Graph Coloring Variants: List Coloring (and a little Correspondence Coloring)

Applied Graph Theory

Abigail Nix & Ari Holcombe Pomerance

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Section outline

List Coloring

Definition

Definition 1

For a graph G, a **list assignment** L assigns each vertex $v \in V(G)$ a set $L(v)$ of colors allowed at v. An L-**coloring** is a proper coloring ϕ of G such that $\phi(v) \in L(v)$ for all v. A graph G is k-**choosable** or **list** k-**colorable** if it has an L-coloring whenever $|L(v)| \geq k$ for all v. The **list chromatic number** or **choice number** or **choosability** $\chi_l(G)$ is the minimum k such that G is k-choosable.

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- In a scheduling application, each committee provides a certain number of times they can meet, where colors are times, and we have an edge between two committees if they share a person.

Example: C_{2m}

Even cycles C_{2m} are 2-choosable.

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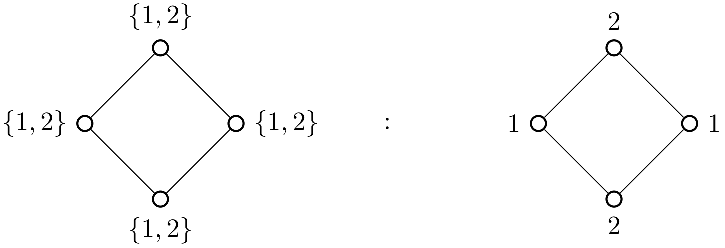
- **Case 1:** If all lists are equal, then we can simply alternate colors along the cycle:



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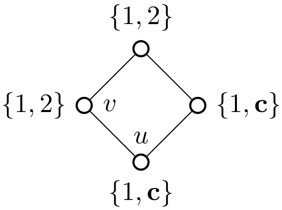
- **Case 2:** There exists a pair of adjacent vertices u, v such that $\mathbf{c} \in L(u)$ and $\mathbf{c} \notin L(v)$.



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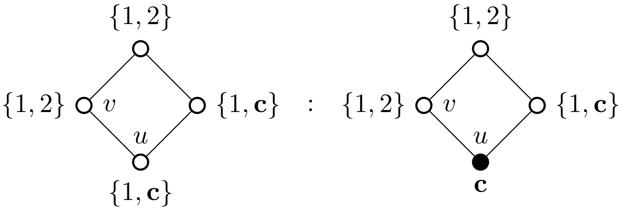

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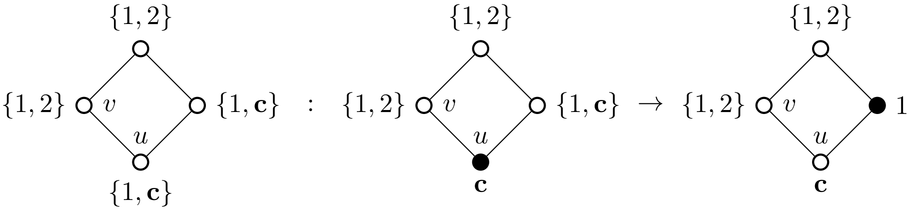
- Then proceed through the rest of the vertices, assigning each of them a color that was not used on the previous vertex.



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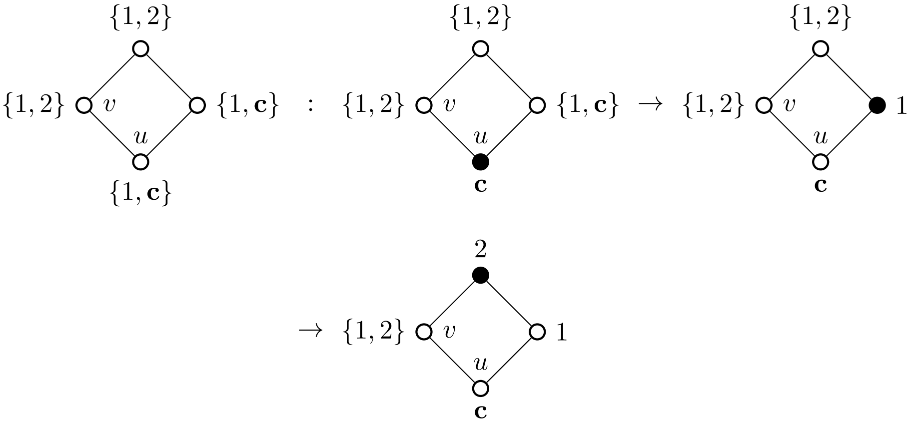

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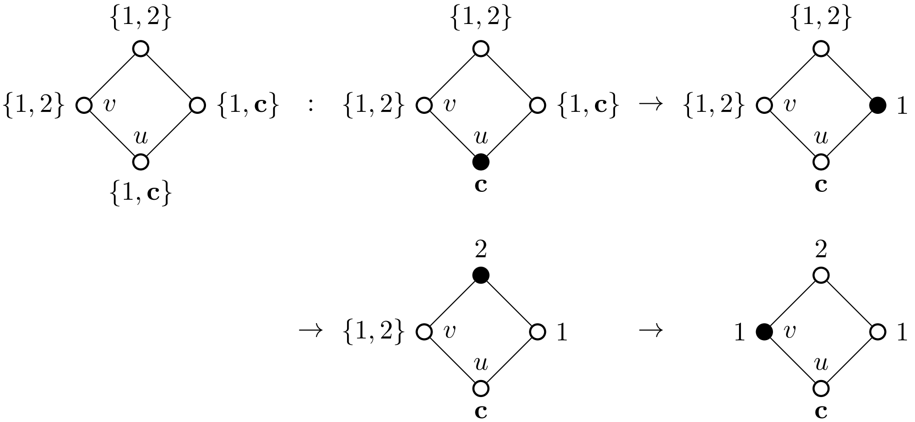
- Assign $\mathbf{c}$ to u.
- Then proceed through the rest of the vertices, assigning each of them a color that was not used on the previous vertex.
- Finally, since $\mathbf{c} \notin L(v)$, we know there is at least one choice of color for v.



Example: $K_{m,m}$

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- $\chi_l(G) \geq \chi(G)$, and $\chi_l(G) > \chi(G)$ for particular graphs
- Complete bipartite graphs $K_{m,m}$ are **not** 2-choosable.

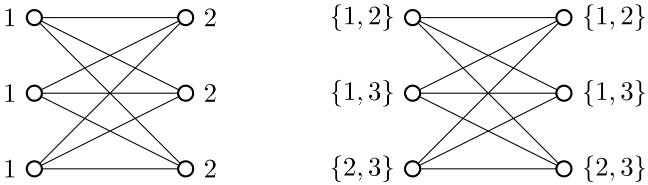

Figure: $K_{3,3}$ is 2-colorable but not 2-choosable.

List Extension of Brooks' Theorem

Degeneracy Bounds

- Recall greedy coloring $\implies \chi(G) \leq \Delta(G) + 1$ because for any ordering of $V(G)$ each vertex has at most $\Delta(G)$ already colored neighbors
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Definition 2

A graph G is k-**degenerate** if every subgraph has a vertex of degree at most k. The **degeneracy** of G is $\max_{H \subseteq G} \Delta(H)$; that is, the minimum k such that G is k-degenerate.

Proposition 1

Every k-degenerate graph is $(k+1)$-choosable. Thus also $\chi_l(G) \leq 1 + \max_{H \subseteq G} \delta(H) \leq 1 + \Delta(G)$.

PreLemmanaries

Lemma 1

Given a connected graph G, let L be a list assignment such that $|L(v)| \geq d(v)$ for all v.

(a) If $|L(y)| > d(y)$ for some vertex y, then G is L-colorable.

(b) If G is 2-connected and some two lists differ, then G is L-colorable.

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Proof.

- For (a), root a spanning tree at y and color towards y (starting with leaves). Each vertex v in this order other than y will have an uncolored neighbor ($< d(v)$ colored neighbors), so v has an available color in its list. Vertex y could have $d(y)$ colored neighbors, but it still has an available color.

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- For (b), find adjacent x, y such that $L(x) - L(y) \neq \emptyset$ (possible since G connected). Choose $c \in L(x) - L(y)$. Define lists for $G - x$ as $L'(v) = L(v)$ if $v \notin N(x)$, $L'(v) = L(v) - c$ if $v \in N(x)$. Then, $|L'(v)| \geq d_{G-x}(v) \forall v \in V(G - x)$ and $|L'(y)| > d_{G-x}(y)$ (since $c \notin L(y)$). By part (a), color $G - x$, then use color c on x to extend to a coloring of G. $\square$

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Definition 3

A graph G is f-**choosable** if it is L-colorable whenever $|L(v)| \geq f(v)$ for each vertex v, where $f : V(G) \to \mathbb{N}$. The graph is **degree-choosable** if it is L-colorable whenever $|L(v)| \geq d(v)$ for each vertex v.

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Lemma 2

If a connected graph G has a degree-choosable induced subgraph H, then G is degree-choosable.

Rubin's Block Theorem

The following structural result will be helpful to prove our final theorem.

Lemma 3 (Erdős-Rubin-Taylor (1979))

Every 2-connected graph G that is not a complete graph or odd cycle has an even cycle with at most one chord.

Graph Blocks

Definition 4

A **block** of a graph G is a maximal connected subgraph of G that has no cut-vertex.

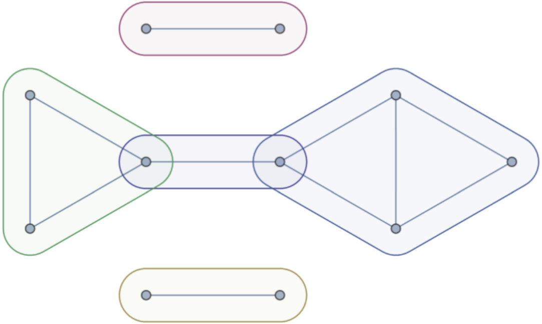

Figure: The blocks of a graph G (Source: Wolfram MathWorld)

Non-Degree Choosable Graphs

Theorem 1 (Borodin (1977), Erdős-Rubin-Taylor (1979))

If graph G is not degree-choosable, then every block of G is a complete graph or an odd cycle.

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Proof idea.

- Any block B that is neither a complete graph nor an odd cycle has a degree-choosable subgraph H, which is an even cycle with at most one chord by Rubin's Block Theorem (3).
- Even cycles are 2-choosable. For even cycles with a chord, color the same way as we color even cycles, using an extra color at one endpoint v of the chord. So H is degree-choosable, since $d(v) = 3$.
- By Lemma 2, this would imply G is degree-choosable. But since we know G is *not* degree-choosable, we conclude that every block must be a complete graph or an odd cycle. □

List Extension of Brooks' Theorem

Corollary 1

(List Extension of **Brooks' Theorem**) If a connected graph G is not a complete graph or an odd cycle, then $\chi_l(G) \leq \Delta(G)$.

Proof.

- Instead, prove the contrapositive. Suppose $\chi_l(G) > \Delta(G)$. By Proposition 1, G is not $(\Delta(G) - 1)$-degenerate, so $\delta(H) \geq \Delta(G)$ for some induced subgraph H, i.e. H is $\Delta(G)$-regular.
- G is connected and vertices of degree $\Delta(G)$ in H cannot have neighbors outside H, so $H = G$.
- By Theorem 1, every block of G is a complete graph or odd cycle. G is regular, so we have one block (cut-vertex separating a block from the rest of the graph would have higher degree) $\implies G$ is a complete graph or an odd cycle. □

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- By Theorem 1, every block of G is a complete graph or odd cycle. G is regular, so we have one block (cut-vertex separating a block from the rest of the graph would have higher degree) $\implies$ G is a complete graph or an odd cycle. □

- $\chi(G) \leq \chi_l(G)$ for every graph G, so this corollary implies Brooks' Theorem, that for connected G, $\chi_l(G) = \Delta(G) + 1$ if and only if G is a complete graph or an odd cycle.

Correspondence Coloring

Correspondence Coloring (Dvořák and Postle, 2016)

- Generalization of list coloring, introduced as a way to allow for vertex identification in proofs
- Instead of two adjacent vertices not being able to receive the **same** color, we establish a correspondence between the lists of vertices to determine what colors are forbidden at each vertex

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A **correspondence assignment** for a graph G consists of a list assignment L and a function C that to every edge $vw \in E(G)$ assigns a partial matching C_{vw} between $L(v)$ and $L(w)$.

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Now G is (L, C)-**colorable** if such an (L, C)-coloring exists.
The **correspondence chromatic number** (or DP chromatic number), $\chi_{corr}(G)$ is the minimum k such that G is (L, C)-colorable whenever $|L(v)| \geq k$ for all $v \in V(G)$.

Example

- If $|L(v)| = k$ for each vertex v, we can find an equivalent correspondence where $L(v) = [k]$ for all $v \in V(G)$

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- Just as we can have $\chi_l(G) > \chi(G)$, we can also have $\chi_{corr}(G) > \chi_l(G)$.

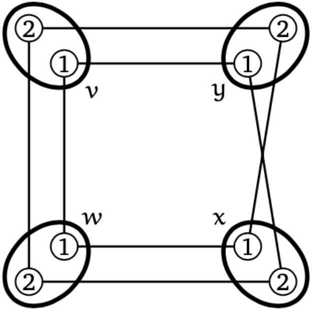

Figure: C_4 has $\chi_{corr} > 2$, even though even cycles are 2-choosable [1].

Thank you! (and references)

[1] Daniel W. Cranston.
Graph Coloring Methods.
[Self-published], 2024.

[2] Douglas B. West.
Combinatorial Mathematics.
Cambridge University Press, 2021.