Intermediate Analytics

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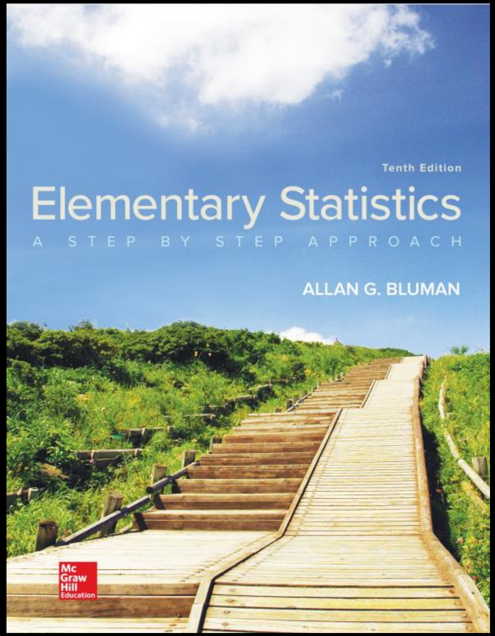
ALY 6015

Slides are mainly borrowed from the textbook:

- ***Elementary Statistics: A Step by-Step Approach***. *10th Edition*, ***Allen Bluman, McGraw Hill***

ANOVA

You will learn in this course:



After completing this chapter, you should be able to:

1. Use the one-way ANOVA technique to determine if there is a significant difference among three or more means.
2. Determine which means differ, using the Scheffé or Tukey test if the null hypothesis is rejected in the ANOVA.
3. Use the two-way ANOVA technique to determine if there is a significant difference in the main effects or interaction.

Introduction

- When an F-test is used to test a hypothesis concerning the means of three or more populations, the technique is called analysis of variance (commonly abbreviated as ANOVA.
- It is used to test claims involving three or more means. (Note: The F-test can also be used to test the equality of two means. But since it is equivalent to the t-test, in this case, the t-test is usually used instead of the F-test when there are only two means.)
- The analysis of variance that is used to compare three or more means is called a one-way analysis of variance since it contains only one variable.
- At first glance, you might think that to compare the means of three or more samples, you can use multiple t-tests comparing two means at a time. But there are several reasons why the t-test should not be done.

One-Way Analysis of Variance

- First, when you are comparing two means at a time, the rest of the means under study are ignored. With the F-test, all the means are compared simultaneously.

- Second, when you are comparing two means at a time and making all pairwise comparisons, the probability of rejecting the null hypothesis when it is true is increased, since the more t-tests that are conducted, the greater the likelihood of getting significant differences by chance alone.

- Third, the more means there are to compare, the more t-tests are needed. For example, for the comparison of 3 means two at a time, 3 t-tests are required.

- For the comparison of 5 means two at a time, 10 tests are required. And for the comparison of 10 means two at a time, 45 tests are required.

One-Way Analysis of Variance

Recall that the characteristics of the F-distribution are as follows:

1. The values of F cannot be negative, because variances are always positive or zero.
2. The distribution is positively skewed.
3. The mean value of F is approximately equal to 1.
4. The F distribution is a family of curves based on the degrees of freedom of the variance of the numerator and the degrees of freedom of the variance of the denominator.

ANOVA

- With the F-test, two different estimates of the population variance are made.
- The first estimate is called the **between-group variance**, and it involves finding the variance of the means.
- The second estimate, the **within-group variance**, is made by computing the variance using all the data and is not affected by differences in the means.
- If there is no difference in the means, the between-group variance estimate will be approximately equal to the within-group variance estimate, and the F-test value will be approximately equal to one. The null hypothesis will not be rejected.
- However, when the means differ significantly, the between-group variance will be much larger than the within-group variance; the F-test value will be significantly greater than one, and the null hypothesis will be rejected. Since variances are compared, this procedure is called **analysis of variance** (ANOVA).

Analysis of the Variance

The formula for the F test is

$$F = \frac{\text{variance between groups}}{\text{variance within groups}}$$

The variance between groups measures the differences in the means that result from the different treatments given to each group. To calculate this value, it is necessary to find the *grand mean* $\overline{X}_{GM}$, which is the mean of all the values in all of the samples. The formula for the grand mean is

$$\overline{X}_{GM} = \frac{\Sigma X}{N}$$

This value is used to find the between-group variance s_B^2. This is the variance among the means using the sample sizes as weights.

The formula for the between-group variance, denoted by s_B^2, is

$$s_B^2 = \frac{\Sigma n_i(\overline{X}_i - \overline{X}_{GM})^2}{k - 1}$$

where k = number of groups
n_i = sample size
$\overline{X}_i$ = sample mean

This formula can be written out as

$$s_B^2 = \frac{n_1(\overline{X}_1 - \overline{X}_{GM})^2 + n_2(\overline{X}_2 - \overline{X}_{GM})^2 + \cdots + n_k(\overline{X}_k - \overline{X}_{GM})^2}{k - 1}$$

Analysis of the Variance

Next find the within group variance, denoted by s_W^2. The formula finds the overall variance by calculating a weighted average of the individual variances. It does not involve using differences of means. The formula for the within-group variance is

$$s_W^2 = \frac{\Sigma(n_i - 1)s_i^2}{\Sigma(n_i - 1)}$$

where n_i = sample size
s_i^2 = variance of sample

This formula can be written out as

$$s_W^2 = \frac{(n_1 - 1)s_1^2 + (n_2 - 1)s_2^2 + \cdots + (n_k - 1)s_k^2}{(n_1 - 1) + (n_2 - 1) + \cdots + (n_k - 1)}$$

Finally, the F test value is computed. The formula can now be written using the symbols s_B^2 and s_W^2.

The formula for the F test for one-way analysis of variance is

$$F = \frac{s_B^2}{s_W^2}$$

where s_B^2 = between-group variance
s_W^2 = within-group variance

Analysis of the Variance

- ✓ The results of the one-way analysis of variance can be summarized by placing them in an ANOVA summary table.
- ✓ The numerator of the fraction of the s_B^2 term is called *the sum of squares between groups*, denoted by *SSB*.

- ✓ The numerator of the s_W^2 term is called *the sum of squares within groups*, denoted by *SSW*. This statistic is also called the sum of squares for the error.

- ✓ *SSB* is divided by d.f.N. to obtain the between-group variance. *SSW* is divided by $N - k$ to obtain the within-group or error variance. These two variances are sometimes called *mean squares*, denoted by *MSB* and *MSW*. These terms are used to summarize the analysis of variance and are placed in a summary table, as shown in Table 12–1.

Analysis of the Variance

TABLE 12–1 Analysis of Variance Summary Table

Source	Sum of squares	d.f.	Mean square	F
Between	SS_B	$k-1$	MS_B	
Within (error)	SS_W	$N-k$	MS_W	
Total				

In the table,

$$SS_B = \text{sum of squares between groups}$$

$$SS_W = \text{sum of squares within groups}$$

$$k = \text{number of groups}$$

$$N = n_1 + n_2 + \cdots + n_k = \text{sum of sample sizes for groups}$$

$$MS_B = \frac{SS_B}{k-1}$$

$$MS_W = \frac{SS_W}{N-k}$$

$$F = \frac{MS_B}{MS_W}$$

Analysis of the Variance

To use the F-test to compare two or more means, the following assumptions must be met:
(In the reference book, the assumptions will be stated in the exercises; however, when encountering statistics in other situations, you must check to see that these assumptions have been met before proceeding. The steps for computing the F-test value for the ANOVA are summarized in this Procedure Table.)

Assumptions for the *F* Test for Comparing Three or More Means

1. The populations from which the samples were obtained must be normally or approximately normally distributed.
2. The samples must be independent of one another.
3. The variances of the populations must be equal.
4. The samples must be simple random samples, one from each of the populations.

Analysis of the Variance

Procedure Table

Finding the *F* Test Value for the Analysis of Variance

Step 1 Find the mean and variance of each sample.

$$(\bar{X}_1, s_1^2), (\bar{X}_2, s_2^2), \ldots, (\bar{X}_k, s_k^2)$$

Step 2 Find the grand mean.

$$\bar{X}_{\text{GM}} = \frac{\Sigma X}{N}$$

Step 3 Find the between-group variance.

$$s_B^2 = \frac{\Sigma n_i(\bar{X}_i - \bar{X}_{\text{GM}})^2}{k - 1}$$

Step 4 Find the within-group variance.

$$s_W^2 = \frac{\Sigma(n_i - 1)s_i^2}{\Sigma(n_i - 1)}$$

Step 5 Find the F test value.

$$F = \frac{s_B^2}{s_W^2}$$

The degrees of freedom are

$$\text{d.f.N.} = k - 1$$

where k is the number of groups, and

$$\text{d.f.D.} = N - k$$

where N is the sum of the sample sizes of the groups

$$N = n_1 + n_2 + \cdots + n_k$$

Example 1– Milles per Gallon

A researcher wishes to see if there is a difference in the fuel economy for city driving for three different types of automobiles: small automobiles, sedans, and luxury automobiles. He randomly samples four small automobiles, five sedans, and three luxury automobiles. The miles per gallon for each are shown.

At $\alpha = 0.05$, test the claim that there is no difference among the means. The data are shown below:

Small	Sedans	Luxury
36	43	29
44	35	25
34	30	24
35	29	
	40	

Source: U.S. Environmental Protection Agency.

Step 1 State the hypotheses and identify the claim.

H_0: $\mu_1 = \mu_2 = \mu_3$ (claim)
H_1: At least one mean is different from the others

Example 1– Milles per Gallon

Step 2 Find the critical value.

$$N = 12 \qquad k = 3$$
$$\text{d.f.N.} = k - 1 = 3 - 1 = 2$$
$$\text{d.f.D.} = N - k = 12 - 3 = 9$$

The critical value from Table H in Appendix A with $\alpha = 0.05$ is 4.26.

Step 3 Compute the test value.

a. Find the mean and variance for each sample. (Use the formulas in Chapter

For the small cars:	$\overline{X} = 37.25$	$s^2 = 20.917$
For the sedans:	$\overline{X} = 35.4$	$s^2 = 37.3$
For the luxury cars:	$\overline{X} = 26$	$s^2 = 7$

Sample Variance (s^2)

$$s^2 = \frac{\sum (x_i - \bar{x})^2}{n-1}$$

b. Find the grand mean.

$$\overline{X}_{GM} = \frac{\Sigma X}{N} = \frac{36 + 44 + 34 + \cdots + 24}{12} = \frac{404}{12} = 33.667$$

c. Find the between-group variance.

$$s_B^2 = \frac{\Sigma n(\overline{X}_i - \overline{X}_{GM})^2}{k-1}$$

$$= \frac{4(37.25 - 33.667)^2 + 5(35.4 - 33.667)^2 + 3(26 - 33.667)^2}{3 - 1}$$

$$= \frac{242.717}{2} = 121.359$$

Example 1– Milles per Gallon

d. Find the within-group variance.

$$s_W^2 = \frac{\Sigma(n_i - 1)s_i^2}{\Sigma(n_i - 1)} = \frac{(4 - 1)(20.917) + (5 - 1)(37.3) + (3 - 1)7}{(4 - 1) + (5 - 1) + (3 - 1)}$$

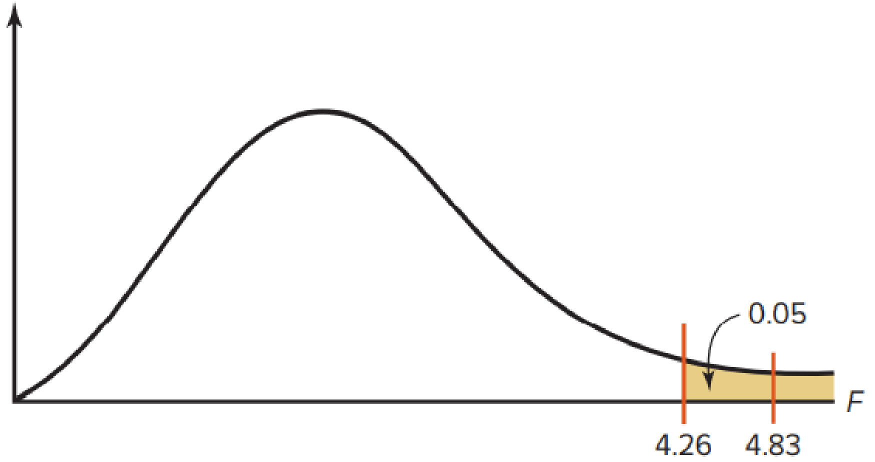
$$= \frac{225.951}{9} = 25.106$$

e. Find the F test value.

$$F = \frac{s_B^2}{s_W^2} = \frac{121.359}{25.106} = 4.83$$

Step 4 Make the decision. The test value 4.83 > 4.26, so the decision is to reject the null hypothesis. See Figure 12–1.

FIGURE 12–1 Critical Value and Test Value for Example 12–1



The F Distribution

TABLE H *(continued)*

d.f.D.: degrees of freedom, denominator	$\alpha = 0.05$ d.f.N.: degrees of freedom, numerator																
	1	2	3	4	5	6	7	8	9	10	12	15	20	24	30	40	60
1	161.4	199.5	215.7	224.6	230.2	234.0	236.8	238.9	240.5	241.9	243.9	245.9	248.0	249.1	250.1	251.1	252.2
2	18.51	19.00	19.16	19.25	19.30	19.33	19.35	19.37	19.38	19.40	19.41	19.43	19.45	19.45	19.46	19.47	19.48
3	10.13	9.55	9.28	9.12	9.01	8.94	8.89	8.85	8.81	8.79	8.74	8.70	8.66	8.64	8.62	8.59	8.57
4	7.71	6.94	6.59	6.39	6.26	6.16	6.09	6.04	6.00	5.96	5.91	5.86	5.80	5.77	5.75	5.72	5.69
5	6.61	5.79	5.41	5.19	5.05	4.95	4.88	4.82	4.77	4.74	4.68	4.62	4.56	4.53	4.50	4.46	4.43
6	5.99	5.14	4.76	4.53	4.39	4.28	4.21	4.15	4.10	4.06	4.00	3.94	3.87	3.84	3.81	3.77	3.74
7	5.59	4.74	4.35	4.12	3.97	3.87	3.79	3.73	3.68	3.64	3.57	3.51	3.44	3.41	3.38	3.34	3.30
8	5.32	4.46	4.07	3.84	3.69	3.58	3.50	3.44	3.39	3.35	3.28	3.22	3.15	3.12	3.08	3.04	3.01
9	5.12	4.26	3.86	3.63	3.48	3.37	3.29	3.23	3.18	3.14	3.07	3.01	2.94	2.90	2.86	2.83	2.79
10	4.96	4.10	3.71	3.48	3.33	3.22	3.14	3.07	3.02	2.98	2.91	2.85	2.77	2.74	2.70	2.66	2.62
11	4.84	3.98	3.59	3.36	3.20	3.09	3.01	2.95	2.90	2.85	2.79	2.72	2.65	2.61	2.57	2.53	2.49
12	4.75	3.89	3.49	3.26	3.11	3.00	2.91	2.85	2.80	2.75	2.69	2.62	2.54	2.51	2.47	2.43	2.38
13	4.67	3.81	3.41	3.18	3.03	2.92	2.83	2.77	2.71	2.67	2.60	2.53	2.46	2.42	2.38	2.34	2.30
14	4.60	3.74	3.34	3.11	2.96	2.85	2.76	2.70	2.65	2.60	2.53	2.46	2.39	2.35	2.31	2.27	2.22
15	4.54	3.68	3.29	3.06	2.90	2.79	2.71	2.64	2.59	2.54	2.48	2.40	2.33	2.29	2.25	2.20	2.16
16	4.49	3.63	3.24	3.01	2.85	2.74	2.66	2.59	2.54	2.49	2.42	2.35	2.28	2.24	2.19	2.15	2.11
17	4.45	3.59	3.20	2.96	2.81	2.70	2.61	2.55	2.49	2.45	2.38	2.31	2.23	2.19	2.15	2.10	2.06
18	4.41	3.55	3.16	2.93	2.77	2.66	2.58	2.51	2.46	2.41	2.34	2.27	2.19	2.15	2.11	2.06	2.02
19	4.38	3.52	3.13	2.90	2.74	2.63	2.54	2.48	2.42	2.38	2.31	2.23	2.16	2.11	2.07	2.03	1.98
20	4.35	3.49	3.10	2.87	2.71	2.60	2.51	2.45	2.39	2.35	2.28	2.20	2.12	2.08	2.04	1.99	1.95

The F Distribution

TABLE H *(continued)*

d.f.D.: degrees of freedom, denominator	$\alpha = 0.025$																
	d.f.N.: degrees of freedom, numerator																
	1	2	3	4	5	6	7	8	9	10	12	15	20	24	30	40	60
1	647.8	799.5	864.2	899.6	921.8	937.1	948.2	956.7	963.3	968.6	976.7	984.9	993.1	997.2	1001	1006	1010
2	38.51	39.00	39.17	39.25	39.30	39.33	39.36	39.37	39.39	39.40	39.41	39.43	39.45	39.46	39.46	39.47	39.48
3	17.44	16.04	15.44	15.10	14.88	14.73	14.62	14.54	14.47	14.42	14.34	14.25	14.17	14.12	14.08	14.04	13.99
4	12.22	10.65	9.98	9.60	9.36	9.20	9.07	8.98	8.90	8.84	8.75	8.66	8.56	8.51	8.46	8.41	8.36
5	10.01	8.43	7.76	7.39	7.15	6.98	6.85	6.76	6.68	6.62	6.52	6.43	6.33	6.28	6.23	6.18	6.12
6	8.81	7.26	6.60	6.23	5.99	5.82	5.70	5.60	5.52	5.46	5.37	5.27	5.17	5.12	5.07	5.01	4.96
7	8.07	6.54	5.89	5.52	5.29	5.12	4.99	4.90	4.82	4.76	4.67	4.57	4.47	4.42	4.36	4.31	4.25
8	7.57	6.06	5.42	5.05	4.82	4.65	4.53	4.43	4.36	4.30	4.20	4.10	4.00	3.95	3.89	3.84	3.78
9	7.21	5.71	5.08	4.72	4.48	4.32	4.20	4.10	4.03	3.96	3.87	3.77	3.67	3.61	3.56	3.51	3.45
10	6.94	5.46	4.83	4.47	4.24	4.07	3.95	3.85	3.78	3.72	3.62	3.52	3.42	3.37	3.31	3.26	3.20
11	6.72	5.26	4.63	4.28	4.04	3.88	3.76	3.66	3.59	3.53	3.43	3.33	3.23	3.17	3.12	3.06	3.00
12	6.55	5.10	4.47	4.12	3.89	3.73	3.61	3.51	3.44	3.37	3.28	3.18	3.07	3.02	2.96	2.91	2.85
13	6.41	4.97	4.35	4.00	3.77	3.60	3.48	3.39	3.31	3.25	3.15	3.05	2.95	2.89	2.84	2.78	2.72
14	6.30	4.86	4.24	3.89	3.66	3.50	3.38	3.29	3.21	3.15	3.05	2.95	2.84	2.79	2.73	2.67	2.61
15	6.20	4.77	4.15	3.80	3.58	3.41	3.29	3.20	3.12	3.06	2.96	2.86	2.76	2.70	2.64	2.59	2.52
16	6.12	4.69	4.08	3.73	3.50	3.34	3.22	3.12	3.05	2.99	2.89	2.79	2.68	2.63	2.57	2.51	2.45
17	6.04	4.62	4.01	3.66	3.44	3.28	3.16	3.06	2.98	2.92	2.82	2.72	2.62	2.56	2.50	2.44	2.38
18	5.98	4.56	3.95	3.61	3.38	3.22	3.10	3.01	2.93	2.87	2.77	2.67	2.56	2.50	2.44	2.38	2.32
19	5.92	4.51	3.90	3.56	3.33	3.17	3.05	2.96	2.88	2.82	2.72	2.62	2.51	2.45	2.39	2.33	2.27
20	5.87	4.46	3.86	3.51	3.29	3.13	3.01	2.91	2.84	2.77	2.68	2.57	2.46	2.41	2.35	2.29	2.22

Example 1– Milles per Gallon

Step 5 Summarize the results. There is enough evidence to conclude that at least one mean is different from the others.

The ANOVA summary table is shown in Table 12–2.

TABLE 12–2 Analysis of Variance Summary Table for Example 12–1

Source	Sum of squares	d.f.	Mean square	F
Between	242.717	2	121.359	4.83
Within (error)	225.954	9	25.106	
Total	468.671	11		

The P-values for ANOVA are found by using the same procedure shown in Section 9–5. For Example 12–1, the F test value is 4.83. In Table H with d.f.N. = 2 and d.f.D. = 9, the F test value falls between $\alpha = 0.025$ with an F value of 5.71 and $\alpha = 0.05$ with an F value of 4.26. Hence, $0.025 < P\text{-value} < 0.05$. In this case, the null hypothesis is rejected at $\alpha = 0.05$ since the P-value < 0.05. The TI-84 P-value is 0.0375.

Example 2 – Tall Buildings

A researcher wishes to see if there is a difference in the number of stories in the tall buildings of Chicago, Houston, and New York City. The researcher randomly selects six buildings in each city and records the number of stories in each building. The data are shown below.

At α = 0.05, can it be concluded that there is a significant difference in the mean number of stories in the tall buildings in each city?

Chicago	Houston	New York City
98	53	85
54	52	67
60	45	75
57	41	52
83	36	94
49	34	42

Source: The World Almanac and Book of Facts

Example 2 – Tall Buildings

SOLUTION

Step 1 State the hypotheses and identify the claim.

H_0: $\mu_1 = \mu_2 = \mu_3$

H_1: At least one mean is different from the others (claim).

Step 2 Find the critical value. Since $k = 3$, $N = 18$, and $\alpha = 0.05$,

d.f.N. $= k - 1 = 3 - 1 = 2$

d.f.D. $= N - k = 18 - 3 = 15$

The critical value is 3.68.

Step 3 Compute the test value.

a. Find the mean and variance of each sample. The mean and variance for each sample are

Chicago	$\overline{X}_1 = 66.8$	$s_1^2 = 371.8$
Houston	$\overline{X}_2 = 43.5$	$s_2^2 = 63.5$
New York	$\overline{X}_3 = 69.2$	$s_3^2 = 387.7$

b. Find the grand mean.

$$\overline{X}_{GM} = \frac{\Sigma X}{N} = \frac{98 + 54 + 60 + \cdots + 42}{18} = \frac{1077}{18} = 59.8$$

Example 2 – Tall Buildings

c. Find the between-group variance.

$$s_B^2 = \frac{\Sigma n_i(\overline{X}_i - \overline{X}_{GM})^2}{k-1}$$

$$= \frac{6(66.8 - 59.8)^2 + 6(43.5 - 59.8)^2 + 6(69.2 - 59.8)^2}{3-1}$$

$$= \frac{2418.3}{2} = 1209.15$$

d. Find the within-group variance.

$$s_W^2 = \frac{\Sigma(n_i - 1)s_i^2}{\Sigma(n_i - 1)}$$

$$= \frac{(6-1)(371.8) + (6-1)(63.5) + (6-1)(387.7)}{(6-1) + (6-1) + (6-1)} = \frac{4115}{15}$$

$$= 274.33$$

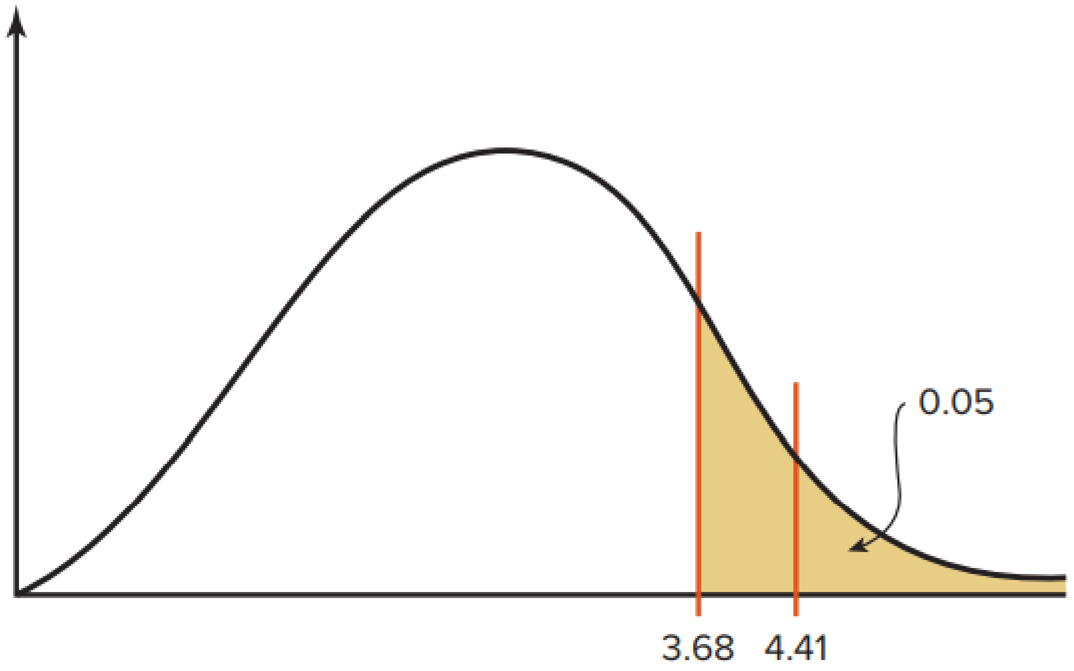
e. Find the F test value.

$$F = \frac{s_B^2}{s_W^2} = \frac{1209.15}{274.33} = 4.41$$

Example 2 – Tall Buildings

Step 4 Make the decision. Since 4.41 > 3.68, the decision is to reject the null hypothesis. See Figure 12–2.

FIGURE 12–2 Critical Value and Test Value for Example 12–2



The Scheffé Test and the Tukey Test

- ✓ When the null hypothesis is rejected using the F-test, the researcher may want to know where the difference between the means is.

- ✓ Several procedures have been developed to determine where the significant differences in the means lie after the ANOVA procedure has been performed. Among the most commonly used tests are the **Scheffé test** and the **Tukey test**.

The Scheffé Test

To conduct the **Scheffé test,** you must compare the means two at a time, using all possible combinations of means. For example, if there are three means, the following comparisons must be done:

$$\overline{X}_1 \text{ versus } \overline{X}_2 \qquad \overline{X}_1 \text{ versus } \overline{X}_3 \qquad \overline{X}_2 \text{ versus } \overline{X}_3$$

Formula for the Scheffé Test

$$F_s = \frac{(\overline{X}_i - \overline{X}_j)^2}{s_W^2[(1/n_i) + (1/n_j)]}$$

where $\overline{X}_i$ and $\overline{X}_j$ are the means of the samples being compared, n_i and n_j are the respective sample sizes, and s_W^2 is the within-group variance.

To find the critical value F' for the Scheffé test, multiply the critical value for the F test by $k - 1$:

$$F' = (k - 1)(\text{C.V.})$$

There is a significant difference between the two means being compared when the F test value, F_S, is greater than the critical value, F'. Example 12–3 illustrates the use of the Scheffé test.

Example

Use the Scheffé test to test each pair of means in Example 12–1 to see if a significant difference exists between each pair of means. Use $\alpha = 0.05$.

SOLUTION

The F critical value for Example 12–1 is 4.26. Then the critical value for the individual tests with d.f.N. = 2 and d.f.D. = 9 is

$$F' = (k - 1)(\text{C.V.}) = (3 - 1)(4.26) = 8.52$$

a. For $\overline{X}_1$ versus $\overline{X}_2$,

$$F_S = \frac{(\overline{X}_1 - \overline{X}_2)^2}{s_W^2[(1/n_1) + (1/n_2)]} = \frac{(37.25 - 35.4)^2}{25.106\left(\frac{1}{4} + \frac{1}{5}\right)} = 0.30$$

Since $0.30 < 8.52$, the decision is that μ_1 is not significantly different from μ_2.

b. For $\overline{X}_1$ versus $\overline{X}_3$,

$$F_S = \frac{(\overline{X}_1 - \overline{X}_3)^2}{s_W^2[(1/n_1) + (1/n_3)]} = \frac{(37.25 - 26)^2}{25.106\left(\frac{1}{4} + \frac{1}{3}\right)} = 8.64$$

Since $8.64 > 8.52$, the decision is that μ_1 is significantly different from μ_3.

c. For $\overline{X}_2$ versus $\overline{X}_3$,

$$F_S = \frac{(\overline{X}_2 - \overline{X}_3)^2}{s_W^2[(1/n_2) + (1/n_3)]} = \frac{(35.4 - 26)^2}{25.106\left(\frac{1}{5} + \frac{1}{3}\right)} = 6.60$$

Since $6.60 < 8.64$, the decision is that μ_2 is not significantly different from μ_3.
Hence, only the mean of the small cars is not equal to the mean of luxury cars.

Tukey Test

- On occasion, when the F-test value is greater than the critical value, the Scheffé test may not show any significant differences in the pairs of means. This result occurs because the difference may actually lie in the average of two or more means when compared with the other mean. The *Scheffé* test can be used to make these types of comparisons, but the technique is beyond the scope of this book.

- The *Tukey* test can also be used after the analysis of variance has been completed to make pairwise comparisons between means **when the groups have the same sample size**. The symbol for the test value in the *Tukey* test is q.

Tukey Test

Formula for the Tukey Test

$$q = \frac{\overline{X}_i - \overline{X}_j}{\sqrt{s_W^2/n}}$$

where $\overline{X}_i$ and $\overline{X}_j$ are the means of the samples being compared, n is the size of the samples, and s_W^2 is the within-group variance.

When the absolute value of q is greater than the critical value for the *Tukey* test, there is a significant difference between the two means being compared. The critical value for the *Tukey* test is found using Table *N* in Appendix *A*, where k is the number of means in the original problem and v is the degrees of freedom for s^2_W , which is $N - k$. The value of k is found across the top row, and v is found in the left column.

Example

Using the Tukey test, test each pair of means in Example 12–2 to see whether a specific difference exists, at $\alpha = 0.05$.

SOLUTION

a. For $\overline{X}_1$ versus $\overline{X}_2$,

$$q = \frac{\overline{X}_1 - \overline{X}_2}{\sqrt{s_W^2/n}} = \frac{66.8 - 43.5}{\sqrt{274.33/6}} = 3.446$$

b. For $\overline{X}_1$ versus $\overline{X}_3$,

$$q = \frac{\overline{X}_1 - \overline{X}_3}{\sqrt{s_W^2/n}} = \frac{66.8 - 69.2}{\sqrt{274.33/6}} = -0.355$$

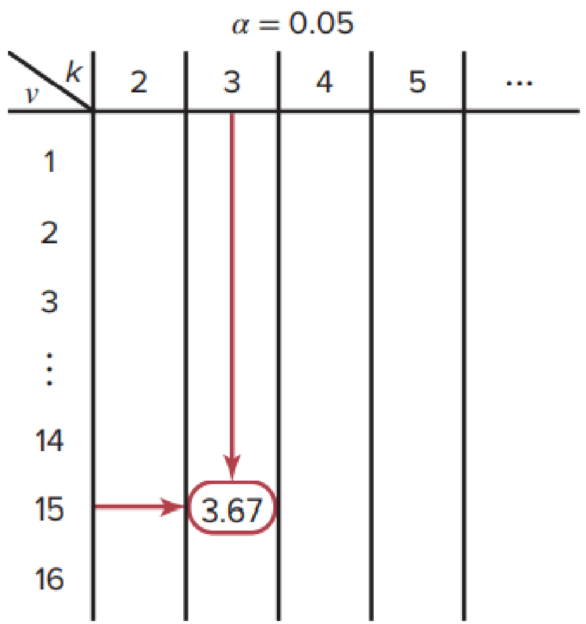
c. For $\overline{X}_2$ versus $\overline{X}_3$,

$$q = \frac{\overline{X}_2 - \overline{X}_3}{\sqrt{s_W^2/n}} = \frac{43.5 - 69.2}{\sqrt{274.33/6}} = -3.801$$

Example

To find the critical value for the *Tukey* test, use Table *N* in Appendix *A*. The number of means *k* is found in the row at the top, and the degrees of freedom for s_W^2 are found in the left column (denoted by *v*). Since k = 3, d.f. = 18 − 3 = 15, and α = 0.05, the critical value is 3.67. See Figure 12–3. Hence, the only *q* value that is greater **<u>in absolute value</u>** than the critical value is the one for the difference between the second and the third means. The conclusion, then, is that there is a significant difference in means for the Houston and the New York City buildings.

FIGURE 12–3 Finding the Critical Value in Table N for the Tukey Test (Example 12–4)



Critical Values for Tukey Test

TABLE N *(continued)*

v \ k	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
									$\alpha = 0.05$										
1	17.97	26.98	32.82	37.08	40.41	43.12	45.40	47.36	49.07	50.59	51.96	53.20	54.33	55.36	56.32	57.22	58.04	58.83	59.56
2	6.08	8.33	9.80	10.88	11.74	12.44	13.03	13.54	13.99	14.39	14.75	15.08	15.38	15.65	15.91	16.14	16.37	16.57	16.77
3	4.50	5.91	6.82	7.50	8.04	8.48	8.85	9.18	9.46	9.72	9.95	10.15	10.35	10.53	10.69	10.84	10.98	11.11	11.24
4	3.93	5.04	5.76	6.29	6.71	7.05	7.35	7.60	7.83	8.03	8.21	8.37	8.52	8.66	8.79	8.91	9.03	9.13	9.23
5	3.64	4.60	5.22	5.67	6.03	6.33	6.58	6.80	6.99	7.17	7.32	7.47	7.60	7.72	7.83	7.93	8.03	8.12	8.21
6	3.46	4.34	4.90	5.30	5.63	5.90	6.12	6.32	6.49	6.65	6.79	6.92	7.03	7.14	7.24	7.34	7.43	7.51	7.59
7	3.34	4.16	4.68	5.06	5.36	5.61	5.82	6.00	6.16	6.30	6.43	6.55	6.66	6.76	6.85	6.94	7.02	7.10	7.17
8	3.26	4.04	4.53	4.89	5.17	5.40	5.60	5.77	5.92	6.05	6.18	6.29	6.39	6.48	6.57	6.65	6.73	6.80	6.87
9	3.20	3.95	4.41	4.76	5.02	5.24	5.43	5.59	5.74	5.87	5.98	6.09	6.19	6.28	6.36	6.44	6.51	6.58	6.64
10	3.15	3.88	4.33	4.65	4.91	5.12	5.30	5.46	5.60	5.72	5.83	5.93	6.03	6.11	6.19	6.27	6.34	6.40	6.47
11	3.11	3.82	4.26	4.57	4.82	5.03	5.20	5.35	5.49	5.61	5.71	5.81	5.90	5.98	6.06	6.13	6.20	6.27	6.33
12	3.08	3.77	4.20	4.51	4.75	4.95	5.12	5.27	5.39	5.51	5.61	5.71	5.80	5.88	5.95	6.02	6.09	6.15	6.21
13	3.06	3.73	4.15	4.45	4.69	4.88	5.05	5.19	5.32	5.43	5.53	5.63	5.71	5.79	5.86	5.93	5.99	6.05	6.11
14	3.03	3.70	4.11	4.41	4.64	4.83	4.99	5.13	5.25	5.36	5.46	5.55	5.64	5.71	5.79	5.85	5.91	5.97	6.03
15	3.01	3.67	4.08	4.37	4.59	4.78	4.94	5.08	5.20	5.31	5.40	5.49	5.57	5.65	5.72	5.78	5.85	5.90	5.96
16	3.00	3.65	4.05	4.33	4.56	4.74	4.90	5.03	5.15	5.26	5.35	5.44	5.52	5.59	5.66	5.73	5.79	5.84	5.90
17	2.98	3.63	4.02	4.30	4.52	4.70	4.86	4.99	5.11	5.21	5.31	5.39	5.47	5.54	5.61	5.67	5.73	5.79	5.84
18	2.97	3.61	4.00	4.28	4.49	4.67	4.82	4.96	5.07	5.17	5.27	5.35	5.43	5.50	5.57	5.63	5.69	5.74	5.79
19	2.96	3.59	3.98	4.25	4.47	4.65	4.79	4.92	5.04	5.14	5.23	5.31	5.39	5.46	5.53	5.59	5.65	5.70	5.75
20	2.95	3.58	3.96	4.23	4.45	4.62	4.77	4.90	5.01	5.11	5.20	5.28	5.36	5.43	5.49	5.55	5.61	5.66	5.71

Two-Way Analysis of Variance

- The analysis of variance technique shown previously is called a **one-way ANOVA** since there is only one independent variable.

- **The two-way ANOVA** is an extension of the one-way analysis of variance; it involves two independent variables. The independent variables are also called **factors**. The two-way analysis of variance is quite complicated, and many aspects of the subject should be considered when you are using a research design involving a two-way ANOVA.

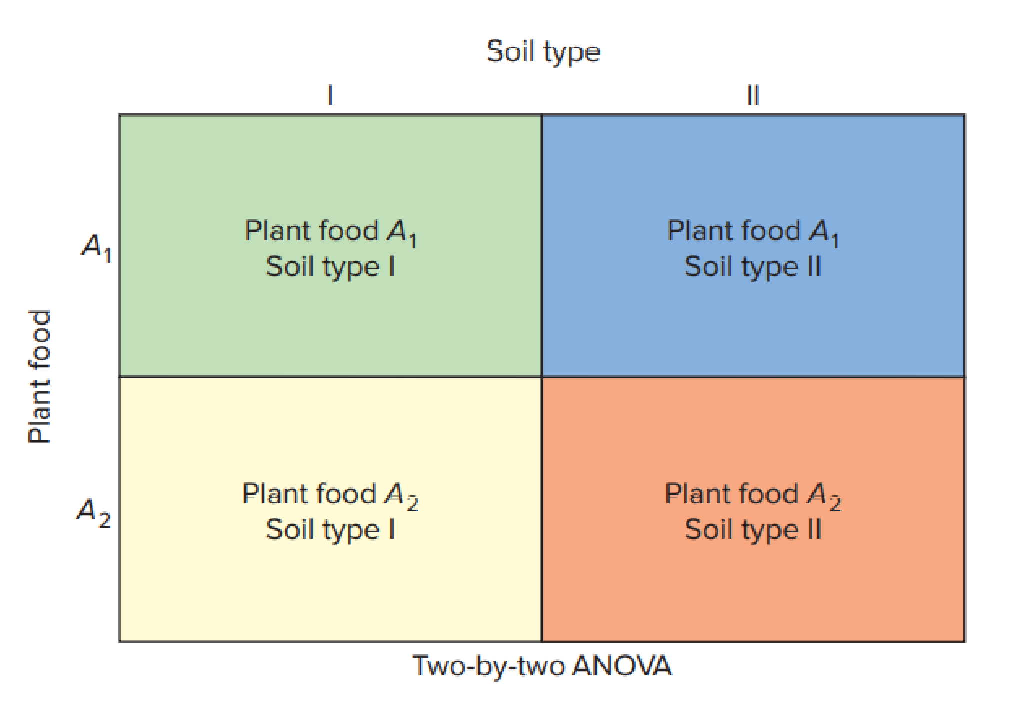
Two-Way Analysis of Variance

- For example, suppose a researcher wishes to test the effects of two different types of plant food and two different types of soil on the growth of certain plants.
- The two independent variables are the type of plant food and the type of soil, while the dependent variable is the plant growth.
- Other factors could be water, temperature, and sunlight, are held constant.
- To conduct this experiment, the researcher sets up four groups of plants. Assume that the plant food type is designated by the letters *A1* and *A2* and the soil type by the Roman numerals *I* and *II*.
- The groups for such a two-way ANOVA are sometimes called ***treatment groups***. The four groups are

Group 1	Plant food A_1, soil type I
Group 2	Plant food A_1, soil type II
Group 3	Plant food A_2, soil type I
Group 4	Plant food A_2, soil type II

Two-Way Analysis of Variance

The plants are assigned to the groups at random. This design is called a 2 × 2 design, since each variable consists of two levels, that is, two different treatments. The two-way ANOVA enables the researcher to test the effects of the plant food and the soil type in a single experiment rather than in separate experiments involving the plant food alone and the soil type alone.



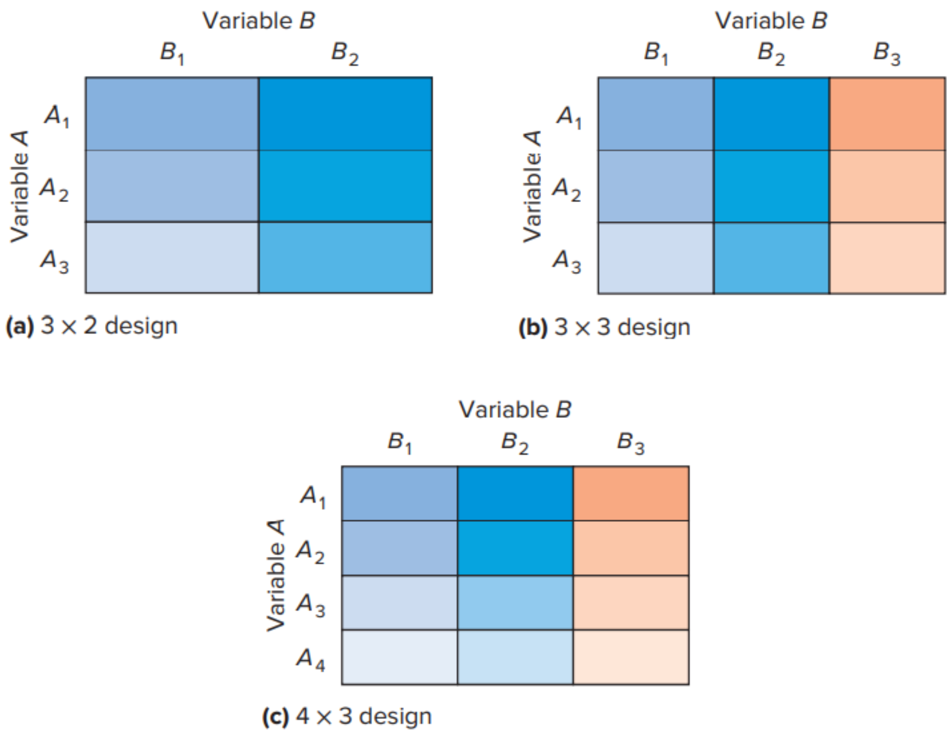
Two-Way Analysis of Variance

- In this case, the effect of the plant food is the change in the response variable that results from changing the level or the type of food.
- The effect of soil type is the change in the response variable that results from changing the level or type of soil.
- These two effects of the independent variable are called the <u>**main effects**</u>. Furthermore, the researcher can test an additional hypothesis about the effect of the **interaction of the two variables**— plant food and soil type—on plant growth.
- For example, is there a difference between the growth of plants using plant food *A1* and soil type *II* and the growth of plants using plant food *A2* and soil type *I*? When a difference of this type occurs, the experiment is said to have a significant <u>**interaction effect**</u>. The interaction effect represents the joint effect of the two factors over and above the effects of each factor considered separately.
- That is, the types of plant food affect plant growth differently in different soil types. When the interaction effect is statistically significant, the researcher should not consider the effects of the individual factors without considering the interaction effect.

Two-Way Analysis of Variance

There are many different kinds of two-way ANOVA designs, depending on the number of levels of each variable. Figure 12–5 shows a few of these designs. As stated previously, the plant food–soil type experiment uses a 2 × 2 ANOVA. The design in Figure 12–5(a) is called a 3 × 2 design, since the factor in the rows has three levels and the factor in the columns has two levels. Figure 12–5(b) is a 3 × 3 design, since each factor has three levels. Figure 12–5(c) is a 4 × 3 design, since the factor in the rows has four levels and the factor in the columns has three levels.

FIGURE 12–5
Some Types of Two-Way ANOVA Designs



Two-Way Analysis of Variance

TABLE 12–4 ANOVA Summary Table for Plant Food and Soil Type

Source	Sum of squares	d.f.	Mean square	F
Plant food				
Soil type				
Interaction				
Within (error)				
Total				

In general, the two-way ANOVA summary table is set up as shown in Table 12–5.

TABLE 12–5 ANOVA Summary Table

Source	Sum of squares	d.f.	Mean square	F
A	SS_A	$a-1$	MS_A	F_A
B	SS_B	$b-1$	MS_B	F_B
$A \times B$	$SS_{A\times B}$	$(a-1)(b-1)$	$MS_{A\times B}$	$F_{A\times B}$
Within (error)	SS_W	$ab(n-1)$	MS_W	
Total				

Two-Way Analysis of Variance

In the table,

$$SS_A = \text{sum of squares for factor } A$$

$$SS_B = \text{sum of squares for factor } B$$

$$SS_{A\times B} = \text{sum of squares for interaction}$$

$$SS_W = \text{sum of squares for within-group term or error term}$$

$$a = \text{number of levels of factor } A$$

$$b = \text{number of levels of factor } B$$

$$n = \text{number of subjects in each group}$$

$$MS_A = \frac{SS_A}{a-1}$$

$$MS_B = \frac{SS_B}{b-1}$$

$$MS_{A\times B} = \frac{SS_{A\times B}}{(a-1)(b-1)}$$

$$MS_W = \frac{SS_W}{ab(n-1)}$$

$$F_A = \frac{MS_A}{MS_W} \qquad \text{with d.f.N.} = a-1,\ \text{d.f.D.} = ab(n-1)$$

$$F_B = \frac{MS_B}{MS_W} \qquad \text{with d.f.N.} = b-1,\ \text{d.f.D.} = ab(n-1)$$

$$F_{A\times B} = \frac{MS_{A\times B}}{MS_W} \qquad \text{with d.f.N.} = (a-1)(b-1),\ \text{d.f.D.} = ab(n-1)$$

Example

A researcher wishes to see whether the type of gasoline used and the type of automobile driven have any effect on gasoline consumption. Two types of gasoline, regular and high-octane, will be used, and two types of automobiles, two-wheel- and all-wheel drive, will be used in each group. There will be two automobiles in each group, for a total of eight automobiles used.

Using a two-way analysis of variance, determine if there is an interactive effect, an effect due to the type of gasoline used, and an effect due to the type of vehicle driven.

The data (in miles per gallon) are shown here, and the summary table is given in Table 12–6.

Example

Gas	Type of automobile	
	Two-wheel-drive	All-wheel-drive
Regular	26.7	28.6
	25.2	29.3
High-octane	32.3	26.1
	32.8	24.2

TABLE 12–6 ANOVA Summary Table for Example 12–5

Source	SS	d.f.	MS	F
Gasoline A	3.920			
Automobile B	9.680			
Interaction ($A \times B$)	54.080			
Within (error)	3.300			
Total	70.980			

Example

SOLUTION

Step 1 State the hypotheses. The hypotheses for the interaction are

H_0: There is no interaction effect between type of gasoline used and type of automobile a person drives on gasoline consumption.

H_1: There is an interaction effect between type of gasoline used and type of automobile a person drives on gasoline consumption.

The hypotheses for the gasoline types are

H_0: There is no difference between the means of gasoline consumption for two types of gasoline.

H_1: There is a difference between the means of gasoline consumption for two types of gasoline.

The hypotheses for the types of automobile driven are

H_0: There is no difference between the means of gasoline consumption for two-wheel-drive and all-wheel-drive automobiles.

H_1: There is a difference between the means of gasoline consumption for two-wheel-drive and all-wheel-drive automobiles.

Example

Step 2 Find the critical values for each F test. In this case, each independent variable, or factor, has two levels. Hence, a 2×2 ANOVA table is used. Factor A is designated as the gasoline type. It has two levels, regular and high-octane; therefore, $a = 2$. Factor B is designated as the automobile type. It also has two levels; therefore, $b = 2$. The degrees of freedom for each factor are as follows:

$$\text{Factor } A: \quad \text{d.f.N.} = a - 1 = 2 - 1 = 1$$

$$\text{Factor } B: \quad \text{d.f.N.} = b - 1 = 2 - 1 = 1$$

$$\begin{aligned}\text{Interaction } (A \times B): \quad \text{d.f.N.} &= (a - 1)(b - 1) \\ &= (2 - 1)(2 - 1) = 1 \cdot 1 = 1\end{aligned}$$

$$\begin{aligned}\text{Within (error)}: \quad \text{d.f.D.} &= ab(n - 1) \\ &= 2 \cdot 2(2 - 1) = 4\end{aligned}$$

where n is the number of data values in each group. In this case, $n = 2$.

The critical value for the F_A test is found by using $\alpha = 0.05$, d.f.N. = 1, and d.f.D. = 4. In this case, $F_A = 7.71$. The critical value for the F_B test is found by using $\alpha = 0.05$, d.f.N. = 1, and d.f.D. = 4; also F_B is 7.71. Finally, the critical value for the $F_{A \times B}$ test is found by using d.f.N. = 1 and d.f.D. = 4; it is also 7.71.

Note: If there are different levels of the factors, the critical values will not all be the same. For example, if factor A has three levels and factor b has four levels, and if there are two subjects in each group, then the degrees of freedom are as follows:

$$\text{d.f.N.} = a - 1 = 3 - 1 = 2 \qquad \text{factor } A$$

$$\text{d.f.N.} = b - 1 = 4 - 1 = 3 \qquad \text{factor } B$$

$$\begin{aligned}\text{d.f.N.} &= (a - 1)(b - 1) = (3 - 1)(4 - 1) \\ &= 2 \cdot 3 = 6 \qquad \text{factor } A \times B\end{aligned}$$

$$\text{d.f.D.} = ab(n - 1) = 3 \cdot 4(2 - 1) = 12 \qquad \text{within (error) factor}$$

The F Distribution

TABLE H *(continued)*

d.f.D.: degrees of freedom, denominator	$\alpha = 0.05$																
	d.f.N.: degrees of freedom, numerator																
	1	2	3	4	5	6	7	8	9	10	12	15	20	24	30	40	60
1	161.4	199.5	215.7	224.6	230.2	234.0	236.8	238.9	240.5	241.9	243.9	245.9	248.0	249.1	250.1	251.1	252.2
2	18.51	19.00	19.16	19.25	19.30	19.33	19.35	19.37	19.38	19.40	19.41	19.43	19.45	19.45	19.46	19.47	19.48
3	10.13	9.55	9.28	9.12	9.01	8.94	8.89	8.85	8.81	8.79	8.74	8.70	8.66	8.64	8.62	8.59	8.57
4	7.71	6.94	6.59	6.39	6.26	6.16	6.09	6.04	6.00	5.96	5.91	5.86	5.80	5.77	5.75	5.72	5.69
5	6.61	5.79	5.41	5.19	5.05	4.95	4.88	4.82	4.77	4.74	4.68	4.62	4.56	4.53	4.50	4.46	4.43
6	5.99	5.14	4.76	4.53	4.39	4.28	4.21	4.15	4.10	4.06	4.00	3.94	3.87	3.84	3.81	3.77	3.74
7	5.59	4.74	4.35	4.12	3.97	3.87	3.79	3.73	3.68	3.64	3.57	3.51	3.44	3.41	3.38	3.34	3.30
8	5.32	4.46	4.07	3.84	3.69	3.58	3.50	3.44	3.39	3.35	3.28	3.22	3.15	3.12	3.08	3.04	3.01
9	5.12	4.26	3.86	3.63	3.48	3.37	3.29	3.23	3.18	3.14	3.07	3.01	2.94	2.90	2.86	2.83	2.79
10	4.96	4.10	3.71	3.48	3.33	3.22	3.14	3.07	3.02	2.98	2.91	2.85	2.77	2.74	2.70	2.66	2.62
11	4.84	3.98	3.59	3.36	3.20	3.09	3.01	2.95	2.90	2.85	2.79	2.72	2.65	2.61	2.57	2.53	2.49
12	4.75	3.89	3.49	3.26	3.11	3.00	2.91	2.85	2.80	2.75	2.69	2.62	2.54	2.51	2.47	2.43	2.38
13	4.67	3.81	3.41	3.18	3.03	2.92	2.83	2.77	2.71	2.67	2.60	2.53	2.46	2.42	2.38	2.34	2.30
14	4.60	3.74	3.34	3.11	2.96	2.85	2.76	2.70	2.65	2.60	2.53	2.46	2.39	2.35	2.31	2.27	2.22
15	4.54	3.68	3.29	3.06	2.90	2.79	2.71	2.64	2.59	2.54	2.48	2.40	2.33	2.29	2.25	2.20	2.16
16	4.49	3.63	3.24	3.01	2.85	2.74	2.66	2.59	2.54	2.49	2.42	2.35	2.28	2.24	2.19	2.15	2.11
17	4.45	3.59	3.20	2.96	2.81	2.70	2.61	2.55	2.49	2.45	2.38	2.31	2.23	2.19	2.15	2.10	2.06
18	4.41	3.55	3.16	2.93	2.77	2.66	2.58	2.51	2.46	2.41	2.34	2.27	2.19	2.15	2.11	2.06	2.02
19	4.38	3.52	3.13	2.90	2.74	2.63	2.54	2.48	2.42	2.38	2.31	2.23	2.16	2.11	2.07	2.03	1.98
20	4.35	3.49	3.10	2.87	2.71	2.60	2.51	2.45	2.39	2.35	2.28	2.20	2.12	2.08	2.04	1.99	1.95

Example

Step 3 Complete the ANOVA summary table to get the test values. The mean squares are computed first.

$$MS_A = \frac{SS_A}{a-1} = \frac{3.920}{2-1} = 3.920$$

$$MS_B = \frac{SS_B}{b-1} = \frac{9.680}{2-1} = 9.680$$

$$MS_{A\times B} = \frac{SS_{A\times B}}{(a-1)(b-1)} = \frac{54.080}{(2-1)(2-1)} = 54.080$$

$$MS_W = \frac{SS_W}{ab(n-1)} = \frac{3.300}{4} = 0.825$$

The F values are computed next.

$$F_A = \frac{MS_A}{MS_W} = \frac{3.920}{0.825} = 4.752 \qquad \text{d.f.N.} = a - 1 = 1 \qquad \text{d.f.D.} = ab(n-1) = 4$$

$$F_B = \frac{MS_B}{MS_W} = \frac{9.680}{0.825} = 11.733 \qquad \text{d.f.N.} = b - 1 = 1 \qquad \text{d.f.D.} = ab(n-1) = 4$$

$$F_{A\times B} = \frac{MS_{A\times B}}{MS_W} = \frac{54.080}{0.825} = 65.552 \qquad \text{d.f.N.} = (a-1)(b-1) = 1 \qquad \text{d.f.D.} = ab(n-1) = 4$$

The completed ANOVA table is shown in Table 12–7.

Example

The completed ANOVA table is shown in Table 12–7.

TABLE 12–7 Completed ANOVA Summary Table for Example 12–5

Source	SS	d.f.	MS	*F*
Gasoline *A*	3.920	1	3.920	4.752
Automobile *B*	9.680	1	9.680	11.733
Interaction ($A \times B$)	54.080	1	54.080	65.552
Within (error)	3.300	4	0.825	
Total	70.980	7		

Step 4 Make the decision. Since $F_B = 11.733$ and $F_{A \times B} = 65.552$ are greater than the critical value 7.71, the null hypotheses concerning the type of automobile driven and the interaction effect should be rejected. Since the interaction effect is statistically significant, no decision should be made about the automobile type without further investigation.

Step 5 Summarize the results. Since the null hypothesis for the interaction effect was rejected, it can be concluded that the combination of type of gasoline and type of automobile does affect gasoline consumption.

Example

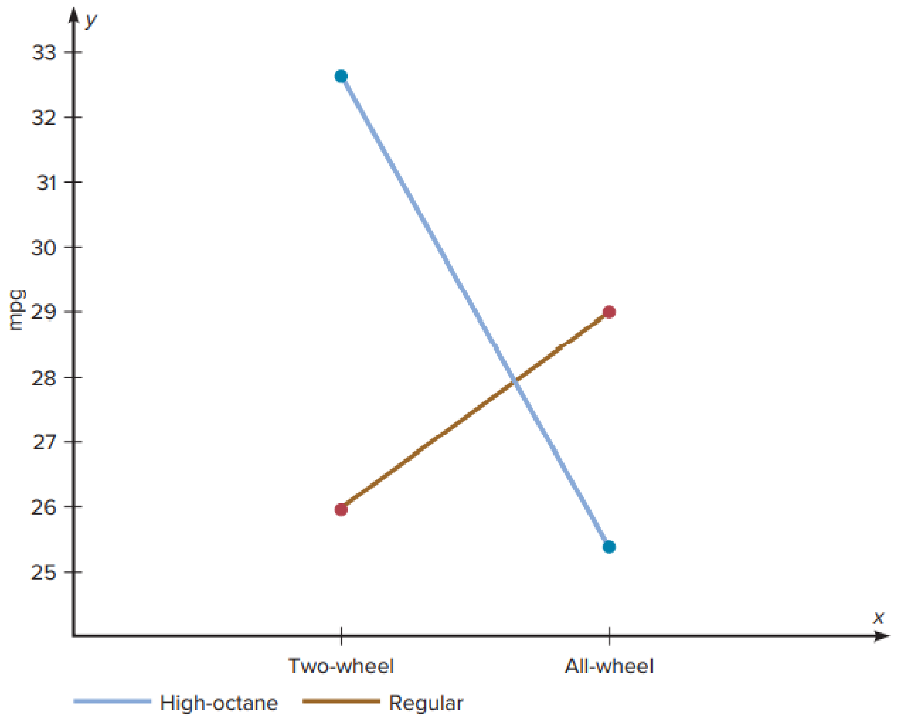
To interpret the results of a two-way analysis of variance, researchers suggest drawing a graph, plotting the means of each group, analyzing the graph, and interpreting the results. In Example 12–5, find the means for each group or cell. The means for each cell are shown in the chart here:

Gas	Type of automobile	
	Two-wheel-drive	All-wheel-drive
Regular	$\overline{X} = \frac{26.7 + 25.2}{2} = 25.95$	$\overline{X} = \frac{28.6 + 29.3}{2} = 28.95$
High-octane	$\overline{X} = \frac{32.3 + 32.8}{2} = 32.55$	$\overline{X} = \frac{26.1 + 24.2}{2} = 25.15$

Disordinal Interaction

The graph of the means for each of the variables is shown in Figure 12– 6. In this graph, the lines cross each other. When such an intersection occurs and the interaction is significant, the interaction is said to be a **disordinal interaction**. When there is a disordinal interaction, you should not interpret the main effects without considering the interaction effect. The other type of interaction that can occur is an **ordinal interaction**. Figure 12–7 shows a graph of means in which an ordinal interaction occurs between two variables.

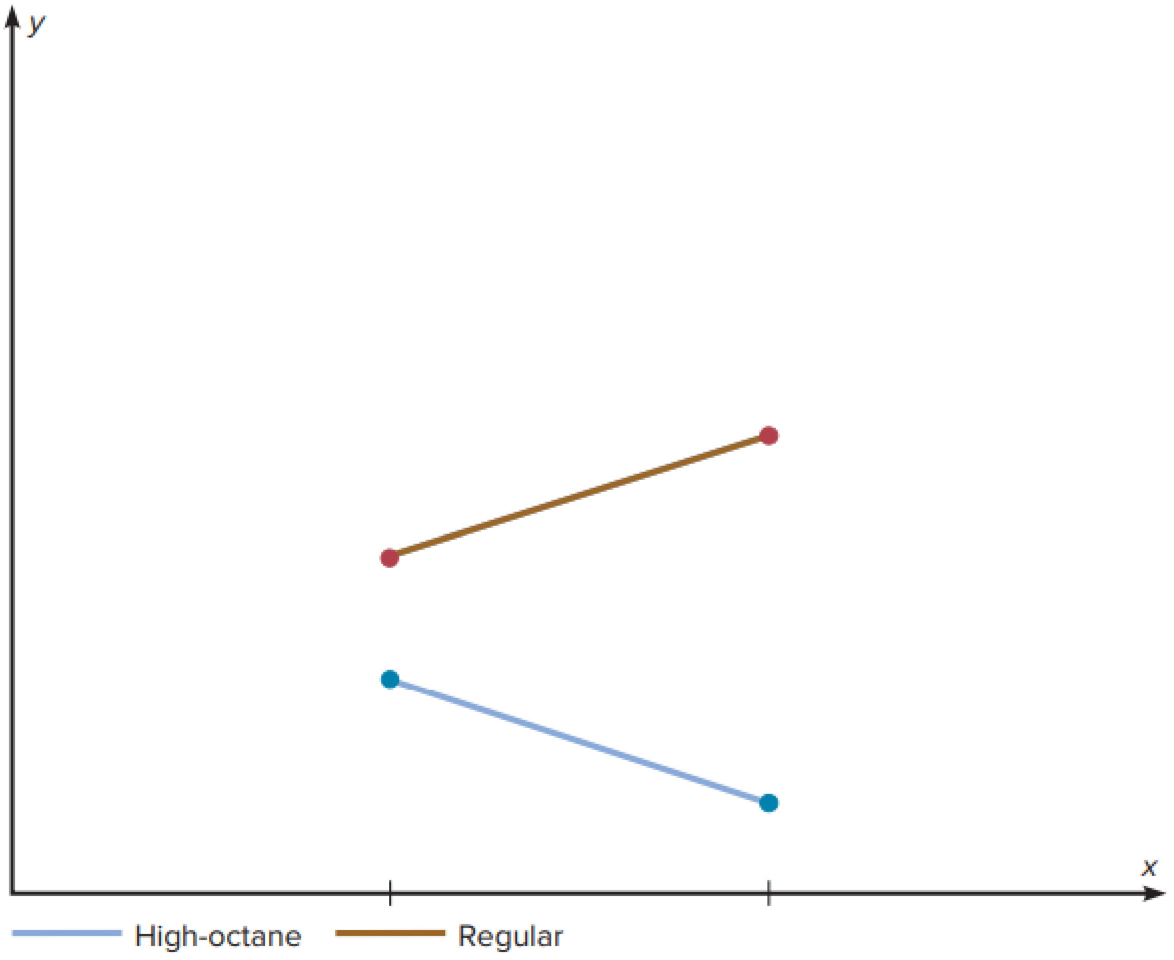
FIGURE 12–6
Graph of the Means of the Variables in Example 12–5



Ordinal Interaction

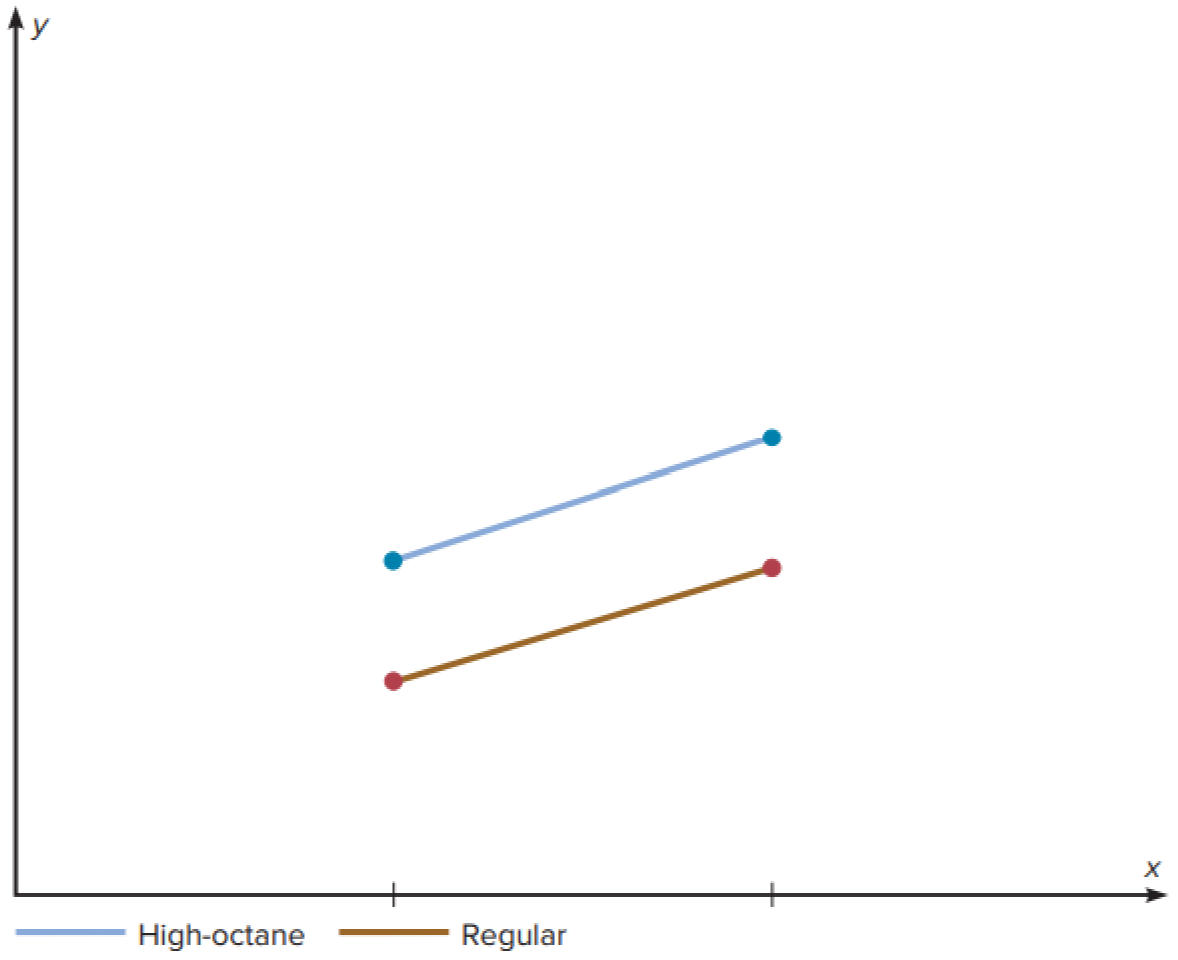
FIGURE 12–7

Graph of Two Variables Indicating an Ordinal Interaction



No Interaction

FIGURE 12–8
Graph of Two Variables
Indicating No Interaction



Two-Way Analysis of Variance

- If the lines do not cross each other, nor are they parallel and if the F-test value for the interaction is significant and the lines do not cross each other, then the interaction is said to be an **ordinal interaction** and the main effects can be interpreted independently of each other.
- Finally, when there is no significant interaction effect, the lines in the graph will be parallel or approximately parallel. When this situation occurs, the main effects can be interpreted independently of each other because there is no significant interaction.
- Figure 12–8 shows the graph of two variables when the interaction effect is not significant; the lines are parallel.
- Example 12–5 was an example of a 2 × 2 two-way analysis of variance since each independent variable had two levels. For other types of variance problems, such as a 3 × 2 or a 4 × 3 ANOVA, the interpretation of the results can be quite complicated.
- Procedures using tests such as the *Tukey* and *Scheffé* tests for analyzing the cell means to exist and are similar to the tests shown for the one-way ANOVA, but they are beyond the scope of this textbook. Many other designs for analysis of variance are available to researchers, such as three-factor designs and repeated-measure designs; they are also beyond the scope of this book.
- In summary, two-way ANOVA is an extension of one-way ANOVA. The former can be used to test the effects of two independent variables and a possible interaction effect on a dependent variable.

References

- **Elementary Statistics: A Step-by-Step Approach, Allen Bluman, 10th Edition, McGraw Hill, 2017, ISBN 13: 978-1-259-755330, Chapters 12.**