Satisfiability Modulo Theories
Lezione 10 - Summary of the course and exercises

(slides revision: Saturday 14^{th} March, 2015, 11:46)

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19 Ottobre 2012

Outline

Solving SMT formulæ by reduction to SAT

Approaches to solve SMT formulæ are based on the observation that SMT can be **reduced** to SAT, i.e., the purely Boolean Satisfiability Problem
Consider for instance the $\mathcal{LIA}$ formula

$$\varphi \equiv (x - y \leq 0) \wedge (y - z \leq 0) \wedge ((z - x \leq -1) \vee (z - x \leq -2))$$

We may use a Boolean variable a to mean "$x - y \leq 0$" evaluates to $\top$ in the model.
Similarly we could use b, c, d for the other $\mathcal{T}$-atoms, obtaining

$$\psi \equiv a \wedge b \wedge (c \vee d)$$

However we are not done with the encoding ! In fact altough $\mu^{\mathbb{B}} \equiv \{a, b, c\}$ is a (Boolean) model for ψ, the correspondent set of $\mathcal{T}$-atoms $\{x - y \leq 0, y - z \leq 0, z - x \leq -1\}$ is not consistent in $\mathcal{LIA}$: we cannot extend $\mu^{\mathbb{B}}$ to a model μ that satisfies φ.
This information can be added to the encoding in the following form

$$\neg(a \wedge b \wedge c)$$

Similarly we may derive **all** the remaining incompatibilities

$$\neg(a \wedge b \wedge d) \qquad \neg(\neg a \wedge \neg b \wedge \neg c) \qquad \neg(\neg a \wedge \neg b \wedge \neg d)$$

Solving SMT formulæ by reduction to SAT

Initial $\mathcal{LIA}$ formula

$$\varphi \equiv (x - y \leq 0) \wedge (y - z \leq 0) \wedge ((z - x \leq -1) \vee (z - x \leq -2))$$

Putting all the conditions together we get the Boolean formula

$$\psi \equiv \; a \wedge b \wedge (c \vee d) \;\wedge\; \neg(a \wedge b \wedge c) \;\wedge\; \neg(a \wedge b \wedge d) \;\wedge\; \neg(\neg a \wedge \neg b \wedge \neg c) \;\wedge\; \neg(\neg a \wedge \neg b \wedge \neg d)$$

Starting from φ we have

- (i) encoded the structure of φ
- (ii) **exhaustively** encoded all incompatible relations between $\mathcal{T}$-atoms

Theorem (Exercise 4 - correctness of the encoding)

φ is $\mathcal{T}$-satisfiable $\Leftrightarrow$ ψ is satisfiable, where ψ is obtained from φ with the steps (i)-(ii)

Exercise 4 - Proof

(a_i is the Boolean variable corresponding to a $\mathcal{T}$-atom P_i)

($\Rightarrow$)

If φ is $\mathcal{T}$-satisfiable, then it means that a model μ exists. A model $\mu^{\mathbb{B}}$ for ψ can be defined with $\mu^{\mathbb{B}}(a_i) = \mu(P_i)$.

($\Leftarrow$)

Suppose that ψ is satisfiable but φ is not. If so, then there is a model $\mu^{\mathbb{B}}$ (e.g., $\{\neg a_1, a_3\}$) such that its encoding (e.g., $(\neg P_1 \wedge P_3)$) represents an incompatible relation of $\mathcal{T}$-atoms. But if it is an incompatible relation, than its negation was added to ψ (e.g, $\neg(a_1 \wedge a_3)$), and it should not be satisfied by $\mu^{\mathbb{B}}$. Contradiction.

Exercize 2

Given the unsatisfiable formula

$$\psi \equiv \; a \wedge b \wedge (c \vee d) \;\wedge\; \neg(a \wedge b \wedge c) \;\wedge\; \neg(a \wedge b \wedge d) \;\wedge\; \neg(\neg a \wedge \neg b \wedge \neg c) \;\wedge\; \neg(\neg a \wedge \neg b \wedge \neg d)$$

show that $\neg(\neg a \wedge \neg b \wedge \neg c)$ and $\neg(\neg a \wedge \neg b \wedge \neg d)$ are redundant.

The last two clauses are redundant if

$$a \wedge b \wedge (c \vee d) \;\wedge\; \neg(a \wedge b \wedge c) \;\wedge\; \neg(a \wedge b \wedge d)$$

is already unsatisfiable on its own. In every model a and b must be assigned to $\top$. This simplifies the formula to

$$(c \vee d) \;\wedge\; \neg(c) \;\wedge\; \neg(d)$$

Now in every model c and d must be assigned to $\bot$. This simplifies the formula to

$$\bot$$

SAT-Solving

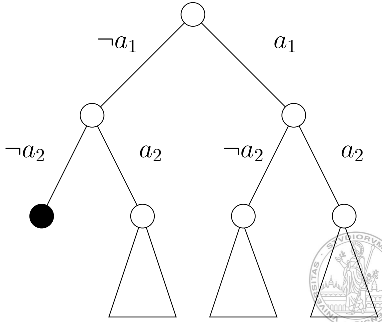
SAT-Solving is (the art of) finding the assignment satisfying all clauses

The assignment evolves **incrementally** by taking **Decisions**, starting from empty { }, but it can be **backtracked** if found wrong

The evolution of the assignment can be represented as a **tree**

$(\neg a_1 \vee a_3 \vee a_9)$
$(\neg a_2 \vee \neg a_3 \vee a_4)$
$(a_1 \vee a_2)$
$(\neg a_4 \vee a_5 \vee a_{10})$

$\{\ldots\}$

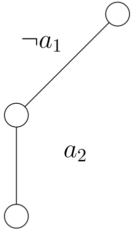


SAT-Solving

There are assignments which can be trivially driven towards the right direction. In the example below, given the current assignment, the third clause can be satisfied only by setting $a_2 \mapsto \top$

$$
\begin{array}{l}
(\neg a_1 \vee a_3 \vee a_9) \\
(\neg a_2 \vee \neg a_3 \vee a_4) \\
(a_1 \vee a_2) \\
(\neg a_4 \vee a_5 \vee a_{10})
\end{array}
$$

$\{\neg a_1, a_2\}$



This step is called **Boolean Constraint Propagation** (BCP). It represents a **forced** deduction. It triggers whenever all literals but one are assigned $\bot$

$$(\neg a_1 \vee a_2 \vee \neg a_3 \vee a_4 \vee \neg a_5)$$

SAT-Solving

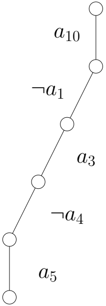
Decision-level: in an assignment, it is the number of decisions taken

Clearly, BCPs do not contribute to increase the decision level

When necessary, we will indicate decision level on top of literals $\overset{5}{a}$ in the assignment

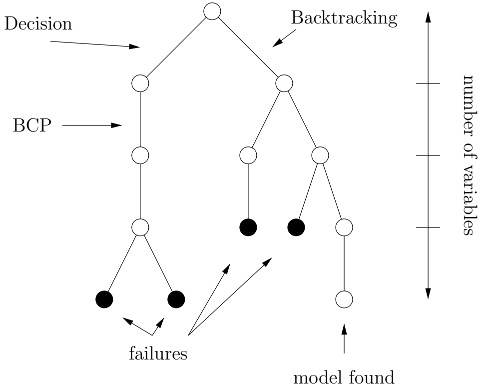
Example:

$$\{\overset{0}{a_{10}}, \overset{1}{\neg a_1}, \overset{2}{a_3}, \overset{3}{\neg a_4}, \overset{3}{a_5}\}$$



SAT-Solving

The process of finding the satisfying assignment is called **search**, and in its basic version it evolves with **Decisions**, **BCPs**, and **backtracking**



Consider the following scenario before and after BCP

$$\begin{array}{l} \ldots \\ (\neg a_{10} \vee \neg a_1 \vee a_4) \\ (a_3 \vee \neg a_1 \vee a_5) \\ (\neg a_4 \vee a_6) \\ (\neg a_5 \vee \neg a_6) \\ \ldots \end{array} \qquad \begin{array}{l} \ldots \\ (\neg a_{10} \vee \neg a_1 \vee a_4) \\ (a_3 \vee \neg a_1 \vee a_5) \\ (\neg a_4 \vee a_6) \\ (\neg a_5 \vee \neg a_6) \\ \ldots \end{array}$$

$$\{\overset{0}{a_{10}}, \overset{1}{\neg a_3}, \overset{2}{a_7}, \overset{3}{\neg a_2}, \overset{4}{a_1}\} \qquad \{\overset{0}{a_{10}}, \overset{1}{\neg a_3}, \overset{2}{a_7}, \overset{3}{\neg a_2}, \overset{4}{a_1}, \overset{4}{a_4}, \overset{4}{a_5}, \overset{4}{a_6}\}$$

BCP leads to a conflict, but, what is the **reason** for it ? Do a_7 and a_2 play any role ?

Does it make sense to consider the assignments (which backtracking would produce) ?
$\{\overset{0}{a_{10}}, \overset{1}{\neg a_3}, \overset{2}{\neg a_7}, \overset{3}{\neg a_2}, \overset{4}{a_1}\}$ — $\{\overset{0}{a_{10}}, \overset{1}{\neg a_3}, \overset{2}{a_7}, \overset{3}{\neg a_2}, \overset{4}{a_1}\}$ — $\{\overset{0}{a_{10}}, \overset{1}{\neg a_3}, \overset{2}{a_7}, \overset{3}{a_2}, \overset{4}{a_1}\}$

No, because *whenever* a_{10}*,* $\neg a_3$ *is assigned, then* a_1 *must not be set to* $\top$. This translates to an additional clause, which can be *learnt*, i.e., it can be added to the formula

$$(\neg a_{10} \vee a_3 \vee \neg a_1)$$

SAT-Solving

In practice, we use **resolution** steps to compute the learnt clause:

- we start from the clause with all literals to $\bot$ (conflicting clause)
- iteratively, we take the clause that propagated the last literal on the trail and we apply resolution
- we stop when only one literal from the current decision level is left in the clause

$$\dfrac{\dfrac{\dfrac{(\neg a_5 \lor \neg a_6) \quad (\neg a_4 \lor a_6)}{(\neg a_5 \lor \neg a_4)} \quad (a_3 \lor \neg a_1 \lor a_5)}{(\neg a_4 \lor a_3 \lor \neg a_1)} \quad (\neg a_{10} \lor \neg a_1 \lor a_4)}{(\neg a_{10} \lor a_3 \lor \neg a_1)}$$

Trail	dl	Reason
a_{10}	0	(a_{10})
$\neg a_3$	1	Decision
a_7	2	Decision
$\neg a_2$	3	Decision
a_1	4	Decision
a_4	4	$(\neg a_{10} \lor \neg a_1 \lor a_4)$
a_5	4	$(a_3 \lor \neg a_1 \lor a_5)$
a_6	4	$(\neg a_4 \lor a_6)$

$$\{\overset{0}{a_{10}}, \overset{1}{\neg a_3}, \overset{2}{a_7}, \overset{3}{\neg a_2}, \overset{4}{a_1}, \overset{4}{a_4}, \overset{4}{a_5}, \overset{4}{a_6}\}$$

We say that $(\neg a_{10} \lor a_3 \lor \neg a_1)$ is the **conflict clause** and it is the one we learn

Consider the following clause set and assignment

$$
\begin{array}{l}
(\neg a_1 \vee a_2) \\
(\neg a_1 \vee a_3 \vee a_9) \\
(\neg a_2 \vee \neg a_3 \vee a_4) \\
(\neg a_4 \vee a_5 \vee a_{10}) \\
(\neg a_4 \vee a_6 \vee a_{11}) \\
(\neg a_5 \vee \neg a_6) \\
(a_1 \vee a_7 \vee \neg a_{12}) \\
(a_1 \vee a_8) \\
(\neg a_7 \vee \neg a_8 \vee \neg a_{13}) \\
\ldots
\end{array}
$$

$$\{\overset{1}{\neg a_9}, \overset{2}{a_{12}}, \overset{2}{a_{13}}, \overset{3}{\neg a_{10}}, \overset{3}{\neg a_{11}}, \ldots, \overset{6}{a_1}\}$$

- Find the correct conflict clause
- Find the correct decision level to backtrack

Lecture 3 - Exercize 1

Trail	dl	Reason
$\neg a_9$	1	Decision
a_{12}	2	Decision
a_{13}	2	(some clause)
$\neg a_{10}$	3	Decision
$\neg a_{11}$	3	(some clause)
$\dots$		
a_1	6	Decision
a_2	6	$(\neg a_1 \vee a_2)$
a_3	6	$(\neg a_1 \vee a_3 \vee a_9)$
a_4	6	$(\neg a_2 \vee \neg a_3 \vee a_4)$
a_5	6	$(\neg a_4 \vee a_5 \vee a_{10})$
a_6	6	$(\neg a_4 \vee a_6 \vee a_{11})$

$(\neg a_1 \vee a_2)$
$(\neg a_1 \vee a_3 \vee a_9)$
$(\neg a_2 \vee \neg a_3 \vee a_4)$
$(\neg a_4 \vee a_5 \vee a_{10})$
$(\neg a_4 \vee a_6 \vee a_{11})$
$(\neg a_5 \vee \neg a_6)$
$(a_1 \vee a_7 \vee \neg a_{12})$
$(a_1 \vee a_8)$
$(\neg a_7 \vee \neg a_8 \vee \neg a_{13})$

$$\{\overset{1}{\neg a_9}, \overset{2}{a_{12}}, \overset{2}{a_{13}}, \overset{3}{\neg a_{10}}, \overset{3}{\neg a_{11}}, \dots, \overset{6}{a_1}\}$$

$$\frac{(\neg a_5 \vee \neg a_6) \qquad (\neg a_4 \vee a_6 \vee a_{11})}{(\neg a_5 \vee \neg a_4 \vee a_{11})}$$

$$\frac{(\neg a_5 \vee \neg a_4 \vee a_{11}) \qquad (\neg a_4 \vee a_5 \vee a_{10})}{(\neg a_4 \vee a_{10} \vee a_{11})}$$

$$\frac{(\neg a_4 \vee a_{10} \vee a_{11}) \qquad (\neg a_2 \vee a_3 \vee a_4)}{(\neg a_2 \vee \neg a_3 \vee a_{10} \vee a_{11})}$$

$$\frac{(\neg a_2 \vee \neg a_3 \vee a_{10} \vee a_{11}) \qquad (\neg a_2 \vee a_3 \vee a_9)}{(\neg a_2 \vee a_9 \vee a_{10} \vee a_{11})}$$

$$\frac{(\neg a_2 \vee a_9 \vee a_{10} \vee a_{11}) \qquad (\neg a_1 \vee a_2)}{(\neg a_1 \vee a_9 \vee a_{10} \vee a_{11})}$$

Conflict clause: $(\neg a_1 \vee a_9 \vee a_{10} \vee a_{11})$
Backtracking level: 3

The Lazy Approach

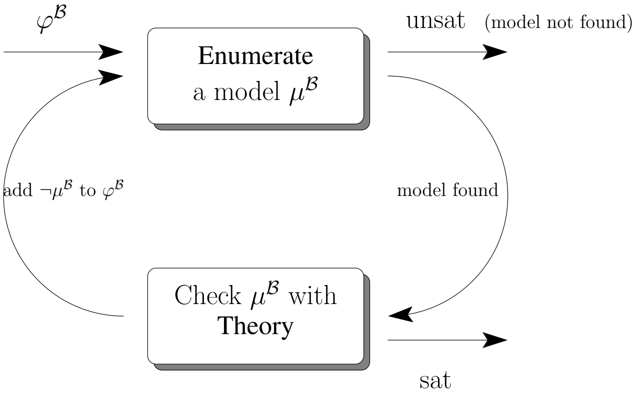
Notice that

- Assignments μ of φ are many (potentially ∞), infeasible to check if any of them is a model **systematically**
- Models $\mu^{\mathcal{B}}$ of $\varphi^{\mathcal{B}}$ are finite in number, and easy to enumerate with a SAT-solver
- A model $\mu^{\mathcal{B}}$ is nothing but a **conjunction of $\mathcal{T}$-atoms**, can be checked efficiently with a $\mathcal{T}$-solver

These observations suggest us a methodology to tackle the SMT($\mathcal{T}$) problem

- Enumerate a Boolean model $\mu^{\mathcal{B}}$ of $\varphi^{\mathcal{B}}$ (abstraction). If no model exist we are done (φ is unsatisfiable)
- Check if $\mu^{\mathcal{B}}$ is satisfiable using the $\mathcal{T}$-solver. If so $\mu^{\mathcal{B}}$ can be extended to a model μ of φ, and so we are done ! (φ is satisfiable)
- It not, we tell the SAT-solver not to enumerate $\mu^{\mathcal{B}}$ again, thus **cutting away systematically an infinite number** of assignments for φ (refinement)
- It can be blocked by adding a clause $\neg\mu^{\mathcal{B}}$. Go up
- It terminates because there are finite Boolean models

The lazy approach falls into the so-called **abstraction-refinement** paradigm



The Lazy Approach

The interaction described naturally falls within the CDCL style, enriched with a $\mathcal{T}$-solver

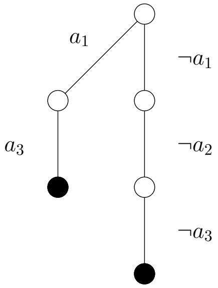
$$\varphi \equiv (x = 3 \vee \neg(x < 3)) \;\wedge\; (x = 3 \vee \neg(x > 3)) \;\wedge\; (x > 3 \vee \neg(x < 3)) \;\wedge\; (x > 3 \vee \neg(x = 3))$$

$$\varphi^{\mathcal{B}} \equiv \begin{array}{l} (a_1 \vee \neg a_2) \\ (a_1 \vee \neg a_3) \\ (a_3 \vee \neg a_2) \\ (a_3 \vee \neg a_1) \\ (\neg a_1 \vee \neg a_3) \\ (\neg a_1) \\ (a_1 \vee a_2 \vee a_3) \\ (\ ) \end{array}$$

$$\begin{array}{l} a_1 \equiv x = 3 \\ a_2 \equiv x < 3 \\ a_3 \equiv x > 3 \end{array}$$

$\mu^{\mathcal{B}}$: $\{\ \ \}$

SAT-solver:	UNS
$\mathcal{T}$-solver:	Idle



$(a_1 \vee \neg a_2)$
$(a_1 \vee \neg a_3)$
$(a_3 \vee \neg a_2)$
$(a_3 \vee \neg a_1)$
$(\neg a_1 \vee \neg a_3)$

Trail	dl	Reason
a_1	1	Decision
a_3	1	$(a_3 \vee \neg a_1)$

$\{\overset{1}{a_1}, \overset{1}{a_3}\}$

$$\frac{(\neg a_1 \vee \neg a_3) \qquad (a_3 \vee \neg a_1)}{(\neg a_1)}$$

Conflict clause: $(\neg a_1)$
Backtracking level: 0

$(a_1 \vee \neg a_2)$
$(a_1 \vee \neg a_3)$
$(a_3 \vee \neg a_2)$
$(a_3 \vee \neg a_1)$
$(\neg a_1 \vee \neg a_3)$
$(\neg a_1)$
$(a_1 \vee a_2 \vee a_3)$

Trail	dl	Reason
$\neg a_1$	0	$(\neg a_1)$
$\neg a_2$	0	$(a_1 \vee \neg a_2)$
$\neg a_3$	0	$(a_1 \vee \neg a_3)$

$\{\neg \overset{0}{a_1}, \neg \overset{0}{a_2}, \neg \overset{0}{a_3}\}$

$$\dfrac{\dfrac{\dfrac{(a_1 \vee a_2 \vee a_3) \qquad (a_3 \vee \neg a_1)}{(a_1 \vee a_2)} \qquad (a_1 \vee \neg a_2)}{(a_1)} \qquad (\neg a_1)}{\bot}$$

Conflict clause: $\bot$
Backtracking level: 0

A $\mathcal{T}$-solver for $\mathcal{IDL}$

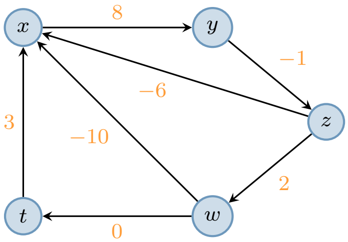
The constraint $x - y \leq c$ says that

"the distance between x and y is at most c"

This can be encoded as $(x, y; c)$

So, a set $\mu^{\mathcal{B}}$ can be encoded as a graph. Concrete example:

$$\begin{array}{rcr} x - y & \leq & 8 \\ y - z & \leq & -1 \\ z - x & \leq & -6 \\ z - w & \leq & 2 \\ w - x & \leq & -10 \\ w - t & \leq & 0 \\ t - x & \leq & 3 \end{array}$$



$G(V, E)$: $V = \{x, y, z, w, t\}$, $E = \{(x, y; 8), (y, z; -1), (z, x; -6), (z, w; 2), (w, x; -10), \ldots\}$

A $\mathcal{T}$-solver for $\mathcal{IDL}$

Theorem (Translation)

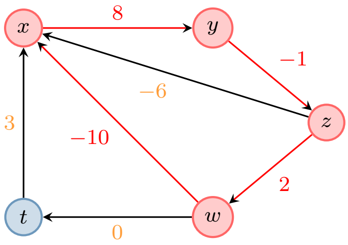
$\mu^{\mathcal{B}}$ is $\mathcal{IDL}$-unsatisfiable

iff

there is a **negative cycle** *in the corresponging graph $G(V, E)$*

E.g.:

$$
\begin{array}{rcr}
x - y & \leq & 8 \\
y - z & \leq & -1 \\
z - x & \leq & -6 \\
z - w & \leq & 2 \\
w - x & \leq & -10 \\
w - t & \leq & 0 \\
t - x & \leq & 3
\end{array}
$$



Lecture 5 - Exercize 4

Let's first recall the notion of **minimality**

A conflict $\nu^{\mathbb{B}}$ is **minimal** if it does not contain redundant $\mathcal{T}$-atoms

A $\mathcal{T}$-atom P in a conflict $\nu^{\mathbb{B}}$ is redundant if $\nu^{\mathbb{B}} \setminus \{P\}$ is still a conflict

So, how do we check, in general, that a conflict $\nu^{\mathbb{B}} = \{P_1, \ldots, P_n\}$ is minimal ? Iteratively for $i = 1, \ldots, n$, we see if $\nu^{\mathbb{B}} \setminus \{P_i\}$ is still a conflict.

In the case of difference logic every conflict is minimal **by construction**. In fact $\nu^{\mathbb{B}}$ is a conflict if and only if it is a **cycle** with negative sum.

Doing $\nu^{\mathbb{B}} \setminus \{P_i\}$ is equivalent to breaking the cycle, no matter what P_i. Therefore all $\mathcal{T}$-atoms are not redundant, and so conflicts are minimal.

Prove

Observation 2

A set of constraints
$\{ x_1 - x_2 \leq c_1,\ x_2 - x_3 \leq c_2,\ \ldots,\ x_{n-1} - x_n \leq c_{n-1} \}$
implies $\quad x_1 - x_n \leq c_n \quad$ iff $\quad c_1 + c_2 + \ldots c_{n-1} \leq c_n$

using

Lemma (Farka's Lemma for $\mathcal{IDL}$)

$\mu^{\mathcal{B}}$ is unsatisfiable iff there exists a subset
$\nu^{\mathcal{B}} = \{ x_1 - x_2 \leq c_1,\ x_2 - x_3 \leq c_2,\ \ldots,\ x_n - x_1 \leq c_n \}$ of $\mu^{\mathcal{B}}$ such that $c_1 + \ldots + c_n < 0$

Lecture 5 - Exercize 5

Observation 2

A set of constraints
$\{\ x_1 - x_2 \leq c_1,\ x_2 - x_3 \leq c_2,\ \dots,\ x_{n-1} - x_n \leq c_{n-1}\ \}$
implies $\quad x_1 - x_n \leq c_n \quad$ iff $\quad c_1 + c_2 + \dots c_{n-1} \leq c_n$

is better formalized as

$$(x_1 - x_2 \leq c_1 \ \wedge\ x_2 - x_3 \leq c_2 \ \wedge\ \dots \ \wedge\ x_{n-1} - x_n \leq c_{n-1}) \rightarrow (x_1 - x_n \leq d_n)$$

is valid iff

$$c_1 + c_2 + \dots c_{n-1} \leq d_n$$

Lemma (Farka's Lemma for $\mathcal{IDL}$)

$\mu^{\mathcal{B}}$ is unsatisfiable iff there exists a subset
$\nu^{\mathcal{B}} = \{\ x_1 - x_2 \leq c_1,\ x_2 - x_3 \leq c_2,\ \dots,\ x_n - x_1 \leq c_n\ \}$ of $\mu^{\mathcal{B}}$ such that $c_1 + \dots + c_n < 0$

is better formalized as

$$x_1 - x_2 \leq c_1 \ \wedge\ x_2 - x_3 \leq c_2 \ \wedge\ \dots \ \wedge\ x_{n-1} - x_n \leq c_{n-1} \ \wedge\ x_n - x_1 \leq c_n$$

is unsatisfiable iff

$$c_1 + c_2 + \dots c_{n-1} + c_n < 0$$

$$(x_1 - x_2 \leq c_1 \ \wedge \ x_2 - x_3 \leq c_2 \ \wedge \ \ldots \ \wedge \ x_{n-1} - x_n \leq c_{n-1}) \rightarrow (x_1 - x_n \leq d_n)$$
is valid iff
$$c_1 + c_2 + \ldots c_{n-1} \leq d_n$$

is equivalent to (using the well-known fact: $\varphi \rightarrow \psi$ is valid iff $\varphi \wedge \neg\psi$ is unsat)

$$x_1 - x_2 \leq c_1 \ \wedge \ x_2 - x_3 \leq c_2 \ \wedge \ \ldots \ \wedge \ x_{n-1} - x_n \leq c_{n-1} \ \wedge \ \neg(x_1 - x_n \leq d_n)$$
is unsatisfiable iff
$$c_1 + c_2 + \ldots c_{n-1} \leq d_n$$

which is equivalent to (using the fact $\neg(x - y \leq c) \iff y - x \leq -c - 1$)

$$x_1 - x_2 \leq c_1 \ \wedge \ x_2 - x_3 \leq c_2 \ \wedge \ \ldots \ \wedge \ x_{n-1} - x_n \leq c_{n-1} \ \wedge \ x_n - x_1 \leq -d_n - 1$$
is unsatisfiable iff
$$c_1 + c_2 + \ldots c_{n-1} \leq d_n$$

which is equivalent to (using the fact $c \leq d \iff c - d \leq 0 \iff c - d - 1 < 0$)

$$x_1 - x_2 \leq c_1 \ \wedge \ x_2 - x_3 \leq c_2 \ \wedge \ \ldots \ \wedge \ x_{n-1} - x_n \leq c_{n-1} \wedge x_n - x_1 \leq -d_n - 1$$
is unsatisfiable iff
$$c_1 + c_2 + \ldots c_{n-1} - d_n - 1 < 0$$

which is Farka's Lemma if we set $c_n \equiv -d_n - 1$

Tableau			*lb*		Bounds		*ub*		μ	
	...		-4	$\leq$	x_1	$\leq$	10	x_1	$\mapsto$	12
x_1	$=$	$3x_2 - 4x_3 + 2x_4 - x_5$	1	$\leq$	x_2	$\leq$	3	x_2	$\mapsto$	1
	...		-4	$\leq$	x_3	$\leq$	-1	x_3	$\mapsto$	-1
			1	$\leq$	x_4	$\leq$	2	x_4	$\mapsto$	2
			-1	$\leq$	x_5	$\leq$	10	x_5	$\mapsto$	-1

which among $\mathcal{N} = \{x_2, x_3, x_4\}$ do I choose for pivoting ? Clearly, the value of $\mu(x_1)$ is too high, I have to decrease it by playing with the values of $\mathcal{N}$:

- $3x_2$ cannot decrease, as $\mu(x_2) = lb(x_2)$ and cannot be moved down
- $-4x_3$ cannot decrease, as $\mu(x_3) = ub(x_3)$ and cannot be moved up
- $2x_4$ can decrease, as $\mu(x_4) = ub(x_4)$, and can be moved down
- $-x_5$ can decrease, as $\mu(x_5) = lb(x_5)$, and can be moved up

both x_4 and x_5 are therefore good candidates for pivoting. To avoid loops, choose variable with smallest subscript (Bland's Rule). This rule is not necessarily efficient, though

There might be cases in which **no suitable variable for pivoting can be found**. This indicates unsatisfiability. Consider the following where we have just asserted $x_1 \leq 9$

Tableau	*lb*	Bounds	*ub*	μ
$\begin{array}{rcl} & \dots & \\ x_1 & = & 3x_2 - 4x_3 + 2x_4 - x_5 \\ & \dots & \end{array}$	$\begin{array}{r} -4 \\ 1 \\ -4 \\ 2 \\ -1 \end{array}$	$\begin{array}{ccc} \leq & x_1 & \leq \\ \leq & x_2 & \leq \\ \leq & x_3 & \leq \\ \leq & x_4 & \leq \\ \leq & x_5 & \leq \end{array}$	$\begin{array}{r} 9 \\ 3 \\ -1 \\ 2 \\ -1 \end{array}$	$\begin{array}{ccr} x_1 & \mapsto & 12 \\ x_2 & \mapsto & 1 \\ x_3 & \mapsto & -1 \\ x_4 & \mapsto & 2 \\ x_5 & \mapsto & -1 \end{array}$

no variable among $\mathcal{N} = \{x_2, x_3, x_4\}$ can be chosen for pivoting. This is because (due to tableau)

$$x_2 \geq 1 \;\wedge\; x_3 \leq -1 \;\wedge\; x_4 \geq 4 \;\wedge\; x_5 \leq -1 \;\Rightarrow\; x_1 \geq 12 \;\Rightarrow\; \neg(x_1 \leq 9)$$

Therefore

$$\{x_2 \geq 1,\; x_3 \leq -1,\; x_4 \geq 4,\; x_5 \leq -1,\; \neg(x_1 \leq 9)\}$$

is a $\mathcal{T}$-conflict (modulo the row $x_1 = 3x_2 - 4x_3 + 2x_4 - x_5$)

A conflict returned by the Simplex involves a row

$$x_1 = a_2 x_2 + \ldots + a_n x_n$$

and exactly n bounds

$$\{x_1 \sim_1 b_1, x_2 \sim_2 b_2, \ldots, x_n \sim_n b_n\}$$

with $\sim_i \in \{\leq, \geq\}$.
This conflict is minimal: we show that we can find a model μ if we remove a bound. W.l.o.g., we remove $x_2 \sim_2 b_2$: then we can set $\mu(x_j) = b_j$ for $j \neq 2$. Then we can pivot the row

$$x_2 = c_1 x_1 + c_3 x_3 + \ldots + c_n x_n$$

and compute a suitable value for $\mu(x_2) = c_1\mu(x_1) + \ldots + c_n\mu(x_n)$.
(we can set the value we want to $\mu(x_2)$ because it is unbounded now!)

A $\mathcal{T}$-solver for $\mathcal{UF}$

The solving phase inside the $\mathcal{UF}$-solver happens in two steps

Given a conjunction of $\mathcal{T}$-literals φ to check for satisfiability, the $\mathcal{UF}$-solver

first it constructs **equivalence classes** using φ^+ (the set of positive $\mathcal{T}$-literals), using, for example, the Union-Find algorithm

and then checks one by one the negative $\mathcal{T}$-literals in φ^-

A $\mathcal{T}$-solver for $\mathcal{UF}$

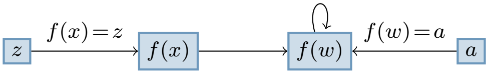
First of all notice that $\mathcal{T}$-conflicts are always of the form

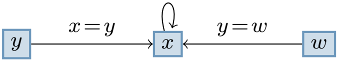
$$\{\ s \neq t, \text{“other equalities that cause } s \text{ and } t \text{ to join the same class”}\}$$

We reconstruct the conflict by storing the reason that caused two classes to collapse. When a $s \neq t$ causes *unsat*, we call a routine Explain(s,t) that collects all the reasons that made s equal to t

Example

$$\{\ x = y,\ y = w,\ f(x) = z,\ f(w) = a,\ a \neq z\ \}$$





On processing $a \neq z$, we call Explain(a, z)

Lecture 7 - Exercize 2

We can store a reason of a Union inside the struct `Node`

```
struct Node {
  ...
  Node *  root;     // Points to class' representant
  Enode * reason;   // T-atom that caused union
};
```

explanation = { } // Global variable

procedure Explain(s, t)

if ($s == t$) return // Nothing to explain ...

while($s.next \neq s$) // Collect reasons while moving to root
  if ($s.reason ==$ NULL) // No reason: union caused by congruence
    // Let $s \equiv f(s_1, \ldots, s_n)$, $s.next \equiv f(t_1, \ldots, t_n)$
    foreach $i \in \{1, \ldots, n\}$
      Explain(s_i, t_i)
  else // Reason exists, save it
    explanation = explanation $\cup$ $\{s.reason\}$
  $s = s.next$

// Same "while" loop for t ...