Fish Harvesting Optimization

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Linear Algebra and Differential Equations

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1 Context

Differential equations have a variety of real-world applications. One of these is to simulate population growth; when fishing in large quantities, fish farmers and fishing companies oftentimes need to determine the optimal amount of fish to harvest every year without permanently damaging the population. Doing otherwise may have environmental, legal, or financial consequences. This paper will demonstrate how math can help with decision-making in the industry by using a basic logistic growth model for fish populations.

2 Working Example

As a demonstrative example, this paper covers a particular scenario. A fish farmer wishes to maximize his harvest without depleting his fish stock in the long run. Suppose $y(t)$ represents the number of fish at time t years, with the fish population growing logistically and governed by differential equation

$$\frac{dy}{dt} = y(2 - 0.01y)$$

We also already know the following relationship between $\frac{dy}{dt}$ and y:

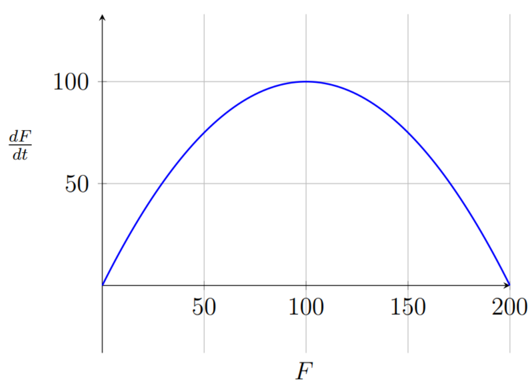

Figure 1: Rate of change in population (dF/dt)with respect to population size F

3 Logistic Growth

Given the information represented in Figure 1, we can infer that pond's carrying capacity is 200 fish. Carrying capacities are seen as the maximum population size and thus our fish farmer will never be able to harvest more than 200 fish in a year if he had less than 200 to begin with.

Equation 1 has more information to offer. With a general understanding of growth rates, we can map the fish population for any number of years given an initial population. Differential equations are far easier to solve after classifying them. Equation 1 is a first-order separable equation, as it can be arranged in the form:

$$N(y)\frac{dy}{dx} = M(x)$$

Observe how, when transformed, equation (1) matches this description:

$$\frac{dy}{dt}\frac{1}{(2y-0.01y^2)} = 1$$

We can solve first order separable differential equations by integrating both sides:

$$\frac{dy}{(2y-0.01y^2)} = 1dt \rightarrow \int \frac{dy}{(2y-0.01y^2)} = \int 1dt \rightarrow \int \frac{100}{(200y-y^2)}dy = t+c$$
$$\rightarrow 100\int \frac{1}{y^2(\frac{200}{y}-1)}dy = t+c$$

Employing u-substitution with $u = \frac{200}{y} - 1$ ...

$$\rightarrow \frac{-1}{2}\int \frac{1}{u}du = t+c \rightarrow \frac{-1}{2}\ln(-1+\frac{200}{y}) = t+c \rightarrow \ln(-1+\frac{200}{y}) = -2t-c, \text{ [if c = 2c]}$$
$$\rightarrow -1+\frac{200}{y} = e^{(-2t-c)} \rightarrow y = \frac{200}{e^{(-2t-c)}+1}$$

The general solution that was calculated represents a series of equations to model the relationship between population and time. However, we must specify what the starting population is to determine a specific solution. This is done by determining the value of constant C. Let's demonstrate this idea by establishing initial populations of 50, 100 and 200 fish...

If $y(0) = 50, 50 = \frac{200}{e^{-c}+1}, 50(e^{-c}+1) = 200, e^{-c}+1 = 4, e^{-c} = 3, c = -ln(3)$
Creating specific solution: $y = \frac{200}{e^{-2t+ln(3)}+1}$, which simplifies to $y = \frac{200e^{2t}}{3+e^{2t}}$

If $y(0) = 100, 100 = \frac{200}{e^{-c}+1}, 100(e^{-c}+1) = 200, e^{-c}+1 = 2, e^{-c} = 1, c = -ln(1)$
Creating specific solution: $y = \frac{200}{e^{-2t+ln(1)}+1}$, which simplifies to $y = \frac{200e^{2t}}{1+e^{2t}}$

If $y(0) = 200, 200 = \frac{200}{e^{-c}+1}, 200(e^{-c}+1) = 200, e^{-c}+1 = 1, e^{-c} = 0, c = -ln(0)$
Creating specific solution: $y = \frac{200}{e^{-2t+ln(0)}+1}$, which simplifies to $y = 200$

The specific solutions we have found given different initial populations can be plotted. In figure 2, they are modeled 2.5 years after $t = 0$.

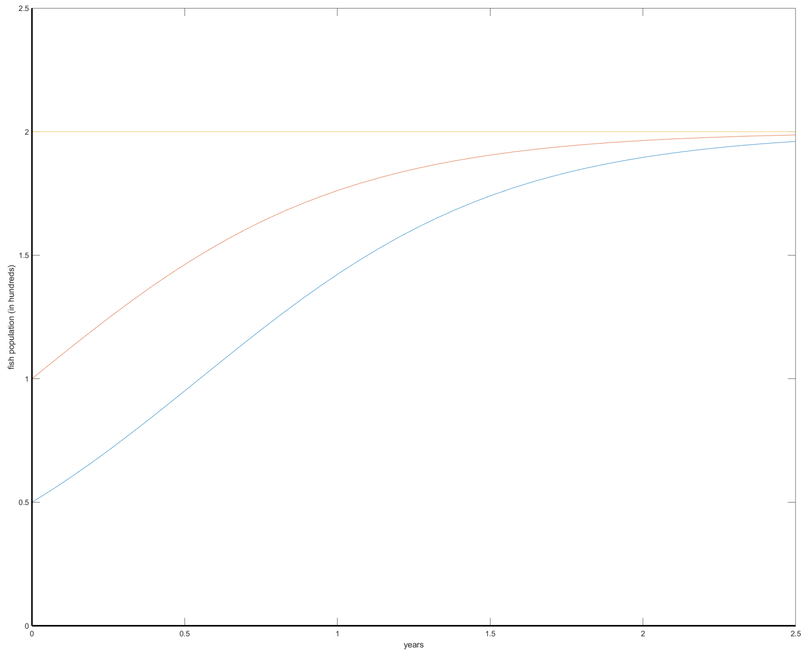

Figure 2: Fish populations in 2.5 years, given initial populations of 50, 100, and 150 fish.

As this graphic demonstrates, these three different initial fish populations approximate 200 within 2.5 years. This confirms our previous assumption about carrying capacity. While obvious, this also serves to indicate that it makes no sense for the farmer to bring in more than 200 fish into the pond, as any surplus will die off till the population stabilizes at 200. Perhaps unsurprisingly, the closer the initial population is to the carrying capacity, the quicker it will approximate it. On the other hand, since there is no fish being taken out of the pond, even a small initial population, like 50 fish, is able to become self-sustaining under this model.

While this basic information is useful and serves as the primary model driver for the fish population in this pond, it says nothing about the impact of harvesting! The next section will update our model to better serve the interest of fish farmers.

4 Logistic Growth With Harvesting

To incorporate harvesting rates, we will update our growth model so that P now represents the number of fish in the pond at time t, and H represents the number of fish harvested annually. The fish population growth is now governed by the differential equation

$$\frac{dP}{dt} = P(2 - 0.01P) - H$$

With MATLAB, we can now plot the slope fields of this differential equation and observe the solution given a number H and an initial starting population. As an example, we will simulate the fish population when harvesting 70,100, and 120 fish/year respectively, given initial fish populations of 42,101, and 150.

As shown in figure 3, if a fish farmer harvests a significantly smaller amount per year, the fish population will grow to approximately 150. However, the fish farmer could be missing out on the opportunity to make more money! On the other extreme, if the fish farmer harvests 120 fish per year, the fish population will eventually reach zero, regardless of how many fish there were to begin with, as shown in Figure 5. A good medium is an annual harvest of 100 fish, represented in Figure 4, where an initial population of 101 fish is shown to be self-sustaining. The same is true for the graph of the fish population with 150 initial fish, however, one may assume that this represents an unnecessary cost to the fish farmer if these fish were not there to begin with!

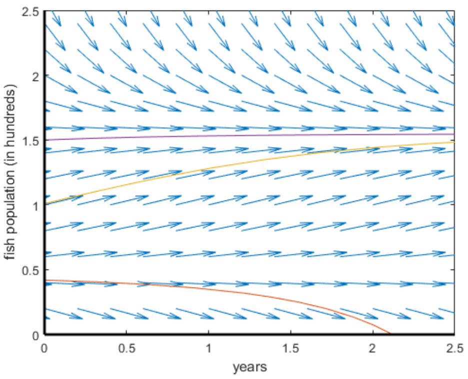

Figure 3: Slope field of fish population given annual harvest of 70 fish; projections population given initial counts of 42 (orange), 101 (yellow), and 150 fish (purple).

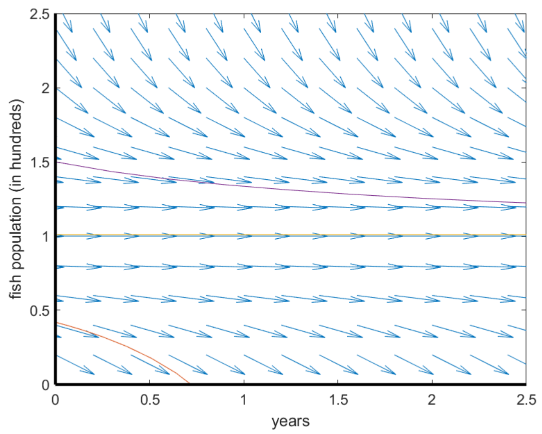

Figure 4: Slope field of fish population given annual harvest of 100 fish; projections population given initial counts of 42 (orange), 101 (yellow), and 150 fish (purple).

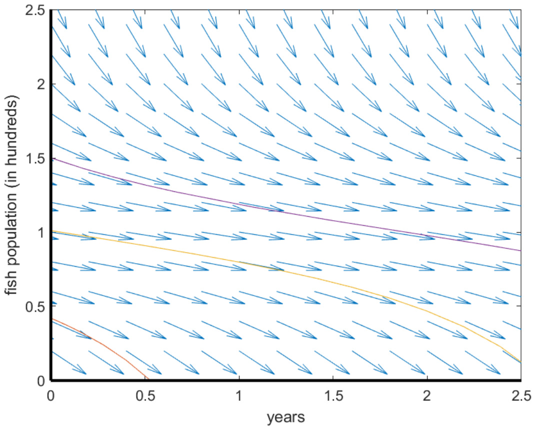

Figure 5: Slope field of fish population given annual harvest of 120 fish; population projections given initial populations of 42 (orange), 101 (yellow), and 150 fish (purple).

With the given relationships, we can also experiment to see what occurs as H approaches 0 and 100. These two annual output values are important because they respectively represent no harvesting occurring, and half of the fish pond's carrying capacity being harvested every year.

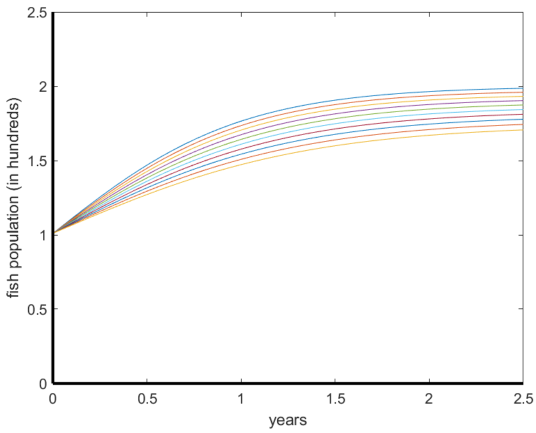

Figure 6: 2.5 year prediction of fish population given initial population of 101 fish. From top to bottom, graphs of solutions with annual harvesting ranging from 0 to 50 fish in intervals of 5 (H=0:5:50).

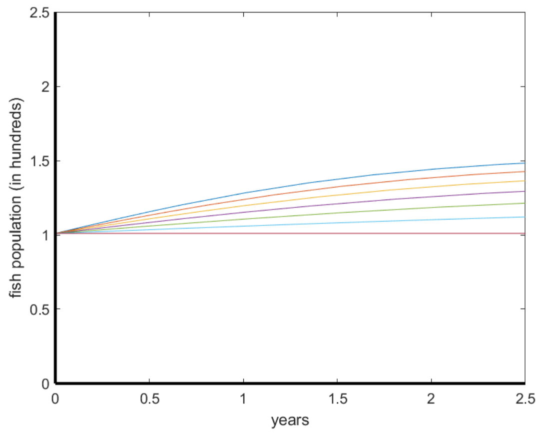

Figure 7: 2.5 year prediction of fish population given initial population of 101 fish. From top to bottom, graphs of solutions with annual harvesting ranging from 70 to 100 fish in intervals of 5 (H=70:5:100).

As figure 6 and 7 demonstrate, the fish population approximates the carrying capacity as H approaches 0, and becomes increasingly smaller over time than its carrying capacity as H approaches 100, even if the fish population is not in decline. In other words, the farmer can expect fish populations to stabilize further below carrying capacity as he or she increases annual harvesting.

The example we've highlighted earlier only considers 3 different annual outputs, and 3 different initial populations. It also required human thinking and analysis to determine the optimal harvesting rate for the farmer. Using software, we could test thousands of solutions and have an intelligent program automatically determine a fish farmer's optimal rate of harvest and starting population in a matter of seconds.