STAT 30850 Final Report

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Introduction

There are many contexts where data is observed sequentially through time. For instance in high frequency stock trading, investment firms have to make rapid decisions in response to new stock evaluations on timescales of a fraction of a second, in A/B testing, technology companies often test the effect of varied advertisements on the "click behavior" of a user which is correlated with the effectiveness of the advertisement, or finally in genomics / or clinical trail applications streaming data can be common (Kohavi et al. 2009, Javanmard and Montanari (2015), Aharoni and Rosset (2014)). The setting in which hypothesis testing must be performed on sequential-streaming data is called "online testing". In online testing controlling the False Discovery Rate (FDR) at a given level has unique challenges as one doesn't observe all the data that could potentially be seen, further in the time series. Here we propose to use and implement a Bayesian model-based approach to control FDR in the online testing setting. We review and contrast our approach to previous commonly used heuristics / algorithms that are effective at controlling FDR but conservative (low power) in online hypothesis testing. We show our approach has higher power when compared to previous methods. Finally, we discuss future extensions and applications of our method.

Background

Broadly speaking, previous methods for controlling FDR in the online testing context use heuristics that increase or decrease the level at which one rejects a test depending on the number of previous discoveries made. Here we review three related, commonly used and well studied approaches to FDR control: α-investing, Levels Based on Number of Discoveries (LBOND), Levels Based on Recent Discoveries (LBORD) (Foster and Stine 2007, Javanmard and Montanari (2015)).

α-investing

Let:

t - be a time index
$w(t)$ - be a wealth function which changes through time
P_t - be a p-value output from an arbitrary test a time t
α - a global level that one would like to control FDR at
α_t - a time specific level

In alpha-investing one defines a wealth function w. We imagine p-values are streaming to the researcher/statistician over time t which are provided by some arbitrary test. We then proceed to run the α-investing procedure:

1. Set $w(t=0) = \alpha$
2. At time t choose $\alpha_t \leq \frac{w(t-1)}{1+w(t-1)}$
3. Reject the null hypothesis if $P_t \leq \alpha_t$
4. Define $w(t)$ as a function of $w(t-1)$

$$w(t) = \begin{cases} w(t-1) + \alpha & P_t \leq \alpha_t \\ w(t-1) - \frac{\alpha_t}{1-\alpha_t} & P_t > \alpha_t \end{cases}$$

5. Repeat the procedure starting back at (2) for each new data point in the time series.

As we can see above when we reject the null, the wealth function grows and when we fail to reject the null the wealth function decays. Specifically at time 0 we set the wealth function to a "global level" alpha. We then proceed to set a time specific α_t. We then reject or fail to reject the p-value P_t from time t and redefine our wealth function w depending on what decision was made. This ensures that the more discoveries we make the less stringent we are through time and reciprocally the fewer discoveries we make the more stringent we are through time. For instance if we fail to reject for many sequential time points the threshold at which we reject the null hypothesis has to be very strong to overcome the current state of the wealth function.

LBOND / LBORD

Let:

t - be a time index
P_t - be a p-value output from an arbitrary test a time t
α - a global level that one would like to control FDR at
β_t - a time specific weight
D_t - count of discoveries made up to time t

In Levels Based on Number of Discoveries (LBOND) we define a series of weights β_t which all sum up to the global level α. We then set a time specific α_t equal to the weight at time t multiplied by the max of 1 and the number of discoveries made up to the last time step D_{t-1}. We reject a p-value P_t if its less that α_t and add to our discovery count.

1. At time t set $\alpha_t = \beta_t \cdot max\{1, D_{t-1}\}$ where $\sum_{t=1}^{\infty} \beta_t = \alpha$
2. Reject if $P_t \leq \alpha_t$
3. If discovery add to D
4. Repeat

Levels Based on Recent Discoveries follows a similar approach but uses weights from the time when the last discovery was made.

1. At time t set $\alpha_t = \beta_t \cdot max\{1, D_{t-\tau t}\}$ where $\sum_{t=1}^{\infty} \beta_t = \alpha$
2. Reject if $P_t \leq \alpha_t$
3. If discovery add to D
4. Repeat

Where $\tau(t)$ is the time of the most recent discovery before time t and $\tau(t)$ starts by being set to zero. LBORD has consistent power over time because the β weight is reset after each discovery.

TODO: Maybe add more interpretation of LBOND / LBORD

Methods

Bayesian FDR

Here we propose to apply a Bayesian approach to FDR control to the online testing setting. Specifically we follow the work of Efron and model our streaming data as test statistics coming from a mixture model (Efron and Tibshirani 2002). A Bayesian approach to FDR control considers an underlying mixture distribution consisting of *null*, coming from the null hypothesis, and *signal*, coming from the alternate hypothesis, components, and controls FDR based on the parameters of this distribution:

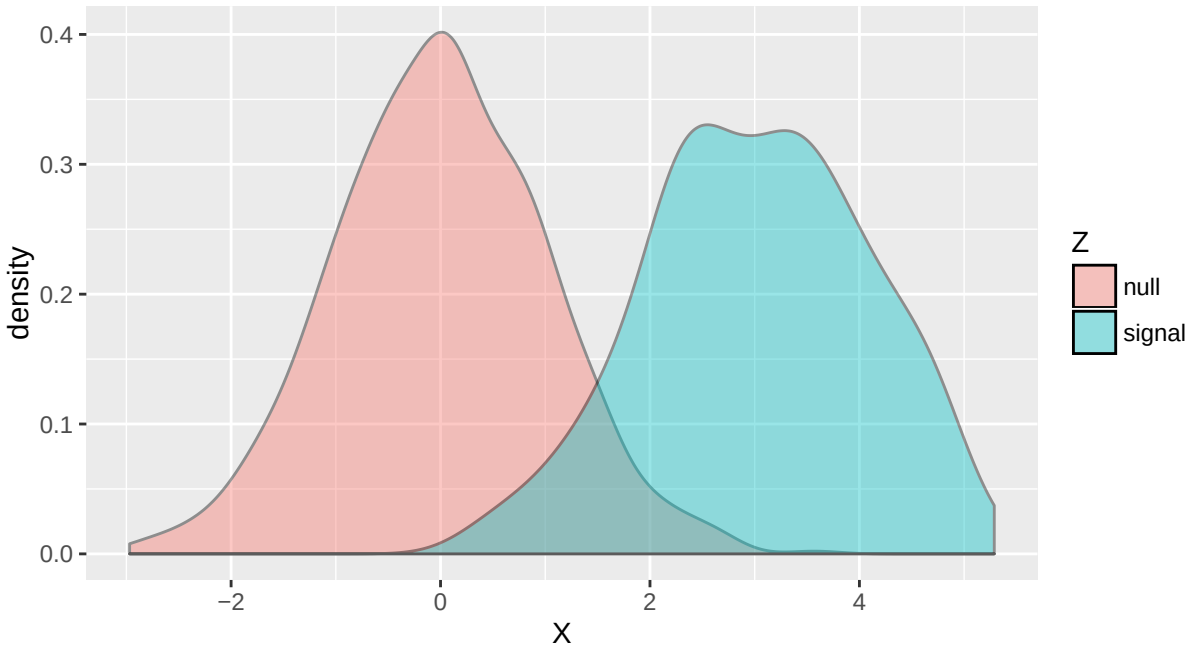


Figure 1*: Density of a mixture distribution of Gaussians simulated with 80% proportion of nulls, the mean of the signal component at 3, variance of the signal component at 1, the mean of the null component at 0 and the variance of the null component at 1*

Let:

X - be a test statistic
π_0 - be the proportion of nulls
μ_1 - be the mean of the signals
σ_1^2 - be the variance of the signals

X can be modeled as a mixture of Gaussians:

$$X \mid \pi_0, \mu_1, \sigma_1^2 \sim \pi_0 N(0,1) + (1-\pi_0)N(\mu_1, \sigma_1^2)$$

In *Figure 1* we can see a plot the resulting density of a simulated mixture model with the underlying parameters $\theta = \{\mu_0 =, \sigma_0^2 = 1, \pi_0 = .8, \mu_1 = 3, \sigma_1^2 = 1\}$. We can see that assuming the data comes from an underlying mixture model with diverged means between the signal and null components can provide valuable information and flexible approaches to controlling FDR. Particularly if we assume that we only know the parameters of the null component. We can find the Bayesian interpretation of FDR by (following homework 2):

$$\hat{FDR}(x) = E[\hat{FDP}(x)]$$

the expectation of the denominator is

$$E[\#X \geq x] = nP(X \geq x)$$

because this should be binomially distributed with size n and $p = P(X \geq x)$ thus one can rewrite the $\hat{FDR}(x)$ as

$$\begin{aligned}\hat{FDR}(x) &= E[\frac{n(1-\Phi(x))}{n(\pi_0(1-\Phi(x)) + (1-\pi_0)(1-\Phi(\frac{x-\mu_1}{\sigma_1}))}] \\ &= E[\frac{(1-\Phi(x))}{(\pi_0(1-\Phi(x)) + (1-\pi_0)(1-\Phi(\frac{x-\mu_1}{\sigma_1}))}]\end{aligned}$$

This is a constant thus

$$= \frac{(1-\Phi(x))}{(\pi_0(1-\Phi(x)) + (1-\pi_0)(1-\Phi(\frac{x-\mu_1}{\sigma_1}))}$$

We can see that if we estimate π_0, μ_1, σ_1^2 then we can control FDR at a given level:

$$\alpha = \frac{\pi_0(1-\Phi(\hat{x}))}{\pi_0(1-\Phi(\hat{x})) + (1-\pi_0)\left(1-\Phi\left(\frac{\hat{x}-\mu_1}{\sigma_1}\right)\right)}$$

Where we reject X if $X > \hat{x}$.

Markov Chain Monte Carlo (Gibbs Sampler)

We apply this mixture model framework to online testing by estimating the unknown parameters of the Gaussian mixture model described above at each time point t. Specifically we use a Markov Chain Monte Carlo approach to sample from the posterior distributions of the unknown parameters $\theta = \{\pi_0, \mu_1, \sigma_1^2\}$.

Let:

t - time index of a test statistic streaming in
X - a vector of t test statistics that have streamed in
X_t - the test statistic at the t^{th} time point
Z - vector of latent states of X_t being a signal or null
Z_t - latent state at time t of X_t being a signal or null
π_0 - proportion of nulls
μ_1 - mean of the signals
σ_1^2 - variance of the signals

As described above we model X_t as a mixture of Gaussians:

$$X_t \mid \pi_0, \mu_1, \sigma_1^2 \sim \pi_0 N(0,1) + (1-\pi_0)N(\mu_1, \sigma_1^2)$$
$$X_t \mid Z_t = 0 \sim N(0,1)$$
$$X_t \mid Z_t = 1, \mu_1, \sigma_1^2 \sim N(\mu_1, \sigma_1^2)$$

We can reparameterize this model in terms of the precision ϕ_1 of the signals and write down the likelihood of the model conditioned on the latent indicators as:

$$L(\pi_0, \mu_1, \sigma_1^2 \mid X, Z) \propto (\pi_0)^{n_0} exp(-\frac{1}{2}\sum_{t:z_t=0} x_t^2) \cdot (1-\pi_0)^{n_1} exp(-\frac{\phi_1}{2}\sum_{t:z_t=1}(x_t-\mu_1)^2)$$

where n_0 and n_1 are the number of observed nulls and signals respectively. We can then set priors on π_0, μ_1, ϕ_1 which satisfy conjugacy:

$$\pi_0 \sim Beta(\alpha, \beta)$$
$$\phi_1 \sim Gamma(\frac{a}{2}, \frac{b}{2})$$
$$\mu_1 \mid \phi_1 \sim Normal(\mu^*, \frac{1}{\alpha^* \phi_1})$$

thus the posterior distributions of these parameters can be written as:

$$\pi_0 \mid X, Z = 0 \sim Beta(\alpha + n_0, \beta + n_1)$$
$$\phi_1 \mid X, Z \sim Gamma(\frac{a+n_1}{2}, b + \sum_{t:z_t=1}(x_t - \mu_1)^2)$$

$$\mu_1 \mid X, Z, \phi_1 \sim Normal(\frac{\alpha^*\mu^* + n_1 + \bar{x_1}}{\alpha^* + n_1}, \frac{1}{(\alpha^* + n_1)\phi_1})$$

We also need to sample from the posterior of Z due to the conditional dependencies above:

$$P(Z_t \mid X_t = x_t, \pi_0, \mu_1, \phi_1) = \frac{\pi_0 exp(-\frac{x_t^2}{2})}{\pi_0 exp(-\frac{x_t^2}{2}) + ((1-\pi_0)\phi_1 exp(-\frac{\phi_1}{2}(x_t - \mu_1)^2))}$$

We proceed to run the Gibbs sampling algorithim using the above posterior distributions:

1. Set $\pi_0^{(0)}$, $\mu_1^{(0)}$ and $\phi_1^{(0)}$
2. Update Z by sampling from its posterior conditioned on X and the current values π_0, μ_1, and ϕ_1
3. Update π_0 by sampling from its posterior conditioned on X
4. Update ϕ_1 by sampling from its posterior conditioned on X and the current value of Z
5. Update μ_1 by sampling from its posterior conditioned on X and ϕ_1

Results

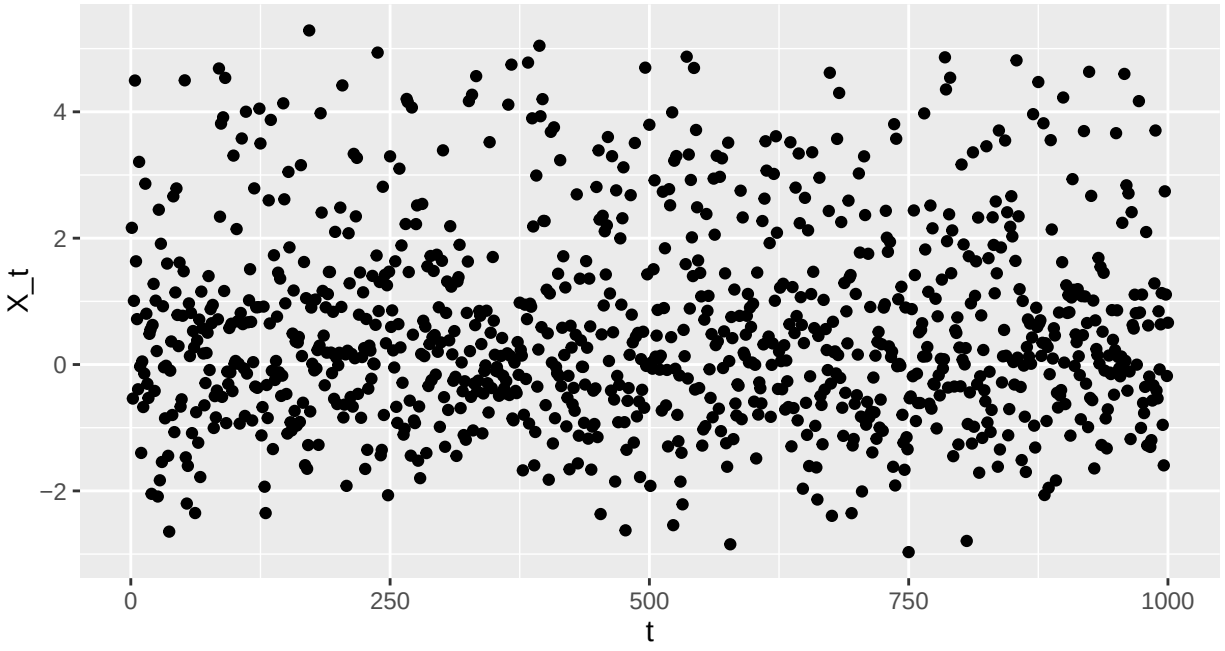
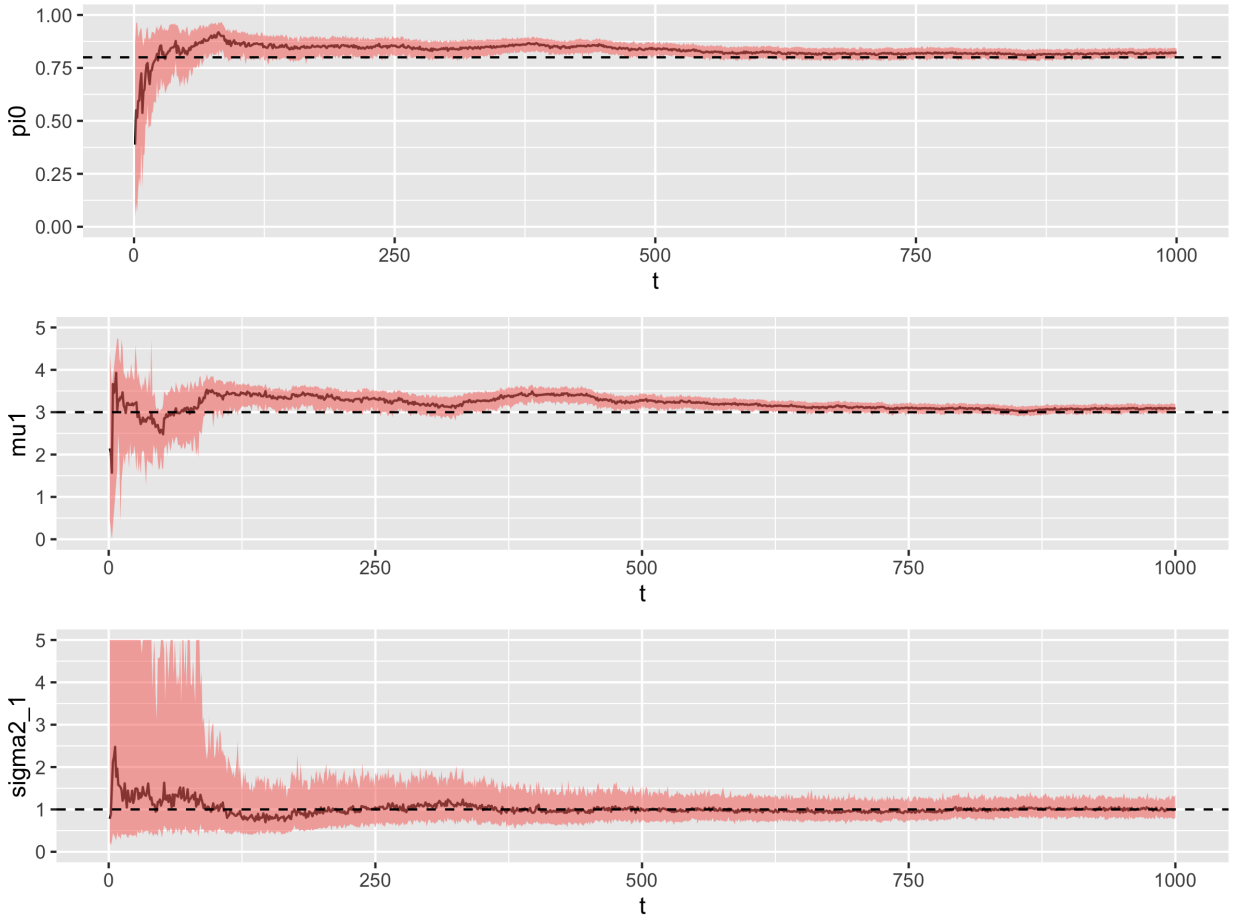


Figure 2:


pi0
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0.50
0.25
0.00
0
250
500
750
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t
mu1
5
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3
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sigma2_1
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***Figure 3**:*

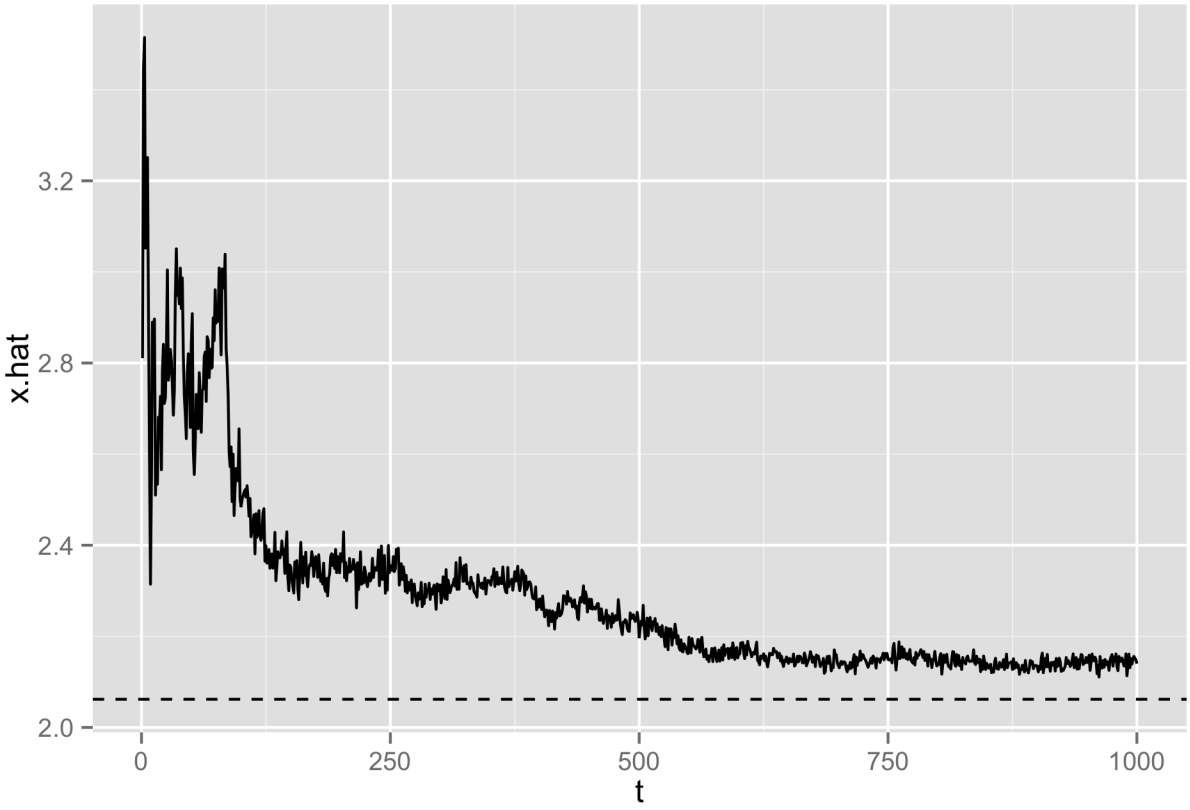

Figure 4:

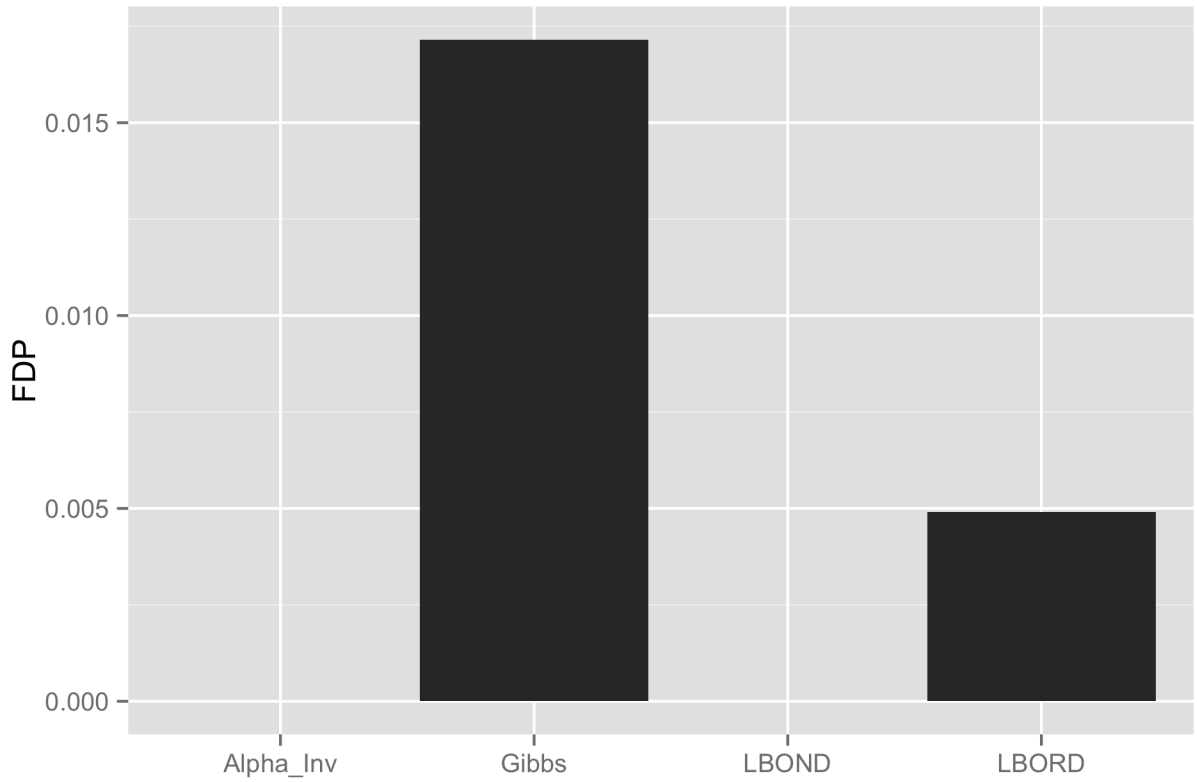

Figure 5:

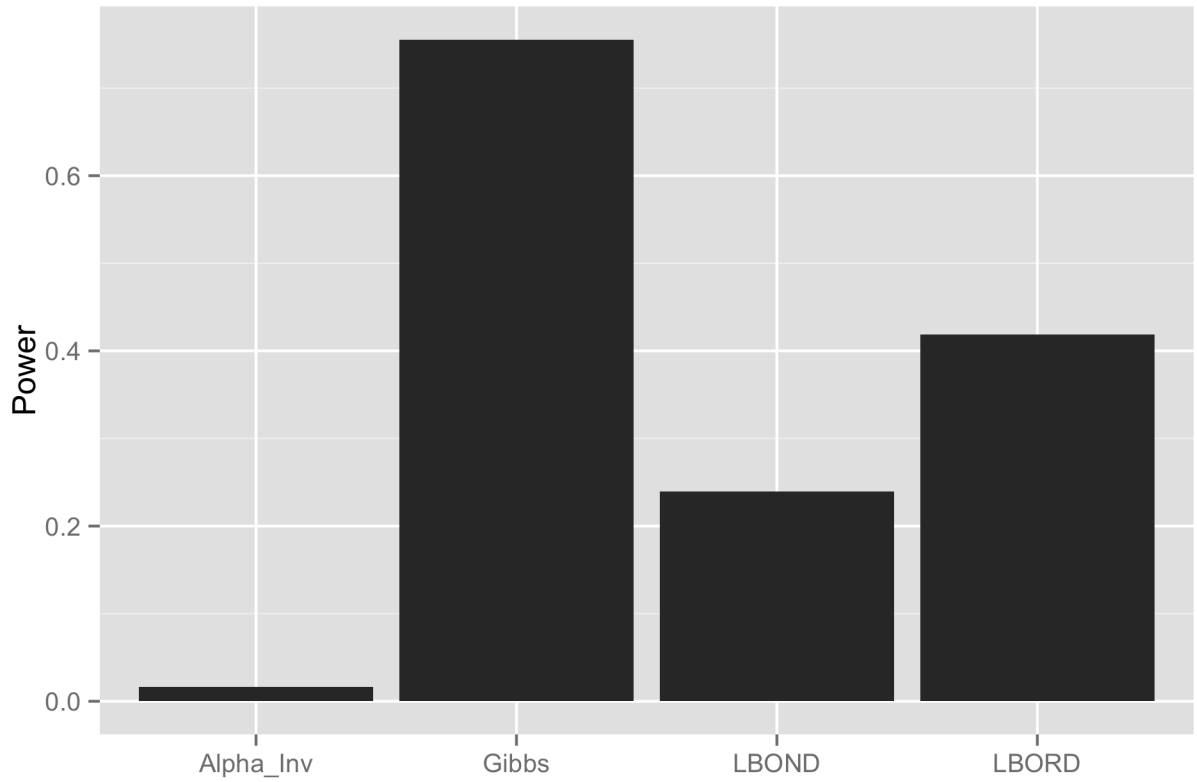

***Figure 6**:*

Conclusion

References

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