sin\sqrt{-\gamma}x=0$

$-\sqrt{-\gamma}A\sin+\sqrt{-\gamma}B\cos\sqrt{-\gamma}x=0$

$B=0$

for $x=2$

$2\sqrt{-\gamma}=n\pi$

$-A\sqrt{-\gamma}\sin 2\sqrt{-\gamma}$

$\sqrt{-\gamma}=\frac{n\pi}{2}$

$X=\sum_{n=1}^{\infty}A_n\cos\frac{n\pi x}{2}$ $\quad \gamma=-\left(\frac{n\pi}{2}\right)^2$

Name ______________________________ Section __________

6. (30 pts) Use separation of variables to separate $\frac{\partial^2 u}{\partial x^2} + D\frac{\partial u}{\partial x} = \frac{\partial u}{\partial t}$ into two ODEs. Do NOT attempt any additional work.

$$\frac{X''T + \cancel{D}X'T}{\cancel{D}XT} = \frac{\cancel{X}T'}{D\cancel{X}T}$$

$$\frac{X'' + X'}{X} = +\lambda \qquad \frac{T'}{DT} = (+)\lambda$$

$$T' + \lambda DT = 0$$

$$X'' + X' - \lambda X = 0$$

Name ______________________________ Section __________

7. (30 points) Construct a boundary value problem (BVP) based on each scenario presented below. You must use one of the three classical PDEs studies in class to model each physical situation. Only set up the BVPs; do not attempt to solve.

(a) A slender rod of length 12, thermal diffusivity of 8, and with an insulated right end and a temperature of 50 at the left end. At time, 0, the system is given an input across the length of the rod or $f(x) = x^2 + 4$

$8u_{xx} = u_t$

B.C. $u(0,t) = 50$

$u_x(12,t) = 0$

I.C. $u(x,0) = x^2 + 4$

(b) A taut string of length π is tied to the x-axis at both ends. The magnitude of the tension divided by the string's linear density is 4. The string is put into motion from rest by the function $f(x) = \sin 5x$

$u(x,0) = 0$

$u_t(x,0) = \sin 5x$ $\quad$ $u_{tt} = 4u_{xx}$

$u(0,t) = 0$

$u(\pi,t) = 0$

(c) A thin rectangular plate of width 2 and height 3 has achieved temperature equilibrium. Three of the plate's edges are insulted, while the right edge temperature is given by the function $g(y) = 2 - (y + 1)$

$u_x(0,y) = 0$

$u_y(x,0) = 0$

$u_y(x,3) = 0$

$u(2,y) = 2 - (y+1)$

TAB G TERM END EXAMINATION

Term End Examinations are kept in a secure location and can be accessed by request to the Mathematical Sciences ABET Coordinator, COL Paul Goethals

TAB H FINAL COURSE GRADES

Department of Mathematical Sciences - MA365

Report Generated Tuesday, 04 Feb 2020 at 1359

Name	Points Earned	Percent	Letter Grade	OM	TEE 600
[redacted]		0.00%			
[redacted]	1941.67	97.08%	A+	1	570.00
[redacted]	1916.00	95.80%	A	2	559.00
[redacted]	1913.00	95.65%	A	3	563.00
[redacted]	1912.00	95.60%	A	4	562.00
[redacted]	1909.00	95.45%	A	5	580.00
[redacted]	1903.33	95.17%	A	6	567.00
[redacted]	1903.00	95.15%	A	7	549.00
[redacted]	1903.00	95.15%	A	7	551.00
[redacted]	1891.00	94.55%	A	9	532.00
[redacted]	1888.67	94.43%	A	10	574.00
[redacted]	1886.00	94.30%	A	11	536.00
[redacted]	1884.00	94.20%	A	12	551.00
[redacted]	1879.30	93.97%	A	13	565.00
[redacted]	1866.67	93.33%	A	14	559.00
[redacted]	1864.00	93.20%	A	15	566.00
[redacted]	1862.50	93.13%	A	16	539.00
[redacted]	1862.33	93.12%	A	17	534.00
[redacted]	1862.00	93.10%	A	18	546.00
[redacted]	1861.67	93.08%	A	19	553.00
[redacted]	1861.00	93.05%	A	20	553.00
[redacted]	1860.00	93.00%	A	21	554.00
[redacted]	1860.00	93.00%	A	21	536.00
[redacted]	1860.00	93.00%	A	21	546.00
[redacted]	1841.67	92.08%	A-	24	549.00
[redacted]	1839.00	91.95%	A-	25	539.00
[redacted]	1839.00	91.95%	A-	25	522.00
[redacted]	1835.34	91.77%	A-	27	537.00
[redacted]	1828.00	91.40%	A-	28	552.00

[redacted]

[redacted]	1824.67	91.23%	A-	29 531.00
[redacted]	1821.00	91.05%	A-	30 545.00
[redacted]	1820.33	91.02%	A-	31 543.00
[redacted]	1820.00	91.00%	A-	32 556.00
[redacted]	1819.67	90.98%	A-	33 521.00
[redacted]	1819.00	90.95%	A-	34 526.00

Report Generated Tuesday, 04 Feb 2020 at 1359

Name	Points Earned	Percent	Letter Grade	OM	TEE 600
[redacted]	1819.00	90.95%	A-	34	550.00
	1817.67	90.88%	A-	36	531.00
	1808.00	90.40%	A-	37	537.00
	1807.00	90.35%	A-	38	534.00
	1804.33	90.22%	A-	39	487.00
	1802.00	90.10%	A-	40	529.00
	1800.34	90.02%	A-	41	533.00
	1782.00	89.10%	B+	42	560.00
	1781.67	89.08%	B+	43	525.00
	1781.00	89.05%	B+	44	550.00
	1780.67	89.03%	B+	45	513.00
	1780.00	89.00%	B+	46	541.00
	1773.34	88.67%	B+	47	520.00
	1763.00	88.15%	B+	48	554.00
	1751.00	87.55%	B+	49	538.00
	1740.00	87.00%	B+	50	569.00
	1735.30	86.77%	B	51	488.00
	1733.67	86.68%	B	52	544.00
	1725.00	86.25%	B	53	515.00
	1720.00	86.00%	B	54	506.00
	1717.00	85.85%	B	55	533.00
	1713.33	85.67%	B	56	563.00
	1710.00	85.50%	B	57	552.00
	1705.66	85.28%	B	58	496.00
	1694.00	84.70%	B	59	545.00
	1692.00	84.60%	B	60	491.00
	1672.67	83.63%	B	61	525.00
	1663.33	83.17%	B	62	520.00
	1662.67	83.13%	B	63	499.00
	1660.00	83.00%	B	64	500.00
	1660.00	83.00%	B	64	507.00

[redacted]	1660.00	83.00%	B	64 540.00
	1600.00	80.00%	B-	67 512.00
	1587.50	79.38%	C+	68 425.00
	1579.00	78.95%	C+	69 467.00

Report Generated Tuesday, 04 Feb 2020 at 1359

Name	Points Earned	Percent	Letter Grade	OM	TEE 600
[redacted]	1434.84	71.74%	C-	70	455.00

Department of Mathematical Sciences

Report Generated Tuesday, 04 Feb 2020 at 1359 hours

AY-T: 2020-1

Course Grading Scale			Cadets
A+	>=	97.00	1
A	>=	93.00	22
A-	>=	90.00	18
B+	>=	87.00	9
B	>=	83.00	16
B-	>=	80.00	1
C+	>=	77.00	2
C	>=	73.00	0
C-	>=	70.00	1
D	>=	65.00	0
F	>=	0.00	0
NC			1

	Points:	Percent:
Highest:	1941.67	97.08%
Average:	1791.07	89.55%
Lowest:	1434.84	71.74%
Standard Deviation:		4.84
Average Letter Grade:		B+
Section GPA:		3.51
Total Number of Cadets:		71

SECTION III

ASSIGNMENTS

TAB I PROBLEM SETS

PROBLEM SET #1

MA365 – Advanced Mathematics for Engineers and Scientists
AY19-1 Problem Set 1

Due Lesson 7 prior to your class hour. **100 Points**

Instructions: This assignment will be submitted in class on the due date provided, with cover sheet. This is an *individual* assignment; however, if you receive assistance, please document any assistance received in accordance with Documentation of Academic Work guidelines. Submit all supporting calculations and analysis. Ensure all work is logical, neat, and organized.

---

1. **(10 points)** If a complex function $f(z) = u(x, y) + iv(x, y)$ is analytic, then the functions $u(x, y)$ and $v(x, y)$ that form its real and imaginary parts are said to be *harmonic.* An interesting and useful property of the *harmonic* functions is that level curves $u(x, y) = c_1$ and $v(x, y) = c_2$ are orthogonal. Show that the level curves $u(x, y) = c_1$ and $v(x, y) = c_2$ are orthogonal. [Hint: consider the gradient of u and the gradient of v. Ignore the case where a gradient vector is the zero vector.]

2. **(15 points)**In the next block of MA365 we will study some of the classic partial differential equations (PDEs), to include Laplace's equation

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0. \tag{1}$$

   Laplace's equation is used to model potentials such as electrostatic, gravitational and velocity in fluid mechanics. A property of Laplace's equation is that all complex analytic functions $f(z) = u(x, y) + iv(x, y)$ are solutions.

   a. Verify that the complex function $f_1(z) = iz$ is analytic.

   b. Following Example 2 on p. 831 of your text, find the streamlines, $(x(t),\ y(t))$, of the flow associated with the complex function $f_1(z) = iz$.

3. **(25 points)** Consider a thin square metallic plate with edges of length 3 meters. The edges of the plate have the boundary conditions $20°\ C$, $20°\ C$, $30°\ C$ and $25°\ C$, for edges $1 - 4$ respectively, numbered in a clock-wise direction. Engineers are interested in knowing the temperature distribution inside the plate during a specific time period in order to determine the thermal stress to which the plate is subjected. Assuming the boundary conditions are held constant during the specified time period, the temperature inside the plate will reach a specific equilibrium temperature distribution after some time has passed. Assuming the plate has reached steady-state, consider four equally spaced points on the plate. Approximate the temperature at these points using a practical application of the Mean-Value Property:

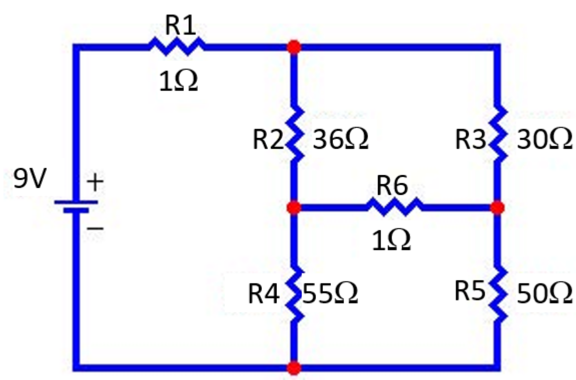
   If a plate has reached a thermal equilibrium, and P is a grid point not on the boundary of the plate, then the temperature at P is the average of the temperature of the four closest grid points to P.

   Using the Mean-Value Property, establish a linear system of four equations relating the temperature of the four nodes $x_1,\ x_2,\ x_3$, and x_4 and the boundary conditions. Create and solve a matrix-vector system $\mathbf{AX}$=$\mathbf{B}$ for the vector of equilibrium temperatures, $\mathbf{X} = (x_1\ x_2\ x_3\ x_4)^T$ using either Gaussian elimination or Gauss-Jordan elimination. You may NOT use technology. SHOW ALL WORK!

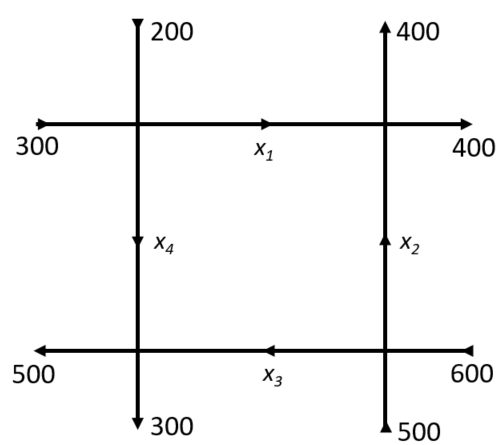
MA365 – Advanced Mathematics for Engineers and Scientists
AY19-1 Problem Set 1

Due Lesson 7 prior to your class hour. **100 Points**

4. **(25 points)** Find the current in each branch of this electrical circuit. You may use technology. Next, determine what the current is in the branch containing R6 if the resistance of R4 is changed to 60Ω. **BONUS (2pt)**: What type of circuit is this commonly known as?



5. **(25 points)** Methods of electrical circuit analysis have applications to other fields. For instance, applying the analogue of Kirchhoff's point rule, find the traffic flow (cars per hour) in the net of one-way streets (in the directions indicated by the arrows) shown in the figure. You should do this problem without the aid of technology. Is the solution unique? How can you tell?



Instructor Solution

1) Since $f(z)$ is analytic, it satisfies Cauchy-Riemann Equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = \frac{-\partial v}{\partial x}$$

Therefore

$$\nabla u = \left\langle \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y} \right\rangle = \left\langle \frac{\partial v}{\partial y} , \frac{\partial u}{\partial y} \right\rangle \quad \text{and}$$

$$\nabla v = \left\langle \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y} \right\rangle = \left\langle \frac{-\partial u}{\partial y}, \frac{\partial v}{\partial y} \right\rangle$$

$\Rightarrow \nabla u \cdot \nabla v = 0$ Therefore ∇u and ∇v are orthogonal for all (x, y)

If this is true and gradient vectors are orthogonal to their respective level curves (contour lines) for all (x, y) by definition, it must be true that level curves of $u(x, y)$ and $v(x, y)$ are orthogonal to one another for any (x, y)

2) a) $f(z) = iz = i(x+iy) = ix - y$

Therefore $u(x,y) = -y$ and $v(x,y) = x$

$$\frac{\partial u}{\partial x} = 0 = \frac{\partial v}{\partial y} \qquad \frac{\partial u}{\partial y} = -1 = -\frac{\partial v}{\partial x}$$

Satisfies Cauchy-Riemann Equations for all (x,y)

Thus $f(z)$ is everywhere analytic

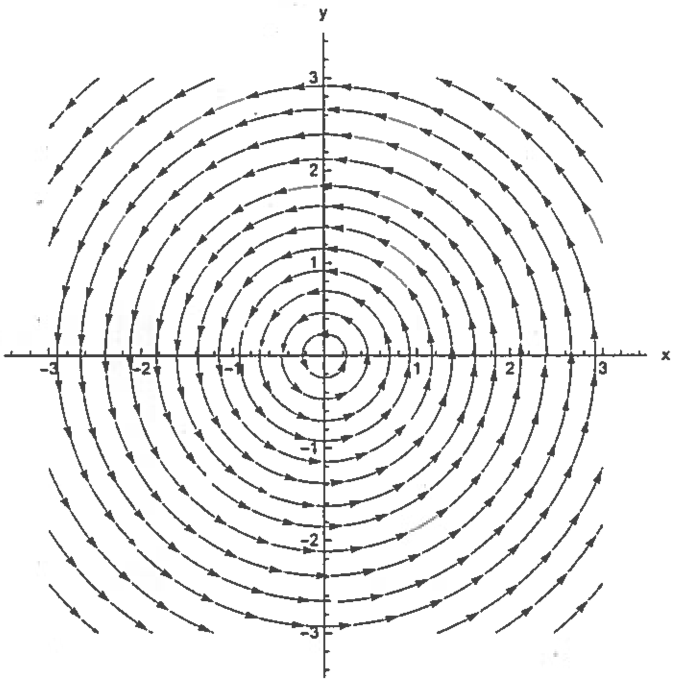
b.) $\frac{dx}{dt} = u(x,y) = -y \quad \frac{dy}{dt} = v(x,y) = x$

MA365 Problem Set 1 Solutions

Problem 2

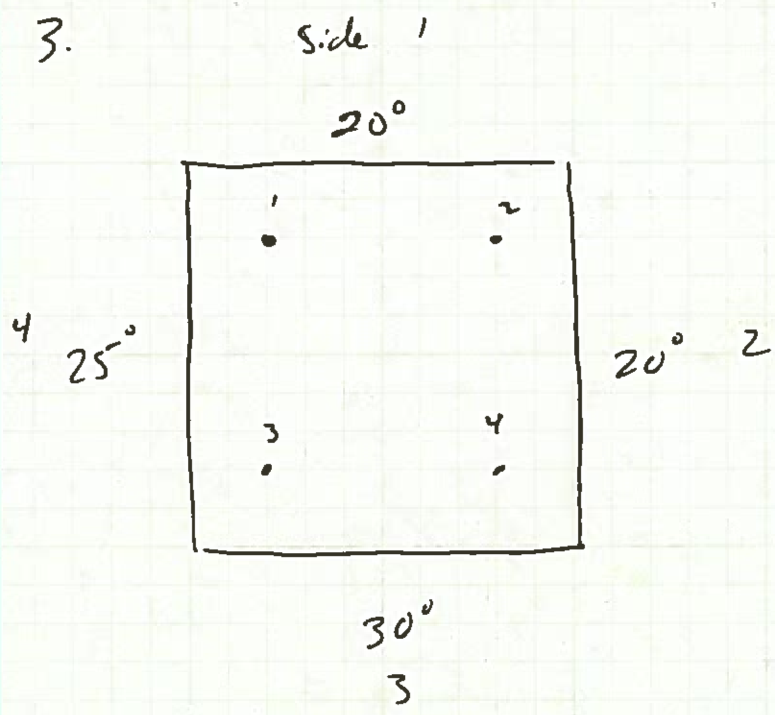
In[4]:=
```
StreamPlot[{-y, x}, {x, -3, 3}, {y, -3, 3}, Frame → False,
  Axes → True, AxesLabel → {"x", "y"}, StreamStyle → Blue]
```

Out[4]=



These show the solution trajectories for the system of differential equations, or streamlines of the complex function.

3.



$$P_1 = \frac{20+25+P_2+P_3}{4} \Rightarrow 45 + P_2 + P_3 = 4P_1 \Rightarrow 45 = 4P_1 - P_2 - P_3$$

$$P_2 = \frac{20+20+P_1+P_4}{4} = 40 + P_1 + P_4 = 4P_2 \Rightarrow -P_1 - P_4 + 4P_2 = 40$$

$$P_3 = \frac{30+25+P_1+P_4}{4} \Rightarrow 55 = 4P_3 - P_1 - P_4$$

$$P_4 = \frac{20 + \overset{30}{\cancel{25}} + P_2 + P_3}{4} \Rightarrow 50 = 4P_4 - P_2 - P_3$$

$$A\vec{x} = \vec{v}$$

$$\begin{bmatrix} 4 & -1 & -1 & 0 \\ -1 & 4 & 0 & -1 \\ -1 & 0 & 4 & -1 \\ 0 & -1 & -1 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 45 \\ 40 \\ 55 \\ 50 \end{bmatrix}$$

3 cont.

$$\left[\begin{array}{cccc|c} 4 & -1 & 1 & 0 & 45 \\ -1 & 4 & 0 & -1 & 40 \\ -1 & 0 & 4 & -1 & 55 \\ 0 & -1 & -1 & 4 & 50 \end{array}\right]$$

$R_2 - R_3 \Rightarrow R_2$

$R_1 + 4R_3 = R_3$

$$\left[\begin{array}{cccc|c} 4 & -1 & -1 & 0 & 45 \\ 0 & 4 & -4 & 0 & -15 \\ 0 & -1 & 15 & -4 & 265 \\ 0 & -1 & -1 & 4 & 50 \end{array}\right]$$

$R_3 - R_4 \Rightarrow R_3$

$R_2 + 4R_4 \Rightarrow R_4$

$$\left[\begin{array}{cccc|c} 4 & -1 & -1 & 0 & 45 \\ 0 & 4 & -4 & 0 & -15 \\ 0 & 0 & 16 & -8 & 215 \\ 0 & 0 & -8 & 16 & 185 \end{array}\right]$$

$2R_4 + R_3 \Rightarrow R_4$.

$$\left[\begin{array}{cccc|c} 4 & -1 & -1 & 0 & 45 \\ 0 & 4 & -4 & 0 & -15 \\ 0 & 0 & 16 & -8 & 215 \\ 0 & 0 & 0 & 24 & 585 \end{array}\right]$$

$R_4/24 \Rightarrow R_4$

$R_3/16 \Rightarrow R_3$

$R_2/4 \Rightarrow R_2$

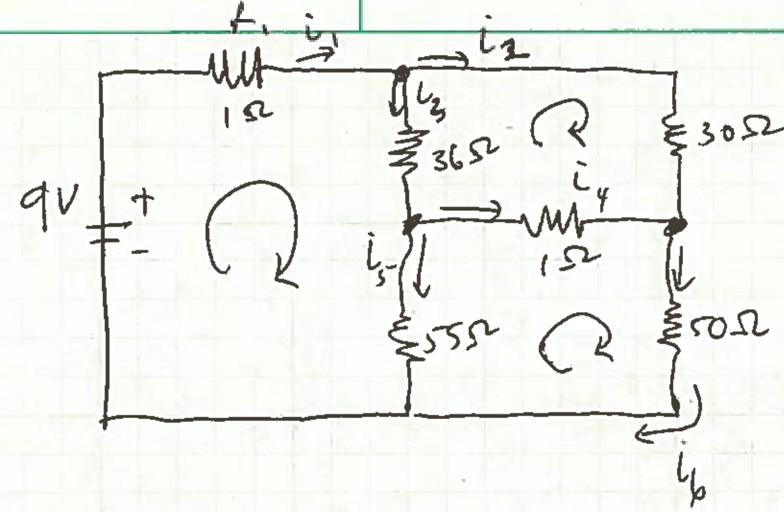
$R_1/4 \Rightarrow R_1$

$$\left[\begin{array}{cccc|c} 1 & -1/4 & -1/4 & 0 & 45 \\ 0 & 1 & -1 & 0 & -15 \\ 0 & 0 & 1 & -1/2 & 215/16 \\ 0 & 0 & 0 & 1 & 195/8 \end{array}\right]$$

Back calculating will thus yield

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 23.125 \\ 21.875 \\ 25.625 \\ 24.375 \end{bmatrix}$$

4.)



applying pt. Rule

$$i_1 - i_2 - i_3 = 0$$

$$i_3 - i_4 - i_5 = 0$$

$$i_2 + i_4 - i_6 = 0$$

applying Loop Rule

L_1: $9 - i_1 - 36 i_3 - 55 i_5 = 0$

$L2$: $36 i_3 - 30 i_2 + i_4 = 0$

$L3$: $55 i_5 - i_4 - 50 i_6 = 0$

$$\begin{bmatrix} 1 & -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ 1 & 0 & 25 & 0 & 55 & 0 \\ 0 & -30 & 36 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 55 & -50 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ i_4 \\ i_5 \\ i_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 9 \\ 0 \\ 0 \end{bmatrix}$$

Solve using technology

$i_1 = (63909/273751) = .23$ A

$i_2 = (34839/273751) = .127$ A

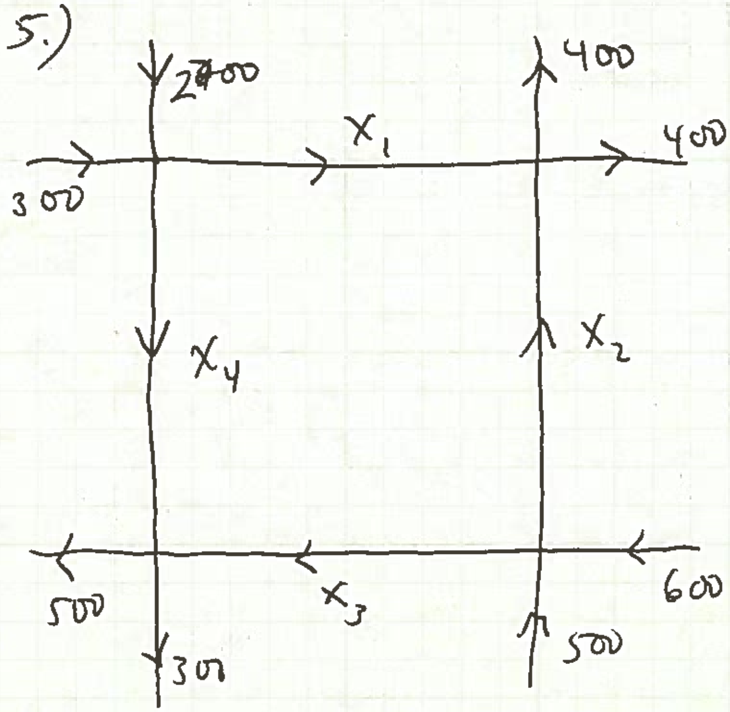
$i_3 = (1710/16103) = .106$ A

$i_4 = (-1350/273751) = -.005$ A

$i_5 = (30420/273751) = .11$ A

$i_6 = (33489/273751) = .122$ A

5.)



a.) $300 + 200 - X_1 - X_4 = 0$

b.) $X_1 + X_2 - 400 - 400 = 0$

c.) $X_3 + X_4 - 300 - 500 = 0$

d.) $600 + 500 - X_2 - X_3 = 0$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix} = \begin{bmatrix} 500 \\ 800 \\ 800 \\ 1100 \end{bmatrix}$$

$$-R_1 + R_2 \Rightarrow R_2 \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 500 \\ 0 & 1 & 0 & -1 & 300 \\ 0 & 0 & 1 & 1 & 800 \\ 0 & 1 & 1 & 0 & 1100 \end{array}\right] \Rightarrow -R_2 + R_4 \Rightarrow R4 \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 500 \\ 0 & 1 & 0 & -1 & 300 \\ 0 & 0 & 1 & 1 & 800 \\ 0 & 0 & 1 & 1 & 800 \end{array}\right]$$

$$-R_3 + R_4 \Rightarrow R_4 \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 500 \\ 0 & 1 & 0 & -1 & 300 \\ 0 & 0 & 1 & 1 & 800 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

System is consistent because rank $(A) =$ rank $(A|B) = 3$ but $3 < n = 4$, Therefore system has infinitely many solutions

Solution is <u>not unique</u>

PROBLEM SET #2

MA365 – Advanced Mathematics for Engineers and Scientists
Problem Set 2: Linear Algebra II

Due Lesson 10 prior to your class hour. **100 Points**

<u>Instructions</u>: This assignment will be submitted in class on the due date provided, with cover sheet. This is an *individual* assignment; if you receive assistance, please document any assistance received in accordance with Documentation of Academic Work guidelines. Submit all supporting calculations and analysis. Ensure all work is logical, neat, and organized.

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1. **(20 points)** In PS1, you considered the temperature of a thin metallic 3 m × 3 m plate. You should have found estimates for the temperature at 4 interior points of this plate using an application of the Mean-Value Property. These points were equally spaced, so that they were all 1 m from the nearest edge of the plate and 1 m from any adjacent points. Suppose that the engineers studying this plate are not satisfied with the precision from this 1 m mesh. They suggest using a 0.5 m mesh instead. They ask for estimates of the temperature at all of the interior points (the nodes of the mesh not on the boundaries of the plate). The boundary conditions remain the same as they were presented in PS1.

   a. How many interior points are there with a 0.5 m mesh?

   b. Create and solve a matrix-vector system **AX**=**B** for the vector of equilibrium temperatures, **X**, just as you did in PS1, but now using the finer mesh. You may use technology.

   c. Compare the four temperatures calculated in PS1 to the temperatures calculated at the corresponding points on the finer mesh and comment on the estimates' sensitivity to using a finer mesh.

2. **(10 points)** Consider the set of column vectors $\mathbf{v}_1 = (4\ 3\ 2\ 1)^T$, $\mathbf{v}_2 = (1\ 2\ 2\ 1)^T$, $\mathbf{v}_3 = (-1\ 1\ 1\ 1)^T$, $\mathbf{v}_4 = (2\ 3\ 4\ 1)^T$ and $\mathbf{v}_5 = (1\ 7\ -5\ 1)^T$. Recall that by definition the vectors are linearly independent if the only coefficients that satisfy

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 + c_4\mathbf{v}_4 + c_5\mathbf{v}_5 = \mathbf{0} \text{ are } c_1 = 0, c_2 = 0, c_3 = 0, c_4 = 0, c_5 = 0.$$

otherwise the set is linearly dependent. Note that the above relationship is the same as the linear system

$$4c_1 + c_2 - c_3 + 2c_4 + c_5 = 0$$
$$3c_1 + 2c_2 + c_3 + 3c_4 + 7c_5 = 0$$
$$2c_1 + 2c_2 + c_3 + 4c_4 - 5c_5 = 0$$
$$c_1 + c_2 + c_3 + c_4 + c_5 = 0.$$

Without doing any further work, explain why we can now conclude that the set of vectors is linearly dependent. In your discussion, ensure to reference proper theorems and terminology from your text.

MA365 – Advanced Mathematics for Engineers and Scientists
Problem Set 2: Linear Algebra II

Due Lesson 10 prior to your class hour. **100 Points**

3. **(20 points)** Suppose your friend bought four bottles of fancy olive oils with the following nutritional facts per serving:

	Fiber	Fructose	Fat	Folic Acid
Alpha oil	1 g	30 g	6 g	2 mg
Beta oil	20 g	60 g	10 g	80 mg
Gamma oil	12 g	6 g	42 g	30 mg
Delta oil	9 g	157 g	38 g	23 mg

She made her purchase after being told in a very convincing seminar that she would receive the proper balance of the four nutrients–fiber, fructose, fat, and folic acid–by using these four bottles in her diet. The bottles cost $29.95 apiece. Are all of the bottles necessary?

4. **(20 points)** For the following coefficient matrices, determine if **AX=0** has only the trivial solution (i.e determine if the following matrices are nonsingular). For each nonsingular matrix, determine the inverse using first Theorem 8.6.2 and second Theorem 8.6.4. SHOW ALL WORK!

a. $\mathbf{A}_a = \begin{pmatrix} -2 & 0 & -1 \\ 6 & 4 & 0 \\ 4 & 4 & -1 \end{pmatrix}$ b. $\mathbf{A}_b = \begin{pmatrix} -3 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 \\ 0 & 0 & -7 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$.

5. **(30 points)** Recall that a simple spring-mass system with damping can be written as

$$mx'' + \beta x' + qx = 0$$

where $x(t)$ is the displacement of the mass, m, at time, t; β is the damping constant; and q is the spring constant. (See Section 3.8 in your text for spring-mass system details.)

a. Show that the system can also be written in the form $\mathbf{X}' = \mathbf{AX}$, where

$$\mathbf{A} = \begin{pmatrix} 0 & 1 \\ -\frac{q}{m} & -\frac{\beta}{m} \end{pmatrix} \quad \text{and} \quad \mathbf{X} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

b. Show that $(\mathbf{A} - \lambda\mathbf{I})\mathbf{K} = \mathbf{0}$ if we assume a solution of the form $\mathbf{X} = \mathbf{K}e^{\lambda t}$.

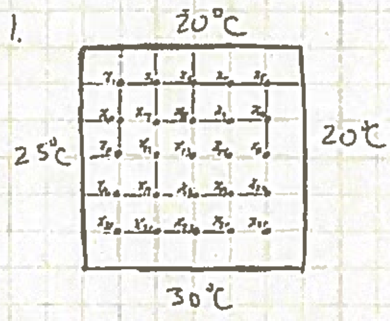
c. Find the eigenvalues, λ, and associated eigenvectors, **K**, of the *overdamped* system with $m = 1$ kg, $\beta = 20\ \frac{\text{kg}}{\text{s}}$ and $q = 36\ \frac{\text{N}}{\text{m}}$.

d. Determine the general solution of the form $\mathbf{X} = C_1\mathbf{K}_1e^{\lambda_1 t} + C_2\mathbf{K}_2e^{\lambda_2 t}$. (See Section 10.2 in your text for the theory of homogenous linear systems.)

e. Use the method of diagonalization to solve the matrix-vector system. Show that your solution is the same as that determined in Problem 5.d.

Instructor Solution

1.



$$x_1 = \frac{x_2 + x_6 + 20 + 25}{4}$$

$$x_2 = \frac{x_1 + x_3 + x_7 + 20}{4}$$

$$x_3 = \frac{x_2 + x_4 + x_8 + 20}{4}$$

$$x_4 = \frac{x_3 + x_5 + x_9 + 20}{4}$$

$$x_5 = \frac{x_4 + x_{10} + 20 + 20}{4}$$

$$x_6 = \frac{x_1 + x_7 + x_{11} + 25}{4}$$

$$x_7 = \frac{x_2 + x_6 + x_8 + x_{12}}{4}$$

$$x_8 = \frac{x_3 + x_7 + x_9 + x_{13}}{4}$$

$$x_9 = \frac{x_4 + x_8 + x_{10} + x_{14}}{4}$$

$$x_{10} = \frac{x_5 + x_9 + x_{15} + 20}{4}$$

$$x_{11} = \frac{x_6 + x_{12} + x_{16} + 25}{4}$$

$$x_{12} = \frac{x_7 + x_{11} + x_{13} + x_{17}}{4}$$

$$x_{13} = \frac{x_8 + x_{12} + x_{14} + x_{18}}{4}$$

$$x_{14} = \frac{x_9 + x_{13} + x_{15} + x_{19}}{4}$$

$$x_{15} = \frac{x_{10} + x_{14} + x_{20} + 20}{4}$$

$$x_{16} = \frac{x_{11} + x_{17} + x_{21} + 25}{4}$$

$$x_{17} = \frac{x_{12} + x_{16} + x_{18} + x_{22}}{4}$$

$$x_{18} = \frac{x_{13} + x_{17} + x_{19} + x_{23}}{4}$$

$$x_{19} = \frac{x_{14} + x_{18} + x_{20} + x_{24}}{4}$$

$$x_{20} = \frac{x_{15} + x_{19} + x_{25} + 20}{4}$$

$$x_{21} = \frac{x_{16} + x_{22} + 25 + 30}{4}$$

$$x_{22} = \frac{x_{17} + x_{21} + x_{23} + 30}{4}$$

$$x_{23} = \frac{x_{18} + x_{22} + x_{24} + 30}{4}$$

$$x_{24} = \frac{x_{19} + x_{23} + x_{25} + 30}{4}$$

$$x_{25} = \frac{x_{20} + x_{24} + 20 + 30}{4}$$

$$\begin{pmatrix}
4 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 4 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 4 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 4 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 4 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 4 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & -1 & 4 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & -1 & 4 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 4 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 4 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 4 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 4 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 4 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 4 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 4 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 4 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 4 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 4 & -1 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 4 & -1 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 4 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 4 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 4 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 4 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 4 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 4
\end{pmatrix}
\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \\ x_9 \\ x_{10} \\ x_{11} \\ x_{12} \\ x_{13} \\ x_{14} \\ x_{15} \\ x_{16} \\ x_{17} \\ x_{18} \\ x_{19} \\ x_{20} \\ x_{21} \\ x_{22} \\ x_{23} \\ x_{24} \\ x_{25} \end{pmatrix}
=
\begin{pmatrix} 45 \\ 20 \\ 20 \\ 20 \\ 40 \\ 25 \\ 0 \\ 0 \\ 0 \\ 20 \\ 25 \\ 0 \\ 0 \\ 0 \\ 20 \\ 25 \\ 0 \\ 0 \\ 0 \\ 20 \\ 55 \\ 30 \\ 30 \\ 30 \\ 50 \end{pmatrix}$$

a. 25 interior points

b. See attached MMA printout

c. x_7 corresponds to x_1 from PS1; x_9 to x_2; x_{17} to x_4; x_{19} to x_3

PS1		PS2	
x_1	23.125°C	x_7	23.106°C
x_2	21.875°C	x_9	21.818°C
x_3	24.375°C	x_{19}	24.394°C
x_4	25.625°C	x_{17}	25.682°C

much less than 1% change in values using finer mesh although cooler temps are cooler and hotter temps are hotter

2) This is an undetermined homogeneous system, so it must have nontrivial solutions. Non-trivial solutions give non-zero values for $c_1, c_2, \ldots c_5$. Therefore vectors are linearly dependent

OR

The maximum rank that the co-efficients matrix has in this system is 4. which is the maximum number of linearly independent column vectors. As there are 5 column vectors and therefore cannot be linearly independent

4a. 0 : Singular

b. $\begin{pmatrix} -\frac{1}{3} & & & \\ & -\frac{1}{6} & & \\ & & -\frac{1}{7} & \\ & & & -1 \end{pmatrix}$

3 convert mg to g to get $\left[\begin{array}{cccc|c} 1 & 20 & 12 & 9 & 0 \\ 30 & 60 & 6 & 57 & 0 \\ 1 & 10 & 42 & 38 & 0 \\ 2 & 80 & 30 & 23 & 0 \end{array}\right]$

→ Row Reduce to Row echlon form, I used Mathematica for this problem to get:

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 5 & \cancel{8} \\ 0 & 1 & 0 & 1/10 & \\ 0 & 0 & 1 & 1/5 & \\ 0 & 0 & 0 & 0 & \end{array}\right]$$

Theorem 8.3.1: rank(A) of # of nonzero rows is 3 Therefore it is not full rank and Thus does not have linear independence. With The last row of all 0's, bottle D is not nescessary

4 a.) Singular

b.) $A_b = \begin{vmatrix} -3 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 \\ 0 & 0 & -7 & 0 \\ 0 & 0 & 0 & -1 \end{vmatrix}$

calculate determinant to be -126 Therefor it is Not Singular

$\operatorname{adj} A = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} \\ C_{21} & C_{22} & C_{23} & C_{24} \\ C_{31} & C_{32} & C_{33} & C_{34} \\ C_{41} & C_{42} & C_{43} & C_{44} \end{bmatrix}$

$C_{11} = 42$ $C_{21} = 0$ $C_{31} = 0$

$C_{12} = 0$ $C_{22} = -21$ $C_{32} = 0$

$C_{13} = 0$ $C_{23} = 0$ $C_{33} = 18$

$C_{14} = 0$ $C_{24} = 0$ $C_{34} = 0$

- remember sign patterns

$+ \quad - \quad + \quad -$

$+ \quad - \quad + \quad -$

and so on

$C_{41} = 0$

$C_{42} = 0$

$C_{43} = 0$

$C_{44} = 126$

calculate matrices of minor

Then assemble adj matrix

$\operatorname{adj} A_b = \begin{bmatrix} 42 & 0 & 0 & 0 \\ 0 & -21 & 0 & 0 \\ 0 & 0 & 18 & 0 \\ 0 & 0 & 0 & 126 \end{bmatrix}$

$Ad^{-1} = \frac{-1}{126} \begin{vmatrix} 42 & 0 & 0 & 0 \\ 0 & -21 & 0 & 0 \\ 0 & 0 & 18 & 0 \\ 0 & 0 & 0 & 126 \end{vmatrix} = \begin{vmatrix} -1/3 & 0 & 0 & 0 \\ 0 & 1/6 & 0 & 0 \\ 0 & 0 & -1/7 & 0 \\ 0 & 0 & 0 & -1 \end{vmatrix}$

use reduced row method to find Ad^{-1}

$\left| \begin{matrix} -3 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 \\ 0 & 0 & -7 & 0 \\ 0 & 0 & 0 & -1 \end{matrix} \right| \begin{matrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{matrix} \left| \begin{matrix} -\frac{1}{3} R_1 \\ \frac{1}{6} R_2 \\ -\frac{1}{7} R_3 \\ -R_4 \end{matrix} \right. \Rightarrow \left| \begin{matrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{matrix} \right| \begin{matrix} -1/3 & 0 & 0 & 0 \\ 0 & 1/6 & 0 & 0 \\ 0 & 0 & -1/7 & 0 \\ 0 & 0 & 0 & -1 \end{matrix} \right|$

5.) $mx'' + \beta x' + qx = 0$

a.) $X' = AX$ where $A = \begin{bmatrix} 0 & 1 \\ -\frac{q}{m} & -\frac{\beta}{m} \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$

expanding

This we get $X_1' = X_2$ and

$X_2' = -\frac{q}{m} X_1 - \frac{\beta}{m} X_2$

as if $x_1' = x_2$

Then $x_2' = x_1''$

So $x'' = -\frac{q}{m} x_1 - \frac{\beta}{m} x_1'$ rearrange to get

$x_1'' + \frac{\beta}{m} x_1' + \frac{q}{m} x_1 = 0$

multiply through by m $\Rightarrow$ $mx'' + \beta x_1' + qx$

15.) $(A - \lambda I)K = 0$ can be written $AK = \lambda K$

$X = Ke^{\lambda t}$ & $X' = \lambda K e^{\lambda t}$ sub into $X' = AX$

$\lambda K e^{\lambda t} = AK \cdot e^{\lambda t} \Rightarrow \lambda K = AK$ which is the same as $(A - \lambda I)K = 0$

c.) $A = \begin{bmatrix} 0 & 1 \\ -36 & -20 \end{bmatrix}$ $\begin{vmatrix} -\lambda & 1 \\ -36 & -20-\lambda \end{vmatrix}$

$-\lambda(-20-\lambda) - -36 = \lambda^2 + 20\lambda + 36$

$(\lambda+2)(\lambda+18)$

$\lambda = -2, -18$

$\lambda = -2$

$\begin{bmatrix} --2 & 1 \\ -36 & -20--2 \end{bmatrix} = \left[\begin{array}{cc|c} 2 & 1 & 0 \\ -36 & -18 & 0 \end{array}\right] = \quad -2x_1 = x_2 \quad \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

$\lambda = -18$

$\left[\begin{array}{cc|c} 18 & 1 & 0 \\ -36 & -2 & 0 \end{array}\right] = -18x_1 = x_2 \Rightarrow \begin{bmatrix} 1 \\ -18 \end{bmatrix}$

d.) according to Sec 10.2 when we have distinct real eigenvalues solution would be

$x = c_1 \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-2t} + c_2 \begin{pmatrix} 1 \\ -18 \end{pmatrix} e^{-18t}$

e) $D = \begin{pmatrix} -2 & 0 \\ 0 & -18 \end{pmatrix}$ $P = \begin{pmatrix} 1 & 1 \\ -2 & -18 \end{pmatrix}$

$P^{-1} = \frac{1}{\det} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{16} \begin{bmatrix} -18 & -1 \\ 2 & 1 \end{bmatrix} =$

$A = P^{-1} D P$

UNITED STATES MILITARY ACADEMY
WEST POINT.

PROBLEM SET #3

MA365 – Advanced Mathematics for Engineers and Scientists
Problem Set 3: Orthogonal Functions and Fourier Series

100 Points

Instructions: This assignment will be submitted in class on the due date provided, with cover sheet. This is an *individual* assignment; if you receive assistance, please document any assistance received in accordance with Documentation of Academic Work guidelines. Submit all supporting calculations and analysis. Ensure all work is logical, neat, and organized.

1. **(20 points)** Show that the given set of functions is orthogonal on the indicated interval. Find the norm of each function in the set. $\{1, \cos\frac{n\pi}{p}x\}, n = 1, 2, 3, \ldots; [0, p]$

2. **(10 points)** Find the Fourier series of $f(x) = x + \pi,\ -\pi < x < \pi$

   BONUS (10 pts) Use the result from above to show $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots$

3. **(30 points)** Suppose a uniform beam of length L is simply supported at $x = 0$ and $x = L$. If the load per unit length is given by $w(x) = w_0x/L$, $0 < x < L$, then the differential equation for the deflection $y(x)$ is

$$EI\frac{d^4y}{dx^4} = \frac{w_0x}{L},$$

   where E, I, and w_0 are constants.

   a. Expand $w(x)$ in a half-range sine series.

   b. Use the method of Example 4 on pp 686-7 of your text to find a particular solution $y(x)$ of the differential equation.

4. **(40 points)** Consider the functions listed in parts a and b below. Let $e_m(x) = f(x) - s_m(x)$ where $s_m(x)$ is the Fourier partial sum. For each part, first plot $|e_m(x)|$ versus x for $0 \le x \le 4$ for several values of m. Next, for each part, find the smallest value of m for which $|e_m(x)| \le 0.01$ for all x in the interval.

   a. $f(x) = x^2/4$

   b. $f(x) = x^2(4 - x)$

Instructor Solution

	MA365	Problem Set 3 Solutions

1. $\left(1, \cos\frac{n\pi}{p}x\right) = \int_0^p \cos\frac{n\pi}{p}x\,dx = \frac{p}{n\pi}\sin\frac{n\pi}{p}x\Big|_0^p = \frac{p}{n\pi}\left(\cancelto{0}{\sin n\pi} - 0\right) = 0$

$$\left(\cos\frac{m\pi}{p}x, \cos\frac{n\pi}{p}x\right) = \int_0^p \cos\frac{m\pi}{p}x\cos\frac{n\pi}{p}x\,dx = \frac12\int_0^p \cos\left(\frac{m\pi}{p}x - \frac{n\pi}{p}x\right) + \cos\left(\frac{m\pi m\pi}{p}x\right)dx$$

$$= \frac12\left(\frac{p}{(m-n)\pi}\sin\left(\frac{(m-n)\pi}{p}x\right) + \frac{p}{(m+n)\pi}\sin\left(\frac{(m+n)\pi}{p}x\right)\right)\Bigg|_0^p = \frac12\left(\cancelto{0}{\frac{p}{(m-n)\pi}\sin(m-n)\pi} + \cancelto{0}{\frac{p}{(m+n)\pi}\sin(m+n)\pi} - 0\right)$$

$$= 0$$

since both inner products are 0, set is orthogonal

$$\|1\| = \sqrt{\int_0^p dx} = \sqrt{x\Big|_0^p} = \underline{\underline{\sqrt{p}}}$$

$$\left\|\cos\frac{n\pi}{p}x\right\| = \sqrt{\int_0^p \cos^2\frac{n\pi}{p}x\,dx} = \sqrt{\frac12\int_0^p \cos\frac{2n\pi}{p}x + 1\,dx} = \sqrt{\frac12\left(\frac{p}{2n\pi}\sin\frac{2n\pi}{p}x + x\right)\Bigg|_0^p}$$

$$= \sqrt{\frac12\left(\cancelto{0}{\frac{p}{2n\pi}\sin 2n\pi} + p - 0\right)} = \underline{\underline{\sqrt{\frac{p}{2}}}}$$

2. $f(x) = x + \pi, \quad -\pi < x < \pi$

$$a_0 = \frac1\pi\int_{-\pi}^{\pi} x + \pi\,dx = \frac1\pi\left(x^2 + \pi x\Big|_{-\pi}^{\pi}\right) = \frac1\pi\left(\pi^2 + \pi^2 - \cancelto{0}{(\pi^2 - \pi^2)}\right) = 2\pi$$

$$a_n = \frac1\pi\int_{-\pi}^{\pi}(x+\pi)\cos nx\,dx \xrightarrow{MMA} = 0$$

$$b_n = \frac1\pi\int_{-\pi}^{\pi}(x+\pi)\sin nx\,dx \xrightarrow{MMA} = \frac{-2(-1)^n}{n} = \frac{2(-1)^{n+1}}{n}$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos(nx) + b_n\sin(nx) \implies \underline{\underline{f(x) = \pi + \sum_{n=1}^{\infty}\frac{2(-1)^{n+1}}{n}\sin(nx)}}$$

BONUS

$$x + \cancel{\pi} = \cancel{\pi} + \sum_{n=1}^{\infty}\frac{2(-1)^{n+1}}{n}\sin(nx)$$

$$x = 2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin(nx)$$

$$\frac{x}{2} = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin(nx) = \frac11\sin x - \frac12\sin 2x + \frac13\sin 3x - \frac14\sin 4x + \frac15\sin 5x - \frac16\sin 6x \ldots$$

BONUS continued

$$\underbrace{\sin x}_{\text{needs to be }1} - \frac{1}{2}\underbrace{\sin 2x}_{\text{needs to be }0} + \frac{1}{3}\underbrace{\sin 3x}_{\text{needs to be }-1} - \frac{1}{4}\underbrace{\sin 4x}_{\text{needs to be }0} + \frac{1}{5}\underbrace{\sin 5x}_{\text{needs to be }1} - \frac{1}{6}\underbrace{\sin 6x}_{\text{needs to be }0} + \frac{1}{7}\sin 7x - \ldots$$

this pattern will emerge if $x = \frac{\pi}{2}$

if $x = \frac{\pi}{2}$ then we've shown $\frac{x}{2} = \sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \cdots$

will be $\frac{\frac{\pi}{2}}{2} = \frac{\pi}{4} = \sin(\frac{\pi}{2}) - \frac{1}{2}\sin 2(\frac{\pi}{2}) + \frac{1}{3}\sin(3\frac{\pi}{2}) - \frac{1}{4}\sin(4\frac{\pi}{2}) \cdots$

$$= 1 - \frac{1}{2}(0) + \frac{1}{3}(-1) - \frac{1}{4}(0) + \frac{1}{5}(1) - \frac{1}{6}(0) + \frac{1}{7}(-1) \cdots$$

$$\therefore \quad \underline{\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots}$$

3.

$$EI\frac{d^4y}{dx^4} = \frac{w_0 x}{L} \qquad w(x) = \frac{w_0 x}{L} \qquad p = L$$

a. $b_n = \frac{2}{L}\int_0^L \frac{w_0 x}{L}\sin\frac{n\pi x}{L}\,dx \xRightarrow{MMA} b_n = \frac{2w_0}{n\pi}(-1)^{n+1}$

$$w(x) = \sum_{n=1}^{\infty}\frac{2w_0}{n\pi}(-1)^{n+1}\sin\left(\frac{n\pi}{L}x\right)$$

b. assume $y_p = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi}{L}x$, then substitute into diff eq

$$y_p' = \frac{\pi}{L}\sum_{n=1}^{\infty} B_n n\cos\frac{n\pi}{L}x$$

$$y_p'' = -\frac{\pi^2}{L^2}\sum_{n=1}^{\infty} B_n n^2\sin\frac{n\pi}{L}x$$

$$y_p''' = -\frac{\pi^3}{L^3}\sum_{n=1}^{\infty} B_n n^3\cos\frac{n\pi}{L}x$$

$$y_p^{(4)} = \frac{\pi^4}{L^4}\sum_{n=1}^{\infty} B_n n^4\sin\frac{n\pi}{L}x$$

$$\Rightarrow \underbrace{EI\sum_{n=1}^{\infty} B_n\frac{n^4\pi^4}{L^4}\sin\frac{n\pi}{L}x = \sum_{n=1}^{\infty}\frac{2w_0}{n\pi}(-1)^{n+1}\sin\left(\frac{n\pi}{L}x\right)}$$

for these two series to be equal, coefficients must be equal

$$(EI)B_n\frac{n^4\pi^4}{L^4} = \frac{2w_0}{n\pi}(-1)^{n+1}$$

$$B_n = \frac{2w_0 L^4(-1)^{n+1}}{n^5\pi^5 EI} \Rightarrow \underline{y_p = \sum_{n=1}^{\infty}\frac{2w_0 L^4(-1)^{n+1}}{n^5\pi^5 EI}\sin\left(\frac{n\pi}{L}x\right)}$$

4. See Appendix

MA365 Problem Set 3 Appendix

Problem 2

In[1]:= $an = \frac{1}{\pi}\int_{-\pi}^{\pi}(x+\pi)\,\text{Cos}[n * x]\,dx$

$bn = \frac{1}{\pi}\int_{-\pi}^{\pi}(x+\pi)\,\text{Sin}[n * x]\,dx$

Out[1]= $\frac{2\,\text{Sin}[n\,\pi]}{n}$

Out[2]= $\frac{-2\,n\,\pi\,\text{Cos}[n\,\pi] + 2\,\text{Sin}[n\,\pi]}{n^2\,\pi}$

Problem 3

In[3]:= $b3n = \frac{2}{L}\int_{0}^{L}\frac{wo * x}{L}\,\text{Sin}[n * \pi * x / L]\,dx$

Out[3]= $\frac{2\,wo\,(-n\,\pi\,\text{Cos}[n\,\pi] + \text{Sin}[n\,\pi])}{n^2\,\pi^2}$

Problem 4

part a)

In[5]:= $f1[x\_] = \frac{x^2}{2}$; (*This is an even function,

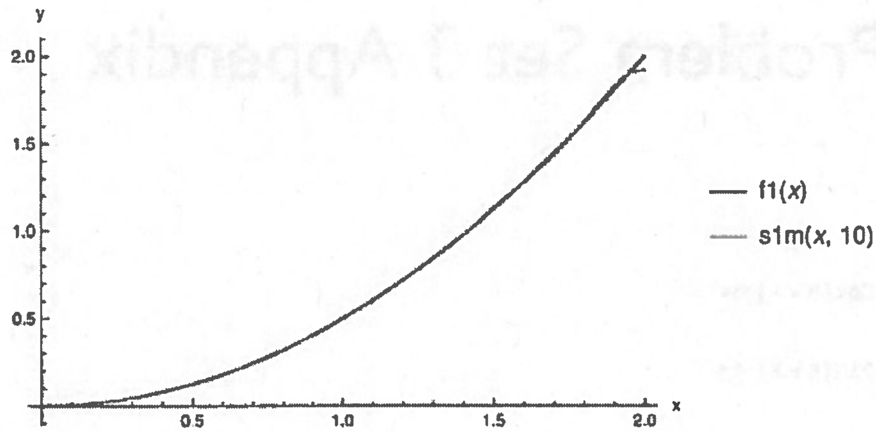
so let's expand in a half-range cosine series *)

$a10 = \int_{0}^{2} f1[x]\,dx$;

$a1n = \int_{0}^{2} f1[x] * \text{Cos}[n * \pi * x / 2]\,dx$;

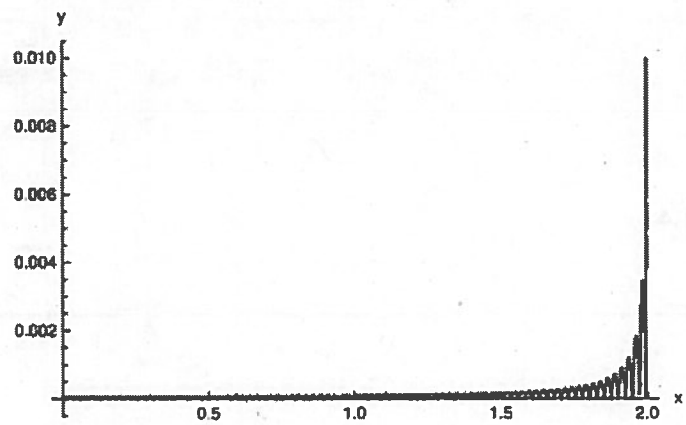
In[6]:= $s1m[x\_, m\_] := \frac{a10}{2} + \sum_{n=1}^{m}(a1n * \text{Cos}[n * \pi * x / 2])$

```
Plot[{f1[x], s1m[x, 10]}, {x, 0, 2}, AxesLabel → {"x", "y"}, PlotLegends -> "Expressions"]
```




```
In[9]:= e1m[x_, m_] := f1[x] - s1m[x, m]
```

```
Plot[Abs[e1m[x, 81]], {x, 0, 2}, AxesLabel → {"x", "y"}, PlotRange → All]
```



We notice the max error value occurs at x=2.

```
In[10]:= Abs[e1m[2, 80] // N]
Out[10]= 0.0100691
```

```
Abs[e1m[2, 81] // N]
0.00994551
```

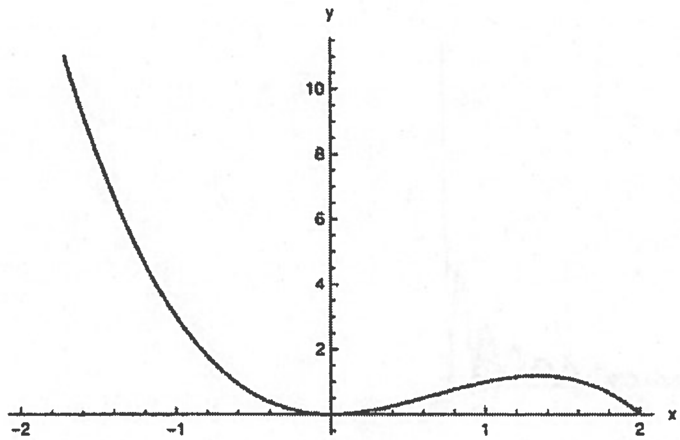
The threshold value is 81. This is the smallest value of m such that the error is less than the tolerance 0.01 for any x on the interval.

part b)

```
In[11]:= f2[x_] = x² (2 - x);
```

In[13]:=
```
Plot[f2[x], {x, -2, 2}, AxesLabel → {"x", "y"}]
```

Out[13]=



This function is neither even nor odd, so we'll just perform a Fourier series expansion. We notice that f(x) = 0 at both ends of the interval, so Gibbs phenomenon should be minimal.

In[14]:=

$$\texttt{a20} = \int_0^2 \texttt{f2[x]}\, dx;$$

$$\texttt{a2n} = \int_0^2 \texttt{f2[x]} * \texttt{Cos}[n * \pi * x]\, dx;$$

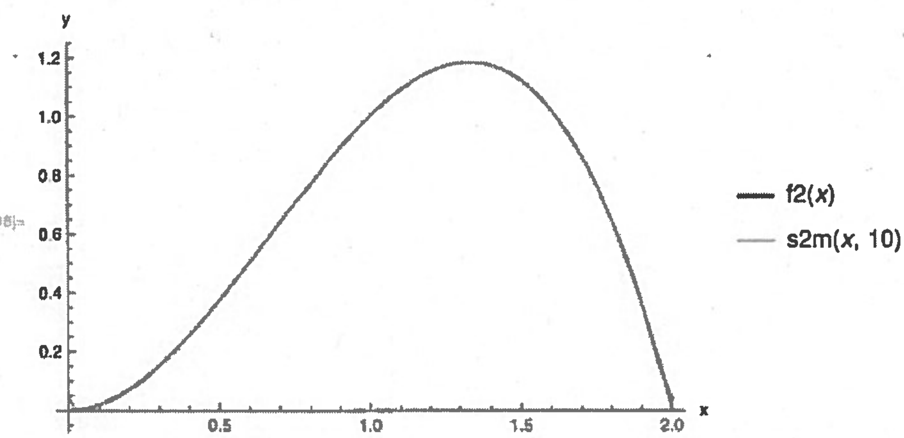
$$\texttt{b2n} = \int_0^2 \texttt{f2[x]} * \texttt{Sin}[n * \pi * x]\, dx;$$

In[17]:=

$$\texttt{s2m[x\_, m\_]} := \frac{\texttt{a20}}{2} + \sum_{n=1}^{m} (\texttt{a2n} * \texttt{Cos}[n * \pi * x] + \texttt{b2n} * \texttt{Sin}[n * \pi * x])$$

In[18]:=
```
Plot[{f2[x], s2m[x, 10]}, {x, 0, 2}, AxesLabel → {"x", "y"}, PlotLegends -> "Expressions"]
```

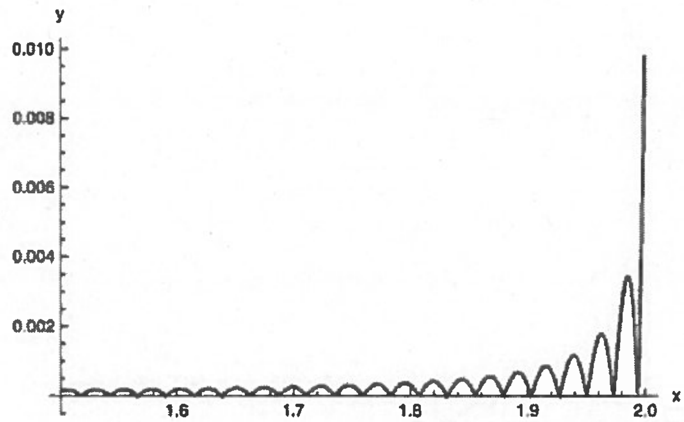
Out[18]=



In[19]:=
```
e2m[x_, m_] := f2[x] - s2m[x, m]
```

In[20]:=
```
Plot[Abs[e2m[x, 41]], {x, 1.5, 2}, AxesLabel → {"x", "y"}, PlotRange → All]
```

Out[20]=



We notice the max error value occurs at x=2.

In[21]:=
```
Abs[e2m[2, 40]] // N
```

Out[21]=
```
0.0100065
```

```
Abs[e2m[2, 41]] // N
```

```
0.00976542
```

The threshold value is 41. This is the smallest value of m such that the error is less than the tolerance 0.01 for any x on the interval.

PROBLEM SET #4

MA365 – Advanced Mathematics for Engineers and Scientists
AY20–1 Problem Set 4: Boundary Value Problems

Due Lesson 29 prior to your class hour. **100 Points**

Instructions: This assignment will be submitted in class on the due date provided, with cover sheet. This is an *individual* assignment; if you receive assistance, please document any assistance received in accordance with Documentation of Academic Work guidelines. Submit all supporting calculations and analysis. Ensure all work is logical, neat, and organized.

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1. **(30 points)** Consider the boundary value problem $y'' + \lambda y = 0$, $y(0) + y'(0) = 0$, $y(1) = 0$.

   a. Find equation(s) that define(s) this problem's eigenvalues.

   b. Use technology to approximate the first four eigenvalues λ_1, λ_2, λ_3, and λ_4. Give the eigenfunctions corresponding to these approximations.

2. **(15 points)** Example 2 starting on p. 708 of your textbook shows how to use separation of variables to solve the PDE $\dfrac{\partial^2 u}{\partial x^2} = 4\dfrac{\partial u}{\partial y}$. For the three cases for λ: zero, negative, positive; corresponding solutions were found. Verify that each of these solutions satisfies the PDE.

   a. $(\lambda = 0)$, $u(x, y) = A_1 + B_1 x$.

   b. $(\lambda = -\alpha^2 < 0)$, $u(x, y) = A_2 e^{\alpha^2 y} \cosh 2\alpha x + B_2 e^{\alpha^2 y} \sinh 2\alpha x$.

   c. $(\lambda = \alpha^2 > 0)$, $u(x, y) = A_3 e^{-\alpha^2 y} \cos 2\alpha x + B_3 e^{-\alpha^2 y} \sin 2\alpha x$.

3. **(30 points)** Consider a uniform bar of length L having an initial temperature distribution given by $u(x, 0) = \sin(\pi x/L)$, $0 \le x \le L$. Assume that both ends of the bar are insulated.

   a. Find the temperature $u(x, t)$.

   b. What is the steady state temperature as $t \to \infty$?

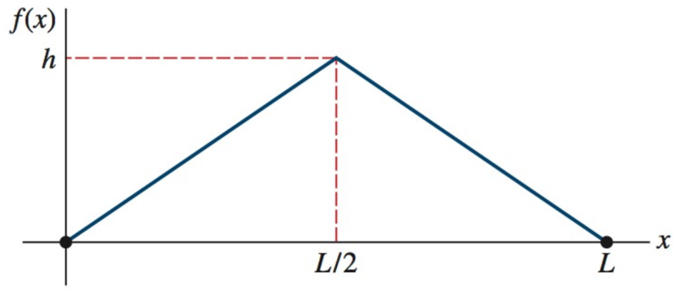
   c. Let $k = 1$ and $L = 40$. Plot u versus x for several values of t. Also plot u versus t for several values of x.

   d. Describe briefly how the temperature in the rod changes as time progresses.

MA365 – Advanced Mathematics for Engineers and Scientists
AY20–1 Problem Set 4: Boundary Value Problems

Due Lesson 29 prior to your class hour. **100 Points**

4. **(25 points)** A string is tied to the x-axis at $x = 0$ and at $x = L$ and its initial displacement $u(x, 0) = f(x)$, $0 < x < L$, is shown in the figure below.



a. The string is released from rest at $t = 0$. Use technology to plot a partial sum of the first six **nonzero** terms of your solution $u(x, t)$, for $t = 0.1k$, $k = 0, 1, 2, ..., 20$. Assume that $a = 1$, $h = 1$, and $L = \pi$.

b. **(10 bonus points)** Use technology to animate the solution to this problem.

Instructor Solution

1. $y'' + \lambda y = 0$, $y(0) + y'(0) = 0$, $y(1) = 0$

case 1: $\lambda = 0$ $\quad y = c_1 x + c_2$

impose BCs: $y(0) + y'(0) = c_2 + c_1 = 0$

$y(1) = c_1 + c_2 = 0$

implies that $c_1 = -c_2$

so $\lambda = 0$ is an eigenvalue
and corresponding eigenfunction $y_0 = x - 1$ (or $cx - c$)

case 2: $\lambda = -\alpha^2 < 0$ $\quad y = c_1 \cosh \alpha x + c_2 \sinh \alpha x$

impose BCs: $y(0) + y'(0) = c_1 \cancelto{1}{\cosh(0)} + c_2 \cancelto{0}{\sinh(0)} + c_1 \alpha \cancelto{0}{\sinh(0)} + c_2 \alpha \cancelto{1}{\cosh(0)} = 0$

$= c_1 + c_2 \alpha = 0$

and $y(1) = c_1 \cosh \alpha + c_2 \sinh \alpha = 0$; Can only hold if $c_1 = c_2 = 0$
so $\lambda < 0$ only trivial solution possible

case 3: $\lambda = \alpha^2 > 0$ $\quad y = c_1 \cos \alpha x + c_2 \sin \alpha x$

impose BCs: $y(0) + y'(0) = c_1 \cancelto{1}{\cos(0)} + c_2 \sin(0) - c_1 \alpha \cancelto{0}{\sin(0)} + c_2 \alpha \cancelto{1}{\cos(0)} = 0$

$= c_1 + c_2 \alpha = 0 \implies c_1 = -c_2 \alpha$

and $y(1) = c_1 \cos \alpha + c_2 \sin \alpha = 0$

$= -c_2 \alpha \cos \alpha + c_2 \sin \alpha = 0$

$= c_2 (\sin \alpha - \alpha \cos \alpha) = 0$

for $c_2 \neq 0$, then $\sin \alpha = \alpha \cos \alpha$ or $\dfrac{\sin \alpha}{\cos \alpha} = \alpha$ or $\tan \alpha = \alpha$

use technology to find the roots of $\tan \alpha = \alpha$

* see attached Mathematica code

$\lambda_1 = 20.19$, $\lambda_2 = 59.68$, $\lambda_3 = 118.9$, $\lambda_4 = 197.9$

$y_1 = -4.493 \cos 4.493x + \sin 4.493x$

$y_2 = -7.725 \cos 7.725x + \sin 7.725x$

$y_3 = -10.904 \cos 10.904x + \sin 10.904x$

$y_4 = -14.066 \cos 14.066x + \sin 14.066x$

2. a) $u(x,y) = A_1 + B_1 x$

$$\frac{\partial}{\partial x}(A_1 + B_1 x) = B_1 \qquad \frac{\partial}{\partial y}(A_1 + B_1 x) = 0$$

$$\frac{\partial}{\partial x} B_1 = 0 \qquad \frac{\partial}{\partial y} 0 = 0$$

$$\therefore \frac{\partial^2 u}{\partial x^2} = 0$$

this solution satisfies PDE, because $0 = 4(0)$

b) $$\frac{\partial}{\partial x}\left(A_2 e^{\alpha^2 y}\cosh 2\alpha x + B_2 e^{\alpha^2 y}\sinh 2\alpha x\right) = 2\alpha A_2 e^{\alpha^2 y}\sinh 2\alpha x + 2\alpha B_2 e^{\alpha^2 y}\cosh 2\alpha x$$

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = 4\alpha^2 A_2 e^{\alpha^2 y}\cosh 2\alpha x + 4\alpha^2 A_2 e^{\alpha^2 y}\sinh 2\alpha x$$

$$\frac{\partial u}{\partial y} = \alpha^2 A_2 e^{\alpha^2 y}\cosh 2\alpha x + \alpha^2 B_2 e^{\alpha^2 y}\sinh 2\alpha x$$

clear to see that $\frac{\partial^2 u}{\partial x^2} = 4\frac{\partial u}{\partial y}$

c) $$\frac{\partial}{\partial x}\left(A_3 e^{-\alpha^2 y}\cos 2\alpha x + B_3 e^{-\alpha^2 y}\sin 2\alpha x\right) = -2\alpha A_3 e^{-\alpha^2 y}\sin 2\alpha x + 2\alpha B_3 e^{-\alpha^2 y}\cos 2\alpha x$$

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = -4\alpha^2 A_3 e^{-\alpha^2 y}\cos 2\alpha x - 4\alpha^2 e^{-\alpha^2 y}\sin 2\alpha x$$

$$\frac{\partial u}{\partial y} = -\alpha^2 A_3 e^{-\alpha^2 y}\cos 2\alpha x - \alpha^2 B_3 e^{-\alpha^2 y}\sin 2\alpha x$$

again, clear to see that $\frac{\partial^2 u}{\partial x^2} = 4\frac{\partial u}{\partial y}$

3. $k\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$, $0 < x < 100$, $t > 0$

$u(0,t) = 0$

$u(100,t) = 0$

$u(x,0) = \begin{cases} 6x/5, & 0 \le x \le 50 \\ 6(100-x)/5 & 50 < x \le 100 \end{cases}$

we know from experience on this type of problem where we have homogenous Dirichlet BCs, that solution form is of type

$$u(x,t) = \sum_{n=1}^{\infty} A_n e^{-k\left(\frac{n^2\pi^2}{L^2}\right)t} \sin\frac{n\pi}{L}x$$

where

$A_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi}{L}x\,dx$ so $A_n = \frac{2}{100}\left(\int_0^{50} 1.2x \sin\frac{n\pi}{L}x\,dx + \int_{50}^{100} 1.2(100-x)\sin\frac{n\pi}{L}x\,dx\right)$

See Mathematica for A_n calculations: $A_n = \frac{480}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)$

$\frac{480}{\pi^2} \approx 48.6342$

a) so, $u(x,t) = \sum_{n=1}^{\infty} \frac{480}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)e^{-k\left(\frac{n^2\pi^2}{10000}\right)t}\sin\frac{n\pi}{100}x$

b) see Mathematica work

4. $a^2\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}$ $L = \pi$ $a = 1$ $f(x) = \begin{cases} \frac{2}{\pi}x, & 0 \le x \le \frac{\pi}{2} \\ \frac{2}{\pi}(\pi - x), & \frac{\pi}{2} < x \le \pi \end{cases}$

$u(0,t) = 0$

$u(L,t) = 0$

$u(x,0) = f(x)$

$\left.\frac{\partial u}{\partial t}\right|_{t=0} = 0$

because $g(x) = 0$, we know

$$u(x,t) = \sum_{n=1}^{\infty} A_n \cos\frac{n\pi}{\pi}t \sin\frac{n\pi}{\pi}x \quad \text{where}$$

$$A_n = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \frac{2}{\pi}x\sin\frac{n\pi}{\pi}x\,dx + \frac{2}{\pi}\int_{\frac{\pi}{2}}^{\pi}\frac{2}{\pi}(\pi - x)\sin\frac{n\pi}{\pi}x\,dx$$

using Mathematica $A_n = \frac{8\sin\left(\frac{n\pi}{2}\right)}{n^2\pi^2}$

but A_n will be 0 for even values of n, so need 11 terms in series to have six non zero terms

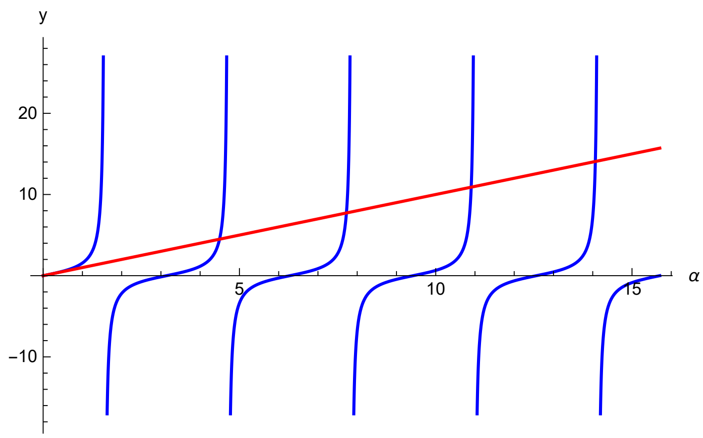
$$S_6(x,t) = \sum_{n=1}^{11}\frac{8\sin\left(\frac{n\pi}{2}\right)}{n^2\pi^2}\cos nt \sin nx$$

Problem 1

We need to find the roots of equation tanα = α. One way is to use the FindRoot command in Mathematica. This command requires us to provide a starting point--or ballpark guess as to the root. We'll supply our guesses by first inspecting the graphs of y = tanα (blue) and y = α (red) and noting where they intersect.

In[3]:=
```
Plot[{Tan[α], α}, {α, 0, 5π}, AxesLabel → {"α", "y"}, PlotStyle → {Blue, Red}]
```

Out[3]=



It appears that α=0 is a root to this equation, and it indeed it is, because we know that tan 0 = 0, but α must be greater than 0, so we'll look at the next four roots.

In[4]:=
```
roots = Table[FindRoot[Tan[α] == α, {α, guess}], {guess, {4.2, 7.6, 10.9, 14}}]
```

Out[4]=
```
{{α → 4.49341}, {α → 7.72525}, {α → 10.9041}, {α → 14.0662}}
```

Eigenvalues are found by squaring the roots that we found.

In[5]:= α^2 /. roots

Out[5]=
```
{20.1907, 59.6795, 118.9, 197.858}
```

Problem 3

In[6]:=

In[7]:= Assuming[n ∈ Integers,

$$An = \frac{2}{100}\left(\int_0^{50} \frac{6}{5}\, x * Sin\left[\frac{n*\pi}{100}\, x\right] dx + \int_{50}^{100} \frac{6}{5}\,(100 - x)\, Sin\left[\frac{n*\pi}{100}\, x\right] dx\right)\Big] \,//\, FullSimplify$$

Out[7]=

$$\frac{480\, Sin\left[\frac{n\pi}{2}\right]}{n^2\,\pi^2}$$

this coefficient will be 0, though, for even values of n. In order to make the partial sum S_5 consist of the the first 5 nonzero terms (for n = 1, 3, 5, 7, 9), we can just set the upper limit of the index to be 9

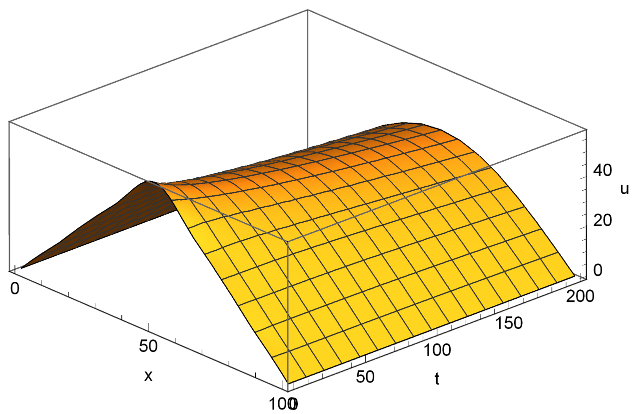
part b

In[8]:=
$$\text{S5}[x\_, t\_, k\_] := \sum_{n=1}^{9} \left(\text{An} * e^{\frac{-k\,(n^2\,\pi^2)}{10\,000} t}\, \text{Sin}\left[\frac{n * \pi}{100} x\right] \right)$$

Assume that k = 1.6352

In[9]:=
```
Plot3D[S5[x, t, 1.6352], {x, 0, 100}, {t, 0, 200},
 AxesLabel → {"x", "t", "u"}, PlotRange → All, ViewPoint → {300, -280, 180}]
```

Out[9]=



This viewpoint gives a good look at the initial temperature.

Problem 4

In[10]:=
$$\text{A4n} = \frac{2}{\pi} \left(\int_0^{\pi/2} \frac{2}{\pi}\, x * \text{Sin}\left[\frac{n * \pi}{\pi} x\right] dx + \int_{\pi/2}^{\pi} \frac{2}{\pi} (\pi - x)\, \text{Sin}\left[\frac{n * \pi}{\pi} x\right] dx \right) \; // \; \text{Simplify}$$

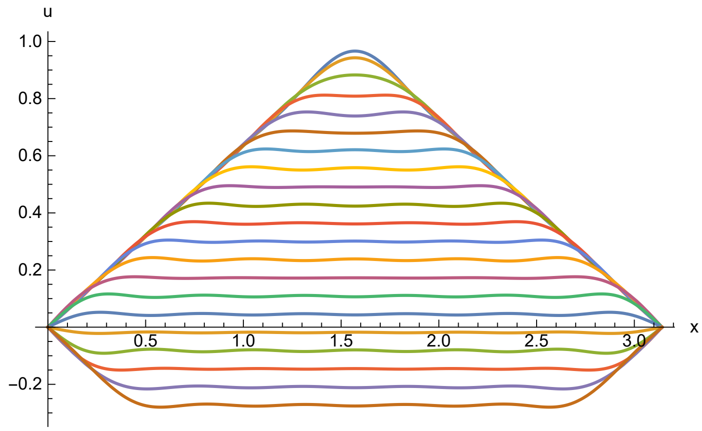
Out[10]=
$$\frac{8\,\text{Sin}\left[\frac{n\,\pi}{2}\right] - 4\,\text{Sin}[n\,\pi]}{n^2\,\pi^2}$$

The coefficient will be 0 for even values of n, so we need 11 terms in the partial sum S_6 to represent the first six nonzero terms.

In[11]:=
$$\text{S6}[x\_, t\_] := \sum_{n=1}^{11} \frac{8\,\text{Sin}\left[\frac{n*\pi}{2}\right]}{n^2\,\pi^2}\, \text{Cos}[n * t]\, \text{Sin}[n * x]$$

In[12]:=
```
Plot[Evaluate[Table[S6[x, t], {t, 0, 2, 0.1}]], {x, 0, π}, AxesLabel → {"x", "u"}]
```

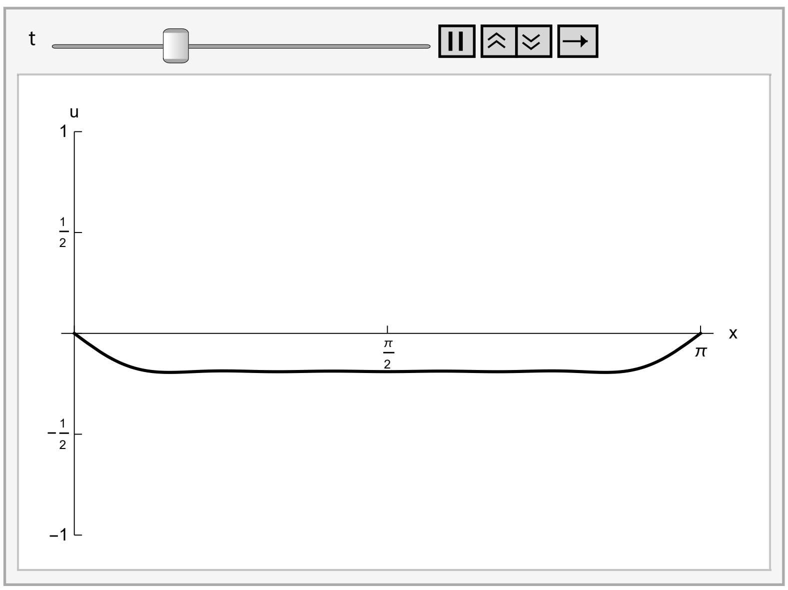
Out[12]=



bonus part

In[13]:=
```
Animate[Plot[S6[x, t], {x, 0, π}, AxesLabel → {"x", "u"}, PlotRange → {-1, 1},
  Ticks → {{0, π/2, π}, {-1, -1/2, 0, 1/2, 1}}, PlotStyle → Black], {t, 0, 2 π}]
```

Out[13]=



even though the first part only specifies 0<t<2, the period is actually 2π, so animation looks better running from 0 to 2π.

UNITED STATES MILITARY ACADEMY
WEST POINT.

TAB J ARTICLE CRITIQUE

Outstanding Student Work

UNITED STATES MILITARY ACADEMY

JOURNAL REVIEW #1

MA365: ADVANCED ENGINEERING MATH

SECTION H10

LTC KRISTOPHER AHLERS

By:

[redacted]

WEST POINT, NY

05 NOV 2019

_____ MY DOCUMENTATION IDENTIFIES ALL SOURCES USED AND ASSISTANCE RECEIVED IN COMPLETING THIS ASSIGNMENT.

_____ I DID NOT USE ANY SOURCES OR ASSISTANCE REQUIRING DOCUMENTATION IN COMPLETING THIS ASSIGNMENT.

SIGNATURE: ____

Article Summary:

My article is entitled “Numerical analysis of the effect of weld-induced residual stress and plastic damage on the ballistic performance of welded steel plate.” This is a mouthful, but the article itself isn’t terribly hard to follow. The core principle addressed by the article is that manufacturing processes often introduce residual stresses into a material. These residual stresses eventually manifest themselves as cracks in the material itself, which lead to stress concentrations and material failure over time. There are different ways of preventing residual stresses in material to elongate the operational life of a manufactured component, but you can never fully remove all the residual stresses from a material. Welding is a manufacturing process that typically introduces a high amount of residual stress to a workpiece. Many studies have been done over the years on the effects of weld-induced residual stress on component performance under cyclic loading. These types of applications are commonly found in machinery where the component undergoes repeated cycles of stressing and destressing.

Flores-Johnson et al, however, identified a need for research about the effects of residual stresses on a material’s ability to withstand impact loading. Specifically, they used finite-element analysis software to model a welded steel plate’s reaction to impact forces. They used a software called Abaqus/Standard to model the initial residual stresses in the plate after welding, and they fed this data into a different software package called Abaqus/Explicit to model the effects of the ballistic testing on the material. The plate used was 18 mm thick of 316L austenitic stainless steel and the weld was a 3-pass TIG weld made to fill an 80 mm long by 6mm deep slot in the plate (0.9 mm ER316L filler wire used). The weld was made with the plate material unclamped so that the plate material could warp freely, and all residual stresses would be solely due to thermal effects of welding. The projectiles were either hemispherical-nosed (HN) or flat-nosed (FN), with a 10 mm diameter and mass of 25.8 g. Testing was done with impact velocities ranging from 300 to 800 m/s. Additionally, the final finite-element (FE) mesh consisted of 398,124 individual elements with an average size of 0.5 x 0.5 x 0.5 mm^3. The FE mesh was altered in two key ways to help the research progress: first, the researchers created a model of only half the plate so that researchers could see what was occurring within the plate upon impact; second, the weld cap that rose from the face of the material was considered to be insignificant and was removed from the FE analysis so that the plate could be treated as a perfect rectangular prism in order to reduce computational complexity.

Flores-Johnson et al relied heavily on previous research when creating their study. For example, they used the existing Johnson-Cook material model and the shear failure fracture criterion model proposed by Dey et al. Once Flores-Johnson et al had validated that the models could be applied accurately my running experimental validation tests, they loaded the two validated models into the Abaqus/Explicit software for final testing. For final testing, ballistic testing was performed on the parent plate, the welded plate from the front, and the welded plate from the back. Both HN and FN projectiles were used, and the plastic strain produced in the material was graphed versus time. Since the welded plates already had a considerable amount of plastic strain present in them before impact due to the residual stresses, they tended to reach fracture failure more quickly than the parent material. A key result of the research was that weld-induced deformation produces strain hardening of the plate material (which is typically good), but the negative effects on ballistic performance versus the parent plate far outweighed any positive benefits introduced by the welding process.

Improvements to Current Research:

As is typical for research that seeks to break ground in new areas of any field, there is much room for improvement to the Flores-Johnson et al study. One area of concern is their over-reliance on previously conducted research that was not originally meant to be applied to their specific focus. I am specifically talking about the Johnson-Cook material model and the shear failure fracture criterion. These two models are used by Flores-Johnson et al after what I would categorize as only preliminary testing to ensure that the models fit the experimental data. Although the published material does show a high amount of fit between experimental and model-predicted behavior, I believe that the credibility of the research would greatly benefit from further testing and validation of the model used. If it turns out that the models used are not a best fit, then a key aspect to future research might include the use or creation of an entirely different model that more closely fits the behavior of the material under ballistic impact loading. Since there is no previous research of this kind, there is nothing available to say that the Johnson-Cook model and the shear failure fracture criterion are the ideal models to use for ballistic impact modeling. These models were simply the best educated guesses made by the researchers, and iterative studies might find that better models exist or can be derived.

Another way that Flores-Johnson et al could improve their research would be to study a more applicable type of round for ballistic testing. Although the two projectiles used were similar in shape, size, and weight to actual ammunition, I think that testing done with actual rounds might provide results that are more beneficial to more people. This could help the military and law enforcement sectors as they seek to understand how their equipment is affected by ammunition impact.

Additionally, I think that varying the size of plate used in testing, as well as the type of weld used in testing, would provide a great deal more information than the vary specific scenario that was analyzed. The bottom-line up front for this train of thought is that the research thus far is very limited in size and scope. Numerical analysis has been shown to work for one specific situation, but in order to validate the technique on a widespread scale, widespread testing must be conducted. This must involve different materials, thicknesses, types of projectiles, types of welds, amount of weld, etc. As the breadth and depth of the research in this field expands, so too will the validity of Flores-Johnson et als' work.

I would like to see future research get rid of one of the simplifying assumptions made by Flores-Johnson et al. The weld cap was ignored in the current research in order to save computational effort for minimal gains in understanding of the reactions of the plate to ballistic impact. However, I believe that some tests should be done with the weld cap included to see just how big of an impact its inclusion makes to the overall results. Especially after more testing is done with different types of weld caps, I think this could be a particularly interesting addition to the research.

Another way to broaden and improve the research would be to use a different software for the finite-element mesh. I am not overly familiar with the industry of finite-element analysis, but I am sure that there must be more than one software available on the market that can meet the needs of the research. There might even be a more accurate software that could be used to refine and improve the accuracy of research results.

Works Cited

Flores-Johnson, E.A., et al. "Numerical Analysis of the Effect of Weld-Induced Residual Stress and Plastic Damage on the Ballistic Performance of Welded Steel Plate." Computational Materials Science, vol. 58, Elsevier B.V., June 2012.

UNITED STATES MILITARY ACADEMY

ARTICLE WRITE-UP

MA365: ADVANCED MATHEMATICS FOR ENGINEERS AND SCIENTISTS

SECTION H1

LTC AHLERS

By

[redacted]

WEST POINT, NEW YORK

05 DECEMBER 2019

[redacted] MY DOCUMENTATION IDENTIFIES ALL SOURCES USED AND ASSISTANCE RECEIVED IN COMPLETING THIS ASSIGNMENT.

____ I DID NOT USE ANY SOURCES OR ASSISTANCE REQUIRING DOCUMENTATION IN COMPLETING THIS ASSIGNMENT.

SIGNATURE: [redacted]________________________________

Article Summary

In the interest of diving into this design optimization better, I'm going to provide my own additional background information to that which is provided in the article. In the rock-climbing style known as traditional, or "trad," climbing, climbers place protection ranging from tiny metal nuts to pitons to larger, sturdier contraptions in cracks and pockets of rock in order to secure their ropes incrementally on their ascent to protect them from falling. Among the most interesting of these contraptions is the "friend" (674.) The friend, often times referred to as a "cam" or "Camelot" device typically uses four "spiral shape" cams, which use "a spring and pulley mechanism" to utilize the friction of the ribbed metal edges of the cams and the pressure of the mechanism against the rock faces to provide a secure anchor point for the climber (Bonney et al., 674.) In their article, Bonney, Coaplen, and Doeff use both a baseline physics understanding, and differential equations mathematically derive the curved shape of this "spiral." As previously described, the positioning of these cams is such that the shape will determine the effectiveness of the friction and mechanical advantage element of the cam edges and how the system functions in different sized cracks.

The authors begin their article by establishing the baseline physical equations which describe the forces on the cam using a free-body diagram, but more specifically draws out the vertical component of the reaction force of the cam using a cotangential value of the angle of incidence of said reaction (675.) Next, the authors start to parameterize the actual shape of the cam lobe by setting a coordinate system to originate at the pivot point of the mechanism, and determining two first-order differential functions $K_1(t)$ and $K_2(t)$, which are based on the tangent vector of the curve being parallel with the rock surface (a good demonstration of a proper cam placement)(675-676.) The authors use the fact that the triangle formed here and in the previous step are similar in order to the cotangential identity to the functions, which leads to a system of these two linear differential equations (1.7 and 1.8) (676.)

At this point, the authors narrow their considerations for finding an equation that functions as an appropriate fit, and denotes the general solution given in equation (1.9) to be an adequate exponential curve which satisfies the proportion requirements of the mechanism (677.) The nature of friends requires them to be able to be inserted as a thinner body, so they can expand to fill the space and provide an anchor. Considering this necessity, the writers discuss how the lobes of the cans are cut and modelled to be of the most minimum radius to maintain full friction contact while adequately placed (677.) The math used by the writers to do this is very nicely simplistic, and as shown in equations 2.1-2.3 involves little more effort than acknowledging that half the width of the whole mechanism must be able to fit in half the crack of the rock, and the other several angles are a little more flexible in terms of author decision-making (678).

Bonney and associates finish their brief illustration of a possible description of this cam body shape by acknowledging that commercial cams are often made with a larger angle "alpha," therefore creating more of an edge on the shorter radius side of the lobe (678.) The team of writers also acknowledge that though they achieved the same result, the creator of the friend, Ray Jardine, interestingly, arrived at this same answer by a whole separate mathematical means involving "an infinitesimal angular displacement (679.)

Article Improvements

There were several aspects of Bonney et al's description of the cam shape and their article in general which can be improved to expand on the the conclusions made and provide more meaningful results to the reader and to mountaineering design at large. Firstly, and most nit-picky, I believe that the model created by the writers would be much more interesting if it was compared with Jardine's original curve equation definition. Secondly, the writers in the article assume that an absolute minimum angle "alpha" is to be applied to the design. Often times this ideal placement attitude of the piece is not possible, and therefore it is interesting when you consider the possibilities of using more or less of the cam lobe. Lastly, while this article never billed itself as something ground-breaking or prototypical, I believe that if a more thorough 3-D modelling of loading conditions was used to expand on the author's work that a fuller understanding of the usefulness of a friend could be found – or even a potential redesign of the tool.

First-off, I had no clue that this piece of climbing protection had only been around since 1973. I think that this relatively recent innovation that has largely dominated how climbers protect themselves on routes since its inception probably has a very interesting origin story, and I think it would serve to highlight Bonney and associates' work if they were to show how the two methods for modelling the design would overlap or differ. This would give the article a little more narrative structure while also increasing the mathematical credibility of it by relating it to the creator of the tool himself.

When placing friends, it is possible to "over-cam" or "under-cam" a piece of protection, indicated by the cams of the friend being either over-rotated and tight or under-activated and potentially loose. This difference in posture effects the amount of the side of the cam that contacts the wall and because this angle of incidence is different, the cam is unable to provide the ideal horizontal normal force on the rock face. Assuming this bare minimum alpha measurement from the pivot point of the cam head works out graphically in an article, but in uneven, or potentially off-width rock channel scenarios, the more material that can provide support the better. The authors should examine different loading conditions and specifically how the system of linear differential equations differs when a cam is positioned in a non-ideal scenario.

Lastly, it is significant to realize that the cam placed inside of rock does not sit two-dimensionally square. In fact, a friend is meant to be pleased at an angle where the pulley and subsequent carabiner faces downward, and inevitable a little out, so that the rope can be attached outside of the crack. This 3-dimensional aspect does not necessarily change the overall structure of the device, and I have no normative claims to make about the results of such an analysis, but I think it would both improve and complicate the mathematical model to look at this third dimension of loading. The linear system of differential equations would require additional substance in order to cover the range of effects experienced in a system with another dimension.

Works Cited

Bonney, Matthew et al. “What Makes a Good Friend? The Mathematics of Rock Climbing.” *Society for Industrial and Applied Mathematics Review,* vol. 40, no. 3, 1998, pdfs.semanticscholar.org/686d/69d97703311f0e058d4aff5b30e5e5b4ce5f.pdf