The Out-of-Kilter Algorithm

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Outline [1]

1. Optimality Conditions

2. Out-of-Kilter Algorithm

3. Correctness and Complexity

[1] Ahuja et al. (1993)

Optimality Conditions

Negative Cycle Optimality Conditions

Theorem

A feasible flow x^* is optimal for a minimum cost flow problem if and only if the residual network $G(x^*)$ contains no negative cycle.

Reduced Cost Optimality Conditions

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Theorem

A feasible flow x^* is optimal for a minimum cost flow problem if and only if there exists a set of node potentials π that satisfy $c_{ij}^{\pi} \geq 0$ for every arc (i, j) in the residual network $G(x^*)$.

Complementary Slackness Optimality Conditions

Theorem

A feasible flow x^* is optimal for a minimum cost flow problem if and only if there exists a set of node potentials π such that:

1. *if* $c^\pi_{ij} > 0$, *then* $x^*_{ij} = 0$
2. *if* $0 < x^*_{ij} < u_{ij}$, *then* $c^\pi_{ij} = 0$
3. *if* $c^\pi_{ij} < 0$, *then* $x^*_{ij} = u_{ij}$

Out-of-Kilter Algorithm

Kilter Numbers

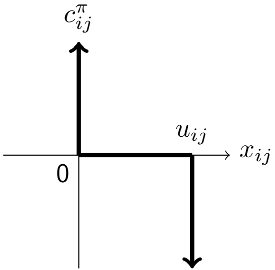

Figure: The kilter diagram for arc (i, j).

Complementary Slackness conditions

x^* is optimal if and only if the following hold:

1. if $c_{ij}^{\pi} > 0$, then $x_{ij}^* = 0$
2. if $0 < x_{ij}^* < u_{ij}$, then $c_{ij}^{\pi} = 0$
3. if $c_{ij}^{\pi} < 0$, then $x_{ij}^* = u_{ij}$

- arcs that satisfy complementary slackness optimality conditions are "in-kilter", those that don't are "out-of-kilter"

Kilter Numbers

Definition

The kilter number of an arc (i, j), k_{ij}, with current flow x_{ij} and reduced cost c^{π}_{ij}, is defined as how much x_{ij} needs to be changed without changing c^{π}_{ij} in order to make (i, j) in-kilter.

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We make a simplifying assumption that the algorithm starts with a feasible flow, i.e., $0 \leq x_{ij} \leq u_{ij}$ for all (i, j). So, with $r_{ij} = u_{ij} - x_{ij}$, we can redefine k_{ij} as

$$k_{ij} = \begin{cases} 0 & \text{if } c^{\pi}_{ij} \geq 0 \\ r_{ij} & \text{if } c^{\pi}_{ij} < 0. \end{cases} \tag{1}$$

The Out-of-Kilter Algorithm

Algorithm 1 Out-Of-Kilter Algorithm for finding a Minimum Cost Flow

procedure OutOfKilter(G)
    $\pi := 0$
    find feasible flow x ▷ using a Max Flow Algorithm
    define residual network $G(x)$
    for $(i, j) \in G(x)$ **do**
        $k_{ij} := \text{Kilter}((i, j))$
    end for
    while there exists $(i, j) \in G(x)$ with $k_{ij} > 0$ **do** ▷ there is an out-of-kilter arc
        select out-of-kilter arc $(p, q) \in G(x)$
        define length of each arc $(i, j) \in G(x)$ as $\max\{0, c^{\pi}_{ij}\}$
        let $d(i)$ denote the shortest path distance from q to i for each $i \in G(x) - \{(q, p)\}$
        let P denote the shortest path from q to p
        $\pi'(i) := \pi(i) - d(i)$ for each node i in $G(x)$
        if $c^{\pi'}_{pq} < 0$ **then**
            $W := P \cup \{(p, q)\}$
            $\delta := \min\{r_{ij} : (i, j) \in W\}$
            augment δ units of flow along W
            update x with new feasible flow, residual network $G(x)$, and reduced costs $c^{\pi'}_{ij}$
        end if
    end while
    return x ▷ x is the minimum cost flow
end procedure

Correctness and Complexity

Correctness

Lemma

Updating node potentials, i.e., setting $\pi'(i) := \pi(i) - d(i)$ for all nodes i, does not increase the kilter number of any arc in $G(x)$.

Correctness

Lemma

Augmenting flow along $W = P \cup \{(q,p)\}$ *does not increase the kilter number of any arc in* $G(x)$*, and strictly decreases* k_{pq}*, the kilter number of arc* (p,q)*.*

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- only arcs whose kilter numbers can change are those along W (or their reversed arcs)
- P is a shortest path with arc lengths $\max\{0, c^\pi_{ij}\}$, so for $(i,j) \in P$,

$$d(j) = d(i) + \max\{0, c^\pi_{ij}\} \geq d(i) + c^\pi_{ij}.$$

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- $\pi' = \pi - d$, so

$$c^{\pi'}_{ij} = c_{ij} - (\pi(i) - d(i)) + (\pi(j) - d(j)) = c^{\pi}_{ij} + d(i) - d(j) \leq 0.$$

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$$d(j) = d(i) + \max\{0, c_{ij}^{\pi}\} \geq d(i) + c_{ij}^{\pi}.$$
- $\pi' = \pi - d$, so
$$c_{ij}^{\pi'} = c_{ij} - (\pi(i) - d(i)) + (\pi(j) - d(j)) = c_{ij}^{\pi} + d(i) - d(j) \leq 0.$$
- Augment flow along W by $\delta = \min\{r_{ij} : (i,j) \in W\}$, so new flow is feasible. Thus, if $c_{ij}^{\pi'} = 0$, k_{ij} stays 0, and if $c_{ij}^{\pi'} < 0$, k_{ij} decreases since r_{ij} decreases.

Correctness

Proof.

- augmenting flow along (i, j) may introduce the arc (j, i) into the new residual network, but since $c_{ij}^{\pi'} \leq 0$, then $c_{ji}^{\pi'} \geq 0$, meaning $k_{ji} = 0$, and (j, i) is in-kilter.

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- For arc (p,q), we only augment flow if $c_{pq}^{\pi'} < 0$.
- Since $c_{pq}^{\pi'} < 0$, $k_{pq} = r_{pq}$. Thus, increasing flow on (p,q) strictly decreases k_{pq} since $r_{pq} = u_{pq} - x_{pq}$ is strictly decreased.

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- Note $c_{qp}^{\pi'} = -c_{pq}^{\pi'} > 0$, so (q,p) is in-kilter.

Therefore, the flow augmentation step of the algorithm does not increase the kilter number of any arc, strictly decreases the kilter number of (p,q), and only adds in-kilter arcs to the updated residual network. □

Complexity

- Maximum possible kilter number is $U = \max\{u_{ij}\}$ since

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- In each iteration, we solve a shortest path problem. Using Dijkstra's, this takes $O(n^2)$, but could be different for different algorithms.
- So, Out-of-Kilter takes at most $O(mUn^2)$ time, pseudopolynomial!

General Out-of-Kilter Algorithm

- We started with a feasible flow, but could also start with flow $x = 0$, whether or not this is feasible.
- In general, OOK algorithm maintains mass balance constraints at every iteration, but may disrespect flow bounds and complementary slackness optimality conditions.
- At each iteration, the algorithm increases both feasibility and optimality.

Thank you!

References

Ahuja, R. K., Magnanti, T. L., and Orlin, J. B. (1993). *Network Flows: Theory, Algorithms, and Applications*. Prentice Hall.

Durbin, E. P. and Kroenke, D. (1967). *The Out-of-Kilter Algorithm: A Primer*. RAND Corporation, Santa Monica, CA.