Invariance and Stability of Deep Convolutional Representations

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Understanding Deep Convolutional Representations

Are they stable to deformations?
How can we achieve invariance to transformation groups?
Do they preserve signal information?
How can we measure model complexity?

Kernel approach: construct functional space containing CNNs.

Why? Separate learning from representation: $f(x) = \langle f, \Phi(x) \rangle$

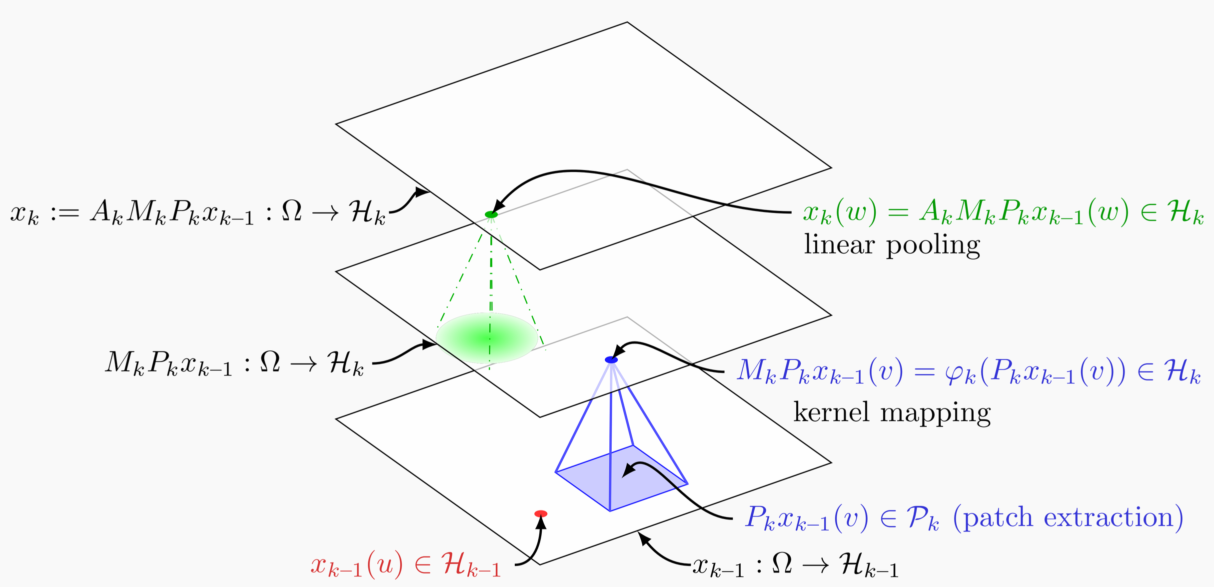
- $\Phi(x)$: CNN **architecture** (stability, invariance, signal preservation)
- f: CNN **model**, learning, generalization through $\|f\|$

$$|f(x) - f(x')| \leq \|f\| \cdot \|\Phi(x) - \Phi(x')\|$$

- $\|f\|$ **controls both stability and generalization**!

 $\rightarrow$ discriminating small deformations requires large $\|f\|$
 $\rightarrow$ learning stable functions is "easier"

Deep Convolutional Kernel Representation based on CKNs



- $x_0 : \Omega \to \mathcal{H}_0$: initial (**continuous**) signal
  - $u \in \Omega = \mathbb{R}^d$: location ($d = 2$ for images)
  - $x_0(u) \in \mathcal{H}_0$: value ($\mathcal{H}_0 = \mathbb{R}^3$ for RGB images)
- $x_k : \Omega \to \mathcal{H}_k$: *feature map* at layer k

$$x_k = A_k M_k P_k x_{k-1}$$

Patch extraction operator P_k.

Extract small patch of feature map x_{k-1} around each point u.

$$\|P_k x\| = \|x\|$$

Non-linear mapping operator M_k.

Pointwise non-linearity φ_k to each patch (kernel map).

$$\|M_k x\| \leq \|x\| \quad \text{and} \quad \|M_k x - M_k x'\| \leq \|x - x'\|$$

Holds for (tractable) **CKN approximations** by projection. [Mairal, 2016]
(Also holds for generic CNNs with spectral norm factor.)

Pooling operator A_k.

Linear Gaussian pooling at scale σ_k (typically exponential in k).

$$\|A_k x\| \leq \|x\|$$

Multilayer construction.

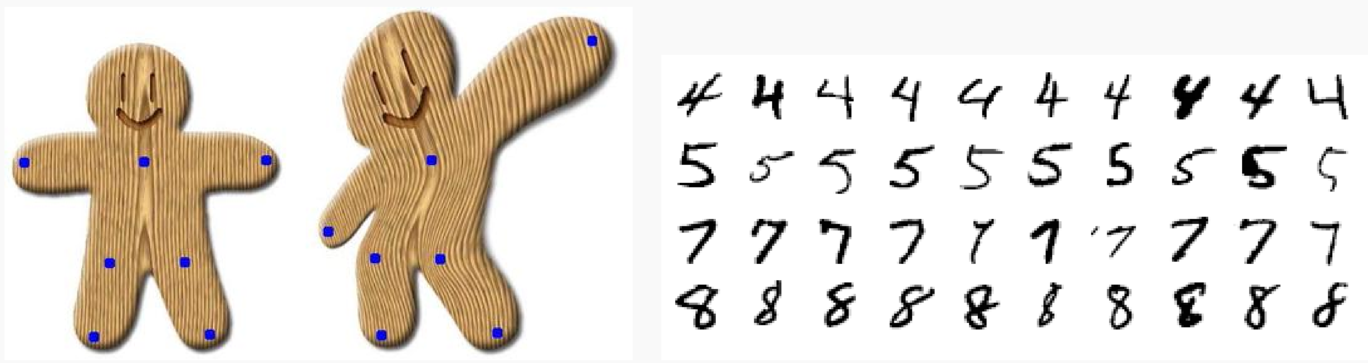
$$x_n := A_n M_n P_n A_{n-1} M_{n-1} P_{n-1} \cdots A_1 M_1 P_1 x_0 \ \in \ L^2(\Omega, \mathcal{H}_n)$$

Assume $x_0 = A_0 x$ (**anti-aliasing**), since x_0 is typically discrete

Invariance and Stability to Deformations

Deformations = ?

- $L_\tau x(u) = x(u - \tau(u))$: action of diffeomorphism $\tau : \Omega \to \Omega$
- Much richer group of transformations than translations



Definition of stability. [Mallat, 2012; Bruna and Mallat, 2013]

$\Phi(\cdot)$ is **stable to deformations** if

$$\|\Phi(L_\tau x) - \Phi(x)\| \leq (\underbrace{C\|\nabla\tau\|_\infty}_{\text{deformation}} + \underbrace{C'\|\tau\|_\infty}_{\text{translation}})\|x\|$$

Translation invariance. $L_c x(u) = x(u - c)$

- P_k, M_k, A_k commute with L_c: $\Box L_c = L_c \Box$

$$\|\Phi(L_c x) - \Phi(x)\| = \|L_c \Phi(x) - \Phi(x)\| \leq \|L_c A_n - A_n\| \cdot \|x\|$$

- Mallat [2012]: $\|L_\tau A_n - A_n\| \leq \frac{C_2}{\sigma_n}\|\tau\|_\infty$
- **Group invariance**: have P_k, A_k commute with $L_g x(u) = x(g^{-1}u)$
  - similar to [Cohen and Welling, 2016]
  - only need global pooling at last layer for global invariance

Stability to deformations.

- P_k and A_k do not commute with L_τ $\rightarrow$ study commutator $[\Box, L_\tau]$
- $[P_k, L_\tau]$ unstable at high frequencies $\rightarrow$ adapt to current resolution
- We show: if $\sup_{u \in S_k} |u| \leq \kappa\sigma_{k-1}$

$$\|[P_k A_{k-1}, L_\tau]\| \leq C_1 \|\nabla\tau\|_\infty$$

- C_1 grows as κ^{d+1} $\implies$ more stable with **small patches** (e.g. 3x3)

Theorem (**Stability**)

Let $\Phi_n(x) := A_n M_n P_n A_{n-1} M_{n-1} P_{n-1} \cdots A_1 M_1 P_1 A_0 x$.
If $\|\nabla\tau\|_\infty \leq 1/2$,

$$\|\Phi_n(L_\tau x) - \Phi_n(x)\| \leq \left(C_1 (1+n) \|\nabla\tau\|_\infty + \frac{C_2}{\sigma_n}\|\tau\|_\infty\right) \|x\|$$

Controlling stability.

- Full kernel method: $\|f\|_{\mathcal{H}_\mathcal{K}}$ (regularizer)
- CKN: $\|w_{n+1}\|_2$, ℓ_2 norm of last layer (regularizer)
- CNN: $\|w_{n+1}\|_2 \cdot \Pi_k \|W_k\|_2$ (??)

Signal Preservation for Kernel Representation

- Signal is preserved if discretized with subsampling $\leq$ patch size
- Recovery via linear measurements (need full kernel representation)
- For CKNs: depends on quality of kernel approximations

Model Complexity of CNNs

- RKHS contains CNNs with smooth homogeneous activations.
- RKHS norm controls generalization (complexity) and stability.

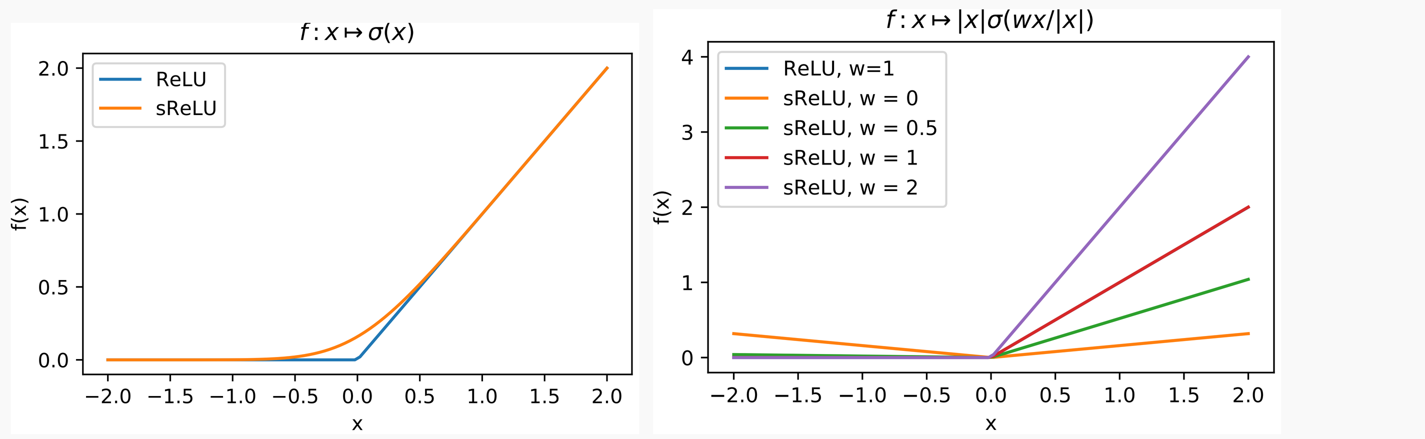
Patch kernels and their RKHS.

$$K_k(z, z') = \|z\|\|z'\|\kappa_k\left(\frac{\langle z, z'\rangle}{\|z\|\|z'\|}\right), \qquad \kappa_k(1) = 1$$

- e.g. Gaussian on sphere, inverse polynomial, etc.
- RKHS $\mathcal{H}_k$ contains, for smooth σ s.t. $C_\sigma^2(\|g\|^2) < \infty$,

$$f_g : z \mapsto \|z\|\sigma(\langle g, x\rangle / \|x\|) \qquad (\star)$$

- $\|f_g\|^2 \leq C_\sigma^2(\|g\|^2)$
- e.g. linear, polynomial, smooth ReLU
- Homogeneous version of [Zhang et al., 2016, 2017]



Construction of a CNN in the final RKHS.

- CNN f_σ with smooth homogeneous activations
- p_k feature maps at layer k, filters $w_k^{ij}(u)$, $W_k(u) := [w_k^{ij}(u)]_{ij}$
- Define intermediate ($\star$) functions (one per feature map)

Theorem (**RKHS norm of CNNs**)

The CNN function f_σ is in the RKHS $\mathcal{H}_\mathcal{K}$, with norm

$$\|f_\sigma\|^2 \leq p_n \sum_{i=1}^{p_n} \|w_{n+1}^i\|_2^2 B_{n,i},$$

where $B_{1,i} = C_\sigma^2(\|w_1^i\|_2^2)$ and $B_{k,i} = C_\sigma^2\left(p_{k-1}\sum_{j=1}^{p_{k-1}} \|w_k^{ij}\|_2^2 B_{k-1,j}\right)$.

Theorem (**RKHS norm using spectral norms**)

The CNN function f_σ is in the RKHS $\mathcal{H}_\mathcal{K}$, with norm

$$\|f_\sigma\|^2 \leq \|w_{n+1}\|^2 \, C_\sigma^2(\|W_n\|_2^2 \dots C_\sigma^2(\|W_2\|_2^2 \, C_\sigma^2(\|W_1\|_F^2)) \dots)$$

$\rightarrow$ **generalization** with Rademacher complexity and margin bounds.

Relevant References

- S. Mallat (2012).
  Group invariant scattering.
- Y. Zhang, P. Liang, and M. J. Wainwright (2017).
  Convexified convolutional neural networks.
- P. Bartlett, D. J. Foster, and M. Telgarsky (2017).
  Spectrally-normalized margin bounds for neural networks.