Intermediate Analytics

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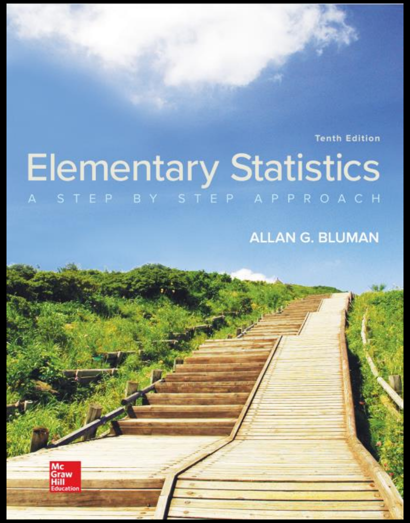
ALY 6015

Nonparametric Statistics

Slides are mainly borrowed from the textbook:

- ***Elementary Statistics: A Step-by-Step Approach**. 10th Edition, **Allen Bluman, McGraw Hill***

You will learn in this course:



After completing this chapter, you should be able to:

1. State the advantages and disadvantages of nonparametric methods.
2. Test hypotheses, using the sign test. Test hypotheses, using the Wilcoxon rank sum test.
3. Test hypotheses, using the signed-rank test. Test hypotheses, using the Kruskal-Wallis test.
4. Compute the Spearman rank correlation coefficient.
5. Test hypotheses, using the runs test.

Introduction

- Statistical tests, such as the *z*, *t*, and *F* tests, are called **<u>parametric tests</u>**. Parametric tests are statistical tests for population parameters such as means, variances, and proportions that involve assumptions about the populations from which the samples were selected.
- One assumption is that these populations are normally distributed. But what if the population in a particular hypothesis-testing situation is not normally distributed?
- Statisticians have developed a branch of statistics known as **<u>nonparametric statistics</u>** or **<u>distribution-free statistics</u>** to use when the population from which the samples are selected is not normally distributed or is distributed in any other particular way. Nonparametric statistics can also be used to test hypotheses that do not involve specific population parameters, such as μ, σ, or p.

Advantages of Nonparametric Statistical Tests

Nonparametric statistical tests are used to test hypotheses about population parameters when the assumption about normality cannot be met.

There are six advantages that nonparametric methods have over parametric methods:

1. They can be used to test population parameters when the variable is not normally distributed.
2. They can be used when the data are nominal or ordinal.
3. They can be used to test hypotheses that do not involve population parameters.
4. In some cases, the computations are easier than those for the parametric counterparts.
5. They are easy to understand.
6. There are fewer assumptions that have to be met, and the assumptions are easier to verify.

Disadvantages of Nonparametric Statistical Tests

There are three disadvantages of nonparametric methods:

1. They are less sensitive than their parametric counterparts when the assumptions of the parametric methods are met. Therefore, larger differences are needed before the null hypothesis can be rejected.
2. They tend to use less information than parametric tests. For example, the sign test requires the researcher to determine only whether the data values are above or below the median, not how much above or below the median each value is.
3. They are less efficient than their parametric counterparts when the assumptions of the parametric methods are met. That is, **larger sample sizes** are needed to overcome the loss of information. For example, the nonparametric sign test is about 60% as efficient as its parametric counterpart, the z test. Thus, a sample size of 100 is needed for use of the sign test, compared with a sample size of 60 for use of the z test to obtain the same results.

Ranking

Many nonparametric tests involve the ranking of data, that is, the positioning of a data value in a data array according to some rating scale. **Ranking** is an ordinal variable. For example, suppose a judge decides to rate five speakers on an ascending scale of 1 to 10, with 1 being the best and 10 being the worst, for categories such as voice, gestures, logical presentation, and platform personality. The ratings are shown in the chart.

Speaker	A	B	C	D	E
Rating	8	6	10	3	1

Speaker	E	D	B	A	C
Rating	1	3	6	8	10
Ranking	1	2	3	4	5

Ranking

Speaker	A	B	C	D	E
Rating	8	6	10	6	3

Speaker	E	D	B	A	C
Rating	3	6	6	8	10
Ranking	1	Tie for 2nd and 3rd		4	5

$$\frac{2+3}{2} = \frac{5}{2} = 2.5$$

Speaker	E	D	B	A	C
Rating	3	6	6	8	10
Ranking	1	2.5	2.5	4	5

The Sign Test

Single-Sample Sign Test

The simplest nonparametric test, the sign test for single samples, is used to test the value of a median for a specific sample.

The **sign test** for a single sample is a nonparametric test used to test the value of a population median.

Test Value for the Sign Test

If $n \leq 25$, the test value is the smaller number of plus or minus signs. When $n > 25$, the test value is

$$z = \frac{(X + 0.5) - 0.5n}{\frac{\sqrt{n}}{2}}$$

where X is the smaller number of plus or minus signs and n is the total number of plus or minus signs.

The Sign Test

- When using the sign test, the researcher hypothesizes the specific value for the median of a population; then he or she selects a random sample of data and compares each value with the conjectured median.
- If the data value is above the conjectured median, it is assigned a plus sign. If the data value is below the conjectured median, it is assigned a minus sign. And if it is exactly the same as the conjectured median, it is assigned a 0.
- Then the numbers of plus and minus signs are compared to determine if they are significantly different.
- If the null hypothesis is true, the number of plus signs should be approximately equal to the number of minus signs.
- If the null hypothesis is not true, there will be a disproportionate number of plus or minus signs. There are two cases for using the sign test. The first case is when the sample size n is less than or equal to 25. The other case is when the sample size n is greater than 25.

The Sign Test

For example, when $n \leq 25$, if there are 8 positive signs and 3 negative signs, the test value is 3. When the sample size is 25 or less, Table J in Appendix A is used to determine the critical value. For a specific α, if the test value is less than or equal to the critical value obtained from the table, the null hypothesis should be rejected. The values in Table J are obtained from the binomial distribution when $p = 0.5$. The derivation is omitted here. When $n > 25$, the normal approximation with Table E can be used for the critical values. In this case, $\mu = np$ or $0.5n$ since $p = 0.5$ and $\sigma = \sqrt{npq}$ or $\sqrt{n}/2$ since p and $q = 0.5$ and $\sqrt{npq} = \sqrt{n(0.5)(0.5)}$ or $0.5\sqrt{n}$ which is the same as $\sqrt{n}/2$.

The Procedure Table for the sign test is given next.

Procedure Table

Performing the Sign Test

Step 1 State the hypotheses and identify the claim.

Step 2 Find the critical value. Use Table J in Appendix A when $n \leq 25$ and Table E when $n > 25$.

Step 3 Compute the test value.

Step 4 Make the decision.

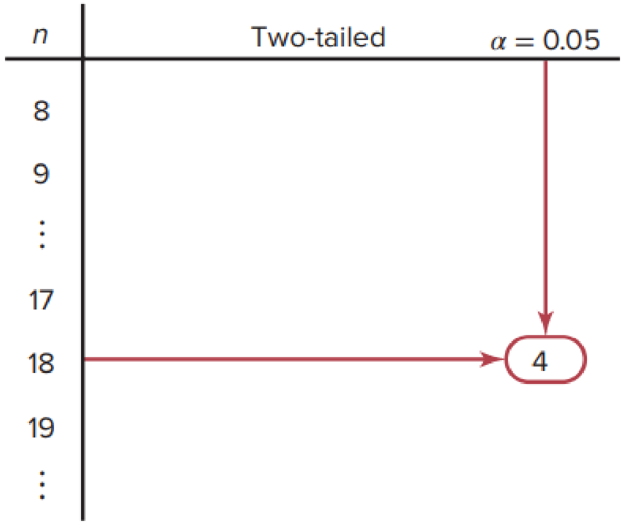
Step 5 Summarize the result.

Example 1 – Patients at a Medical Center

The manager of Green Valley Medical Center claims that the median number of patients seen by doctors who work at the center is 80 per day. To test this claim, 20 days are randomly selected and the number of patients seen is recorded and shown.
At $\alpha = 0.05$, test the claim.

82	85	93	81	80
86	95	89	74	62
72	84	88	81	83
105	80	86	81	87

FIGURE 13–1
Finding the Critical Value in Table J for Example 13–1



Example 1 – Patients at a Medical Center

SOLUTION

Step 1 State the hypotheses and identify the claim.

H_0: Median = 80 (claim).
H_1: Median $\neq$ 80.

Step 2 Find the critical value.
Subtract the hypothesized median, 80, from each data value. If the data value falls above the hypothesized median, assign the value a + sign. If the data value falls below the hypothesized median, assign the data value a − sign. If the data value is equal to the median, assign it a 0.

82 − 80 = +2, so 82 is assigned a + sign.
86 − 80 = +6, so 86 is assigned a + sign.
72 − 80 = −8, so 72 is assigned a − sign.
etc.

The completed table is shown.

+	+	+	+	0
+	+	+	−	−
−	+	+	+	+
+	0	+	+	+

Since $n \leq 25$, refer to Table J in Appendix A. In this case, $n = 20 - 2 = 18$ (There are two zeros) and $\alpha = 0.05$. The critical value for a two-tailed test is 4. See Figure 13–1.

The Sign Test
Table J

TABLE J Critical Values for the Sign Test

Reject the null hypothesis if the smaller number of positive or negative signs is less than or equal to the value in the table.

n	One-tailed, $\alpha = 0.005$ Two-tailed, $\alpha = 0.01$	$\alpha = 0.01$ $\alpha = 0.02$	$\alpha = 0.025$ $\alpha = 0.05$	$\alpha = 0.05$ $\alpha = 0.10$
8	0	0	0	1
9	0	0	1	1
10	0	0	1	1
11	0	1	1	2
12	1	1	2	2
13	1	1	2	3
14	1	2	3	3
15	2	2	3	3
16	2	2	3	4
17	2	3	4	4
18	3	3	4	5
19	3	4	4	5
20	3	4	5	5
21	4	4	5	6
22	4	5	5	6
23	4	5	6	7
24	5	5	6	7
25	5	6	6	7

Note: Table J is for one-tailed or two-tailed tests. The term n represents the total number of positive and negative signs. The test value is the number of less frequent signs.

Example 1 – Patients at a Medical Center

Step 3 Compute the test value. Count the number of + and − signs in step 2, and use the smaller value as the test value. In this case, there are 15 plus signs and 3 minus signs, so the test value is 3.

Step 4 Make the decision. Compare the test value 3 with the critical value 4. If the test value is less than or equal to the critical value, the null hypothesis is rejected. In this case, the null hypothesis is rejected since $3 < 4$.

Step 5 Summarize the results. There is enough evidence to reject the null hypothesis that the median of the number of patients seen per day is 80.

Example 2 - Wave Heights

An oceanographer wishes to test the claim that the median height of waves in a resort town on the Atlantic Ocean is 2.4 feet. A random sample of 50 days shows the heights of the waves on 20 days were at least 2.4 feet.

At $\alpha = 0.05$, test the claim that the median height of the waves is at least 2.4 feet

SOLUTION

Step 1 State the hypotheses and identify the claim.

$$H_0\text{: MD} = 2.4 \qquad \text{and} \qquad H_1\text{: MD} < 2.4$$

Step 2 Find the critical value. Since $\alpha = 0.05$ and $n = 50$, and since the test is left-tailed, the critical value is -1.65, obtained from Table E.

Step 3 Compute the test value.

$$z = \frac{(x + 0.5) - 0.5n}{\sqrt{n}/2} = \frac{(20 + 0.5) - 0.5(50)}{\sqrt{50}/2} = \frac{-4.5}{3.536} = -1.27$$

Step 4 Make the decision. Since the test value of -1.27 is greater than -1.65, the decision is to not reject the null hypothesis.

Step 5 Summarize the results. There is not enough evidence to reject the claim that the median height of the waves is at least 2.4 feet.

The Sign Test
Table E

a = 0.05 →
z= -1.64 Or -1.65

TABLE E The Standard Normal Distribution

Cumulative Standard Normal Distribution

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
−3.4	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0002
−3.3	.0005	.0005	.0005	.0004	.0004	.0004	.0004	.0004	.0004	.0003
−3.2	.0007	.0007	.0006	.0006	.0006	.0006	.0006	.0005	.0005	.0005
−3.1	.0010	.0009	.0009	.0009	.0008	.0008	.0008	.0008	.0007	.0007
−3.0	.0013	.0013	.0013	.0012	.0012	.0011	.0011	.0011	.0010	.0010
−2.9	.0019	.0018	.0018	.0017	.0016	.0016	.0015	.0015	.0014	.0014
−2.8	.0026	.0025	.0024	.0023	.0023	.0022	.0021	.0021	.0020	.0019
−2.7	.0035	.0034	.0033	.0032	.0031	.0030	.0029	.0028	.0027	.0026
−2.6	.0047	.0045	.0044	.0043	.0041	.0040	.0039	.0038	.0037	.0036
−2.5	.0062	.0060	.0059	.0057	.0055	.0054	.0052	.0051	.0049	.0048
−2.4	.0082	.0080	.0078	.0075	.0073	.0071	.0069	.0068	.0066	.0064
−2.3	.0107	.0104	.0102	.0099	.0096	.0094	.0091	.0089	.0087	.0084
−2.2	.0139	.0136	.0132	.0129	.0125	.0122	.0119	.0116	.0113	.0110
−2.1	.0179	.0174	.0170	.0166	.0162	.0158	.0154	.0150	.0146	.0143
−2.0	.0228	.0222	.0217	.0212	.0207	.0202	.0197	.0192	.0188	.0183
−1.9	.0287	.0281	.0274	.0268	.0262	.0256	.0250	.0244	.0239	.0233
−1.8	.0359	.0351	.0344	.0336	.0329	.0322	.0314	.0307	.0301	.0294
−1.7	.0446	.0436	.0427	.0418	.0409	.0401	.0392	.0384	.0375	.0367
−1.6	.0548	.0537	.0526	.0516	.0505	.0495	.0485	.0475	.0465	.0455
−1.5	.0668	.0655	.0643	.0630	.0618	.0606	.0594	.0582	.0571	.0559
−1.4	.0808	.0793	.0778	.0764	.0749	.0735	.0721	.0708	.0694	.0681
		.0951	.0934	.0918	.0901	.0885	.0869	.0853	.0838	.0823
		.1131	.1112	.1093	.1075	.1056	.1038	.1020	.1003	.0985
		.1335	.1314	.1292	.1271	.1251	.1230	.1210	.1190	.1170
		.1562	.1539	.1515	.1492	.1469	.1446	.1423	.1401	.1379
		.1814	.1788	.1762	.1736	.1711	.1685	.1660	.1635	.1611
		.2090	.2061	.2033	.2005	.1977	.1949	.1922	.1894	.1867
		.2389	.2358	.2327	.2296	.2266	.2236	.2206	.2177	.2148
		.2709	.2676	.2643	.2611	.2578	.2546	.2514	.2483	.2451
		.3050	.3015	.2981	.2946	.2912	.2877	.2843	.2810	.2776

- **Left**-tailed: critical value is the `α`-**th** quantile of the standard normal distribution N(0,1).
- **Right**-tailed: critical value is the `(1-α)`-**th** quantile.

3. Two-tailed test: critical value equals `±(1-α/2)`-**th** quantile of N(0,1).

The Sign Test

Paired-Sample Sign Test

The sign test can also be used to test sample means in a comparison of two dependent samples, such as a before-and-after test. Recall that when dependent samples are taken from normally distributed populations, the t-test is used (Section 9–4). When the condition of normality cannot be met, the nonparametric sign test can be used.

The **paired-sample sign test** is a nonparametric test that is used to test the difference between two population medians when the samples are dependent.

Two Assumptions for the Paired-Sign Test

1. The sample is random.
2. The variables are dependent or paired.

Example – Ear Infections in Swimmers

A medical researcher believed the number of ear infections in swimmers can be reduced if the swimmers use earplugs. A sample of 10 people was selected, and the number of infections for a four-month period was recorded. During the first two months, the swimmers did not use the earplugs; during the second two months, they did. At the beginning of the second two-month period, each swimmer was examined to make sure that no infection was present. The data are shown here.

At α = 0.05, can the researcher conclude that using earplugs reduced the number of ear infections?

	Number of ear infections	
Swimmer	Before, X_B	After, X_A
A	3	2
B	0	1
C	5	4
D	4	0
E	2	1
F	4	3
G	3	1
H	5	3
I	2	2
J	1	3

Example – Ear Infections in Swimmers

SOLUTION

Step 1 State the hypotheses and identify the claim.

H_0: The number of ear infections will not be reduced.

H_1: The number of ear infections will be reduced (claim).

Step 2 Find the critical value. Subtract the after values X_A from the before values X_B, and indicate the difference by a positive or negative sign or 0, according to the value, as shown in the table.

Swimmer	Before, X_B	After, X_A	Sign of difference
A	3	2	+
B	0	1	−
C	5	4	+
D	4	0	+
E	2	1	+
F	4	3	+
G	3	1	+
H	5	3	+
I	2	2	0
J	1	3	−

From Table J, with $n = 9$ (the total number of positive and negative signs; the 0 is not counted) and $\alpha = 0.05$ (one-tailed), at most 1 negative sign is needed to reject the null hypothesis because 1 is the smallest entry in the $\alpha = 0.05$ column of Table J.

The Sign Test

TABLE J Critical Values for the Sign Test

Reject the null hypothesis if the smaller number of positive or negative signs is less than or equal to the value in the table.

n	One-tailed, $\alpha = 0.005$ / Two-tailed, $\alpha = 0.01$	$\alpha = 0.01$ / $\alpha = 0.02$	$\alpha = 0.025$ / $\alpha = 0.05$	$\alpha = 0.05$ / $\alpha = 0.10$
8	0	0	0	1
9	0	0	1	1
10	0	0	1	1
11	0	1	1	2
12	1	1	2	2
13	1	1	2	3
14	1	2	3	3
15	2	2	3	3
16	2	2	3	4
17	2	3	4	4
18	3	3	4	5
19	3	4	4	5
20	3	4	5	5
21	4	4	5	6
22	4	5	5	6
23	4	5	6	7
24	5	5	6	7
25	5	6	6	7

Note: Table J is for one-tailed or two-tailed tests. The term n represents the total number of positive and negative signs. The test value is the number of less frequent signs.

Example – Ear Infections in Swimmers

Step 3 Compute the test value. Since $n \leq 25$, we will count the number of positive and negative signs found in step 2 and use the smaller value as the test value. There are 7 positive signs and 2 negative signs, so the test value is 2.

Step 4 Make the decision. Compare the test value 2 with the critical value 1. If the test value is less than or equal to the critical value, the null hypothesis is rejected. In this case, $2 > 1$, so the decision is not to reject the null hypothesis.

Step 5 Summarize the results. There is not enough evidence to support the claim that the use of earplugs reduced the number of ear infections.

The Wilcoxon Rank Sum Test

The two tests considered in this section and in Section 13–4 are the Wilcoxon rank sum test, which is used for independent samples, and the Wilcoxon signed-rank test, which is used for dependent samples. Both tests are used to compare distributions. The parametric equivalents are the *z* and *t*-tests for independent samples (Sections 9 –1 and 9–2) and the *t*-test for dependent samples (Section 9 –3). For the parametric tests, as stated previously, the samples must be selected from approximately normally distributed populations, but the assumptions for the Wilcoxon tests are different.

The **Wilcoxon rank sum test** is a nonparametric test that uses ranks to determine if two independent samples were selected from populations that have the same distributions.

Assumptions for the Wilcoxon Rank Sum Test

1. The samples are random and independent of one another.
2. The size of each sample must be greater than or equal to 10.

The Wilcoxon Rank Sum Test

Formula for the Wilcoxon Rank Sum Test When Samples Are Independent

$$z = \frac{R - \mu_R}{\sigma_R}$$

where

$$\mu_R = \frac{n_1(n_1 + n_2 + 1)}{2}$$

$$\sigma_R = \sqrt{\frac{n_1 n_2(n_1 + n_2 + 1)}{12}}$$

R = sum of ranks for smaller sample size (n_1)
n_1 = smaller of sample sizes
n_2 = larger of sample sizes
$n_1 \geq 10$ and $n_2 \geq 10$

Note that if both samples are the same size, either size can be used as n_1.

The Wilcoxon Rank Sum Test

Procedure Table

Wilcoxon Rank Sum Test

Step 1 State the hypotheses and identify the claim.

Step 2 Find the critical value(s). Use Table E.

Step 3 Compute the test value.

a. Combine the data from the two samples, arrange the combined data in order, and rank each value.

b. Sum the ranks of the group with the smaller sample size. (*Note:* If both groups have the same sample size, either one can be used.)

c. Use these formulas to find the test value.

$$\mu_R = \frac{n_1(n_1 + n_2 + 1)}{2}$$

$$\sigma_R = \sqrt{\frac{n_1 n_2(n_1 + n_2 + 1)}{12}}$$

$$z = \frac{R - \mu_R}{\sigma_R}$$

where R is the sum of the ranks of the data in the smaller sample and n_1 and n_2 are each greater than or equal to 10.

Step 4 Make the decision.

Step 5 Summarize the results.

Example - Times to Complete an Obstacle Course

Two independent random samples of the army and marine recruits are selected, and the time in minutes it takes each recruit to complete an obstacle course is recorded, as shown in the table.

At $\alpha = 0.05$, is there a difference in the times it takes the recruits to complete the course?

Army	15	18	16	17	13	22	24	17	19	21	26	28
Marines	14	9	16	19	10	12	11	8	15	18	25	

Example - Times to Complete an Obstacle Course

SOLUTION

Step 1 State the hypotheses and identify the claim.

H_0: There is no difference in the times it takes the recruits to complete the obstacle course.

H_1: There is a difference in the times it takes the recruits to complete the obstacle course (claim).

Step 2 Find the critical value. Since $\alpha = 0.05$ and this test is a two-tailed test, use the critical values of $+1.96$ and -1.96 from Table E.

Step 3 Compute the test value.

a. Combine the data from the two samples, arrange the combined data in ascending order, and rank each value. Be sure to indicate the group.

Step 4 Make the decision. The decision is to reject the null hypothesis, since $-2.41 < -1.96$.

Step 5 Summarize the results. There is enough evidence to support the claim that there is a difference in the times it takes the recruits to complete the course.

Example - Times to Complete an Obstacle Course

Time	8	9	10	11	12	13	14	15	15	16	16	17
Group	M	M	M	M	M	A	M	A	M	A	M	A
Rank	1	2	3	4	5	6	7	8.5	8.5	10.5	10.5	12.5

Time	17	18	18	19	19	21	22	24	25	26	28
Group	A	M	A	A	M	A	A	A	M	A	A
Rank	12.5	14.5	14.5	16.5	16.5	18	19	20	21	22	23

b. Sum the ranks of the group with the smaller sample size. (*Note:* If both groups have the same sample size, either one can be used.) In this case, the sample size for the marines is smaller.

$$R = 1 + 2 + 3 + 4 + 5 + 7 + 8.5 + 10.5 + 14.5 + 16.5 + 21 = 93$$

c. Substitute in the formulas to find the test value.

$$\mu_R = \frac{n_1(n_1 + n_2 + 1)}{2} = \frac{(11)(11 + 12 + 1)}{2} = 132$$

$$\sigma_R = \sqrt{\frac{n_1 n_2(n_1 + n_2 + 1)}{12}} = \sqrt{\frac{(11)(12)(11 + 12 + 1)}{12}}$$

$$= \sqrt{264} = 16.2$$

$$z = \frac{R - \mu_R}{\sigma_R} = \frac{93 - 132}{16.2} = -2.41$$

The Wilcoxon Signed-Rank Test

When the samples are dependent, as they would be in a before-and-after test using the same subjects, the Wilcoxon signed-rank test can be used in place of the *t*-test for dependent samples. Again, this test does not require the condition of normality.

The **Wilcoxon signed-rank test** is a nonparametric test used to test whether two dependent samples have been selected from two populations having the same distributions.

Assumptions for the Wilcoxon Signed-Rank Test

1. The paired data have been obtained from a random sample.
2. The population of differences has a distribution that is approximately symmetric.

To find the test value for the Wilcoxon signed-rank test, denoted by w_s, when $n \leq 30$, rank the absolute values of the differences of each pair of data values. Assign either a + or a − sign to each rank according to the original value of the difference. Then sum the positive ranks and the negative ranks separately. Finally, when $n \leq 30$, select the smaller of the absolute value of the sums as the test value w_s.

The Wilcoxon Signed-Rank Test

Procedure Table

Wilcoxon Signed-Rank Test

Step 1 State the hypotheses and identify the claim.

Step 2 Find the critical value from Table K when $n \leq 30$ and from Table E when $n > 30$.

Step 3 Compute the test value.

When $n \leq 30$:

a. Make a table, as shown.

Before, X_B	After, X_A	Difference $D = X_B - X_A$	Absolute value $\|D\|$	Rank	Signed rank

b. Find the differences (before − after), denoted by $X_B - X_A$, and place the values in the Difference column.

c. Find the absolute value of each difference, and place the results in the Absolute value column.

d. Rank each absolute value from lowest to highest, and place the rankings in the Rank column.

e. Give each rank a positive or negative sign, according to the sign in the Difference column.

f. Find the sum of the positive ranks and the sum of the negative ranks separately.

g. Select the smaller of the absolute values of the sums, and use this absolute value as the test value w_s.

When $n > 30$, use Table E and the test value

$$z = \frac{w_s - \dfrac{n(n+1)}{4}}{\sqrt{\dfrac{n(n+1)(2n+1)}{24}}}$$

where

n = number of pairs where difference is not 0

w_s = smaller sum in absolute value of signed ranks

Step 4 Make the decision. Reject the null hypothesis if the test value is less than or equal to the critical value.

Step 5 Summarize the results.

Example - Shoplifting Incidents

In a large department store, the owner wishes to see whether the number of shoplifting incidents per day will change if the number of uniformed security officers is doubled. A random sample of 7 days before security is increased and 7 days after the increase shows the number of shoplifting incidents.

Is there enough evidence to support the claim, at $\alpha = 0.05$, that there is a difference in the number of shoplifting incidents before and after the increase in security?

Day	Number of shoplifting incidents	
	Before	After
Monday	7	5
Tuesday	2	3
Wednesday	3	4
Thursday	6	3
Friday	5	1
Saturday	8	6
Sunday	12	4

Example - Shoplifting Incidents

SOLUTION

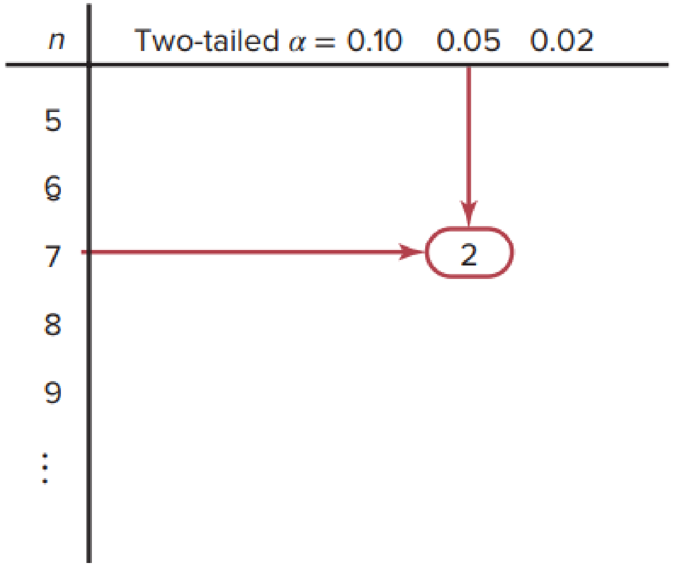
Step 1 State the hypotheses and identify the claim.

H_0: There is no difference in the number of shoplifting incidents before and after the increase in security.

H_1: There is a difference in the number of shoplifting incidents before and after the increase in security (claim).

Step 2 Find the critical value from Table K because $n \leq 30$. Since $n = 7$ and $\alpha = 0.05$ for this two-tailed test, the critical value is 2. See Figure 13–2.

FIGURE 13–2
Finding the Critical Value in Table K for Example 13–5



Step 3 Find the test value.

a. Make a table as shown.

Critical Values for Wilcoxon Signed-Rank Test

TABLE K Critical Values for the Wilcoxon Signed-Rank Test

Reject the null hypothesis if the test value is less than or equal to the value given in the table.

n	One-tailed, $\alpha = 0.05$	$\alpha = 0.025$	$\alpha = 0.01$	$\alpha = 0.005$
	Two-tailed, $\alpha = 0.10$	$\alpha = 0.05$	$\alpha = 0.02$	$\alpha = 0.01$
5	1	–	–	–
6	2	1	–	–
7	4	2	0	–
8	6	4	2	0
9	8	6	3	2
10	11	8	5	3
11	14	11	7	5
12	17	14	10	7
13	21	17	13	10
14	26	21	16	13
15	30	25	20	16
16	36	30	24	19
17	41	35	28	23
18	47	40	33	28
19	54	46	38	32
20	60	52	43	37
21	68	59	49	43

Example - Shoplifting Incidents

Step 3 Find the test value.

a. Make a table as shown.

Day	Before, X_B	After, X_A	Difference $D = X_B - X_A$	Absolute value $\lvert D \rvert$	Rank	Signed rank
Mon.	7	5				
Tues.	2	3				
Wed.	3	4				
Thurs.	6	3				
Fri.	5	1				
Sat.	8	6				
Sun.	12	4				

b. Find the differences (before minus after), and place the values in the Difference column.

$$7 - 5 = 2 \qquad 6 - 3 = 3 \qquad 8 - 6 = 2$$
$$2 - 3 = -1 \qquad 5 - 1 = 4 \qquad 12 - 4 = 8$$
$$3 - 4 = -1$$

Example - Shoplifting Incidents

c. Find the absolute value of each difference, and place the results in the Absolute value column. (*Note:* The absolute value of any number except 0 is the positive value of the number. Any differences of 0 should be ignored.)

$$|2| = 2 \qquad |3| = 3 \qquad |2| = 2$$
$$|-1| = 1 \qquad |4| = 4 \qquad |8| = 8$$
$$|-1| = 1$$

d. Rank each absolute value from lowest to highest, and place the rankings in the Rank column. In the case of a tie, assign the values that rank plus 0.5.

Value	2	1	1	3	4	2	8
Rank	3.5	1.5	1.5	5	6	3.5	7

e. Give each rank a plus or minus sign, according to the sign in the Difference column. The completed table is shown here.

Example - Shoplifting Incidents

Day	Before, X_B	After, X_A	Difference $D = X_B - X_A$	Absolute value $\lvert D \rvert$	Rank	Signed rank
Mon.	7	5	2	2	3.5	+3.5
Tues.	2	3	−1	1	1.5	−1.5
Wed.	3	4	−1	1	1.5	−1.5
Thurs.	6	3	3	3	5	+5
Fri.	5	1	4	4	6	+6
Sat.	8	6	2	2	3.5	+3.5
Sun.	12	4	8	8	7	+7

f. Find the sum of the positive ranks and the sum of the negative ranks separately.

Positive rank sum	$(+3.5) + (+5) + (+6) + (+3.5) + (+7)$	$= +25$
Negative rank sum	$(-1.5) + (-1.5)$	$= -3$

g. Select the smaller of the absolute values of the sums ($\lvert -3 \rvert$), and use this absolute value as the test value w_s. In this case, $w_s = \lvert -3 \rvert = 3$.

Step 4 Make the decision. Reject the null hypothesis if the test value is less than or equal to the critical value. In this case, $3 > 2$; hence, the decision is to not reject the null hypothesis.

Step 5 Summarize the results. There is not enough evidence at $\alpha = 0.05$ to support the claim that there is a difference in the number of shoplifting incidents before and after the increase in security. Hence, the security increase probably made no difference in the number of shoplifting incidents.

The Kruskal-Wallis Test

The analysis of variance uses the *F*-test to compare the means of three or more populations. The assumptions for the ANOVA test are that the populations are normally distributed and that the population variances are equal. When these assumptions cannot be met, the nonparametric *Kruskal-Wallis* test, sometimes called the *H*-test, can be used to compare three or more means.

The **Kruskal-Wallis test** is a nonparametric test that is used to determine whether three or more samples came from populations with the same distributions.

Assumptions for the Kruskal-Wallis Test

1. There are at least three random samples.
2. The size of each sample must be at least 5.

The Kruskal-Wallis Test

Formula for the Kruskal-Wallis Test

$$H = \frac{12}{N(N+1)}\left(\frac{R_1^2}{n_1} + \frac{R_2^2}{n_2} + \cdots + \frac{R_k^2}{n_k}\right) - 3(N+1)$$

where

R_1 = sum of ranks of sample 1
n_1 = size of sample 1
R_2 = sum of ranks of sample 2
n_2 = size of sample 2
⋮
R_k = sum of ranks of sample k
n_k = size of sample k
$N = n_1 + n_2 + \cdots + n_k$
k = number of samples

The Kruskal-Wallis Test

Procedure Table

Kruskal-Wallis Test

Step 1 State the hypotheses and identify the claim.

Step 2 Find the critical value. Use the chi-square table, Table G, with d.f. $= k - 1$ ($k =$ number of groups).

Step 3 Compute the test value.

a. Arrange the data from lowest to highest and rank each value.

The Kruskal-Wallis Test

b. Find the sum of the ranks of each group.

c. Substitute in the formula.

$$H = \frac{12}{N(N+1)}\left(\frac{R_1^2}{n_1} + \frac{R_2^2}{n_2} + \cdots + \frac{R_k^2}{n_k}\right) - 3(N+1)$$

where

$$N = n_1 + n_2 + \cdots + n_k$$
$$R_k = \text{sum of ranks for } k\text{th group}$$
$$k = \text{number of groups}$$

Step 4 Make the decision.

Step 5 Summarize the results.

Example – Hospital Infections

A researcher wishes to see if the total number of infections that occurred in three groups of randomly selected hospitals is the same. The data are shown in the table. At $\alpha = 0.05$, is there enough evidence to reject the claim that the number of infections in the three groups of hospitals is the same?

Group A	Group B	Group C
557	476	105
315	232	110
920	80	167
178	116	155

Source: Pennsylvania Health Care Cost Containment Council.

EXAMPLE – Hospital Infections

SOLUTION

Step 1 State the hypotheses and identify the claim.

H_0: There is no difference in the number of infections in the three groups of hospitals (claim).

H_1: There is a difference in the number of infections in the three groups of hospitals.

Step 2 Find the critical value. Use the chi-square table (Table G) with d.f. $= k - 1$, where k = the number of groups. With $\alpha = 0.05$ and d.f. $= 3 - 1 = 2$, the critical value is 5.991.

Step 3 Compute the test value.

a. Arrange all the data from the lowest value to the highest value and rank each value.

Amount	Group	Rank
80	B	1
105	C	2
110	C	3
116	B	4
155	C	5
167	C	6
178	A	7
232	B	8
315	A	9
476	B	10
557	A	11
920	A	12

Table G

TABLE G The Chi-Square Distribution

Degrees of freedom	α									
	0.995	0.99	0.975	0.95	0.90	0.10	0.05	0.025	0.01	0.005
1	—	—	0.001	0.004	0.016	2.706	3.841	5.024	6.635	7.879
2	0.010	0.020	0.051	0.103	0.211	4.605	5.991	7.378	9.210	10.597
3	0.072	0.115	0.216	0.352	0.584	6.251	7.815	9.348	11.345	12.838
4	0.207	0.297	0.484	0.711	1.064	7.779	9.488	11.143	13.277	14.860
5	0.412	0.554	0.831	1.145	1.610	9.236	11.071	12.833	15.086	16.750
6	0.676	0.872	1.237	1.635	2.204	10.645	12.592	14.449	16.812	18.548
7	0.989	1.239	1.690	2.167	2.833	12.017	14.067	16.013	18.475	20.278
8	1.344	1.646	2.180	2.733	3.490	13.362	15.507	17.535	20.090	21.955
9	1.735	2.088	2.700	3.325	4.168	14.684	16.919	19.023	21.666	23.589
10	2.156	2.558	3.247	3.940	4.865	15.987	18.307	20.483	23.209	25.188
11	2.603	3.053	3.816	4.575	5.578	17.275	19.675	21.920	24.725	26.757
12	3.074	3.571	4.404	5.226	6.304	18.549	21.026	23.337	26.217	28.300
13	3.565	4.107	5.009	5.892	7.042	19.812	22.362	24.736	27.688	29.819
14	4.075	4.660	5.629	6.571	7.790	21.064	23.685	26.119	29.141	31.319
15	4.601	5.229	6.262	7.261	8.547	22.307	24.996	27.488	30.578	32.801
16	5.142	5.812	6.908	7.962	9.312	23.542	26.296	28.845	32.000	34.267
17	5.697	6.408	7.564	8.672	10.085	24.769	27.587	30.191	33.409	35.718
18	6.265	7.015	8.231	9.390	10.865	25.989	28.869	31.526	34.805	37.156
19	6.844	7.633	8.907	10.117	11.651	27.204	30.144	32.852	36.191	38.582
20	7.434	8.260	9.591	10.851	12.443	28.412	31.410	34.170	37.566	39.997

EXAMPLE – Hospital Infections

b. Find the sum of the ranks for each group.

Group A	$7 + 9 + 11 + 12 = 39$
Group B	$1 + 4 + 8 + 10 = 23$
Group C	$2 + 3 + 5 + 6 = 16$

c. Substitute in the formula.

$$H = \frac{12}{N(N+1)}\left(\frac{R_1^2}{n_1} + \frac{R_2^2}{n_2} + \frac{R_3^2}{n_3}\right) - 3(N+1)$$

where

$$N = 12 \qquad R_1 = 39 \qquad R_2 = 23 \qquad R_3 = 16$$
$$n_1 = n_2 = n_3 = 4$$

Therefore,

$$H = \frac{12}{12(12+1)}\left(\frac{39^2}{4} + \frac{23^2}{4} + \frac{16^2}{4}\right) - 3(12+1)$$
$$= 5.346$$

Step 4 Make the decision. Since the test value of 5.346 is less than the critical value of 5.991, the decision is to not reject the null hypothesis.

Step 5 Summarize the results. There is not enough evidence to reject the claim that there is no difference in the number of infections in the groups of hospitals. Hence, the differences are not significant at $\alpha = 0.05$.

The Spearman Rank Correlation Coefficient

The techniques of regression and correlation were explained in Chapter 10. To determine whether two variables are linearly related, you use the Pearson product-moment correlation coefficient. Its values range from +1 to −1. One assumption for testing the hypothesis that $\rho = 0$ for the Pearson coefficient is that the populations from which the samples are obtained are normally distributed. If this requirement cannot be met, the nonparametric equivalent, called the **Spearman rank correlation coefficient** (denoted by r_s), can be used when the data are ranked.

> The Spearman rank correlation coefficient is a nonparametric statistic that uses ranks to determine if there is a relationship between two variables.

Rank Correlation Coefficient

The computations for the rank correlation coefficient are simpler than those for the *Pearson* coefficient and involve ranking each set of data. The difference in ranks is found, and *rs* is computed by using these differences. If both sets of data have the same ranks, *rs* will be +1. If the sets of data are ranked in exactly the opposite way, *rs* will be −1. If there is no relationship between the rankings, *rs* will be near 0. The assumptions for the *Spearman* rank correlation coefficients are given next.

Assumptions for Spearman's Rank Correlation Coefficient

1. The sample is a random sample.
2. The data consist of two measurements or observations taken on the same individual.

Formula for Computing the Spearman Rank Correlation Coefficient

$$r_s = 1 - \frac{6\Sigma d^2}{n(n^2 - 1)}$$

where

d = difference in ranks

n = number of data pairs

Rank Correlation Coefficient

Procedure Table

Finding and Testing the Value of Spearman's Rank Correlation Coefficient

Step 1 State the hypotheses.

Step 2 Find the critical value.

Step 3 Find the test value.

a. Rank the values in each data set.

b. Subtract the rankings for each pair of data values ($X_1 - X_2$).

c. Square the differences.

d. Find the sum of the squares.

e. Substitute in the formula.

$$r_s = 1 - \frac{6\Sigma d^2}{n(n^2 - 1)}$$

where

d = difference in ranks

n = number of pairs of data

Step 4 Make the decision.

Step 5 Summarize the results.

Example – Bank Branches and Deposits

A researcher wishes to see if there is a relationship between the number of branches a bank has and the total number of deposits (in billions of dollars) the bank receives. A sample of eight regional banks is selected, and the number of branches and the amount of deposits is shown in the table.

At $\alpha = 0.05$, is there a significant linear correlation between the number of branches and the amount of deposits?

Bank	Number of branches	Deposits (in billions)
A	209	$23
B	353	31
C	19	7
D	201	12
E	344	26
F	132	5
G	401	24
H	126	4

Source: SNL Financial.

Example – Bank Branches and Deposits

SOLUTION

Step 1 State the hypotheses.

H_0: $\rho = 0$ and H_1: $\rho \neq 0$

Step 2 Find the critical value. Use Table L to find the value for $n = 8$ and $\alpha = 0.05$. It is ± 0.738. See Figure 13–3.

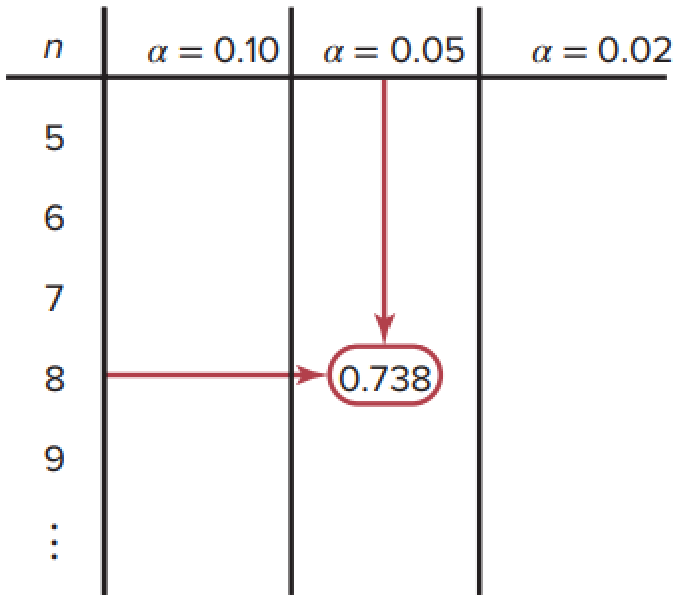


FIGURE 13–3
Finding the Critical Value in Table L for Example 13–7

Table L

TABLE L Critical Values for the Rank Correlation Coefficient

Reject H_0: $\rho = 0$ if the absolute value of r_S is greater than the value given in the table.

n	$\alpha = 0.10$	$\alpha = 0.05$	$\alpha = 0.02$	$\alpha = 0.01$
5	0.900	–	–	–
6	0.829	0.886	0.943	–
7	0.714	0.786	0.893	0.929
8	0.643	0.738	0.833	0.881
9	0.600	0.700	0.783	0.833
10	0.564	0.648	0.745	0.794
11	0.536	0.618	0.709	0.818
12	0.497	0.591	0.703	0.780
13	0.475	0.566	0.673	0.745
14	0.457	0.545	0.646	0.716
15	0.441	0.525	0.623	0.689
16	0.425	0.507	0.601	0.666
17	0.412	0.490	0.582	0.645
18	0.399	0.476	0.564	0.625
19	0.388	0.462	0.549	0.608
20	0.377	0.450	0.534	0.591
21	0.368	0.438	0.521	0.576
22	0.359	0.428	0.508	0.562
23	0.351	0.418	0.496	0.549
24	0.343	0.409	0.485	0.537
25	0.336	0.400	0.475	0.526

Example – Bank Branches and Deposits

Step 3 Find the test value.

a. Rank each data set as shown in the table.

Bank	Branches	Rank	Deposits	Rank
A	209	5	23	5
B	353	7	31	8
C	19	1	7	3
D	201	4	12	4
E	344	6	26	7
F	132	3	5	2
G	401	8	24	6
H	126	2	4	1

Let X_1 be the rank of the branches and X_2 be the rank of the deposits.

b. Subtract the ranking $(X_1 - X_2)$.

$$5 - 5 = 0 \qquad 7 - 8 = -1 \qquad 1 - 3 = -2 \qquad \text{etc.}$$

c. Square the differences.

$$0^2 = 0 \qquad (-1)^2 = 1 \qquad (-2)^2 = 4 \qquad \text{etc.}$$

d. Find the sum of the squares.

$$0 + 1 + 4 + 0 + 1 + 1 + 4 + 1 = 12$$

Example – Bank Branches and Deposits

The results can be summarized in a table as shown.

X_1	X_2	$d = X_1 - X_2$	d^2
5	5	0	0
7	8	−1	1
1	3	−2	4
4	4	0	0
6	7	−1	1
3	2	1	1
8	6	2	4
2	1	1	1
			$\Sigma d^2 = 12$

e. Substitute in the formula for r_s.

$$r_s = 1 - \frac{6\Sigma d^2}{n(n^2 - 1)} \quad \text{where } n = \text{number of pairs}$$

$$r_s = 1 - \frac{6 \cdot 12}{8(8^2 - 1)} = 1 - \frac{72}{504} = 0.857$$

Example – Bank Branches and Deposits

Step 4 Make the decision. Reject the null hypothesis since $r_s = 0.857$, which is greater than the critical value of 0.738.

Step 5 Summarize the results. There is enough evidence to say that there is a linear relationship between the number of branches a bank has and the deposits of the bank.

The Runs Test

When samples are selected, you assume that they are selected at random. How do you know if the data obtained from a sample are truly random? One way to answer this question is to use the **runs test**. Before you can use the runs test, you must be able to determine the number of runs in a sequence of events.

A **run** is a succession of identical letters preceded or followed by a different letter or no letter at all, such as the beginning or end of the succession.

Consider the following situations for a researcher interviewing 20 people for a survey. Let their gender be denoted by *M* for male and *F* for female. Suppose the participants were chosen as follows:

Situation 1: *M M M M M M M M M M F F F F F F F F F F*

It does not look as if the people in this sample were selected at random, since 10 males were selected first, followed by 10 females. Consider a different selection:

Situation 2: *F M F M F M F M F M F M F M F M F M F M*

The Runs Test

In this case, it seems as if the researcher selected a female, then a male, etc. This selection is probably not random either. Finally, consider the following selection:

Situation 3: *F F F M M F M F M M F F M M F F M M M F*

This selection of data looks as if it may be random, since there is a mix of males and females and no apparent pattern to their selection. Let's examine each of the three situations to determine the number of runs. The first situation presented has two runs:

Run 1: *M M M M M M M M M M*

Run 2: *F F F F F F F F F F*

The Runs Test

The second situation has 20 runs. (Each letter constitutes one run.)

The third situation has 11 runs.

Run 1	F F F	Run 5	F	Run 9	F F
Run 2	M M	Run 6	M M	Run 10	M M M
Run 3	F	Run 7	F F	Run 11	F
Run 4	M	Run 8	M M		

The number of runs is used to determine the value of the test statistic. Here, n_1 represents the number of items in category 1 and n_2 represents the number of items in category 2.

In the preceding situation, there were 10 males and 10 females; hence, $n_1 = 10$ and $n_2 = 10$. Also, G is used to denote the number of runs. In situation 1, $G = 2$; in situation 2, $G = 20$; and in situation 3, $G = 11$.

Example

Determine the number of runs and the values of n_1 and n_2 in each sequence.

a. $M\ M\ F\ F\ F\ M\ F\ F$

b. $H\ T\ H\ H\ H$

c. $A\ B\ A\ A\ A\ B\ B\ A\ B\ B\ B$

SOLUTION

a. There are four runs, as shown.

$$\underbrace{\mathrm{M\ M}}_{1}\quad \underbrace{\mathrm{F\ F\ F}}_{2}\quad \underbrace{\mathrm{M}}_{3}\quad \underbrace{\mathrm{F\ F}}_{4}$$

There are 3 M's and 5 F's, so $n_1 = 3$ and $n_2 = 5$.

b. There are three runs, as shown.

$$\underbrace{\mathrm{H}}_{1}\quad \underbrace{\mathrm{T}}_{2}\quad \underbrace{\mathrm{H\ H\ H}}_{3}$$

There are 4 heads and 1 tail, so $n_1 = 4$ and $n_2 = 1$.

c. There are six runs, as shown.

$$\underbrace{\mathrm{A}}_{1}\quad \underbrace{\mathrm{B}}_{2}\quad \underbrace{\mathrm{A\ A\ A}}_{3}\quad \underbrace{\mathrm{B\ B}}_{4}\quad \underbrace{\mathrm{A}}_{5}\quad \underbrace{\mathrm{B\ B\ B}}_{6}$$

There are 5 A's and 6 B's, so $n_1 = 5$ and $n_2 = 6$.

The Runs Test

The **runs test for randomness** is a nonparametric test that is used to determine if a sequence of data values occurs at random.

Assumptions for the Runs Test for Randomness

1. The data from the sample are arranged in the order in which they were selected.
2. Each letter, number, or event can be classified into one or two mutually exclusive categories.

Formulas for the Test Statistic Value for the Runs Test

When $n_1 \leq 20$ and $n_2 \leq 20$, use the number of runs denoted by G, as the test statistic value. When $n_1 > 20$ or $n_2 > 20$—or when $n_1 > 20$ *and* $n_2 \geq 20$—use

$$z = \frac{G - \mu_G}{\sigma_G}$$

where

$$\mu_G = \frac{2n_1 n_2}{n_1 + n_2} + 1$$

$$\sigma_G = \sqrt{\frac{2n_1 n_2(2n_1 n_2 - n_1 - n_2)}{(n_1 + n_2)^2(n_1 + n_2 - 1)}}$$

The Runs Test

Procedure Table

The Runs Test

Step 1 State the hypotheses and identify the claim.

Step 2 Find the critical values.

a. Use Table M when $n_1 \leq 20$ and $n_2 \leq 20$.

b. Use Table E when $n_1 > 20$ or $n_2 > 20$ or when $n_1 > 20$ *and* $n_2 > 20$.

Step 3 Find the test value.

a. Use the number of runs if $n_1 \leq 20$ and $n_2 \leq 20$.

b. Use the formula when $n_1 > 20$ or $n_2 > 20$ or when $n_1 > 20$ *and* $n_2 > 20$.

$$z = \frac{G - \mu_G}{\sigma_G}$$

Step 4 Make the decision.

Step 5 Summarize the results.

The Runs Test

To determine whether the number of runs is within the random range, use Table *M* in Appendix *A* when $n_1 \leq 20$ and $n_2 \leq 20$. The values are for a two-tailed test with $\alpha = 0.05$. For a sample of 12 males and 8 females, the table values shown in Figure 13 – 4 mean that any number of runs from 7 to 15 would be considered random. If the number of runs is 6 or less or is 16 or more, then the sample is probably not random and the null hypothesis should be rejected.

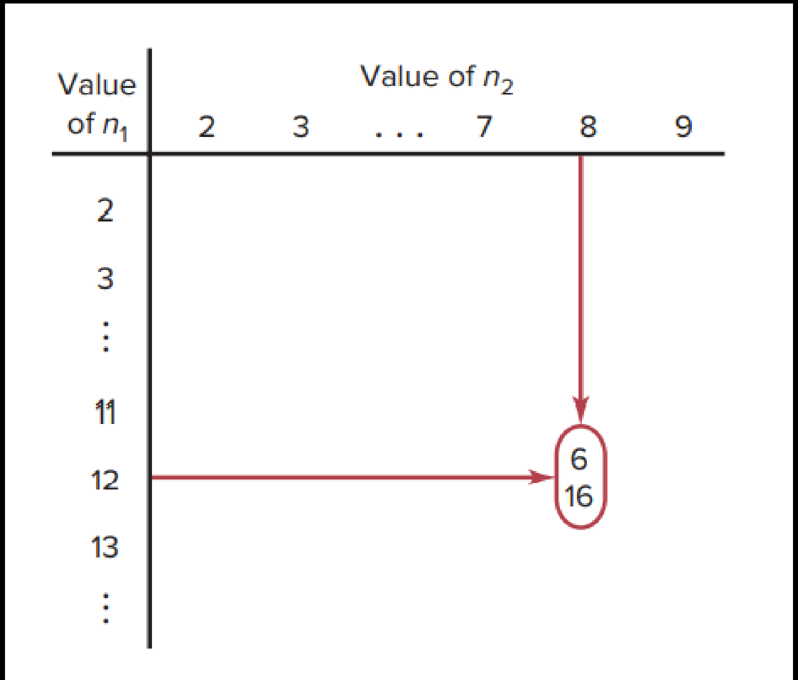


Table M

TABLE M Critical Values for the Number of Runs

This table gives the critical values at $\alpha = 0.05$ for a two-tailed test. Reject the null hypothesis if the number of runs is less than or equal to the smaller value or greater than or equal to the larger value.

Value of n_1	Value of n_2: 2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
2	1	1	1	1	1	1	1	1	1	1	2	2	2	2	2	2	2	2	2
	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6
3	1	1	1	1	2	2	2	2	2	2	2	2	2	3	3	3	3	3	3
	6	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8
4	1	1	1	2	2	2	3	3	3	3	3	3	3	3	4	4	4	4	4
	6	8	9	9	9	10	10	10	10	10	10	10	10	10	10	10	10	10	10
5	1	1	2	2	3	3	3	3	3	4	4	4	4	4	4	4	5	5	5
	6	8	9	10	10	11	11	12	12	12	12	12	12	12	12	12	12	12	12
6	1	2	2	3	3	3	3	4	4	4	4	5	5	5	5	5	5	6	6
	6	8	9	10	11	12	12	13	13	13	13	14	14	14	14	14	14	14	14
7	1	2	2	3	3	3	4	4	5	5	5	5	5	6	6	6	6	6	6
	6	8	10	11	12	13	13	14	14	14	14	15	15	15	16	16	16	16	16
8	1	2	3	3	3	4	4	5	5	5	6	6	6	6	6	7	7	7	7
	6	8	10	11	12	13	14	14	15	15	16	16	16	16	17	17	17	17	17
9	1	2	3	3	4	4	5	5	5	6	6	6	7	7	7	7	8	8	8
	6	8	10	12	13	14	14	15	16	16	16	17	17	18	18	18	18	18	18
10	1	2	3	3	4	5	5	5	6	6	7	7	7	7	8	8	8	8	9
	6	8	10	12	13	14	15	16	16	17	17	18	18	18	19	19	19	20	20
11	1	2	3	4	4	5	5	6	6	7	7	7	8	8	8	9	9	9	9
	6	8	10	12	13	14	15	16	17	17	18	19	19	19	20	20	20	21	21
12	2	2	3	4	4	5	6	6	7	7	7	8	8	8	9	9	9	10	10
	6	8	10	12	13	14	16	16	17	18	19	19	20	20	21	21	21	22	22

Example – Gender of Train Passengers

On a commuter train, the conductor wishes to see whether the passengers enter the train at random. He observes the first 25 people, with the following sequence of males (M) and females (F).

$F\ F\ F\ M\ M\ F\ F\ F\ F\ M\ F\ M\ M\ M\ F\ F\ F\ F\ M\ M\ F\ F\ F\ M\ M$

Test for randomness at $\alpha = 0.05$.

EXAMPLE – Gender of Train Passengers

SOLUTION

Step 1 State the hypotheses and identify the claim.

H_0: The passengers board the train at random, according to gender (claim).

H_1: The passengers do not board the train at random, according to gender.

Step 2 Determine the critical value. There are 10 M's and 15 F's, so $n_1 = 10$ and $n_2 = 15$. Using Table M and $\alpha = 0.05$, the critical value is $\begin{smallmatrix}7\\18\end{smallmatrix}$ which means do not reject H_0 if the number of runs is between 7 and 18.

Step 3 Find the test value. Determine the number of runs. Arrange the letters according to runs of males and females, as shown.

Run	Gender
1	F F F
2	M M
3	F F F F
4	M
5	F
6	M M M
7	F F F F
8	M M
9	F F F
10	M M

There are 10 runs.

Step 4 Make the decision. Compare these critical values with the number of runs. Since the number of runs is 10 and 10 is between 7 and 18, do not reject the null hypothesis.

Step 5 Summarize the results. There is not enough evidence to reject the hypothesis that the passengers board the train at random according to gender.

Example – Ages of Substance Abuse Program Participants

Twenty people enrolled in a substance abuse program. Test the claim that the ages of the people, according to the order in which they enroll, occur at random, at $\alpha = 0.05$.

The data are 18, 36, 19, 22, 25, 44, 23, 27, 27, 35, 19, 43, 37, 32, 28, 43, 46, 19, 20, 22.

SOLUTION

Step 1 State the hypotheses and identify the claim.

H_0: The ages of the people, according to the order in which they enroll in a substance abuse program, occur at random (claim).

H_1: The ages of the people, according to the order in which they enroll in a substance abuse program, do not occur at random.

Example – Ages of Substance Abuse Program Participants

Step 2 Find the critical value.

Find the median of the data. Arrange the data in ascending order.

18 19 19 19 20 22 22 23 25 27 27
28 32 35 36 37 43 43 44 46

The median is 27.

Replace each number in the original sequence as written in the example with an A if it is above the median and with a B if it is below the median. Eliminate any numbers that are equal to the median.

Recall the original sequence is 18, 36, 19, 22, . . . , 22. Then

18 is below the median, so it is B;

36 is above the median, so it is A;

19 is below the median, so it is B;

etc.

The sequence of letters, then, is

B A B B B A B A B A A A A A A B B B

There are 9 A's and 9 B's. Table M shows that with $n_1 = 9$, $n_2 = 9$, and $\alpha = 0.05$, the number of runs should be between 5 and 15.

Step 3 Find the test value. Determine the number of runs from the sequence of letters.

Run	Letters
1	B
2	A
3	B B B
4	A
5	B
6	A
7	B
8	A A A A A A
9	B B B

The number of runs $G = 9$.

Step 4 Make the decision. Since there are 9 runs and 9 falls between the critical values 5 and 15, the null hypothesis is not rejected.

Step 5 Summarize the results. There is not enough evidence to reject the hypothesis that the ages of the people who enroll occur at random.

Example – Baseball All-Star Winners

The data show the winners of the baseball all-star games (N = National League, A = American League) from 1962 to 2012. At $\alpha = 0.05$, can it be concluded that the sequence of winners is random?

A	N	N	N	N	N	N	N	N	A
N	N	N	N	N	N	N	N	N	N
N	A	N	N	A	N	A	A	A	A
A	A	N	N	N	A	A	A	A	A
A	A	A	A	A	A	A	N	N	N

(Note: The tie in 2002 has been omitted.)

Example – Baseball All-Star Winners

SOLUTION

Step 1 State the hypotheses and identify the claim.

H_0: The winners occur at random (claim).

H_1: The winners do not occur at random.

Step 2 Determine the actual values.

Since $n_1 > 20$ and $n_2 > 20$, Table E is used. At $\alpha = 0.05$, the critical values are ± 1.96.

Step 3 Find the test value.

$$n_1 \text{ (National)} = 28 \qquad n_2 \text{ (American)} = 22$$

The number of runs is

1. A
2. NNNNNNNN
3. A
4. NNNNNNNNNNN
5. A
6. NN
7. A
8. N
9. AAAAAA
10. NNN
11. AAAAAAAAAAAA
12. NNN

There are $G = 12$ runs.

The Sign Test
Table E

1- a/2 =
1- 0.05/2 =
1- 0.025 = 9750

TABLE E *(continued)*

Cumulative Standard Normal Distribution

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
0.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
0.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
0.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
0.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
0.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
0.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
0.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
0.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
0.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
		.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
		.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
		.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
		.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
		.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
		.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974

- **Left**-tailed: critical value is the `α`-**th** quantile of the standard normal distribution N(0,1).
- **Right**-tailed: critical value is the `(1-α)`-**th** quantile.

3. Two-tailed test: critical value equals `±(1-α/2)`-**th** quantile of N(0,1).

Example – Baseball All-Star Winners

$$\mu_G = \frac{2n_1n_2}{n_1 + n_2} + 1$$

$$= \frac{2(28)(22)}{28 + 22} + 1 = 25.64$$

$$\sigma_G = \sqrt{\frac{2n_1n_2(2n_1n_2 - n_1 - n_2)}{(n_1 + n_2)^2(n_1 + n_2 - 1)}}$$

$$= \sqrt{\frac{2(28)(22)[2(28)(22) - 28 - 22]}{(28 + 22)^2(28 + 22 - 1)}} = 3.448$$

$$z = \frac{G - \mu_G}{\sigma_G}$$

$$= \frac{12 - 25.64}{3.448}$$

$$= -3.96$$

Step 4 Make the decision. Since $-3.96 < -1.96$, the decision is to reject the null hypothesis.

Step 5 Summarize the results. There is enough evidence to reject the claim that the sequence of winners occurs at random.

References

- **Elementary Statistics: A Step-by-Step Approach, Allen Bluman, 10th Edition, McGraw Hill, 2017, ISBN 13: 978-1-259-755330, Chapters 13.**